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Degrees of Freedom of the Interference Channel with a Cognitive Helper

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***Abstract*—In this letter, we characterize the degrees of freedom (DoF) of the $K \geq 3$ user Gaussian interference network with a cognitive helper where each node is equipped with only one antenna. Specifically, each user sends one independent message to its corresponding receiver through its own antenna and via the help of the cognitive helper. For this network, we show that the sum DoF value is outer bounded by $(K+1)/2$ when K is odd and $K^2/(2(K+1))$ when K is even, respectively. The new DoF outer bounds are derived based on the fact that collaboration among users does not decrease the capacity region and increasing the number of users does not increase the capacity per user. In addition, we provide a new achievable scheme to achieve a total of $(K+1)/2$ DoF for any $K \geq 3$. Thus, the exact DoF value of the network is characterized with the total DoF given as $(K+1)/2$, whenever K is odd. The new achievable scheme is based on interference neutralization and asymptotic interference alignment.**

I. Introduction

Exploring the fundamental capacity limit is a key objective in the request for understanding wireless communications networks, and pointing to the development direction of practical techniques. In order to capture the essence of the capacity limit, we are primarily interested in the degrees of freedom (DoF) characterization of wireless networks. The notion of DoF, also known as the number of independent signaling dimensions, is of great significance for understanding the high signal-to-noise power ratio (SNR) behavior of the capacity. By revealing the most essential aspects of the communication problem, DoF investigations have generated many fundamental ideas such as interference alignment [11], [6], deterministic channel models [10], rational dimensions [15], [7], aligned interference neutralization [9], subspace alignment chains [3], and genie chains [4]. While DoF characterizations have recently been obtained for a wide variety of wireless networks, in this letter we are primarily interested in an interference network with a cognitive helper, a setup which has not been considered before.

For the K-user interference channel where each node has only one antenna, Cadambe, et. al. have shown in [6] that for time-varying/frequency-selective channels the total DoF is $K/2$. While the DoF outer bound is the simple cooperation outer bound, the most significant contribution of [6] is an linear asymptotic interference alignment scheme, i.e., the [CJ08] scheme. For constant-valued channels on the other hand, the DoF remained open until recently Motahari el. al. show in [7] that it has a total of $K/2$ DoF as well, by essentially mimicking the [CJ08] linear alignment scheme over the rational dimensions. Therefore, no matter if the channel coefficients are constant-valued or not, the K-user interference channel with single antenna at each node has a total of $K/2$ DoF.

A. Motivation

Many advances in wireless transmission have rested on the use of multiple antennas for transmission and reception. Multiple antenna systems fundamentally provide an increase in the number of DoF that can be exploited by a system for transmission. While the DoF result of single-user setting are known, we are interested in the multiuser setting. For example, for the K-user SIMO/MISO Gaussian interference channels where there are 2 *independent* antennas at each receiver/transmitter, the channel has a total of $2K/3$ DoF for the $K > 2$ setting [1], which is strictly greater than $K/2$.

In contrast to having multiple (and thus centralized) antennas at each node, a multiple antenna scenario can also be created by sharing antennas among users. If there are dependencies among the channel coefficients aroused by antenna sharing, it remains open in general whether the DoF value of the wireless network is affected. The DoF problems of this scenario have been investigated in [14] [13], etc. In particular, Sreekanth et. al. in [13] consider the DoF of the K-user interference channel with cognitive messages sharing and clustered decoding. For example, for the $K = 4$ user SISO interference network where each message is transmitted from one and the next indexed transmitters and each message is decoded from the observation of its own received signal, the network has a total of $8/3$ DoF [13], which are identical to the MISO setting [1], i.e., channel dependencies do not translate to DoF loss. However, if the value of K is beyond 4, a simple cooperation DoF outer bound implies that the DoF value is strictly less than $2K/3$, i.e., DoF are lost due to channel dependencies aroused by antenna sharing.

On the other hand, let us consider another scenario where antenna sharing is not among ordered pairwise users, but through an additional helping node. Specifically, consider a K-user interference network, as shown in Fig. 1, where each message is available at its own transmitter and all messages are available at the cognitive helper through noiseless orthogonal links. Equivalently, this model can be seen as a special MISO interference channel where every transmitter, having two antennas, shares one *common* antenna of the helper while the other is a *private* antenna. The question of interest is whether the channel dependencies aroused by antenna sharing in such a way affect the DoF of this network.

B. Prior Work

Recently, Chaaban et. al. considered the DoF for the $K = 3$ setting of the network in Fig. 1 in [8], and have shown that each user is able to achieve $2/3$ DoF, i.e., for a total of $2K/3$ DoF with $K = 3$. The general $K > 3$ user case is also considered in [8], for which an outer bound on the total DoF value given by $2K/3$ is derived. The DoF outer bound essentially follows the K-user SIMO/MISO interference channel DoF outer bound and it is claimed in [8] that this outer bound is achievable, although there are correlations among channel coefficients due to the antenna sharing at the cognitive helper. That is to say, the aroused channel correlation does not give rise to the DoF loss, compared to SIMO/MISO interference channels.

C. Contribution

In this letter, we disprove the claim in [8] that the outer bound of $2K/3$ is achievable in general, by showing that for the $K \geq 3$ user Gaussian interference network with a cognitive helper, as shown in Fig. 1, the sum DoF value is bounded above by $\frac{K+1}{2}$ and $\frac{K^2}{2(K+1)}$ when K is odd and even, respectively. Note that when $K > 4$, the values of our new DoF outer bounds are strictly less than $2K/3$ which is claimed to be achievable in [8]. Our new DoF outer bounds are derived using the facts that collaboration among users does not decrease the capacity region and increasing the number of users does not increase the capacity per user. On the other hand, we provide a new achievable scheme, which is based on interference neutralization and interference alignment, to achieve $\frac{K+1}{2}$ sum DoF for any $K \geq 3$. Therefore, we establish the DoF result of the network in Fig. 1 when K is odd.

II. System Model and Definitions

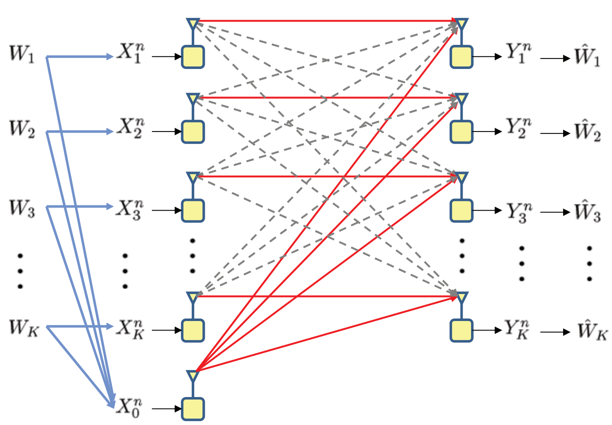

Fig. 1. K-user interference channel with a cognitive helper (The solid red and dashed lines denote the desired message and interference carrying links, respectively.)

We begin by specifying the assumptions for the K-user interference channel with a cognitive helper. As shown in Fig. 1, each node has only one antenna. Transmitter k sends one independent message W_k to its desired receiver where $k \in \{1, 2, 3 \cdots, K\} \triangleq \mathcal{K}$. Transmitter 0 is a cognitive helper where all the messages of other users $W_k,\ k \in \mathcal{K}$ are available through orthogonal noiseless links (blue lines) in Fig. 1. At the receiver side, receiver k wants to decode its own message W_k. While our results are valid regardless of whether the channel coefficients are constant or varying in time/frequency, in this letter we assume block fading channels, meaning that the channel coefficients are drawn from a continuous distribution in each block and change independently to other values in the next. For constant channels where the channel coefficients remain fixed during the entire transmission, all the results we obtain in this letter remain the same by using rational alignment framework[1]. We assume that all the channel coefficients are perfectly known to all nodes in the network.

At the time index $t \in \mathbb{Z}^+$, for $i \in \mathcal{K} \cup \{0\}$, transmitter i sends a complex-valued signal $X_i(t)$, which satisfies an average power constraint $\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}[|X_i(t)|^2] \leq \rho$ for T channel uses where ρ is referred to as the SNR. At the receiver side, receiver j observes an complex signal $Y_j(t)$ at time index t, which is given by:

$$Y_j(t) = \sum_{i=0}^{K} H_{ji}(t)X_i(t) + Z_j(t), \qquad j \in \mathcal{K} \tag{1}$$

where $H_{ji}(t)$ is the channel coefficient from transmitter i to receiver j at time t. The term $Z_j(t)$ represents the independent identically distributed (i.i.d.) zero mean unit variance circularly symmetric complex Gaussian noise at receiver j.

The capacity region $\mathcal{C}(\rho)$ of the network in Fig. 1 is a set of achievable rate tuples $\mathbf{R}(\rho) = (R_1(\rho), \cdots, R_K(\rho))$ such that each user can simultaneously decode its own message with arbitrarily small error probability. The maximum sum rate of this channel is defined as $R_\Sigma(\rho) = \max_{\mathbf{R}(\rho)\in\mathcal{C}(\rho)} \sum_{k=1}^{K} R_k(\rho)$. The capacity in the high SNR regime can be characterized through the DoF, i.e., $d_k \triangleq \lim_{\rho\to\infty} R_k(\rho)/\log\rho$, and the total DoF $d_\Sigma = \sum_{k=1}^{K} d_k$. In this letter, we define the message set $\mathcal{W}_\mathcal{S} \triangleq \{W_k : k \in \mathcal{S} \subset \mathcal{K}\}$. Also, we denote $\bar{X}_\mathcal{S}$ as the signal vector

$$\bar{X}_\mathcal{S} = \left[X_{\mathcal{S}_1}\ X_{\mathcal{S}_2} \cdots X_{\mathcal{S}_{|\mathcal{S}|}}\right]^T \tag{2}$$

where the subscript $\mathcal{S}_k$ represents the k^{th} element of the ordered set $\mathcal{S}$. For brevity, we let $\mathbb{Z}_+ = \mathbb{Z}^+ \cup \{0\}$, and we define $\mathcal{S}^c$ as the complement of $\mathcal{S}$ in the set $\mathcal{K}$. Moreover, we use $o(x)$ to represent any function $f(x)$ such that $\lim_{x\to\infty} f(x)/x = 0$.

III. Main Results

In this section, we present our DoF results of the network that we defined in Section II, and defer all the technical proofs to the next two sections. Since the result of $K = 2$ user case has been shown by Sridharan et. al. in [12], we consider the $K \geq 3$ setting in this letter.

[1]The notion of rational alignment is introduced in [7], which essentially mimics the linear alignment over the vector subspaces in rational dimensions.

Lemma 1: (Outer Bound) For the $K \geq 3$ user interference channel with a cognitive helper that we defined in Section II, the total DoF value is outer bounded by

$$d_\Sigma \leq \begin{cases} \frac{K+1}{2} & \text{if } K \text{ is odd,} \\ \frac{K^2}{2(K-1)} & \text{if } K \text{ is even.} \end{cases} \tag{3}$$

Proof: The converse proof is based on the fact that collaborating users do not decrease the channel capacity region and increasing the number of users does not increase the DoF value per user. The proof is presented in detail in Section IV.

Lemma 2: (Inner Bound) For the $K \geq 3$ user interference channel with a cognitive helper that we defined in Section II, a total of $\frac{K+1}{2}$ DoF are achievable almost surely.

Proof: The achievability proof is based on interference neutralization and asymptotic interference alignment. The proof is presented in Section V in detail.

Theorem 1: For the $K \geq 3$ user interference channel with a cognitive helper that we defined in Section II, if K is odd, then this network has a total number of DoF $d_\Sigma = \frac{K+1}{2}$ almost surely.

Proof: This theorem follows directly from Lemma 1 and Lemma 2.

IV. DoF Converse: Proof of Lemma 1

In this section, we provide a new information theoretic DoF outer bound of the network that we defined in Section II. We first show the intuition of the converse and then translate it into an information theoretic statement.

A. Intuition of the Converse

First, suppose that K is an odd number. In this case, we allow collaboration among the receivers $1, 2, \cdots, \frac{K+1}{2}$ as one node, denoted by $\mathcal{S} = \{1, 2, \cdots, \frac{K+1}{2}\}$. Since collaborating among antennas does not decrease the capacity region, the DoF outer bound of the new network still applies to the original channel that we consider. Given the reliable communication assumption, each receiver is able to decode its own message with arbitrarily small error probability. Therefore, receivers in $\mathcal{S}$ first decode messages $\mathcal{W}_\mathcal{S}$ and then remove the signals caused by $\bar{X}_\mathcal{S}$ from the received signal vector $\bar{Y}_\mathcal{S} = [Y_1 \cdots Y_{\frac{K+1}{2}}]^T$, to obtain the $\frac{K+1}{2}$ dimensional observation from the transmitters $\mathcal{S}^c \cup \{0\}$, i.e., a total of $\frac{K+1}{2}$ antennas. Therefore, by inverting the channel matrix from these antennas to the receivers in $\mathcal{S}$, we are able to reconstruct the signals sent from those transmit antennas, such that the receivers in $\mathcal{S}$ can decode the messages $\mathcal{W}_{\mathcal{S}^c}$ subject to the noise distortion[2]. Since receivers $\mathcal{S}$ having a total of $\frac{K+1}{2}$ antennas decode all messages subject to the noise distortion, we have the sum DoF outer bound $d_\Sigma \leq |\mathcal{S}| = \frac{K+1}{2}$.

Next, suppose that K is an even number. We have argued that the sum DoF are outer bounded by $\frac{K}{2}$ for the $K-1$ user case. Also, note that increasing the number of users cannot increase the DoF per user. Thus, for the K user case, the sum DoF $d_\Sigma \leq \frac{K}{2} \times \frac{K}{K-1} = \frac{K^2}{2(K-1)}$.

B. Information Theoretic DoF Converse Proof

We are going to translate the intuition given above into an information theoretic statement in this section.

First consider that K is odd. For each receiver k, using Fano's inequality and owing to the reliable communications assumption, we have

$$H(W_k|Y_k^n) \leq n\ o(n), \qquad k \in \mathcal{K}. \tag{4}$$

Consider an arbitrary subset $\mathcal{S} \subset \mathcal{K}$ with cardinality $|\mathcal{S}| = (K+1)/2$. According to the entropy chain rule and the fact that conditioning does not increase entropy, we obtain

$$H(\mathcal{W}_\mathcal{S}|\bar{Y}_\mathcal{S}^n) \leq \sum_{k\in\mathcal{S}} H(W_k|Y_k^n) \leq |\mathcal{S}|n\ o(n), \tag{5}$$

which means that the messages $\mathcal{W}_\mathcal{S}$ can be decoded by receivers $\mathcal{S}$. In addition, the messages $\mathcal{W}_{\mathcal{S}^c}$ can be decoded using all the received signals:

$$H(\mathcal{W}_{\mathcal{S}^c}|\mathcal{W}_\mathcal{S}, \bar{Y}_\mathcal{K}^n) \leq \sum_{k\in\mathcal{S}^c} H(W_k|Y_k^n) \leq |\mathcal{S}^c|n\ o(n). \tag{6}$$

Our goal is to show that the messages $\mathcal{W}_{\mathcal{S}^c}$ can be decoded by the receivers in $\mathcal{S}$ as well subject to the noise distortion. In order to do this, we will show that the transmit signals $\bar{X}_{\mathcal{S}^c}$ and X_0, which carry the messages $\mathcal{W}_{\mathcal{S}^c}$, can be reconstructed at the receivers in $\mathcal{S}$ subject to the noise distortion. Specifically, given the received signals $\bar{Y}_\mathcal{S}^n$ and messages $\mathcal{W}_{\mathcal{S}^c}$ as the side information, we consider the entropy contributed by the messages $\mathcal{W}_{\mathcal{S}^c}$ as follows:

$$\begin{aligned} & H(\mathcal{W}_{\mathcal{S}^c}|\mathcal{W}_\mathcal{S}, \bar{Y}_\mathcal{S}^n) \\ \leq\ & H(\mathcal{W}_{\mathcal{S}^c}|\mathcal{W}_\mathcal{S}, \bar{Y}_\mathcal{S}^n) - H(\mathcal{W}_{\mathcal{S}^c}|\mathcal{W}_\mathcal{S}, \bar{Y}_\mathcal{K}^n) + |\mathcal{S}^c|n\ o(n) && (7) \\ \leq\ & I(\mathcal{W}_{\mathcal{S}^c}; \bar{Y}_{\mathcal{S}^c}^n|\mathcal{W}_\mathcal{S}, \bar{Y}_\mathcal{S}^n) + |\mathcal{S}^c|n\ o(n) && (8) \\ \leq\ & h(\bar{Y}_{\mathcal{S}^c}^n|\mathcal{W}_\mathcal{S}, \bar{Y}_\mathcal{S}^n) - h(\bar{Y}_{\mathcal{S}^c}^n|\mathcal{W}_{\mathcal{S}^c}, \mathcal{W}_\mathcal{S}, \bar{Y}_\mathcal{S}^n) + |\mathcal{S}^c|n\ o(n) && (9) \\ \leq\ & h(\bar{Y}_{\mathcal{S}^c}^n|\bar{X}_\mathcal{S}^n, \bar{Y}_\mathcal{S}^n) - h(\bar{Z}_{\mathcal{S}^c}^n) + |\mathcal{S}^c|n\ o(n) && (10) \end{aligned}$$

where (7) follows from (6), and (10) is obtained because the signals $\bar{Y}_{\mathcal{S}^c}^n$ can be reconstructed by using all messages $\mathcal{W}_\mathcal{K}$, subject to distortion by the noise term $\bar{Z}_{\mathcal{S}^c}^n$.

Note that for each channel use, we have the following signal relation:

$$\bar{Y}_\mathcal{S} = \mathbf{H}_{\mathcal{S},\mathcal{S}}\bar{X}_\mathcal{S} + \mathbf{H}_{\mathcal{S},\mathcal{S}^c\cup\{0\}}\bar{X}_{\mathcal{S}^c\cup\{0\}} + \bar{Z}_\mathcal{S}, \tag{11a}$$

$$\bar{Y}_{\mathcal{S}^c} = \mathbf{H}_{\mathcal{S}^c,\mathcal{S}}\bar{X}_\mathcal{S} + \mathbf{H}_{\mathcal{S}^c,\mathcal{S}^c\cup\{0\}}\bar{X}_{\mathcal{S}^c\cup\{0\}} + \bar{Z}_{\mathcal{S}^c}, \tag{11b}$$

where $\mathbf{H}_{\mathcal{A},\mathcal{B}}$ denotes the $|\mathcal{A}| \times |\mathcal{B}|$ channel matrix from the transmitters $\mathcal{B}$ to receivers $\mathcal{A}$. Note that since $|\mathcal{S}| = |\mathcal{S}^c \cup \{0\}| = \frac{K+1}{2}$, the chancel matrix $\mathbf{H}_{\mathcal{S},\mathcal{S}^c\cup\{0\}}$ is square. Since it contains generic random variables, it is invertible almost surely. Therefore, with the first term in (10) in mind, we eliminate the contributions of $\bar{X}_\mathcal{S}$ and $\bar{Y}_\mathcal{S}$ from $\bar{Y}_{\mathcal{S}^c}$ and obtain:

$$\begin{aligned} \bar{Z}^* \triangleq\ & \bar{Y}_{\mathcal{S}^c} - \mathbf{H}_{\mathcal{S}^c,\mathcal{S}}\bar{X}_\mathcal{S} \\ & -[\mathbf{H}_{\mathcal{S}^c,\mathcal{S}^c}\ \mathbf{H}_{\mathcal{S},0}][\mathbf{H}_{\mathcal{S},\mathcal{S}^c}\ \mathbf{H}_{\mathcal{S},0}]^{-1}(\bar{Y}_\mathcal{S} - \mathbf{H}_{\mathcal{S},\mathcal{S}}\bar{X}_\mathcal{S}) \end{aligned}$$

[2]We use the phrase "subject to noise distortion" to indicate the widely used DoF outer bound argument whereby reducing noise at a node by an amount that is SNR independent (and therefore inconsequential for DoF) allows it to decode a message.

$$= \bar{Z}_{\mathcal{S}^c} - [\mathbf{H}_{\mathcal{S}^c,\mathcal{S}^c}\ \mathbf{H}_{\mathcal{S},0}][\mathbf{H}_{\mathcal{S},\mathcal{S}^c}\ \mathbf{H}_{\mathcal{S},0}]^{-1}\bar{Z}_{\mathcal{S}} \tag{12}$$

which is a noise term independent of the SNR. Thus, substituting (12) into (10), we obtain:

$$H(\mathcal{W}_{\mathcal{S}^c}|\mathcal{W}_{\mathcal{S}}, \bar{Y}^n_{\mathcal{S}}) \leq h(\bar{Z}^*) - h(\bar{Z}^n_{\mathcal{S}^c}) + |\mathcal{S}^c|n\ o(n). \tag{13}$$

Moreover, we have the following inequality:

$$\begin{aligned} H(\mathcal{W}_{\mathcal{K}}) &= I(\mathcal{W}_{\mathcal{S}}, \mathcal{W}_{\mathcal{S}^c}; \bar{Y}^n_{\mathcal{S}}) + H(\mathcal{W}_{\mathcal{S}}, \mathcal{W}_{\mathcal{S}^c}|\bar{Y}^n_{\mathcal{S}}) && (14)\\ &\leq h(\bar{Y}^n_{\mathcal{S}}) - h(\bar{Y}^n_{\mathcal{S}}|\mathcal{W}_{\mathcal{K}}) + H(\mathcal{W}_{\mathcal{S}}|\bar{Y}^n_{\mathcal{S}}) \\ &\quad + H(\mathcal{W}_{\mathcal{S}^c}|\mathcal{W}_{\mathcal{S}}, \bar{Y}^n_{\mathcal{S}}) && (15)\\ &\leq h(\bar{Y}^n_{\mathcal{S}}) - h(\bar{Z}^n_{\mathcal{S}}) + |\mathcal{S}|n\ o(n) + h(\bar{Z}^*) \\ &\quad - h(\bar{Z}^n_{\mathcal{S}^c}) + |\mathcal{S}^c|n\ o(n) && (16)\\ &\leq n|\mathcal{S}|\log(\rho) + n\ o(\log(\rho)) + Kn\ o(n) && (17) \end{aligned}$$

where (16) follows from (5) and (13). In other words, we obtain the sum rate inequality

$$nR_{\Sigma} \leq n|\mathcal{S}|\log(\rho) + n\ o(\log(\rho)) + Kn\ o(n). \tag{18}$$

By dividing by n on both sides of the inequality (18) and letting $n \to \infty$, we have:

$$R_{\Sigma} \leq |\mathcal{S}|\log(\rho) + o(\log(\rho)). \tag{19}$$

Finally, dividing the $\log(\rho)$ term on both sides of the inequality above and letting $\rho \to \infty$ produces the DoF outer bound:

$$d_{\Sigma} \leq |\mathcal{S}| = \frac{K+1}{2}. \tag{20}$$

Next suppose that K is even. Note that we already proved that the sum DoF value of arbitrary $K-1$ out of K users is outer bounded by $K/2$. That is,

$$\sum_{k\in\mathcal{S}'} d_k \leq \frac{K}{2}, \qquad \forall\ \mathcal{S}' \subset \mathcal{K} \text{ with } |\mathcal{S}'| = K-1. \tag{21}$$

By selecting $\mathcal{S}' = \{k\}^c$ for $k \in \mathcal{K}$, we have the following inequalities:

$$d_{\Sigma} - d_k \leq \frac{K}{2}, \qquad k \in \mathcal{K}. \tag{22}$$

Adding up the K inequalities above, we have

$$\begin{aligned} \sum_{k=1}^{K}(d_{\Sigma} - d_k) &= Kd_{\Sigma} - d_{\Sigma} \leq \frac{K^2}{2} \\ \Longrightarrow d_{\Sigma} &\leq \frac{K^2}{2(K-1)}. && (23) \end{aligned}$$

So far, we finished the proof of the DoF outer bound given in Lemma 1.

V. DoF Achievability: Proof of Lemma 2

In this section, we present our DoF achievability result given in Lemma 2. The achievable scheme we propose in this letter is based on interference neutralization and asymptotic interference alignment. In the following, we first introduce the interference neutralization step to convert the original channel $\{H_{ji}\}$ to an effective channel $\{G_{ji}\}$, and then present the interference alignment scheme for the effective channel to achieve the desired DoF.

A. Interference Neutralization Step

As introduced in section II, each message $W_k,\ k \in \mathcal{K}$ is available at two transmitters, i.e., transmitter k and the cognitive helper 0. We first consider the $K-1$ messages $\mathcal{W}_{\{1\}^c}$, i.e., from W_2 to W_K, at receiver 1. Our goal in this step is to neutralize the signals carrying these messages at receiver 1. Basically, this can be done by sending each message W_k from transmitter k and the cognitive helper 0 using a linear beamforming vector such that the signals carrying the message W_k has null projection at receiver 1. Specifically, suppose $X_{0k}(W_k)$ is the signal carrying the message W_k sent from helper 0, then it means that

$$H_{1k}X_k(W_k) + H_{10}X_{0k}(W_k) = 0, \quad k \in \{1\}^c \tag{24}$$

which produces $X_{0k}(W_k) = -H_{1k}/H_{10}X_k(W_k)$ for $k \neq 1$. After these operations, receiver 1 is interference free, and the desired signals carrying the message W_1 are sent from transmitter 1 and helper 0. Note that at helper 0, we use superposition coding such that

$$X_0 = \sum_{i=1}^{K} X_{0i}(W_i). \tag{25}$$

Next, let us consider signals associated with user $i,\ i \in \{1\}^c$. Each receiver i hears the desired signal X_i from transmitter i and X_{0i} from helper 0. According to (24), the signal desired at each receiver i is given by

$$\begin{aligned} H_{ii}X_i + H_{i0}X_{0i} &= H_{ii}X_i + H_{i0}\left(-\frac{H_{1i}}{H_{10}}X_i\right) \\ &= \left(H_{ii} - \frac{H_{i0}H_{1i}}{H_{10}}\right)X_i. && (26) \end{aligned}$$

Similarly, for $j \in \{1\}^c$, each receiver j observes interference X_i from transmitter i and X_{0i} from helper 0 where $i \in \{1, j\}^c$. That is to say, the interference carrying the message W_i at receiver j is given by

$$\begin{aligned} H_{ji}X_i + H_{j0}X_{0i} &= H_{ji}X_i + H_{j0}\left(-\frac{H_{1i}}{H_{10}}X_i\right) \\ &= \left(H_{ji} - \frac{H_{j0}H_{1i}}{H_{10}}\right)X_i. && (27) \end{aligned}$$

Notice that besides messages $\mathcal{W}_{\{1,j\}^c}$, each receiver j also sees interference carrying the message W_1 from transmitter 1 and helper 0, i.e., X_1 and $X_{01}(W_1)$.

Now for $j \in \mathcal{K}$ and $i \in \mathcal{K} \cup \{0\}$, we define an effective channel $\{G_{ji}\}$ such that

$$\begin{cases} G_{ji} \triangleq H_{ji} & j \in \mathcal{K},\ i = 0, 1, \\ G_{ii} \triangleq H_{ii} - \frac{H_{i0}H_{1i}}{H_{10}} & i \in \{1\}^c, \\ G_{ji} \triangleq H_{ji} - \frac{H_{j0}H_{1i}}{H_{10}} & j \in \{1\}^c,\ i \in \{1, j\}^c. \end{cases} \tag{28}$$

Using this effective channel, the received signal at time t at receiver 1 is given by

$$Y_1(t) = \underbrace{G_{11}(t)X_1(t) + G_{10}X_{01}(t)}_{\text{desired signal}} + Z_1(t), \tag{29}$$

while at each receiver $j \in \{1\}^c$ we have

$$Y_j(t) = \underbrace{G_{jj}(t)X_j(t)}_{\text{desired signal}} + \underbrace{\sum_{i=1, i\neq j}^{K} G_{ji}X_i(t) + G_{j0}X_{01}}_{\text{interference}} + Z_j(t). \quad (30)$$

So far, we converted the original channel $\{H_{ji}\}$ to the effective channel $\{G_{ji}\}$ as shown in Fig. 2, where each transmit signal $X_i(W_i),\ i \in \mathcal{K}$ is a mapping function of only its own message W_i. The signal of the cognitive helper is $X_{01}(W_1)$ which is a function of the message W_1 only.

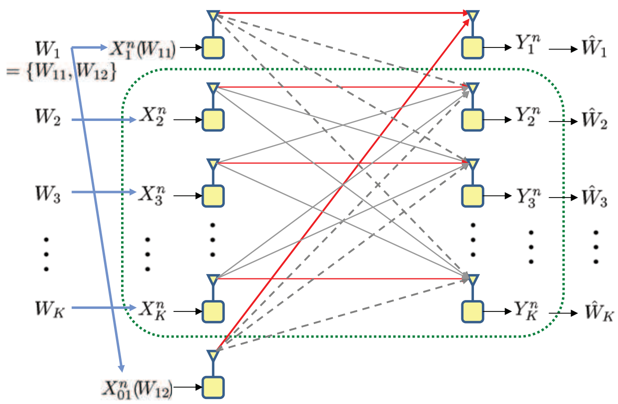

Fig. 2. The Effective Network after Interference Neutralization

B. Interference Alignment Step

In this section, we will show that the effective channel $\{G_{ji}\}$ that we obtain in (28) by using interference neutralization offers a total of $(K+1)/2$ DoF.

As shown in Fig. 2, let us first consider the users sending messages $\mathcal{W}_{\{1\}^c}$, i.e., those included in the green dashed rectangle. Note that in the absence of the message W_1, the remaining $K-1$ users comprise of a fully connected $K-1$ user interference channel. Thus, intuitively each user is able to achieve $1/2$ DoF by using interference alignment. Moreover, we split the message W_1 into two sub-messages W_{11} and W_{12}, such that the signal $X_1(W_1) = X_1(W_{11})$ is a coding function of W_{11} only, and the signal $X_{01}(W_1) = X_{01}(W_{12})$ is a coding function of W_{12} only. We let each of the messages W_{11} and W_{12} carry $1/2$ DoF. Since receiver 1 is free of interference, it is able to decode W_{11} and W_{12} as long as X_1 and X_{01} are distinguishable. In addition, at receiver $j \in \{1\}^c$, in order to protect the desired signals, the interference X_1 and X_{01} need to be aligned into the interference subspaces spanned by the interference of users i where $i \in \{1, j\}^c$. Intuitively, this can be done by using the [CJ08] asymptotic alignment proposed in [6]. Since each of the messages $W_2, W_3, \cdots, W_K$ carries $1/2$ DoF and $W_1 = \{W_{11}, W_{12}\}$ carries 1 DoF, we can achieve a total of $(K+1)/2$ DoF. Based on this intuition, we will present the rigorous proof as follows.

Let $N = (K-1)K$ and consider $L = L_1 + L_2$ symbol extensions in the time domain where

$$L_1 = \binom{n-1+N}{N}, \qquad L_2 = \binom{n+N}{N}, \quad (31)$$

then each user has an L dimensional signal space. Our goal is to show that each transmitter is able to send L_1 symbols in the L dimensional signal space. In order to do this, we use the same $L \times L_1$ beamforming matrix $\mathbf{V}$ to send each message from the corresponding transmitter, i.e., for transmitter $k \in \mathcal{K} \cup \{0\}$, we have

$$\mathbf{X}_k = \mathbf{V}[x_{k1} \ \cdots \ x_{kL_1}]^T \quad (32)$$

where $x_{kl},\ l \in \{1, 2, \cdots, L_1\}$ is the l^{th} data stream sent from transmitter k. At the receiver side, since receiver 1 is interference free, we only need to consider interference alignment at receivers 2 to K. Denoting by span$(\mathbf{U})$ the column space of the $L \times L_2$ matrix $\mathbf{U}$, we require that the interference at each receiver spans a signal subspace which is contained in span$(\mathbf{U})$. Thus, we have the interference alignment constraints:

$$\text{span}(\mathbf{G}_{ji}\mathbf{V}) \subset \text{span}(\mathbf{U}), \qquad j \in \{1\}^c,\ i \neq j \quad (33a)$$

$$\lim_{n\to\infty} \frac{|\mathbf{V}|}{|\mathbf{U}|} = 1. \quad (33b)$$

Using the asymptotic alignment scheme proposed by Cadambe et. al. in [6], the solution of (33) is given by:

$$\mathbf{V} = \left\{ \prod_{j=2}^{K}\prod_{i\neq j} \mathbf{G}_{ji}^{\alpha_{ji}}\mathbf{1} : \ \sum_{j=2}^{K}\sum_{i\neq j}\alpha_{ji} \leq n-1,\ \alpha_{ji} \in \mathbb{Z}_+ \right\},$$

$$\mathbf{U} = \left\{ \prod_{j=2}^{K}\prod_{i\neq j} \mathbf{G}_{ji}^{\alpha_{ji}}\mathbf{1} : \ \sum_{j=2}^{K}\sum_{i\neq j}\alpha_{ji} \leq n,\ \alpha_{ji} \in \mathbb{Z}_+ \right\}. \quad (34)$$

It can be easily verified that the solution in (34) satisfies the alignment conditions in (33).

What remains to be shown is that: (a) at each receiver, the interference signal subspace and the desired signal subspace have null intersection almost surely; (b) at receivers $j \neq 1$, the L_1 desired symbols are distinguishable, and at receiver $j = 1$, the $2L_1$ desired symbols are distinguishable.

Regarding the first issue, since receiver 1 is interference free, we only need to consider receiver $j \neq 1$. At receiver $j \neq 1$, the desired signal subspace is given by the column space of

$$\mathbf{G}_{jj}\mathbf{V} = (\mathbf{H}_{jj} - \mathbf{H}_{j0}\mathbf{H}_{1j}\mathbf{H}_{10}^{-1})\mathbf{V}, \qquad j \neq 1. \quad (35)$$

Notice that each effective channel matrix $\mathbf{G}_{ji}$ is a diagonal matrix (due to symbol extension in the time domain) and can be represented as

$$\mathbf{G}_{ji} = \begin{bmatrix} G_{ji}(1) & & & \\ & G_{ji}(2) & & \\ & & \ddots & \\ & & & G_{ji}(L) \end{bmatrix}$$

where $G_{jj}(n) = H_{jj}(n) - \frac{H_{j0}(n)H_{1j}(n)}{H_{10}(n)}$ for $1 \leq n \leq L$, and $\mathbf{H}_{ji}$ is also with the same type, i.e., a diagonal matrix with $H_{ji}(n)$ as its n^{th} diagonal entry. Notice that $\mathbf{H}_{jj}$ in (35) is with generic diagonal entries and does not appear in the alignment conditions (33). Thus, the column spaces of $\mathbf{G}_{22}\mathbf{V}$ and $\mathbf{U}$ have null intersection almost surely.

Regarding the second issue, it suffices to show that the desired signal subspace has full rank almost surely. Recall that the construction of $\mathbf{V}$ is associated with $G_{ji}(n),\ j \neq 1,\ i \neq j$ only. Owing to the effective channel construction in (28) and since $H_{ji}(n),\ j \neq 1,\ i \neq j$ are generic, the matrix $\mathbf{V}$ has full rank almost surely. Thus, we establish that the L_1 desired symbols are distinguishable at receivers $j \neq 1$. Moreover, at receiver 1, we need to ensure that the desired signal sent from transmitter 1 and helper 0, i.e., the column spaces of $\mathbf{G}_{11}\mathbf{V}$ and $\mathbf{G}_{10}\mathbf{V}$ have null intersection almost surely. Essentially, this can be easily verified as $\mathbf{G}_{11} = \mathbf{H}_{11}$ which does not appear in any other $\mathbf{G}_{ji}$ where $(j,i) \neq (1,1)$.

With this, we established the achievability of $(K+1)/2$ sum DoF, as shown in Lemma 2.

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References

[1] T. Gou and S. A. Jafar, "Degrees of Freedom of the K User $M \times N$ MIMO Interference Channel", *IEEE Transactions on Information Theory*, Dec. 2010, Vol. 56, Issue: 12, pp. 6040–6057.

[2] A. Ghasemi, A. Motahari, A. Khandani, "Interference Alignment for the K User MIMO Interference Channel," *arXiv:0909.4604*, Sep. 2009.

[3] C. Wang, T. Gou, S. Jafar, "Subspace Alignment Chains and the Degrees of Freedom of the Three-User MIMO Interference Channel", *e-print arXiv:1109.4350*, Sept. 2011.

[4] C. Wang, H. Sun, S. Jafar, "Genie Chains and the Degrees of Freedom of the K-User MIMO Interference Channel", *in preparation.*

[5] S. Jafar, M. Fakhereddin, "Degrees of Freedom for the MIMO Interference Channel," *IEEE Transactions on Information Theory*, July 2007, Vol. 53, No. 7, Pages: 2637-2642.

[6] V. Cadambe and S. Jafar, "Interference alignment and the degrees of freedom of the K user interference channel", *IEEE Trans. on Information Theory*, vol. 54, pp. 3425–3441, Aug. 2008.

[7] A. Motahari, S. Gharan, M. A. Maddah-Ali, A, Khandani, "Real Interference Alignment: Exploiting the Potential of Single Antenna Systems", *arxiv.org/pdf/0908.2282*, August 2009.

[8] A. Chaaban and A. Sezgin, "The DoF of the K-user Interference Channel with a Cognitive Relay", *Proceedings of IEEE ISIT 2012*, July 2012.

[9] Tiangao Gou, Syed A. Jafar, Sang-Woon Jeon, Sae-Young Chung, "Aligned Interference Neutralization and the Degrees of Freedom of the 2x2x2 Interference Channel", *e-print arXiv:1012.2350*, December 2010.

[10] S. Avestimehr, S. Diggavi, and D. Tse, "deterministic approach to wireless relay networks", *Allerton Conf. Commun., Control, Comput.*, Monticello, IL, Sep. 2007.

[11] M.A. Maddah-Ali, A.S. Motahari , and A.K. Khandani, "Communication Over MIMO X Channels: Interference Alignment, Decomposition, and Performance Analysis," IEEE Transaction on Information Theory, Vol. 54, No. 8, pp. 3457-3470, Aug. 2008.

[12] S. Sridharan, S. Vishwanath, S. A. Jafar, and S. Shamai, "On the capacity of cognitive relay assisted Gaussian interference channel," *in Proc. of IEEE ISIT*, Toronto, Ontario, Canada, Jul. 2008.

[13] V. S. Annapureddy, Aly El Gamal and V. V. Veeravalli, "Degrees of Freedom of Interference Channels with CoMP Transmission and Reception," *e-print arXiv:1109.5396v1*, September 2011.

[14] A. Lapidoth, N. Levy, S. Shamai, and M. Wigger, "A cognitive network with clustered decoding," *in Proc. IEEE ISIT 2009*, Seoul, Korea, 2009, pp. 596 C 600.

[15] R. Etkin and E. Ordentlich, "On the Degrees-of-Freedom of the K-User Gaussian Interference Channel", *http://arxiv.org/abs/0901.1695*, Jan. 2009.