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# Area Law Unification and the Holographic Event Horizon

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## Abstract

We prove a new, large family of area laws in general relativity, which apply to certain classes of untrapped surfaces that we dub generalized holographic screens. Our family of area laws contains, as special cases, the area laws for marginally-trapped surfaces (holographic screens) and the event horizon (Hawking's area theorem). In addition to these results in general relativity, we show that in the context of holography the geometry of a generalized holographic screen is related to the outer entropy of the screen. Specifically, we show for spherically-symmetric spacetimes that the area of the largest HRT surface consistent with the outer wedge can be computed in terms of the geometry of the general (not necessarily marginally-trapped) codimension-two surface defining the wedge. This outer entropy satisfies a second law of thermodynamics, growing monotonically along the generalized holographic screen. In particular, this result provides the holographic dual for the geometry of the event horizon for spherically-symmetric spacetimes.

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# 1 Introduction

Area laws for dynamical surfaces in spacetime have, both historically and recently, been important drivers of progress in theoretical physics. Under certain positivity conditions for the flow of energy-momentum, general relativity constrains the dynamics of certain surfaces such that their area only increases. The most well known example is Hawking’s area law for black holes [1, 2], which mandates that the area of the event horizon always grows with time. This provided the basis for the thermodynamic understanding of black holes [2–6]; in turn, black hole thermodynamics helped inspire the development of holography [7–9]. In the context of string theory, the AdS/CFT correspondence [10–13] has provided the prime example of a tractable holographic model that can be explored in detail. Holography thus gives us powerful tools with which to understand quantum gravity.

The areas of extremal surfaces in asymptotically-AdS spacetimes have proved to be of significance beyond their geometrical interpretation. They correspond to entanglement entropies of regions in the boundary CFT, given by the Ryu-Takayanagi formula [14–16] for static slices and more generally by the Hubeny-Rangamani-Takayanagi (HRT) prescription [17–19]. An understanding of the dynamics of these surfaces can shed light on the entanglement structure of the boundary and vice versa, with the question being actively researched from both the gravitational and field theory perspectives [20–28].

Building on earlier work [29,30], an interesting area theorem in general relativity was recently proved [31–33] for holographic screens, a substantive extension of apparent horizons to timelike or spacelike objects whose slices are marginally-trapped or -antitrapped surfaces [34]. Such screens can be found in many spacetimes of interest, such as expanding universes and inside of black holes, and it was shown that these surfaces have areas that grow in a particular direction along the screen. While geometrically interesting in their own right, such surfaces—as their name implies—have been suggested to have a holographic interpretation, as the surfaces on which to formulate a “boundary” theory in general spacetimes beyond AdS [35–38], though at present no explicit boundary theory is known for this more general conjectured form of holography. Moreover, an entropic interpretation of the area of the holographic screen has been demonstrated [39]: the area of an apparent horizon equals the area of the largest HRT surface compatible with the domain of dependence of the spacetime outside the apparent horizon. That is, the apparent horizon area can be viewed as an “outer entropy” of the spacetime. In contrast, despite the success of Hawking’s area theorem in sparking black hole thermodynamics and the holographic revolution, a valid holographic interpretation of the event horizon itself has remained

elusive [40, 41].

In this paper, we will show that both the holographic screen and the event horizon are special cases of a much more general class of surfaces, which we will call *generalized holographic screens*, all of which satisfy an area law. Thus, we will unify the area law discovered in Refs. [31, 32] and Hawking’s area law [1]. These generalized holographic screens extend the concept of holographic screens to surfaces that are not marginally trapped; these new surfaces sweep out large portions of the interior of a black hole and can also be constructed in cosmological spacetimes. These results are proved purely in general relativity and are independent of holography.

Furthermore, we will show, for spherically-symmetric spacetimes, that the outermost space-like portion of generalized holographic screens have an entropic interpretation analogous to that of apparent horizons given in Ref. [39]. In particular, we will prove another new general relativity result, giving the area of the largest HRT surface compatible with the outer wedge of a slice of the generalized holographic screen. The area of this maximal HRT surface is given by a geometric quantity computable in terms of the area and curvature of the generalized holographic screen. Viewed as a holographic statement, we compute the outer entropy of a non-marginally-trapped surface inside a black hole. This implies that we find a new entry in the holographic dictionary: the entropic interpretation of the event horizon (in terms of its area and curvature), for spherically-symmetric spacetimes. Comparing the evolution of the maximal HRT area associated with different slices, we show that the outer entropy satisfies a second law, despite being a complicated function of geometric quantities on the generalized holographic screen.

The remainder of this paper is organized as follows. In Sec. 2, we define our terminology and give the definition of generalized holographic screens, in particular proving in Sec. 2.3 that they satisfy an area law. In Sec. 3, we review the definition of outer entropy and show that, for the generalized holographic screen, it is upper bounded by the area of the screen in Planck units. In Sec. 4 we compute the outer entropy for spherically-symmetric spacetimes; we discuss several special cases of interest in Sec. 4.3 and prove the second law for the outer entropy in Sec. 4.4. We conclude and discuss future directions in Sec. 5.

## 2 Generalized Holographic Screens

In this section, we derive our results based on classical general relativity. First, we will discuss some differential geometry formalism and review the notion of (marginally-trapped) holographic screens. We will then introduce the notion of generalized holographic screens and establish our family of area laws, illustrating how Hawking’s area theorem for event horizons arises as a special

case.

## 2.1 Formalism and Review

Throughout the paper, we will consider a smooth spacetime  $(M, g_{ab})$  of dimension  $D \geq 3$  that is globally hyperbolic (or, in the asymptotically-AdS case, with appropriate boundary conditions [42]). We will also assume the Einstein equations and the null energy condition (NEC),  $T_{ab}k^ak^b \geq 0$  for energy-momentum tensor  $T_{ab}$  and any null vector  $k^a$ ; equivalently, we could assume the null curvature condition (NCC)  $R_{ab}k^ak^b \geq 0$ . We will use mostly-plus metric signature and sign conventions  $R_{ab} = R^c{}_{acb}$  and  $R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ce}\Gamma^e{}_{bd} - \Gamma^a{}_{de}\Gamma^e{}_{bc}$ . We follow the standard differential geometry notation, defining the chronological future (respectively, past) of a set  $S$  as  $I^\pm(S)$ , the future (respectively, past) domains of dependence  $D^\pm(S)$  as the set of points  $p \in M$  such that every past (respectively, future) inextendible causal curve through  $p$  in  $M$  intersects  $S$ , and the domain of dependence  $D(S)$  as the union  $D^+(S) \cup D^-(S)$ . We use a dot  $\dot{S}$ , circle  $\mathring{S}$ , and bar  $\bar{S}$  to denote the boundary, interior, and closure of a set  $S$ , respectively. In our conventions,  $S \not\subset I^\pm(S)$ , but  $S \subset D^\pm(S)$ .

Let us first review some results of Ref. [32]. We define a *future holographic screen*  $H$  to be a smooth (codimension-one) hypersurface for which one can define a *foliation* (i.e., a partition of  $H$ ) into marginally-trapped codimension-two compact acausal surfaces called *leaves*. From a leaf  $\sigma$ , we will call the two future-directed orthogonal null geodesic congruences  $k$  and  $l$ ; the marginally-trapped condition stipulates that, on  $\sigma$ ,

$$\theta_k = 0 \quad \text{and} \quad \theta_l < 0, \tag{1}$$

where  $\theta_k = \nabla_a k^a$  and  $\theta_l = \nabla_a l^a$  are the null expansions for  $k$  and  $l$ , respectively. Defining an area element  $\delta A$ , the expansions can equivalently be written as  $\theta_k = \nabla_k \log \delta A$  and  $\theta_l = \nabla_l \log \delta A$ , where  $\nabla_k = k^a \nabla_a$  and  $\nabla_l = l^a \nabla_a$  are the covariant derivatives along the congruences. Throughout, we will extend the definition of  $k$  and  $l$  to null vector fields over the entire spacetime  $M$ . For a given  $H$ , this choice of  $k$  and  $l$  over all of  $M$  is not unique, but our results will hold for all such choices.

We consider the case in which each leaf  $\sigma$  splits some Cauchy surface  $\Sigma$  into two disjoint subsets,  $\Sigma = \Sigma^+ \cup \sigma \cup \Sigma^-$ , where  $\sigma = \mathring{\Sigma}^\pm$  and we label  $\Sigma^-$  as the outer portion (which we take to be in the  $k$  direction) and  $\Sigma^+$  as the inner portion (which we take to be in the  $l$  direction); we choose this notation and the  $\pm$  convention for outer versus inner to match that of Ref. [32]. We note that  $\sigma$  divides the spacetime into four disjoint portions,  $I^\pm(\sigma)$  and  $D(\Sigma^\pm)$ ; in particular,  $I^\pm(\sigma)$  and  $\bar{D}(\Sigma^\pm) - \sigma$  together constitute a four-part partition of  $M - \sigma$ , as shown in Fig. 2 of

Ref. [43]. As proved in Ref. [43], the boundaries of these regions can be characterized by the geodesic congruences  $k$  and  $l$ , truncating at any conjugate points (i.e., caustics) or intersections of finitely-separated geodesics. This fact will be used frequently in our arguments that follow.

One can define a real parameter  $\tau$  on  $H$  such that each leaf  $\sigma$  is a surface of constant, unique  $\tau$ . We can also write the tangent vector field  $h^a$  parallel to the leaf-orthogonal curves within  $H$  as

$$h^a = \alpha l^a + \beta k^a \quad (2)$$

for some real parameters  $\alpha$  and  $\beta$ , normalized so that  $h^a(d\tau)_a = 1$ .

We can then make the following definitions of null surfaces:

$$\begin{aligned} N_{+k}(\sigma) &= \dot{I}^+(\Sigma^+) - \Sigma^+ = \dot{D}^+(\Sigma^-) - I^-(D^+(\Sigma^-)) \\ N_{-k}(\sigma) &= \dot{I}^-(\Sigma^-) - \Sigma^- = \dot{D}^-(\Sigma^+) - I^+(D^-(\Sigma^+)) \\ N_{+l}(\sigma) &= \dot{I}^+(\Sigma^-) - \Sigma^- = \dot{D}^+(\Sigma^+) - I^-(D^+(\Sigma^+)) \\ N_{-l}(\sigma) &= \dot{I}^-(\Sigma^+) - \Sigma^+ = \dot{D}^-(\Sigma^-) - I^+(D^-(\Sigma^-)). \end{aligned} \quad (3)$$

The result of Ref. [43] implies that the expressions on the right-hand side are independent of the choice of Cauchy surface  $\Sigma$  and are indeed defined only by the leaf  $\sigma$ ; that is,  $N_{\pm k}$  and  $N_{\pm l}$  are *light sheets*, null surfaces defined up to caustics and nonlocal intersections of null geodesics. We further define  $N_k(\sigma) = N_{+k}(\sigma) \cup N_{-k}(\sigma)$  and  $N_l(\sigma) = N_{+l}(\sigma) \cup N_{-l}(\sigma)$  and note that  $\sigma = N_{+k}(\sigma) \cap N_{-k}(\sigma) = N_{+l}(\sigma) \cap N_{-l}(\sigma)$ . Given the Cauchy-surface-independence, we define the spacetime regions

$$\begin{aligned} K^+(\sigma) &= I^+(\Sigma^+) \cup D^-(\Sigma^+) - N_{-k}(\sigma) \\ K^-(\sigma) &= I^-(\Sigma^-) \cup D^+(\Sigma^-) - N_{+k}(\sigma) \\ L^+(\sigma) &= I^+(\Sigma^-) \cup D^-(\Sigma^-) - N_{-l}(\sigma) \\ L^-(\sigma) &= I^-(\Sigma^+) \cup D^+(\Sigma^+) - N_{+l}(\sigma), \end{aligned} \quad (4)$$

so  $N_k(\sigma) = \dot{K}^+(\sigma) = \dot{K}^-(\sigma)$  and  $N_l(\sigma) = \dot{L}^+(\sigma) = \dot{L}^-(\sigma)$ ; see Fig. 1.

Finally, as in Ref. [32] we will take  $R_{ab}k^ak^b + \varsigma_k^2$  to be strictly positive on  $H$ , where  $\varsigma_k$  is the shear tensor of the  $k$  congruence as defined in Ref. [44]. Along with the Raychaudhuri equation

$$\nabla_k \theta_k = -\frac{1}{D-2} \theta_k^2 - \varsigma_k^2 - R_{ab}k^ak^b \quad (5)$$

and the NEC, this genericity assumption implies that  $\theta_k$  is strictly positive (negative) to the past (respectively, future) of  $\sigma$ . Note that the term involving the twist tensor is absent in Eq. (5) because the congruence is surface-orthogonal.

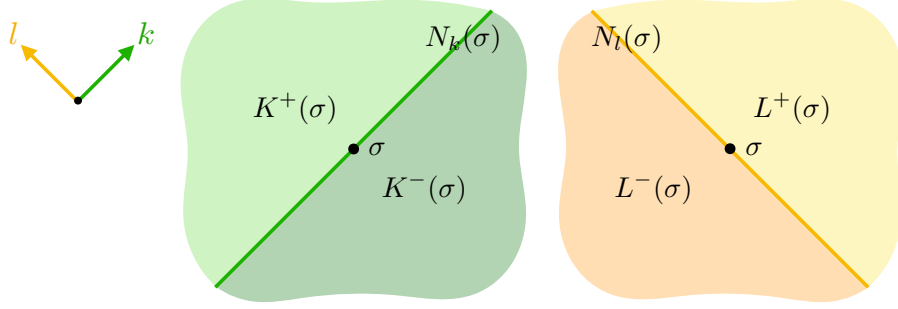


Figure 1: Generic Penrose diagrams for spacetime regions  $K^\pm(\sigma)$  and  $L^\pm(\sigma)$  defined in Eq. (4), divided by the light sheets  $N_k(\sigma)$  and  $N_l(\sigma)$ , respectively, defined in Eq. (3).

Given these conditions and additional technical assumptions,<sup>1</sup> Ref. [32] then shows that  $\alpha < 0$  everywhere on  $H$ . That is,  $h^a$  points either (timelike) to the past or (spacelike) outwards. As a result, the sets of  $K^\pm$  are monotonic under inclusion: writing  $K^\pm(\tau) = K^\pm(\sigma(\tau))$ , one obtains the inclusion relations

$$\begin{aligned} \bar{K}^+(\tau_1) &\subset K^+(\tau_2) & (\alpha < 0, \text{ any } \beta) \\ \bar{K}^-(\tau_2) &\subset K^-(\tau_1) & (\alpha < 0, \text{ any } \beta) \end{aligned} \tag{6}$$

for  $\tau_1 < \tau_2$ . Analogously, if we can choose a region where  $\beta$  is constant throughout a leaf, with the same sign at  $\sigma(\tau_1)$  and  $\sigma(\tau_2)$ , the sets  $L^\pm(\tau) = L^\pm(\sigma(\tau))$  are also monotonic under inclusion:

$$\begin{aligned} \bar{L}^+(\tau_2) &\subset L^+(\tau_1) & (\alpha < 0, \beta > 0) \\ \bar{L}^-(\tau_1) &\subset L^-(\tau_2) & (\alpha < 0, \beta > 0) \\ \bar{L}^+(\tau_1) &\subset L^+(\tau_2) & (\alpha < 0, \beta < 0) \\ \bar{L}^-(\tau_2) &\subset L^-(\tau_1) & (\alpha < 0, \beta < 0). \end{aligned} \tag{7}$$

Finally, Ref. [32] shows that the holographic screen  $H$  satisfies an area law:  $A[\sigma(\tau_1)] < A[\sigma(\tau_2)]$ , so  $dA/d\tau > 0$ .<sup>2</sup> By reversing the time direction and swapping past for future in all of the definitions, one can define *past holographic screens*, which are foliated by marginally-antitrapped surfaces and which also satisfy an area law.

<sup>1</sup>Ref. [32] also assumes that every inextendible portion of  $H$  contains either a complete leaf or is completely timelike and that the sets of points in  $H$  for which  $\alpha$  is positive and negative share a boundary on which  $\alpha$  vanishes.

<sup>2</sup>Throughout, we will use round brackets for scalar arguments. For objects that take a set of points in  $M$  as an argument, we will use round brackets if the object being defined is itself a subset of the spacetime (e.g.,  $D(S)$ ), while we will use square brackets in the case of a quantity defined on the spacetime (e.g.,  $A[S]$  for the area of a surface  $S$ ).



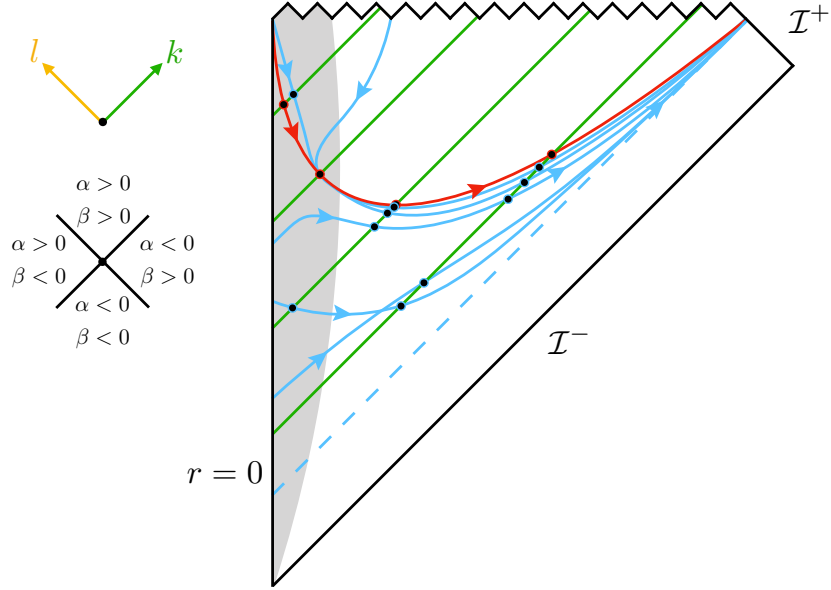


Figure 2: Generalized (future) holographic screens inside a black hole formed from collapse. The holographic screen  $H$  is shown in red and has both a timelike and spacelike portion. Several examples of generalized holographic screens  $H'$  are illustrated by the blue curves. In this example, they can have both timelike and spacelike portions or can be purely spacelike. The limiting case of the event horizon (blue dashed line) also corresponds to a generalized holographic screen. A few representative light sheets  $N_k(\sigma)$  are illustrated by the green lines and on these light sheets the codimension-two leaves  $\sigma$  of  $H$  (on which  $\theta_k = 0$ ) and the leaves  $\sigma'$  of  $H'$  (on which  $\theta_k \neq 0$ ) are represented by the black dots; for each  $\sigma' \subset H'$  there exists  $\sigma \subset H$  for which  $\sigma' \subset N_k(\sigma)$ . We will show that all of these screens obey an area law, with increasing area toward the past and outward directions, as illustrated by the arrows pointing in the direction of increasing  $\tau$ .

## 2.2 Definition of Generalized Holographic Screens

We will show that there is a much larger family of surfaces, beyond the holographic screens discussed in Sec. 2.1, that also satisfy an area law. In particular, we are interested in relaxing the requirement that the leaves be marginally trapped. Given a future holographic screen  $H$  as described in Sec. 2.1, we will define a *generalized future holographic screen*  $H'$  as a surface to the past (future) of  $H$  when  $H$  is spacelike (respectively, timelike), with  $H'$  being spacelike if and only if the corresponding section of  $H$  is spacelike. A few examples of generalized future holographic screens are shown in Fig. 2. We will later prove that  $H'$  satisfies an area law, but before that let us first specify the conditions defining  $H'$  more precisely.

Formally, we define a generalized future holographic screen as a (codimension-one) hypersurface  $H'$  with a foliation into codimension-two leaves  $\sigma'$  and tangent vector  $h'^a = \alpha l^a + \beta k^a$

(for some  $\alpha$  and  $\beta$ ) satisfying the following criteria:

1. For each  $\sigma' \subset H'$ , there exists  $\sigma \subset H$  for which  $\sigma' \subset N_k(\sigma)$ . For each  $p \in \sigma$ , we can identify some  $p' \in \sigma'$  satisfying  $p' \in N_k(p)$ . If  $h$  is spacelike or null, then we require  $p' \in N_{-k}(p)$ , while if  $h$  is timelike or null,  $p' \in N_{+k}(p)$ .
2. The signature and orientation of  $h'$  at  $p' \in \sigma'$  matches that of  $h$  at  $p \in \sigma$  for which  $p' \in N_k(p)$ .
3.  $\theta_l < 0$  on  $H'$ .

For *generalized past holographic screens*,  $N_{\pm k}$  are simply swapped in condition 1, while condition 3 becomes  $\theta_l > 0$ . By Eq. (6),  $N_k(\sigma(\tau_1)) \cap N_k(\sigma(\tau_2)) = \emptyset$  for  $\tau_1 \neq \tau_2$ , so the  $\sigma$  for which  $\sigma' \subset N_k(\sigma)$  is unique. That is, there is a function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  for which  $\sigma'(\tau) \subset N_k(\sigma(\phi(\tau)))$ . Note that  $\phi$  is not necessarily injective or surjective: there may be more than one slice  $\sigma' \subset H'$  in the same  $N_k(\sigma)$  and there may be some  $\sigma \subset H$  for which  $N_k(\sigma) \cap H' = \emptyset$ .

Note that the event horizon itself is a generalized future holographic screen, corresponding to the limit in which  $\phi(\tau)$  maps all numbers to infinity, where the leaves of the original holographic screen  $\sigma(\tau)$  go to  $\mathcal{I}^+$  as  $\tau \rightarrow \infty$ . While the event horizon is teleologically defined (i.e., it requires knowledge of the entire future history of the spacetime), the holographic screen is defined quasilocally, in terms of metric and its derivatives measurable at a point, in a particular Cauchy slicing. The generalized holographic screen shares characteristics of both of these definitions: it is defined in terms of the holographic screen, but using past- or future-directed light sheets. Hence, the event horizon is a generalized holographic screen in the particular limit in which all  $\sigma'$  are in  $N_{-k}(\sigma)$  for the leaf  $\sigma = H \cap \mathcal{I}^+$  on the boundary of the spacetime.

### 2.3 Area Law

We now show that there is an area law on the generalized holographic screen  $H'$ . By condition 1 in Sec. 2.2, for the region where  $H$  is timelike (respectively, spacelike), we have  $\theta_k < 0$  (respectively,  $\theta_k > 0$ ) on  $H'$  by the Raychaudhuri equation (5) and the fact that  $\theta_k = 0$  on  $H$ . By condition 2, we thus have  $\theta_k \leq 0$  when  $H'$  is timelike (or null) and  $\theta_k \geq 0$  when  $H'$  is spacelike (or null).

Moreover, condition 2 implies that  $\alpha < 0$  on  $H'$ , since  $\alpha < 0$  on  $H$ . That is,  $h'^a$  is either past- or outward-directed, so  $\beta < 0$  when  $H'$  is timelike and  $\beta > 0$  when  $H'$  is spacelike. Hence,  $\beta\theta_k \geq 0$  on  $H'$ . By condition 3,  $\theta_l < 0$  on  $H'$ , so  $\alpha\theta_l > 0$ . That is, we have shown that the

general covariant definition of  $H'$  given in conditions 1 through 3 implies

$$\alpha\theta_l + \beta\theta_k > 0 \tag{8}$$

everywhere on  $H'$ .

We can now adapt the zigzag argument of Ref. [32] to prove an area law on  $H'$ . Let us first consider the case in which  $\sigma'$  is smooth; we will subsequently extend our result to the more general case of non-smooth  $\sigma'$ . Since  $\theta_l < 0$  on  $H$ , by continuity there always exists a surface near  $H$  with  $\theta_l < 0$  satisfying conditions 1 and 2, so choices of  $H'$  always exist. Given smooth  $\sigma$ , there always exists a smooth surface  $\sigma' \subset N_k(\sigma)$  by taking  $\sigma'$  sufficiently near  $\sigma$ , since by a theorem of Ref. [45], geodesics cannot exit the boundary of the future or past of  $\sigma$  instantaneously. From  $\sigma'(\tau) \subset H'$ , consider the light sheet  $N_{-l}(\sigma'(\tau))$  going in the past  $l$  direction. From  $\sigma'(\tau + d\tau) \subset H'$ , consider the null hypersurface  $N_k(\sigma'(\tau + d\tau))$ , following both the past and future  $k$  directions. Since  $\alpha < 0$ ,  $\tilde{\sigma}(\tau, \tau + d\tau) = N_{-l}(\sigma'(\tau)) \cap N_k(\sigma'(\tau + d\tau))$  is nonempty; see Fig. 3. For regions of  $H'$  that are spacelike ( $\beta > 0$ ),  $\tilde{\sigma}(\tau, \tau + d\tau) \subset N_{-k}(\sigma'(\tau + d\tau))$ . Conversely, for parts of  $H'$  that are timelike ( $\beta < 0$ ),  $\tilde{\sigma}(\tau, \tau + d\tau) \subset N_{+k}(\sigma'(\tau + d\tau))$ .

Since  $\tilde{\sigma}(\tau, \tau + d\tau) \subset N_{-l}(\sigma'(\tau))$ , we have

$$A[\tilde{\sigma}(\tau, \tau + d\tau)] - A[\sigma'(\tau)] = A[\tilde{\sigma}(\tau, \tau + d\tau)]\alpha\theta_l d\tau \tag{9}$$

for infinitesimal  $d\tau$ , recalling the definition of  $\theta_l = \nabla_l \log \delta A$ . Similarly, the change in area from  $\tilde{\sigma}(\tau, \tau + d\tau)$  to  $\sigma'(\tau + d\tau)$  is

$$A[\sigma'(\tau + d\tau)] - A[\tilde{\sigma}(\tau, \tau + d\tau)] = A[\tilde{\sigma}(\tau, \tau + d\tau)]\beta\theta_k d\tau \tag{10}$$

since  $\theta_k = \nabla_k \log \delta A$ . Hence,

$$A[\sigma'(\tau + d\tau)] - A[\sigma'(\tau)] = A[\tilde{\sigma}(\tau, \tau + d\tau)](\alpha\theta_l + \beta\theta_k)d\tau. \tag{11}$$

By Eq. (8), we therefore have

$$A[\sigma'(\tau + d\tau)] - A[\sigma'(\tau)] > 0, \tag{12}$$

leading to an area law along  $H'$ :

$$\frac{dA[\sigma'(\tau)]}{d\tau} > 0. \tag{13}$$

Specifically, writing the induced metric on  $\sigma'(\tau)$  as  $\gamma_{ab}^{\sigma'(\tau)}$ , the area grows at the rate

$$\frac{dA[\sigma'(\tau)]}{d\tau} = \int_{\sigma'(\tau)} \sqrt{\gamma^{\sigma'(\tau)}} (\alpha\theta_l[\sigma'(\tau)] + \beta\theta_k[\sigma'(\tau)]). \tag{14}$$

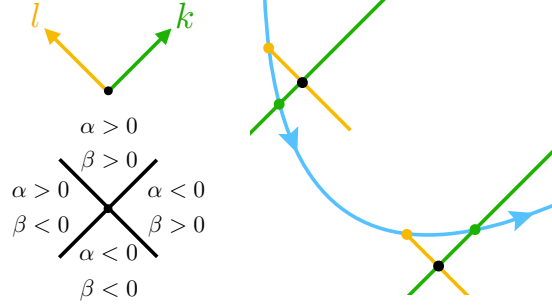


Figure 3: Zigzag construction proving the area law on a generalized holographic screen  $H'$  (blue curve). From  $\sigma'(\tau)$  (examples given by orange dots), we take the past  $l$  light sheet  $N_{-l}(\sigma'(\tau))$  (orange lines), while from  $\sigma'(\tau + d\tau)$  (green dots), we take the  $k$  light sheet  $N_k(\sigma'(\tau + d\tau))$  (green lines). The intersection  $\tilde{\sigma}(\tau, \tau + d\tau)$  (black dots) is always nonempty. We have  $A[\tilde{\sigma}(\tau, \tau + d\tau)] > A[\sigma'(\tau)]$  and  $A[\sigma'(\tau + d\tau)] > A[\tilde{\sigma}(\tau, \tau + d\tau)]$ , so area increases along  $H'$ .

We thus have a general covariant geometric formulation of a generalized holographic screen that is not a marginally-trapped surface but that nonetheless satisfies an area law.

Let us now generalize our result by relaxing the requirement that  $\sigma'$  is so close to  $\sigma$  as to be smooth. In particular,  $\sigma'$  can now contain portions of caustics or nonlocal intersections in  $N_k(\sigma)$ , where null geodesics can enter or exit the light sheet defining the past or future of  $\sigma$  [43]. Even in this case, an area law can be proved. For a spacelike part of  $H'$ , between  $\tilde{\sigma}(\tau, \tau + d\tau)$  and  $\sigma'(\tau + d\tau)$ , it is possible for future-directed null geodesics to enter  $N_{-k}(\sigma'(\tau + d\tau))$ , but not to leave it (see, e.g., Refs. [44, 45]). Similarly, for a timelike part of  $H'$ , it is possible for future-directed null geodesics to leave but not enter  $N_{+k}(\sigma'(\tau + d\tau))$  between  $\sigma'(\tau + d\tau)$  and  $\tilde{\sigma}(\tau, \tau + d\tau)$ . Hence, in both cases  $A[\sigma'(\tau + d\tau)] - A[\tilde{\sigma}(\tau, \tau + d\tau)]$  is lower-bounded by the right-hand side of Eq. (10) and is therefore still positive.

We next consider the other light sheets defining  $\tilde{\sigma}(\tau, \tau + d\tau)$ , i.e.  $N_{-l}(\sigma'(\tau))$ . Future-directed null geodesics cannot leave  $N_{-l}(\sigma'(\tau))$ ; however, they can enter  $N_{-l}(\sigma'(\tau))$  only when they encounter a caustic or a nonlocal intersection with a null geodesic originating from elsewhere on  $\sigma'$  [43]. If they entered through a caustic, one would find that, moving from past to future, their expansion  $\theta_l$  jumps discontinuously from  $-\infty$  to  $+\infty$  at the entry point and then decreases continuously toward  $\sigma'(\tau)$ . This implies that since  $\theta_l$  is by definition negative on  $\sigma'(\tau)$ , we can always choose  $d\tau$  sufficiently small that a caustic is not encountered between  $\sigma'(\tau)$  and  $\tilde{\sigma}(\tau, \tau + d\tau)$  on  $N_{-l}(\sigma'(\tau))$ . It is also clear that  $d\tau$  can always be chosen small enough that the generators of  $N_{-l}(\sigma'(\tau))$  do not encounter any nonlocal intersections between  $\sigma'(\tau)$  and  $\tilde{\sigma}(\tau, \tau + d\tau)$ . Hence, Eq. (9) still holds, and the right-hand side of Eq. (14) gives a lower bound on the rate of area increase. We thus find that the area increase rate is still positive. Namely,

we have an area law on the generalized holographic screen  $H'$  even if  $\sigma'$  is not close to  $\sigma$ .

The original holographic screen  $H$  is a special case of our family of generalized holographic screens  $H'$ , taking the limit in which  $\sigma' \rightarrow \sigma$  for all  $\tau$ , so that  $\theta_k \rightarrow 0$ . Hence, the area law for  $H'$  reduces smoothly to the area law for the holographic screen  $H$  derived in Ref. [32].

Moreover, Hawking's area theorem [1] is also a special case of our area law for generalized holographic screens. In the case of a holographic screen  $H$ , the marginally-trapped condition prescribes a particular foliation into leaves  $\sigma$ . For a region of a generalized holographic screen  $H'$  where the mapping between leaves  $\sigma' \subset H'$  and  $\sigma \subset H$  is one-to-one (i.e., the function  $\phi$  is injective),  $H'$  inherits the foliation of  $H$ . However, if we choose  $H'$  to have a finite null region, then multiple leaves in  $H'$  lie within  $N_k(\sigma)$  for the same  $\sigma \subset H$ . In this region, the foliation of  $H$  does not prescribe a foliation of  $H'$ ; under any foliation of a null portion of  $H'$  into leaves  $\sigma'$ , the area law proved above still applies by virtue of the positivity of  $\theta_k$ . Similarly, Hawking's area theorem is independent of the spacelike Cauchy slicing: for any two spacelike Cauchy slices  $\Sigma_1$  and  $\Sigma_2$  where  $\Sigma_2 \subset I^+(\Sigma_1)$ , the event horizon  $\dot{I}^-(\mathcal{I}^+)$  grows in area, so  $A[\dot{I}^-(\mathcal{I}^+) \cap \Sigma_1] \leq A[\dot{I}^-(\mathcal{I}^+) \cap \Sigma_2]$  [45]. Hence, for any spacelike Cauchy slicing of the spacetime, we can define a foliation of a null portion of  $H'$  simply via its intersection with the Cauchy slices. In Hawking's area theorem, the area law follows from proving that the expansion on the horizon is nonnegative in a spacetime satisfying the NCC. In our present context, assuming an asymptotically-stationary spacetime, so that the horizon is asymptotically marginally trapped, implies that there exists a holographic screen  $H$  that asymptotes to the horizon. We can thus define the horizon itself as a generalized holographic screen  $H'$ , on which  $\theta_k$  is positive by the Raychaudhuri equation (5).

Our family of generalized holographic screens thus unifies two previously known area laws associated with black holes, namely, those of the holographic screen and the event horizon. This unification is nontrivial: while it is true that a convex combination of two monotonic functions is itself monotonic, such intuition does not readily apply to spacetime geometries, in which the notion of taking a combination of two surfaces is not in general well defined without specifying additional geometric information for how to determine the new surface. Our definition in Sec. 2.2 provides precisely the requisite specifications, guaranteeing, as we have shown in this section, an area law for the generalized holographic screen.

## 2.4 Alternate Construction of Screens

The definition of generalized holographic screens in Sec. 2.2 leads to immense freedom in choosing  $H'$ . The only requirements are those given in conditions 1 through 3.

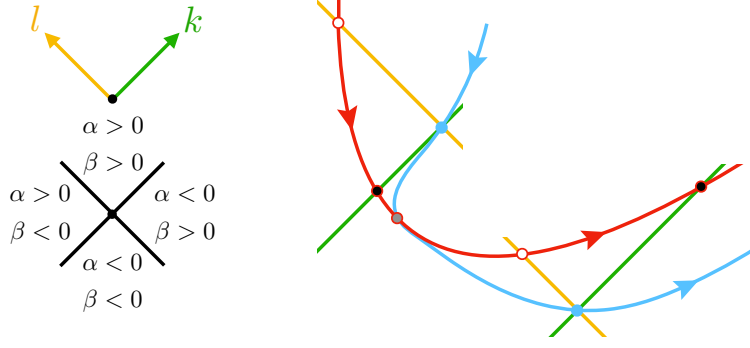


Figure 4: Illustration of the intersection construction of generalized holographic screens. Arrows indicate the direction of increasing  $\tau$ . Examples of leaves  $\sigma(\tau)$  in the holographic screen  $H$  (red curve) are given by the white dots and the corresponding  $\sigma(f(\tau))$  is given by the immediately succeeding black dot. A leaf  $\sigma'$  (blue dot) of the generalized holographic screen  $H'$  (blue curve) is given by the intersection of  $N_l(\sigma(\tau))$  (yellow line) and  $N_k(\sigma(f(\tau)))$  (green line) as shown in Eq. (15). The function  $f(\tau) \geq \tau$  equals  $\tau$  precisely when  $H$  has null tangent (gray dot).

However, we can formulate an elegant alternative way of defining a particular subset of generalized future holographic screens parameterized by a single real function. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function with  $df/d\tau > 0$  and  $f(\tau) \geq \tau$ , with equality if and only if  $\sigma(\tau) \subset H$  has null tangent  $h^a$ . In this subsection, we will also assume for simplicity that each leaf of  $H$  is entirely timelike, spacelike, or null. Then we can remove conditions 1 and 2 and instead simply define  $H'$  to be the hypersurface foliated by leaves

$$\sigma'(\tau) = N_l(\sigma(\tau)) \cap N_k(\sigma(f(\tau))). \quad (15)$$

We still require condition 3 that  $\theta_l < 0$ . See Fig. 4 for an illustration of this construction. The analogous construction for generalized past holographic screens can be defined similarly.

On the spacelike part of  $H$  (on which  $\beta > 0$ ), for  $\tau_1 < \tau_2$ , Eq. (7) implies  $\sigma'(\tau_2) \subset N_{-l}(\sigma(\tau_2)) \subset \bar{L}^+(\sigma(\tau_2)) \subset L^+(\sigma(\tau_1))$ , while by Eq. (6),  $\sigma'(\tau_2) \subset N_{-k}(\sigma(f(\tau_2))) \subset \bar{K}^-(\sigma(f(\tau_2))) \subset K^-(\sigma(f(\tau_1)))$ , since  $f(\tau_1) < f(\tau_2)$  by definition of  $f$ . Now, for any cross section  $\hat{\sigma}$  of  $N_k(\sigma)$ ,  $K^\pm(\sigma) = K^\pm(\hat{\sigma})$ , while for any cross section  $\hat{\sigma}$  of  $N_l(\sigma)$ ,  $L^\pm(\sigma) = L^\pm(\hat{\sigma})$ . Hence,  $K^-(\sigma(f(\tau_1))) = K^-(\sigma'(f(\tau_1)))$  and  $L^+(\sigma(\tau_1)) = L^+(\sigma'(\tau_1))$ . Again by Eq. (6), along with the property  $f(\tau) > \tau$ , we have therefore shown that

$$\sigma'(\tau_2) \subset L^+(\sigma'(\tau_1)) \cap K^-(\sigma'(f(\tau_1))) \subset L^+(\sigma'(\tau_1)) \cap K^-(\sigma'(\tau_1)) = \mathring{D}(\Sigma^-(\sigma'(\tau_1))), \quad (16)$$

where  $\Sigma(\sigma'(\tau_1))$  is a Cauchy surface split (into  $\Sigma^\pm$ ) by  $\sigma'(\tau_1)$ . It will be convenient to define the *outer wedge*  $O_W(\sigma') = \mathring{D}[\Sigma^-(\sigma')]$ . We thus find that every point in  $\sigma'(\tau_2)$  is spacelike separated

from every point in  $\sigma'(\tau_1)$ . Hence, using this alternative definition of the generalized holographic screen, we automatically have that  $H'$  is spacelike and directed outward when the corresponding portion of  $H$  is spacelike.

Similarly, on the timelike part of  $H$  (on which  $\beta < 0$ ), we have  $\sigma'(\tau_2) \subset N_{-l}(\sigma(\tau_2)) \subset \overline{L}^-(\sigma(\tau_2)) \subset L^-(\sigma(\tau_1))$  and further  $\sigma'(\tau_2) \subset N_{+k}(\sigma(f(\tau_2))) \subset \overline{K}^-(\sigma(f(\tau_2))) \subset K^-(\sigma(f(\tau_1)))$ , again by the condition  $df/d\tau > 0$ . We further have in this case  $K^-(\sigma(f(\tau_1))) = K^-(\sigma'(f(\tau_1)))$  and  $L^-(\sigma(\tau_1)) = L^-(\sigma'(\tau_1))$ . Hence, again using Eq. (6) and that  $f(\tau) > \tau$ , we have

$$\sigma'(\tau_2) \subset L^-(\sigma'(\tau_1)) \cap K^-(\sigma'(f(\tau_1))) \subset L^-(\sigma'(\tau_1)) \cap K^-(\sigma'(\tau_1)) = I^-(\sigma'(\tau_1)), \quad (17)$$

so every point in  $\sigma'(\tau_2)$  is in the chronological past of every point in  $\sigma'(\tau_1)$ . Thus, we automatically have that  $H'$  is timelike and past-directed when the corresponding portion of  $H$  is timelike.

Therefore, with the simple requirements that  $df/d\tau > 0$  and  $f(\tau) \geq \tau$  (with equality when  $H$  is null), we have an elegant construction of a generalized holographic screen  $H'$ , defined by its leaves as in Eq. (15), that automatically has the correct tangent and thus, by the argument in Sec. 2.3, satisfies an area law.

### 3 Outer Entropy

Having established the general relativity results of Sec. 2, we now would like to understand their holographic interpretation. In AdS/CFT [10–12], certain geometric quantities in the bulk have interpretations in terms of properties of the boundary CFT state. The most celebrated example of this is the Ryu-Takayanagi relation [14, 15] and its generalization to dynamical spacetimes by Hubeny, Rangamani, and Takayanagi [17], which relates the area of certain extremal surfaces in the bulk to the von Neumann entropy

$$S[\rho] = -\text{tr } \rho \log \rho \quad (18)$$

of the reduced density matrix  $\rho$  on the homologous region on the boundary. In particular, the HRT prescription implies that, for a boundary state  $\rho$  corresponding to some classical bulk geometry with an extremal surface  $X_{\text{HRT}}$ , the von Neumann entropy satisfies

$$S[\rho] = \frac{A[X_{\text{HRT}}]}{4G\hbar}. \quad (19)$$

For a two-sided geometry in AdS/CFT described by a pure state, this entropy gives a measure of the entanglement between the boundary regions corresponding to the two sides of the spacetime

split by the HRT surface. An extremal surface is defined to be a surface whose area is a local extremum as a functional over all surfaces in the bulk. The HRT surface is chosen to be homologous to the boundary and an extremal surface of minimal area; such a surface can be identified using the maximin prescription [18]. One can show that the HRT surface is a surface on which  $\theta_k = \theta_l = 0$  and that there exists some Cauchy slice on which the area of the surface equals the minimal cross section of the slice. While the HRT form of the entropy (19) has been extensively tested in AdS/CFT [16, 19], our results in this section will not need all of the structure of AdS/CFT for validity. Instead, our conclusions will carry over under the assumption that the identification (19) can be made in any spacetime, that is, that there is some maximal extremal surface inside the black hole to which one can associate a fine-grained entropy for the ensemble. This is the same set of assumptions used in Ref. [39]. Moreover, if the holographic screen does indeed provide a boundary description of the spacetime in terms of a pure state, then this entropy would again equal the entanglement entropy between the boundary regions corresponding to the two sides of the spacetime split by the HRT surface.

There are compelling reasons why it is desirable to seek some entropic interpretation of the generalized holographic screens we considered in Sec. 2. It has been conjectured that holographic screens play the role of the boundary of AdS in AdS/CFT for non-asymptotically-AdS spacetimes, enabling a suitable generalization of holography to arbitrary geometries [35–38], although the details of this duality, including the explicit boundary theory, are not yet known. If this is the case, then it is well motivated to ask whether there is a sense of renormalization in these holographic theories. In AdS/CFT, renormalization group flow can be cast as motion in the bulk direction; formulating the theory on a surface at finite bulk coordinate yields a coarse-grained version of the original CFT [46–50]. Thus, it is well motivated to ask whether the generalized holographic screens of Sec. 2 play any similar coarse- or fine-grained role. Indeed, one can view the area law discovered in Sec. 2.3 as evidence for some second law interpretation.

Furthermore, the fact that the event horizon itself is encompassed in the family of generalized holographic screens makes the quest for an entropic interpretation of these surfaces especially interesting. The laws of black hole mechanics [2–6] describing the dynamics of the event horizon  $\mathcal{H}$  have direct thermodynamic interpretations, including Hawking’s area theorem corresponding to the second law of thermodynamics and the Bekenstein-Hawking entropy,

$$S_{\text{BH}} = \frac{A[\mathcal{H}]}{4G\hbar}. \quad (20)$$

Black hole thermodynamics was historically one of the original motivations for holography. Despite this connection, however, there has previously been no direct interpretation of the event



horizon itself from a holographic perspective. Indeed, there are arguments showing that certain straightforward possibilities involving the area of the event horizon (i.e., the causal holographic information [40]) cannot have a simple information-theoretic dual [41].

Previously, it was shown that the outermost spacelike portion of the holographic screen  $H$  does possess a dual in terms of the von Neumann entropy [39]. Specifically, let us consider an outermost marginally-trapped surface (i.e., an *apparent horizon*  $\sigma$ ), where by *outermost* we require that  $\sigma$  is homologous to the boundary, with a partial spacelike Cauchy surface connecting  $\sigma$  with the boundary such that any surface circumscribing  $\sigma$  has area greater than that of  $\sigma$ . Moreover, let us define the *outer entropy* associated with a codimension-two surface  $\chi$ ,

$$S^{(\text{outer})}[\chi] = \max_{\tilde{\rho}}(S[\tilde{\rho}] : O_W(\chi)), \quad (21)$$

as the entanglement entropy of one side of the entire boundary (computed via the HRT prescription) associated with the geometry described by the holographic state  $\tilde{\rho}$ , maximized over all possible  $\tilde{\rho}$  corresponding to spacetimes  $\tilde{M}$ , satisfying the NCC, for which the *outer wedge*  $O_W(\chi)$  is held fixed. In this sense, the outer entropy can be viewed as arising from the coarse-graining of the degrees of freedom associated with the (fine-grained) von Neumann entropy; equivalently, it can be viewed as the maximum holographic entanglement entropy for one side of the boundary consistent with the outer wedge. We recall from Sec. 2.4 that the outer wedge is defined as the set of points in  $M$  spacelike separated from  $\chi$  on the outer side, that is,  $O_W(\chi) = \mathring{D}(\Sigma^-(\chi))$ , where  $\Sigma(\chi)$  is a Cauchy surface split by  $\chi$ . With these definitions, the main result of Ref. [39] is that the outer entropy for the apparent horizon (the outermost spacelike part of the holographic screen) is given by its area:

$$S^{(\text{outer})}[\sigma] = \frac{A[\sigma]}{4G\hbar}. \quad (22)$$

We wish to relate the geometrical properties of leaves  $\sigma'$  of the outermost spacelike or null part of the generalized holographic screen  $H'$  defined in Sec. 2 to their outer entropy. We will show that  $S^{(\text{outer})}[\sigma']$  is bounded from above by the area of  $\sigma'$ . Moreover, for the special case of spherically-symmetric spacetimes, we will provide an explicit formula for  $S^{(\text{outer})}[\sigma']$  in terms of the geometry of  $\sigma'$  (its area, curvature, etc.).

For the remainder of this paper, we will implicitly restrict ourselves to the outermost spacelike or null part of a generalized holographic screen  $H'$ , which we will write simply as  $H'$ . That is, for any leaf  $\sigma' \subset H'$  we consider in this and the following sections, we will take  $\sigma'$  to be in  $N_{-k}(\sigma)$  for  $\sigma \subset H$  such that  $\sigma$  is an outermost marginally-trapped surface in the sense of Ref. [39]. Furthermore, in addition to the NEC, we will also impose the cosmological-constant-subtracted

dominant energy condition ( $\Lambda$ DEC). That is, writing the Einstein equation as

$$R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}, \quad (23)$$

we allow  $\Lambda$  to take either sign but impose the dominant energy condition (DEC) on  $T_{ab}$ :  $-T^a_b t^b$  is a causal, future-directed vector for all causal, future-directed vectors  $t^a$ .<sup>3</sup> This is essentially a causality requirement, enforcing that the positive flux of null energy not be superluminal as seen in any inertial frame. Finally, we will assume a generic condition on  $\sigma'$ , requiring that  $\theta_k$  be strictly positive (rather than merely  $\geq 0$ ) on  $\sigma'$ , so  $\sigma' \not\subset H$ , thus making the generalized holographic screen distinct from the original holographic screen. That is, the generalized holographic screen we consider in this and the next sections is foliated by leaves that are each *normal surfaces* (i.e., for which  $\theta_l < 0$  and  $\theta_k > 0$ ).

Let us first upper bound  $S^{(\text{outer})}[\sigma']$  for some leaf  $\sigma' \subset H'$ . We can choose the spacetime in the complement of  $O_W(\sigma')$  to be the one that maximizes the area of the HRT surface  $X_{\text{HRT}}$ . By definition,  $\theta_k = \theta_l = 0$  on  $X_{\text{HRT}}$  and further there exists some Cauchy surface  $\Sigma$  on which  $X_{\text{HRT}}$  is a surface of minimal cross-sectional area. If  $\sigma' \subset \Sigma$ , we have  $A[\sigma'] \geq A[X_{\text{HRT}}]$  by definition of  $\Sigma$ . Moreover, if  $\sigma' \subset I^+(\Sigma)$ , then  $N_{-k}(\sigma')$  intersects  $\Sigma$  on some codimension-two surface  $X^+$ , while if  $\sigma' \subset I^-(\Sigma)$ , then  $N_{+l}(\sigma')$  intersects  $\Sigma$  on some codimension-two surface  $X^-$ .<sup>4</sup> Since  $X_{\text{HRT}}$  is a surface of minimal cross-sectional area on  $\Sigma$ , it follows that  $A[X_{\text{HRT}}] \leq A[X^+]$  and  $A[X_{\text{HRT}}] \leq A[X^-]$ . By the Raychaudhuri equation (5) in the  $k$  direction and the fact that  $\theta_k > 0$  on  $\sigma'$ , it follows that  $\theta_k > 0$  on the entire segment of  $N_{-k}(\sigma')$  between  $X^+$  and  $\sigma'$ , so  $A[X^+] < A[\sigma']$ . Similarly, the Raychaudhuri equation in the  $l$  direction is

$$\nabla_l \theta_l = -\frac{1}{D-2} \theta_l^2 - \zeta_l^2 - R_{ab} l^a l^b, \quad (24)$$

where  $\zeta_l$  is the shear of the  $l$  congruence and  $\nabla_l = l^a \nabla_a$ . As a result, since  $\theta_l < 0$  on  $\sigma'$ , we have  $\theta_l < 0$  on the entire segment of  $N_{+l}(\sigma')$  between  $\sigma'$  and  $X^-$ , so  $A[X^-] < A[\sigma']$ . Since we have been considering the spacetime in which the area of  $X_{\text{HRT}}$  is maximal for fixed  $O_W(\sigma')$ , we have

<sup>3</sup>Note that this is similar to, but somewhat stronger than, the null dominant energy condition (NDEC), which requires the NEC plus the stipulation that  $-T^a_b k^b$  be a causal vector for all null  $k$ . While the NDEC allows for the cosmological constant contribution, for either sign of  $\Lambda$ , to be folded into  $T_{ab}$ , it does not bound the sign of  $T_{kl}$  on its own, which the  $\Lambda$ DEC does.

<sup>4</sup>An intersection of  $N_l$  with  $\Sigma$  is guaranteed by a no-go theorem for topology change in general relativity: since  $M$  is by hypothesis globally hyperbolic, it has a Cauchy surface and  $M \simeq \Sigma \otimes \mathbb{R}$  [51], so any causal hypersurface that completely divides the spacetime—such as  $N_l(\sigma')$  or  $N_{-k}(\sigma') \cup N_{+l}(\sigma')$  [43]—must intersect any Cauchy surface in a codimension-two surface of finite area.

$S^{(\text{outer})}[\sigma'] = A[X_{\text{HRT}}]/4G\hbar$ . We thus obtain an upper bound on the outer entropy of  $\sigma'$ :<sup>5</sup>

$$S^{(\text{outer})}[\sigma'] \leq \frac{A[\sigma']}{4G\hbar}. \quad (25)$$

## 4 Holographic Dual for Spherically-Symmetric Spacetimes

Beyond the upper bound in Eq. (25), we would like to have an explicit expression for the outer entropy  $S^{(\text{outer})}[\sigma']$ , defined in Sec. 3, for the generalized holographic screen constructed in Sec. 2. While there are subtleties for general spacetimes, we can derive an explicit expression in the case of spherically-symmetric surfaces  $\sigma'$ .

Before assuming spherical symmetry, let us first establish some intermediate results. First, we note that, for the  $\sigma \subset H$  for which  $\sigma' \subset N_{-k}(\sigma)$ , there exists (since by hypothesis  $\sigma$  is an outermost marginally-trapped surface) a partial Cauchy surface  $\Sigma \subset O_W(\sigma)$  such that for any slice  $\rho$  of  $\Sigma$ , which by definition subtends  $\sigma$ ,  $A[\rho] > A[\sigma]$ . Such a partial Cauchy surface also exists for  $\sigma'$ , since  $\theta_k \geq 0$  between  $\sigma'$  and  $\sigma$  and is positive at  $\sigma'$ : simply take the union of  $\Sigma$  and  $N_{-k}(\sigma) \cap N_{+k}(\sigma')$ . Thus, there exists a Cauchy surface  $\Sigma' \supset \sigma'$  for which  $\Sigma'^-$  connects  $\sigma'$  with the boundary and such that every slice  $\rho \subset \Sigma'^-$  satisfies  $A[\rho] > A[\sigma']$ .

We can prove that  $X_{\text{HRT}}$  is in  $\overline{D}(\Sigma'^+)$ , the closure of the domain of dependence of  $\Sigma'^+$ , the interior partial Cauchy surface ending on  $\sigma'$ . We recall that  $I^\pm(\sigma')$  and  $\overline{D}(\Sigma'^\pm) - \sigma'$  form a partition of  $M - \sigma'$ . Suppose that  $X_{\text{HRT}} \not\subset \overline{D}(\Sigma'^+)$ . Then either  $N_{-l}(X_{\text{HRT}})$  or  $N_{+k}(X_{\text{HRT}})$  intersects  $\Sigma'^-$  on some surface  $\zeta$ . We have  $A[\zeta] > A[\sigma']$ . Moreover, by the Raychaudhuri equation along  $N_{-l}(X_{\text{HRT}})$  and  $N_{+k}(X_{\text{HRT}})$ , we have  $A[X_{\text{HRT}}] \geq A[\zeta]$ . Hence,  $A[X_{\text{HRT}}] > A[\sigma']$ , in contradiction with the result established in Sec. 3 that  $A[\sigma'] \geq A[X_{\text{HRT}}]$ . We therefore must have  $X_{\text{HRT}} \subset \overline{D}(\Sigma'^+)$ .

### 4.1 Construction

In order to place a lower bound on  $S^{(\text{outer})}[\sigma']$  for a spherically-symmetric  $\sigma'$ , it suffices to analyze spacetimes that are also spherically symmetric in the interior of  $\sigma'$ ; for these geometries, we can find the maximal HRT surface and calculate its area. We will do this presently and subsequently argue that our construction is optimal over *all* geometries, producing the HRT surface of maximal area for fixed  $O_W(\sigma')$ , so our lower bound is in fact saturated.

To construct our spacetime outside of  $O_W(\sigma')$ , we will use the *characteristic initial data formalism* [53–59] as in Ref. [39]. Given a Cauchy surface formed by light sheets, the characteristic

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<sup>5</sup>Ref. [52] reaches a similar conclusion.

initial data formalism implies that a spacetime exists for self-consistent initial data satisfying the *constraint equations*. For the null portion of a Cauchy surface in the  $k$  direction, the constraint equations are [29, 30, 60–64]

$$\begin{aligned}
\nabla_k \theta_k &= -\frac{1}{D-2} \theta_k^2 - \varsigma_k^2 - G_{kk} && \text{[Raychaudhuri]} \\
\mathcal{L}_k \omega_i &= -\theta_k \omega_i + \frac{D-3}{D-2} \mathcal{D}_i \theta_k - (\mathcal{D} \cdot \varsigma_k)_i + G_{ik} && \text{[Damour-Navier-Stokes]} \\
\nabla_k \theta_l &= -\frac{1}{2} \mathcal{R} - \theta_k \theta_l + \omega^2 + \mathcal{D} \cdot \omega + G_{kl}, && \text{[Cross-focusing]}
\end{aligned} \tag{26}$$

while for a null portion of a Cauchy surface in the  $l$  direction, the constraint equations become

$$\begin{aligned}
\nabla_l \theta_l &= -\frac{1}{D-2} \theta_l^2 - \varsigma_l^2 - G_{ll} && \text{[Raychaudhuri]} \\
\mathcal{L}_l \omega_i &= -\theta_l \omega_i - \frac{D-3}{D-2} \mathcal{D}_i \theta_l + (\mathcal{D} \cdot \varsigma_l)_i - G_{il} && \text{[Damour-Navier-Stokes]} \\
\nabla_l \theta_k &= -\frac{1}{2} \mathcal{R} - \theta_k \theta_l + \omega^2 - \mathcal{D} \cdot \omega + G_{kl}. && \text{[Cross-focusing]}
\end{aligned} \tag{27}$$

Here,  $\mathcal{R}$  is the intrinsic Ricci curvature of the codimension-two slices at constant affine parameter and  $G_{ab}$  is the Einstein tensor,  $R_{ab} - \frac{1}{2} R g_{ab}$ . The twist one-form gauge field (the Hájíček one-form) is  $\omega_i = \frac{1}{2} q_{ib} \mathcal{L}_k l^b$ , where  $q_{ab} = g_{ab} + k_a l_b + l_a k_b$  is the induced metric. Lie derivatives are denoted by  $\mathcal{L}$ , while  $\mathcal{D}$  is the transverse covariant derivative within the codimension-two surface. We use letters  $a, b$  for  $D$ -dimensional spacetime indices,  $i, j$  for  $(D-2)$ -dimensional transverse spatial indices in the codimension-two surface, and indices  $k$  and  $l$  for a  $D$ -dimensional spacetime index contracted into null vectors  $k^a$  and  $l^a$ , respectively.

The junction conditions mandate continuity of  $\theta_k$ ,  $\theta_l$ , and  $\omega_i$ , while  $\varsigma_k$  and  $\varsigma_l$  can change discontinuously via an appropriate shock wave in the Weyl tensor [44] (i.e., gravitational waves [65, 66]). We choose  $k$  and  $l$  to be affinely parameterized tangent vectors to null geodesic congruences originating orthogonally from the codimension-two surfaces we consider. We further specify the relative normalization of these vectors to be  $k \cdot l = -1$ , so  $g_{kl} = -1$ . These choices eliminate other terms that could have appeared in the Damour-Navier-Stokes equations in Eqs. (26) and (27) [62]. On the  $k$  and  $l$  congruences, we can define affine parameters  $\nu$  and  $\mu$ , respectively, normalized such that  $\nabla_\nu = \nabla_k$  and  $\nabla_\mu = \nabla_l$ . Using the Einstein equation (23), we can replace  $G_{kl}$  by  $8\pi G T_{kl} - \Lambda g_{kl} = 8\pi G T_{kl} + \Lambda$ ,  $G_{kk}$  by  $8\pi G T_{kk}$ , and  $G_{ll}$  by  $8\pi G T_{ll}$ . The transverse coordinates  $x^i$  are chosen to always lie within the codimension-two surface of constant affine parameter.

For now, we restrict to a spherically-symmetric spacetime in the interior of  $\sigma'$ , i.e., in  $\overline{D}(\Sigma^+)$ . Requiring the energy-momentum tensor to respect the  $SO(D-1)$  invariance of spherical sym-

metry, we must have  $T_{ik} = T_{il} = 0$ . Similarly, the shears  $\varsigma_k$  and  $\varsigma_l$  both vanish, as does the twist one-form  $\omega_i$ . Hence, for spherical spacetimes satisfying the Einstein equation, the constraint equations (26) and (27) become

$$\begin{aligned}\nabla_k \theta_k &= -\frac{1}{D-2} \theta_k^2 - 8\pi G T_{kk} && \text{[Raychaudhuri]} \\ \nabla_k \theta_l &= -\frac{1}{2} \mathcal{R} - \theta_l \theta_k + 8\pi G T_{kl} + \Lambda && \text{[Cross-focusing]}\end{aligned}\tag{28}$$

and

$$\begin{aligned}\nabla_l \theta_l &= -\frac{1}{D-2} \theta_l^2 - 8\pi G T_{ll} && \text{[Raychaudhuri]} \\ \nabla_l \theta_k &= -\frac{1}{2} \mathcal{R} - \theta_l \theta_k + 8\pi G T_{kl} + \Lambda. && \text{[Cross-focusing]}\end{aligned}\tag{29}$$

While the NEC requires that  $T_{kk}$  and  $T_{ll}$  be nonnegative, the  $\Lambda$ DEC imposes similar condition on  $T_{kl}$ . We can rewrite the  $\Lambda$ DEC as the requirement that  $T_{ab} t_1^a t_2^b \geq 0$  for all causal, future-directed vectors  $t_1$  and  $t_2$ . Making the particular choice  $t_1 = k$  and  $t_2 = l$ , we have  $T_{kl} \geq 0$ .

For a spherically-symmetric spacetime, there is a nice relation between the intrinsic Ricci curvature  $\mathcal{R}$  and the null expansion. For a  $(D-2)$ -sphere of radial coordinate  $r$ ,

$$\mathcal{R} = \frac{(D-2)(D-3)}{r^2},\tag{30}$$

which implies  $\nabla_k(\log \mathcal{R}) = -(2/r)(dr/d\nu)$ . Writing  $A \propto r^{D-2}$  for the area of the constant- $\nu$  cross section of  $N_{-k}(\sigma')$ , we therefore have

$$\theta_k = \frac{\nabla_k A}{A} = \frac{D-2}{r} \frac{dr}{d\nu} = -\frac{D-2}{2} \nabla_k \log \mathcal{R}.\tag{31}$$

Given  $O_W(\sigma')$ , let us now construct a particular spacetime and compute its HRT surface. On  $N_{-k}(\sigma')$ , we will choose data with  $T_{kk} = 0$ . Hence, we can solve the Raychaudhuri equation in Eq. (28) to compute  $\theta_k(\nu)$  on  $N_{-k}(\sigma')$ :

$$\theta_k(\nu) = \left[ \frac{1}{\theta_k[\sigma']} + \frac{\nu}{D-2} \right]^{-1},\tag{32}$$

where we define  $\sigma'$  to correspond to the  $\nu = 0$  surface. Thus,  $N_{-k}(\sigma')$  encounters a caustic at affine parameter

$$\nu_c = -\frac{D-2}{\theta_k[\sigma']}.\tag{33}$$

Using the relation (31), we have

$$\mathcal{R}(\nu) = \mathcal{R}[\sigma'] \exp \left[ -\frac{2}{D-2} \int_0^\nu \theta_k(\nu) d\nu \right] = \frac{\mathcal{R}[\sigma']}{\left[ 1 + \frac{\theta_k[\sigma'] \nu}{D-2} \right]^2} = \left[ \frac{\theta_k(\nu)}{\theta_k[\sigma']} \right]^2 \mathcal{R}[\sigma'].\tag{34}$$

Note that if  $\mathcal{R}(\nu)$  and  $\theta_k(\nu)$  diverge to  $+\infty$ , they do so together, as  $r \rightarrow 0$ . However, there exist spacetimes that do not have  $r \rightarrow 0$  accessible along  $N_{-k}(\sigma')$  and hence do not possess a caustic.

Let us define a surface  $X \subset N_{-k}(\sigma')$  on which  $\theta_l = 0$ . For  $X$  to exist, we must choose our data on  $N_{-k}(\sigma')$  such that the affine parameter  $\nu_0$  on which  $\theta_l$  vanishes satisfies  $\nu_0 > \nu_c$ . We choose  $T_{kl}$  to vanish on  $N_{-k}(\sigma')$ .<sup>6</sup> Without loss of generality, let us write  $\theta_l(\nu)$  on  $N_{-k}(\sigma')$  as

$$\theta_l(\nu) = \frac{\theta_k[\sigma']\theta_l[\sigma']q(\nu)}{\theta_k(\nu)} \quad (35)$$

for some function  $q(\nu)$  that satisfies  $q(\nu = 0) = 1$  and  $q(\nu = \nu_0) = 0$  on  $X$ , that is, for some  $\nu_0 \in (\nu_c, 0)$ . Since we seek the first time  $\theta_l$  vanishes when going from  $\sigma'$  along the  $-k$  congruence, without loss of generality we can take  $q(\nu) > 0$  for  $\nu \in (\nu_0, 0]$ . The cross-focusing equation in Eq. (28), combined with Eqs. (32) and (34), then becomes

$$(a\nu + b)^3 q' + (a\nu + b)^2 (cq + d) = e, \quad (36)$$

where the constants  $a, b, c, d, e$  are given by

$$\begin{aligned} a &= \frac{1}{D-2}, & b &= \frac{1}{\theta_k[\sigma']}, & c &= \frac{D-1}{D-2}, \\ d &= -\frac{\Lambda}{\theta_k[\sigma']\theta_l[\sigma']}, & e &= -\frac{\mathcal{R}[\sigma']}{2(\theta_k[\sigma'])^3\theta_l[\sigma']}. \end{aligned} \quad (37)$$

The general solution is

$$q(\nu) = \frac{e}{(c-2a)(a\nu+b)^2} + m(a\nu+b)^{-\frac{c}{a}} - \frac{d}{c}, \quad (38)$$

where  $m$  is a constant of integration that we fix by demanding  $q(\nu = 0) = 1$ . That is,

$$\begin{aligned} q(\nu) &= \left[1 + \frac{\theta_k[\sigma']\nu}{D-2}\right]^{-(D-1)} + \frac{D-2}{D-1} \frac{\Lambda}{\theta_k[\sigma']\theta_l[\sigma']} \left\{1 - \left[1 + \frac{\theta_k[\sigma']\nu}{D-2}\right]^{-(D-1)}\right\} \\ &\quad + \frac{1}{2} \frac{D-2}{D-3} \frac{\mathcal{R}[\sigma']}{\theta_k[\sigma']\theta_l[\sigma']} \left\{\left[1 + \frac{\theta_k[\sigma']\nu}{D-2}\right]^{-(D-1)} - \left[1 + \frac{\theta_k[\sigma']\nu}{D-2}\right]^{-2}\right\}. \end{aligned} \quad (39)$$

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<sup>6</sup>We can make the choice of  $T_{kl}$  and  $T_{kk}$  vanishing on  $N_{-k}(\sigma')$  consistently with energy-momentum conservation  $\nabla^a T_{ab} = 0$ , the NEC, the  $\Lambda$ DEC, and smoothness via a regularization procedure, in which we consider a shell of matter occupying a thin slice of  $N_{-k}(\sigma')$  adjacent to  $\sigma'$ , then take the limit as the shell thickness goes to zero.

Defining

$$\begin{aligned}\xi(\nu) &= \frac{\theta_k(\nu)}{\theta_k[\sigma']}, \\ \rho &= -\frac{1}{2} \frac{D-2}{D-3} \frac{\mathcal{R}[\sigma']}{\theta_k[\sigma']\theta_l[\sigma']}, \\ \lambda &= \frac{D-2}{D-1} \frac{\Lambda}{\theta_k[\sigma']\theta_l[\sigma']},\end{aligned}\tag{40}$$

we can rewrite  $q$  simply as

$$q(\nu) = (1 - \rho - \lambda)\xi(\nu)^{D-1} + \rho\xi(\nu)^2 + \lambda.\tag{41}$$

By definition,  $\rho > 0$ . For now, we will take  $D \geq 4$ , postponing a discussion of the special case of  $D = 3$  to Sec. 4.3.3. The polynomial in Eq. (41) will have a single zero at some real value of  $\xi = \xi_0 > 1$  if and only if

$$\rho + \lambda > 1.\tag{42}$$

See Fig. 5 for an illustration of  $q$  as a polynomial in  $\xi$ . This zero corresponds to the surface  $X$  on which  $\theta_l = 0$ , at affine parameter

$$\nu_0 = \frac{D-2}{\theta_k[\sigma']} \left( \frac{1}{\xi_0} - 1 \right).\tag{43}$$

Since by assumption  $\xi_0 \in (1, \infty)$ , we have  $\nu_0 \in (\nu_c, 0)$ , so  $X$  indeed exists with  $\theta_k$  having no caustic along  $N_{+k}(X) \cap N_{-k}(\sigma') = \Sigma_1$ . The area of  $X$  is

$$A[X] = A[\sigma'] \exp \left[ \int_0^{\nu_0} \theta_k(\nu) d\nu \right] = \frac{A[\sigma']}{\xi_0^{D-2}}.\tag{44}$$

For general  $\rho$ ,  $\lambda$ , and  $D$ , there is no closed-form expression for the zero of Eq. (41), even if it exists. For the present, we will continue to write the zero as  $\xi_0$  and will later consider the cases in which either  $\rho$  or  $\lambda$  is negligible, allowing the zero to be analytically expressed.

From  $X$ , we will follow  $N_{+l}(X)$ , holding  $\theta_l = 0$  fixed, so that the area is stationary along the light sheet. This requires setting  $T_{ll} = 0$  to satisfy the Raychaudhuri equation in Eq. (29). We also set  $T_{kl} = 0$  and hold  $\mathcal{R}$  fixed.

Consider the polynomial  $q(\xi) = (1 - \rho - \lambda)\xi^{D-1} + \rho\xi^2 + \lambda$ . From the fact that  $q(\xi = 1) = 1$ , that  $\xi_0$  gives the unique real zero of  $q(\xi)$  for  $\xi_0 > 1$ , and that  $q(\xi) < 0$  for sufficiently large  $\xi$ , we must have  $dq/d\xi < 0$  at  $\xi = \xi_0$ . By Eq. (41), this requirement implies

$$(D-3)\rho\xi_0^2 + (D-1)\lambda > 0.\tag{45}$$

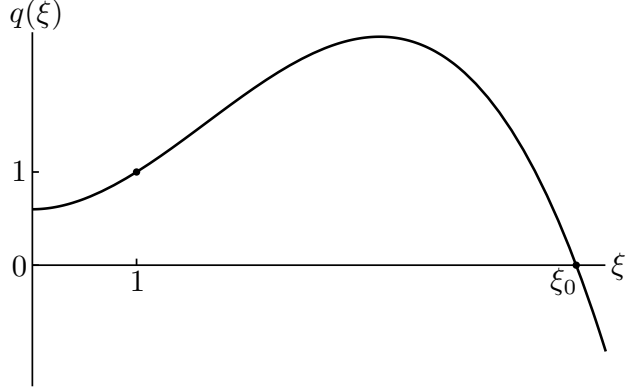


Figure 5: Example of the polynomial  $q(\xi) = (1 - \rho - \lambda)\xi^{D-1} + \rho\xi^2 + \lambda$ . By definition,  $q(\xi = 1) = 1$  and  $\rho > 0$ , but  $\lambda$  is allowed to take either sign. For  $D \geq 4$ , there is exactly one real zero at  $\xi_0 > 1$  if and only if  $\rho + \lambda > 1$ . For small  $\xi$ , the polynomial behaves like  $\rho\xi^2 + \lambda$ , while for large  $\xi$ , the dominant contribution is  $(1 - \rho - \lambda)\xi^{D-1}$ .

Using Eq. (45), along with the definition of  $\xi$  in Eq. (40) and its relation to  $\mathcal{R}(\nu)$  in Eq. (34), we therefore have

$$-\frac{1}{2}\mathcal{R}[X] + \Lambda = -\frac{1}{2}\xi_0^2\mathcal{R}[\sigma'] + \Lambda = \frac{\theta_k[\sigma']\theta_l[\sigma']}{D-2}[(D-3)\rho\xi_0^2 + (D-1)\lambda] < 0. \quad (46)$$

Hence, from the cross-focusing equation in Eq. (29), we find that  $\nabla_l\theta_k < 0$  on  $N_{+l}[X]$ , so there will be some value  $\mu_0$  of the affine parameter  $\mu$  for which  $\theta_k$  vanishes. The surface  $\tilde{X}$  at  $\mu = \mu_0$  satisfies  $\theta_k = \theta_l = 0$ .

We can complete the entire spacetime by CPT reflection about  $\tilde{X}$ . Furthermore, defining  $\Sigma_2 = N_{-l}(\tilde{X}) \cap N_{+l}(X)$ , we observe that  $\tilde{X}$  is a minimal cross section on the Cauchy slice  $\tilde{\Sigma}$  formed by  $\Sigma'^- \cup \Sigma_1 \cup \Sigma_2$  and its CPT reflection. As a result, any other extremal surface  $\hat{X}$  will have greater area than  $\tilde{X}$ , following the argument in Ref. [39]: by the Raychaudhuri equation any slice of  $N_k(\hat{X})$  has area upper bounded by that of  $\hat{X}$  and furthermore the intersection of  $N_k(\hat{X})$  with  $\tilde{\Sigma}$  will have area lower bounded by that of  $\tilde{X}$ , so  $A[\tilde{X}] \leq A[\hat{X}]$ . Hence,  $\tilde{X}$  is an HRT surface, which we will henceforth label as  $X_{\text{HRT}}$ . The area of  $X_{\text{HRT}}$  equals  $A[X]$  by construction. We have thus constructed a lower bound for  $S^{(\text{outer})}[\sigma']$ :

$$S^{(\text{outer})}[\sigma'] \geq \frac{A[X_{\text{HRT}}]}{4G\hbar} = \frac{A[X]}{4G\hbar} = \frac{A[\sigma']}{4G\hbar\xi_0^{D-2}}. \quad (47)$$

Our construction is summarized in Fig. 6.



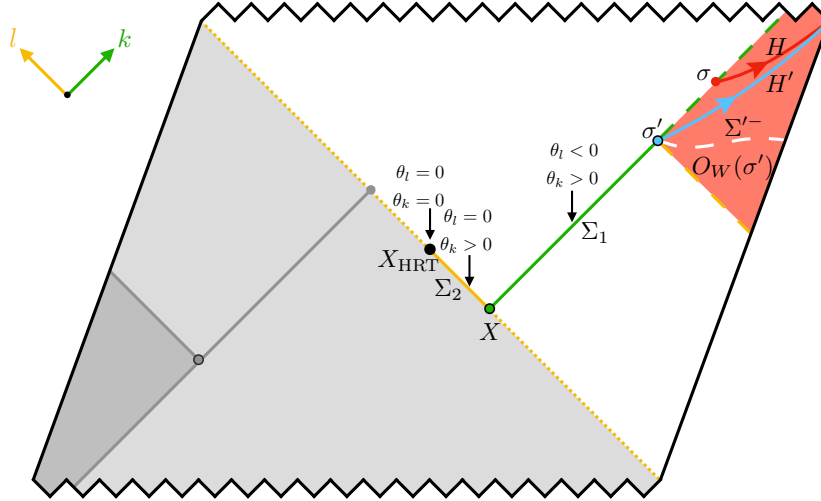


Figure 6: Illustration of the construction of the HRT surface using the characteristic initial data formalism. The outer wedge  $O_W(\sigma')$  (red shading) of  $\sigma'$  (blue dot) is held fixed. We flow along  $N_{-k}(\sigma')$  until we reach a marginally antitrapped surface  $X$ . We then flow along  $N_{+l}(X)$  while keeping cross sections of the light sheet stationary, until we reach a surface  $X_{\text{HRT}}$  where  $\theta_k = \theta_l = 0$  (black dot). The spacetime is completed (gray shading) by CPT reflection across  $N_l(X_{\text{HRT}})$  (orange solid and dotted lines). The partial Cauchy surface  $\Sigma'^-$  (white dashed line) connecting  $\sigma'$  with the boundary by hypothesis satisfies  $A[\rho] > A[\sigma']$  for all cross sections  $\rho \subset \Sigma'^-$ . We note that  $X_{\text{HRT}}$  has minimal cross-sectional area on the Cauchy slice formed by the union of  $\Sigma'^-$ ,  $\Sigma_1 = N_{-k}(\sigma') \cap N_{+k}(X)$  (green solid line), and  $\Sigma_2 = N_{+l}(X) \cap N_{-l}(X_{\text{HRT}})$  (orange solid line), along with their CPT reflections, so  $X_{\text{HRT}}$  is indeed an HRT surface.

## 4.2 Optimization

We now argue that our construction in Sec. 4.1 is in fact optimal. Namely, for a spherically-symmetric  $\sigma'$  with its outer wedge fixed, the construction produces the spacetime that has the HRT surface with the largest possible area (subject to the NEC and  $\Lambda$ DEC). This implies that our lower bound in Eq. (47) is actually an equality.

We begin by considering an *arbitrary* spacetime satisfying our energy conditions and with the outer wedge of  $\sigma'$  fixed. Since  $X_{\text{HRT}} \subset \overline{D}(\Sigma'^+)$ ,  $N_{-l}(X_{\text{HRT}}) \cap N_{-k}(\sigma')$  is nonempty and, in particular, is some codimension-two surface  $Y$ ; see Fig. 7. Now,  $A[Y] \leq A[\sigma']$ , since  $\theta_k > 0$  along  $N_{-k}(\sigma')$ . The fact that  $\theta_l = 0$  on  $X_{\text{HRT}}$  implies  $\theta_l[Y] \geq 0$ , so since  $\theta_l[\sigma'] < 0$  by construction,  $Y \neq \sigma'$  and  $A[Y] < A[\sigma']$ . By continuity, there must be some surface  $Z \subset N_{+k}(Y) \cap N_{-k}(\sigma')$  for which  $\theta_l[Z] = 0$ . We have  $A[Z] < A[\sigma']$  and, if  $Z \neq Y$ ,  $A[Z] > A[Y]$ .

Recalling the definition of  $\Sigma$  as a Cauchy surface on which  $X_{\text{HRT}}$  has minimal cross-sectional area, we can define the codimension-two surface  $W = N_l(Z) \cap \Sigma$ , which by definition satisfies

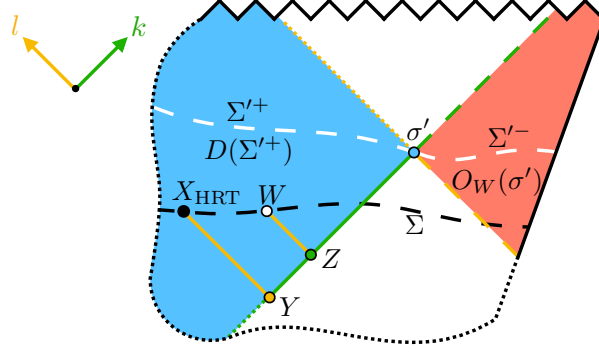


Figure 7: Illustration of various definitions appearing in the procedure for maximizing the area of the HRT surface while keeping the outer wedge  $O_W(\sigma')$  (red shading) of  $\sigma'$  (blue dot) held fixed. The HRT surface  $X_{\text{HRT}}$  (black dot) must appear in the closure of the inner domain of dependence  $D(\Sigma'^+)$  (blue shading) of a Cauchy surface  $\Sigma'$  passing through  $\sigma'$  (white dashed line), so the surface  $Y = N_{-l}(X_{\text{HRT}}) \cap N_{-k}(\sigma')$  exists, on which  $\theta_l \geq 0$ . By continuity, there must exist a surface  $Z \subset N_{+k}(Y) \cap N_{-k}(\sigma')$  on which  $\theta_l = 0$ . By definition, there exists a Cauchy surface  $\Sigma \supset X_{\text{HRT}}$  for which  $X_{\text{HRT}}$  has the minimal cross-sectional area. Since  $A[Z] \geq A[W]$ , where  $W = N_l(Z) \cap \Sigma$ , it follows that  $A[Z] \geq A[X_{\text{HRT}}]$ .

$A[W] \geq A[X_{\text{HRT}}]$ . Since  $\theta_l = 0$  on  $Z$ , it follows from the Raychaudhuri equation (24) that slices of  $N_l(Z)$  have areas upper bounded by  $A[Z]$ , so  $A[W] \leq A[Z]$  and hence  $A[Z] \geq A[X_{\text{HRT}}]$ .

To compute  $S^{(\text{outer})}[\sigma']$ , we must maximize the area of the HRT surface or, equivalently, minimize the quantity

$$\Delta A = A[\sigma'] - A[X_{\text{HRT}}] \quad (48)$$

over all spacetimes with the geometry of  $\sigma'$  held fixed. Let us write  $\Delta A$  as the sum of  $\Delta A_1$  and  $\Delta A_2$ , where

$$\begin{aligned} \Delta A_1 &= A[\sigma'] - A[Z] \\ \Delta A_2 &= A[Z] - A[X_{\text{HRT}}]. \end{aligned} \quad (49)$$

We note that  $\Delta A_1 > 0$  and  $\Delta A_2 \geq 0$ . A sufficient condition for minimizing  $\Delta A$  is to simultaneously minimize  $\Delta A_1$  and  $\Delta A_2$ .

While we have taken  $\sigma'$  to be spherically symmetric, the quantity  $S^{(\text{outer})}[\sigma']$  is in general maximized over all possible spacetimes with  $O_W(\sigma')$  held fixed; in particular,  $N_{-k}(\sigma')$  could a priori break spherical symmetry. Even if this happens, we would take the affine parameter  $\nu$ , which is now defined separately for each generator of  $N_{-k}(\sigma')$ , to respect spherical symmetry at  $\sigma'$ . Specifically, we choose  $\nu = 0$  at  $\sigma'$  and take the normalization of  $\nu$  such that  $\theta_k$  is uniform over  $\sigma'$ .

Let us first choose the data on  $N_{-k}(\sigma') \cap N_{+k}(Z)$  to minimize  $\Delta A_1$ . Because of the Ray-

chaudhuri equation (5) and the NEC, a given area element  $\delta A$  can only decrease toward the  $-k$  direction (recalling that  $\theta_k = d \log \delta A / d\nu$ ). Hence, we optimize the area of  $Z$  by taking  $\varsigma_k = T_{kk} = 0$  along each null geodesic generating  $N_{-k}(\sigma')$ . This implies that without a priori assuming spherical symmetry, we have deduced that the area elements at surfaces of constant  $\nu$  are maximized if they are all given by a simple rescaling of the original area element:

$$\delta A(\nu) = \delta A[\sigma'] \exp \left[ \int_0^\nu \theta_k(\nu) d\nu \right] = \delta A[\sigma'] \left[ 1 + \frac{\nu \theta_k[\sigma']}{D-2} \right]^{D-2}. \quad (50)$$

In particular, the metric on a constant- $\nu$  surface is given simply by conformally rescaling that on  $\sigma'$ , so it is spherically symmetric. Hence, our optimization of  $\Delta A_1$  implies, given a spherically-symmetric surface  $\sigma'$ , that  $N_{-k}(\sigma')$  is also spherical on surfaces of constant affine parameter.<sup>7</sup>

Since we now know that the geometry on  $N_{-k}(\sigma')$  respects spherical symmetry, we expect to have  $\omega_i = T_{ik} = 0$  there as well. This conclusion can also be understood as a consequence of the  $\Lambda$ DEC and our choice of  $T_{kk} = 0$ , via the following argument. By the  $\Lambda$ DEC,  $-T^a_b t^b$  is a causal vector for all causal  $t$ , so in particular  $v^a = -T^a_k$  is causal. By choosing  $T_{kk} = 0$  along  $N_{-k}(\sigma')$ , we have  $v \cdot k = 0$ , so  $v \propto k$ . Since the transverse coordinates are by definition orthogonal to  $k$ ,  $v_i$  vanishes, so  $T_{ik} = 0$ ; see Ref. [67]. By our choice  $T_{kk} = \varsigma_k = 0$ , the Raychaudhuri equation implies that  $\mathcal{D}_i \theta_k = 0$ , as seen in Eq. (50). Hence, the Damour-Navier-Stokes equation in Eq. (26) becomes simply  $\mathcal{L}_k \omega_i = -\theta_k \omega_i$ , which, given the initial condition that  $\omega_i[\sigma'] = 0$  (by spherical symmetry of  $\sigma'$ ), implies that  $\omega_i = 0$  along the entirety of  $N_{-k}(\sigma')$  as expected.

The above choice of the data,  $\varsigma_k = T_{kk} = 0$ , only minimizes  $\nabla_k \theta_k$ . To actually minimize  $\Delta A_1$ , we must also make  $\nabla_k \theta_l$  as large and negative as possible, in order to bring the  $\theta_l = 0$  surface,  $Z$ , to its minimum affine distance from  $\sigma'$ ; see Eq. (50). By Eq. (28), this can be done by taking  $T_{kl} = 0$  along  $N_{-k}(\sigma') \cap N_{+k}(Z)$ . Strictly speaking, we have thus far minimized  $\Delta A_1$  by optimizing each free term of definite sign in the Raychaudhuri and cross-focusing equations in Eq. (26), which is consistent with taking  $Z$  to be a surface of constant affine parameter. The remaining term in the cross-focusing equation,  $\mathcal{D} \cdot \omega$ , has indefinite sign and one could a priori imagine using this term to bring  $Z$  closer to  $\sigma'$  along some generators of  $N_{-k}(\sigma')$ . However, taking  $\omega$  to be nonzero along  $N_{-k}(\sigma')$  requires turning on  $(\mathcal{D} \cdot \varsigma_k)_i$  or  $T_{ik}$  by the Damour-Navier-Stokes equation, which in turn implies positive  $\varsigma_k^2$  or  $T_{kk}$ , which take  $\delta A(\nu)$  away from its optimal profile (50). Moreover, since  $\mathcal{D} \cdot \omega$  integrates to zero over any slice of  $N_{-k}(\sigma')$ , taking this term to be nonzero shifts some areas of  $Z$  closer to  $\sigma'$  and some farther away, in a manner that averages to zero for small  $\omega$ . Since  $\delta A(\nu)$  is convex in  $\nu$ , integrating  $\delta A(\nu)$  over the angular directions for

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<sup>7</sup>This conclusion is closely related to the light-cone theorem [67], which uses stronger assumptions about the energy conditions but a more general geometric setup.

a distribution of  $\nu$  values averaging to  $\bar{\nu}$  always gives a smaller quantity than integrating  $\delta A(\bar{\nu})$  for constant  $\bar{\nu}$ . Hence, a nonzero  $\mathcal{D} \cdot \omega$  term only increases  $\Delta A_1$ , so our procedure thus far has indeed achieved the minimum value of  $\Delta A_1$  consistent with our energy conditions and spherical symmetry of  $\sigma'$ .

We next consider  $\Delta A_2$ . The constraint equations in Eq. (27) imply that we can achieve the optimal configuration of  $\Delta A_2 = 0$  by taking  $Y = Z$ , so that  $\theta_l$  vanishes at  $Y$ , and setting  $\varsigma_l = T_{ll} = 0$  along  $N_{+l}(Y)$  until we reach a surface with  $\theta_k = 0$ . That is, we hold constant affine parameter slices of  $N_{+l}(Y)$  to be stationary, so that each slice has the same area, while keeping  $\omega_i = T_{kl} = 0$ . This part of our setup is the time-reversed and  $k \leftrightarrow l$  analogue of the construction in Ref. [39].

We have now minimized  $\Delta A_1$  and  $\Delta A_2$  simultaneously, producing the HRT surface of maximal area consistent with the outer wedge for spherically-symmetric  $\sigma'$ . The generality of the argument implies that this construction is indeed optimal. Since the construction is precisely what we followed in deriving Eq. (47) in Sec. 4.1, the inequality there is in fact an equality:

$$S^{(\text{outer})}[\sigma'] = \frac{A[\sigma']}{4G\hbar\xi_0^{D-2}}. \quad (51)$$

In particular, this implies that any successful algorithm for maximizing the area of the HRT surface, not necessarily that of Sec. 4.1, would be guaranteed to reproduce Eq. (51).<sup>8</sup>

We emphasize that  $\xi_0$  in Eq. (51) can be computed entirely from geometrical data on  $\sigma'$ . We therefore have a new entry in the holographic dictionary: the spherical outer entropy of  $\sigma'$  is a holographic quantity defined by the geometry of this leaf of the generalized holographic screen. The outer entropy expression in Eq. (51) is one of the main results of this work, giving an entropic interpretation to the generalized holographic screen. This is especially interesting in the case in which  $\sigma'$  corresponds to the event horizon: Eq. (51) provides the first valid interpretation of the event horizon in terms of an entropic, holographic quantity computable from the horizon geometry.

### 4.3 Cases of Interest

Though it is not possible to obtain an analytic expression for  $\xi_0$  from Eq. (41) in complete generality, we can compute it in several cases of interest. The first is the case of negligible  $\lambda$ ,

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<sup>8</sup>For example, had we instead followed  $N_{+l}(\sigma')$  to a surface  $X'$  on which  $\theta_k = 0$  and then followed  $N_{-k}(X')$  to an HRT surface, the optimal construction would have yielded a surface of the same area as given by Eqs. (44) and (51); this follows from the manifest symmetry of Eqs. (44) and (51) under swapping  $k \leftrightarrow l$ :  $\xi_0$  is a zero of the polynomial given in Eq. (41), with coefficients given in Eq. (40) that are invariant under  $k \leftrightarrow l$ .

which corresponds to three possible situations: (i) an asymptotically-flat spacetime with  $\Lambda = 0$ , (ii) a black hole in which  $\mathcal{R} \gg |\Lambda|$ , i.e., a black hole much smaller than the (A)dS scale, and (iii) folding  $\Lambda$  into  $T_{ab}$  and, instead of the  $\Lambda$ DEC requirement, simply requiring the DEC on this entire  $T_{ab}$ . Another case of interest is that of negligible  $\rho$ , corresponding to a black hole much larger than the length scale of the cosmological constant. Other particular situations to consider are three-dimensional spacetimes and surfaces in pure (A)dS or Minkowski space. We will compute  $S^{(\text{outer})}[\sigma']$  for each of these cases in turn.

### 4.3.1 Small $\Lambda$

Let us first consider the case in which  $\Lambda$  is negligible in the polynomial in Eq. (41), i.e., cases (i), (ii), or (iii) above. We can then drop  $\lambda$ , so the zero in  $q$  occurs at

$$\xi_0 = (1 - \rho^{-1})^{-\frac{1}{D-3}}. \quad (52)$$

Note that  $\rho \rightarrow \infty$  corresponds to the apparent horizon, where  $\theta_k[\sigma'] \rightarrow 0$ . Since  $\mathcal{R}[\sigma'] > 0$ ,  $\theta_k[\sigma'] > 0$ , and  $\theta_l[\sigma'] < 0$ , we have  $\rho > 0$ . Moreover, the condition (42) for the zero requires  $\rho > 1$  (which is automatically satisfied for a spherically-symmetric normal surface), so  $\xi_0 > 1$ . Therefore, for generalized holographic screens with a geometry on  $\sigma'$  satisfying  $\rho > 1$ , the spherical outer entropy is

$$S^{(\text{outer})}[\sigma'] = \frac{A[\sigma']}{4G\hbar} \left(1 - \frac{1}{\rho}\right)^{\frac{D-2}{D-3}}. \quad (53)$$

This provides us with an explicit entropic formula for the geometry of a generalized holographic screen, including the event horizon, for any outer wedge associated with a spherically-symmetric normal surface on which the cosmological constant is negligible. It is then straightforward to compute  $\rho$  for various spacetimes of interest and substitute into Eq. (53) to yield the outer entropy.

### 4.3.2 Large $\Lambda$

Let us now consider the opposite limit, in which the cosmological constant dominates over the intrinsic curvature of the generalized holographic screen. Since our construction in Sec. 4.1 required  $\rho + \lambda > 1$ , in the limit in which  $\Lambda$  dominates we must consider a negative cosmological constant  $\Lambda < 0$  in order to have  $\lambda > 0$  (by Eq. (40), recalling that  $\theta_k[\sigma'] > 0$  and  $\theta_l[\sigma'] < 0$ ), so we are in an asymptotically-AdS spacetime. We consider a black hole much larger than the AdS length. In this case, we can drop  $\rho$  from the polynomial in Eq. (41) and solve for  $\xi_0$ :

$$\xi_0 = (1 - \lambda^{-1})^{-\frac{1}{D-1}}. \quad (54)$$

Note that  $\lambda \rightarrow \infty$  corresponds to the apparent horizon,  $\theta_k[\sigma'] \rightarrow 0$ , for fixed  $\Lambda$ . The condition (42) for the zero requires  $\lambda > 1$ , so  $\xi_0 > 1$ . We thus have the outer entropy given by the geometry on  $\sigma'$  in the large black hole limit:

$$S^{(\text{outer})}[\sigma'] = \frac{A[\sigma']}{4G\hbar} \left(1 - \frac{1}{\lambda}\right)^{\frac{D-2}{D-1}}. \quad (55)$$

This is an entropic dual of the geometry of the generalized holographic screen, including the event horizon, for a black hole large compared to the AdS scale.

### 4.3.3 $D = 3$

If  $D = 3$ , the analysis above needs to be modified. In particular, in three spacetime dimensions, the polynomial in Eq. (41) becomes

$$q(\nu) = (1 - \lambda)\xi(\nu)^2 + \lambda, \quad (56)$$

so the terms involving  $\rho$  cancel. Note that, despite the factor of  $D - 3$  in the denominator of  $\rho$  in Eq. (40), there is also a factor of  $D - 3$  in the numerator arising from the intrinsic Ricci curvature given in Eq. (30), so the cancellation of  $\rho$  is well defined. We therefore have

$$\xi_0 = (1 - \lambda^{-1})^{-\frac{1}{2}}, \quad (57)$$

so that the solution behaves like the  $\Lambda$ -dominated case of Sec. 4.3.2. This implies that a surface with  $\theta_l = 0$  can only be reached in  $D = 3$  for  $\Lambda < 0$ .

We can understand what is happening here from the cross-focusing equation for  $\nabla_k \theta_l$  in Eq. (28). Even without assuming spherical symmetry,  $\mathcal{R}$  vanishes in  $D = 3$ , since  $\sigma'$  is simply a curve, which does not have intrinsic curvature. Hence, the only term in Eq. (28) that can be negative—and thus allow  $\theta_l$  to reach zero somewhere on  $N_{-k}(\sigma')$ —is  $\Lambda$ . This requirement of negative cosmological constant accords with the fact that in  $D = 3$  there are no black holes in asymptotically-flat or asymptotically-dS spacetimes, but there do exist BTZ black holes in asymptotically-AdS spacetimes [68].

### 4.3.4 Vanishing entropy for (A)dS

Suppose that  $\sigma'$  is in a region of pure AdS for a black hole formed from collapse; for example,  $\sigma'$  can be in the innermost region of AdS-Vaidya spacetime. In Ref. [41], this spacetime was given as a counterexample to show that the area of the causal surface in this region cannot have a straightforward holographic interpretation as a von Neumann entropy. This conclusion

follows from rigidity of the bulk vacuum, which implies that any spacetime one can construct with  $O_W(\sigma')$  fixed would have no HRT surface for  $\sigma'$  located in a pure AdS region.

We can see how the expression of our entropy in Eq. (51) remains consistent in this setup. For pure AdS spacetime,<sup>9</sup> the metric is given by

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right) dt^2 + \frac{1}{1 + \frac{r^2}{L^2}} dr^2 + r^2 d\Omega_{D-2}^2, \quad (58)$$

where

$$\Lambda = -\frac{(D-1)(D-2)}{2L^2}. \quad (59)$$

For the radial null vectors  $k$  and  $l$ , with  $k \cdot l = -1$ , we can choose the relative normalization to be equal:

$$k^a, l^a = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{1 + \frac{r^2}{L^2}}}, \pm \sqrt{1 + \frac{r^2}{L^2}}, \vec{0} \right). \quad (60)$$

With this choice,  $dr/d\nu = \sqrt{(1 + (r^2/L^2))/2}$  and

$$\theta_k = -\theta_l = \frac{D-2}{\sqrt{2}r} \sqrt{1 + \frac{r^2}{L^2}}. \quad (61)$$

From the definitions in Eq. (40), along with Eqs. (30), (59), and (61), we find that for a spherically-symmetric leaf  $\sigma'$  in a pure (A)dS region,

$$\begin{aligned} \rho &= \left(1 + \frac{r_0^2}{L^2}\right)^{-1}, \\ \lambda &= \frac{r_0^2}{L^2} \left(1 + \frac{r_0^2}{L^2}\right)^{-1}, \end{aligned} \quad (62)$$

where  $r_0$  represents the location of  $\sigma'$ . We thus find that for spacetimes locally AdS, dS, or Minkowski around  $\sigma'$ , spherical light sheets obey Eq. (41) with

$$\rho + \lambda = 1. \quad (63)$$

In these special cases,  $q(\nu)$  does not have a zero for  $\xi(\nu) > 1$ , since the requirement in Eq. (42) is violated. In particular,  $\theta_l \rightarrow -\infty$  when  $\theta_k \rightarrow +\infty$  as  $\nu \rightarrow \nu_c$ , which corresponds to the light sheets converging to a point at  $r = 0$ . This implies that there is no HRT surface, so  $S^{(\text{outer})}[\sigma'] = 0$ . Formally, setting  $\rho + \lambda = 1$  in Eq. (41),  $q(\xi)$  becomes  $\rho\xi^2 + 1 - \rho$ , which has

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<sup>9</sup>For the straightforward extension to dS spacetime, one can simply take  $L^2$  to be negative.

no zero in  $(1, \infty)$  for positive  $\rho$ ; in this case, as  $\xi \rightarrow \infty$  (as  $\theta_k \rightarrow \infty$ ), Eq. (35) implies that  $\theta_l \rightarrow -\infty$ . If we instead take the limit as  $\rho + \lambda \rightarrow 1$ , the zero satisfies  $\xi_0 \rightarrow \infty$ , so Eq. (51) implies that  $S^{(\text{outer})}[\sigma'] \rightarrow 0$ . Thus, the outer entropy we derived in Eq. (51) does not suffer from the problem that the causal holographic information (which was given simply by the area, i.e., Eq. (51) without the  $\xi_0^{D-2}$  factor) had encountered.

## 4.4 The Second Law

Let us now compute how  $S^{(\text{outer})}[\sigma']$  changes as we evaluate it for different leaves  $\sigma'(\tau)$  along the generalized holographic screen  $H'$ . By definition, the outer wedges for consecutive leaves along  $H'$  are nested,  $O_W(\sigma'(\tau_1)) \supset O_W(\sigma'(\tau_2))$  for  $\tau_1 < \tau_2$ . This implies that the spacetime region held fixed when we scan possible spacetimes in finding the HRT surface of maximal area becomes progressively smaller. Since a maximum evaluated on consecutively larger domains can only grow, it follows that we should have  $\nabla_\tau S^{(\text{outer})}[\sigma'(\tau)] \geq 0$ . We will now see explicitly how this comes about for the spherical outer entropy given by Eq. (51), which will serve as a nontrivial check on our result. Note that the area law computed for  $H'$  in Sec. 2.3 does not a priori guarantee a second law for Eq. (51), since  $S^{(\text{outer})}[\sigma'(\tau)]$  is *not* simply the area of  $\sigma'$ ; instead, we will find that the increase in the area of  $\sigma'(\tau)$ , along with the behavior of  $\xi_0(\tau)$ , will combine to give a second law for  $S^{(\text{outer})}[\sigma'(\tau)]$ .

Even though the root  $\xi_0$  of the polynomial in Eq. (41) cannot be expressed in closed form for general  $D$ ,  $\rho$ , and  $\lambda$ , we can still prove the second law for  $S^{(\text{outer})}[\sigma'(\tau)]$ . Recalling that the tangent vector along  $H'$  is  $h'^a = \alpha l^a + \beta k^a$ , we have

$$\begin{aligned} \nabla_\tau \log S^{(\text{outer})}[\sigma'(\tau)] &= \alpha \nabla_l \log S^{(\text{outer})}[\sigma'] + \beta \nabla_k \log S^{(\text{outer})}[\sigma'] \\ &= \alpha[\theta_l - (D-2)\nabla_l \log \xi_0] + \beta[\theta_k - (D-2)\nabla_k \log \xi_0], \end{aligned} \tag{64}$$

where for the rest of this section, we will suppress the implicit argument of  $\sigma'(\tau)$  in variables on the right-hand side. Let us take the  $\nabla_k$  derivative of

$$q(\nu_0) = (1 - \rho - \lambda)\xi_0^{D-1} + \rho\xi_0^2 + \lambda = 0 \tag{65}$$

to get

$$\begin{aligned} (1 - \rho - \lambda) [(D-3)\rho\xi_0^2 + (D-1)\lambda] \nabla_k \log \xi_0 \\ = [\xi_0^2 + \lambda(1 - \xi_0^2)] \nabla_k \rho + [1 - \rho(1 - \xi_0^2)] \nabla_k \lambda, \end{aligned} \tag{66}$$

where we have used the condition (65) again to write  $\xi_0^{D-1}$  in terms of  $\xi_0^2$ . The analogous equation also holds for the  $\nabla_l$  derivative.



From the definitions in Eq. (40), using the constraint equations in Eq. (28) along with Eq. (31), we find

$$\begin{aligned}
\nabla_k \rho &= \rho \left( \mathcal{R}^{-1} \nabla_k \mathcal{R} - \theta_k^{-1} \nabla_k \theta_k - \theta_l^{-1} \nabla_k \theta_l \right) \\
&= \rho \left( \frac{D-3}{D-2} \theta_k + \frac{\mathcal{R}}{2\theta_l} - \frac{\Lambda}{\theta_l} + \frac{8\pi G}{\theta_k} T_{kk} - \frac{8\pi G}{\theta_l} T_{kl} \right) \\
&= \rho \left[ \frac{D-3}{D-2} \theta_k (1-\rho) - \frac{D-1}{D-2} \theta_k \lambda + 8\pi G \left( \frac{T_{kk}}{\theta_k} - \frac{T_{kl}}{\theta_l} \right) \right]
\end{aligned} \tag{67}$$

and

$$\begin{aligned}
\nabla_k \lambda &= -\lambda \left( \theta_k^{-1} \nabla_k \theta_k + \theta_l^{-1} \nabla_k \theta_l \right) \\
&= \lambda \left( \frac{D-1}{D-2} \theta_k + \frac{\mathcal{R}}{2\theta_l} - \frac{\Lambda}{\theta_l} + \frac{8\pi G}{\theta_k} T_{kk} - \frac{8\pi G}{\theta_l} T_{kl} \right) \\
&= \lambda \left[ \frac{D-1}{D-2} \theta_k (1-\lambda) - \frac{D-3}{D-2} \theta_k \rho + 8\pi G \left( \frac{T_{kk}}{\theta_k} - \frac{T_{kl}}{\theta_l} \right) \right].
\end{aligned} \tag{68}$$

Hence, from Eq. (66) we obtain, again using the definition of the zero in Eq. (65) and after some rearrangement,

$$\nabla_k \log \xi_0 = \frac{8\pi G}{\theta_k \theta_l} \frac{\xi_0^{D-1}}{(D-3)\rho\xi_0^2 + (D-1)\lambda} (T_{kl}\theta_k - T_{kk}\theta_l) + \frac{\theta_k}{D-2}. \tag{69}$$

Using this relation and the analogous one for  $\nabla_l$ , Eq. (64) becomes

$$\begin{aligned}
\nabla_\tau S^{(\text{outer})}[\sigma'(\tau)] &= -\frac{8\pi G(D-2)\xi_0^{D-1}S^{(\text{outer})}[\sigma'(\tau)]}{\theta_k\theta_l[(D-3)\rho\xi_0^2 + (D-1)\lambda]} [(\alpha\theta_l + \beta\theta_k)T_{kl} - \alpha T_{ll}\theta_k - \beta T_{kk}\theta_l] \\
&= -\frac{2\pi(D-2)\xi_0 A[\sigma']}{\hbar\theta_k\theta_l[(D-3)\rho\xi_0^2 + (D-1)\lambda]} [(\alpha\theta_l + \beta\theta_k)T_{kl} - \alpha T_{ll}\theta_k - \beta T_{kk}\theta_l].
\end{aligned} \tag{70}$$

Let us consider the signs of the factors appearing in Eq. (70) in turn. The term in brackets in the denominator,  $(D-3)\rho\xi_0^2 + (D-1)\lambda$ , is guaranteed to be positive by Eq. (45). Moreover, by Eq. (8),  $\alpha\theta_l + \beta\theta_k > 0$ . In particular, we have  $\alpha < 0$  and  $\theta_l < 0$  on  $\sigma'$  from the definition of a generalized holographic screen given in Sec. 2.2, while  $\beta > 0$  and  $\theta_k > 0$  since we are considering the outermost spacelike portion of  $H'$ . Together with  $\xi_0 > 1$ , we thus conclude that the entire prefactor in front of the last set of square brackets in Eq. (70) is positive. Now, the NEC requires that  $T_{kk}$  and  $T_{ll}$  are both nonnegative, while the  $\Lambda$ DEC implies that  $T_{kl} \geq 0$ . Thus, all the terms in the last set of square brackets in Eq. (70) are nonnegative. This proves that the outer entropy given in Eq. (51) obeys the second law of thermodynamics,

$$\nabla_\tau S^{(\text{outer})}[\sigma'(\tau)] \geq 0, \tag{71}$$

along the generalized holographic screen. Interestingly, Eq. (70) is reminiscent of a Clausius relation, with  $dS \propto dQ$  for some flow of energy-momentum.

## 5 Conclusions

In this work, we identified a large new class of codimension-one surfaces, the generalized holographic screens, that extend the concept of holographic screens [31] to surfaces that are not marginally trapped. The family of generalized holographic screens connect the concept of holographic screens with event horizons, as both are members of this larger class of geometric objects. We showed in Sec. 2 that all generalized holographic screens satisfy an area theorem (13), thus relating the previously known area theorems of Ref. [1] and Ref. [32] (as well as the related area laws of Refs. [38, 69–71]).

Further, we showed in Secs. 3 and 4 that generalized holographic screens have an entropic interpretation. In Eq. (51), we calculated the outer entropy—the largest von Neumann entropy, computed via the HRT formula, for fixed outer wedge—for leaves of the generalized holographic screen for spherically-symmetric spacetimes and subsequently showed that this entropy obeys the second law of thermodynamics.

The interpretation of the event horizon geometry through some relation to the von Neumann entropy—via a well defined holographic prescription—has hitherto been unknown in AdS/CFT. In this paper, we have found such a connection, expressing a particular geometric quantity defined on the event horizon—notably, not simply the area—in terms of the outer entropy. This outer entropy gives the maximum area of the HRT surface for the collection of geometries with fixed causal wedge; equivalently, this expresses the maximal entanglement entropy between the two sides of the black hole for a pure boundary state.

We note that the specific details of the construction of the generalized holographic screen in Sec. 2 are in fact not necessary to obtain the area law result in Eq. (13) or the second law result in Eq. (71).<sup>10</sup> Instead, it is sufficient to require that the outer wedges of infinitesimally separated leaves  $\sigma'(\tau)$  be nested in the outer spacelike direction ( $\alpha < 0$  and  $\beta > 0$ ) and that  $\sigma'(\tau)$  is a normal surface ( $\theta_k > 0$  and  $\theta_l < 0$ ). This is possible, e.g., even if  $\sigma'(\tau)$  is not entirely within  $N_{-k}(\sigma)$  for some single  $\sigma \subset H$  as required for a general holographic screen in Sec. 2. In fact, a weaker set of conditions guaranteeing  $\alpha\theta_l + \beta\theta_k > 0$  is sufficient to obtain the area law of Eq. (13), while the second law of Eq. (71) requires the related condition of positivity of Eq. (70). A related example is the monotonicity theorem for renormalized leaf areas given in Ref. [38].

The generalized holographic screen  $H'$  and holographic screen  $H$  are related to each other by a network of coarse- and fine-graining relationships. As illustrated in panels a) and b) of Fig. 8, the second law on  $H$  associated with increase of the outer entropy can be understood from the nesting

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<sup>10</sup>We thank Raphael Bousso for discussion on this point.

of outer wedges of leaves  $\sigma \subset H$ , i.e., coarse-graining of the data held fixed in the direction of increasing  $\tau$ , and similarly for  $H'$ . Meanwhile, each leaf  $\sigma' \subset H'$  is by definition in  $N_k(\sigma)$  for some leaf  $\sigma \subset H$ . For spacelike  $H'$ ,  $\sigma' \subset N_{-k}(\sigma)$  and we can therefore view the process of going from  $H$  to  $H'$  as a fine-graining (i.e., more data is being held fixed), since  $O_W(\sigma') \supset O_W(\sigma)$ , as shown in panel c) of Fig. 8, illustrating the upper bound  $S^{(\text{outer})}[\sigma'] \leq A[\sigma]/4G\hbar$ . Finally, in the case of a spacelike generalized holographic screen formed via the intersection construction of Sec. 2.4, for each leaf  $\sigma' \subset H'$  there is also a leaf in  $H$  for which  $\sigma'$  is on the  $-l$  light sheet and for which the outer wedge contains  $O_W(\sigma')$ , as illustrated in panel d) of Fig. 8; in this direction, going from  $H$  to  $H'$  can be viewed as a coarse-graining. In this case, the area of the corresponding leaf on  $H$  provides a lower bound on  $S^{(\text{outer})}[\sigma']$ .

This work leaves numerous avenues for future research. Investigation of the explicit boundary formulation of the outer entropy for non-marginally-trapped surfaces, in terms of boundary operators (cf. Ref. [39]) and the boundary density matrix, could prove fruitful. Moreover, it would be very interesting to explore the meaning and utility of the outer entropy of generalized holographic screens in more general spacetimes as a compelling geometric quantity in the context of classical general relativity.

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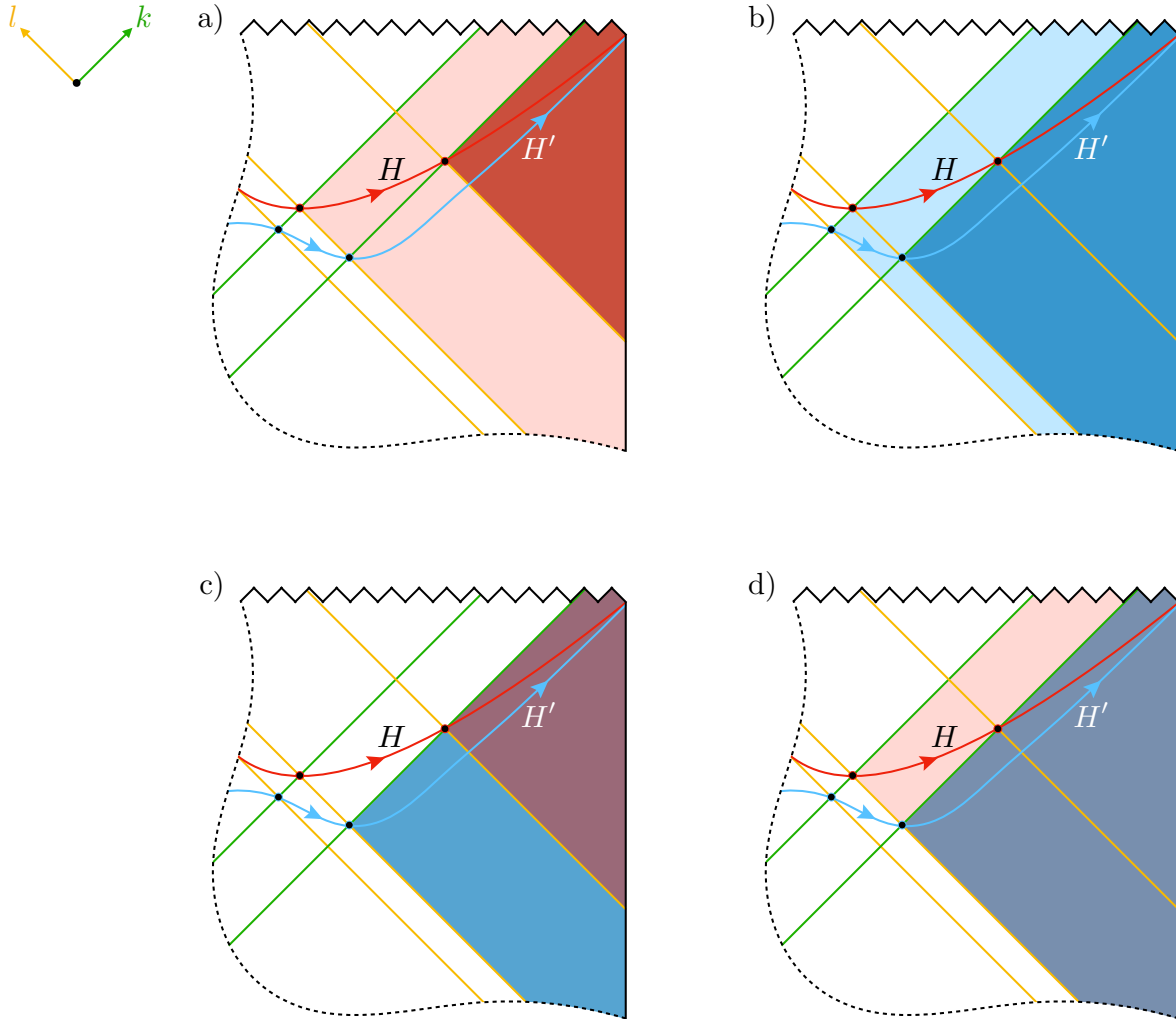


Figure 8: Generic Penrose diagrams illustrating the relationship between the outer wedges of the holographic screen  $H$  (red line) and generalized holographic screen  $H'$  (blue line), in the spacelike case. In the direction of increasing  $\tau$  (arrows), outer wedges of leaves of  $H$  are nested, as shown in panel a). Similarly, wedges of  $H'$  are nested as  $\tau$  increases, as shown in panel b). This nesting mandates an increase in outer entropy on  $H$  and  $H'$ . For spacelike screens, each leaf  $\sigma' \subset H'$  is in  $N_{-k}(\sigma)$  for some leaf  $\sigma \subset H$ , leading to the nesting  $O_W(\sigma') \supset O_W(\sigma)$  illustrated in panel c). In the case of a generalized holographic screen constructed via intersections as in Sec. 2.4, the opposite nesting also occurs, as shown in panel d). Outer wedges attached to leaves on  $H$  ( $H'$ ) are shown in translucent red (respectively, blue), with darker shades indicating increasing  $\tau$ .

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