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The Law of Demand Versus Diminishing Marginal Utility



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# *The Law of Demand versus Diminishing Marginal Utility*

Bruce R. Beattie and Jeffrey T. LaFrance

*Diminishing marginal utility (DMU) is neither necessary nor sufficient for downward-sloping demand. Yet upper-division undergraduate and beginning graduate students often presume otherwise. This paper provides two simple counter-examples that can be used to help students understand that the Law of Demand does not depend on diminishing marginal utility. The examples are accompanied with the geometry and basic mathematics of the utility functions and the implied ordinary/Marshallian demands.*

**Key words:** Convex preferences, diminishing marginal utility, downward-sloping demand

In a combined total of more than a half century of university teaching experience, many students in our advanced undergraduate, master's, and beginning PhD level courses have come to us convinced that the principle of diminishing marginal utility (DMU) is a primary explanation for and cause of downward-sloping demand (DSD) in the theory of consumer behavior. It has been generally accepted since the beginning of the 20th Century that the Law of Demand does not require cardinal utility and the strong assumption of DMU (Samuelson, p. 93). Yet, clearly explaining why this is true continues to be a challenge in teaching consumption theory.

This paper presents two valid utility-function/applied-demand models that can be used by teachers of upper-division and beginning graduate courses to convince students that DMU is neither necessary nor sufficient for DSD. DMU is also not necessary for convex preferences (downward-sloping convex indifference curves, CIC). We have found the counter-examples, and, in particular, the graphics provided in this paper to be most helpful in teaching the fundamentals of consumer behavior and demand. Most agricultural economics courses seriously tackle the theory of consumer behavior as foundation for applied demand and price analysis generally at the upper-division undergraduate and beginning graduate level.

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First, we show that DMU is *not necessary* for CIC. Although not essential for DSD, convex preferences are commonly presumed in the theory of consumer behavior. Next, we establish that DMU is *not necessary* for DSD. Last, we show that DMU is not *sufficient* for DSD.

Our approach is to construct simple counter-examples for each case. To develop the counter-examples,[1] we use two valid utility functions that both satisfy the usual and accepted preference axioms of consumer theory (Varian; Henderson and Quandt; Silberberg and Suen; Mas-Colell, Whinston and Green). We present the essential mathematical results (utility function specifications, marginal utility equations, marginal utility slope equations, indifference curve equations, indifference curve slope and curvature equations, ordinary Marshallian demand equations and demand slope equations) and geometric interpretation. The final section of the paper contains a summary and conclusions.

## DMU, Downward-sloping CIC, and DSD

*Result 1*. DMU is *not necessary* for negatively-sloped CICs.

Assume a simple two-good ($q_1$, $q_2$) Stone-Geary utility ($u$) function:[2]

$$u(q_1, q_2) = q_1^2 q_2^2 . \tag{1}$$

The marginal utility for good one, $MU_1$, for example, is given by

$$\frac{\partial u}{\partial q_1} = 2q_1 q_2^2 . \tag{2}$$

The slope of (2) is

$$\frac{\partial^2 u}{\partial q_1^2} = 2q_2^2 , \tag{3}$$

which is strictly positive for all $q_1$, $q_2 > 0$; i.e., $MU_1$ is everywhere increasing**.**

The indifference curve equation obtained by rearranging (1) is

$$q_2 = \sqrt{u} \big/ q_1 , \tag{4}$$

a rectangular hyperbola in $q_1$. The slope and curvature of (4) is given by (5) and (6), respectively:

$$\left. \frac{dq_2}{dq_1} \right|_u = -\frac{\sqrt{u}}{q_1^2} = -\frac{q_2}{q_1} ; \tag{5}$$

$$\left. \frac{d^2 q_2}{dq_1^2} \right|_u = \frac{2\sqrt{u}}{q_1^3} = \frac{2q_2}{q_1^2} . \tag{6}$$

Clearly (5) is strictly negative and (6) is strictly positive for all $q_1$, $q_2 > 0$.

Figures 1a and 1b show computer-generated, three-dimensional and two-dimensional graphs of the utility function (1) and indifference curve map (4), respectively. It is readily seen in figure

**Figure 1a. Surface plot for $u(q_1, q_2) = q_1^2 q_2^2$.**

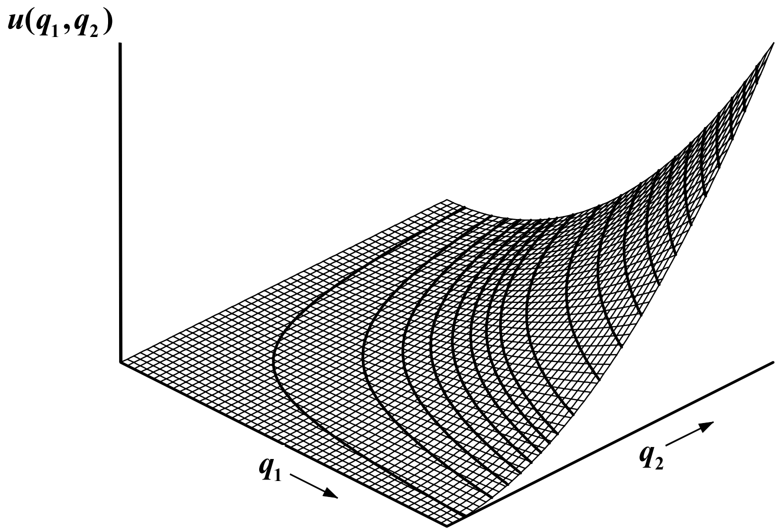
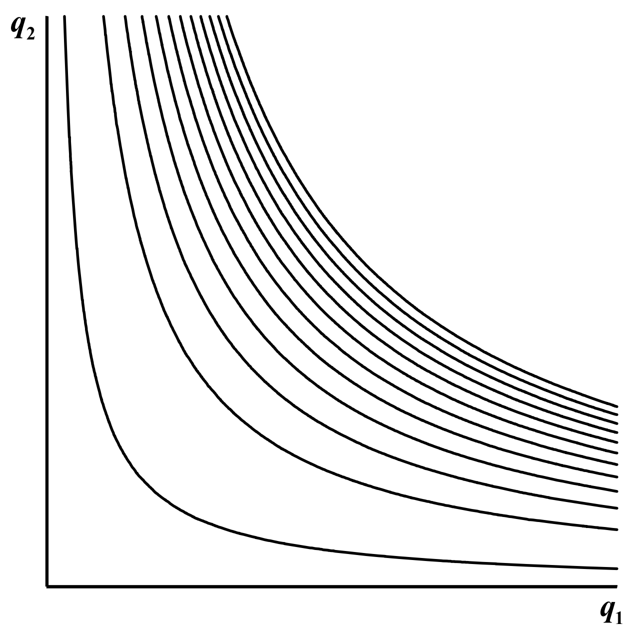



**Figure 1b. Contour plot for $u(q_1, q_2) = q_1^2 q_2^2$.**



1a that marginal utility (MU) increases everywhere on the utility surface for both $q_1$ and $q_2$. And in figure 1b, it is clear that the indifference curves become more dense as one moves across the graph parallel to the $q_1$ axis, increasing $q_1$ while holding $q_2$ constant, or vice versa — again, reflecting increasing MU. Yet, we observe in both figures that the indifference curves are everywhere negatively sloped per equation (5) and convex to the origin per equation (6).

*Upshot:* Despite the fact that the marginal utilities for both goods are everywhere increasing, the indifference curves are everywhere negatively sloped and convex to the origin. This simple utility function clearly shows that DMU is *not necessary* for CIC. More generally, we know that the convexity of the level curves for a two-variable model depends on an expression involving all first and second partial derivatives of the function. In the words of Silberberg and Suen, "…convexity of the indifference curves in no way implies, or is implied by, 'diminishing marginal utility,' . . . diminishing marginal utility and convexity of indifference curves are two entirely independent concepts. And that is how it must be: Convexity of an indifference curve relates to how marginal evaluations change *holding utility* (the dependent variable) *constant.* The concept of diminishing marginal utility refers to changes in total utilities, i.e., movements from one indifference level to another" (pp. 52-53).

*Result 2*. DMU is not necessary for DSD.

To establish that DMU is not necessary for downward-sloping demand, we continue with the Stone-Geary example used in establishing "Result 1" (depicted in figures 1a and 1b). The equation of the demand function for $q_1$, obtained from the solution of the first-order necessary conditions of the budget-constrained maximization of (1), is given by

$$q_1 = m/2p_1 , \tag{7}$$

where $m$ is income and $p_1$ is the price of $q_1$.[3] The slope of (7) is

$$\frac{\partial q_1}{\partial p_1} = -\frac{m}{2p_1^2} . \tag{8}$$

Clearly, the own-price effect from (8) is strictly negative.

*Upshot:* This example depicts that the indifference curves are negatively sloped and convex as $q_1$ increases given $u$ fixed. In consumer theory, the behavior of the marginal utility relationship is immaterial. It is the convexity of the indifference curves, not DMU, that is crucial for DSD in this case.[4] Suffice it to say DMU *is not necessary* for DSD.

Results 1 and 2 follow from the fact that a utility function is unique only up to a monotonic transformation (Varian). An implication of non-uniqueness, in the context of our example Stone-Geary utility function, is that the implied demand functions for the two goods are the same irrespective of the exponents on $q_1$ and $q_2$ in equation (1), as long as they are positive. This, of course, is another way of saying that it does not matter whether marginal utility is decreasing (exponents <1), constant (exponents =1), or increasing (exponents >1).

*Result 3*. DMU is not sufficient for DSD.

To establish that DMU is not sufficient for DSD, we use as our example a utility function that gives rise to a linear upward-sloping demand for $q_1$ and exhibits DMU for both $q_1$ and $q_2$ (LaFrance)[5], viz.,

(9) $$u(q_1, q_2) = (1 - q_1)\exp\left\{\frac{q_2 - 100}{1 - q_1}\right\}.$$

The demand function for $q_1$ implied by (9) is

(10) $$q_1 = 101 + \frac{p_1}{p_2} - \frac{m}{p_2}$$

where $m$ and $p_1$ are as defined previously and $p_2$ is the price of $q_2$. The slope of (10) with respect to $p_1$ is

(11) $$\frac{\partial q_1}{\partial p_1} = \frac{1}{p_2},$$

which is strictly positive rather than negative. The appendix presents complete mathematical derivations for a generalized version of equation (9), including establishment of DMU.

Figures 2a and 2b present the essential geometry of this case. Like the previous figures, figure 2a is the three-dimensional representation of the utility function and figure 2b is the two-space indifference map. In figure 2a, there is DMU for $q_1$ and for $q_2$. The curvature of the utility function in both the $q_1$ and $q_2$ direction is concave to the $q_1q_2$ plane. And in figure 2b, the indifference curves become less dense as $q_1$ increases given $q_2$ and vice versa. Also, we see clearly that the indifference curves are negatively sloped and convex to the origin in the regular region.

*Upshot:* DMU, in addition to being unnecessary, is not sufficient for DSD as sometimes alleged.[6] In this example, we have a perfectly acceptable (well-behaved) utility function giving rise to an upward-sloping demand function even when the marginal utility of that good is diminishing—the long-known *Giffen good* case (Spiegel).

## Conclusion

This paper provides examples of how to convince students of something that often must be unlearned, namely the idea that diminishing marginal utility is the principal rationale for convex indifference curves and downward-sloping demand. The paper presents two simple utility functions to demonstrate the algebra and geometry of why:

1. Diminishing marginal utility is *not necessary* for convex indifference curves.
2. Diminishing marginal utility is *neither necessary nor sufficient* for downward- sloping demand.

Downward-sloping demand can be motivated by appealing to students' common sense. When asked, students will confess that when the price of a good rises, other things constant, they typically reduce their purchases of that good. The instructor can then proceed to explain that they do what it is they know they do, for two reasons: First, they seek and find relatively less expensive substitutes. And, second, an increased price reduces effective purchasing power for all goods. For most (normal) goods, the income effect reinforces the negative substitution effect, contributing further to reduced consumption of the subject good (Stigler, pp. 60-61; Slutsky). The result is unambiguous DSD. There is no need to burden students with something (DMU) that is unnecessary and insufficient for making the case.

When students are ready for a formal treatment of consumer choice (typically at the intermediate level), the why and why not of DSD should be taught and learned drawing upon the assumptions and framework of ordinal utility maximization and the ideas of substitution and

**Figure 2a. Surface plot for** $u(q_1, q_2) = (1 - q_1)\exp\left\{\frac{q_2 - 100}{1 - q_1}\right\}$**.**

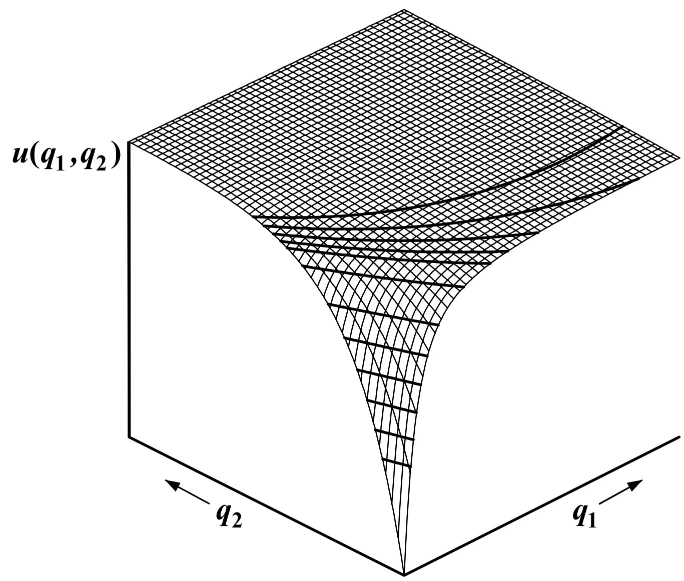
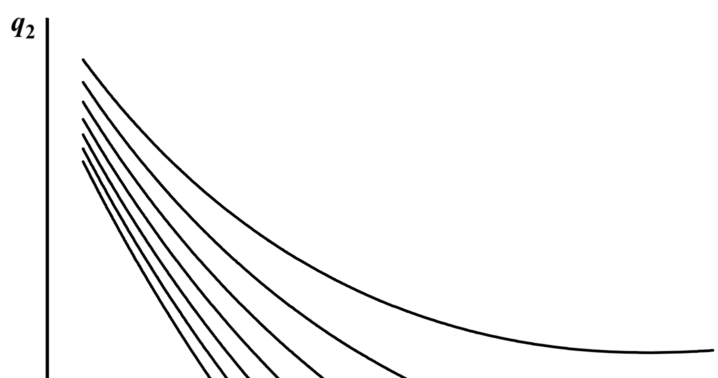



**Figure 2b. Contour plot for** $u(q_1, q_2) = (1 - q_1)\exp\left\{\frac{q_2 - 100}{1 - q_1}\right\}$**.**



income effects. We have found the two utility functions and associated graphs in this paper most helpful in dissuading students who still want to believe that the underlying motivation for DSD is the cardinal utility idea of DMU.

## Appendix: Insufficiency of DMU for DSD

The following utility function gives rise to a positively-sloped linear demand function for $q_1$ despite DMU for both $q_1$ and $q_2$:

(A1) $$u(q_1,q_2)=\frac{(\beta+\gamma q_1)}{\gamma^2}\exp\left\{\frac{\gamma(\alpha+\gamma q_2-q_1)}{\beta+\gamma q_1}\right\}$$

assuming $\alpha>m/p_1>0$, $\beta>0$, and $\gamma<0$. The first-order partial derivatives of (A1) are

(A2) $$\frac{\partial u}{\partial q_1}=-\left(\frac{\alpha+\gamma q_2-q_1}{\beta+\gamma q_1}\right)\exp\left\{\frac{\gamma(\alpha+\gamma q_2-q_1)}{\beta+\gamma q_1}\right\}$$

$$>0\ \forall\ \alpha+\gamma q_2>q_1>-\beta/\gamma.$$

(A3) $$\frac{\partial u}{\partial q_2}=\exp\left\{\frac{\gamma(\alpha+\gamma q_2-q_1)}{\beta+\gamma q_1}\right\}>0\ \forall\ q_1\neq-\beta/\gamma.$$

The condition $q_1\neq-\beta/\gamma$ is necessary for the utility function to be well-defined, while the conditions $\alpha+\gamma q_2>q_1>-\beta/\gamma$ are necessary for the utility function to be increasing in both goods. The ratio $-\beta/\gamma>0$ may be arbitrarily small, but is not necessarily so. The utility function has a pole (can equal any real number) at the point

$$(q_1,q_2)=\left(-\beta/\gamma,\ -(\alpha\gamma+\beta)/\gamma^2\right).$$

Since $\dfrac{\partial u/\partial q_1}{\partial u/\partial q_2}=-\left(\dfrac{\alpha+\gamma q_2-q_1}{\beta+\gamma q_1}\right)=\dfrac{p_1}{p_2}$ and $q_2=\dfrac{m}{p_2}-\dfrac{p_1}{p_2}q_1$ at an interior solution for the demand equations, we obtain the demand for $q_1$ as

(A4) $$q_1=\alpha+\beta\frac{p_1}{p_2}+\gamma\frac{m}{p_2}.$$

Note that the demand for good one is upward sloping with respect to its own price and downward sloping with respect to income, the classic case of a Giffen good. This property holds for all values of $(p_1,p_2,m)$ that lead to an interior utility maximizing solution.

We now show that in the range where both goods are purchased in positive quantities and where preferences are strictly increasing in both goods, this utility function (A1) exhibits diminishing marginal utility in $q_1$ and in $q_2$. The second-order partial derivatives of (A1) are

(A5) $$\frac{\partial^2 u}{\partial q_1^2}=\frac{\left[\beta+\gamma(\alpha+\gamma q_2)\right]^2}{(\beta+\gamma q_1)^3}\exp\left\{\frac{\gamma(\alpha+\gamma q_2-q_1)}{\beta+\gamma q_1}\right\}\leq 0\ \forall\ q_1>-\beta/\gamma.$$

(A6) $$\frac{\partial^2 u}{\partial q_2^2}=\frac{\gamma^2}{(\beta+\gamma q_1)}\exp\left\{\frac{\gamma(\alpha+\gamma q_2-q_1)}{\beta+\gamma q_1}\right\}\leq 0\ \forall\ q_1>-\beta/\gamma.$$

Both (A5) and (A6) are strictly negative throughout the region of strict monotonicity, $\alpha + \gamma q_2 > q_1 > -\beta/\gamma$. In fact, $u$ is concave and a simple transformation of this particular normalization (in particular, $-u^2$) is jointly strongly concave in $(q_1, q_2)$, i.e.

(A7) $$\frac{\partial^2 u}{\partial q_1 \partial q_2} = \frac{-\gamma\left[\beta + \gamma\left(\alpha + \gamma q_2\right)\right]}{\left(\beta + \gamma q_1\right)^2} \exp\left\{\frac{\gamma\left(\alpha + \gamma q_2 - q_1\right)}{\beta + \gamma q_1}\right\}$$

(A8) $$|\boldsymbol{\mathcal{H}}| = \left(\frac{\partial^2 u}{\partial q_1^2}\right)\left(\frac{\partial^2 u}{\partial q_2^2}\right) - \left(\frac{\partial^2 u}{\partial q_1 \partial q_2}\right)^2 \equiv 0 \ \forall \ (q_1, q_2).$$

The reason for condition (A8) is as follows. At any point $(q_1, q_2)$ in the two-dimensional plane, define the constant $\gamma\left(\alpha + \gamma q_2 - q_1\right)/\left(\beta + \gamma q_1\right) = c$. Then the utility function is linear in $q_1$ (equivalently, linear in $q_2$, or jointly linear in $q_1$ and $q_2$) on the line defined by

$$q_2 = \left[-\alpha\gamma + \beta c + \gamma q_1(1+c)\right]/\gamma^2 .$$

Note that this line passes through the point $(-\beta/\gamma, (\alpha\gamma + \beta)/\gamma)$, the pole of the utility function. Even so, the preference function is jointly concave in $(q_1, q_2)$, and it is easy enough to show that the monotonic transformation $-u^2$ is a strictly concave function of the original $u$, which is strongly concave (has a strictly negative Hessian) throughout the region of regularity for $u$.

Thus, this utility function (or a simple transformation of it) possesses the property of diminishing marginal utility in both goods, yet generates a demand for one of the goods that violates the Law of Demand.

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## Endnotes

[1]The advantage of counter-examples is, of course, that the *general* validity of a proposition can be refuted with a single-counter example.

[2]The Stone-Geary utility function first appeared in the literature in the late 1940s, after its production economics counterpart – the Cobb-Douglas production function. Owing to its simplicity and tractability, numerous textbook authors have used the Stone-Geary functional form to provide a concrete demonstration of convex indifference curves and the derivation of consumer demand functions. See, for example, Silberberg and Suen, Henderson and Quandt, Mas-Colell, Whinston and Green, and Varian.

[3]While unnecessary for our purpose here, a more general version of the Stone-Geary utility function, $u(q_1, q_2) = (q_1 - \alpha_1)^{\beta}(q_2 - \alpha_2)^{1-\beta}$, yields demands that are functions of both product prices, $q_i = \alpha_i + \beta_i(m - \alpha_1 p_1 - \alpha_2 p_2)/p_i$, $i = 1,2$, where $\beta_2 = 1 - \beta_1$.

[4]We see when we get to "Result 3" that even convexity of the indifference curves does not guarantee DSD.

[5]This form of utility function generates a single linear demand equation (Hausman). This type of utility model is commonplace among applied researchers wanting to estimate systems of linear demands (Burt and Brewer; Cicchetti, Fisher, and Smith; LaFrance; LaFrance and deGorter; von Haefen).

[6]The untidy suggestion that DMU gives rise to (is sufficient for) DSD may trace to Friedman. In his influential *Price Theory* (1962, p.39), Friedman unfortunately stated,

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