Title

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RESULTS IN MODAL CORRESPONDENCE THEORY FOR POSSIBILITY SEMANTICS

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1. Introduction

Possibility semantics [19] (based on [20]) is a generalization of standard Kripke semantics that makes use of a concept of *possibility frames*.[1] Like Kripke frames, possibility frames have a set of states and binary accessibility relations for modalities. In addition, possibility frames have a refinement relation, which is a partial order between states. Some states in a possibility model may only partially determine the atomic propositions, in contrast to worlds in Kripke models, which completely determine each atomic proposition. Consequently, possibility frames have a close connection with intuitionistic modal frames, but the former yield *classical* modal logic. As is the case for intuitionistic modal semantics, a key issue for possibility semantics is the interaction between the refinement and accessibility relations. In this setting, modal axioms express properties not only of the accessibility relation but also of the interaction between accessibility and refinement.

While standard Kripke frames are semantically equivalent to complete, atomic and completely additive Boolean algebras with operators (BAOs), possibility frames are semantically equivalent to complete and completely additive, but not necessarily atomic, BAOs. As shown in [19, Theorem 5.27], for any complete and completely additive BAO, there exists a possibility frame that validates the same modal formulae as the BAO does, and vice versa, just as there exists such a modally equivalent Kripke frame for any complete, atomic and completely additive BAO. It follows from this and other results [21] that more normal modal logics are sound and complete with respect to some class of possibility frames than with respect to some class of Kripke frames. For other recent results on possibility semantics and mention related work, see [4, 5, 17, 16].

In the present paper, we show how aspects of correspondence theory, as studied for standard Kripke semantics [1], can be extended to the more general setting of possibility semantics. In Section 2, we introduce possibility semantics briefly, referring to [19] for a more detailed account of the semantics. We define key concepts such as possibility frames, possibility models and the standard translation. In Section 3, we study syntactic sufficient conditions for local correspondence. In particular, we prove the analogue of Sahlqvist's Theorem for possibility semantics, namely, that every Sahlqvist

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[1] What we call "possibility frames" in the present paper are essentially the "*full* possibility frames" of [19].

formula locally corresponds to a first-order formula with respect to possibility frames. This extends a result [19, Proposition 6.23] which states that Lemmon-Scott formulae $\Diamond_{\bar{a}}\Box_{\bar{b}}p \to \Box_{\bar{c}}\Diamond_{\bar{d}}p$ have first-order correspondents over possibility frames. In Section 4, we study more model-theoretic aspects of correspondence theory. We prove a counterpart of van Benthem's characterization [2] of first-order definable modal formulae in terms of preservation by ultrapowers. Finally, in Section 5 we state an open problem for future research and related work.

2. Possibility semantics

2.1. **Introduction to the semantics.** Fix an enumeration $\Phi = \{p_i \mid i \in \kappa\}$ ($\kappa = |\Phi|$) of propositional variables (whose indices we sometimes identify with the variables themselves) and a nonempty set I of modal operator indices. Then the modal language $\mathscr{L}(\Phi, I)$ is generated by the following grammar:

$$\phi ::= p \mid \phi \wedge \phi \mid \neg\phi \mid \phi \to \phi \mid \Box_a \phi,$$

where $\phi \in \mathscr{L}(\Phi, I)$, $p \in \Phi$ and $a \in I$. We assume that $\phi_1 \vee \phi_2$ and $\Diamond_a \phi$ are shorthand for $\neg(\neg\phi_1 \wedge \neg\phi_2)$ and $\neg\Box_a\neg\phi$, respectively.

We view a partially ordered set $\mathbb{P}$ as a topological space whose open sets are the downward closed sets. This is an Alexandrov topology. We denote by $\overline{X}$ and X° the closure and the interior of a set $X \subseteq \mathbb{P}$, so $\overline{X} = \{x \in \mathbb{P} \mid \exists x' \sqsubseteq x\, x' \in X\}$ and $X^\circ = \{x \in \mathbb{P} \mid \forall x' \sqsubseteq x\, x' \in X\}$, where $\sqsubseteq$ is the partial order of $\mathbb{P}$. We write $\mathrm{RO}(\mathbb{P})$ for the set of regular open subsets of $\mathbb{P}$, i.e., those subsets $X \subseteq \mathbb{P}$ such that $\overline{X}^\circ = X$. For $X \subseteq \mathbb{P}$, the least regular open set containing X is $\overline{(\Downarrow X)}^\circ$, where $\Downarrow X$ denotes the least downward closed set containing X. We write X^{ro} for $\overline{(\Downarrow X)}^\circ$. Note that $(\cdot)^{\mathrm{ro}}$ satisfies the axioms of closure operators: for any $X, Y \subseteq \mathbb{P}$, we have $X \subseteq X^{\mathrm{ro}}$, $X^{\mathrm{ro}} = (X^{\mathrm{ro}})^{\mathrm{ro}}$ and $X \subseteq Y \Rightarrow X^{\mathrm{ro}} \subseteq Y^{\mathrm{ro}}$. For $x, y \in \mathbb{P}$, we also write $x \between y$ to indicate that x and y are *compatible*, i.e., $\exists z\,(z \sqsubseteq x \wedge z \sqsubseteq y)$. We write $x \perp y$ to indicate that it is not the case that $x \between y$.

We give a definition of possibility frames in the following. Note that, in [19, Definition 2.21], the term "possibility frame" is used for a kind of general frame version of the structures defined in Definition 2.1.(i) below, which are essentially the "*full* possibility frames" of [19, Definition 2.21]. The structures in Definition 2.1.(i) are the possibility-semantic analogues of Kripke frames.

Definition 2.1.

(i) A *possibility frame* is a triple $\mathfrak{F} = (F, \sqsubseteq, (R_a)_{a \in I})$ where $(F, \sqsubseteq)$ is a partially ordered set, each R_a is a binary relation on F, and the set $\mathrm{RO}(\mathfrak{F}) := \mathrm{RO}(F, \sqsubseteq)$ is closed under the map

$$l_a : X (\subseteq F) \mapsto \{y \in F \mid R_a[y] \subseteq X\}$$

for each $a \in I$. We refer to elements of F as *states* of the frame. We call $\sqsubseteq$ and each R_a the *refinement relation* and an *accessibility relation* of $\mathfrak{F}$, respectively.

(ii) A *possibility model* is a pair $\mathfrak{M} = (\mathfrak{F}, \pi)$ where π is a map $\Phi \to \mathrm{RO}(\mathfrak{F})$, called a *valuation* on the frame $\mathfrak{F}$.

When considering a possibility frame $\mathfrak{F}$, we regard l_a as a map $\mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$.

Definition 2.2. Let $\mathfrak{M} = (\mathfrak{F}, \pi)$ be a possibility model and $\phi \in \mathscr{L}(\Phi, I)$.

(i) For $w \in \mathfrak{M}$, define the relation $\mathfrak{M}, w \Vdash \phi$ recursively as follows:[2]

$$\mathfrak{M}, w \Vdash p \iff w \in \pi(p) \quad (p \in \Phi);$$
$$\mathfrak{M}, w \Vdash \phi_1 \wedge \phi_2 \iff \mathfrak{M}, w \Vdash \phi_1 \text{ and } \mathfrak{M}, w \Vdash \phi_2;$$
$$\mathfrak{M}, w \Vdash \neg\phi \iff \forall v \sqsubseteq w\,(\mathfrak{M}, v \nVdash \phi);$$
$$\mathfrak{M}, w \Vdash \phi_1 \to \phi_2 \iff \forall v \sqsubseteq w\,(\mathfrak{M}, v \Vdash \phi_1 \Rightarrow \mathfrak{M}, v \Vdash \phi_2);$$
$$\mathfrak{M}, w \Vdash \Box\phi \iff \forall v\,(Rwv \Rightarrow \mathfrak{M}, v \Vdash \phi).$$

(ii) Let $[\![\phi]\!]^{\mathfrak{M}} = \{w \in \mathfrak{M} \mid \mathfrak{M}, w \Vdash \phi\}$. Call this the *truth set* of ϕ in $\mathfrak{M}$.
(iii) For $w \in \mathfrak{F}$, we write $\mathfrak{F}, w \Vdash \phi$ and say that v *forces* ϕ in $\mathfrak{F}$ if and only if for every possibility model $(\mathfrak{F}, \pi)$, we have $(\mathfrak{F}, \pi), w \Vdash \phi$. $\mathfrak{F}$ *validates* ϕ if and only if for every $v \in \mathfrak{F}$, the formula ϕ is forced by v in $\mathfrak{F}$.

Note that since we define $\vee$ in terms of $\wedge$ and $\neg$, we have the following:

$$\mathfrak{M}, w \Vdash \phi_1 \vee \phi_2 \iff \forall w' \sqsubseteq w\, \exists w'' \sqsubseteq w' (\mathfrak{M}, w'' \Vdash \phi_1 \text{ or } \mathfrak{M}, w'' \Vdash \phi_2).$$

Remark. Let (W,R) be an arbitrary Kripke frame. Let $\mathbb{P} = (W, \sqsubseteq)$ be the discrete partial order on W, i.e., $x \sqsubseteq y \iff x = y$ for $x, y \in W$. Then $\mathfrak{F} = (\mathbb{P}, R)$ is a possibility frame. Let $\mathfrak{M}_0$ be a Kripke model that is an expansion of (W,R), and let $\mathfrak{M}$ be the possibility model that is an expansion of $\mathfrak{F}$ with the same valuation as $\mathscr{M}$. Let $\mathfrak{M}_0$ be a Kripke that is an expansion of (W,R), and let $\mathfrak{M}$ be a possibility model that is an expansion of $\mathfrak{F}$ with the same valuation as $\mathfrak{M}_0$. Then we have

$$\mathfrak{M}_0 \Vdash_{\text{Kripke}} \phi \iff \mathfrak{M} \Vdash \phi$$

for any modal formula ϕ, where $\Vdash_{\text{Kripke}}$ is the forcing relation of Kripke semantics.

In the present paper, we are interested in the relationship between the validity of a modal formula over a possibility frame and the first-order properties of the accessibility and refinement relations in the frame. To see how familiar correspondences from Kripke semantics must be reconsidered in the setting of possibility semantics, it helps to consider a concrete example, such as the following.

Example 2.3. Consider the possibility frame $\mathfrak{F} = (F, \sqsubseteq, R)$ of Figure 1. It can be checked that $\mathfrak{F}$ satisfies the axioms for a possibility frame.[3] Note that for each state w in $\mathfrak{F}$ there exists exactly one v such that Rwv. This property of partial functionality is defined by the F axiom $\Diamond p \to \Box p$ over standard Kripke frames. However, it can be seen that for the state y we have $\mathfrak{F}, y \nVdash \Diamond p \to \Box p$. To see this, observe that the forcing clause for the defined operator $\Diamond$ works out to (see Figure 2):

$$(\mathfrak{F}, \pi), y \Vdash \Diamond\phi \iff \forall v' \sqsubseteq y\, \exists w' (Rv'w' \wedge \exists u \sqsubseteq w' (\mathfrak{F}, \pi), u \Vdash \phi). \tag{1}$$

Consider the valuation π also shown in Figure 1. (It is easy to check that this is indeed a valuation on $\mathfrak{F}$, i.e., $\pi(p)^{\text{ro}} = \pi(p)$.) Then we know $(\mathfrak{F}, \pi), y \Vdash \Diamond p$: in (1), the only possible value of v' is y itself, and one can pick w' to be t so that the right hand side holds. However, we also have $(\mathfrak{F}, \pi), y \nVdash \Box p$, since $t \notin \pi(p)$.

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[2] Note that the clauses for $\neg$ and $\to$ scan the partial order *downward*. This is in line with a convention used in weak forcing (see, e.g., [23]), to which the present semantics is related. In contrast, in the literature on semantics for intuitionistic logic, the convention of going upward is more common.

[3] The possibility frame $\mathfrak{F}$ is constructed from a Kripke frame $(\{0,1,2\}, R)$, where R is the symmetric closure of $\{(1,0),(1,2),(0,0),(2,2)\}$, by *functional powerset possibilization* as in [19, p. 53]. The observations made here follow from the construction.

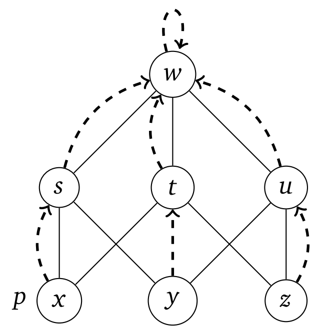
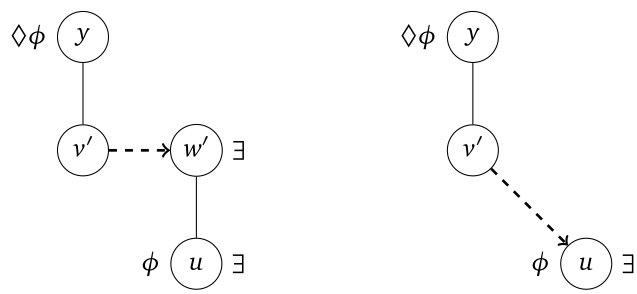


FIGURE 1. A possibility frame $\mathfrak{F}$ and a valuation π on it. The refinement relation of $\mathfrak{F}$ is shown by solid lines as in Hasse diagrams and the accessibility relation is shown by dashed arrows. The valuation π is such that $\pi(p) = \{x\}$.



(1) (4)

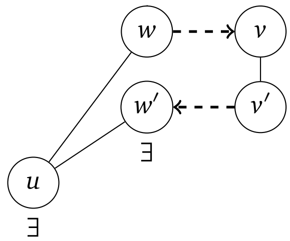
FIGURE 2. Forcing conditions for $\Diamond$. The same conventions as in Figure 1 apply.

Example 2.4. Another example is the B axiom $p \to \Box\Diamond p$. This defines the symmetry of the accessibility relation over standard Kripke frames. The accessibility relation R of $\mathfrak{F}$ from Figure 1 is not symmetric. However, the B axiom is validated by $\mathfrak{F}$; indeed, as we will see later, $p \to \Box\Diamond p$ is validated by $\mathfrak{F}$ if and only if for every $u, v, v' \in \mathfrak{F}$:

$$(Rwv \wedge v' \sqsubseteq v) \Rightarrow \exists w' \, (Rv'w' \wedge w' \between w). \tag{2}$$

(See Figure 3.) All the states in $\mathfrak{F}$ are compatible with one another except that x, y, z are pairwise incompatible. These states are not in the range of R, so (2) holds.

To see why (2) is equivalent to the validity of B, suppose that (2) holds in $\mathfrak{F}$ and that $(\mathfrak{F}, \pi), w \Vdash p$, i.e., $w \in \pi(p)$. We show $(\mathfrak{F}, \pi), w \Vdash \Box\Diamond p$. It suffices to show $(\mathfrak{F}, \pi), v \Vdash \Diamond p$, for an arbitrary v such that Rwv. With (1) in mind, take an arbitrary $v' \sqsubseteq v$. By (2), there exist w' and u such that $Rv'w'$, $u \sqsubseteq w'$ and $u \sqsubseteq w$. Since $\pi(p)$ is open, i.e., downward closed, $u \in \pi(p)$. Then by (1), we have $(\mathfrak{F}, \pi), v \Vdash \Diamond p$. Conversely, suppose that (2) does not hold. For $w \in \mathfrak{F}$, let π be a valuation such that $\pi(p) = \{w\}^{\mathrm{ro}}$. Then $(\mathfrak{F}, \pi), w \Vdash p$. However, we see $(\mathfrak{F}, \pi), w \nVdash \Box\Diamond p$. Indeed, by the failure of (2), there exists v such that $(\mathfrak{F}, \pi), v \nVdash \Diamond p$. This is because if $w' \perp w$ then for all $u \sqsubseteq w'$ we have $u \perp w'$ and thus $u \notin \pi(p) = \{w\}^{\mathrm{ro}}$; for $u \in \{w\}^{\mathrm{ro}}$ if and only if $\forall u' \sqsubseteq u \, u' \between w$.



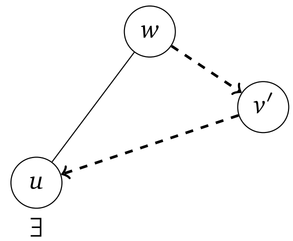


FIGURE 3. Conditions equivalent to the validity of the B axiom. The same conventions as in Figure 1 apply.

It is often the case that conditions on a possibility frame that are equivalent to validity of modal formulae can be simplified by imposing additional conditions on the interaction of the accessibility and the refinement relation in possibility frames. For instance, if we assume

(R-down) $$(Rwv \wedge v' \sqsubseteq v) \Rightarrow Rwv',$$ [4]

it is easily seen that (2) is equivalent to

(3) $$Rwv' \Rightarrow \exists u\,(Rv'u \wedge u \sqsubseteq w),$$

which is much closer to the symmetry of R, the property that the B axiom defines over standard Kripe frames (see again Figure 3). In fact, many familiar modal axioms without $\Diamond$ define the same property over possibility frames satisfying (R-down) as over Kripke frames; for instance, the 4 axiom $\Box p \to \Box\Box p$ is validated by a possibility frame $(F, \sqsubseteq, R)$ satisfying (R-down) if and only if R is transitive. Moreover, (1) can be simplified if $\mathfrak{F}$ satisfies (R-down):

(4) $$(\mathfrak{F}, \pi), y \Vdash \Diamond\phi \iff \forall v' \sqsubseteq y\, \exists u\,(Rv'u \wedge (\mathfrak{F}, \pi), u \Vdash \phi).$$

(See Figure 2.) We refer to [19, Section 2.3] for further discussion of (R-down) and other similar conditions.

A few points should be made about these conditions. First, in Definition 2.1.(i) we stated a condition for a structure $(F, \sqsubseteq, (R_a)_{a\in I})$ to be a possibility frame in terms of RO$(F, \sqsubseteq)$ and l_a; we will see in Section 2.2 that this condition, like (R-down), can be stated in a first-order manner. Second, as shown in [19, Section 2.3], we can assume (R-down) and other conditions on the interaction of R and $\sqsubseteq$ without loss of generality. That is, given a possibility frame $\mathfrak{F}$, we can construct a modally-equivalent possibility frame $\mathfrak{F}'$ that satisfies (R-down) and other interaction conditions (see also Example 3.13). Third, the main results of the present paper hold without imposing these conditions; unless otherwise stated, we do not assume (R-down) and other interaction conditions on possibility frames, beyond those that follow from the definition of possibility frames (again see Section 2.2).

To develop correspondence theory for possibility semantics, we will take an algebraic perspective on possibility frames. An important consequence of the definitions above is that truth sets in an arbitrary possibility model $\mathfrak{M} := (\mathfrak{F}, \pi)$ are always in RO$(\mathfrak{F})$. As is

[4] This condition is often assumed for frames for intuitionistic modal logic (see, e.g., [24]) with the refinement relation flipped. See also Footnote 2.

the case for $\mathrm{RO}(\mathbb{P})$ where $\mathbb{P}$ is an arbitrary partial order, $\mathrm{RO}(\mathfrak{F})$ is a complete Boolean algebra with respect to set inclusion, where the meet is the intersection, the complement is the interior of the set-theoretic complement, and the join is the interior of the closure of the union. One can show that $[\![\phi_1 \wedge \phi_2]\!]^{\mathfrak{M}} = [\![\phi_1]\!]^{\mathfrak{M}} \wedge [\![\phi_2]\!]^{\mathfrak{M}}$, $[\![\neg\phi]\!]^{\mathfrak{M}} = -[\![\phi]\!]^{\mathfrak{M}}$ and $[\![\phi_1 \to \phi_2]\!]^{\mathfrak{M}} = (-[\![\phi_1]\!]^{\mathfrak{M}}) \vee [\![\phi_2]\!]^{\mathfrak{M}}$, where $\wedge$, $-$ and $\vee$ on the right hand sides denote the meet, the complement and the join in $\mathrm{RO}(\mathfrak{F})$, respectively. We trust that no confusion will arise in using the same symbols for the logical connectives and the algebraic operations.

Definition 2.5.

(i) A map $f\colon \mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$ is *completely additive* if it preserves arbitrary joins, i.e., for every family $S \subseteq \mathrm{RO}(\mathfrak{F})$ we have $f(\bigvee S) = \bigvee\{f(X) \mid X \in S\}$. We also say that a map $f\colon \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ is *completely additive in i-th coordinate* for $i \in \{1,\dots,n\}$ if for every $X_1,\dots,X_{i-1},X_{i+1},\dots,X_n \in \mathrm{RO}(\mathfrak{F})$ the map

$$\begin{gathered}\mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})\\ (X_1,\dots,X_{i-1},X,X_{i+1},\dots,X_n) \mapsto f(X_1,\dots,X_{i-1},X,X_{i+1},\dots,X_n)\end{gathered}$$

is completely additive. $f\colon \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ is *completely additive* if it is completely additive in i-th coordinate for every $i \in \{1,\dots,n\}$. *Completely multiplicative* maps are defined similarly, but with joins replaced by meets.

(ii) We say that f is a *left adjoint* of g and that g is a *right adjoint* of f, if $f,g\colon \mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$ satisfy, for $X,Y \in \mathrm{RO}(\mathfrak{F})$,

$$f(X) \subseteq Y \iff X \subseteq g(Y).$$

Note that completely additive maps are order-preserving, and that if f and g both have left adjoints, so does the composite $f \circ g$.

The complete Boolean algebra $\mathrm{RO}(\mathfrak{F})$ becomes a BAO when equipped with the operators l_a for $a \in I$, which are completely multiplicative operators (see [19, Section 2] for more on the duality theory relating possibility frames and BAOs). It is easy to see that, in general, a completely multiplicative map g over a complete lattice $(L,\leq)$ has a left adjoint f of the form $X \mapsto \min\{Z \in L \mid Y \leq g(Z)\}$. In our setting, this implies that each l_a has a left adjoint of the form $Y \mapsto \min\{Z \in \mathrm{RO}(\mathfrak{F}) \mid Y \subseteq l_a(Z)\} = (R_a[Y])^{\mathrm{ro}}$.

2.2. **Translation to classical logic.** Let the signature $\tau = \{\sqsubseteq\} \cup \{R_a \mid a \in I\}$, where $\sqsubseteq$ is a first-order binary relation symbol and each R_a is a first-order binary relation symbol. We write $\mathscr{L}^1(\tau)$ for the first-order τ-language and $\mathscr{L}^2(\tau)$ for the monadic second-order counterpart. $\mathscr{L}^1(\tau)$ will be our first-order correspondence language. We use x,y,z,ξ,η,ζ, etc. for first-order variables and P,Q, etc. for second-order monadic ones. In particular, let $\{P_i\}$ be a set of distinct monadic second-order variables, each P_i corresponding to the propositional variable p_i. Let $\bar{\tau}$ be the signature $\tau \cup \{P_i \mid i \in \kappa\}$.

We regard a possibility frame $\mathfrak{F} = (F,\sqsubseteq,(R_a)_{a\in I})$ as a structure for $\mathscr{L}^1(\tau)$, by letting $\operatorname{dom}\mathfrak{F} = F$, $\sqsubseteq^{\mathfrak{F}} = \sqsubseteq$ and $R_a^{\mathfrak{F}} = R_a$ for each $a \in I$. Likewise, we regard a possibility model $\mathfrak{M} = (\mathfrak{F},\pi)$ as a structure $(\mathfrak{F},(\pi(p))_{p\in\Phi})$ for $\mathscr{L}^1(\bar{\tau})$, as an expansion of $\mathfrak{F}$ with $P_i^{\mathfrak{M}} = \pi(p_i)$. In general, for a structure $\mathfrak{N}$, we use $\models$ for the satisfaction relation for first-order languages, and for parameters $a_1,\dots,a_m \in \mathfrak{N}$ and a first-order formula $\beta(x;y_1,\dots,y_m)$, we write $\beta(\mathfrak{N};a_1,\dots,a_m)$ for the set $\{b \in \mathfrak{N} \mid \mathfrak{N} \models \beta(b;a_1,\dots,a_m)\}$.

We can view a possibility frame $\mathfrak{F}$ as a structure for $\mathscr{L}^2(\tau)$ in two different ways. In one view, which is employed in the rest of this section and Section 3, we consider a possibility frame $\mathfrak{F}$ as a *general prestructure* for $\mathscr{L}^2(\tau)$, with its one-place relational universe being $\mathrm{RO}(\mathfrak{F})$.[5] In the other view, which appears in Section 4, we consider a possibility frame $\mathfrak{F}$ as an (ordinary) structure for $\mathscr{L}^2(\tau)$, with no limitation on values that bound second-order monadic variables can assume. In each case, we again write $\models$ for the corresponding appropriate satisfaction relation for $\mathscr{L}^2(\tau)$.

Having defined classical languages and satisfaction relations, we can see, as in [19, Section 2.2], that the various conditions imposed on possibility frames are actually first-order. First, we can show that there exists a formula $\beta^{Q}_{\mathrm{ro}}(x) \in \mathscr{L}^1(\tau \cup \{Q\})$, where Q is a unary relation symbol, such that for every $X \subseteq \mathfrak{F}$, we have $\beta^{Q}_{\mathrm{ro}}((\mathfrak{F}, X)) = X^{\mathrm{ro}}$, where $(\mathfrak{F}, X)$ is an expansion of $\mathfrak{F}$ that interprets Q as X. Concretely, $\beta^{Q}_{\mathrm{ro}}(x)$ is the formula

$$\forall y \sqsubseteq x\, \exists z \sqsubseteq y\, \exists z' \sqsupseteq z\, Qz'$$

where $\sqsupseteq$ is the inverse of $\sqsubseteq$. With this in mind, it can further be shown [19, Proposition 2.30] that a structure $\mathfrak{F}$ for $\mathscr{L}^1(\tau)$ is a possibility frame if and only if it satisfies (in addition to the axiom of partial orders) the following pair of sentences in $\mathscr{L}^1(\tau)$ for each $a \in I$:

$$\beta^{a}_{R\text{-rule}} :\equiv U((x' \sqsubseteq x \wedge R_a x' y' \wedge y' \between z) \to \exists y\, (R_a x y \wedge y \between z));$$
$$\beta^{a}_{R \Rightarrow \mathrm{win}} :\equiv U(R_a x y \to \forall y' \sqsubseteq y\, \exists x' \sqsubseteq x\, \forall x'' \sqsubseteq x'\, \exists y'' \between y\, R_a x'' y''),$$

where $U(\cdot)$ denotes the universal closure. Understanding the details of these conditions will not be necessary for the purposes of this paper; what will be important for us in this paper is just that the class of possibility frames is first-order definable. We refer to [19, Section 2.3] and [4] for further discussion of these conditions, as well as simpler versions that can be assumed without loss of generality.

We now give the analogue for possibility semantics of the standard translation of modal formulae into first-order formulae.

Definition 2.6. For $\phi \in \mathscr{L}(\Phi, I)$ and a variable x, we define $\mathrm{ST}_x(\phi) \in \mathscr{L}^2(\tau)$ inductively as follows:

$$\begin{aligned} \mathrm{ST}_x(p_i) &= P_i x, \\ \mathrm{ST}_x(\neg\phi) &= \forall y \sqsubseteq x\, \neg\, \mathrm{ST}_y(\phi), \\ \mathrm{ST}_x(\phi_1 \wedge \phi_2) &= \mathrm{ST}_x(\phi_1) \wedge \mathrm{ST}_x(\phi_2), \\ \mathrm{ST}_x(\phi_1 \to \phi_2) &= \forall y \sqsubseteq x\, (\mathrm{ST}_y(\phi_1) \to \mathrm{ST}_y(\phi_2)), \\ \mathrm{ST}_x(\Box_a \phi) &= \forall y\, (R_a x y \to \mathrm{ST}_y(\phi)). \end{aligned}$$

Recall that we are viewing a possibility frame as a general prestructure as explained above. The following definition is standard [1], and the lemmas following it can be proved in the usual way.

Definition 2.7. For $\phi \in \mathscr{L}(\Phi, I)$ and $\alpha(x) \in \mathscr{L}^1(\tau)$, we say that ϕ *locally corresponds* to $\alpha(x)$, or that $\alpha(x)$ is a *local correspondent* of ϕ, if for every possibility frame $\mathfrak{F}$ and $w \in \mathfrak{F}$, we have

$$\mathfrak{F}, w \Vdash \phi \iff \mathfrak{F} \models \alpha(w).$$

For a first-order sentence $\tilde{\alpha} \in \mathscr{L}^1(\tau)$, we say that ϕ *globally corresponds* to $\tilde{\alpha}$, or that $\tilde{\alpha}$ is a *global correspondent* of ϕ, if for every possibility frame $\mathfrak{F}$ we have

$$\mathfrak{F} \Vdash \phi \iff \mathfrak{F} \models \tilde{\alpha}.$$

[5] Our treatment of second-order logic follows [12].

Lemma 2.8. Given a possibility frame $\mathfrak{F}$, $w \in \mathfrak{F}$ and $\phi \in \mathscr{L}(\Phi, I)$, we have

$$\mathfrak{F}, w \Vdash \phi \iff \mathfrak{F} \models U^2(\mathrm{ST}_w(\phi)),^{6}$$

where $U^2(\phi)$ denotes the universal quantification by the monadic second-order variables P_i occurring in ϕ. (Recall that in this section the domain of monadic second-order quanitification is $\mathrm{RO}(\mathfrak{F})$ for a possibility frame $\mathfrak{F}$.)

Lemma 2.9. For $\phi \in \mathscr{L}(\Phi, I)$ and $\alpha(x) \in \mathscr{L}^1(\tau)$, the following are equivalent:

(i) ϕ locally corresponds to $\alpha(x)$.
(ii) For arbitrary possibility frame $\mathfrak{F}$ and $w \in \mathfrak{F}$, we have

$$\mathfrak{F} \models U^2(\mathrm{ST}_w(\phi)) \iff \mathfrak{F} \models \alpha(w).$$

3. Sahlqvist theory

In this section, we prove the possibility-semantic version of Sahlqvist's Theorem.

For $\mathscr{L}(\Phi, I)$, *positive* and *negative* occurrences of propositional variables, and *positive* and *negative* formulae are defined recursively as follows. For $p \in \Phi$, the occurrence of p in $p \in \mathscr{L}(\Phi, I)$ is positive. Suppose an occurrence of p in $\phi \in \mathscr{L}(\Phi, I)$ is positive (respectively, negative) and $\psi \in \mathscr{L}(\Phi, I)$. Then the corresponding occurrences of p in $\phi \wedge \psi$, $\psi \wedge \phi$, $\psi \to \phi$ and $\Box_a \phi$ are positive (respectively, negative); and the corresponding occurrences of p in $\neg\phi$ and $\phi \to \psi$ are negative (respectively, positive). A modal formula is positive (respectively, negative) if all occurrences of all propositional variables in it are positive (respectively, negative).

We define *Sahlqvist antecedents*, *Sahlqvist implications* and *Sahlqvist formulae* in the standard way (see, e.g., [6]). More concretely, they are specified by the following grammar:

$B ::= p_i \mid \Box_a B$	(boxed atoms)
$A ::= B \mid \langle\text{negative formula}\rangle \mid \Diamond A \mid A \wedge A \mid A \vee A$	(Sahlqvist antecedents)
$I ::= A \to \langle\text{positive formula}\rangle$	(Sahlqvist implications)
$F ::= I \mid F \wedge F \mid F \vee F \mid \Box_a F$	(Sahlqvist formulae)

where $i \in \Phi$, $a \in I$, and in the last clause the disjuncts do not have shared variables.

The following is the main theorem of the present section:

Theorem 3.1. Every Sahlqvist formula locally corresponds to a first-order formula in the setting of possibility semantics. Moreover, one can effectively calculate the first-order correspondent from a Sahlqvist formula.

The rest of the present section is devoted to developing a theory necessary to prove the theorem. The argument will be based on *algebraic correspondence theory* [11], although there will be slight changes in terminology and convention.

The key observation is as follows. Call a class function $\mathscr{V}$ a *definably enumerable class* if the domain of $\mathscr{V}$ is the class of possibility frames and there exists a formula $\beta(x; z_1, \ldots, z_k) \in \mathscr{L}^1(\tau)$ such that for every $\mathfrak{F}$ we have

$$\mathscr{V}(\mathfrak{F}) = \{\beta(\mathfrak{F}; w_1, \ldots, w_k) \mid w_1, \ldots, w_k \in \mathfrak{F}\} \cup \{\emptyset\}.$$

[6] By $\mathfrak{F} \models U^2(\mathrm{ST}_w(\phi))$, we mean $U^2(\mathrm{ST}_x(\phi))$ is satisfied by $\mathfrak{F}$ and a variable assignment sending x to w.

Lemma 3.2. Let $\phi(p_0,\dots,p_{n-1}) \in \mathscr{L}(\Phi, I)$ and $\mathscr{V}_0,\dots,\mathscr{V}_{n-1}$ be definably enumerable classes.[7] Assume for every possibility frame $\mathfrak{F}$ and $w \in \mathfrak{F}$, the following are equivalent:

$$\mathfrak{F} \models U^2(\mathrm{ST}_w(\phi)); \tag{5}$$

$$\forall P_0 \in \mathscr{V}_0(\mathfrak{F}) \cdots \forall P_{n-1} \in \mathscr{V}_{n-1}(\mathfrak{F})\,(\mathfrak{F}, P_0,\dots,P_{n-1}) \models \mathrm{ST}_w(\phi). \tag{6}$$

Then, ϕ locally corresponds to a first-order formula.

Proof. Let $\beta_i(x; z^i_1,\dots,z^i_{k_i})$ witness $\mathscr{V}_i$ being definably enumerable. Let $\alpha(x)$ be the first-order formula obtained by replacing, in $U^2(\mathrm{ST}_x(\phi))$, each quantifier $\forall P_i$ by $\forall z^i_1 \cdots \forall z^i_{k_i}$ and each occurrence of $P_i x$ by $\beta_i(x; z^i_1,\dots,z^i_{k_i})$, for each $i \in n$, where z^i_j are fresh variables. Moreover, let $\alpha_\emptyset(x)$ be the formula obtained by replacing, in $\mathrm{ST}_x(\phi)$, each occurrence of $P_i x$ with $x \neq x$. It can easily be seen that ϕ indeed locally corresponds to $\alpha(x) \wedge \alpha_\emptyset(x)$. □

Remark. A statement similar to Lemma 3.2 is true of Kripke semantics, as proved by van Benthem [3, p. 9.15]. Modal formulae that satisfy the hypothesis of the Kripke-semantic version of the lemma belong to a class of formulae that van Benthem called M_1^{sub}, which is now commonly called the class of *van Benthem formulae* [9, Definition 30]. An anonymous reviewer remarked that this class, which includes the class of Sahlqvist formulae, the class of inductive formulae [15] and many more, is beyond the reach of current algorithmic correspondence techniques.

In what follows, by $\mathfrak{F}$ we mean a possibility frame.

Definition 3.3. A modal formula is *normative* if, for each $p \in \Phi$, the number of positive occurrences of p in it is at most one.[8]

In the following, we assume, without loss of generality, that negative propositional variables in a normative Sahlqvist antecedent are all towards the end of the enumeration $p_0, p_1, \dots$ of the propositional variables occurring in the formula.

We will later associate with a normative Sahlqvist antecedent a certain kind of map, a *Sahlqvist map*, between partial orders. Below, n will be the number of propositional variables in a normative antecedent, and m will be the number of those that occur positively.

Definition 3.4. Let $n, m, l \in \omega$ $(m \le n)$ and $\bar{a}_1,\dots,\bar{a}_m \in I^{<\omega}$. A *Sahlqvist map of type* $(n, m, l; \bar{a}_1,\dots,\bar{a}_m)$ is a map of the form $f \circ \langle (g_1 \times \cdots \times g_m) \circ \pi_m, h_1,\dots,h_l\rangle\colon \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ where

(i) $f\colon \mathrm{RO}(\mathfrak{F})^{m+l} \to \mathrm{RO}(\mathfrak{F})$ is completely additive;
(ii) $\pi_m\colon \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})^m$ is the projection onto the first m coordinates, i.e., $\pi_m(X_0,\dots,X_{n-1}) = (X_1,\dots,X_{m-1})$;
(iii) each $g_i\colon \mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$ has a left adjoint of the form
$$Y \mapsto R^{\mathrm{ro}}_{\bar{a}_i}[Y] := (R_{\bar{a}_i(0)}[(R_{\bar{a}_i(1)}[\cdots(R_{\bar{a}_i(|a_i|-1)}[Y])^{\mathrm{ro}}\cdots])^{\mathrm{ro}}])^{\mathrm{ro}};$$
(iv) each $h_i : \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ is order-reversing.

[7] By the notation like $\phi(p_0,\dots,p_{n-1})$, we understand hereafter that all propositional variables occurring in the formula are present in the parentheses.

[8] In [11], a related but slightly different concept of 1-implications is used.

Note that for a formula $\phi(p_0,\dots,p_{n-1}) \in \mathscr{L}(\Phi, I)$ and possibility models $(\mathfrak{F},\pi)$ and $(\mathfrak{F},\pi')$, we have $[\![\phi]\!]^{(\mathfrak{F},\pi)} = [\![\phi]\!]^{(\mathfrak{F},\pi')}$ if $\pi \restriction n = \pi' \restriction n$, where we identify propositional variables with their indices. Write $[\![\phi]\!]^{\mathfrak{F}}$ for the map $\mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ that maps $\tilde{\pi} \in \mathrm{RO}(\mathfrak{F})^n$ to the unique value of $[\![\phi]\!]^{(\mathfrak{F},\pi)}$ where $\pi\colon \Phi \to \mathrm{RO}(\mathfrak{F})$ extends $\tilde{\pi}$.

Lemma 3.5. Let $\phi \in \mathscr{L}(\Phi, I)$ be a positive (respectively, negative) formula. Then $[\![\phi]\!]^{\mathfrak{F}}$ is order-preserving (respectively, order-reversing).

Proof. By simultaneous induction. □

For a sequence of modal indices $\bar{a} \in I^{<\omega}$ and a modal formula ϕ, we define the expression $\Box_{\bar{a}}\phi$ recursively as $\Box_{\langle\rangle}\phi = \phi$ and $\Box_{\bar{a}b}\phi = \Box_{\bar{a}}\Box_b\phi$. Let $r : I^{<\omega} \to I^{<\omega}$ be the string reversal; i.e., $r(\langle\rangle) = \langle\rangle$ and $r(b\bar{a}) = r(\bar{a})b$.

Lemma 3.6. If $\phi(p_0,\dots,p_{n-1})$ is a normative Sahlqvist antecedent, then $[\![\phi]\!]^{\mathfrak{F}}$ is a Sahlqvist map of type $(n, m, l; \bar{a}_0, \dots, \bar{a}_{m-1})$ for some $l \in \omega$, where m is the number of variables that occur positively in ϕ and, for each $i \in m$, the unique positive occurrence of p_i in ϕ follows $\Box_{r(\bar{a}_i)}$.

Proof. By induction. The properties used in the proof are that $\mathrm{RO}(\mathfrak{F})$, the underlying BAO of $\mathfrak{F}$, is a complete and completely additive BAO, making $\wedge$, $\vee$ and the operators for $\Diamond_a$ completely additive; that the operators l_a have left adjoints; and that if $l\colon \mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$ has a left adjoint of the form $Y \mapsto R^{\mathrm{ro}}_{\bar{a}}[Y]$, then $l_b \circ l$ has a left adjoint of the form $Y \mapsto R^{\mathrm{ro}}_{\bar{a}b}[Y]$. □

For $X \in \mathrm{RO}(\mathfrak{F})$, we write $Y \leq_1 X$ if $Y = \{y\}^{\mathrm{ro}}$ for some $y \in X$. Note that if $Y \leq_1 X$ then $Y \subseteq X$.

Lemma 3.7. For $X \in \mathrm{RO}(\mathfrak{F}) \setminus \{\emptyset\}$, we have $X = \bigvee_{Y \leq_1 X} Y$.

Proof. Since $Y \leq_1 X \Rightarrow Y \subseteq X$, we have $\bigvee_{Y \leq_1 X} Y \subseteq X$.

Let $x \in X$ be arbitrary. Then $\{x\}^{\mathrm{ro}} \leq_1 X$, whence $x \in \{x\}^{\mathrm{ro}} \subseteq \bigvee_{Y \leq_1 X} Y$. Therefore, $X \subseteq \bigvee_{Y \leq_1 X} X$. □

Note that the lemma above is a consequence of $(\cdot)^{\mathrm{ro}}$ being a closure operator and $\mathrm{RO}(\mathfrak{F})$ being the set of fixed points of $(\cdot)^{\mathrm{ro}}$.

For $\bar{a} \in I^{<\omega}$, write $\mathbf{V}^{\bar{a}}_1(\mathfrak{F})$ for the family of regular open sets that are either empty or of the form $R^{\mathrm{ro}}_{\bar{a}}[\{z\}^{\mathrm{ro}}]$ where $z \in \mathfrak{F}$. Also, let $\mathbf{V}_0(\mathfrak{F}) := \mathbf{V}_0 := \{\emptyset\}$. For $\bar{a} \in I^{<\omega}$, write $R_{\bar{a}}xy$ if and only if there exist $z_1, \dots, z_{|a|-1} \in \mathfrak{F}$ such that

$$R_{\bar{a}(0)}xz_1 \wedge R_{\bar{a}(1)}z_1z_2 \wedge \cdots \wedge R_{\bar{a}(|a|-1)}z_{|a|-1}y.$$

Lemma 3.8. For each $X \in \mathrm{RO}(\mathfrak{F})$ and for each $\bar{a}$,

$$R^{\mathrm{ro}}_{\bar{a}}[X] = (R_{\bar{a}}[X])^{\mathrm{ro}}.$$

Therefore, $\mathbf{V}^{\bar{a}}_1$ is a definably enumerable class as witnessed by the first-order formula $\beta^{\bar{a}}_1(x;z)$[9]

$$[\exists z' (\lambda y\, R_{\bar{a}}z'y \wedge [\lambda y'\, y' = z/Q]\beta^{Q}_{\mathrm{ro}}(z'))/Q]\beta^{Q}_{\mathrm{ro}}(x).$$

(Recall that $\beta^{Q}_{\mathrm{ro}}(x)$ is a first-order formula that defines X^{ro} in the expansion of $\mathfrak{F}$ that interprets Q as X.)

[9] We use the λ-notation and the notation for syntactic substitution as in [6].

Proof. For $S \subseteq F \times F$, let us define the map l_S by, for $X \subseteq \mathfrak{F}$,

$$l_S(X) = \{x \in \mathfrak{F} \mid \forall y\,(Sxy \Rightarrow y \in X)\}.$$

Then $l_{R_a} = l_a$, and it can be shown that $l_{R_{b\bar{a}}} = l_b \circ l_{R_{\bar{a}}}$. Hence, by induction, we may regard $l_{R_{\bar{a}}}$ as a map $\mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$, for every $\bar{a} \in I^{<\omega}$.

It can easily be seen that $Y \mapsto R^{\mathrm{ro}}_{\bar{a}}[Y]$ is a left adjoint of $l_{a(|a|-1)} \circ \cdots \circ l_{a(0)}$. By reasoning similar to that in the case of l_a, we see that $Y \mapsto R_{\bar{a}}$ is a left adjoint of $l_{R_{r(\bar{a})}}$. Since $l_{a(|a|-1)} \circ \cdots \circ l_{a(0)} = l_{R_{r(\bar{a})}}$, we conclude $R^{\mathrm{ro}}_{\bar{a}}[X] = (R_{\bar{a}}[X])^{\mathrm{ro}}$ for any $X \in \mathrm{RO}(\mathfrak{F})$, by the uniqueness of the left adjoint. □

$\mathbf{V}_0$ is also a definably enumerable class trivially.

Lemma 3.9. Let $f: \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ be a Sahlqvist map of type $(n, m, l; \bar{a}_0, \dots, \bar{a}_{m-1})$ and $G: \mathrm{RO}(\mathfrak{F})^n \to \mathrm{RO}(\mathfrak{F})$ be order-preserving. Then for $w \in \mathfrak{F}$, the following are equivalent:

$$\begin{aligned} &\forall P_0 \in \mathrm{RO}(\mathfrak{F}) \cdots \forall P_{n-1} \in \mathrm{RO}(\mathfrak{F}) \\ &\qquad (w \in f(P_0, \dots, P_{n-1}) \Rightarrow w \in G(P_0, \dots, P_{n-1})); \end{aligned} \tag{7}$$

$$\begin{aligned} &\forall P_0 \in \mathbf{V}_1^{\bar{a}_0}(\mathfrak{F}) \cdots \forall P_{m-1} \in \mathbf{V}_1^{\bar{a}_{m-1}}(\mathfrak{F})\, \forall P_m \in \mathbf{V}_0 \cdots \forall P_{n-1} \in \mathbf{V}_0 \\ &\qquad (w \in f(P_0, \dots, P_{n-1}) \Rightarrow w \in G(P_0, \dots, P_{n-1})). \end{aligned} \tag{8}$$

Proof. For simplicity, assume $n = 2$, $m = 1$, and $l = 2$; it is straightforward to adapt the proof below for the general case.

($\Rightarrow$) is clear. Assume (8). Suppose $f = f_0 \circ \langle g \circ \pi_1, h\rangle$ where $f_0: \mathrm{RO}(\mathfrak{F})^2 \to \mathrm{RO}(\mathfrak{F})$ is completely additive, $g: \mathrm{RO}(\mathfrak{F}) \to \mathrm{RO}(\mathfrak{F})$ is the right adjoint of the map $Y \mapsto R^{\mathrm{ro}}_{\bar{a}_0}[Y]$ and $h: \mathrm{RO}(\mathfrak{F})^2 \to \mathrm{RO}(\mathfrak{F})$ is order-reversing. Take arbitrary $P_0, P_1 \in \mathrm{RO}(\mathfrak{F})$ and assume $w \in f(P_0, P_1)$. We will show $w \in G(P_0, P_1)$.

By the adjunction, we can show that if $P_0 = \emptyset$ then $g(P_0) = \emptyset$. Assume $g(P_0) = \emptyset$. Then $w \in f(P_0, P_1) = f_0(g(P_0), h(P_0, P_1)) = f_0(\emptyset, h(P_0, P_1)) = f_0(g(\emptyset), h(P_0, P_1))$. Since h is order-reversing and f_0 is order-preserving, $w \in f_0(g(\emptyset), h(\emptyset, P_1)) = f(\emptyset, P_1)$. By $\emptyset \in \mathbf{V}_1^{\bar{a}_0}$ and (10), we have $w \in G(P_0, P_1)$.

Assume $g(P_0) \neq \emptyset$. Since h is order-reversing, f_0 is completely additive, and $g(P_0) = \bigvee_{X \leq_1 g(P_0)} X$ (by Lemma 3.7), we have

$$\begin{aligned} w &\in f(P_0, P_1) \\ &= f_0(g(P_0), h(P_0, P_1)) \\ &\subseteq f_0(\bigvee_{X \leq_1 g(P_0)} X, h(P_0, \emptyset)) \\ &= \bigvee_{\{x\}^{\mathrm{ro}} \subseteq g(P_0)} f_0(\{x\}^{\mathrm{ro}}, h(P_0, \emptyset)) \\ &= \bigvee_{R^{\mathrm{ro}}_{\bar{a}_0}[\{x\}^{\mathrm{ro}}] \subseteq P_0} f_0(\{x\}^{\mathrm{ro}}, h(P_0, \emptyset)), \end{aligned}$$

where the last equality follows because g's left adjoint is $Y \mapsto R^{\mathrm{ro}}_{\bar{a}_0}[Y]$. For each $x \in \mathfrak{F}$, let $Q_x = R^{\mathrm{ro}}_{\bar{a}_0}[\{x\}^{\mathrm{ro}}]$. Note that $Q_x \in \mathbf{V}_1^{\bar{a}_0}(\mathfrak{F})$ and that $g(Q_x) \supseteq \{x\}^{\mathrm{ro}}$ (the latter is by the general fact that the composite of a right adjoint after its left adjoint is inflating). Then

$$
\begin{aligned}
w &\in \bigvee_{Q_x \subseteq P_0} f_0(\{x\}^{\mathrm{ro}}, h(P_0, \emptyset)) \\
&\subseteq \bigvee_{Q_x \subseteq P_0} f_0(g(Q_x), h(Q_x, \emptyset)) && (9) \\
&\subseteq \bigvee_{Q_x \subseteq P_0} G(Q_x, \emptyset) && (10) \\
&\subseteq \bigvee_{Q_x \subseteq P_0} G(P_0, P_1) && (11) \\
&= G(P_0, P_1). && (12)
\end{aligned}
$$

The inclusion (9) is by the order-reversing property of h and the order-preserving property of f_0; (10) is by (8); and (11) is because G is order-preserving. ☐

Corollary 3.10. Let $\phi(p_0, \dots, p_{n-1})$ be a normative Sahlqvist antecedent and $\psi(p_0, \dots, p_{n-1})$ be positive. Assume that m is the number of propositional variables that occur positively in ϕ, and that for each $i \in m$ the unique positive occurrence of p_i in ϕ follows $\Box_{\bar{a}_i}$. Then for $w \in \mathfrak{F}$, the following are equivalent:

$$\mathfrak{F} \models U^2(\mathrm{ST}_w(\phi \to \psi)); \tag{13}$$

$$\forall P_0 \in \mathbf{V}_1^{\bar{a}_0}(\mathfrak{F}) \cdots \forall P_{m-1} \in \mathbf{V}_1^{\bar{a}_{m-1}}(\mathfrak{F})\, \forall P_m \in \mathbf{V}_0 \cdots \forall P_{n-1} \in \mathbf{V}_0 \quad (\mathfrak{F}, P_0, \dots, P_{n-1}) \models \mathrm{ST}_w(\phi \to \psi). \tag{14}$$

Proof. Note that, for $w \in \mathfrak{F}$, we have $(\mathfrak{F}, P_0, \dots, P_{n-1}) \models \mathrm{ST}_w(\phi \to \psi)$ if and only if

$$\forall w' \sqsubseteq w (w' \in [\![\phi]\!]^{\mathfrak{F}}(P_0, \dots, P_{n-1}) \Rightarrow w' \in [\![\psi]\!]^{\mathfrak{F}}(P_0, \dots, P_{n-1})).$$

By Lemma 3.6, $[\![\phi]\!]^{\mathfrak{F}}$ is a Sahlqvist map of type $(n, m, l; \bar{a}_0, \dots, \bar{a}_{m-1})$ for some $l \in \omega$. By Lemma 3.5, $[\![\psi]\!]^{\mathfrak{F}}$ is order-preserving. By applying Lemma 3.9 to each $w' \sqsubseteq w$, we obtain the equivalence between (13) and (14). ☐

Corollary 3.11. For any Sahlqvist implication χ with a normative antecedent, there exists a first-order formula $\alpha(x)$ such that χ corresponds to $\alpha(x)$.

Proof. By Corollary 3.10 and Lemma 3.2. ☐

We will now see that the general case reduces to that of normative formulae. For $V, V' \subseteq \mathrm{RO}(\mathfrak{F})$, write $V + V'$ for the family of regular open sets of the form $P \vee P'$, where $P \in V$ and $P' \in V$. Note that if both $\mathscr{V}$ and $\mathscr{V}'$ are definably enumerable classes, so is the class $\mathscr{V} + \mathscr{V}'$ which is defined by $(\mathscr{V} + \mathscr{V}')(\mathfrak{F}) = \mathscr{V}(\mathfrak{F}) + \mathscr{V}'(\mathfrak{F})$.

Lemma 3.12. Let $m \leq n$. Suppose $\phi(p_0, \dots, p_{n-1})$ is a modal formula such that each p_i is positive for $i = 0, \dots, m-1$. Let $\psi(p_m, \dots, p_{n-1})$ be positive. Assume that for definably enumerable classes $\mathscr{V}_0, \dots, \mathscr{V}_{m-1}$ the following are equivalent for each $w \in \mathfrak{F}$:

$$\forall P_0 \in \mathrm{RO}(\mathfrak{F}) \cdots \forall P_{m-1} \in \mathrm{RO}(\mathfrak{F}) \quad (w \in [\![\sigma(\phi)]\!]^{\mathfrak{F}}(P_0, \dots, P_{m-1}) \Rightarrow w \in [\![\sigma(\psi)]\!]^{\mathfrak{F}}(P_0, \dots, P_{m-1})); \tag{15}$$

$$\forall P_0 \in \mathscr{V}_0(\mathfrak{F}) \cdots \forall P_{m-1} \in \mathscr{V}_{m-1}(\mathfrak{F}) \quad (w \in [\![\sigma(\phi)]\!]^{\mathfrak{F}}(P_0, \dots, P_{m-1}) \Rightarrow w \in [\![\sigma(\psi)]\!]^{\mathfrak{F}}(P_0, \dots, P_{m-1})), \tag{16}$$

where

$$\sigma = \left[\bigvee_{0 \le i < m} p_i \middle/ p_m, \dots, \bigvee_{0 \le i < m} p_i \middle/ p_{n-1} \right].$$

Then the following are also equivalent for each $w \in \mathfrak{F}$:

$$\forall P \in \mathrm{RO}(\mathfrak{F}) (w \in [\![\sigma_0(\phi)]\!]^{\mathfrak{F}}(P) \Rightarrow w \in [\![\sigma_0(\psi)]\!]^{\mathfrak{F}}(P)); \tag{17}$$

$$\forall P \in (\mathscr{V}_0 + \cdots + \mathscr{V}_{m-1})(\mathfrak{F}) (w \in [\![\sigma_0(\phi)]\!]^{\mathfrak{F}}(P) \Rightarrow w \in [\![\sigma_0(\psi)]\!]^{\mathfrak{F}}(P)), \tag{18}$$

where $\sigma_0 = [p_0/p_0, \dots, p_0/p_{n-1}]$.

Proof. (17) $\Rightarrow$ (18) is clear. We will see (18) $\Rightarrow$ (16) $\Rightarrow$ (15) $\Rightarrow$ (17). (16) $\Rightarrow$ (15) is by assumption. (15) $\Rightarrow$ (17) is by instantiating (15) with $P_1 \dots, P_{n-1} := P_0$, by $[\![\bigvee_{0 \le i < m} p_i]\!]^{\mathfrak{F}}(P_0, \dots, P_{n-1}) = \bigvee_{0 \le i < m} P_i$, and by the definition of σ.

We show (18) $\Rightarrow$ (16). For simplicity, assume $n = 3$ and $m = 2$ (the proof can be adapted for other cases straightforwardly). Take arbitrary $P_0 \in V_0$ and $P_1 \in V_1$. Then

$$\begin{aligned}
& w \in [\![\sigma(\phi)]\!]^{\mathfrak{F}}(P_0, P_1) \\
& \quad \subseteq [\![\sigma(\phi)]\!]^{\mathfrak{F}}(P_0 \vee P_1, P_0 \vee P_1) && (19) \\
& \quad = [\![\sigma_0(\phi)]\!]^{\mathfrak{F}}(P_0 \vee P_1) && (20) \\
& \quad \Rightarrow && (21) \\
& w \in [\![\sigma_0(\psi)]\!]^{\mathfrak{F}}(P_0 \vee P_1) \\
& \quad = [\![\sigma(\psi)]\!]^{\mathfrak{F}}(P_0, P_1). && (22)
\end{aligned}$$

(19) holds because $\sigma(\phi)$ is positive in p_0, and p_1 and $\sigma(p_2) = p_0 \vee p_1$. (20) holds by the definition of σ and σ_0. (21) follows from (18). (22) holds because neither p_0 nor p_1 occurs in ψ. □

By the lemma above, correspondence theory for a Sahlqvist implication in which the only propositional variable is p_0 reduces to that for a Sahlqvist implication with normative antecedents. More concretely, the case for such an implication χ reduces to that for the formula one obtains by replacing in χ the positive occurrences of p_0 in the antecedent by distinct propositional variables and, simultaneously, the other occurrences of p_0 by the disjunction of those distinct variables. We can further show a similar lemma for multiple variables to reduce the case for general Sahlqvist implications to that for Sahlqvist implications with normative antecedents.

We are now ready to prove the main theorem of this section.

Proof of Theorem 3.1. As in the correspondence theory for the standard Kripke semantics, one can show that the set of modal formulae that locally correspond to first-order formulae are closed under these operations:

$$\begin{aligned}
\chi &\mapsto \Box_{\bar{a}} \chi && (\bar{a} \in I^{<\omega}) \\
(\chi, \chi') &\mapsto \chi \wedge \chi' \\
(\chi, \chi') &\mapsto \chi \vee \chi' && \text{(if no propositional variables occur in both } \chi \text{ and } \chi')
\end{aligned}$$

Also by the observation above one only needs to prove the theorem for a Sahlqvist implication whose antecedent is normative. This follows from Corollary 3.11. □

For a better understanding of the methods of this section, let us apply them to a concrete example.

Example 3.13. Assume that I is a singleton, denote its only element by $*$, and let $R = R_*$ and $x \rhd y \leftrightarrow Ryx$. The B axiom from Example 2.4 has an equivalent form

$$\mathrm{B}^{\mathrm{op}} :\equiv \Diamond\Box p_0 \to p_0$$

which is a Sahlqvist implication. We will calculate a local correspondent of B^{op} as an example, by using the theorems in this section.

As we saw in Example 2.4, we can assume extra conditions on the interaction of R and $\sqsubseteq$ to make correspondents simpler, without loss of generality. In fact, something additional is true here: often, for an interaction condition C, if a first-order formula $\alpha(x)$ is a local correspondent of a modal formula ϕ over the possibility frames that satisfy C, i.e., for any possibility frame $\mathfrak{F} \models C$ and $w \in \mathfrak{F}$,

$$\mathfrak{F}, w \Vdash \phi \iff \mathfrak{F} \models \alpha(w),$$

then one can effectively obtain a first-order $\tilde{\alpha}(x)$ which is a local correspondent of ϕ. See [19, Section 6.3] for the details. To compute a local correspondent of B^{op} it is convenient to assume the following conditions, alongside (R-down):

(separativity) $$x \sqsubseteq y \leftrightarrow \forall x' \sqsubseteq x\, x' \between y;$$

(R-dense) $$(\forall y' \sqsubseteq y\, \exists y'' \sqsubseteq y'\, Rxy'') \to Rxy;$$

(up-R) $$(Rx'y \wedge x' \sqsubseteq x) \to Rxy.$$

Again, we can assume these conditions without loss of generality, in the strong sense stated above. One of the major consequences of the extra conditions is

(23) $$R^{\mathrm{ro}}[\{x\}^{\mathrm{ro}}] = R[\{x\}].$$

We are now ready to calculate a local correspondent of B^{op}. Using the simplified forcing relation (4) for $\Diamond$, we see that $\mathrm{ST}_x(\mathrm{B}^{\mathrm{op}})$ is equivalent to

$$\forall x_1 \sqsubseteq x\, ((\forall x_2 \sqsubseteq x_1\, \exists x_3 \rhd x_2\, \forall x_4 \rhd x_3\, P_0 x_4) \to P_0 x_1).$$

Since p_0 follows exactly one $\Box$ in the antecedent of B^{op}, one can apply Lemma 3.9 where the range of $\forall P_0$ is restricted to $\mathbf{V}_1^*$. This class is defined by the first-order formula $\beta_1^*(x;z)$, where

$$\beta_1^*(x;z) \leftrightarrow Rzx$$

by (23). A local correspondent of B^{op} is then obtained by applying Lemma 3.2: $\alpha_{\mathrm{B}^{\mathrm{op}}}(x) \wedge \alpha_{\emptyset}(x)$ is a local correspondent of B^{op}, where $\alpha_{\mathrm{B}^{\mathrm{op}}}(x)$ is the first-order formula obtained by replacing

$$\forall P_0 \cdots P_0 x \cdots$$

by

$$\forall z_0 \cdots \underbrace{Rz_0 x}_{\text{equivalent to } \beta_1^*(x;z_0)} \cdots$$

in $U^2(\mathrm{ST}_x(\mathrm{B}^{\mathrm{op}}))$, and $\alpha_{\emptyset}(x) = [\lambda x\, x \neq x / P_0]\, \mathrm{ST}_x(\mathrm{B}^{\mathrm{op}})$. $\alpha_{\mathrm{B}^{\mathrm{op}}}(x)$ can be calculated to be

$$\forall z_0\, \forall x_1 \sqsubseteq x\, ((\forall x_2 \sqsubseteq x_1\, \exists x_3 \rhd x_2\, \forall x_4 \rhd x_3\, Rz_0 x_4) \to Rz_0 x_1),$$

and $\alpha_{\emptyset}(x)$ is

$$\forall x_1 \sqsubseteq x\, ((\forall x_2 \sqsubseteq x_1\, \exists x_3 \rhd x_2\, \forall x_4 \rhd x_3\, x_4 \neq x_4) \to x_1 \neq x_1).$$

One can check that, under the assumption of the extra conditions above, $\forall x\, (\alpha_{\mathrm{B}^{\mathrm{op}}}(x) \wedge \alpha_{\emptyset}(x))$ is equivalent to (3), the global correspondent of the B axiom given in Example 2.4.

Given that the analogue of the Sahlqvist Correspondence Theorem holds for possibility semantics, it is natural to ask whether an analogue of the Sahlqvist Completeness Theorem holds for possibility semantics as well. We will briefly discuss this question in Section 5.

4. MODEL-THEORETIC CHARACTERIZATION

In this section, we examine model-theoretic aspects of correspondence theory for possibility frames, extending and adapting the classical work of van Benthem [2]. We will see that the standard results for Kripke semantics smoothly extend to the setting of possibility semantics.

First, we investigate a model-theoretic characterization of modal formulae that *globally* correspond to first-order formulae. Unlike in the previous sections, we now regard possibility frames as (ordinary) structures for $\mathscr{L}^2(\tau)$, i.e., with no restriction on the range of second-order variables. In this section, we use the term "structures" without qualifications to refer to this kind of structure for $\mathscr{L}^2(\tau)$. We also assume in this section that I, the set of modal indices, is finite.

Let $\mathrm{FR}(\phi)$ denote the set of possibility frames $\mathfrak{F}$ such that for every possibility model $\mathfrak{M} = (\mathfrak{F}, \pi)$ and every $w \in \mathfrak{F}$, we have $\mathfrak{M}, w \Vdash \phi$. Equivalently, $\mathrm{FR}(\phi)$ is the set $\mathrm{Mod}(\mathrm{SOT}(\phi))$ of structures that models the monadic second-order formula $\mathrm{SOT}(\phi)$, where:

- $\mathrm{SOT}(\phi) := \tilde{U}^2(\mathrm{ST}_x(\phi)) \wedge \beta_{\mathrm{po}} \wedge \bigwedge_{a\in I} \beta^a_{R\Rightarrow\mathrm{win}} \wedge \beta^a_{R\text{-rule}}$;
- β_{po} states $\sqsubseteq$ is a partial order;
- $\tilde{U}^2(\chi)$ denotes the universal quantification by the second-order monadic variables occurring in χ, but with the domain of the quantification restricted to $\mathrm{RO}(\mathfrak{F})$; concretely, $\tilde{U}^2(\chi) := \chi$ for $\chi \in \mathscr{L}^2(\tau)$ with no monadic second-order free variables and $\tilde{U}^2(\chi) := \tilde{U}^2(\forall P\,(\beta^P_{\mathrm{val}} \to \chi))$ for χ with a monadic second-order free variable P; and
- β^P_{val} is a sentence in $\mathscr{L}^1(\tau \cup \{P\})$ that says that P is a regular open set within a possibility frame; i.e.,

$$\beta^P_{\mathrm{val}} :\equiv \forall x\,(Px \leftrightarrow \beta^P_{\mathrm{ro}}(x)).$$

Definition 4.1. Let $\mathfrak{F}$ be a structure. A *generated substructure* $\mathfrak{G}$ of $\mathfrak{F}$ is a substructure of $\mathfrak{F}$ such that if $x \in \mathfrak{G}$ and $\mathfrak{F} \models \heartsuit xy$ for some $\heartsuit \in \{\sqsupseteq\} \cup \{R_a \mid a \in I\}$, then $y \in \mathfrak{G}$.

It can be shown that a generated substructure of a possibility frame as a structure is again a possibility frame (see [19, Proposition 5.50.2]).

Lemma 4.2. Let $\mathfrak{F}$ be a structure and $\mathfrak{G}$ be a generated substructure of $\mathfrak{F}$. Let π be an interpretation of P_i ($i \in \kappa$). Then for each modal formula ϕ and each $w \in \mathfrak{G}$, we have $(\mathfrak{F}, \pi) \models \mathrm{ST}_w(\phi) \Leftrightarrow (\mathfrak{G}, \pi) \models \mathrm{ST}_w(\phi)$.

Proof. Obvious. □

The following result is originally due to Goldblatt [14]. For a family $(\mathfrak{N}_i)_{i\in J}$ of structures and an ultrafilter U over J, we write $\prod_{i\in J} N_i/U$ for the ultraproduct of the family using J (see, e.g., [22]).

Lemma 4.3. Let $(\mathfrak{F}_i)_{i\in J}$ and $(\mathfrak{G}_i)_{i\in J}$ be families of structures. Assume that each $\mathfrak{F}_i$ is a generated substructure of $\mathfrak{G}_i$. Let U be an ultrafilter over J. Then $\mathfrak{F} := \prod_i \mathfrak{F}_i/U$ is a generated substructure of $\mathfrak{G} := \prod_i \mathfrak{G}_i/U$.

Proof. This can be proved in the same way as over Kripke frames whose accessibility relations are $\heartsuit$'s as in Definition 4.1. □

Recall that an ultrapower $\mathfrak{F}^J/U$ is the ultraproduct $\prod_{i \in J} \mathfrak{F}_i/U$ of the family $(\mathfrak{F}_i)_{i \in J}$ where $\mathfrak{F}_i = \mathfrak{F}$ for all $i \in J$. Given a family $(\mathfrak{F}_i)_{i \in J}$ of structures, one can think of a new structure $\bigoplus_{i \in J} \mathfrak{F}_i$, their *disjoint union*, since the signature τ is relational. Note that, if $(\mathfrak{F}_i)_{i \in J}$ is a family of *possibility frames*, then $\bigoplus_{i \in J} \mathfrak{F}_i$ is again a possibility frame (see [19, Proposition 5.54.2]).

Corollary 4.4. Let $(\mathfrak{F}_i)_{i \in J}$ be a family of structures and $\mathfrak{F} := \bigoplus_{i \in J} \mathfrak{F}_i$. Let U be an ultrafilter over J and $\mathfrak{G} = \prod_i \mathfrak{F}_i/U$. Then $\mathfrak{G}$ is isomorphic to some generated substructure of the ultrapower $\mathfrak{F}^J/U$.

Lemma 4.5. For $\phi \in \mathscr{L}(\Phi, I)$, we have that $\mathrm{FR}(\phi) = \mathrm{Mod}(\forall x\, \mathrm{SOT}(\phi))$ is closed under generated substructures.

Proof. By induction on the complexity of ϕ. □

Lemma 4.6. For $\phi \in \mathscr{L}(\Phi, I)$, if $\mathrm{FR}(\phi)$ is closed under ultrapowers, then it is closed under ultraproducts.

Proof. Obvious from the preceding lemmas, since $\mathrm{FR}(\phi)$ is closed under disjoint unions. □

We can now see that van Benthem's [2] characterization of basic elementary classes of Kripke frames can be extended to possibility frames as well. Recall that a class $\mathscr{K}$ of structures is *basic elementary* if $\mathscr{K} = \mathrm{Mod}(\alpha)$ for some first-order α. By definition, for a modal formula ϕ, we have that $\mathrm{FR}(\phi)$ is basic elementary if and only if ϕ has a global correspondent

Theorem 4.7. For $\phi \in \mathscr{L}(\Phi, I)$, we have $\mathrm{FR}(\phi)$ is basic elementary if and only if it is closed under ultrapowers.

Proof. By a general model-theoretic fact (see, e.g., [7, Corollary 6.1.16 (ii)]), $\mathrm{FR}(\phi) = \mathrm{Mod}(\forall x\, \mathrm{SOT}(\phi))$ is basic elementary if and only if $\mathrm{Mod}(\forall x\, \mathrm{SOT}(\phi))$ and its complement are closed under ultraproducts. Since $\forall x\, \mathrm{SOT}(\phi)$ is Π^1_1 for any $\phi \in \mathscr{L}(\Phi, I)$, we know that $\mathrm{Mod}(\neg \forall x\, \mathrm{SOT}(\phi))$, the complement of $\mathrm{Mod}(\forall x\, \mathrm{SOT}(\phi))$, is always closed under ultraproducts, since Σ^1_1 sentences are preserved under ultraproducts. Then by the previous lemma, $\mathrm{Mod}(\forall x\, \mathrm{SOT}(\phi))$ is basic elementary if and only if it is closed under ultrapowers. □

Let us now turn to *local* correspondence. An analogous result for Kripke semantics was also proved by van Benthem [2].

Theorem 4.8. For $\phi \in \mathscr{L}(\Phi, I)$, we have that ϕ locally corresponds to a first-order formula if and only if for every possibility frame $\mathfrak{F}$, every index set J and an ultrafilter U over J, we have

$$(\dagger) \qquad \forall i \in J\, \mathfrak{F} \models \mathrm{SOT}(\phi)(w_i) \Rightarrow \mathfrak{F}^J/U \models \mathrm{SOT}(\phi)((w_i)_i/U).$$

Proof. First observe that a modal formula ϕ locally corresponds to a first-order $\alpha(x)$ if and only if $\mathrm{Mod}([c/x]\,\mathrm{SOT}(\phi)) = \mathrm{Mod}(\alpha(c))$ where Mod is defined analogously for the language $\mathscr{L}^2(\tau \cup \{c\})$, and c is a new constant symbol. In addition, the quantifier-wise syntactic complexity of the sentence $[c/x]\,\mathrm{SOT}(\phi)$ remains Π^1_1 in the new language. Thus, a proof similar to the one above applies to this theorem.

To be more precise, one can show the following analogue of Corollary 4.4:

Claim. Let $((\mathfrak{F}_i, w_i))_{i\in J}$ be a family of structures for $\mathscr{L}^1(\tau \cup \{c\})$ and U be an ultrafilter over J. Then $\prod_i \mathfrak{F}_i/U$ can be embedded in the ultraproduct

$$(*) \qquad \prod_{i\in J}(\bigoplus_{j\in J}\mathfrak{F}_j, w_i)/U,$$

and its image is a generated substructure of $(\bigoplus_{j\in J} F_j)^J/U$.

Moreover, if $(\mathfrak{F}, w) \models [c/x]\,\mathrm{SOT}(\phi)$, then for a generated substructure $\mathfrak{G}$ of $\mathfrak{F}$ containing w we have $(\mathfrak{G}, w) \models [c/x]\,\mathrm{SOT}(\phi)$, and if $(\mathfrak{F}_i, w_i) \models [c/x]\,\mathrm{SOT}(\phi)$ for all $i \in J$, an index set, then $(\bigoplus_{j\in J} F_j, w_i) \models [c/x]\,\mathrm{SOT}(\phi)$ for all i. Thus, $\mathrm{Mod}(\mathrm{SOT}(\phi)[c/x])$ is closed under ultraproducts of the form $(*)$ if and only if ϕ locally corresponds to a first-order formula. This can easily seen to be equivalent to the condition (†). □

A standard application of a result like Theorem 4.8 is to obtain a syntactic closure property of the set of formulae having first-order correspondents, as follows.

Theorem 4.9. If $\Box_a\phi$ locally corresponds to a first-order formula, so does ϕ.

Proof. For simplicity, we omit modal operator indices. Suppose that ϕ does not locally correspond to a first-order formula. Then by Theorem 4.8, there exist a structure $\mathfrak{F} = (W, R, \sqsubseteq)$, an index set J, an ultrafilter U over J, and $(w_i)_i \in \mathfrak{F}^J$ such that for every $i \in I$ we have $\mathfrak{F} \models \mathrm{SOT}(\phi)(w_i)$ but $\mathfrak{F}^J/U \not\models \mathrm{SOT}(\phi)((w_i)_i/U)$. Let π be the valuation that witnesses the latter fact. Let v be an object not in W. For each $i \in I$, let $\mathfrak{F}_i = (W \sqcup \{v\}, R \sqcup \{(v, w_i)\}, \sqsubseteq \sqcup \{(v, v)\})$. Since for every $i \in J$ we have $\mathfrak{F}_i \models \exists! x\, Rvx$ and $\mathfrak{F}_i \models Rvw_i$, by Łoś's Theorem, we know that $\prod_i \mathfrak{F}_i/U \models \exists! x\, R((v)_i/U)x$ and that the unique witness to the preceding formula is $(w_i)_i/U$. Note that $\mathfrak{F}^J/U$ is a generated substructure of $\prod_i \mathfrak{F}_i/U$. The valuation π is also a valuation for $\prod_i \mathfrak{F}_i/U$. Hence, $(\prod_i \mathfrak{F}_i/U, \pi) \not\models \mathrm{ST}_x(\phi)((w_i)_i/U)$ and $(\prod_i \mathfrak{F}_i/U, \pi) \not\models \mathrm{ST}_x(\Box\phi)((v)_i/U)$. However, by construction, for every $i \in J$ we have $\mathfrak{F}_i \models \mathrm{SOT}(\Box\phi)(v)$. Therefore, by Theorem 4.8, we know that $\Box\phi$ does not locally correspond to a first-order formula. □

5. Conclusion

We have seen that despite the richer structure of possibility frames, involving not only the accessibility relation but also the refinement relation, central results of standard correspondence theory continue to hold in this more general setting. A natural question raised by our results is this: does *every* formula that has a first-order correspondent in the setting of Kripke semantics also have a first-order correspondent in the setting of possibility semantics?

A second open problem suggested by our results concerns the Sahlqvist Completeness Theorem, which states that every Sahlqvist formula is canonical. A natural question to ask here is how this theorem can be extended to our general setting of possibility semantics. In [18, Section 5.6], a possibility-theoretic view of canonical extensions of BAOs is developed, according to which, for a normal modal logic Λ, there is a *canonical possibility frame*[10] whose modal theory is included in Λ. Unlike a canonical Kripke frame, built from the *ultra*filters in the Lindenbaum algebra of a logic, a canonical possibility frame is built from proper filters in the Lindenbaum algebra. The possibility frame constructed from a BAO in this way is called a *filter frame*. Even for an uncountable modal language, the construction of the latter does not require the ultrafilter axiom, or equivalently, the Boolean prime ideal axiom. The possibility-semantic version

[10] Again, we are dropping the word "full" from the technical term defined in [19].

of canonicity of a modal formula ϕ is then defined so that ϕ is *filter-canonical* if and only if, for every normal modal logic Λ containing ϕ, the logic's canonical possibility frame validates ϕ. Assuming the ultrafilter axiom, ϕ is filter-canonical if and only if ϕ is canonical in the standard Kripke-semantic sense [18, p. 122]. He also proves [18], without use of the Boolean prime ideal axiom, that the underlying BAO of the filter possibility frame of a BAO A coincides with what Gehrke and Harding [13] construct as the "canonical extension" of A and what Conradie and Palmigiano [10] call the *constructive canonical extension* of A. Consequences [18, Theorem 7.20] of Conradie and Palmigiano's results [10] are that every *inductive formula* is filter-canonical, and that every normal modal logic axiomatized by inductive formulae is sound and complete with respect to its canonical possibility frame.

Another question raised by our results concerns the potential applicability to possibility semantics of ALBA [8], a group of general syntactic algorithms that can calculate correspondents of formulae in many non-classical logics that are interpreted in lattices with operators. A possible research direction is to adapt the techniques of ALBA in order to reformulate the argument in Section 3 as a variant of ALBA and to prove more general correspondence results for possibility semantics.

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