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Harvesting a Renewable Resource under Uncertainty[1]

(with Erratum at the end)

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Abstract

This paper presents a theory of harvesting that allows for partial harvests and accounts for the risk of extinction, for biological assets with size-dependent stochastic growth. The harvesting decision is formulated as a disinvestment problem in continuous time and generalized Faustmann formulas are derived. The probability of extinction is then analyzed for a wide class of growth functions. An illustration based on the logistic Brownian motion shows that both optimal biomass at harvest and harvest size do not vary monotonically with uncertainty. More generally, this paper illustrates the importance of properly accounting for barriers in stochastic investment problems.

Key words: renewable resources; extinction; uncertainty; irreversibility; real options.

JEL classification: D92, D81, Q20.

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I. Introduction

Biomass uncertainty is pervasive for renewable resources, and it can play a central role in their management. Biomass uncertainty can result, for example, from random fluctuations in nutrient availability, stochastic changes in natural conditions that affect reproduction or migration, or from random predator encroachments. Yet simple rules that account for biological uncertainty and the risk of extinction are still not available to resource managers. This paper addresses this question in a continuous time framework for resources with density-dependent growth. [1] This allows us to derive harvesting rules that generalize the Faustmann formula and to analyze the probability of extinction for a wide class of stochastic growth functions. Moreover, our formulation provides an alternative to the standard real options approach, which has been shown by Saphores (2002) to yield incorrect decision rules in the presence of absorbing boundaries (here for example, the resource becomes extinct when its biomass reaches a minimum size).

While the analysis of density–dependent growth models has received some attention (e.g., see Willassen 1998, or Reed and Clarke 1990, and the references therein), the focus in the literature has mostly been on the single harvest case. An exception is Reed and Clarke (1990). They analyze multiple total harvests with regeneration (the forestry case) when the biomass evolves stochastically, the net resource price follows a geometric Brownian motion (GBM), and there are no fixed harvesting costs.[2] More recently, Willassen (1998) applies stochastic impulse control to rigorously generalize the Faustmann formula to forests experiencing stochastic growth. Sødal (2002) extends Willassen's results using a simpler methodology (the markup approach; see Dixit, Pindyck, and Sødal 1999). He obtains a closed-form rotation formula and shows how to account for endogenous rotation costs. In the context of fisheries, Li (1998) shows the importance of uncertainty and harvest irreversibility using real options. He allows for partial harvests but he does not consider the impact of a harvest on future harvests. Moreover, his analysis is specialized to the case where the fish biomass follows a GBM, thus not accounting for environmental carrying capacity. In summary, partial harvesting rules for biological assets are not yet available in the literature, and the possibility of resource extinction, intuitively an important factor in the decision to harvest, does not seem to have received much attention.

In this context, we derive general harvesting rules incorporating the possibility of resource extinction for repeated, partial and total harvests of biological assets with size-dependent stochastic growth. We suppose that harvesting is instantaneous, which is reasonable in the presence of fixed costs or when the growth rate of a resource decreases with its size. To account for harvest irreversibility and to model uncertainty concisely, we cast the decision to harvest as a disinvestment problem in continuous time and show that harvesting is akin to exercising a real option. This approach is intuitive, as capital theoretic concepts permeate the resource literature, and it provides access to the powerful tools of continuous-time finance.

We then apply our methodology to a wide class of stochastic growth functions to understand the impact of their specification on the probability of extinction. A numerical illustration for the logistic Brownian motion shows that expected net rents, the optimal biomass at harvest, and the harvest size are not monotonic functions of uncertainty; furthermore, we show analytically that a total harvest is optimum when uncertainty is high enough. These results highlight the importance of taking into account the probability of resource extinction.

This paper is organized as follows. Section 2 presents a general framework for analyzing multi-period harvesting problems under uncertainty. Section 3 analyzes the probability of extinction for a wide class of stochastic growth functions. Section 4 illustrates numerically the impact of uncertainty on the decision to harvest when the resource biomass follows a logistic Brownian motion. Section 5 summarizes our conclusions. Two appendices contain additional analytical results and various proofs.

II. Models of harvest under biomass uncertainty

Consider a valuable renewable resource whose biomass X varies randomly due to natural factors (e.g., predators or availability of food) according to the autonomous diffusion process:

$$dX = G(X)dt + v(X)dz. \quad (1)$$

In (1), $G(.)$ and $v(.)$ are continuous, dt is an infinitesimal time increment, and dz is an increment of a standard Wiener process. We suppose that 0 is an absorbing barrier, i.e., if X ever becomes zero, the resource becomes extinct. Moreover, let $K>0$ designate the maximum carrying capacity of the resource.[3] $G(.)$ is assumed strictly positive on $(0,K)$, strictly negative on $(K,+\infty)$ and $G(K)=0$. In addition, $v(.)$ is strictly positive on $(0,+\infty)$.

We suppose that this resource can be harvested instantaneously, which is reasonable if harvest duration is relatively short compared to the growing season. Instantaneous (as opposed to continuous) harvests are intuitively optimal in the presence of fixed costs, but also when the growth rate of a resource decreases with biomass size. In the latter case, harvesting stimulates biomass growth and increases expected profits.

We focus here on the optimal social management of this resource. We suppose that the resource manager's objective is to maximize the present value of the stream of expected utility from successive harvests, or the present value of the expected utility of harvesting once the entire resource, whichever is largest. For successive harvests, we consider two types of problems: either only part of the resource is harvested each time, as for fisheries, or all of the biomass is harvested but a small, fixed amount to permit regeneration, as in forestry. Let us start with the former. A one-time-only, total harvest is just a special case, as shown below.

II.1 Partial harvests.

For partial harvests, the resource manager's objective is:

$$V(x_0) = \underset{\{h_i, x_i^-,\ 0 \le h_i \le x_i^-\}_{i=1}^{+\infty}}{Max} E\left(\sum_{i=1}^{+\infty} U\left(\pi(h_i, x_i^-)\right) e^{-\rho T_i} + e^{-\rho\tau} U(L) \mid x_i^+ = x_i^- - h_i, X(0) = x_0 \right), \qquad (2)$$

subject to Equation (1) for the evolution of *X*. In the above:

- $V(x_0)$ is the unknown value function when the resource biomass is x_0; by construction, $V(x_0)$ is also the present value of the expected utility of harvesting rents.[4]
- *E*(.) is the expectation operator with respect to *X* for the information available at time 0.
- *U*(.) is the resource manager's utility function, assumed strictly concave, increasing, and C^2 (i.e., twice continuously differentiable and with a continuous second derivative).
- $\pi(h,x)$ is the net rent from harvesting biomass *h* when the resource stock is $x \ge h$. π is C^2 and increasing in *x* and *h*.
- h_i, for $i \ge 1$, is the amount of biomass harvested during harvest *i*.
- x_i^- and x_i^+ are respectively the stocks of biomass just before and just after harvest *i*.
- T_1 is the random stopping time at which *X* reaches x_1^- for the first time after starting from $X(0)=x_0$; for $i>1$, T_i-T_{i-1} is the random elapsed time between consecutive harvests *i-1* and *i*.
- $\tau \le +\infty$ is the random stopping time at which the resource becomes extinct, if ever. If $\tau < +\infty$,

then $T_i=+\infty$ for all $T_i \geq \tau$.

- L is the payoff resulting from extinction. L can be positive (e.g., the value of bare land after clear-cutting in forestry) or negative (e.g., the loss of existence value for a species).
- Finally, ρ is the resource manager's risk-free discount rate.

A simplified expression of the objective function

To motivate our approach, it is fruitful to emphasize the parallel between harvesting and investing. First, each of these decisions is (at least partly) irreversible: the harvested stock cannot usually be returned to its environment, nor can much of a bad investment usually be recovered. Second, these decisions are made under uncertainty. Finally, both investing and harvesting can be delayed under unfavorable conditions. These characteristics call for the use of real options.

If we see a renewable natural resource as an asset from which it is possible to disinvest by harvesting, the manager of this resource is holding a perpetual compound option. This option gives the right but not the obligation to harvest, and it never expires if the resource manager's time horizon is infinite, which we assume. By construction, the value of the option equals the present value of the expected utility of harvesting rents.

Moreover, just as in the Faustmann problem in forestry, the resource manager faces the same problem after each harvest. We can thus invoke Bellman's optimality principle to infer that, if there is a unique solution, the optimal harvest policy is simply to harvest a constant amount h^* as soon as the biomass reaches size x^*.[5] x^* and h^* are chosen to maximize the present value of the expected utility from the next and all future harvests. At harvest the resource manager receives the utility $U(\pi(h^*, x^*))$ plus $V(x^* - h^*)$, the option to harvest at the new value of the biomass. Harvesting takes place at randomly spaced intervals (T_i, i=1…+∞), but since X is Markovian, the T_is are independent and identically distributed. In addition, it is necessary to account for the risk of resource extinction. Whereas in a deterministic problem we know with certainty whether a growing biomass will reach a given level, in a stochastic problem with an absorbing barrier it may happen in some cases and never in others. The value of the option to harvest thus has two components, discounted and adjusted for the possibility of extinction: the first one is the utility from harvesting h^* when the biomass is x^* plus the value of the option to harvest in the future; the second one is the utility from extinction. If x_0 is the current stock of biomass, this can be written

$$V(x_0) = D_{0,x^*|x_0}\left\{p_{0;x^*|x_0}\left[U(\pi(h^*,x^*)) + V(x^* - h^*)\right] + p_{x^*;0|x_0}U(L)\right\}, \qquad (3)$$

where:

- $D_{0,x^*|x_0} = E\left(e^{-\rho T_{0,x^*|x_0}}\right)$ is the expected discount factor; $T_{0,x^*|x_0}$ is the random duration between the moment where X equals x_0 and the first time where X hits either 0 or x^*; and
- $p_{0;x^*|x_0}$ is the probability that X first hits x^* before 0 starting from x_0; conversely $p_{x*;0|x_0}$ is the probability that X hits 0 before x^* starting from x_0.

Likewise, the value function immediately after harvest is

$$V(x^* - h^*) = D_{0,x^*|x^*-h^*}\left\{p_{0;x^*|x^*-h^*}\left[U(\pi(h^*,x^*)) + V(x^* - h^*)\right] + p_{x^*;0|x^*-h^*}U(L)\right\}. \qquad (4)$$

Isolating $V(x^* - h^*)$ in (4) and plugging it into (3) allows us to rewrite (2) as follows:[6]

$$V(x_0) = \underset{\{h,x,\ 0\le h\le x\}}{Max}\left[UH(h,x\,|\,x_0) + \frac{p_{0;x|x_0}D_{0,x|x_0}}{1 - p_{0;x|x-h}D_{0,x|x-h}}UH(h,x\,|\,x-h)\right], \qquad (5)$$

where

$$UH(h,x\,|\,\zeta) \equiv D_{0,x|\zeta}\left\{p_{0;x|\zeta}U(\pi(h,x)) + p_{x;0|\zeta}U(L)\right\} \quad (6)$$

is the discounted utility from the next harvest, taking into account the risk of extinction, when the starting biomass is ζ. The first term on the right side of (5) is thus the discounted utility from the first harvest, and the other terms represents the discounted utility from all other future harvests, accounting for the risk of extinction, and conditional on the first harvest taking place.

Note that Equation (5) includes the case where the resource is harvested once to extinction. Indeed, $p_{0;x|0} = 0$, $T_{0,x|0} = 0$, and $D_{0,x|0} = 1$, so $UH(h,x\,|\,0) = U(L)$, and (5) becomes

$$V(x_0) = \underset{\{x,\ 0\le x\}}{Max}\, D_{0,x|x_0}\left\{p_{0;x|x_0}U(\pi(x,x)) + U(L)\right\}. \quad (7)$$

If there is no risk of extinction, 0 cannot be reached before $x > 0$ so for $x_0 \in (0,x)$, $p_{0;x|x_0} = 1$. The discount factor $D_{0,x|x_0}$ then becomes $D_{x|x_0} = E\left(e^{-\rho T_{x|x_0}}\right)$ where $T_{x|x_0}$ is the random elapsed time between X=x_0 and the first time X hits x. Hence, $UH(h,x\,|\,\zeta) = D_{x|\zeta}U(\pi(h,x))$ and (5) simplifies to

$$V(x_0) = \underset{\{h,x,\ 0\le h\le x\}}{Max}\left\{\frac{D_{x|x_0}}{1 - D_{x|x-h}}U(\pi(h,x))\right\}. \quad (8)$$

Likewise, for a single, total harvest without risk of involuntary extinction, Equation (7) becomes

$$V(x_0) = \underset{\{x,\ 0\le x\}}{Max}\, D_{x|x_0}\left\{U(\pi(x,x)) + U(L)\right\}. \quad (9)$$

Section IV highlights numerically the importance of accounting for the risk of extinction.

First order necessary conditions

Let us now look for necessary conditions for an interior solution. As both conditions can be written similarly, let ζ designate either *h* or *x*. When we take the first derivative with respect to ζ of Equation (5), equate it to zero, and rearrange terms, we find

$$\frac{\zeta \dfrac{\partial UH(h,x\,|\,x_0)}{\partial \zeta}}{\dfrac{UH(h,x\,|\,x-h)}{1-p_{0;x|x-h}D_{0,x|x-h}}} = \varepsilon_{\zeta}^{CD} + \varepsilon_{\zeta}^{UH}, \tag{10}$$

which generalizes the Faustmann rotation formula to partial harvests with stochastic resource growth and the risk of extinction.

In (10), $\varepsilon_{\zeta}^{CD} \equiv -\dfrac{\partial Ln\left(\dfrac{p_{0;x|x_0}D_{0,x|x_0}}{1-p_{0;x|x-h}D_{0,x|x-h}}\right)}{\partial Ln(\zeta)}$ and $\varepsilon_{\zeta}^{UH} \equiv -\dfrac{\partial Ln(UH(h,x\,|\,x-h))}{\partial Ln(\zeta)}$ are respectively the elasticity of the sum of discount factors incorporating the risk of extinction during the initial and all subsequent harvests, and the elasticity of the discounted utility from a harvest with starting biomass *x-h*. The left side of Equation (10) is the average utility contribution of the next harvest (numerator) divided by the present value of the utility from all future harvests (denominator); it scales the impact on utility of a marginal change in the next harvest with the discounted utility of all future harvests. Equation (10) thus balances three effects of a change in harvest size or biomass at harvest: 1) on the next harvest; 2) on delaying all future harvests; and 3) on the utility from all future harvests. We also note the first order condition with respect to *x* is slightly more complex because a change in the biomass at harvest also has implications for the length of the wait until the first harvest and for the initial probability of extinction.

It is essential to note that (10) holds for $x=x^*$, $h=h^*$, and $x_0=x^*$. Indeed, since *X* is a diffusion it is Markovian (all relevant information about X is contained in its current state), so the first order conditions for $x_0 \neq x^*$ merely indicate whether or not the maximum of (5) has been attained. They are, however, zero at the optimum, which explains the condition $x_0 = x^*$.

Instead of repeated harvests, however, it may be optimal to harvest the whole biomass as soon as X reaches x^*. In this case, the first derivative with respect to h of the resource manager's objective function is necessarily non-negative at $h^* = x^*$, so in (10) with $\zeta = h$, "≥" should replace "=". The relevant necessary first order condition for x^* may then be obtained from (7).

For practical purposes, we still need to show how to derive $p_{0;x|x_0}$ and $D_{0,x|x_0}$. This is done in Appendix I, where results for three common stochastic processes are also provided.

II.2 Total harvest with regeneration.

In a number of harvesting problems (e.g., in forestry), the entire stock of biomass is harvested each time, except possibly for a small amount $S>0$ to allow for regeneration. With the notations defined above, the resource manager's objective is

$$V(x_0) = \underset{\{x_i^-,\, S \le x_i^-\}_{i=1}^{+\infty}}{Max} E\left(\sum_{i=1}^{+\infty} U\left(\pi(x_i^- - S, x_i^-) \right) e^{-\rho T_i} + e^{-\rho \tau} U(L) \mid x_i^+ = S, X(0) = x_0 \right), \qquad (11)$$

subject to Equation (1) for the evolution of X. S is the starting biomass after harvest, assumed fixed and known, so (11) is just a special case of (2) with $h = x - S$. As a consequence, results obtained in the previous subsection are valid provided h, which is no longer a separate decision variable, is replaced with $x - S$. This includes first order necessary condition (10).

In the same context, it is often possible to restart the growth process at a cost C_R if the starting biomass S fails to grow. In this case, (3) and (4) become respectively

$$\begin{cases} V(x_0) = D_{0,x^*|x_0}\left\{p_{0;x^*|x_0}U(\pi(x^*-S,x^*)) + p_{x^*;0|x_0}U(C_R) + V(S)\right\}, \\ V(S) = D_{0,x^*|S}\{p_{0;x^*|S}U(\pi(x^*-S,x^*)) + p_{x^*;0|S}U(C_R) + V(S)\}. \end{cases} \tag{12}$$

Inserting *V*(*S*) into *V*(x_0) in (12), the resource manager's problem becomes[7]

$$V(x_0) = \underset{\{x,\ 0\leq x\}}{Max}\left[UH_R(x\,|\,x_0) + \frac{D_{0,x|x_0}}{1-D_{0,x|S}}UH_R(x\,|\,S)\right], \tag{13}$$

where $UH_R(x\,|\,\zeta) = D_{0,x|\zeta}\left\{p_{0;x|\zeta}U(\pi(x-S,x)) + p_{x;0|\zeta}U(C_R)\right\}$ is the discounted utility from the next harvest, taking into account the risk of having to restart failed crops, when the starting biomass is ζ. The interpretation of (13) is similar to that of (5).

The first order necessary condition for (13) can be written

$$\frac{x\dfrac{\partial UH_R(x\,|\,x_0)}{\partial x}}{\dfrac{UH_R(x\,|\,S)}{1-D_{0,x|S}}} = \varepsilon_x^{CD_1} + \varepsilon_x^{UH_R}, \tag{14}$$

where $\varepsilon_x^{CD_1} \equiv -\dfrac{\partial Ln(D_{0,x|x_0}[1-D_{0,x|S}]^{-1})}{\partial Ln(x)}$ and $\varepsilon_x^{UH_R} \equiv -\dfrac{\partial Ln(UH_R(x\,|\,S))}{\partial Ln(x)}$ are respectively the biomass elasticity of the discount factor $D_{0,x|x_0}[1-D_{0,x|S}]^{-1}$ and the biomass elasticity of the discounted utility from a harvest when the starting biomass is *S*. As for (10), (14) should be evaluated at $x=x_0=x^*$. Equation (14) generalizes the Faustmann rotation formula to the case where a resource grows stochastically and there is a risk of extinction.

To link our results with previous work, let us also assume that:

- There is no risk of extinction so the discount factor is $D_{x|x_0} = E\left(e^{-\rho T_{x|x_0}}\right)$ (see above);
- The resource manager's risk preferences are reflected in the discount rate;

- The value of a unit of biomass is $1; and
- Net harvesting profits including replanting costs are $\pi(x\text{-}S,x)=x\text{-}C$.

Then, using that $D_{x|S} = D_{x|x_0} D_{x_0|S}$ (see Dixit, Pindyck, and Sødal 1999), (14) becomes

$$\frac{x^* - C}{x^*} = \frac{1 - D_{x^*}(S)}{\varepsilon^D_{x^*}}, \tag{15}$$

where $\varepsilon^D_{x^*} = -\frac{x}{D_x(S)} \frac{dD_x(S)}{dx}|_{x=x^*}$ is the elasticity of the discount factor with respect to the value of the biomass (since here the unit value of biomass is $1). Equation (15) is the generalized Faustmann formula derived by Sødal (2002), who extends Willassen's results (1999) using an approach similar to the one adopted in this paper, but without considering the risk of extinction.[8]

III. Results for a class of growth functions.

To explore the implications for resource extinction of the growth function specification, let us apply the methodology detailed above to the partial harvest problem when $G(.)$ and $v(.)$ satisfy

$$G(x) = xg(x) \text{ and } v(x) = \sigma \mathrm{x}^{\beta}, \tag{16}$$

with $\sigma > 0$; $\beta \geq 0$; $\forall x \in (0,K), g(x) > 0$, $g(K)=0$; $\forall x > K,\ g(x) < 0$ and $g'(x) \leq 0$; and

- Assumption III.1: $g(.)$ is continuous and $\lim_{x \to 0^+} g(x) > 0$.
- Assumption III.2: if $\lim_{x \to 0^+} g(x) = +\infty$, then $\forall \eta > 0,\ \lim_{x \to 0^+} x^{\eta} g(x) = 0$.[9]

This specification encompasses a wide class of processes in population biology (e.g., see Clark 1990). It includes stochastic versions of such popular models as the logistic and Gompertz laws and it is more general than the class of processes analyzed by Reed and Clarke (1990). The importance of β is more apparent with the derivation of the risk of extinction. Results are summarized in two propositions whose proofs can be found in Appendix II.

Proposition 1. Let x and x_0 be two real numbers such that $0<x_0<x$.

- If $\beta \in [0,1)$ or if $\beta = 1$ and $\sigma > \sqrt{2g(0)}$, then there is a risk of extinction and

$$p_{0;x|x_0} = \frac{S(x_0) - S(0)}{S(x) - S(0)} \in (0,1). \tag{17}$$

In (17), the expression of the scale function $S(.)$ is

$$S(x) = \int_{x_1}^{x} \exp\left[\int_{x_2}^{\eta} \frac{-2}{\sigma^2} \frac{g(\xi)}{\xi^{2\beta-1}} d\xi \right] d\eta. \tag{18}$$

This means, by definition (see Karlin and Taylor 1981), that 0 is an attracting barrier for X.

- Otherwise, there is no risk of extinction and $p_{0;x|x_0} = 1$. □

To better understand the impact of β on X, let us analyze the expected time it takes X to reach either 0 or x^*. Indeed, a positive probability of reaching 0 does not imply that extinction will happen in finite time. If extinction happens only over an infinite time interval, 0 is said to be unattainable; 0 is attainable if extinction happens in finite time (Karlin and Taylor, 1981).

Proposition 2. Let a, b, and x_0 be three real numbers such that $0 \leq a < x_0 < b$. Let $v_{a,b|x_0}$ be the expected value of the minimum time it takes X to reach either a or b starting from x_0. Similarly, let $v_{b|x_0}$ be the expected value of the minimum time it takes X to reach b starting from x_0. Then:

- If $\beta \in [0, 0.5)$, 0 is attainable and

$$v_{0|x_0} = 2\left(\left[S(x_0) - S(0)\right] \int\limits_{x_0}^{+\infty} m(\eta)d\eta + \int\limits_{0}^{x_0} \left[S(\eta) - S(0)\right] m(\eta)d\eta \right) < +\infty. \tag{19}$$

- If $\beta \in [0.5, 1)$ or if $\beta = 1$ and $\sigma > \sqrt{2g(0)}$, 0 is unattainable and

$$v_{0,b|x_0} = +\infty. \tag{20}$$

- If $\beta = 1$ and $\sigma < \sqrt{2g(0)}$ or if $\beta > 1$, we already know from Proposition 1 that there is no risk of extinction, so $v_{0|x_0} = +\infty$ and

$$v_{b|x_0} = 2\left(\int\limits_{x_0}^{b} [S(b) - S(\eta)] m(\eta)d\eta + [S(b) - S(x_0)] \int\limits_{0}^{x_0} m(\eta)d\eta \right). \tag{21}$$

Here, *S*(.) is given by (18) and *m*(.) is the speed density of the process,

$$m(\eta) = \frac{1}{\sigma^2 \eta^{2\beta}} \exp\left\{ \int\limits_{x_2}^{\eta} \frac{2g(\xi)}{\sigma^2 \xi^{2\beta - 1}} d\xi \right\}, \tag{22}$$

where x_2 is the same arbitrary constant that appears in the definition of *S*(.) to guarantee that $v_{b|x_0}$ and $v_{a,b|x_0}$ are independent from x_1 and x_2 (see (I.5)). □

The intuition behind these results is simple: the larger is β, the smaller are the stochastic increments of *X* when *X* is close to 0. For $\beta \in [0, 0.5)$, the stochastic increments in (1) can overcome the deterministic component of *X* that tends to bring *X* back towards *K*, so extinction is possible. For $\beta = 1$ and $\sigma < \sqrt{2g(0)}$ or for $\beta > 1$, the opposite is true. When $\beta \in [0.5, 1)$ or when $\beta = 1$ and $\sigma > \sqrt{2g(0)}$, however, the deterministic trend and the stochastic components in

(1) pull X in opposite directions and neither dominates. Suppose for example that $\beta = 1$ and $\sigma > \sqrt{2g(0)}$. For Y=Ln(X), Ito's lemma gives $dY = \left[g\left(e^Y\right) - 0.5\sigma^2 \right] dt + \sigma dz$. As X approaches 0, Y approaches -∞ and behaves like a Brownian motion with trend $g(0) - 0.5\sigma^2 < 0$. It is intuitive here that -∞ is an attractive barrier for Y. As diffusions have finite variations with probability 1 in finite time, Y cannot reach -∞ (and X cannot reach 0) in finite time. Allowing for extinction in infinite time only may not seem very realistic from a biological point of view, but it is a useful simplification when the minimum biomass is small and there is a risk of extinction. This is the case considered below.

IV. Illustration.

Given the popularity of logistic distributions in bioeconomics, let us assume that X follows the logistic Brownian motion process:

$$dX = rX(K - X)dt + \sigma X dz. \tag{23}$$

To keep our derivations as simple as possible, we also suppose that:

- There are no losses from resource extinction, so L=0.
- Measurement units of the biomass are such that K=1.
- The discount rate, ρ, is adjusted to reflect risk preferences so we drop U(.).
- As in the Schaefer model, the harvested biomass h is proportional to the stock of biomass at harvest x and to the harvest effort $E = h/(qx)$, where q is a positive constant.
- All harvested biomass can be sold at a fixed unit price p, and the currency is such that p=1.
- Finally, there are no fixed costs, and variable harvest costs are proportional to harvest effort,

E; c_v denotes the per unit effort cost.

As a result, the net rent from harvesting h when the biomass equals x is $\pi = \left(p - \frac{c_v}{qx} \right) h$.

For more generality and to better understand parameter interactions, we conduct a dimensionless analysis. To this aim, we define the dimensionless parameters κ, ω, and η:

$$\kappa \equiv \frac{2rK}{\sigma^2}, \ \omega = \frac{\rho}{rK}, \ \eta = \frac{c_v}{qpK}. \tag{24}$$

κ gives biomass growth at carrying capacity scaled by the variance parameter of X; ω is the discount rate divided by biomass growth at carrying capacity; and η is the dimensionless variable harvesting cost. To make biomass and harvest variables dimensionless, we simply divide them by K (=1 here). Hence, the net rent from harvesting h at biomass x becomes:

$$\tilde{\pi}(h,x) = \left(1 - \frac{\eta}{x} \right) h. \tag{25}$$

Let us now analyze the risk of extinction. With the notations of the previous section, β=1 and $g(0)$=rK so we know from Proposition 1 that if κ>1 (i.e., if $\sigma < \sqrt{2rK}$), the resource manager does not need to worry about extinction. Using Proposition 2, if we change variables in (21), incorporate parameters defined in (24), and integrate by parts, we can show that the product of ρ by the expected time from the initial state x_0 until X=x is $\rho v_{x|x_0} = \kappa \omega w_{x|x_0}$, where

$$w_{x|x_0} = \int_{\kappa x_0}^{\kappa x} \left(\int_0^{\xi} \zeta^{\kappa-2} e^{-\zeta} d\zeta \right) \xi^{-\kappa} e^{\xi} d\xi = \frac{1}{\kappa - 1} \int_{\kappa x_0}^{\kappa x} \frac{e^{\xi}}{\xi} \varphi(\kappa - 1, \kappa, -\xi) d\xi. \tag{26}$$

$w_{x|x_0}$ is dimensionless and $\varphi(.)$ is the confluent hypergeometric function of the first kind.[10]

If κ<1, however, there is a risk of extinction but not in finite time, so $v_{0,x|x_0} = +\infty$.

Taking the limit of (I.9) when $a \to 0$ leads to

$$p_{0;x|x_0} = \int_0^{\kappa x_0} \xi^{-\kappa} e^{\xi} d\xi \left(\int_0^{\kappa x} \xi^{-\kappa} e^{\xi} d\xi \right)^{-1} = \left(\frac{x_0}{x} \right)^{1-\kappa} \frac{\varphi(1-\kappa, 2-\kappa, \kappa x_0)}{\varphi(1-\kappa, 2-\kappa, \kappa x)}. \tag{27}$$

To derive the dimensionless discount factor, we take the limit of (I.8) with (I.12) when $a \to 0$ and find (like Willassen (1998) and Sødal (2002) for $D_{x|x_0}$)

$$D_{0,x|x_0} = D_{x|x_0} = \left(\frac{x_0}{x} \right)^{\theta} \frac{\varphi(\theta, 2\theta+\kappa, \kappa x_0)}{\varphi(\theta, 2\theta+\kappa, \kappa x)}, \tag{28}$$

where

$$\theta = \frac{1}{2} - \frac{\kappa}{2} + \sqrt{\left(\frac{1}{2} - \frac{\kappa}{2} \right)^2 + \kappa\omega}. \tag{29}$$

Combining (25), (27), and (28), the dimensionless version of the resource manager's problem can then be written

$$\tilde{V}(x_0) = \underset{0 \le h \le x}{Max} \left\{ \frac{p_{0;x|x_0} D_{0,x|x_0}}{1 - p_{0;x|x-h} D_{0,x|x-x}} \tilde{\pi}(h, x) \right\}, \tag{30}$$

with $p_{0;h|\zeta} = 1$ and $D_{0,x|\zeta} = D_{x|\zeta}$ when $\kappa > 1$ because then there is no risk of extinction. Let us first analyze (30) when uncertainty (i.e., σ^2) is large. We have:

Proposition 3. When σ^2 is large enough, a total harvest is optimal, $x^* = h^* \approx 2\eta$, and $\tilde{V}(x_0) \approx \frac{x_0^2}{4\eta}$. □

Proof. When σ^2 is large, $\kappa \equiv \frac{2rK}{\sigma^2} \approx 0$ and $\theta \approx 1$. As a result, $D_{0,x|x_0} \approx \frac{x_0}{x} \frac{\varphi(1,2,0)}{\varphi(1,2,0)} = \frac{x_0}{x}$ and

$p_{0;x|x_0} \approx \frac{x_0}{x}\frac{\varphi(1,2,0)}{\varphi(1,2,0)} = \frac{x_0}{x}$, so (30) becomes $\underset{0\le h\le x}{Max}\left\{\frac{x_0^2}{2x-h}\left(1-\frac{\eta}{x}\right)\right\}$. From this expression, we see that h=x at the optimum, so x^* solves $\underset{0\le x}{Max}\left\{\frac{x_0^2}{x}\left(1-\frac{\eta}{x}\right)\right\}$ (which is also Equation (7)). From the first order condition, $x^*\approx 2\eta$; the rest follows. □

Solving (30) for finite values of σ^2 requires numerical methods. From (27) and (28), terms in x_0 can be factored out of (30), so we can maximize (30) directly. Optimal values of h and x, obtained with Mathcad on a personal computer, are presented in Figures 1a, 1b, and 2.

Figures 1a and 1b give an overview of the impact of uncertainty on the optimal biomass at harvest (x^*) and the amount harvested (h^*) for η=0.1 and for a wide range of values of $\omega = \frac{\rho}{rK}$ and $\frac{1}{\kappa} = \frac{\sigma^2}{2rK}$. Solid lines reflect the risk of extinction, when it is present, whereas dotted lines ignore it. First, we see (Figure 1a) that as ω increases, x^* shifts downwards: since future revenues are more heavily discounted (holding rK constant), harvests take place earlier (at a lower value of the biomass). Second, we observe that with the possibility of extinction, x^* and h^* are not monotonic functions of σ^2. In fact, we can distinguish three regions for x^* and h^*.

When $\kappa^{-1} \in (0,1]$, extinction is not possible; x^* and h^* both increase with κ^{-1}, although h^* initially increases more slowly than x^* because of the non-linear behavior of $D_{x|x_0}$ as a function of σ^2. Indeed, while $T_{x_2|x_1}$ increases with σ^2 (and thus with κ^{-1}) for x_1 and x_2 fixed, $D_{x_2|x_1}$ may first sharply and then slowly decrease with σ^2 for low values of ω ($10^{-5}\le\omega\le 0.3$ for the parameters explored); by contrast, for higher values of ω ($\omega \in (0.7,0.9)$), $D_{x_2|x_1}$ may first sharply and then slowly increase with σ^2.

The second region starts at $\kappa^{-1}=1$ and extends until total harvest is optimum (this is indicated by a diamond on the figures). At $\kappa^{-1}=1$, the probability of extinction becomes non-zero and starts increasing with κ^{-1}; a kink appears for both x^* and h^*, and x^* starts declining with κ^{-1}. The resource manager thus waits less and harvests proportionately more of the biomass because delaying harvest increases the risk of extinction and leaving more biomass for future harvests increases the magnitude of a potential loss.

In the third region (for values of κ^{-1} to the right of the diamond) the risk of extinction is so high that a total harvest is optimal (the curves x^* and h^* merge): it is not worthwhile leaving biomass for future harvests because it may just disappear. Both x^* and h^* decrease as the probability of extinction increases with κ^{-1}.

Figure 2 shows the optimal present value of expected rents for the harvesting rules presented on Figures 1a and 1b. First, with no risk of extinction, we observe that the present value of expected harvesting rents may increase with uncertainty for low values of κ^{-1} and ω because higher values of h^* and x^* more than compensate for decreases in the discount factor. More importantly, we see that the risk of extinction (for $\kappa^{-1}>1$) causes expected harvesting rents to dip (the solid lines on Figure 3). Ignoring extinction leads to overestimating the net present value of expected rents (the dotted lines on Figure 3). Indeed, suppose that the resource manager solves (8) whereas the correct problem is (5) with $L=0$. Simple calculations (not shown) indicate that the resulting loss varies quasi linearly with κ^{-1}, from 0 at $\kappa^{-1}=1$ to ~40% of the total value of the resource at κ^{-1}=2, for $\omega \in (10^{-5}, 0.7)$ and η=0.1. Ignoring the risk of extinction may thus be costly, although the practical implications need to be evaluated empirically. Finally, we note that the present value of expected harvesting rents tends asymptotically towards $\frac{x_0^2}{4\eta}$ (≈0.156 here), as predicted in Proposition 3.

Similar results were obtained for different values of η, the dimensionless parameter for variable harvesting costs. Our methodology can also easily be extended to other specifications of the profit function or to endogenous rotation costs in forestry as in Sødal (2002), for example.

V. Conclusions.

This paper presents a theory of harvesting that explicitly accounts for extinction and includes partial and total harvests of biological assets with size-dependent stochastic growth. This generalizes the existing literature, where continuous-time analytical results are currently available only for one-time harvests or for repeated, total harvests (the forestry case), without consideration for the risk of extinction.

To account for harvest irreversibility and to model uncertainty concisely, we use concepts from the theory of real options and cast the decision to harvest as a disinvestment problem in continuous time, assuming instantaneous harvests. This allows us to derive a generalized version of the Faustmann formula for general growth functions and harvesting cost specifications. The availability of better stochastic harvesting rules is a necessary step for managing renewable resources in a sustainable way.

We also analyze a wide class of stochastic growth functions to understand the impact of the specification of stochasticity on the probability of extinction. A numerical illustration for the logistic Brownian motion shows the importance of taking the probability of resource extinction into account. Moreover, we show analytically that total harvests (and extinction) become optimal when uncertainty is high enough and there is no existence value. Furthermore, we show numerically that the net present value of expected harvesting rents, the optimal biomass at which to harvest, and the optimal harvest size are not monotonic functions of uncertainty. More generally, this paper illustrates the importance of properly accounting for barriers in stochastic investment problems.

Appendix I

The derivations below could also be useful for more general investment problems where the possibility of investing vanishes when the state variable reaches a barrier (see Saphores 2002).

Expression of $p_{0;x|x_0}$

Let *a*, *b*, and x_0 be three real numbers such that $0<a<x_0<b$. Let us first derive $p_{a;b|x_0}$, the probability that *X* first reaches *b* before *a* starting from x_0. To find $p_{0;x|x_0}$, we just take the limit of $p_{a;b|x_0}$ when *a* goes to 0 and substitute *x* for *b*. Karlin and Taylor (1981) show that:

$$p_{a;b|x_0} = \frac{S(x_0) - S(a)}{S(b) - S(a)}, \tag{I.1}$$

where *S*(.) is the scale function of *X*, defined by

$$S(x) = \int_{x_1}^{x} \exp\left[\int_{x_2}^{\eta} \frac{-2G(\xi)}{v^2(\xi)} d\xi\right] d\eta. \tag{I.2}$$

In (I.2), $x_1>0$ and $x_2>0$ are arbitrary integration constants; it is easy to see, however, that $p_{a;b|x_0}$ does not depend on x_1 and x_2. Also from (I.2), $S'(x)\geq 0$ so $p_{a;b|x_0}$ is comprised between 0 and 1.

$D_{0,x|x_0}$ *and other useful functions*

In addition to $D_{0,x|x_0} = E\left(e^{-\rho T_{0,x|x_0}}\right)$ and $D_{x|x_0} = E\left(e^{-\rho T_{x|x_0}}\right)$, it may be useful for managerial purposes to know $v_{0,x|x_0} \equiv E\left(T_{0,x|x_0}\right)$, the expected time it takes *X* to reach either 0 or *x* starting from x_0, or $v_{x|x_0} \equiv E\left(T_{x|x_0}\right)$, when there is no risk of extinction. For 0<a<x$_0$<b, consider

$$W(x_0) = E\left[f\left(\int_0^{T_{a,b|x_0}} h(X(\tau))d\tau \right) \middle| X(0) = x_0 \right], \tag{I.3}$$

where f(.) is C^2, and h(.) is continuous and bounded. Using a Taylor expansion of W(.), the law of total probabilities and the Markov property, Karlin and Taylor (1981) show that W(.) solves:

$$\begin{cases} \dfrac{v^2(\xi)}{2}\dfrac{d^2W(\xi)}{d\xi^2} + G(\xi)\dfrac{dW(\xi)}{d\xi} + h(\xi)E\left[f'\left(\int_0^{T_{a,b|\xi}} X(\tau)d\tau \right)\right] = 0, \\ W(a) = W(b) = f(0). \end{cases} \tag{I.4}$$

If $\forall\xi\geq 0$, $f(\xi)=\xi$ and $h(\xi)=1$, then $W(x_0) = v_{a,b|x_0} = E\left(T_{a,b|x_0}\right)$. Solving (I.4) gives

$$v_{a,b|x_0} = 2\left[p_{a;b|x_0}\int_{x_0}^{b}\left[S(b)-S(\eta)\right]m(\eta)d\eta + p_{b;a|x_0}\int_{a}^{x_0}\left[S(\eta)-S(a)\right]m(\eta)d\eta \right], \tag{I.5}$$

where, by definition, $p_{b;a|x_0} = 1 - p_{a;b|x_0}$, and $m(\eta)$ (the "speed density" of X) is

$$m(\eta) = \frac{1}{v^2(\eta)}\exp\left\{ \int_{x_2}^{\eta} \frac{2G(\xi)}{v^2(\xi)}d\xi \right\}. \tag{I.6}$$

In (I.6), x_2 is the same constant as in S(.) to guarantee that $v_{a,b|x_0}$ does not depend on x_1 or x_2.

If $\forall\xi\geq 0$, $f(\xi)=e^{-\rho\xi}$ and $h(\xi)=1$, then $W(x_0) = D_{a,b|x_0} = E\left(e^{-\rho T_{a,b|x_0}}\right)$ and (I.4) becomes

$$\rho V(x) = G(x)V'(x) + \frac{v^2(x)}{2}V''(x). \tag{I.7}$$

If $\phi_0(.)$ and $\phi_1(.)$ are two independent solutions of (I.7), $D_{a,b|x_0}$ can be written

$$D_{a,b|x_0} = \frac{\phi_1(b)-\phi_1(a)}{\phi_0(a)\phi_1(b)-\phi_0(b)\phi_1(a)}\phi_0(x_0) + \frac{\phi_0(a)-\phi_0(b)}{\phi_0(a)\phi_1(b)-\phi_0(b)\phi_1(a)}\phi_1(x_0). \tag{I.8}$$

A few examples

- If $dX = rX(K-X)dt + \sigma X dz$ (a Logistic Brownian motion process), then

$$p_{a;b|x_0} = \int_{z(a)}^{z(x_0)} \xi^{-\kappa} e^{\xi} d\xi \left(\int_{z(a)}^{z(b)} \xi^{-\kappa} e^{\xi} d\xi \right)^{-1}, \tag{I.9}$$

$$\phi_0(\zeta) = \varsigma^{\theta} \varphi\left(\theta, 2\theta+\kappa, z(\zeta)\right),\ \phi_1(\zeta) = \varsigma^{\theta}\left[z(\zeta)\right]^{1-2\theta-\kappa} \varphi\left(1-\theta-\kappa, 2-2\theta-\kappa, z(\zeta)\right), \tag{I.10}$$

where $z(\zeta) = 2r\sigma^{-2}\zeta$. κ and θ are defined by (24) and (29).

- If $dX = r(K-X)dt + \sigma dz$ (an Ornstein-Uhlenbeck process), then

$$p_{a;b|x_0} = \int_{z_1(a)}^{z_1(x_0)} e^{\xi^2} d\xi \left(\int_{z_1(a)}^{z_1(b)} e^{\xi^2} d\xi \right)^{-1}, \tag{I.11}$$

$$\phi_0(\zeta) = \varphi\left(\frac{\rho}{2r}, \frac{1}{2}, z_1^2(\zeta)\right),\ \phi_1(\zeta) = z_1(\zeta)\varphi\left(\frac{\rho}{2r}+\frac{1}{2}, \frac{3}{2}, z_1^2(\zeta)\right), \tag{I.12}$$

with $z_1(\zeta) = \sqrt{r}\sigma^{-1}(\zeta - K)$.

- Finally, if $dX = rX(Ln(K) - Ln(X))dt + \sigma X dz$ (a Gompertz Brownian motion process), (I.11) and (I.12) still apply provided $z_1(\zeta) = \sqrt{r}\sigma^{-1}\left[Ln(\zeta) - Ln(K) + \sigma^2(2r)^{-1}\right]$.

φ(.) is the confluent hypergeometric function of the first kind: given three real numbers α, β>0, and z, $\varphi(\alpha,\beta,z) \equiv \sum_{n-0}^{+\infty} \frac{(\alpha)_n}{(\beta)_n}\frac{z^n}{n!}$, where $(\zeta)_n=\zeta(\zeta+1)\ldots(\zeta+n-1)$.

When an analytical solution is impossible, numerical methods, based for example on Chebychev polynomials, may be used to compute W(.) in (I.4).

Appendix II

Proof of Proposition 1. Let a, b, and x_0 such that $0<a<x_0<b$. Let us find a finite bound for S(a) or show that $\lim_{a\to 0^+} S(a) = +\infty$. Let $s(\eta) \equiv \exp\left[\frac{2}{\sigma^2}\int_\eta^{x_2} \frac{g(\xi)}{\xi^{2\beta-1}} d\xi\right]$, so $S(x) = \int_{x_1}^{x} s(\eta)d\eta$. $\lim_{x\to 0+} S(x)$ is finite if and only if $s(.)$ is integrable on $[0,x_1]$, $x_1>0$. Now consider different values of β.

- Case 1: $\beta \in [0,1)$. Here $\lim_{\eta\to 0+} s(\eta) < +\infty$ by Assumptions III.1 and III.2 (the latter is useful when β is close to 1 and $\lim_{x\to 0^+} g(x) = +\infty$), so $\lim_{a\to 0+} S(a) = S(0)$ is finite; (17) follows.
- Case 2: $\beta = 1$ and $\sigma > \sqrt{2g(0)}$. By construction, $g(0) < 0.5\sigma^2$, so since $g(.)$ is continuous (Assumption III.1), there exists x_2 and $\overline{B} < 0.5\sigma^2$ such that $\overline{B}$ is an upper bound for $g(.)$ on $[0,x_2]$. Then, for $\eta\in[0,x_2]$, $s(\eta) \le \exp\left[\frac{2}{\sigma^2}\int_\eta^{x_2} \frac{\overline{B}}{\xi} d\xi\right] = \left(\frac{\eta}{x_2}\right)^{-\frac{2\overline{B}}{\sigma^2}} \equiv \overline{s}(\eta)$. Here $-\frac{2\overline{B}}{\sigma^2} > -1$, so $\overline{s}(\eta)$ is integrable on $[0,x_2]$. As a result, $\lim_{a\to 0+} S(a) = S(0)$ is finite and (17) follows.
- Case 3: $\beta = 1$ and $0 < \sigma < \sqrt{2g(0)}$. Similar proof as for Case 4 below.
- Case 4: $\beta > 1$. From Assumption III.1, there exist $L>0$ and $x_2>0$ such that L is a lower bound to $g(.)$ on $[0,x_2]$. Then, $s(\eta) \ge \exp\left[\frac{2}{\sigma^2}\int_\eta^{x_2} \frac{L}{\xi^{2\beta-1}} d\xi\right] = \exp\left[\frac{2L}{\sigma^2}\frac{x_2^{2-2\beta} - \eta^{2-2\beta}}{2-2\beta}\right] \equiv \underline{s}(\eta)$ for any $\eta\in[0,x_2]$. Since $2-2\beta<0$, a change of variable shows that $\underline{s}(\eta)$ is not integrable on $[0,x_2]$, so neither is $s(\eta)$. Hence, $\lim_{a\to 0+} S(a) = -\infty$ and $\lim_{a\to 0+} p_{a;b|x_0} = 1$. □

Before proving Proposition 2, a lemma is needed.

Lemma 1. If $\beta \in [0,0.5) \cup (1,+\infty)$ or if $\beta = 1$ and $\sigma < \sqrt{2g(0)}$, then $\forall x > 0, \int_0^x m(\eta)d\eta < +\infty$.

Conversely, if $\beta \in [0.5,1)$ or if $\beta = 1$ and $\sigma > \sqrt{2g(0)}$, then $\forall x > 0, \lim_{a \to 0+} \int_a^x m(\eta)d\eta = +\infty$. Here,

$m(\eta) \equiv [\sigma^2 \eta^{2\beta} s(\eta)]^{-1}$ is the speed density of X (also see (22)). Let $\tilde{m}(\eta) \equiv \sigma^{-2}\eta^{-2\beta}$.

Proof. For simplicity, let x_2=K. Let x>0. Now consider each case in turn.

- Case 1: $\beta \in [0,0.5)$. As $2\beta \in [0,1)$, $m(.)$ is integrable on [0,K].

- Case 2: $\beta \in [0.5,1)$. As $2\beta > 1$, $\tilde{m}(.)$ is not integrable on [0,K], so $\lim_{a \to 0+} \int_a^x m(\eta)d\eta = +\infty$.

- Case 3: $\beta = 1$ and $\sigma > \sqrt{2g(0)}$. Here $g(0) < 0.5\sigma^2$, so by continuity of g(.) (Assumption III.1), there exists C>0 such that if $\overline{g}$ is the maximum of g(.) on [0,C], $\overline{g} < 0.5\sigma^2$. Then,

$$\forall \eta \in (0,C),\ m(\eta) \geq \frac{C^{\frac{-2\overline{g}}{\sigma^2}}}{\sigma^2} \exp\left\{ \int_C^K \frac{-2g(\xi)}{\sigma^2 \xi} d\xi \right\} \eta^{\frac{2\overline{g}}{\sigma^2}-2} \equiv \underline{m}(\eta).$$ As $\frac{2\overline{g}}{\sigma^2} - 2 < -1$, m(.) is again not integrable on [0,K].

- Case 4: $\beta = 1$ and $\sigma < \sqrt{2g(0)}$. Here $g(0) > 0.5\sigma^2$ so by continuity of g(.) (Assumption III.1), there exists C>0 such that $\min\{g(\xi), \xi \in [0,C]\} \equiv \underline{g} > 0.5\sigma^2$. Then, for $\eta \in (0,C)$,

$$m(\eta) \leq \frac{C^{\frac{-2\underline{g}}{\sigma^2}}}{\sigma^2} \exp\left\{ \int_C^K \frac{-2g(\xi)}{\sigma^2 \xi} d\xi \right\} \eta^{\frac{2\underline{g}}{\sigma^2}-2} \equiv \overline{m}(\eta).$$ As $\frac{2\underline{g}}{\sigma^2} - 2 > -1$, $\int_0^x m(\eta)d\eta < +\infty$.

- Case 5: $\beta > 1$. Similar proof as for Case 4 with $\underline{g} > 0$. □

Proof of Proposition 2. To simplify our derivations, let again x_2=K.

- Case 1: $\beta \in [0, 0.5)$. Let $x_0 \in (0, b)$. Take a→0+ in (I.5) using Proposition 1 and Lemma 1 to get $v_{0,b|x_0} = 2\left[p_{0;b|x_0} \int_{x_0}^{b} [S(b) - S(\eta)] m(\eta) d\eta + p_{b;0|x_0} \int_{0}^{x_0} [S(\eta) - S(0)] m(\eta) d\eta \right] < +\infty$. Then introduce the denominator of $p_{0;b|x_0} = \dfrac{S(x_0) - S(0)}{S(b) - S(0)}$ in the first integral of $v_{0,b|x_0}$ and take b→+∞ to get $[S(x_0) - S(0)] \int_{x_0}^{+\infty} m(\eta) d\eta < +\infty$ ($\lim_{b \to +\infty} S(b) = +\infty$ from Assumption III.1); the second part of $v_{0,b|x_0}$ goes to $\int_{0}^{x_0} [S(\eta) - S(0)] m(\eta) d\eta$ as $\lim_{b \to +\infty} p_{b;0|x_0} = 1$. This gives (19).

- Case 2: $\beta \in [0.5, 1)$ or $\beta = 1$ and $\sigma > \sqrt{2g(0)}$. From Proposition 1, S(.) is bounded on [0,K] but from Lemma 1, m(.) is not integrable on [0,K]. Taking a→0+ in (I.5) gives (20).

- Case 3: $\beta > 1$ or $\beta = 1$ and $0 < \sigma < \sqrt{2g(0)}$. Only the second integral on the right side of (I.5) depends on a. From Proposition 1, $\lim_{a \to 0+} S(a) = -\infty$ and $p_{0;x|x_0} = 1$ for $0<x_0\leq x$; so inserting the denominator of $p_{b;a|x_0} = \dfrac{S(b) - S(x_0)}{S(b) - S(a)}$ into the second integral of (I.5), taking a→0+ and using Lemma 1 gives $2\left(\int_{x_0}^{b} [S(b) - S(\eta)] m(\eta) d\eta + [S(b) - S(x_0)] \int_{0}^{x_0} m(\eta) d\eta \right) < +\infty$.

  Moreover, $p_{0;b|x_0} = 1$ implies that 0 is never reached before b, so $v_{0,b|x_0} = v_{b|x_0}$, which proves (21). □

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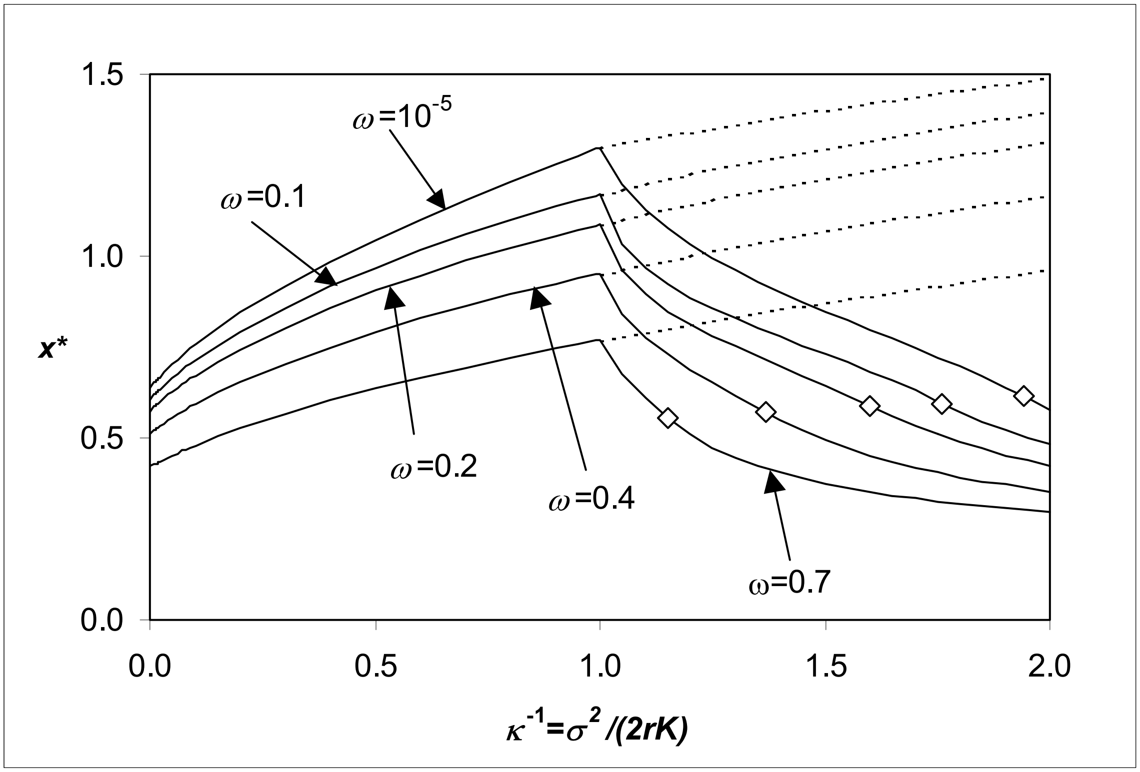

Figure 1a: Optimal biomass at harvest versus κ^{-1}, for η=0.1.

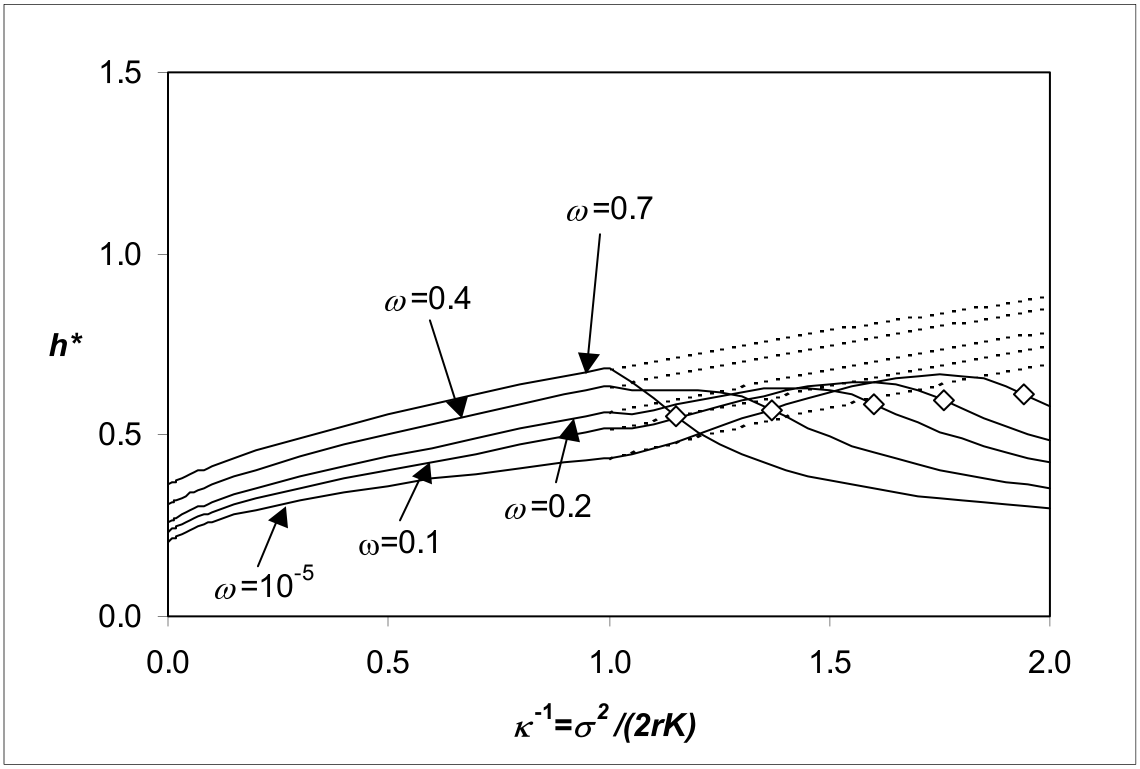

Figure 1b: Optimal biomass harvested versus κ^{-1}, for η=0.1.

Notes for Figures 1a and 1b: Solid lines give x^* (the optimal biomass at harvest) on Figure 1a or h^* (the optimal harvest size) on Figure 1b; they account for the risk of extinction when it is present. By contrast, dotted lines give the biomass at harvest or the harvest size when the risk of extinction is ignored. The diamond indicates where the optimal harvest becomes total. κ gives biomass growth at carrying capacity scaled by the variance parameter of X; ω is the discount rate divided by biomass growth at carrying capacity; and η is the dimensionless variable harvesting cost.

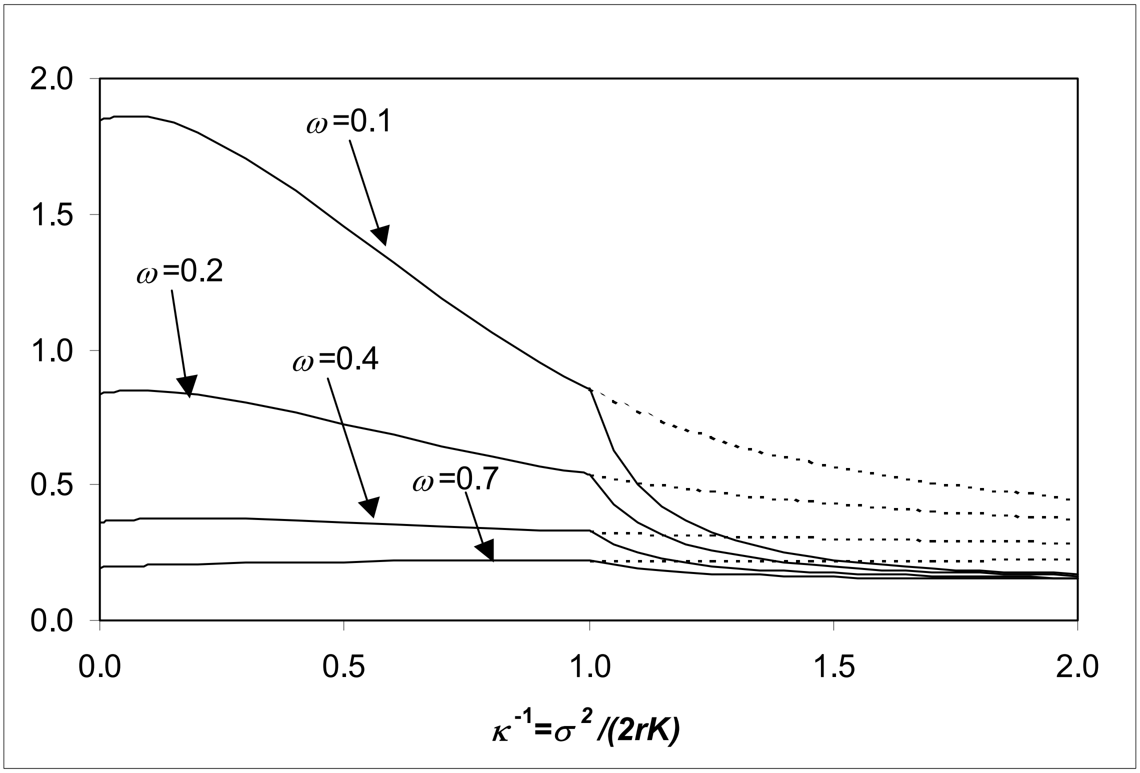

Figure 2: Optimal net present value of expected rents versus κ^{-1}, for η=0.1.

Notes: The optimal net present value of expected harvesting rents, $\tilde{V}(x_0)$, is calculated at x_0=0.25, where x_0 is the initial biomass. For definitions of ω, η, and κ, see (24).

[1] As emphasized by Clarke and Reed (1989) and Reed and Clarke (1990), it is useful to distinguish between two classes of harvesting models. In the first one, growth is density-dependent so the decision to harvest depends on biomass size. It is typically more suitable for wilder resources, such as undisturbed forests, wildlife, and natural fish or shellfish populations. In the second class of models, growth is age-dependent as the impact of environmental factors is more limited, so harvest takes place at fixed time intervals. This situation usually applies to husbanded biological assets, such as livestock or cultivated trees.

[2] In Reed and Clarke (1990), the resource biomass follows the process $dX = Xf(X)dt + \sigma X dz$, where *f*(.) is decreasing, *f*(0)>0, σ>0, and *dz* is an increment of a standard Wiener process.

[3] In this paper, the minimum stock of biomass for long-term survival, denoted by K_0, is assumed to be zero for simplicity, but our approach can easily be extended to tackle the case $K_0>0$.

[4] Throughout this paper, *X* is a random variable and *x* is one of its realizations.

[5] For simplicity, we assume that it is never interesting to harvest to extinction for some low value of *X*. However, it is easy to accommodate this possibility. Suppose that if *X* reaches $\underline{x}$, it is optimal to harvest the whole resource and get utility $\underline{U}(\underline{x})$. Then (3) becomes $V(x_0) = D_{\underline{x},x^*|x_0}\left\{p_{\underline{x};x^*|x_0}\left[U(\pi(h^*,x^*)) + V(x^* - h^*)\right] + p_{x^*;\underline{x}|x_0}\underline{U}(\underline{x})\right\}$ and (4) can be obtained by replacing x_0 with $x^* - h^*$ We can then derive the resource manager's objective function and the corresponding first order necessary conditions. $\underline{x}$ could be endogenous (leading to a problem with three decision variables) or exogenous, in which case $\underline{U}(\underline{x})$ could simply be written $\underline{U}$.

[6] Our approach can also be adapted to the case where the harvesting region is not a simple

interval (i.e., when the resource manager's problem has more than one solution). Suppose for example that the solution is such that there exist $0 < x_1^* < x_2^* < x_3^*$, $0 < h_1^* < x_1^*$ and $0 < h_2^* < x_3^* - x_2^*$, where it is optimal to wait for $X \in (0, x_1^*)$, harvest h_1^* for $X \in (x_1^*, x_2^*)$, wait for $X \in (x_2^*, x_3^*)$, and harvest h_2^* for $X \geq x_3^*$. In this case, if $x_0 \in (0, x_1^*)$, X will stay in this interval; however, if $x_0 \in (x_2^*, x_3^*)$, X may reach $(0, x_1^*)$ and stay there. To solve, we would pick $x_0 \in (x_2^*, x_3^*)$, write equalities such as (3), and formulate the resource manager's problem. Characterizing analytically the functions $G(.)$, $v(.)$, $\pi(.,.)$ and $U(.)$ that lead to a unique solution is beyond the scope of this paper. For now, we resort to showing uniqueness numerically.

[7] For simplicity, this formulation assumes that the resource manager waits until X=x^* or X=0 after the growth process has been restarted so that $x_0 \in (0, x^*)$. It would be easy to introduce a lower bound $\underline{x}$ ($0 < \underline{x} < S$) where the growth process is again restarted. In this case, (12) becomes $V(x_0) = D_{\underline{x},x^*|x_0} \left\{ p_{\underline{x};x^*|x_0} U(\pi(x^* - S, x^*)) + p_{x^*;\underline{x}|x_0} U(C_R) + V(S) \right\}$, where the second equation is obtained by substituting S for x_0. It is straightforward to write the resource manager's problem and to derive the first order necessary condition.

[8] Sødal (2002) and Willassen (1999) conduct their analyses in terms of the value of the biomass of the resource considered. It is straightforward to adapt the derivations of this paper to that case.

[9] This assumption allows us to deal with models such as $dx=rx[Ln(K)-ln(x)]dt+\sigma x^{\beta}dz$, based on Gompertz's law ($dx=rx[Ln(K)-ln(x)]dt$).

[10] See the end of Appendix I for a definition.

Harvesting a Renewable Resource under Uncertainty: Erratum [1]

Jean-Daniel Saphores[2]

Abstract

This paper revisits the problem of harvesting a renewable resource under uncertainty to correct an error in Saphores (2003) linked to the treatment of absorbing barriers. Implications for general, autonomous harvesting problems are then briefly explored.

Key words: renewable resources; extinction; uncertainty; irreversibility; real options.

JEL classification: D92, D81, Q20.

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[1] The helpful comments of Sigbjørn Sødal and Hervé Roche are gratefully acknowledged. I am, of course, responsible for all remaining errors.

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In a recent paper, I present a theory of harvesting that allows for partial harvests and accounts for the risk of extinction, for biological assets with size-dependent stochastic growth (Saphores 2003). However, my analysis contains a formulation error linked to the treatment of absorbing barriers. After tackling this error, implications for more general, autonomous harvesting problems are briefly explored.

Let us first consider a very simple harvesting problem where a risk neutral resource manager needs to decide when, if ever, to totally harvest a valuable renewable resource whose biomass *X* varies randomly according to the diffusion

$$dX = rX(\ln(K) - \ln(X))dt + \sigma X dw. \qquad (1)$$

In (1), $r>0$ indicates how quickly *X* reverts towards $K>0$, $\sigma>0$ is the volatility parameter, *dt* is a time increment, and *dw* is an increment of a standard Wiener process (Karlin and Taylor 1981). This specification is convenient for at least two reasons: first, *X* cannot become negative; and second, $X(t)$ given $X(0)$ has a known, lognormal distribution, which is handy for simulations. Indeed, an application of Ito's lemma to $Z \equiv \ln(X/K)$ leads to the Ornstein-Uhlenbeck process

$$dZ = -r(Z+\kappa)dt + \sigma dw. \qquad (2)$$

The dimensionless parameter κ scales the propensity of *X* (or *Z*) to vary randomly by its propensity to revert to *K* (0 for *Z*). It is defined by

$$\kappa \equiv \frac{\sigma^2}{2r}. \qquad (3)$$

From Karlin and Taylor (1981), Z_t is normally distributed with mean $\left(Z_0+\kappa\right)e^{-rt}-\kappa$ and variance $\left(\frac{1-e^{-2rt}}{2r}\right)\sigma^2$. For simplicity, let us also suppose that:

- The resource can be harvested instantaneously;
- $M \geq 0$ is the minimum viable biomass: as soon as X (Z) reaches L ($m \equiv \ln(L/K)$), the resource becomes extinct but there are no losses from extinction;
- As in the Schaefer model (Clark 1990), the harvested biomass h is proportional to the stock of biomass x and to the harvest effort E: $h=qEx$, where $q>0$ is constant. Here, $h=x$ so $E=1/q$.
- There are no fixed costs and variable harvesting costs are proportional to harvest effort, E; c_v denotes the per unit effort cost;
- The relevant discount rate is ρ; and finally,
- All harvested biomass can be sold at a fixed unit price p.

As a result, the net rent when the biomass equals $x = Ke^z$ is $\pi = pK\left(e^z - \eta\right)$, where η is the dimensionless variable harvesting cost defined by

$$\eta = \frac{c_v}{qpK}. \tag{4}$$

According to Saphores (2003), the objective of a risk-neutral resource manager can be written (after dividing by the unit price of biomass, p, and by K, both assumed constant)

$$V(z_0) = \underset{\{z\}}{Max}\, D_{m,z|z_0}\, p_{m;z|z_0}\left(e^z - \eta\right), \tag{5}$$

where

- $z_0 = \ln\left(x_0 / K\right)$ (x_0 is the current stock of biomass) and $m \equiv \ln\left(M / K\right)$;
- $D_{m,z|z_0} = E\left(e^{-\rho T_{m,z|z_0}}\right)$ is the expected discount factor for $T_{m,z|z_0}$, the random duration between $Z=z_0$ and the first time Z hits either m or z; and

- $p_{m;z|z_0}$ is the probability that *Z* hits *z* before *m* starting from *z₀*.

From Appendix I in Saphores (2003),

$$p_{m;z|z_0} = \int_m^{z_0} \mathrm{e}^{\frac{(\xi+\kappa)^2}{2\kappa}} d\xi \left(\int_m^{z} \mathrm{e}^{\frac{(\xi+\kappa)^2}{2\kappa}} d\xi \right)^{-1}, \tag{6}$$

and

$$D_{m,z|z_0} = \frac{[\phi_2(z)-\phi_2(m)]\phi_1(z_0)+[\phi_1(m)-\phi_1(z)]\phi_2(z_0)}{\phi_1(m)\phi_2(z)-\phi_1(z)\phi_2(m)}, \tag{7}$$

with

$$\phi_1(\xi) = (z+\kappa)\Phi\left(\omega+\frac{1}{2},\frac{3}{2},\frac{(z+\kappa)^2}{2\kappa}\right),\ \phi_2(\xi) = \Phi\left(\omega,\frac{1}{2},\frac{(z+\kappa)^2}{2\kappa}\right). \tag{8}$$

ω is the dimensionless ratio of the discount factor by twice the mean reversion parameter:

$$\omega = \frac{\rho}{2r}, \tag{9}$$

and $\Phi(a,c;z) = \sum_{k=0}^{+\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}$, with $(a)_0=1$, $(a)_k=a\cdot(a+1)\cdots(a+k-1)$, is the confluent hypergeometric function of the first kind. Equations (5) to (8) summarize Model 1.

Using simulations, let us show instead that the objective of a risk-neutral resource manager should be written (after dividing by the unit price of biomass, *p*, and by *K*)

$$V(z_0) = \underset{\{z\}}{Max}\, D_{z|z_0;m}\left(e^z - \eta\right), \tag{10}$$

where $D_{z|z_0;m} = E\left(e^{-\rho T_{z|z_0;m}}\right)$ is the expected discount factor for $T_{z|z_0;m}$, the random duration between $Z=z_0$ and the first time *Z* hits *z*, conditional on hitting *z* before *m*. From Karlin and

Taylor (1981), $W(\xi) \equiv D_{z|\xi;m}$ verifies $\frac{\sigma^2}{2} W''(\xi) - r(\xi+\kappa)W'(\xi) - \rho W(\xi) = 0$. In addition, $W(z)$=1 and $W(m)$=0 (the resource becomes extinct as soon as Z hits m, so $T_{z|m;m} = +\infty$). Hence

$$D_{z|z_0;m} = \frac{\phi_1(m)\phi_2(z_0) - \phi_2(m)\phi_1(z_0)}{\phi_1(m)\phi_2(z) - \phi_1(z)\phi_2(m)}, \tag{11}$$

with $\phi_1(\xi)$ and $\phi_2(\xi)$ defined in (8). Equations (8) to (11) summarize Model 2.

As explained in Saphores (2003), to find the solutions of Model 1 or Model 2, denoted respectively by z_1^* and z_2^*, the first order necessary condition needs to be written at $z_0 = z_i^*$.

To assess which of these two models is correct, I simulate paths of Z that start from z_0=0 (i.e., x_0=K) and go either through m or through the z_i^*s using the exact discretization of (2), for a small time increment Δt:

$$Z_{t+\Delta t} = (Z_t + \kappa)e^{-r\Delta t} - \kappa + \sqrt{\frac{1-e^{-2r\Delta t}}{2r}}\sigma\varepsilon_t,\ \varepsilon_t \sim N(0,1). \tag{12}$$

For $i \in \{1,2\}$, if a path hits m before z_i^*, it contributes nothing to the expected present value for Model i; otherwise, it contributes $e^{-\rho T_i}(e^{z_i^*} - \eta)$, where T_i is the first time that Z hits z_i^* starting from z_0=0, conditional on not hitting m. Results for η=0.75, ω=0.25, Δt=0.0025, and κ between 0.2 and 1.4 are presented in Table 1. It shows that simulated expected profits from Model 2 are consistently higher than for Model 1; the difference between the two is nil for small values of κ but it grows with κ (with the volatility σ of the biomass if all other parameters are held constant). Furthermore, simulated profits for Model 2 are very close to their theoretical values, whereas expected profits predicted by Model 1 are systematically under-estimated (they are overestimated for η=0.1 and ω=0.5 for example). These results and others (not shown here) for different values of η, ω, κ, and z_0 show that Model 2 is correct while Model 1 is not. The underlying reason is that Model 1 uncouples expected profits from the effective time it takes Z to reach z_2^*.

Let us now consider general autonomous, repeated harvesting problems where the resource manager's objective is to maximize the expected present value of the utility of all future harvests when the stock of biomass X follows a diffusion such as

$$dX = G(X)dt + v(X)dw. \tag{13}$$

$G(.)$ and $v(.)$ are continuous, dt is an infinitesimal time increment, and dw is an increment of a standard Wiener process. $G(.)$ is assumed strictly positive on $(0,K)$, strictly negative on $(K,+\infty)$ and $G(K)=0$. In addition, $v(.)$ is strictly positive on $(0,+\infty)$.

Following the logic detailed in Saphores (2003), the resource manager's problem is equivalent to choosing the biomass threshold x^* at which biomass h^* should be harvested to maximize the value function $V(x_0)$ (x_0 is the current stock of biomass) given by

$$V(x_0) = D_{x^*|x_0;M}\left[U(\pi(h^*,x^*)) + V(x^*-h^*)\right] + D_{M|x_0;x^*}U(L), \tag{14}$$

where:

- $U(.)$ is the resource manager's utility function; $U(.)$ is strictly concave, increasing, and C^2;
- $\pi(h,x)$ is the net rent from harvesting biomass h when the resource stock is $x \geq h$. π is C^2 and increasing in x and h;
- $D_{x^*|x_0;M} = E\left(e^{-\rho T_{x^*|x_0;M}}\right)$ is the expected discount factor for $T_{x^*|x_0;M}$, the random duration between $X=x_0$ and the first time X hits x^* conditional on hitting x^* before M, the value of the

minimum biomass; ρ is the appropriate discount rate; conversely $D_{M|x_0;x^*} = E\left(e^{-\rho T_{M|x_0;x^*}}\right)$ is the expected discount factor for $T_{M|x_0;x^*}$, the random duration between X=x_0 and the first time X hits M conditional on hitting M before x^*; and

- L is the payoff resulting from extinction.

Writing (14) at x_0=x^*-h^* to get $V(x^* - h^*)$ and inserting it into (14) gives

$$V(x_0) = \underset{\{h,x,\ M \le h \le x\}}{Max} \left[UH(h,x \mid x_0) + \frac{D_{x|x_0;M}}{1 - D_{x|x-h;M}} UH(h,x \mid x-h) \right], \tag{15}$$

where $UH(h,x \mid \zeta) \equiv D_{x|\zeta;M} U(\pi(h,x)) + D_{M|\zeta;x} U(L)$ is the discounted utility from the next harvest, taking into account the risk of extinction, when the starting biomass is ζ.

The rest of the analysis in Saphores (2003), including first order necessary conditions and special cases, holds once the formulation correction detailed above has been made.

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Table 1: Predicted and Simulated Results for Models 1 and 2.

$\kappa = \frac{\sigma^2}{2r}$	Predicted z^*		Predicted V^*		Simulated V^*	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
0.2	0.3147	0.3147	0.3554	0.3555	0.3539	0.3539
0.4	0.4642	0.4881	0.4238	0.4329	0.4302	0.4307
0.6	0.4984	0.6079	0.4413	0.4840	0.4752	0.4803
0.8	0.5185	0.6976	0.4493	0.5173	0.4996	0.5157
1.0	0.5474	0.7695	0.4605	0.5403	0.5172	0.5378
1.2	0.5810	0.8301	0.4735	0.5575	0.5347	0.5601
1.4	0.6159	0.8829	0.4868	0.5713	0.5492	0.5695

Notes: These results were obtained for η=0.75, ω=0.25, z_0=0, M/K=0.1 (m=ln(M/K)=-2.303), Δt=0.0025, and 50,000 simulations. With 25,000 simulations or a time step of Δt=0.01, results are similar and differ by less than 1.3%. Predicted values were calculated using MathCad.