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Rigorous results on unexpected conductance of certain low-dimensional materials

Author
Zhu, Xiaowen

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Rigorous results on unexpected conductance of certain low-dimensional materials

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Xiaowen Zhu

Dissertation Committee:
Professor Svetlana Jitomirskaya, Chair
Professor Anton Gorodetski
Professor Katya Krupchyk

2022

DEDICATION

To Chende Zhao.

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VITA

Xiaowen Zhu

EDUCATION

PhD in Mathematics. Advisor: Jitomirskaya Svetlana **2022**
University of California, Irvine

Bachelor of Mathematics **2016**
Nanjing University

Exchange student **2015**
University of California, Berkeley

HONORS AND AWARDS

Outstanding TA Award - UCI	2020
Euler Award for outstanding promise as a graduate student - UCI	2019
Honors at Graduation. Outstanding Graduate Award - Nanjing University	2016
The 14TH Research Institute of CETC Scholarship - Nanjing University	2015
First Prize Elite Program - Nanjing University	2014
Outstanding Student of NJU - Nanjing University	2014
National Scholarship of China - Nanjing University	2013

PAPERS AND PREPRINTS

- Large Deviations of the Lyapunov Exponent and Localization for the 1D Anderson Model (with S. Jitomirskaya). Comm. Math. Phys. 370, 311–324 (2019).
- Localization for random CMV matrices. Preprint. arXiv:2110.11386 (2021).
- Magnetic response of twisted bilayer graphene (with S. Becker and J. Kim). Preprint. arXiv:2201.02170 (2022).
- Dynamical localization for the singular Anderson model in $\mathbb{Z}^d$ (with N. Rangamani). Manuscript.

BOOK IN PREPARATION

- Introduction to Group Theory *Learning by Doing*

– with C. Davis, N. Donaldson, M. Hehmann, J. Kramer-Miller, A. Pantano.

TEACHING EXPERIENCE

- Directing Undergraduate research 2021 - 2022
- Teaching Assistant - University of California, Irvine 2017 - 2021
  - Graduate Real Analysis, Graduate Complex Analysis, Introduction to Graduate Analysis,
  - PDE, Probability, ODE, Calculus, Linear Algebra, Abstract Algebra, Putnam preparation seminar.
- Teaching Assistant - Jumpstart program for incoming graduate students in UCI.
- Lecturer - University of California, Irvine.
  - Linear Algebra.

INVITED CONFERENCE AND SEMINAR TALKS

- ”New directions in disordered systems” on the occasion of the 75th birthday of Abel Klein - online 2022
- Spectral Theory of Ergodic Quantum Systems - online 2022
- Applied Math Colloquium in Columbia University - Online 2022
- AMS Spring Western Sectional Meeting, Special Session on Localization and deocalization in ergodic quantum systems - Online 2021
- 38th Annual Western States Mathematical Physics Meeting - Online 2020
- 37th Annual Western States Mathematical Physics Meeting - Irvine, CA 2019
- AMS sectional meeting, “Dynamical Systems and Ergodic Theory” - Riverside, CA 2019
- BIRS Workshop, “Random Matrix Products and Anderson Localization” - BIRS, Canada 2019
- Dynamical system and Ergodicity Seminar - Nanjing, China 2018
- Ergodic Theory and Spectral Theory Seminar - Riverside, CA 2018

- 36th Annual Western States Meeting of Mathematical Physics, - Irvine, CA 2018

OUTREACH

- DECADE student representative - UCI math department 2021 - 2022
- UCI: math circle : volunteer 2019 - 2020

JOURNALS REFEREED FOR

- Communication in Mathematical Physics.
- International Mathematics Research Notices.
- Journal of Spectral Theory.
- Journal of Functional Analysis.

SEMINARS at UCI

- Magnetic TBG 2022
- Floquet theory 2022
- Asymptotic expansion for the d.o.s. (density of states) of the magnetic Schrödinger operators 2021
- Semiclassical approach towards the magnetic Schrödinger operators 2020
- Localization for the 2D Anderson model with Bernoulli potential 2019
- Fractional moment method 2019
- Orthogonal polynomials on the unit circle 2018
- Multiscale analysis (MSA) for the Anderson tight binding model 2017
- An elementary formula about the average expansion of certain products of cocycles 2016

ABSTRACT OF THE DISSERTATION

Rigorous results on unexpected conductance of certain low-dimensional materials

By

Xiaowen Zhu

Doctor of Philosophy in Mathematics

University of California, Irvine, 2022

Professor Svetlana Jitomirskaya, Chair

In this thesis, we will study the conductance of several models originating from condensed matter physics, including the Anderson model for random systems and the Bistritzer-MacDonald (BM) model for twisted bilayer graphene (TBG). In fact, in their time, both models exhibited unexpected conductance properties which bewildered mathematicians and physicists. The Anderson model, developed in 1958 by physicist P.W. Anderson, exhibited unexpected localization/insulating phenomena in the 1D and 2D cases, while TBG was discovered experimentally in 2018 [20] to have unconventional superconductance at certain "magic angles" with relatively flat bands. This thesis has two primary parts. In the first part, we prove different types of localization results in the Anderson model and other related models. In the second part, we study the BM model in various magnetic fields from spectral, semi-classical and physical perspectives; in particular, we focus on the existence and persistence of flat bands which, though mysterious, is believed to be related to the superconductance of TBG [62].

More specifically, for the first part, we initially provide a short non-perturbative proof of Anderson localization and dynamical localization for the 1D Anderson model with

arbitrary disorder (e.g. including Bernoulli potential). After that, we derive the dynamical localization in expectation in a related random CMV model with arbitrary disorder. Finally, we work with 2D Anderson model with Bernoulli potential and prove strong dynamical localization in expectation in this setting.

We start the second part by first discussing the influence of different magnetic and electric potentials on the existence/persistence of flat bands for TBG. After the general discussion, we divert our attention the strong constant magnetic fields and provide the explicit asymptotic expansion of the density of states (DOS). In particular, we point out the intrinsically different roles that chiral and anti-chiral potentials play in the magnetic response of TBG. Finally, from the expansion of the DOS, we are able to study the physical phenomena, including magnetic oscillations and quantum Hall effect of the TBG. We find that the chiral potential enhances these phenomena, while the anti-chiral potential diminishes them.

Chapter 0

Introduction

This thesis addresses work in both the localization of disordered systems and the relatively flat bands of twisted bi-layer graphene (TBG) in magnetic fields, both motivated by unexpected conductance of certain low-dimensional materials in condensed matter physics. In this thesis, we report on some recent mathematical progress on these topics.

0.1 Localization in random systems

This section contains the introduction for Part I, including Chapter 1, 2, 3.

0.1.1 Physical Motivation

Inspired by Fuller and Feher's experiment on the influence of donor impurities (such as phosphorus [P] and arsenic [As]) in pure silica [Si], in 1958, physicist P.W. Anderson published the seminal paper "Absence of Diffusion in Certain Random Lattices". In this paper, he argued that weak disorder allows diffusion; while strong disorder leads to localization of electrons. Thus there is a metal-insulator transition of 3D disordered systems. Later it was also realized that 1D and 2D[1] systems exhibit localization even for arbitrarily weak disorder.

Such phenomenon is called "Anderson localization" and has been widely accepted nowadays as one of the fundamental theories in condensed matter physics. Anderson, together with his advisor Vleck and his collaborator Mott, was also awarded the Nobel prize in 1977 in part for this pioneering work. However, it took almost 20 years for the physics community to accept his idea and for mathematicians to get invloved in the theoretical part. To quote Anderson from his Nobel banquet speech,

> "Localization was a different matter: very few believed it at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it."

As Anderson has expected, over the next several decades, various mathematical tools have been implemented in this problem and fruitful results have been obtained. To state them more explicitly, we first introduce the model.

[1]Localization for arbitrary disorder in 2D remains partly open mathematically, but is widely accepted by physicists.

0.1.2 Anderson model and Anderson localization

Let $H_\omega : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ be

$$(H_\omega \phi)(n) := \sum_{|m-n|=1} (\phi(m) - \phi(n)) + \lambda V_\omega(n)\phi(n).$$

where $\{V_\omega(n)\}_{n\in\mathbb{Z}^d}$ are independent and identically distributed (i.i.d.) random variables with a common probability distribution μ, $\lambda > 0$ is the coupling constant representing the strength of randomness.

We say that H_ω exhibits Anderson localization if for a.e. ω, H_ω has pure point spectrum and the eigenfunctions decay exponentially.

0.1.3 Mathematical Development

Overall results and various methods developed to prove localization in different regimes can be categorized by three main factors: the dimension d, the regularity[2] of the common distribution μ, and the strength of randomness (also called coupling constant) λ. Some methods are only one dimensional; Most methods only work for regular enough μ; in 1D and 2D when localization is expected to hold for all λ, larger λ is more accessible.

Known results and open problems.

- When $d = 1$, for any λ, any nontrivial[3] μ, H_ω exhibits Anderson localization.

[2] Absolutely continuous, or at least Hölder continuous
[3] supported on more than one point

- When $d = 2$, H_ω is widely expected to be localized by physicists for all λ and μ. But Anderson localization is only proved for large enough λ and arbitrary μ.
- When $d \geq 3$, H_ω is expected to be localized when λ is large and delocalized when λ is small and a sharp phase transition is expected. For regular μ, localization for large enough λ is proved but delocalization for small λ and the existence of phase transition are not proved. As for singular μ, localization is only proved in $d = 3$ with large enough λ.

Methods summarized.

The first rigorous proof of Anderson localization in 1D continnum model with regular μ and arbitrary λ was given by Goldshield, Molchanov and Pastur in 1977. In 1980, the same results is obtained using another method by Kunz and Soulliard [59]. Since then, several methods are developed in order to resolve the problem in other regimes of d, μ, λ.

Proofs that work for arbitrary dimension d include multi-scale analysis (MSA), developed in 1983 by Frohlich and Spencer [37], largely improved in 1989 by von Dreifus and Klein [82]; and fractional moment methods (FFM), developed in 1994 by Aizenman and Molchanov [1]. However, both methods required regular μ and large enough λ. Since this thesis focuses on the low-dimensional cases, we refer to [2] for more details in higher dimension.

In the 1D case with possibly singular potential μ, the first proof was given in 1987 by Carmona, Klein and Martinelli [21] in $d = 1$. Their proof has made use of the positivity and regularity of Lyapunov exponent for the 1D model together with the MSA developed in [37], which is iterative and therefore relatively complicated. Recently, several new methods were developed for proving localization in this regime. In 2019, inspired by the non-perturbative proof of localization for the quasi-periodic almost Mathieu operator [53], we provide a new proof of localization for 1D Anderson model with arbitrary μ and λ. As in [53], the idea is to make full use of positivity and subharmonicity of Lyapunov exponent to replace the iterative argument of MSA. (This method will be introduced in Chapter 1.) We mention further that [18] also provided another deterministically inspired proof in 2020; in the mean time, [44] developed a parametric version of Furstenberg theorem which allowed them to provide a purely dynamical proof under the same regime.

When $d = 2, 3$, Anderson localization for singular μ with large λ was proved in 2020 [32] and 2021 [63] respectively.

0.1.4 Outline of results

- In Chapter 1, we provide the new, short, non-perturbative proof of Anderson and dynamical localization in 1D for arbitrary μ with bounded support. We also provide a relatively independent uniform version of Craig-Simon results in Section 1.5. We mention that over the past three years, the method was taken in conjunction with [41], [69], [68], [65], [88], [27], and is expected to apply to models introduced in [34], [36], illustrates the flexibility of this general scheme for proving localization in random one-dimensional frameworks. Indeed, these techniques provide the most direct route to localization in addition to providing proofs of the strongest known localization results for such models.

- Notice that in fact, the original physical concept, localization, can be defined in many different ways: exponential localization of time-independent wave function as Anderson localization above, or absence of transport of the time-dependent solutions for $a.e.\omega$ (dynamical localization), or the absence of transport of the time-dependent solution in expectation (dynamical localization in expectation). In particular, in Chapter 3, we improved the results in [32] and [63] from Anderson localization to the strongest dynamical localization in expectation.

- In Chapter 2, we apply our new method developed in Chapter 1 to a related random CMV model to derive for the first time Anderson localization for arbitrary μ. We also correct the errors in the formulas found in [58], [18], [74] and [73] in Appendix B.

0.2 Magnetic response of twisted bilayer graphene

This section contains the introduction for Part II, Chapter 4

0.2.1 Physical Motivation

It is arguably one of the most exciting recent discoveries in condensed matter physics that by twisting two sheets of graphene at certain "magic angles", the electronic structure undergoes a transition from a Mott-insulating to a superconducting phase [20]. Such experiments are built on earlier theoretical work [33, 12] which introduced the continuum model for the study of TBG. From this model they predicted the first magic angle by observing the appearance of a relatively flat spectral band at some small angle. To discuss our study of TBG in magnetic fields, we first briefly introduce the BM model (see §4.2.1, [12]):

0.2.2 BM model

The BM model is an effective 4×4 matrix-valued Hamiltonian $\begin{pmatrix} H_D^{\theta} & T^{\theta}(x) \\ (T^{\theta}(x))^* & \tilde{H}_D^{-\theta} \end{pmatrix}$, $x \in \mathbb{R}^2$, composed of two twisted-Dirac-operators $H_D^{\theta}, H_D^{-\theta}$ representing two isolated graphene sheets [84] respectively, and a tunneling potential term

$$T^{\theta}(x) = \begin{pmatrix} \alpha_0 V(x/\lambda_\theta) & \alpha_1 \overline{U}(-x/\lambda_\theta) \\ \alpha_1 U(-x/\lambda_\theta) & \alpha_0 V(x/\lambda_\theta) \end{pmatrix}$$

where the diagonal and off-diagonal terms represent two different types of interlayer tunneling potentials. In fact, when two layers of graphene are twisted at an angle θ, a macroscopic honeycomb structure of scale λ_θ, called the moiré pattern, is formed (by a purely geometrical superposition of two sheets of graphene; see Fig.4.1). Then the two different types of interlayer tunnelings (see Fig.4.1) are respectively:

1. the chiral tunnelings $U(x/\lambda_\theta)$ and $\overline{U}(-x/\lambda_\theta)$ localized near the vertices of each moiré hexagon, with tunneling strength α_1 and a stacking similar to AB' and BA'-stacking;

2. the anti-chiral tunneling $V(x/\lambda_\theta)$, localized near the centers of moiré hexagon, with a tunneling strength α_0 and a stacking similar to AA'/BB'-stacking.

Here A and B label the equivalence classes of vertices on the honeycomb lattice and atoms on the lower lattice are indicated by a prime, cf. Figure 4.1. We refer to the BM model as the *chiral* or *anti-chiral* model in the limit of purely chiral ($\alpha_0 = 0$) or anti-chiral ($\alpha_1 = 0$) tunneling interaction, respectively.

While in the full BM model, the bands close to zero appear only approximately flat,

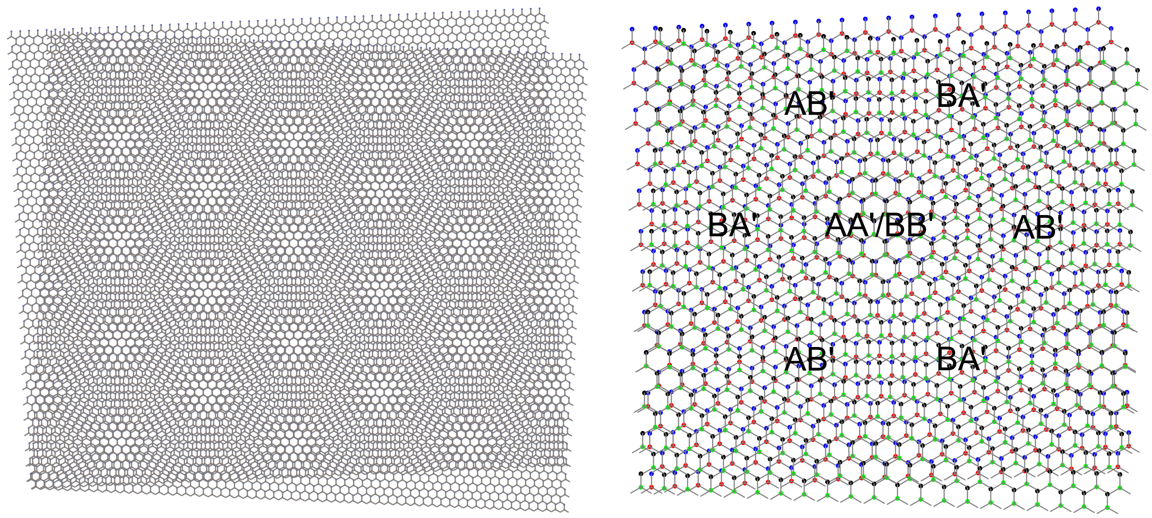


Figure 1: Left: Visible moiré pattern at $\theta = 5°$. Right: Single moiré hexagon, with (A=red, B=blue) and (A'=green, B'=black) denote vertices of two sheets of graphene respectively.

it has been shown in [79, 6, 7] that the chiral model exhibits a perfectly flat band at the magic angle [79, 6] while the anti-chiral model does not [7]. In our study of the magnetic response, we find that chiral and anti-chiral tunnelings play intrinsically different roles not only in spectral properties (see §4.3), but also in the asymptotic expansion of DOS in strong magnetic field (see §4.5), which results in very different physical behaviours (see §4.6).

More specifically, in §4.3, we dicuss the influence of different magnetic potentials on the flat bands in the chiral and anti-chiral model. In §4.5, we derive the explicit asymptotic expansion of the DOS in strong magnetic fields for both chiral and anti-chiral model. We find that the magnetic anti-chiral model has a similar behavior as the magnetic Schrödinger operator, where Landau levels split under perturbations of anti-chiral electric potential, while the magnetic chiral model has stable Landau levels especially at energy zero. Thus, chiral tunneling enhances the peaks of the DOS at Landau levels which leads to an enhancement of physical phenomena including magnetic oscillations and the quantum hall effect, which we discuss in §4.6, while anti-chiral tunneling weakens them.

We also mention that our study of asymptotic behavior in the strong magnetic fields originates naturally, in the physics perspective, from the interest in small twisting angles. In fact, as the twisting angle θ decreases to zero, the scale of the moiré hexagon $\lambda_\theta \sim (\sin\theta)^{-1}$ increases dramatically. Thus, in the spirit of keeping a fixed magnetic flux through each moiré hexagon, the study of a fixed magnetic field with small twisting angle can be rescaled to a fixed twisting angle with a strong magnetic field. We denote them as adiabatic (see §4.2.3) and semiclassical (see §4.2.4) scalings respectively).

In particular, this means we provide the theoretical background for the study of the dependence of Landau levels on small twisting angle that have been studied for a simplified model in [24] and numerically in [67] for a tight-binding model. Furthermore, combining with the study of chiral and anti-chiral tunnelings, we put the substantially pronounced peaks of the DOS for small twisting angles at the Landau levels in [67, Fig. 2,3] on a rigorous footing. Furthermore, our results apply to strong pseudo-magnetic fields generated by physical strain.

0.2.3 Outline of results

We summarize all our main results with an outline of the paper below:

- In Section 4.2, we introduce the BM model with external magnetic field for TBG.
- In Section 4.3, we proved that
  - periodic magnetic fields do not affect the presence of flat bands in Theorem

4.1.

  - flat bands are persisted under rational magnetic flux in Theorem 4.2, 4.3.
  - lots of quasimodes are located close to, and squeezing towards the zero energy level in Theorem 4.4.

- In Section 4.4, we discuss general properties of the DOS including
- In Section 4.5, we derive asymptotic formulae for the DOS:
  - of the chiral model: Theorem 4.5;
  - of the anti-chiral model: Theorem4.6;
  - is termwise-deferentiable w.r.t. B: Prop 4.5.9).
- In Section 4.6, we discuss physical applications of our semiclassical formulae.
- The article also contains two technical appendices to which some of the computations and auxiliary results for the derivation of the DOS are outsourced.

0.2.4 Comment on methods

Finally, we would like to comment on related results and techniques. In this paper, we first perform a spectral and symmetry analysis of our model for various magnetic perturbations. This includes the existence and absence of perfectly flat bands at magic angles for different interlayer potentials and magnetic fields, see also [11] for related results. We then discuss the existence and absence of the phenomenon of exponential squeezing of bands which generalizes results obtained in the non-magnetic setting [6, 7]. In this row of mathematically rigorous results, we also want to mention the computer-assisted proof of the existence of a real magic angle by Luskin and Watson [87].

Our approach to studying these physical phenomena is a thorough asymptotic analysis of the DOS. Here, our approach is inspired by ideas of Helffer and Sjöstrand [47] who studied the perturbation theory of periodic Schrödinger operators in strong magnetic fields and Wang [86], who studied fine spectral asymptotics for random Schrödinger operators in strong magnetic fields. While Helffer and Sjöstrand stopped at studying the spectral perturbation for strong magnetic fields, we obtain a full asymptotic expansion of the DOS. This has also been obtained by Helffer and Sjöstrand for weak magnetic fields [48] where the analysis relied on the semiclassical eigenvalue distribution close to a potential well. In our case, there is no natural well-structure and the asymptotic expansion relies on an asymptotic expansion of the parametrix with a splitting argument to overcome non-elliptic regions close to the real axis. Unlike in previous works by Helffer and Sjöstrand [48] and an article on single-layer graphene by the first author and Zworski [9], we resolve the issue of differentiability of the asymptotic expansion with respect to the semiclassical parameter by relating the asymptotic expansion with the one of the differentiated symbol, here. This expansion is needed for the rigorous analysis of the DOS when differentiated with respect to the magnetic field which is relevant for both the de-Haas van Alphen as well as the quantum Hall effect.

Part I

Localization in random systems

Chapter 1

Anderson localization for 1D Anderson model

1.1 Introduction

Anderson localization for the Anderson model can be proved in several different ways if the common distribution of the i.i.d.r.v's is absolutely continuous. Without that condition (or at least some Hölder regularity) it remains an open question for $d \geq 2$, and the number of approaches that work for $d = 1$ also drops dramatically. Such is the situation, for example, for the Bernoulli-Anderson model. Anderson localization for *arbitrary* 1D disorder was first proved in [21]. The approach was based on certain regularity of the Lyapunov exponents coming from the (analysis around) the Furstenberg theorem to obtain an analogue of Wegner's lemma (automatic in the absolutely continuous case). After that the proof was reduced to multi-scale analysis, with initial scale coming again from the positive Lyapunov exponent. Another argument was later presented in [72], where an approach to positivity and regularity of the Lyapunov exponent using replica trick was given, again reducing the proof to multi-scale analysis. Multi-scale analysis is a method that allows to achieve Green's function decay and ultimately localization from high probability of decay at the initial scale. It works in a variety of settings. Originally developed by Frohlich and Spencer [37], it was significantly simplified in [82] but remains somewhat involved. It should be noted that in the multidimensional case no shortcuts such as Furstenberg theorem or replica trick are available, and the multi-scale analysis is used to reach conclusions analogous to the positivity of the Lyapunov exponent simultaneously with the proof of localization. Yet in the one-dimensional case positivity of the Lyapunov exponent essentially provides the averaged decay statement, thus a large portion of the *conclusion* of the multi-scale analysis, making its machinery seem redundant.

A method to effectively exploit positive Lyapunov exponent for a localization proof based on the analysis of the large deviation set for the Lyapunov exponent was first developed in [50] for the almost Mathieu operator, initiating what was later called a non-perturbative approach, in contrast with earlier proofs based on some form of multi-scale analysis [38, 76]. A robust method based on subharmonic function theory and the theory of semianalytic sets was then developed in [16] and other papers summarized in [15], to conclude localization from positive Lyapunov exponents for analytic quasiperiodic and some other deterministic potentials. The fact that those ideas can be applicable also to the Anderson model was mentioned in some talks by one of the authors circa 2000, but the details were never developed. One goal of this chapter is to obtain a proof of Anderson localization for the 1D Anderson model in the spirit of [50] but with appropriate simplifications due to randomness.

Another proof, also based on large deviations and also avoiding multi-scale analysis was recently developed in [18]. The proof of [18] is based on deterministic ideas close to the ones in [17], which we believe may be somewhat more complicated than needed for the random case. We mention that yet another, purely dynamical, proof of localization for the 1D Anderson model in [44].

One ingredient in our simple argument for spectral localization, Theorem 1.4, is Craig-Simon's upper bound based on subharmonicity of the Lyapunov exponent [25], a statement that holds for any ergodic potential. In order to prove dynamical localization we need a uniform in energy and quantitative version of this statement, that we prove for general ergodic potentials satisfying certain large deviation bounds, a result that could be of independent interest. We note that our proof does not explicitly use subharmonicity.

The rest of this chapter is organized as follows. Section 1.2 contains the preliminaries, the statement of the spectral localization result, Theorem 1.1, and its quick reduction to Theorem 1.2. We then prove the preparatory Lemmas 1.3.1, 1.3.2, 1.3.4, and Corollary 1.3.3 in Section 1.3. Then we complete the proof of Theorem 1.2 in Section 1.4. Our proof effectively establishes a more precise result, Theorem 1.5, which in turn immediately implies the Lyapunov behavior at all eigenvalues, Theorem 1.6. We formulate and prove the general uniform Craig-Simon-type statement in Section 1.5, and use it in Section 1.6 to prove dynamical localization.

1.2 Preliminaries

The one dimensional Anderson model is given by a discrete Schrödinger operators H_ω

$$(H_\omega \Psi)(n) = \Psi(n+1) + \Psi(n-1) + \omega_n \Psi(n),$$

where $\omega_n \in \mathbb{R}$ are independent identically distributed random variables with common Borel probability distribution μ. We will assume that $S \subset \mathbb{R}$, the topological support of μ, is compact , and contains at least two points. We will denote the probability space $\Omega = S^{\mathbb{Z}}$, with elements $\{\omega_n\}_{n\in\mathbb{Z}} \in \Omega$. Denote $\mu^{\mathbb{Z}}$ as $\mathbb{P}$. Let $\mathbb{P}_{[a,b]}$ be $\mu^{[a,b]\cap\mathbb{Z}}$ on $S^{[a,b]\cap\mathbb{Z}}$. Also let T be the shift $T\omega_i = \omega_{i-1}$. Finally, we denote Lebesgue measure on $\mathbb{R}$ by m. We say that H_ω has spectral localization in I if for a.e. ω, H_ω has only pure point spectrum in I and its eigenfunctions $\Psi(n)$ decay exponentially in n.

Definition 1.1. *We call E a generalized eigenvalue (g.e.), if there exists a nonzero polynomially bounded function $\Psi(n)$ such that $H_\omega \Psi = E\Psi$. We call $\Psi(n)$ a generalized eigenfunction.*

Since the set of g.e. supports the spectral measure of H_ω (e.g. [26]), we only need to show:

Theorem 1.1. *For a.e. ω, for every g.e. E, the corresponding generalized eigenfunction $\Psi_{\omega,E}(n)$ decays exponentially in n.*

For $[a,b]$ an interval, $a, b \in \mathbb{Z}$, define $H_{[a,b],\omega}$ to be operator H_ω resticted to $[a,b]$ with zero boundary conditions outside $[a,b]$. Note that it can be expressed as a "$b-a+1$"-dimensional matrix. The Green's function for H_ω restricted to $[a,b]$ with energy $E \notin \sigma_{[a,b],\omega}$ is

$$G_{[a,b],E,\omega} = (H_{[a,b],\omega} - E)^{-1}.$$

Note that this can also be expressed as a "$b-a+1$"-dimensional matrix. Denote its (x,y) entry as $G_{[a,b],E,\omega}(x,y)$.

It is well known that

$$\Psi(x) = -G_{[a,b],E,\omega}(x,a)\Psi(a-1) - G_{[a,b],E,\omega}(x,b)\Psi(b+1), \quad x \in [a,b] \tag{1.1}$$

and we have

$$\sigma := \sigma(H_\omega) = [-2,2] + S \quad a.e.\omega. \tag{1.2}$$

Definition 1.2. *For $c>0, n \in \mathbb{Z}$, we say $x \in \mathbb{Z}$ is (c,n,E,ω)-regular, if*

$$G_{[x-n,x+n],E,\omega}(x,x-n) \le e^{-cn}$$

$$G_{[x-n,x+n],E,\omega}(x,x+n) \le e^{-cn}$$

Otherwise, we call it (c,n,E,ω)-singular.

By (1.1) and definition 2, Theorem 1.1 follows from

Theorem 1.2. *There exists Ω_0 with $\mathbb{P}(\Omega_0)=1$, such that for every $\tilde{\omega} \in \Omega_0$, for any g.e. $\tilde{E}$ of $H_{\tilde{\omega}}$, there exist $N = N(\tilde{E},\tilde{\omega}), C = C(\tilde{E})$, such that for every $n > N$, $2n,\ 2n+1$ are $(C,n,\tilde{E},\tilde{\omega})$-regular.*

Some other standard basic settings are below. Denote

$$P_{[a,b],E,\omega} = det(H_{[a,b],E,\omega} - E), a \le b$$

If $a > b$, let $P_{[a,b],E,\omega} = 1$. Then

$$\left|G_{[a,b],E,\omega}(x,y)\right| = \frac{\left|P_{[a,x-1],E,\omega}P_{[y+1,b],E,\omega}\right|}{\left|P_{[a,b],E,\omega}\right|}, \quad x \leq y \tag{1.3}$$

If we denote the transfer matrix $T_{[a,b],E,\omega}$ as the matrix such that

$$\begin{pmatrix} \Psi(b) \\ \Psi(b-1) \end{pmatrix} = T_{[a,b],E,\omega} \begin{pmatrix} \Psi(a) \\ \Psi(a-1) \end{pmatrix}$$

where Ψ solves $H_\omega \Psi = E\Psi$, then

$$T_{[a,b],E,\omega} = \begin{pmatrix} P_{[a,b],E,\omega} & -P_{[a+1,b],E,\omega} \\ P_{[a,b-1],E,\omega} & -P_{[a+1,b-1],E,\omega} \end{pmatrix}$$

The Lyapunov exponent exists by Kingman's subadditive ergodic theorem and is given by

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \int_0^1 \log \|T_{[0,n],E,\omega}\| d\mathbb{P}(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|T_{[0,n],E,\omega}\|, \quad a.e.\omega.$$

Let $\nu = \inf_{E\in\sigma} \gamma(E)$. By the Furstenberg's theorem $\nu > 0$. It follows from (1.3) that the desired exponential decay of the Green's function can be achieved if all the $P_{[a,b]}$ in (1.3) behave as $e^{(b-a)\gamma(E)}$, thus leading to the study of deviations of $\ln P_{[a,b]}$ from its mean. In fact, the key estimates underlying the analysis of [21] are precisely large deviation bounds for the Lyapunov exponent due to Le Page [61]. Here we will use a corresponding statement for the matrix elements [81]

Lemma 1.2.1 ("uniform-LDT")**.** *For any $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that,* *there exists $N_0 = N_0(\varepsilon)$, such that for every $b-a > N_0$, and any E in a compact set,*

$$\mathbb{P}\left\{\omega : \left|\frac{1}{b-a+1}\log\|P_{[a,b],E,\omega}\| - \gamma(E)\right| \geq \varepsilon\right\} \leq e^{-\eta(b-a+1)}$$

It will also be convenient to use the general subharmonicity upper bound due to Craig-Simon [25]

Theorem 1.3 (Craig-Simon [25]). *For a.e. ω for all E, we have*

$$\overline{\lim_{n\to\infty}} \frac{\log\|T_{[0,n],E,\omega}\|}{n+1} \leq \gamma(E)$$

1.3 Main lemmas

Denote

$$B^+_{[a,b],\varepsilon} = \left\{(E,\omega) : |P_{[a,b],E,\omega}| \geq e^{(\gamma(E)+\varepsilon)(b-a+1)}\right\} \tag{1.4}$$

$$B^-_{[a,b],\varepsilon} = \left\{(E,\omega) : |P_{[a,b],E,\omega}| \leq e^{(\gamma(E)-\varepsilon)(b-a+1)}\right\} \tag{1.5}$$

and denote $B^{\pm}_{[a,b],\varepsilon,E} = \{\omega : (E,\omega) \in B^{\pm}_{[a,b],\varepsilon}\}$, $B^{\pm}_{[a,b],\varepsilon,\omega} = \{E : (E,\omega) \in B^{\pm}_{[a,b],\varepsilon}\}$, $B_{[a,b],*} = B^+_{[a,b],*} \cup B^-_{[a,b],*}$.

Let $E_{j,(\omega_a,\cdots,\omega_b)}$ be eigenvalues of $H_{[a,b],\omega}$ with $\omega|_{[a,b]} = (\omega_a,\cdots,\omega_b)$.

Large deviation theorem gives us the estimate that for all E, a, b, ε

$$\mathbb{P}(B^{\pm}_{[a,b],\varepsilon,E}) \leq e^{-\eta(b-a+1)} \tag{1.6}$$

Assume $\varepsilon = \varepsilon_0 < \frac{1}{8}\nu$ is fixed for now, so we omit it from the notations until Lemma

1.3.4. Let $\eta_0 = \eta(\varepsilon_0)$ be the corresponding parameter from Lemma 1.2.1

Lemma 1.3.1. *For $n \geq 2$, if x is $(\gamma(E) - 8\varepsilon_0, n, E, \omega)$-singular, then*

$$(E, \omega) \in B^-_{[x-n,x+n]} \cup B^+_{[x-n,x-1]} \cup B^+_{[x+1,x+n]}$$

Remark 1.1. Note that from (1.6), for all $E, x, n \geq 2$,

$$\mathbb{P}(B^-_{[x-n,x+n],E} \cup B^+_{[x-n,x-1],E} \cup B^+_{[x+1,x+n],E}) \leq 3e^{-\eta_0(n+1)}$$

Proof. Follows imediately from the definition of singularity and (1.3). □

Now we will use the following three lemmas to find the proper Ω_0 for Theorem 1.2.

Lemma 1.3.2. *Let $0 < \delta_0 < \eta_0$. For a.e. ω (we denote this set as Ω_1), there exists $N_1 = N_1(\omega)$, such that for every $n > N_1$,*

$$\max\{m(B^-_{[n+1,3n+1],\omega}), m(B^-_{[-n,n],\omega})\} \leq e^{-(\eta_0-\delta_0)(2n+1)}$$

Proof. By (1.6),

$$m \times \mathbb{P}(B^-_{[n+1,3n+1]}) \leq m(\sigma)e^{-\eta_0(2n+1)}$$

$$m \times \mathbb{P}(B^-_{[-n,n]}) \leq m(\sigma)e^{-\eta_0(2n+1)}$$

If we denote

$$\Omega_{\delta_0,n,+} = \left\{\omega : m(B^-_{[n+1,3n+1],\omega}) \leq e^{-(\eta_0-\delta_0)(2n+1)}\right\}$$

$$\Omega_{\delta_0,n,-} = \left\{\omega : m(B^-_{[-n,n],\omega}) \leq e^{-(\eta_0-\delta_0)(2n+1)}\right\},$$

We have by Tchebyshev,

$$\mathbb{P}(\Omega^c_{\delta_0,n,\pm}) \leq m(\sigma)e^{-\delta_0(2n+1)}. \tag{1.7}$$

By Borel-Cantelli lemma, we get for *a.e.* ω,

$$\max\{m(B^-_{[n+1,3n+1],\omega}), m(B^-_{[-n,n],\omega})\} \leq e^{-(\eta_0-\delta_0)(2n+1)},$$

for $n > N_1(\omega)$. □

Remark 1.2. Note that we can actually shift the operator and use center point l instead of 0. Then we will get $\Omega_1(l)$ instead of Ω_1, $N_1(l,\omega)$ instead of $N_1(\omega)$. And if we pick $N_1(l,\omega)$ in the theorem as the smallest integer satisfying the conclusion, we can estimate when we will have $N_1(l,\omega) \leq \ln^2|l|$, which is very useful in the proof for dynamical localization in section 6.

The next results follows from :

Theorem 1.4. *For a.e. ω(we denote this set as Ω_2), for all E, we have*

$$\max\left\{\overline{\lim_{n\to\infty}}\frac{\log\|T_{[-n,0],E,\omega}\|}{n+1}, \overline{\lim_{n\to\infty}}\frac{\log\|T_{[0,n],E,\omega}\|}{n+1}\right\} \leq \gamma(E) \tag{1.8}$$

$$\max\left\{\overline{\lim_{n\to\infty}}\frac{\log\|T_{[n+1,2n+1],E,\omega}\|}{n+1}, \overline{\lim_{n\to\infty}}\frac{\log\|T_{[2n+1,3n+1],E,\omega}\|}{n+1}\right\} \leq \gamma(E) \tag{1.9}$$

Remark 1.3. (1.8) is a direct reformulation of the result of [25], Theorem 1.3, while (1.9) follows by exactly the same proof.

Corollary 1.3.3. *For every $\omega \in \Omega_2$, for every E, there exists $N_2 = N_2(\omega, E)$, such* *that for every $n > N_2$,*

$$\max\{\|T_{[-n,0],E,\omega}\|, \|T_{[0,n],E,\omega}\|\} < e^{(\gamma(E)+\varepsilon)(n+1)}$$
$$\max\{\|T_{[n+1,2n+1],E,\omega}\|, \|T_{[2n+1,3n+1],E,\omega}\|\} < e^{(\gamma(E)+\varepsilon)(n+1)}$$

Lemma 1.3.4. *Let $\varepsilon > 0, K > 1$, For a.e. ω(we denote this set as $\Omega_3 = \Omega_3(\varepsilon, K)$), there exists $N_3 = N_3(\omega)$, so that for every $n > N_3$, for every $E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}$, for every y_1, y_2 satisfying $-n \le y_1 \le y_2 \le n$, $|-n - y_1| \ge \frac{n}{K}$, and $|n - y_2| \ge \frac{n}{K}$, we have $E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})} \notin B_{[-n,y_1],\varepsilon,\omega} \cup B_{[y_2,n],\varepsilon,\omega} \cup B_{[-n,n],\varepsilon,\omega}$.*

Remark 1.4. Note that ε and K are not fixed yet, we're going to determine them later in section 1.4.

Proof. Let $\bar{\mathbb{P}}$ be the probability that there are some y_1, y_2, j with

$$E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})} \in B_{[-n,y_1],\varepsilon,\omega} \cup B_{[y_2,n],\varepsilon,\omega} \cup B_{[-n,n],\varepsilon,\omega}.$$

Note that for any fixed $\omega_c, \cdots, \omega_d$, with $[c,d] \cap [a,b] = \emptyset$, by independence,

$$\mathbb{P}_{[c,d]^c}(B_{[a,b],\varepsilon,E_{j,(\omega_c,\cdots,\omega_d)}}) = \mathbb{P}_{[a,b]}(B_{[a,b],\varepsilon,E_{j,(\omega_c,\cdots,\omega_d)}}) \le e^{-\eta_0(b-a+1)}$$

Applying to $[a,b] = [-n, y_1]$ or $[y_2, n]$, $[c,d] = [n+1, 3n+1]$ and integrating over $\omega_{-n}, \cdots, \omega_{y_1}$ or $\omega_{y_2}, \cdots, \omega_n$, we get

$$\mathbb{P}(B_{[-n,y_1],\varepsilon,E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}} \cup B_{[y_2,n],\varepsilon,E_{j,(\omega_{n+1},\cdots,\omega_{3n+1})}}) \le 2e^{-\eta_0(\frac{n}{K}+1)},$$

and

$$\mathbb{P}(B_{[-n,n],\varepsilon,\omega}) \le e^{-\eta_0(2n+1)}$$

so

$$\bar{\mathbb{P}} \leq (2n+1)^3 2e^{-\eta_0(\frac{n}{K}+2)}$$

Thus by Borel-Cantelli, we get the result. □

Remark 1.5. Similar to remark 1.2, we can get $\Omega_3(l)$, $N_3(l,\omega)$ for an operator shifted by ℓ instead, and get the result that for $a.e.\omega$ (we denote this set as Ω_{N_3}), there exists $L_3(\omega)$, such that for any $|l| > L_3$, $N_3(l,\omega) \leq \ln^2|l|$. This will be of use in section 6 for proving dynamical localization.

1.4 Proof of Theorem 2.2

We will only provide a proof that $2n+1$ is (c,n,E,ω)-regular, the argument for $2n$ being similar.

Proof. Let ε be small enough such that

$$\varepsilon < \min\{(\eta_0 - \delta_0)/3, \nu\}. \tag{1.10}$$

Now let

$$L := e^{(\eta_0 - \delta_0 - \varepsilon)} > 1,$$

and note that since S is bounded, by (1.2) we have there exists $M > 0$, such that

$$|P_{[a,b],E,\omega}| < M^{(b-a+1)}, \quad \forall E \in \sigma, \omega$$

Pick K big enough such that

$$M^{\frac{1}{K}} < L$$

Let $\tau > 0$ be such that

$$M^{\frac{1}{K}} \leq L - \tau < L \tag{1.11}$$

Let $\Omega_0 = \Omega_1 \cap \Omega_2 \cap \Omega_3(\varepsilon, K)$. Pick $\tilde{\omega} \in \Omega_0$, and take $\tilde{E}$ a *g.e.* for $H_{\tilde{\omega}}$, with Ψ the corresponding generalized eigenfunction. Without loss of generality assume $\Psi(0) \neq 0$. Then there exists N_4, such that for every $n > N_4$, 0 is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \tilde{\omega})$-singular.

For $n > N_0 = \max\{N_1(\tilde{\omega}), N_2(\tilde{\omega}, \tilde{E}), N_3(\tilde{\omega}), N_4(\tilde{\omega}, \tilde{E})\}$, assume $2n+1$ is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \tilde{\omega})$-singular. Then both 0 and $2n+1$ is $(\gamma(\tilde{E}) - 8\varepsilon_0, n, \tilde{E}, \tilde{\omega})$-singular. So by Lemma 1.3.1, $\tilde{E} \in B^-_{[n+1,3n+1],\varepsilon_0,\tilde{\omega}} \cup B^+_{[n+1,2n],\varepsilon_0,\tilde{\omega}} \cup B^+_{[2n+2,3n+1],\varepsilon_0,\tilde{\omega}}$. By Corollary 1.3.3 and (1.4), $\tilde{E} \notin B^+_{[n+1,2n],\varepsilon_0,\tilde{\omega}} \cup B^+_{[2n+2,3n+1],\varepsilon_0,\tilde{\omega}}$, so it can only lie in $B^-_{[n+1,3n+1],\varepsilon_0,\tilde{\omega}}$.

Note that in (1.5), $P_{[n+1,3n+1],E,\tilde{\omega}}$ is a polynomial in E that has $2n+1$ real zeros (eigenvalues of $H_{[n+1,3n+1],\tilde{\omega}}$), which are all in $B = B^-_{[n+1,3n+1],\varepsilon,\tilde{\omega}}$. Thus B consists of less than or equal to $2n+1$ intervals around the eigenvalues. $\tilde{E}$ should lie in one of them. By Lemma 1.3.2, $m(B) \leq Ce^{-(\eta_0-\delta_0)(2n+1)}$. So there is some e.v. $E_{j,[n+1,3n+1],\tilde{\omega}}$ of $H_{[n+1,3n+1],\omega}$ such that

$$|\tilde{E} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leq e^{-(\eta_0-\delta_0)(2n+1)}$$

By the same argument, there exists $E_{i,[-n,n],\tilde{\omega}}$, such that

$$|\tilde{E} - E_{i,[-n,n],\tilde{\omega}}| \leq e^{-(\eta_0-\delta_0)(2n+1)}$$

Thus $|E_{i,[-n,n],\tilde{\omega}} - E_{j,[n+1,3n+1],\tilde{\omega}}| \leq 2e^{-(\eta_0-\delta_0)(2n+1)}$. However, by Theorem 1.3.4, one has $E_{j,[n+1,3n+1],\tilde{\omega}} \notin B_{[-n,n],\varepsilon,\tilde{\omega}}$, while $E_{i,[-n,n],\tilde{\omega}} \in B_{[-n,n],\varepsilon,\tilde{\omega}}$ This will give us a contradiction below.

Since $|E_{i,[-n,n],\tilde{\omega}} - E_{j,[n+1,3n+1],\tilde{\omega}}| \le 2e^{-(\eta_0-\delta_0)(2n+1)}$ and $E_{i,[-n,n],\tilde{\omega}}$ is the e.v. of $H_{[-n,n],\tilde{\omega}}$,

$$\left\| G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}} \right\| \ge \frac{1}{2} e^{(\eta_0-\delta_0)(2n+1)}$$

Thus there exist $y_1, y_2 \in [-n, n]$ and such that

$$\left| G_{[-n,n],E_{j,[n+1,3n+1],\tilde{\omega}},\tilde{\omega}}(y_1, y_2) \right| \ge \frac{1}{2n} e^{(\eta_0-\delta_0)(2n+1)}$$

Let $E_j = E_{j,[n+1,3n+1],\tilde{\omega}}$. We have $E_j \notin B_{[-n,n],\varepsilon,\tilde{\omega}}$, thus

$$|P_{[-n,n],\varepsilon,E_j,\tilde{\omega}}| \ge e^{(\gamma(E_j)-\varepsilon)(2n+1)}$$

so by (1.3),

$$\left\| P_{[-n,y_1],\varepsilon,E_j,\tilde{\omega}} P_{[y_2,n],\varepsilon,E_j,\tilde{\omega}} \right\| \ge \frac{1}{2n} e^{(\eta_0-\delta_0)(2n+1)} e^{(\gamma(E_j)-\varepsilon)(2n+1)} \tag{1.12}$$

Then for the left hand side of (1.12), there are three cases:

1. both $|-n-y_1| > \frac{n}{K}$ and $|n-y_2| > \frac{n}{K}$
2. one of them is large, say $|-n-y_1| > \frac{n}{K}$ while $|n-y_2| \le \frac{n}{K}$
3. both small.

For (1),

$$\frac{1}{2n} e^{(\eta_0-\delta_0+\gamma(E_j)-\varepsilon)(2n+1)} \le e^{2n(\gamma(E_j)+\varepsilon)}$$

Since by our choice (1.10), $\eta_0 - \delta_0 + \gamma(E_j) - \varepsilon > \gamma(E_j) + \varepsilon$, for n large enough, we get a contradiction.

For (2),

$$\frac{1}{2n}e^{(\eta_0-\delta_0+\gamma(E_j)-\varepsilon)(2n+1)} \leq e^{(\gamma(E_j)+\varepsilon)(2n+1)}(M)^{\frac{n}{K}}$$

is in contradiction with (1.10) and (1.11)

For (3), with (1.10) and (1.11)

$$\frac{1}{2n}e^{(\eta_0-\delta_0+\gamma(E_j)-\varepsilon)(2n+1)} \leq M^{\frac{2n}{K}} \leq (L-\tau)^{2n} \leq (e^{(\eta_0-\delta_0+\gamma(E_j)-\varepsilon)}-\tau)^{2n},$$

also a contradiction.

Thus our assumption that $2n+1$ is not $(\gamma(\tilde{E})-8\varepsilon_0, n, \tilde{E}, \tilde{\omega})$-regular is false. Theorem 1.2 follows. □

Note that we have established the following more precise version of Theorem 1.2

Theorem 1.5. *There exists Ω_0 with $\mathbb{P}(\Omega_0)=1$, such that for every $\tilde{\omega}\in\Omega_0$, for any g.e. $\tilde{E}$ of $H_{\tilde{\omega}}$, and $\varepsilon>0$, there exists $N=N(\tilde{E},\tilde{\omega},\varepsilon)$, such that for every $n>N$, $2n,\ 2n+1$ are $(\gamma(E)-\varepsilon, n, \tilde{E}, \tilde{\omega})$-regular.*

It is a standard patching argument (e.g. proof of Theorem 3 in [50]) that this implies $|\Psi_E(n)| \leq C_{E,\varepsilon}e^{-(\gamma(E)-\varepsilon)n}$ for any $\varepsilon>0$. Combined with Theorem 1.3, this immediately implies that we have Lyapunov behavior at every generalized eigenvalue.

Theorem 1.6. *For a.e. ω for all generalized eigenvalues E, we have*

$$\lim_{n\to\infty}\frac{\log\|T_{[0,n],E,\omega}\|}{n+1}=\gamma(E)$$

1.5 Uniform and Quantitative Craig-Simon

Craig-Simon theorem 1.3 implies that for a.e. ω and every $E \in \sigma$ there exists $N(\omega, E)$ such that for $n > N$, $\|T_{[0,n],E,\omega}\| \leq e^{(n+1)(\gamma(E)+\varepsilon)}$. For the proof of dynamical localization one however needs a statement of this type with N uniform in E. Such a statement is the goal of this section. We will show that it holds for any ergodic dynamical system satisfying the uniform LDT (Large Deviation Type) condition: Lemma 1.2.1. Thus this result has more general nature than the rest of the chapter and may be of independent interest. In particular, it is applicable to quasiperiodic dynamics with Diophantine frequencies and analytic sampling functions. We note that uniform LDT condition can also be replaced by a combination of a pointwise LDT condition and continuity of the Lyapunov exponent.

We have:

Theorem 1.7. *Let the ergodic family H_ω satisfy Lemma 1.2.1. Fix $\varepsilon_0 > 0$. For a.e. ω (we denote this set as $\Omega_2 = \Omega_2(\varepsilon_0)$), there exists $N_2(\omega)$, such that for any $n > N_2(\omega)$, $E \in \sigma$,*

$$|P_{[0,n],E,\omega}| \leq e^{(\gamma(E)+\varepsilon_0)(n+1)}$$

An immediate corollary is

Corollary 1.5.1. *Let H_ω, ε_0 be as above. Then there exists Ω_2 with $\mathbb{P}(\Omega_2) = 1$, such that for $\omega \in \Omega_2$, there exists $N_2(\omega)$ such that*

$$\max \left\{ |P_{[0,n],E,\omega}|, |P_{[-n,0],E,\omega}|, |P_{[n+1,2n+1],E,\omega}|, |P_{[2n+1,3n+1],E,\omega}| \right\} \leq e^{(\gamma(E)+3\varepsilon_0)(n+1)}.$$

Thus we can replace Corollary 1.3.3 with this uniform version.

Proof. We start with the following

Lemma 1.5.2. *Let $Q(x)$ be a polynomial of degree $n-1$. Let $x_i = \cos\frac{2\pi(i+\theta)}{n}$, $0<\theta<1/2$, $i = 1, 2, \cdots, n$. If $Q(x_i) \le a^n$, for all i, then $Q(x) \le Cna^n$, for all $x \in [-1, 1]$, where $C = C(\theta)$ is a constant.*

Proof. By Lagrange interpolation, we have

$$Q(x) = \sum_{i=1}^{n} Q(x_i) \prod_{j\neq i} \frac{x - x_j}{x_i - x_j}$$

Note that

$$\sum_{j\neq i} \ln|x_i - x_j| = \sum_{j\neq i} \left\{ \ln\left|\sin\frac{\pi(i+j+2\theta)}{n}\right| + \ln\left|\sin\frac{\pi(i-j)}{n}\right| + \ln 2 \right\}$$
$$=: A + B + (n-1)\ln 2.$$

We will use the following lemma without giving a proof.

Lemma 1.5.3 (Lemma 9.6 in [5])**.** *Let p and q be relatively prime. Let $1 \le k_0 \le q$ be such that*

$$|\sin 2\pi(x + k_0 p/(2q))| = \min_{1\le k\le q} |\sin 2\pi(x + kp/(2q))|.$$

Then

$$\ln q + \ln(2/\pi) < \sum_{\substack{k=1 \\ k\neq k_0}}^{q} \ln|\sin 2\pi(x + kp/(2q))| + (q-1)\ln 2 \le \ln q.$$

For B, we take $p = 1,\ q = n,\ x = -i/(2n),\ k = j$. Then $k_0 = i$, and we get

$$B \ge \ln n + \ln(2/\pi) - (n-1)\ln 2.$$

For A, we estimate by Lemma 1.5.3 with $p = 1,\ q = n,\ x = (i+2\theta)/2n,\ k = j$. If

$k_0 = j_0$ is the minimum term of $\ln|\sin\frac{\pi(i+j+2\theta)}{n}|$, then

$$A \geq \ln n + \ln(2/\pi) - (n-1)\ln 2 - \ln\left|\sin\frac{\pi(2i+2\theta)}{n}\right| + \ln\left|\sin\frac{\pi(i+j_0+2\theta)}{n}\right|$$

For $0 < \theta < 1/4$, we have

$$\frac{|\sin\frac{\pi(2i+2\theta)}{n}|}{|\sin\frac{\pi(i+j_0+2\theta)}{n}|} = \frac{|\sin\frac{\pi(2i+2\theta)}{n}|}{|\sin\frac{\pi\cdot 2\theta}{n}|} \leq \frac{1}{|\sin\frac{\pi\cdot 2\theta}{n}|} = O(n)$$

Thus

$$\sum_{j\neq i}\ln|x_i - x_j| \geq -(n-1)\ln 2 + \ln n + C$$

Writing $x = \cos\frac{2\pi a}{n}$, by Lemma 1.5.3, we get

$$\sum_{j\neq i}\ln|x - x_j| \leq -(n-1)\ln 2 + 2\ln n + C$$

Thus

$$\prod_{j\neq i}\frac{x-x_j}{x_i-x_j} \leq Cn$$

and we have

$$Q(x) \leq Cna^n$$

□

Now we can finish the proof of Theorem 1.7.

We know that σ is compact, so contained in some bounded closed interval. Assume we are dealing with $[a, a+A]$. Unifrom LDT implies that γ is a continuous function of E [35]. Since $\gamma(E)$ is uniformly continuous, for any ε_0, there exists δ_0 such that

$$|\gamma(E_x) - \gamma(E_y)| \leq \varepsilon_0, \quad if\ |E_x - E_y| \leq \delta_0. \tag{1.13}$$

Divide the interval $[a, a+A]$ into length-δ_0 sub-intervals. There are $K = [A/\delta_0] + 1$ of them (the last one may be shorter). Denote them as I_k, for $k = 1, \cdots, K$. For $I_k = [E_{k,n}, E_{k+1,n}]$, let $E_{k1,n}, \cdots, E_{kn,n}$ be distributed as in Lemma 5.3. Namely, set $E_{ki,n} = E_{k,n} + (x_i + 1)\delta_0/2$, where x_i are as in Lemma 5.3, $0 < \theta < 1/2$. Note that for any E_x, $E_y \in [E_{k1,n}, E_{kn,n}]$, $|\gamma(E_x) - \gamma(E_y)| \leq \varepsilon_0$. Since by the uniform-LDT condition

$$\mathbb{P}\left(\left\{\omega : \exists i = 1, \cdots, n,\ s.t.\ |P_{[0,n],E_{ki,n},\omega}| \geq e^{(\gamma(E_{ki,n})+\varepsilon_0)(n+1)}\right\}\right) \leq ne^{-\eta_0(n+1)},$$

by Borel-Cantelli, for a.e. ω, (we denote this set as $\Omega(k)$), there exists $N(k,\omega)$, such that for all $n > N(k,\omega)$,

$$|P_{[0,n],E_{ki,n},\omega}| \leq e^{(\gamma(E_{ki,n})+\varepsilon_0)(n+1)}, \quad \forall i = 1, \cdots, n.$$

If we denote $\gamma_{k,n} = \inf_{E \in [E_{k1,n}, E_{kn,n}]} \gamma(E)$, then by (1.13)

$$|P_{[0,n],E_{ki,n},\omega}| \leq e^{(\gamma(E_{ki,n})+\varepsilon_0)(n+1)} \leq e^{(\gamma_{k,n}+2\varepsilon_0)(n+1)}, \quad \forall i = 1, \cdots, n.$$

Let M be big enough such that, for any $n > M$, $C(n+2)e^{\gamma(E)+2\varepsilon_0} \leq e^{\varepsilon_0(n+1)}$. Thus by Lemma 1.5.2, applied to $Q(x) = P_{[0,n],E,\omega}|_{E=E_{k,n}+\frac{(x+1)\delta_0}{2}}$, a polynomial of degree $n+1$, we have that, if $n > \max\{N(k,\omega), M\}$, for $E \in [E_{k,n}, E_{k+1,n}]$

$$|P_{[0,n],E,\omega}| \leq C(n+2)e^{(\gamma_{k,n}+2\varepsilon_0)(n+2)} \leq C(n+2)e^{(\gamma(E)+2\varepsilon_0)(n+2)} \leq e^{(\gamma(E)+3\varepsilon_0)(n+1)}$$

Let $\Omega_2 = \bigcap_k \Omega(k)$, $\tilde{N}(\omega) = \max_k\{N(k,\omega), M\}$. Then for any $n > \tilde{N}(\omega)$,

$$|P_{[0,n],E,\omega}| \leq e^{(\gamma(E)+3\varepsilon_0)(n+1)}, \quad \forall E \in [a, a+A]$$

□

This allows us to also obtain a quantitative version of Theorem 1.7. Assume the $N_2(\omega)$ in Theorem 1.7 is chosen to be the smallest satisfying the condition. Let $l \in \mathbb{Z}$, $N_2(l,\omega) = N_2(T^l\omega)$. Let $\bar{\Omega}_2 = \bigcap_{l\in\mathbb{Z}} T^l\Omega_2$.

Lemma 1.5.4. *For a.e. ω (we denote this set as $\tilde{\Omega}_2$), there exists $L_2 = L_2(\omega)$, such that for all $|l| > L_2$, $N_2(l,\omega) \leq \ln^2|l|$. In particular, if $n > \ln^2|l|$, then*

$$|P_{[l,l+n],E,\omega}| \leq e^{(\gamma(E)+\varepsilon_0)(n+1)}, \ \textit{for all } E \in \sigma$$

Proof. Let $\omega \in \bar{\Omega}_2$, $l \in \mathbb{Z}$, $k \in \mathbb{N}$. By Theorem 1.7, $\bar{\Omega}$ has full measure. We have

$$\begin{aligned} \mathbb{P}\{\omega : N_2(l,\omega) \geq k\} &\leq \sum_{n=k}^{\infty} \mathbb{P}\{\omega : N_2(l,\omega) = n\} \leq \sum_{n=k}^{\infty} \mathbb{P}(B^+_{[l,l+n-1],E}) \\ &\leq \sum_{n=k}^{\infty} Ce^{-(\gamma(E)+\varepsilon_0)n} \leq Ce^{-(\gamma(E)+\varepsilon_0)k} \end{aligned}$$

Thus

$$\mathbb{P}\{\omega : N_2(l,\omega) \geq \ln^2|l|\} \leq Ce^{-(\gamma(E)+\varepsilon_0)(\ln^2|l|)}$$

By Borel-Cantelli lemma, we get the result and the corresponding $\tilde{\Omega}_2$. □

1.6 Dynamical Localization

Now we have established the spectral localization for 1-d Anderson model. With some more effort, we can get the dynamical localization. We say that H_ω exhibits dynamical localization if for *a.e.* ω, for any $\varepsilon > 0$, there exists $\alpha = \alpha(\omega) > 0$, $C = C(\varepsilon, \omega)$, such that for all $x, y \in \mathbb{Z}$:

$$\sup_t |\langle \delta_x, e^{-itH_\omega} \delta_y \rangle| \leq C_\varepsilon e^{\varepsilon |y|} e^{-\alpha |x-y|}$$

According to [28], we only need to prove that for a.e. ω, H_ω has SULE (Semi-Uniformly Localized Eigenfunction). We say H has SULE if H has a complete set $\{\varphi_E\}$ of orthonormal eigenfunctions, such that there is $\alpha > 0$, and for each $\varepsilon > 0$, a C_ε such that for any eigenvalue E, there exists $l = l_E \in \mathbb{Z}$, such that

$$|\varphi_E(x)| \leq C_\varepsilon e^{\varepsilon |l_E|} e^{-\alpha |x - l_E|}, \quad x \in \mathbb{Z}$$

In fact, we will prove that $|\varphi_E(x)| \leq C_\varepsilon e^{C \ln^2(1+|l_E|)} e^{-\alpha|x-l_E|}$, see (1.16), (1.18). In order to do this, we need to modify Lemma 1.3.2, Lemma 1.3.4 using the same method as in Lemma 1.5.4. Assume the $N_i(\omega),\ i = 1, 3$ in Lemmas 1.3.2, 1.3.4 are chosen to be the smallest parameters satisfying the condition. Let $l \in \mathbb{Z}$, $N_i(l, \omega) = N_i(T^l \omega)$. Let $\bar{\Omega}_i = \bigcap_{l \in \mathbb{Z}} T^l \Omega_i$, $i = 1, 3$.

Lemma 1.6.1. *For a.e.* ω *(we denote this set as* $\tilde{\Omega}_{1,3}$*), there are* $L_1(\omega), L_3(\omega)$ *such that for any* $|l| > \max\{L_1, L_3\}$,

$$\max\{N_1(l, \omega), N_3(l, \omega)\} \leq \ln^2 |l|$$

Proof. Let $\omega \in \bar{\Omega}_1$, $l \in \mathbb{Z}$, $k \in \mathbb{N}$, then by (1.7)

$$\mathbb{P}\{\omega : N_1(l,\omega) > k\} \leq \sum_{n=k}^{\infty} \mathbb{P}(\Omega_{\delta,n,\pm}) \leq \sum_{n=k}^{\infty} 2m(\sigma)e^{-\delta_0(2n+1)} \leq Ce^{-\delta_0(2k+1)}$$

Thus

$$\mathbb{P}\{\omega : N_1(l,\omega) > \ln^2 |l|\} \leq Ce^{-\delta_0(2\ln^2 |l|+2)}$$

By Borel-Cantelli lemma, we can get the result. The same argument works for N_3. □

Then we rebuild the criteria for regularity around a singular point l.

Lemma 1.6.2. *For a.e. ω (we denote this set as $\tilde{\Omega}$), for any l, there exists $N(l,\omega)$, such that for any $n > N(l,\omega)$ and for all $E \in \sigma$ either l or $l+2n+1$, and either l or $l-2n-1$ are $(\gamma(E)-8\varepsilon_0, n, E, \omega)$-regular.*

Proof. In section 4, we proved that either 0 or $2n+1$ is $(\gamma(E)-8\varepsilon_0, n, E, \omega)$-singular for all $n > N(\omega)$, with and modify $\tilde{\Omega}$ accordingly. □

Now, take $\tilde{\Omega} = \tilde{\Omega}_2 \cup \tilde{\Omega}_{1,3}$ and fix $\omega \in \tilde{\Omega}$. We omit ω from notations from now on.

By Lemma 1.6.1 and Lemma 1.5.4, there exist L_1, L_2, L_3 such that for all $|l| > \max\{L_1, L_2, L_3\}$,

$$N_i(l) \leq \ln^2 |l|, \quad \forall i = 1,2,3$$

for all $E \in \sigma$.

Let l_E be a position of the maximum point of φ_E. Take L_4 with $\ln^2 L_4 \geq [\frac{\ln 2}{\gamma(E)-8\varepsilon_0}]+1$. For any $n \geq \ln^2 L_4$, and any e.v. E, l_E is naturally $(\mu - 8\varepsilon_0, n, E)$-singular by (1.1).

Let $L = \max\{L_1, L_2, L_3, L_4\}$, $N(l) := \max\{N_1(l), N_2(l), N_3(l), \frac{\ln 2}{\gamma(E)-8\varepsilon_0}\}$. Then for any

$|l| > L$,

$$N(l) \leq \ln^2 |l| \tag{1.14}$$

If $|l_E| > L$, then for any $n \geq N(l_E)$, l_E is $(\gamma(E)-8\varepsilon_0, n, E)$-singular, so $x = l_E \pm (2n+1)$ is $(\gamma(E) - 8\varepsilon_0, n, E)$-regular. By (1.1), for any $|x - l_E| \geq N(l_E)$

$$|\varphi_E(x)| \leq 2e^{-(\gamma(E)-8\varepsilon_0)|x-l_E|} \tag{1.15}$$

Since φ_E is normalized, in fact for all x,

$$|\varphi_E(x)| \leq 2e^{(\gamma(E)-8\varepsilon_0)N(l_E)}e^{-(\gamma(E)-8\varepsilon_0)|x-l_E|}$$

By (1.14), for any ε,

$$|\varphi_E(x)| \leq 2e^{(\gamma(E)-8\varepsilon_0)\ln^2(1+|l_E|)}e^{-(\gamma(E)-8\varepsilon_0)|x-l_E|} \tag{1.16}$$

If $|l_E| \leq L$, for any ε, for $n \geq N(l_E)$, we use the same argument as (1.15) and get

$$|\varphi_E(x)| \leq 2e^{-(\gamma(E)-8\varepsilon_0)|x-l_E|} \leq 2e^{\varepsilon \ln^2(1+|l_E|)}e^{-(\gamma(E)-8\varepsilon_0)|x-l_E|} \tag{1.17}$$

While for $n \leq N_{l_E}$, set $M_{2\varepsilon} = \min_{k\in[-L,L],\ |x-k|<N(k)}\{e^{\varepsilon \ln^2(1+|k|)}e^{-(\gamma(E)-8\varepsilon_0)|x-k|}\}$ and $C_{2\varepsilon} = M_{2\varepsilon}^{-1}$. Then for all $|x - l_E| < N(l_E)$,

$$|\varphi_E(x)| \leq 1 \leq C_{2\varepsilon}e^{\varepsilon \ln^2(1+|l_E|)}e^{-(\gamma(E)-8\varepsilon_0)|x-l_E|} \tag{1.18}$$

Thus for $C_\varepsilon = \max\{2, C_{2\varepsilon}\}$, (1.16) (1.17) and (1.18) provide SULE. $\square$

Chapter 2

Anderson localization in random CMV matrix

2.1 Introduction

The aim of this chapter is to establish Anderson localization and dynamical localization in expectation (see Definition 2.1 and 2.2) for random CMV matrices with arbitrary distribution.

CMV matrices were introduced by Cantero, L. Moral, L. Vel ([19]) in 2003 and play an important role in the study of orthogonal polynomials on the unit circle (OPUC). See [73], [75], [43] for a concise and elegant report of the main results and [74] for a detailed monograph on this subject.

The study of random CMV matrices was motivated by random Anderson models for Schrödinger operators. When the distribution is absolutely continuous, Anderson localization for CMV matrices has been proved in [80], [42], [74, Sec. 12.6] using the spectral averaging method, but these techniques cannot be applied in the singular case. For one-dimensional Anderson model, the first proof that can handle arbitrary randomness was given in [21], based on the multi-scale analysis. In 2019, [52] provided a short proof of Anderson localization and dynamical localization (for the one-dimensional Anderson model with arbitrary distribution) using positive Lyapunov exponents together with uniform large deviation type (LDT) estimates and uniform Craig-Simon results. In 2020, strong dynamical localization was proved in [41] following this method. In this chapter, we exploit the techniques in [52] and [41] to prove Anderson localization and strong dynamical localization for random CMV matrices with arbitrary distribution. In particular, our results apply in the singular case. The main novelties of the proof are the large-deviation estimates of determinants with modified boundary conditions (Lemma 4.2) and a streamlined approach to the localization proof in comparison with [52] and [41], so that EDL follows directly from our key statement (Theorem 2.4).

It is important to mention that the singular potential random CMV model was also studied in [18] in 2019 as a close relative of the Anderson model, for which a new proof of localization was also given in [18]. The CMV proof in [18] relies on certain results in [58]. However those contain a significant number of misprints and minor errors (some of those stemming from small misprints in [74] and [73]). Article [18] inherits those errors, certain steps of the proof in [18] no longer work as claimed when the relevant expressions are corrected. In particular, a crucial part of the argument of [18] on the elimination of double resonances does not work as intended although we believe may be corrected. We discuss this in Appendix B.

Finally, this chapter when taken in conjunction with [52], [41], [69], [68], and [65], illustrates the flexibility of this general scheme for proving localization in random one-dimensional frameworks. Indeed, these techniques provide the most direct route to localization in addition to providing proofs of the strongest known localization results for such models (EDL).

The remainder of the paper is organized as follows:

- In Section 2.2, we present the model and the main results (AL and EDL).
- In Section 2.3, we present a key theorem on regularity of Green functions from which AL and EDL are derived.
- In Section 2.4, we present uniform large deviation theorem (Lemma 2.4.2) and uniform Craig-Simon estimates (Lemma 2.4.4).
- In Section 2.5, we first provide an outline of the proof and prove our key Theorem 2.3.
- Finally, Appendix A provides technical details needed for subsection 2.3.3 and Appendix B corrects the errors in the formulas found in [58], [18], [74] and [73]. It is our hope that these corrections provide clarification for other readers working on CMV matrices.

2.2 Model and main results

2.2.1 OPUC

Let η be a probability measure which is supported on an infinite subset of $\partial\mathbb{D}$ where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. Let $\Phi_n(z)$ be the monic polynomial of degree n s.t.

$$\langle\Phi_m(z), \Phi_n(z)\rangle = \int_{\partial\mathbb{D}} \overline{\Phi}_m(z)\Phi_n(z)d\eta(z) = \delta_{mn}, \quad \forall m, n \in \mathbb{N}.$$

The $\Phi_n(z)$'s are called the orthogonal polynomials on the unit circle (OPUC) w.r.t. η. Let $\phi_n(z) = \frac{\Phi_n(z)}{\|\Phi_n(z)\|}$ where $\|\cdot\|$ is the $L^2(\partial\mathbb{D}; d\eta)$ norm.

It is clear that given η, we can compute $\Phi_n(z)$ and $\phi_n(z)$ inductively from $\Phi_0(z) = \phi_0(z) = 1$. Moreover, there is a recurrence relation for $\Phi_n(z)$ which we state here without proof (see [74, Theorem 1.5.2]):

Proposition 2.2.1 (Szegő's Recurrence)**.** *Given η, there is a sequence of $\alpha_n \in \mathbb{D}$ s.t.*

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n\Phi_n^*(z)$$
$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z),$$

*where $Q(z)^* := z^n\overline{Q(1/\bar{z})}$ for polynomials $Q(z)$ of degree n. The terms $\{\alpha_n\}_{n=0}^{\infty}$ are called **Verblunsky coefficients**. Furthermore, let $\rho_n = (1 - |\alpha_n|^2)^{1/2}$. We have*

$$\|\Phi_n\|^2 = \|\Phi_n^*\|^2 = \rho_n^2\|\Phi_{n-1}\|^2 = \prod_{k=0}^{n-1}\rho_k^2.$$

Thus, for the normalized ϕ_n, we have

$$\begin{pmatrix} \phi_{n+1} \\ \phi^*_{n+1} \end{pmatrix} = \frac{1}{\rho_n} \begin{pmatrix} z & -\overline{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix} \begin{pmatrix} \phi_n \\ \phi^*_n \end{pmatrix}.$$

By Szegő's recurrence, each η corresponds to a sequence of $\{\alpha_n\}_{n=0}^{\infty} \in \mathbb{D}^{\mathbb{N}}$. It turns out that this correspondence is bijective (e.g. [74, Theorem 1.7.11]).

Proposition 2.2.2 (Verblunsky's Theorem)**.** *There is a bijection between nontrivial (supported on an infinite set) probability measures η on $\partial\mathbb{D}$ and $\{\alpha_n\}_{n=0}^{\infty} \in \mathbb{D}^{\mathbb{N}}$.*

2.2.2 CMV matrices

A CMV matrix is a matrix representation of the multiplication-by-z operator on $L^2(\partial\mathbb{D}; d\eta)$ w.r.t a basis which is obtained from orthonormalizing the set

$$\{1, z, z^{-1}, z^2, z^{-2}, \cdots\}.$$

It is important to understand the relation between η and α_n, especially under perturbations. On the one hand, the definition implies that η is a spectral measure of the CMV matrix. On the other hand, the CMV matrices can be expressed by the Verblunsky coefficients α_n and $\rho_n = (1 - |\alpha_n|^2)^{\frac{1}{2}} > 0$ (See [74, Sec 4.3] for more details):

$$
\mathcal{C} = \begin{bmatrix}
\overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & & & & & \\
\rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & & & & & \\
& \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & & & \\
& \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & & & \\
& & & \overline{\alpha_4}\rho_3 & -\overline{\alpha_4}\alpha_3 & \overline{\alpha_5}\rho_4 & \rho_5\rho_4 & \\
& & & \rho_4\rho_3 & -\rho_4\alpha_3 & -\overline{\alpha_5}\alpha_4 & -\rho_5\alpha_4 & \\
& & & & \ddots & \ddots & \ddots &
\end{bmatrix} \tag{2.1}
$$

We will study a two-sided version of the above matrix. The two-sided version depicted below is called an *extended CMV matrix.*

$$
\mathcal{E} = \begin{bmatrix}
\ddots & \ddots & \ddots & & & & & \\
\overline{\alpha_0}\rho_{-1} & -\overline{\alpha_0}\alpha_{-1} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & & & & \\
\rho_0\rho_{-1} & -\rho_0\alpha_{-1} & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & & & & \\
& & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & & \\
& & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & & \\
& & & & \overline{\alpha_4}\rho_3 & -\overline{\alpha_4}\alpha_3 & \overline{\alpha_5}\rho_4 & \rho_5\rho_4 \\
& & & & \rho_4\rho_3 & -\rho_4\alpha_3 & -\overline{\alpha_5}\alpha_4 & -\rho_5\alpha_4 \\
& & & & & \ddots & \ddots & \ddots
\end{bmatrix} \tag{2.2}
$$

The relationship between $\mathcal{C}$ and $\mathcal{E}$ is explained in Remark 2.3.

2.2.3 Random CMV matrices

As with the Anderson model, we are interested in the random extended CMV matrix $\mathcal{E}_\omega$ where $\alpha_n = \omega_n \in \mathbb{D}$ are i.i.d. random variables with common Borel probability distribution μ supported on a compact subset S of $\mathbb{D}$. We assume μ is non-trivial in the sense that it contains at least two points and as we introduced in the introduction, there are no regularity requirements on μ. Let the probability space be $\Omega = S^{\mathbb{Z}}$, with elements $\omega = \{\omega_n\}_{n\in\mathbb{Z}} \in \Omega$. Denote $\mu^{\mathbb{Z}}$ by $\mathbb{P}$. Let $\mathbb{P}_{[m,n]}$ be $\mu^{[m,n]\cap\mathbb{Z}}$ on $\Omega_{[m,n]} := S^{[m,n]\cap\mathbb{Z}}$. Hence whenever we write $[m, n]$ in this paper, we mean $[m, n] \cap \mathbb{Z}$. Also let T be the shift on Ω, i.e. $(T\omega)_i = \omega_{i-1}$. Finally, we denote Lebesgue measure on the unit circle by m.

By the classical ergodicity argument for random operators (e.g. [26, Chapter 9]), we see that the spectrum of $\mathcal{E}_\omega$ is almost surely deterministic, i.e. there is $\Sigma \subset \partial\mathbb{D}$ s.t. for a.e. ω, $\sigma(\mathcal{E}_\omega) = \Sigma$. Furthermore, the pure point spectrum, a.c. spectrum and s.c. spectrum are all a.s. deterministic, i.e. $\sigma_*(\mathcal{E}_\omega) = \Sigma_*$, $* \in \{p.p., a.c., s.c.\}$.

2.2.4 Main results

We can now introduce our main results.

Definition 2.1 (AL)**.** *We say $\mathcal{E}_\omega$ exhibits Anderson localization (AL, also called spectral localization) on $\mathcal{I}$ if for a.e. ω, $\mathcal{E}_\omega$ has only pure point spectrum in $\mathcal{I}$ and its eigenfunctions $\Psi_\omega(n)$ decay exponentially in n.*

Definition 2.2 (EDL)**.** *We say $\mathcal{E}_\omega$ exhibits dynamical localization in expectation (EDL,* *also known as strong dynamical localization) on $\mathcal{I}$ if there is $C, \eta > 0$ s.t.*

$$\sup_{t\in\mathbb{R}} \mathbb{E}\left(|\langle \delta_p, e^{-it\mathcal{E}_\omega}\chi_{\mathcal{I}}(\mathcal{E}_\omega)\delta_q\rangle|\right) \leq Ce^{-\eta|p-q|}$$

where $\chi_{\mathcal{I}}$ is the characteristic function of $\mathcal{I}$.

We will prove in this paper that

Theorem 2.1 (AL)**.** *There is a set $\mathcal{D} \subset \partial\mathbb{D}$ which contains at most three points such that, $\mathcal{E}_\omega$ exhibits AL on any compact interval $\mathcal{I} \subset \partial\mathbb{D} \setminus \mathcal{D}$.*

Remark 2.1. The existence of this exceptional set is due to the failure of Fürstenberg's Theorem (see Subsection 2.4.3).

Theorem 2.2 (EDL)**.** *There is a set $\mathcal{D} \subset \partial\mathcal{D}$ which contains at most three points s.t. $\mathcal{E}_\omega$ exibihits EDL on any compact interval $\mathcal{I} \subset \partial\mathbb{D} \setminus \mathcal{D}$.*

2.3 Theorem 2.3 implies AL and EDL

Below, we will formulate the key theorem, Theorem 2.3. We then prove AL (Theorem 2.1) and EDL (Theorem 2.2) from it. To do so, we make some preparations in Subsection 2.3.1-2.3.3, state Theorem 2.3 in Subsection 2.3.4 and prove Theorem 2.1 and 2.2 in Subsection 2.3.5.

2.3.1 Decomposition of CMV matrices

We start with a decomposition of a CMV matrix which helps us to deal with its more complicated five-diagonal nature. Let $\alpha_n \in \mathbb{D}$, $\rho_n = (1 - |\alpha_n|^2)^{\frac{1}{2}}$. Define the unitary matrix acting on $\ell^2(\{n, n+1\})$ by

$$\Theta_n = \begin{pmatrix} \overline{\alpha_n} & \rho_n \\ \rho_n & -\alpha_n \end{pmatrix}.$$

Define

$$\mathcal{L} = \bigoplus_{n \ even} \Theta_n, \quad \mathcal{M} = \bigoplus_{n \ odd} \Theta_n.$$

Then one can check directly by computation that the extended CMV matrix satisfies

$$\mathcal{E} = \mathcal{L}\mathcal{M}.$$

By definition of Θ_n, α_n and ρ_n, it is easy to see that $\mathcal{L}$ and $\mathcal{M}$ are unitary on $\ell^2(\mathbb{Z})$. Thus $\mathcal{E}$ is also unitary. (More details can be found in [74, Theorem 4.2.5].)

Let $P_{[a,b]} : \ell^2(\mathbb{Z}) \to \ell^2([a,b])$ be the natural projection. Let $X_{[a,b]} = P_{[a,b]} X (P_{[a,b]})^*$ for $X \in \{\mathcal{E}, \mathcal{L}, \mathcal{M}\}$. Then it is easily verified that

$$\mathcal{E}_{[a,b]} = \mathcal{L}_{[a,b]} \mathcal{M}_{[a,b]}. \tag{2.3}$$

2.3.2 Modification of the Boundary Conditions

Notice that $\mathcal{E}_{[a,b]}$, $\mathcal{L}_{[a,b]}$ are not always unitary due to the fact that the "boundary terms" α_{a-1} and α_b satisfy $|\alpha_{a-1}| < 1$ and $|\alpha_b| < 1$. Thus we can instead manually create unitary operators by modifying these boundary conditions. Let $\beta, \gamma \in \partial\mathbb{D}$.

Define

$$\tilde{\alpha}_n = \begin{cases} \alpha_n, & n \neq a-1, b \\ \beta, & n = a-1 \\ \gamma, & n = b \end{cases}.$$

Denote the extended CMV matrix with Verblunsky coefficients $\tilde{\alpha}_n$ by $\tilde{\mathcal{E}}$. Then define

$$\mathcal{E}_{[a,b]}^{\beta,\gamma} = P_{[a,b]}\tilde{\mathcal{E}}P_{[a,b]}.$$

$\mathcal{L}_{[a,b]}^{\beta,\gamma}$ and $\mathcal{M}_{[a,b]}^{\beta,\gamma}$ are defined correspondingly. Now $\mathcal{E}_{[a,b]}^{\beta,\gamma}$, $\mathcal{L}_{[a,b]}^{\beta,\gamma}$ and $\mathcal{M}_{[a,b]}^{\beta,\gamma}$ are all unitary.

Remark 2.2. Notice that this modification is only a formal modification of the boundary value $|\alpha_{a-1}| < 1$ to $|\beta| = 1$ and $|\alpha_b| < 1$ to $|\gamma| = 1$. So, all the formulas for $\mathcal{E}_{[a,b]}$ with α_{a-1} and α_b still hold for $\mathcal{E}_{[a,b]}^{\beta,\gamma}$ with β and γ. For example, $\mathcal{E}_{[a,b]}^{\beta,\gamma} = \mathcal{L}_{[a,b]}^{\beta,\gamma}\mathcal{M}_{[a,b]}^{\beta,\gamma}$ follows from (2.3).

Remark 2.3. We will use $\mathcal{E}_{[a,b]}^{\beta,\cdot}$, $\mathcal{E}_{[a,b]}^{\beta,\cdot}$ to denote single-sided boundary condition modification. By comparing (2.1) and (2.2), it is easy to see that $\mathcal{C} = \mathcal{E}_{[0,+\infty]}^{-1,\cdot}$.

2.3.3 Green's functions, Generalized eigenfunctions, Poisson formula

Now we can define the Green's function. Usually it is defined to be $G_{[a,b],z} = (\mathcal{E}_{[a,b],\omega} - z)^{-1}$. However, since $\mathcal{E}_\omega$ is five-diagonal, it is more complicated than a Jacobi matrix, and the restriction to $[a, b]$ is not unitary. Thus we can modify the boundary and rewrite the characteristic function $(\mathcal{E}_{[a,b]}^{\beta,\gamma} - z)\Psi = 0$ as $(z(\mathcal{L}_{[a,b]}^{\beta,\gamma})^* - \mathcal{M}_{[a,b]}^{\beta,\gamma})\Psi = 0$. Then $A_{[a,b],z}^{\beta,\gamma} := (z(\mathcal{L}_{[a,b]}^{\beta,\gamma})^* - \mathcal{M}_{[a,b]}^{\beta,\gamma})$ is tri-diagonal (see Lemma A.0.1 in the appendix B) and it is natural to define the Green's function to be

$$G^{\beta,\gamma}_{[a,b],z} = (A^{\beta,\gamma}_{[a,b],z})^{-1} = \left(z\left(\mathcal{L}^{\beta,\gamma}_{[a,b]}\right)^* - \mathcal{M}^{\beta,\gamma}_{[a,b]}\right)^{-1}$$

for $|\beta| = |\gamma| = 1$, $z \notin \sigma(\mathcal{E}^{\beta,\gamma}_{[a,b]})$.

Exponential decay of the off-diagonal entries of the Green's function turns out to be essential in the study of localization phenomenons. It is closely related to the exponential decay of (generalized) eigenfunctions through Poisson formula.

Definition 2.3 (Generalized eigenvalues and generalized eigenfunctions)**.** *Fix ω. We call z_ω a generalized eigenvalue (g.e.) of $\mathcal{E}_\omega$, if there exists a nonzero, polynomially bounded function $\Psi_\omega(n)$ such that $\mathcal{E}_\omega\Psi_\omega = z_\omega\Psi_\omega$. We call $\Psi_\omega(n)$ a generalized eigenfunction (g.e.f.).*

Lemma 2.3.1 (Poisson formula)**.** *Let Ψ be a g.e.f. of $\mathcal{E}$ w.r.t. a g.e. z, i.e. $\mathcal{E}\Psi = z\Psi$. Let $|\beta| = |\gamma| = 1$. Then for $a < x < b$,*

$$\begin{aligned}\Psi(x) = &- G^{\beta,\gamma}_{[a,b],z}(x,a)\begin{cases}\Psi(a)(z\bar{\beta} - z\bar{\alpha}_{a-1}) + \Psi(a-1)z\rho_{a-1}, & a \text{ odd},\\ \Psi(a)(\alpha_{a-1} - \beta) - \Psi(a-1)\rho_{a-1}, & a \text{ even},\end{cases}\\ &- G^{\beta,\gamma}_{[a,b],z}(x,b)\begin{cases}\Psi(b)(-\bar{\alpha}_b + \bar{\gamma}) - \Psi(b+1)\rho_b, & b \text{ odd},\\ \Psi(b)(z\alpha_b - z\gamma) + \Psi(b+1)z\rho_b, & b \text{ even}.\end{cases}\end{aligned}$$

We give a proof in Lemma A.0.2 in the appendix.

2.3.4 Schnol's theorem, Regularity, Key statement

Recall that Schnol's theorem (see [56, Theorem 7.1], or [26, Sec. 2.4]) says that the spectral measures are supported on the set of g.e.'s. Thus, to show Anderson localization it is enough to show that for a.e. ω, for any g.e. z_ω of $\mathcal{E}_\omega$, the corresponding g.e.f. Ψ_ω decays exponentially, because this would imply that each g.e. is indeed an eigenvalue, so $\mathcal{E}_\omega$ has only pure point spectrum.

Thus for a g.e.f. Ψ_ω which is polynomially bounded, if we can show the Green's function $|G^{\beta,\gamma}_{[n+1,3n+1],\omega,z}(2n+1, n+1)|$ and $|G^{\beta,\gamma}_{[n+1,3n+1],\omega,z}(2n+1, 3n+1)|$ are exponentially small, then $|\Psi_\omega(2n+1)|$ will decay exponentially due to the Poisson formula. This idea inspires us to define regularity as follows:

Definition 2.4 (Regularity). *Let β, $\gamma \in \partial\mathbb{D}$. For fixed ω, $z \notin \sigma(\mathcal{E}^{\beta,\gamma}_{[a,b],\omega})$, $c > 0, n \in \mathbb{Z}$, we say $x \in \mathbb{Z}$ is (c, n, ω, z)-regular, if*

$$|G^{\beta,\gamma}_{[x-n,x+n],\omega,z}(x, x-n)| \leq e^{-cn},$$
$$|G^{\beta,\gamma}_{[x-n,x+n],\omega,z}(x, x+n)| \leq e^{-cn}.$$

Otherwise, we call it (c, n, ω, z)-singular.

2.3.5 Proof of AL and EDL

We can now formulate our key statement:

Theorem 2.3. *There is a set $\mathcal{D} \subset \partial\mathbb{D}$ which contains at most three points such that, for any compact interval $\mathcal{I} \subset \partial\mathbb{D} \setminus \mathcal{D}$, if we let $\nu := \inf_{z\in\mathcal{I}} \gamma(z) > 0$, then for any $0 < \varepsilon < \nu/2$, there is $N = N(\varepsilon), \eta = \eta(\varepsilon) > 0$ s.t. $\forall n > N$, $\forall x \in \mathbb{Z}$, there is a subset $\Omega_{x,n} \subset \Omega_{[x-n,x+n]}$ s.t.*

1. $\mathbb{P}(\Omega_{x,n}) \geq 1 - e^{-\eta(2n+1)}$.

2. $\forall \omega \in \Omega_{x,n}$, *either* x *or* $x + 2n + 1$ *is* $(\gamma(z) - 2\varepsilon, n, \omega, z)$*-regular for any* $z \in \mathcal{I}$.

This Theorem will be proved in the next two sections. We first show Theorem 2.3 implies Theorem 2.1 and Theorem 2.2 before proving Theorem 2.3.

Proof of Theorem 2.1 (AL). Find $\mathcal{D}$, $\mathcal{I}$, ν from Theorem 2.3. For any $0 < \varepsilon < \nu/2$, find $N(\varepsilon)$, $\eta(\varepsilon)$, $\Omega_{x,n}$ from Theorem 2.3. For any $x \in \mathbb{Z}$, since $\sum_n \mathbb{P}\left((\Omega_{x,n})^c\right) < \infty$, by the Borel-Cantelli Lemma, for a.e. ω, eventually either x or $x+2n+1$ is $(\gamma(z)-2\varepsilon, n, \omega, z)$-regular.

On the other hand, for a.e. ω, take any g.e. $z \in \mathcal{I}$. Let $\Psi_\omega(m)$ be the corresponding g.e.f.. WLOG assume $\Psi_\omega(x) \neq 0$. Thus by Lemma A.0.2, we claim that for such x, ω, z and Ψ_ω, x is eventually $(\gamma(z)-2\varepsilon, n, \omega, z)$-singular. For if x is $(\gamma(z)-2\varepsilon, n, \omega, z)$-regular infinitely often, then $\Psi_\omega(x) = 0$.

Since x is eventually $(\gamma(z)-2\varepsilon, n, \omega, z)$-singular, $x+2n+1$ is $(\gamma(z)-2\varepsilon, n, \omega, z)$-regular. Thus $\Psi_\omega(x + 2n + 1)$ decays exponentially as $n \to \infty$. A similar argument applies to $\Psi_\omega(x+2n+2)$. Therefore, for a.e. ω, all of the g.e.f.'s $\Psi_\omega(n)$ decay exponentially. $\square$

Proof of Theorem 2.2 (EDL). By Theorem 2.1, for a.e. ω, there is an orthonormal basis $\{\Psi_{k,\omega}\}$ of eigenfunctions of $\mathcal{E}_\omega$. Denote the corresponding eigenvalues by $z_{k,\omega}$. Define the localization center as the left-most $c_{k,\omega} \in \mathbb{Z}$ s.t.

$$|\Psi_{k,\omega}(c_{k,\omega})| = \max_{n\in\mathbb{Z}} |\Psi_{k,\omega}(n)|.$$

We will employ the following lemma from [51] which provides a sufficient condition for EDL:

Lemma 2.3.2 ([51])**.** *If there are $\tilde{C} > 0$, $\tilde{\gamma} > 0$, s.t. for any $x, y \in \mathbb{Z}$*

$$\mathbb{E}\Big(\sum_{k: c_{k,\omega} = y} |\Psi_{k,\omega}(x)|^2\Big) \le \tilde{C} e^{-\tilde{\gamma}|x-y|}, \tag{2.4}$$

Then there are $C > 0$, $\gamma > 0$, s.t.

$$\sup_{t \in \mathbb{R}} \mathbb{E}(|\langle \delta_p, e^{it\mathcal{E}_\omega} \chi_I(\mathcal{E}_\omega)\rangle|) \le C(|x-y|+1)e^{-\gamma|x-y|}.$$

Thus, we only need to show (2.4). To do so, observe that for any ω, $c_{k,\omega}$ as a localization center is always $(\gamma(z) - 2\varepsilon, n, \omega, z)$-singular for those n s.t. $e^{-(\gamma-2\varepsilon)n} < \frac{1}{2}$. By Theorem 2.3, if $\omega \in \Omega_{y,n}$, where we take $n = \max\{N, \frac{\log(2)}{\gamma(z)-2\varepsilon}\} + 1$, then $|x - c_{k,\omega}| = n > N$ and thus x is $(\gamma(z) - 2\varepsilon, n, \omega, z)$-regular. Then we have

$$|\Psi_{k,\omega}(x)| \le 2|\Psi_{k,\omega}(y)|e^{-(\gamma(z)-2\varepsilon)(|x-y|)}, \quad \forall |x-y| > n, \omega \in \Omega_{y,n}.$$

Since $\Psi_{k,\omega}$ is an orthonormal basis, by Bessel's Inequality, we have

$$\sum_{k: c_{k,\omega}=y} |\Psi_{k,\omega}(x)|^2 \le 4 \sum_{k: c_{k,\omega}=y} |\Psi_{k,\omega}(y)|^2 e^{-(\nu-2\varepsilon)|x-y|} \le 4e^{-(\nu-2\varepsilon)|x-y|}.$$

Thus if $|x-y| > n$, we have

$$\begin{aligned}
\mathbb{E}\Big(\sum_{k: c_{k,\omega}=y} |\Psi_{k,\omega}(x)|^2\Big) \le & \int_{\Omega_{\Lambda_{2N}(y)}} \sum_{k: c_{k,\omega}=y} |\Psi_{k,\omega}(x)|^2 d\mathbb{P}(\omega) \\
& + \int_{(\Omega_{\Lambda_{2N}(y)})^c} \sum_{k: c_{k,\omega}=y} |\Psi_{k,\omega}(x)|^2 d\mathbb{P}(\omega) \\
\le & 1 * 4e^{-(\nu-2\varepsilon)|x-y|} + e^{-\eta|x-y|} * 1 \\
\le & 5e^{-\tilde{\gamma}|x-y|}
\end{aligned} \tag{2.5}$$

with $\tilde{\gamma} = \min\{\nu - 2\varepsilon, \eta\}$. If $|x - y| \leq n$, since $\Psi_{k,\omega}$ is orthonormal basis, we have

$$\sum_{k:c_{k,\omega}=y} |\Psi_{k,\omega}(x)|^2 \leq 1, \quad \text{thus} \quad \mathbb{E}(\sum_{k:c_{k,\omega}=y} |\Psi_{k,\omega}(x)|^2) \leq 1. \tag{2.6}$$

Since there are finitely many x's in the case when $|x - y| \leq n$, combining (2.5) and (2.6), we see that there is $\tilde{C}$, s.t. for any x

$$\sum_{k:c_{k,\omega}=y} |\Psi_{k,\omega}(x)|^2 \leq \tilde{C} e^{-\tilde{\gamma}|x-y|}$$

Having shown (2.4), EDL follows by Lemma 2.3.2. □

2.4 Uniform LDT Estimates and Uniform Craig-Simon Results

In this section, we introduce the uniform large-deviation-type estimates (uniform LDT) and uniform Craig-Simon results which are preliminary results needed for the proof of Theorem 2.3. We begin by connecting the Green's function with determinants of box-restrictions, transfer matrices and Lyapunov exponents.

2.4.1 Determinants with boundary conditions

Let

$$\begin{aligned} \mathcal{P}^{\beta,\gamma}_{[a,b],\omega,z} &:= \det(z - \mathcal{E}^{\beta,\gamma}_{[a,b],\omega}) = \det(A^{\beta,\gamma}_{[a,b]}), \\ P^{\beta,\gamma}_{[a,b],\omega,z} &:= (\rho_{a-1} \cdots \rho_b)^{-1} \mathcal{P}^{\beta,\gamma}_{[a,b],\omega,z}. \end{aligned} \tag{2.7}$$

If $a > b$, let $P^{\beta,\gamma}_{[a,b],\omega,z} = 1$. Note that although we have modified the boundary conditions in $\mathcal{P}^{\beta,\gamma}_{[a,b],\omega,z}$, we keep ρ_{a-1} and ρ_b unchanged in the second formula above. Moreiver, $P^{\beta,\cdot}_{[a,b],\omega,z}$ and $P^{\cdot,\gamma}_{[a,b],\omega,z}$ are defined similarly.

By Cramer's rule, we have

$$\begin{aligned} \left|G^{\beta,\gamma}_{[a,b],\omega,z}(x,y)\right| &= \frac{|\mathcal{P}^{\beta,\cdot}_{[a,x-1],\omega,z}\mathcal{P}^{\cdot,\gamma}_{[y+1,b],\omega,z}|}{\mathcal{P}^{\beta,\gamma}_{[a,b],\omega,z}} \prod_{k=x}^{y-1} \rho_k \\ &= \frac{\left|P^{\beta,\cdot}_{[a,x-1],\omega,z}P^{\cdot,\gamma}_{[y+1,b],\omega,z}\right|}{\left|P^{\beta,\gamma}_{[a,b],\omega,z}\right|}, \quad a \le x \le y \le b \end{aligned} \tag{2.8}$$

2.4.2 Transfer Matrix and Lyapunov Exponents

Recall by Theorem 2.2.1,

$$\begin{pmatrix} \phi_{n+1}(z) \\ \phi^*_{n+1}(z) \end{pmatrix} = \frac{1}{\rho_n} \begin{pmatrix} z & -\overline{\alpha_n} \\ -\alpha_n z & 1 \end{pmatrix} \begin{pmatrix} \phi_n(z) \\ \phi_n(z)^* \end{pmatrix}$$

Denote

$$S_z(\alpha) = \frac{1}{\rho_\alpha} \begin{pmatrix} z & -\overline{\alpha} \\ -\alpha z & 1 \end{pmatrix},$$

Let $T_{[a,b]} = S_z(\alpha_b) \cdots S_z(\alpha_a)$ be the transfer matrix, then

$$\begin{pmatrix} \phi_{b+1}(z) \\ \phi^*_{b+1}(z) \end{pmatrix} = T_{[a,b]} \begin{pmatrix} \phi_a(z) \\ \phi^*_a(z) \end{pmatrix}.$$

By [85, Theorem 1] together with Remark 2.3, we have:

$$T_{[a,b]} = \frac{1}{\rho_a \cdots \rho_b} \begin{bmatrix} z\mathcal{P}_{[a+1,b],z} & \mathcal{P}^{-1,\cdot}_{[a,b],z} - z\mathcal{P}_{[a+1,b],z} \\ z(\mathcal{P}^{-1,\cdot}_{[a,b],z} - z\mathcal{P}_{[a+1,b],z})^* & (\mathcal{P}_{[a+1,b],z})^* \end{bmatrix}$$

or

$$T_{[a,b]} = \begin{bmatrix} zP_{[a+1,b],z} & \rho_{a-1}P^{-1,\cdot}_{[a,b],z} - zP_{[a+1,b],z} \\ z(\rho_{a-1}P^{-1,\cdot}_{[a,b],z} - zP_{[a+1,b],z})^* & (P_{[a+1,b],z})^* \end{bmatrix} \tag{2.9}$$

where $Q(z)^* = z^n\overline{Q(1/\bar{z})}$ if $Q(z)$ is a polynomial of degree n.

Note that $\frac{1}{\sqrt{z}}S_z(\alpha) \in SU(1,1)$, where

$$SU(1,1) = \left\{ \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} : u, v \in \mathbb{C}, |u|^2 - |v|^2 = 1 \right\},$$

and for $\sqrt{z}$, we take the principal branch. Note also that $SU(1,1) = Q^{-1} \cdot SL(2,\mathbb{R}) \cdot Q$, where $Q = -\frac{1}{2}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$. Thus, the definitions of Lyapunov exponents for $SL(2,\mathbb{R})$-cocycles and the corresponding properties (positivity and continuity, large deviation and subharmonicity results) generalize to $SU(1,1)$-cocycles. Moreover, $\|S_z(\alpha)\| = \|\frac{1}{\sqrt{z}}S_z(\alpha)\|$ when $|z| = 1$. Thus, when the α_n's are i.i.d., by Kingman's subadditive theorem ([55]), the Lyapunov exponent $\gamma(z)$ is well-defined:

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{n}\int_0^1 \log \|T_{[0,n],\omega,z}\| d\mathbb{P}(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|T_{[0,n],\omega,z}\|, \quad \text{a.e. } \omega.$$

2.4.3 Positivity and Continuity of Lyapunov Exponent

By Fürstenberg Theorem, random Schödinger operators have positive Lyapunov exponent: $\gamma(z) > 0$ for any $z \in \mathbb{R}$. At the same time, random CMV matrices may have an exceptional set $\mathcal{D} \subset \partial\mathbb{D}$ which contains at most three points s.t. $\gamma(E) > 0$ on $\partial\mathbb{D} \setminus \mathcal{D}$. In fact, depending on the support of μ, either $\mathcal{D} = \emptyset$ or $\mathcal{D} = \{1, -1\}$ or $\mathcal{D} = \{1, \theta_0, \overline{\theta_0}\}$, for some $\theta_0 \in \partial\mathbb{D}$. The reason is, roughly speaking, the positivity get destroyed when $S_z(\alpha_i)$ and $S_z(\alpha_j)$ have a common invariant measure. This would happen only if α_i, α_j and z satisfy certain algebraic conditions which characterize the exceptional set. See [74, Theorem 12.6.3. and 10.4.18] for more details.

Continuity of Lyapunov exponents on $\mathcal{I} \subset \partial\mathbb{D} \setminus \mathcal{D}$ can be proved using the general method (e.g. [22, Sec.V.4.2], [14]) originally developed by Fürstenberg and Kifer [40, Theorem B] for self-adjoint random matrices, which by conjugation, extend to $SU(1,1)$ random matrices naturally. We also refer to [18, Sec.2], [46, Sec. 7] for a review of the proof of continuity of Lyapunov exponents.

We fix an interval $\mathcal{I} \subset \partial\mathbb{D} \setminus \mathcal{D}$ from now on and denote $\nu := \inf_{z \in \mathcal{I}} \gamma(z) > 0$.

2.4.4 Uniform Large-deviation-type estimates

We can now introduce the uniform large-deviation-type estimates, a crutial component of the proof of Theorem 2.3. These LDT type estimates for $\|T_{[a,b],\omega,z}\|$ were proved in [61]. Here we use a matrix-entry version from [81, Theorem 5]. The result was proved for $SL(2,\mathbb{R})$-cocycle. Here by conjugation, we rewrite it for our $SU(1,1)$-cocycle $T_{[a,b],\omega,z}$. So under the same assumption for positivity and continuity of Lyapunov exponent, which, in particular, holds for any compact interval $\mathcal{I} \subset \partial\mathbb{D}$, we have the following lemma:

Lemma 2.4.1 ("uniform-LDT"). *Given a compact interval $\mathcal{I} \subset \partial\mathbb{D} \setminus \mathcal{D}$. For any $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon, \mathcal{I})$, $N = N(\varepsilon, \mathcal{I}) > 0$, such that*

$$\mathbb{P}\left\{\omega : \left|\frac{1}{b-a+1}\log|\langle T_{[a,b],\omega,z}u, v\rangle| - \gamma(z)\right| \geq \varepsilon\right\} \leq e^{-\eta(b-a+1)}$$

for any $b - a > N$, for any unit vector u, v and any $z \in \mathcal{I}$.

Thus for our model, we have

Lemma 2.4.2. *Given a compact interval $\mathcal{I} \subset \partial\mathbb{D} \setminus \mathcal{D}$. For any $\varepsilon > 0$, there is an $\tilde{\eta} = \tilde{\eta}(\varepsilon, \mathcal{I}), \tilde{N}_1 = \tilde{N}_1(\varepsilon, \mathcal{I}) > 0$ s.t.*

$$\mathbb{P}\left\{\omega : \left|\frac{1}{b-a+1}\log|P^{-1,\cdot}_{[a,b],\omega,z}| - \gamma(z)\right| \geq \varepsilon\right\} \leq e^{-\eta(b-a+1)} \tag{2.10}$$

$$\mathbb{P}\left\{\omega : \left|\frac{1}{b-a+1}\log|P^{\cdot,1}_{[a,b],\omega,z}| - \gamma(z)\right| \geq \varepsilon\right\} \leq e^{-\eta(b-a+1)} \tag{2.11}$$

$$\mathbb{P}\left\{\omega : \left|\frac{1}{b-a+1}\log|P^{-1,1}_{[a,b],\omega,z}| - \gamma(z)\right| \geq \varepsilon\right\} \leq e^{-\eta(b-a+1)} \tag{2.12}$$

for every $b - a > \tilde{N}_0$, and any $z \in \mathcal{I}$.

Proof. First recall that α_k is supported on a compact subset of $\mathbb{D}$, $\rho_k = \sqrt{1 - |\alpha_k|^2}$. Thus there is $\delta > 0$ s.t.

$$\begin{cases} |\alpha_k| \leq 1 - \delta < 1, \\ 0 < \delta \leq |\rho_k| \leq 1 - \delta < 1. \end{cases} \tag{2.13}$$

(2.10), (2.11) and (2.12) above each require seperate consideraions.

To prove (2.10), let $u = (1, 0)^T$, $v = (1, 1)^T$ in Lemma 2.4.1. By (2.9), the $zP_{[a+1,b],z}$

term cancels and we get

$$\mathbb{P}\left\{\omega : \left|\frac{1}{b-a+1}\log|\rho_{a-1}P^{-1,\cdot}_{[a,b],\omega,z}| - \gamma(z)\right| \geq \varepsilon\right\} \leq e^{-\eta(b-a+1)}.$$

By (2.13), ρ_{a-1} can be absorbed by large enough $b-a+1$ and a modified ε.

As for (2.11), the inequality follows from (2.10) by setting $\widetilde{\alpha}_j = -\overline{\alpha}_{a+b-1-j}$ for $a-1 \leq j \leq b$ and observing that $P^{\cdot,1}_{[a,b],z}(\widetilde{\alpha}_j) = P^{-1,\cdot}_{[a,b],z}(\alpha_j)$.

Lastly, to prove (2.12), notice that by (2.7) and Lemma A.0.1, we have

$$\left|P^{-1,1}_{[a,b],z}\right| = \frac{\left|\det(A^{-1,1}_{[a,b],z})\right|}{\rho_{a-1}\cdots\rho_b} = \frac{\left|A^{\alpha_b=1}_{b,b}\mathcal{P}^{-1,\cdot}_{[a,b-1]} - A_{b,b-1}\prod_{n=a}^{b-1}A_{n,n+1}\right|}{\rho_{a-1}\cdots\rho_b}.$$

And by Lemma A.0.1 and (2.13), we have $\delta < |A^{\alpha_b=1}_{b,b}| < 2$. Thus by Lemma A.0.1, the first and second terms are bounded respectively by

$$\frac{\delta\left|P^{-1,\cdot}_{[a,b-1]}\right|}{\delta} \leq \frac{\left|A^{\alpha_b=1}_{b,b}\mathcal{P}^{-1,\cdot}_{[a,b-1]}\right|}{\rho_{a-1}\cdots\rho_b} = \frac{\left|A^{\alpha_b=1}_{b,b}P^{-1,\cdot}_{[a,b-1]}\right|}{\rho_b} \leq \frac{2\left|P^{-1,\cdot}_{[a,b-1]}\right|}{\delta}$$

$$\delta \leq \frac{\left|A_{b,b-1}\prod_{n=a}^{b-1}A_{n,n+1}\right|}{\rho_{a-1}\cdots\rho_b} = \frac{\rho_{b-1}\prod_{n=a}^{b-1}\rho_n}{\rho_{a-1}\cdots\rho_b} = \frac{\rho_{b-1}}{\rho_{a-1}\rho_b} \leq \frac{1}{\delta^2}.$$

Hence,

$$\left|P^{-1,\cdot}_{[a,b-1]}\right| - \frac{1}{\delta^2} \leq \left|P^{-1,1}_{[a,b-1]}\right| \leq \frac{2}{\delta}\left|P^{-1,\cdot}_{[a,b-1]}\right| + \delta.$$

and the third inequality follows. □

2.4.5 "Bad sets" and Singularity

To simplify the notation, we introduce "bad sets" and use them to characterize "singularity" in Defintion 2.4. Denote

$$B^{\beta,\gamma,+}_{[a,b],\varepsilon} = \left\{(\omega, z) \in \mathcal{I} \times \Omega : |P^{\beta,\gamma}_{[a,b],\omega,z}| \geq e^{(\gamma(z)+\varepsilon)(b-a+1)}\right\}$$
$$B^{\beta,\gamma,-}_{[a,b],\varepsilon,} = \left\{(\omega, z) \in \mathcal{I} \times \Omega : |P^{\beta,\gamma}_{[a,b],\omega,z}| \leq e^{(\gamma(z)-\varepsilon)(b-a+1)}\right\}$$

Let $B^{\beta,\gamma,\pm}_{[a,b],\varepsilon,z}$ and $B^{\beta,\gamma,\pm}_{[a,b],\varepsilon,\omega}$ be the z and ω sections of $B^{\beta,\gamma,\pm}_{[a,b],\varepsilon}$. Let $B^{\beta,\gamma}_{[a,b],\varepsilon,*} = B^{\beta,\gamma,+}_{[a,b],\varepsilon,*} \cup B^{\beta,\gamma,-}_{[a,b],\varepsilon,*}$. All of these sets have corresponding definitions for the single-sided boundary case. Thus, (2.10), (2.11), (2.12) can be rewritten as

$$\mathbb{P}(B^{*}_{[a,b],\varepsilon,z}) \leq e^{-\eta(b-a+1)}$$

where $*$ can be any of the three kinds of boundary conditions β, γ or $\beta, \cdot$, or $\cdot, \gamma$.

We can characterize singular points using the bad sets:

Lemma 2.4.3. *For any $\varepsilon < \nu/2$, for $n \geq 2$, if x is $(\gamma(z) - 2\varepsilon, n, \omega, z)$-singular, then*

$$(\omega, z) \in B^{\beta,\gamma,-}_{[x-n,x+n],\varepsilon} \cup B^{\beta,\cdot,+}_{[x-n,x-1],\varepsilon} \cup B^{\cdot,\gamma,+}_{[x+1,x+n],\varepsilon}$$

Proof. The result follows imediately from the definition of singularity and (2.8). □

2.4.6 Uniform Craig-Simon results

We will also use a uniform version of Craig-Simon's results. The Craig-Simon estimates [25] are a general subharmonicity upper bound estimate. It is extended in [52, Theorem

5.1] to the uniform version. See [52, Section 5] for more details.

Lemma 2.4.4 (Uniform Craig-Simon)**.** *Let $\mathcal{E}_\omega$ satisfy the uniform-LDT condition in Lemma 2.4.1. Then for any ε, there is $\tilde{\eta} = \tilde{\eta}(\varepsilon), \tilde{N}_2 = \tilde{N}_2(\varepsilon)$ s.t. for any $x \in \mathbb{Z}$, $n > N_1$, there is $\tilde{\Omega}_{x,n}$ s.t.*

1. $\mathbb{P}(\tilde{\Omega}_{x,n}) \geq 1 - ne^{-\eta(n+1)}$,
2. *for any $\omega \in \tilde{\Omega}_{x,n}$, we have for every $z \in \mathcal{I}$.*

$$\max\{|P^{\beta,\cdot}_{[x+1,x+n],z}|, |P^{\cdot,\gamma}_{[x-n,x-1],z}|\} \leq e^{(\gamma(z)+\varepsilon)(n+1)}, \quad i.e.$$

$$(\omega, z) \notin B^{\beta,\cdot,+}_{[x+1,x+n],\varepsilon} \cup B^{\cdot,\gamma,+}_{[x-n,x-1],\varepsilon}.$$

Proof. The deterministic result is a direct reformulation of [52, Theorem 5.1], while the probabilistic results can be extracted from the last line on Page 9 in [52]. □

Remark 2.4. We mention in particular that $\tilde{\eta}$ in Lemma 2.4.2 and Lemma 2.4.4 for the same ε are the same. In fact, the $\tilde{\eta}$ in Lemma 2.4.4 comes from applying 2.4.2. (See [52]).

2.5 Proof of Theorem 2.3

We will prove Theorem 2.3 in this section. Heuristically, Theorem 2.3 says, with high probability, one of two points will be regular if they are far enough from each other. The idea is that with high probability, if x is a $(\gamma(z) - 2\varepsilon, n, \omega, z)$-singular point, then z will be exponentially close to $\sigma(\mathcal{E}_{[x-n,x+n],\omega})$. We will denote this set by $\Omega^{(1)}_{x,n}$. So, if we have two far away singular points x, y, then $\sigma(\mathcal{E}_{[x-n,x+n],\omega})$ and $\sigma(\mathcal{E}_{[x-n,x+n],\omega})$

are also exponentially close to the same z. However, we can also show that with high probability $\sigma(\mathcal{E}_{[x-n,x+n],\omega})$ and $\sigma(\mathcal{E}_{[x-n,x+n],\omega})$ cannot be exponentially close. We will denote this set by $\Omega^{(2)}_{x,n}$. Then $\Omega^{(1)}_{x,n}\cap\Omega^{(2)}_{x,n}$ will be the set of high probability where one of these two points must be regular.

For convenience, we will omit ω, z from the subscript of $T_{[a,b],\omega,z}$, $P^*_{[a,b],\omega,z}$, $G^*_{[a,b],\omega,z}$ and $A_{[a,b],\omega,z}$ in this section unless it is necessary.

2.5.1 The first set $\Omega^{(1)}_{x,n}$

As we mentioned above, we choose $\Omega^{(1)}_{x,n}$ s.t. singularity implies exponential closeness to the spectrum:

Lemma 2.5.1. *For any $0<\varepsilon<\nu$, there are $\eta_1=\eta_1(\varepsilon)$, $N_1=N_1(\varepsilon_1)$ s.t. for any $n>N_1$, $x\in\mathbb{Z}$, $0<\delta<\eta_1$, there is $\Omega^{(1)}_{x,n}=\Omega^{(1)}_{x,n}(\delta)$, s.t.*

1. $\mathbb{P}(\Omega^{(1)}_{x,n})\geq 1-m(\mathcal{I})e^{-(\eta_1-\delta)(2n+1)}-ne^{-\eta_1(2n+1)}$,
2. *For $\omega\in\Omega^{(1)}_{x,n}$, if x is $(\gamma(z)-2\varepsilon,n,\omega,z)$-singular, then*

$$\operatorname{dist}(z,\sigma(\mathcal{E}_{[x-n,x+n],\omega}))\leq e^{-\delta(2n+1)}.$$

Proof. Fix any $0<\varepsilon<\nu/2$. Let $\tilde{\eta}(\varepsilon)$, $\tilde{N}_1(\varepsilon)$ be as in Lemma 2.4.2. Let $\tilde{N}_2(\varepsilon)$, $\tilde{\Omega}_{x,n}$ be as in Lemma 2.4.4. Then let $\eta:=\tilde{\eta}$, $N:=\max\{N_1,N_2\}$, and

$$\Omega^{(1)}_{x,n}:=\left\{\omega: m(B^{\beta,\gamma,-}_{[x-n,x+n],\omega})\leq e^{-\delta_1(2n+1)}\right\}\cap\widetilde{\Omega}_{x,n}. \tag{2.14}$$

By Chebyshev's Inequality and Fubini's Theorem, we obtain part (1):

$$\mathbb{P}\left((\Omega^{(1)}_{x,n})^c\right) \leq m \times \mathbb{P}\left\{(\omega, z) : (\omega, z) \in B^{\beta,\gamma,-}_{[x-n,x+n]}, z \in \mathcal{I})\right\} + \mathbb{P}(\widetilde{\Omega}_{x,n})$$
$$\leq m(\mathcal{I})e^{-(\eta_1-\delta)(2n+1)} + ne^{-\eta_1(2n+1)}.$$

Now for part (2), take any $\omega \in \Omega^{(1)}_{x,n}$, and any $(\gamma(z) - 2\varepsilon, n, \omega, z)$-singular point x. By Lemma 2.4.3,

$$(\omega, z) \in B^{\beta,\gamma,-}_{[x-n,x+n],\varepsilon} \cup B^{\beta,\cdot,+}_{[x-n,x-1],\varepsilon} \cup B^{\cdot,\gamma,+}_{[x+1,x+n],\varepsilon},$$

However, since $\omega \in \widetilde{\Omega}_{x,n}$, by Lemma 2.4.4,

$$(\omega, z) \notin B^{\beta,\cdot,+}_{[x-n,x-1],\varepsilon} \cup B^{\cdot,\gamma,+}_{[x+1,x+n],\varepsilon}.$$

We see that $(\omega, z) \in B^{\beta,\gamma,-}_{[x-n,x+n],\varepsilon}$. Thus

$$z \in B^{\beta,\gamma,-}_{[x-n,x+n],\varepsilon,\omega} \quad \text{with} \quad m(B^{\beta,\gamma,-}_{[x-n,x+n],\varepsilon,\omega}) \leq e^{-\delta(2n+1)},$$

where the latter is due to (2.14). Notice further that

$$B^{\beta,\gamma,-}_{[x-n,x+n],\varepsilon,\omega} = \{z : |P^{\beta,\gamma}_{[x-n,x+n],\omega,z}| \leq e^{(\gamma(z)-\varepsilon)(2n+1))}\}$$

where for each ω, $|P^{\beta,\gamma}_{[x-n,x+n],\omega,z}|$ is a polynomial in z with roots $\sigma(\mathcal{E}^{\beta,\gamma}_{[x-n,x+n],\omega})$. Thus $B^{\beta,\gamma,-}_{[x-n,x+n],\varepsilon,\omega}$ is a finite union of intervals, each centered around points of $\sigma(\mathcal{E}^{\beta,\gamma}_{[x-n,x+n],\omega})$, of overall length less than $e^{-\delta(2n+1)}$. Thus,

$$\text{dist}(z, \sigma(\mathcal{E}^{\beta,\gamma}_{[x-n,x+n],\omega})) \leq e^{-\delta(2n+1)}.$$

□

2.5.2 The second set $\Omega^{(2)}_{x,n}$

As mentioned above, the aim of choosing $\Omega^{(2)}_{x,n}$ is to make sure $\sigma(\mathcal{E}^{\beta,\gamma}_{[x-n,x+n],\omega})$ and $\sigma(\mathcal{E}^{\beta,\gamma}_{[x+n+1,x+3n+1],\omega})$ are not exponentially close for $\omega \in \Omega^{(2)}_{x,n}$.

Lemma 2.5.2. *For any $\delta > 0$, there is $\eta_2(\delta)$, $N_2(\delta)$ s.t. for any $n > N_2$, $x \in \mathbb{Z}$, there is $\Omega^{(2)}_{x,n}$, s.t.*

1. $\mathbb{P}(\Omega^{(2)}_{x,n}) \geq 1 - 2(2n+2)^3 e^{-\eta_2(2n+1)}$,

2. *If $\omega \in \Omega^{(2)}_{x,n}$, then*

$$\operatorname{dist}\left(\sigma(\mathcal{E}^{\beta,\gamma}_{[x-n,x+n],\omega}), \sigma(\mathcal{E}^{\beta,\gamma}_{[x+n+1,x+3n+1],\omega})\right) \geq 2e^{-\delta(2n+1)}$$

Proof. Since each entry in $\mathcal{E}$ is bounded, there is M s.t.

$$|P_{[a,b],z}| \leq M^{b-a+1}, \quad \forall a \leq b \in \mathbb{Z}, \forall z \in \mathcal{I}.$$

Choose $\varepsilon' < \delta/2$. Apply Lemma 2.4.2 to get $\tilde{\eta}(\varepsilon')$, $\tilde{N}_1(\varepsilon')$. Choose $K \geq \frac{2\log M}{\delta - 2\varepsilon'}$. Let $\eta_2 := \frac{\tilde{\eta}}{2K}$, $\tilde{N}_2 := K\tilde{N}_1$ and

$$(\Omega^{(2)}_{x,n})^c := \bigcup_{z_i \in Z(\omega)} \bigcup_{(y_1,y_2)\in Y} \left(B^{\beta,\cdot}_{[x-n,x+y_1-1],\varepsilon',z_i} \cup B^{\cdot,\gamma}_{[x+y_2+1,x+n],\varepsilon',z_i}\right) \cup B^{\beta,\gamma}_{[x-n,x+n],\varepsilon',z_i}$$

where

$$Y = \{(y_1,y_2) : x-n \leq y_1 \leq y_2 \leq x+n, |y_1 - (-n)|, |n - y_2| \geq \frac{n}{K}\},$$

$$Z = Z(\omega) = Z(\omega_{[x+n+1,x+3n+1]}) = \sigma(\mathcal{E}_{[x+n+1,x+3n+1],\omega}).$$

We remark here that while $z_i(\omega)$ and $Z(\omega)$ depend on ω, they actually only depend on $\Omega_{[x+n+1,x+3n+1]}$ which is independent from $\Omega_{[x-n,x+n]}$. Thus $z_i = z_i(\omega) = z_i(\omega_{[x+n+1,x+3n+1]})$ in $B^{\beta,\gamma}_{[x-n,x+n],z_i}$ operates like any other fixed z that does not depend on ω. A rigorous argument is as follows:

For any fixed $\omega_c, \cdots, \omega_d$, with $[c,d] \cap [a,b] = \emptyset$, assume $d-c, b-a \geq \tilde{N}_1$. By independence,

$$\mathbb{P}_{[c,d]^c}(B^*_{[a,b],\varepsilon',z_{i,(\omega_c,\cdots,\omega_d)}}) = \mathbb{P}_{[a,b]}(B^*_{[a,b],\varepsilon',z_{i,(\omega_c,\cdots,\omega_d)}}) \leq e^{-\eta_2(b-a+1)}$$

where $*$ represents corresponding boundary conditions, $z_{i,(\omega_c,\cdots,\omega_d)} \in \sigma(\mathcal{E}_{[c,d]})$. Applying to $[a,b] = [x-n, x+y_1-1]$ or $[x+y_2+1, x+n]$ or $[x-n, x+n]$, $[c,d] = [x+n+1, x+3n+1]$ and integrating over $\omega_a, \cdots, \omega_b$, we obtain for $n \geq \tilde{N}_2$,

$$\mathbb{P}(B^{\beta,\cdot}_{[x-n,x+y_1-1],\varepsilon',z_i} \cup B^{\cdot,\gamma}_{[x+y_2+1,x+n],\varepsilon',z_i}) \leq 2e^{-\eta_2(\frac{n}{K}+1)},$$
$$\mathbb{P}(B^{\beta,\gamma}_{[x-n,x+n],\varepsilon',z_i}) \leq e^{-\eta_2(2n+1)}.$$

Thus we obtain part (1):

$$\mathbb{P}(\Omega^{(2)}_{x,n}) \geq 1 - (2n+1)((2n+1)^2+1)2e^{-\eta_2\frac{n}{K}} \geq 1 - 2(2n+2)^3 e^{-\eta_2\frac{n}{K}}$$

We prove part (2) by contradiction. Let $\omega \in \Omega^{(2)}_{x,n}$, we assume that there are $z_i \in \sigma(\mathcal{E}_{[x+n+1,x+3n+1]})$, $z_j \in \sigma(\mathcal{E}_{[x-n,x+n]})$ s.t.

$$|z_i - z_j| \leq 2e^{-\delta(2n+1)}.$$

Then

$$\|G^{\beta,\gamma}_{[x-n,x+n],\omega,z_i}\| \geq \frac{1}{2}e^{\delta(2n+1)}.$$

Thus there are $x-n \le y_1 \le y_2 \le x+n$ s.t.

$$\frac{|P_{[x-n,x+y_1-1],\omega,z_i} P_{[x+y_2+1,x+n],\omega,z_i}|}{|P_{[x-n,x+n],\omega,z_i}|} = |G^{\beta,\gamma}_{[x-n,x+n],\omega,z_i}(y_1,y_2)| \ge \frac{1}{2n} e^{\delta(2n+1)}.$$

There are three cases, and we claim that each leads to a contradiction.

1. If $|y_1-(-n)| \ge \frac{n}{K}$, $|n-y_2| \ge \frac{n}{K}$, since

$$\omega \notin B^{\beta,\cdot}_{[x-n,x+y_1-1],\varepsilon',z_i} \cup B^{\cdot,\gamma}_{[x+y_2+1,x+n],\varepsilon',z_i} \cup B^{\beta,\gamma}_{[x-n,x+n],\varepsilon',z_i},$$

   if $K>1$, we have

$$\frac{1}{2n} e^{\delta(2n+1)} \le e^{(\gamma(z_i)+\varepsilon')\frac{2n}{K} - (\gamma(z_i)-\varepsilon')(2n+1)} \le e^{(2n+1)(2\varepsilon')}.$$

   But $\delta > 2\varepsilon'$. Thus when n is large enough, say, $n > \tilde{N}_3$, there will be a contradiction.

2. If one of $|y_1-(-n)|$ and $|n-y_2| \ge \frac{n}{K}$, then if $K>1$, we have

$$\frac{1}{2n} e^{\delta(2n+1)} \le M^{\frac{n}{K}} e^{n\frac{(\gamma(z_i)+\varepsilon')}{K} - (\gamma(z_i)-\varepsilon')(2n+1)} \le e^{(2n+1)(\frac{\log M}{2K}+2\varepsilon')}$$

   By our choice of $K \ge \frac{2\log M}{\delta-2\varepsilon'}$, we have $\delta > \frac{\log M}{2K} + 2\varepsilon'$. Thus again, when n is large enough, say, $n > \tilde{N}_4$, we arrive at a contradiction.

3. If both $|y_1-(-n)| \le \frac{n}{K}$, $|n-y_2| \le \frac{n}{K}$, then

$$\frac{1}{2n} e^{\delta(2n+1)} \le M^{\frac{2n}{K}} e^{-(\gamma(z_i)-\varepsilon')(2n+1)} \le e^{(2n+1)(\frac{\log M}{2K}+\varepsilon')}$$

   By our choice of K, we have $\delta > \frac{\log M}{2K} + \varepsilon'$. Thus when n is large enough, say,

$n > \tilde{N}_5$, again we arrive at a contradiction.

Take $N_2 = \max\{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4, \tilde{N}_5\}$. Then for any $n > N_2$, we have a contradiction for all three cases, and hence

$$\text{dist}\left(\sigma(\mathcal{E}^{\beta,\gamma}_{[x-n,x+n],\omega}), \sigma(\mathcal{E}^{\beta,\gamma}_{[x+n+1,x+3n+1],\omega})\right) \geq 2e^{-\delta(2n+1)}$$

□

We now prove Theorem 2.3:

Proof of Theorem 2.3. By Lemma 2.5.1, for any $\varepsilon > 0$, we can find $\eta_1(\varepsilon), N_1(\varepsilon)$ and $\delta = \eta_1/2$, s.t. (1) and (2) of Lemma 2.5.1 hold. For such δ, apply Lemma 2.5.2 to find η_2, N_2 and $\Omega^{(2)}_{x,n}$ for any $x \in \mathbb{Z}$, $n > N_2$. Now let $\eta := \min\{\eta_1, \eta_2/2\}$, $N := \max\{N_1, N_2\}$. Set

$$\Omega_{x,n} := \Omega^{(1)}_{x,n} \cap \Omega^{(1)}_{x+2n+1,n} \cap \Omega^{(2)}_{x,n}.$$

Then we obtain part (1):

$$\begin{aligned}\mathbb{P}(\Omega_{x,n}) &\geq 1 - 2m(\mathcal{I})e^{-\eta_1(2n+1)/2} - 2ne^{-\eta_1(2n+1)} - 2(2n+2)^3 e^{-\eta_2(2n+1)} \\ &\geq 1 - Ce^{-\eta(2n+1)}.\end{aligned}$$

As for part (2), let $\omega \in \Omega_{x,n}$. Assume both x and $x+2n+1$ are $(\gamma(z_i) - 2\varepsilon, n, \omega, z)$-singular. Then by Lemma 2.5.1, we have

$$\text{dist}(z, \sigma(\mathcal{E}_{[x-n,x+n],\omega})) \leq e^{-\delta(2n+1)},$$

$$\text{dist}(z, \sigma(\mathcal{E}_{[x+n+1,x+3x+1],\omega})) \leq e^{-\delta(2n+1)}.$$

Thus

$$\operatorname{dist}\left(\sigma(\mathcal{E}^{\beta,\gamma}_{[x-n,x+n],\omega}), \sigma(\mathcal{E}^{\beta,\gamma}_{[x+n+1,x+3n+1],\omega})\right) \le 2e^{-\delta(2n+1)}.$$

However, Lemma 2.5.2 guarantees that if $\omega \in \Omega_{x,n}$, then

$$\operatorname{dist}\left(\sigma(\mathcal{E}^{\beta,\gamma}_{[x-n,x+n],\omega}), \sigma(\mathcal{E}^{\beta,\gamma}_{[x+n+1,x+3n+1],\omega})\right) > 2e^{-\delta(2n+1)}.$$

which is a contradiction. Thus at least one of the two points x or $x + 2n + 1$ must be regular. □

Chapter 3

Dynamical Localization for the Singular Anderson Model in $\mathbb{Z}^d$

3.1 Introduction

We consider the d-dimensional Anderson model, a random Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ given by:

$$(H_\omega\phi)(n) := \sum_{|m-n|=1} (\phi(m) - \phi(n)) + V_\omega(n)\phi(n).$$

Here, the $V_\omega(n)$ are independent and identically distributed ($i.i.d.$) real-valued random variables with common distribution μ, $\forall n \in \mathbb{Z}^d$. We will assume that $S \subset \mathbb{R}$, the topological support of μ, is compact and contains at least two points. The underlying probability space here is $\Omega = S^{\mathbb{Z}^d}$, with elements $\{\omega_n\}_{n\in\mathbb{Z}^d} \in \Omega$. The probability measure $\mu^{\mathbb{Z}}$ will be denoted by $\mathbb{P}$ and its restriction to subsets $\Omega_\Lambda = S^{\Lambda\cap\mathbb{Z}^d}$ (where

$\Lambda \subset \mathbb{Z}^d$) by $\mathbb{P}_\Lambda$.

This paper will provide a comprehensive, self-contained proof from the end of an appropriate multi-scale analysis (MSA) result to Anderson and (strong) dynamical localization.

In order to properly contextualize this paper and detail exactly what it provides, it is necessary to briefly describe some chronology and background related to the results alluded to above. Soon after the MSA had taken a firm foothold in the literature and community, Germinet and Klein provided an axiomatic treatment of conditions needed in an MSA to obtain dynamical results. Previous work by the same authors showed that these axioms held in the so-called regular potential regime and thus provided a proof of dynamical localization in such situations. Bourgain and Koenig then used the multiscale framework to prove localization (again at the bottom of the spectrum) in the continuum Bernoulli case. It is worth noting that this paper represented a significant leap in technique and process demonstrated by their use of unique continuation principles as well as the Peierls argument. One unfortunate consequence of the MSA in the singular potential regime is the weaker than usual probability estimates that result. In the presence of regularity, it is possible to use the same probability estimates for spectral localization coupled with a short Borel-Cantelli argument to quickly obtain dynamical results. While the above argument does not apply in the singular regime, Klein and Germinet nevertheless proved that dynamical localization continues to hold.

Recently, inspired by a unique continuation principle developed for the $\mathbb{Z}^2$ lattice (Malinkova et. Al), Smart and Ding obtained the ingredients for a multiscale analysis and the corresponding spectral results. This work was then extended to the $\mathbb{Z}^3$ lattice by Li and Zhang. These results represent the analog of Bourgain and Koenig's work in the continuum. It is thus expected that dynamical localization and the various other consequences obtained by Germinet and Klein should hold here. Again, the issue is that Bernoulli (and more generally singular) potentials imply very weak probability estimates and additional considerations are needed to obtain dynamical localization. This paper provides these additional steps.

While the techniques presented here closely follow the work of Germinet and Klein, there are some features worth mentioning. Firstly, the treatment by Germinet and Klein is carried out in the continuum; thus, there are technical issues that can be avoided as well as estimates that can be simplified and improved in the discrete setting. In this regard, we are able to improve certain probability estimates by using the fact that points on the lattice are at least a unit distance from each other. While these improved estimates eventually get eroded by other requirements of the MSA, it is our hope that such considerations prove useful in other contexts, especially when translating results from the continuum to discrete setting. Secondly, unique continuation in the discrete setting is even weaker than in the continuum and this is reflected in the even weaker probability estimates that are used as a starting point for the MSA. As such, we provide the necessary modifications needed to obtain dynamical localization under these very weak conditions in a general ($\mathbb{Z}^d$) lattice setting. This not only provides a proof of dynamical localization (at the bottom of the spectrum) in $\mathbb{Z}^2$ and $\mathbb{Z}^3$, but also hopefully provides a simple reference in the event corresponding initial estimates (e.g. MSA inputs) are obtained in dimension $d > 3$.

The remainder of this paper is organized as follows:

- Section 3.2 contains the preliminary definitions and notations needed to state the main results,

- Section 3.3 summarizes properties of the generalized eigenfunction expansion and the main concepts, (3.2), "normalized generalized eigenfunction" used to prove localization.

- Section 3.4 provides two main Lemmas needed to pass from the MSA to the precursor for localization.

- Section 3.5 provides the two spectral reductions needed for the proof of the key Theorem 3.3.

- Section 3.6 contains the proof of dynamical localization in expectation.

3.2 Preliminaries and Main results

3.2.1 Preliminaries

We first provide the necessary notations and definitions needed to state the main theorem.

For $x \in \mathbb{Z}^d$, let $|x| = \max_{i=1}^{d} |x_i|$. Let $\langle x \rangle = (1 + |x|^2)^{1/2}$. Fix $\nu > d$ for the rest of the paper and denote $T_{x_0}\phi(x) = \langle x - x_0 \rangle^{\nu} \phi(x)$, where $x_0 \in \mathbb{Z}^d$. When $x_0 = 0$, we will simply write T_ν. Let $||\phi|| = \sqrt{\sum_{n \in \mathbb{Z}^d} \phi(n)^2}$ denote the ℓ^2 norm of ϕ.

For an operator $A : \ell^2 \to \ell^2$, let $\|A\|$ denote the operator norm. Let $\|A\|_p = Tr(|A|^p)^{1/p}$ denote the Schattern norm. In particular, $\|A\|_1$ and $\|A\|_2$ denote the trace norm and Hilbert-Schmidt norm respectively.

For $x \in \mathbb{Z}^d$, let $\Lambda_L(x) = \{y \in \mathbb{Z}^d : |y - x| < L/2\}$. Let $\Lambda_{L_2,L_1}(x) = \Lambda_{L_2}(x) \setminus \Lambda_{L_1}(x)$.

We omit x when it is clear. Let $P_\Lambda : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ be a projection to $\ell^2(\Lambda)$. Let $H_{\omega,\Lambda} = P_\Lambda H_\omega P_\Lambda$ and $G_{\omega,\Lambda,E} = (H_{\omega,\Lambda} - E)^{-1}$. Let χ_I denote the characteristic function on $\mathbb{R}$.

We start with the concept of a "good" box or scale.

Definition 3.1 (Good boxes and scales)**.** *We say that:*

1. $\Lambda = \Lambda_L(x_0)$ *is* (ω, E, m)*-regular if* $\forall x, y \in \Lambda$ *with* $|y - x| \geq \frac{L}{100}$*, we have*

$$|G_{\omega,\Lambda,E}(x, y)| \leq e^{-m|y-x|};$$

2. $\Lambda = \Lambda_L(x_0)$ *is* (ω, E, m, η)*-good if* Λ *is* (ω, m, E)*-regular and*

$$||G_{\omega,\Lambda,E}|| \leq e^{L^{1-\eta}}.$$

3. *The scale* $L \in \mathbb{Z}$ *is* (E, m, η, p)*-good if for any* $x \in \mathbb{Z}^d$*, we have*

$$\mathbb{P}\{\omega : \Lambda_L(x) \text{ is } (\omega, E, m, \eta)\text{-good}\} \geq 1 - L^{-pd}.$$

Remark 3.1. We fix $0 \leq \eta < 1$ and omit it in the statement of most theorems.

We are interested in the following types of localization:

Definition 3.2. *We say* H_ω *exhibits*

1. *Anderson localization (AL) in an interval* $I \subset \mathbb{R}$ *if for a.e.* ω*,* H_ω *has pure point spectrum and the eigenfunctions decay exponentially.*

2. *Dynamical localization (DL) of order p in I if for a.e. ω,*

$$\sup_{t>0} ||\langle X\rangle^p e^{-itH_\omega}\chi_I(H_\omega)\delta_0|| < \infty$$

3. *Strong dynamical localization in expectation of order (p, s) in I if*

$$\mathbb{E}\left\{\sup_{t\geq 0} ||\langle X\rangle^p e^{-itH_\omega}\chi_I(H_\omega)\delta_0||^s\right\} < \infty$$

We define the generalized eigenfunction (g.e.f.) and the generalized eigenvalue (g.e.v.) as follows:

Definition 3.3. *Fix ν. If $H\phi = E\phi$ and $\phi(x) \leq C\langle x\rangle^\nu$ for some $C > 0$, then we call ϕ a g.e.f. of H w.r.t. the g.e.v. E. Let $\Theta^\nu_\omega(E) = \{\phi : \mathbb{Z}^d \to \mathbb{R} \ : H_\omega\phi = E\phi, \ ||T_\nu^{-1}\phi|| < \infty\}$ denote the set of g.e.f.'s with fixed ω, ν and E.*

Fix $\nu > d$ for the rest of the paper and omit it.

Let $T_{x_0}\phi(x) = \langle x - x_0\rangle^\nu \phi(x)$, where $x_0 \in \mathbb{Z}^d$. We denote T_{x_0} by T when $x_0 = 0$. One can check that $\langle y\rangle \leq \sqrt{2}\langle x\rangle\langle x - y\rangle$. Thus $\|T_x^{-1}T_y|| \leq 2^{\frac{\nu}{2}}\langle x - y\rangle^\nu$, which means

$$\|T_x^{-1}\phi\| < \infty \ \Leftrightarrow \ \|T_y^{-1}\phi\| < \infty.$$

Then we can introduce our key definition:

Definition 3.4. *Given $\omega \in \Omega$, $n \in \mathbb{Z}^d$ and $E \in \mathbb{R}$, define the quantity*

$$W_\omega(x; E) := \sup_{\phi\in\Theta_\omega(E)} \frac{|\phi(x)|}{||T_x^{-1}\phi||}, \ \textit{if } \Theta_\omega(E) \neq \emptyset$$
$$W_\omega(x; E) := \sup_{\phi\in\Theta_\omega(E)} \frac{||P_{\Lambda_{2L,L}(x)}\phi||}{||T_x^{-1}\phi||}, \ \textit{if } \Theta_\omega(E) \neq \emptyset$$

Remark 3.2. Roughly speaking, $W_\omega(x;E)$ (or $W_{\omega,L}(x;E)$) quantifies the highest extent of localization at x (or the band $\Lambda_{2L,L}(x)$) among all g.e.f.'s w.r.t. the same E. This quantity will be very useful in the proof of dynamical localization because it provides a "uniform-in-g.e.f." upper bound.

3.2.2 Main results

Theorem 3.3 provide the estimates of the decay of $W_\omega(x;E)$ from "the end of MSA", while Theorem 3.4 extract the localization results from this estimates.

By **"The end of MSA"**, we mean that there is $m > 0, s > 0, p > 0, I \subset \mathbb{R}$ and $L_0 > 0$ s.t. any scale $L \geq L_0$ is (E,m,s,p)-good-scale for any $E \in I$.

In particular, as we mentioned, when $d = 2,3$, by rewriting [31, Theorem 1.4],[63, Theorem 2.4], we derived the following "the end of MSA" results (by taking $\eta_0 = m_0 = \frac{\varepsilon}{2}, E_0 = \varepsilon$ and $L_0 = L_0(\eta_0, m_0) > \alpha$ large enough).

Theorem 3.1. *Let $d = 2,3$, for any $0 < p_0 < 1/2$, there are $m_0, \eta_0, E_0 > 0$ and $L_0 = L_0(m_0, \eta_0)$ s.t. if $E \in (0, E_0)$ and $L \geq L_0$, then L is (E, m_0, η_0, p_0)-good scale.*

Our first results, which works for all d is:

Theorem 3.2. *Assume $\mathcal{I} \subset \mathbb{R}$ is an open bounded interval. Assume further that there is L_0 s.t. $\forall L > L_0$, scale L is (E,m,η,p)-good for some m, $p > 0$, $0 \leq \eta < 1$ and for all $E \in \mathcal{I}$. Then there is $0 < \tilde{p} < p$,*

Theorem 3.3. *Let $\mathcal{I} \subset \mathbb{R}$ be a bounded open interval, $m > 0$, $p > 0$, $s \in (0,1)$, and assume there is a scale $\mathcal{L}$, s.t. any $L \geq \mathcal{L}$ is (E,m,s,p)-good for all energies $E \in \mathcal{I}$. Let $M = m/30^{\hat{n}+2}$, where $\hat{n} = \hat{n}(p) := \{n \in \mathbb{N} : 2^{1/n} - 1 < p\}$. Fix $\tilde{p} \in (0,p)$, let*

$\mu = \beta/2$, *with* $\beta = \rho^{n_1}$ *where* $\rho > 0$ *and* $n_1 \in \mathbb{N}$ *s.t.*

$$(1+p)^{-1} < \rho < 1 \quad \textit{and} \quad (n_1+1)\beta < p - \tilde{p}.$$

Let

$$\mathcal{I}_L := \{E \in \mathcal{I} : \ \textit{dist}\ (E, \mathbb{R} \setminus \mathcal{I}) > e^{-ML^{\mu}}\}.$$

Then given a sufficiently large L, *for any* $x_0 \in \mathbb{Z}^d$, *there exists an event* $\mathcal{U}_{L,x_0}$ *s.t.*

1. $\mathcal{U}_{L,x_0} \in \mathcal{F}_{\Lambda_L}(x_0)$ *and* $\mathbb{P}\{\mathcal{U}_{L,x_0}\} \geq 1 - L^{-\tilde{p}d}$.

2. *If* $\omega \in \mathcal{U}_{L,x_0}$, $E \in \mathcal{I}_L$, *we have*

$$W_{\omega,x_0}(E) > e^{-ML^{\mu}} \Rightarrow W_{\omega,x_0,L}(E) \leq e^{-ML}$$

   and thus

$$W_{\omega,x_0}(E)W_{\omega,x_0,L}(E) < e^{-\frac{s}{2}ML^{\mu}}, \quad \textit{for large enough } L.$$

This theorem is proved in Section 6. Based on this, we derived the following localization results:

Theorem 3.4. *Let* $\mathcal{I} \subset \mathbb{R}$ *be a bounded open interval. Assume that there is a scale* $\mathcal{L}$ *s.t. any* $L \geq \mathcal{L}$ *are* (E, m, s, p)*-good scale. Then for all* $x_0 \in \mathbb{Z}^d$ *and* $I \subset \bar{I} \subset \mathcal{I}$, *we have:*

1. *for any* $x \in \mathbb{Z}^d$, $L \geq 1$, *and* $s \in \left(0, \frac{\tilde{p}d}{\nu}\right)$, *there is* C *s.t.*

$$\mathbb{E}\left\{||W_\omega(x;E)W_\omega(x;E)||^s_{L^\infty(I,d\mu_\omega(E))}\right\} \leq CL^{-(\tilde{p}d - s\nu)}.$$

2. H_ω *has Anderson localization in* $\mathcal{I}$ *a.e.* ω.

3. *Given $b > 0$, for all $s \in (0, \frac{\tilde{p}d}{bd+\nu})$ we have strong dynamical localization of order (bd, s), i.e.*

$$\mathbb{E}\left\{\sup_{t\geq 0}\left(\left\|\langle X - x_0\rangle^{bd} e^{-itH_\omega} P_\omega(I)\delta_{x_0}\right\|_1\right)^s\right\} \leq C < \infty.$$

3.3 Genralized eigenfunction expansion

In this section, we provide a brief introduction to generalized eigenfunction expansions. This is necessary for several reasons: first, not every self-adjoint operator has eigenfunctions (i.e. those with continuous spectrum) and second, these eigenfunctions are needed to define the $W_\omega(x; E)$ below and the $W_\omega(x; E)$ play a crucial role in the estimates from section 6 and hence in the proofs of the localization results from section 7.

In what follows, we will enlarge the domain (using rigged Hilbert spaces) and employ the Bochner integral to ensure we have access to the appropriate decompositions. We note that this presentation, while self-contained, is brief and further details can be found in sections 15.1 and 15.2 of [10].

Let $\mathcal{H}_+, \mathcal{H}_-$ be the rigged-Hilbert spaces:

$$\mathcal{H}_+ = \ell^2(\mathbb{Z}^d, \langle x\rangle^{2\nu} dx), \quad \mathcal{H} = \ell^2(\mathbb{Z}^d, dx), \quad \mathcal{H}_- = \ell^2(\mathbb{Z}^d, \langle x\rangle^{-2\nu} dx).$$

We have $\|T^{-1}f\|_{\mathcal{H}_+} = ||f||_{\mathcal{H}} = ||Tf||_{\mathcal{H}_-}$.

Define two natural embeddings $i_+ : \mathcal{H}_+ \to \mathcal{H}$, and $i_- : \mathcal{H} \to \mathcal{H}_-$. View T Let $T_+ = Ti_+$, $T_- = i_-T$. Then T_+, T_- are isomorphisms between $\mathcal{H}_+$ and $\mathcal{H}$, and $\mathcal{H}$ and

$\mathcal{H}_+$.

For any self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$, we could form a chain and derive correspondingly $C := T_- A T_+ : \mathcal{H}_+ \to \mathcal{H}_-$ as follows:

$$\mathcal{H}_+ \xrightarrow{i_+} \mathcal{H} \xrightarrow{T} \mathcal{H} \xrightarrow{A} \mathcal{H} \xrightarrow{T} \mathcal{H} \xrightarrow{i_-} \mathcal{H}_-$$

Then $A \to C$ is a Banach Isomorphism between $\mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$ and $\mathcal{B}(\mathcal{H}, \mathcal{H})$. Consider the Schatten norm $||A||_p = (Tr(|A|^p))^{1/p}$ in $\mathcal{B}(\mathcal{H}, \mathcal{H})$. We could then define the Schatten norm $|| \cdot ||_{p,\pm}$ in $\mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$ induced from $\mathcal{B}(\mathcal{H}, \mathcal{H})$ by the isomorphism:

$$\|C\|_{p,\pm} := \left\|T_-^{-1} C T_+^{-1}\right\|_p$$

The Bochner Theorem below will help us obtain the desired generalized eigenfunction decomposition later. We provide it without proof here:

Theorem 3.5 ([10, Theorem 15.1.1] footnote here)**.** *Let $\theta : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H}_+, \mathcal{H}_-)$ be an operator-valued measure with finite trace, i.e.*

1. *$\theta(B)$ is non-negative for any Borel $B \subset \mathbb{R}$,*
2. *$Tr_\pm(\theta(\mathbb{R})) < \infty$,*
3. *$\theta(\bigsqcup_j B_j) = \sum_j \theta(B_j)$, converging in the weak sense.*

Then θ can be differentiated w.r.t. trace measure $\rho(B) = Tr_\pm(\theta(B))$. This implies that $\exists Q(E) : \mathcal{H}_+ \to \mathcal{H}_-$, s.t.

$$\theta(B) = \int_B Q(E) d\rho(E)$$

where

$$\begin{cases} 0 \le Q(E) = Tr_{\pm}(Q(E)) = 1, \quad \rho\text{-}a.e.\ E, \\ Q(E) \text{ is weakly measurable w.r.t. } \mathcal{B}(\mathbb{R}), \\ \text{The integral holds in the Hilbert-Schmidt norm.} \end{cases}$$

In particular, for the projection-valued spectral measure $P_\omega(B) := \chi_B(H_\omega) : \mathcal{H} \to \mathcal{H}$, we apply the theorem above to $\theta_\omega(B) := i_- P_\omega(B) i_+ : \mathcal{H}_+ \to \mathcal{H}_-$, which satisfy the conditions:

$$Tr_{\pm}(i_- P_\omega(\mathbb{R}) i_+) = Tr(T^{-1} P_\omega(\mathbb{R}) T^{-1}) \le \sum_{n \in \mathbb{Z}^d} ||T^{-1}\delta_n||^2 \le \sum_{n \in \mathbb{Z}^d} \frac{1}{(1+n^2)^\nu} < \infty.$$

Thus there are $Q_\omega(E) : \mathcal{H}_+ \to \mathcal{H}_-$, and $\mu_\omega(B) := Tr(T^{-1} P_\omega(B) T^{-1}) = ||T^{-1} P_\omega(B)||_2^2$, s.t.

$$i_- P_\omega(B) i_+ = \int_B Q_\omega(E) d\mu_\omega(E)$$

with $Tr_{\pm}(Q(E)) = 1,$ for μ_ω-a.e. E.

Furthermore, for $u \in \mathcal{H}_+$, for $f \in \mathcal{B}_{1,b}(\mathbb{R})$ be a bounded Borel function. We have:

$$\begin{aligned} P_\omega(B) u &= \left(\int_B Q_\omega(E) d\mu_\omega(E) \right) u, \\ f(H_\omega) P_\omega(B) u &= \left(\int_B f(E) Q_\omega(E) d\mu_\omega(E) \right) u, \end{aligned} \tag{3.1}$$

and

$$\text{Range}(Q_\omega(E)) \subset \tilde{\Theta}\omega(E), \qquad \text{for } \mu_\omega\text{-a.e.} E.$$

The proof can be found in [10, Theorem 15.2.1]. Thus if $\text{Range}(Q_\omega(E)) \subset \mathcal{H}$, μ_ω-a.e.E, then H_ω has p.p. spectrum.

Define $W_\omega(x;E), W_\omega(x;E), \mathbf{W}_\omega(x;E), \mathbf{W}_{\omega,L}(x;E)$ to estimates the extent of localization for the g.e.f.s

$$W_\omega(x;E) = \begin{cases} \sup\limits_{\phi\in\Theta_\omega(E)} \frac{|\phi(x)|}{||T_x^{-1}\phi||}, & \Theta_\omega(E) \neq \emptyset \\ 0, & otherwise. \end{cases}$$
$$W_\omega(x;E) = \begin{cases} \sup\limits_{\phi\in\Theta_\omega(E)} \frac{||\chi_{x,L}\phi||}{||T_x^{-1}\phi||}, & \Theta_\omega(E) \neq \emptyset \\ 0, & otherwise. \end{cases}$$
$$\mathbf{W}_\omega(x;E) = \begin{cases} \sup\limits_{Q_\omega(E)\phi\neq 0} \frac{|Q_\omega(E)\phi(x)|}{||T_x^{-1}Q_\omega(E)\phi||}, & Q_\omega(E) \neq 0 \\ 0, & otherwise, \end{cases}$$
$$\mathbf{W}_{\omega,L}(x;E) = \begin{cases} \sup\limits_{Q_\omega(E)\phi\neq 0} \frac{||\chi_{x,L}Q_\omega(E)\phi||}{||T_x^{-1}Q_\omega(E)\phi||}, & Q_\omega(E) \neq 0 \\ 0, & otherwise. \end{cases} \quad (3.2)$$

The estimates provided in the following lemma are the key to estimating the kernel present in the dynamical localization statements. The W_ω above allow us to avoid the analysis of one generalized eigenfunction at a time which necessarily entails passing to centers of localization to obtain global estimates. Thus, the price one pays for the notational burden is significantly outweighed by the uniform control they afford.

Lemma 3.3.1. *We have*

$$||\chi_{x,L}f(H_\omega)P_\omega(I)\delta_x||_1 \leq \int_I f(E)||\chi_{x,L}Q_\omega(E)\delta_x||_1 d\mu_\omega(E),$$
$$\begin{aligned} ||\chi_{x,L}Q_\omega(E)\delta_x||_1 &\leq ||\chi_{x,L}Q_\omega(E)||_2||\delta_x Q_\omega(E)||_2 \\ &\leq \boldsymbol{W}_\omega(x;E)\,\boldsymbol{W}_{\omega,L}(x;E)||T_x^{-1}Q_\omega(E)||_2||T_x^{-1}Q_\omega(E)||_2. \end{aligned}$$

Definition 3.5 (Gerneralized eigenfunction)**.** *A non-zero vector $\phi \in \mathcal{H}_-$, $\phi \neq 0$ is* *called a generalized eigenfunction (g.e.f.) of $H_\omega : \mathcal{H} \to \mathcal{H}$ if there is $E \in \mathbb{C}$, called generalized eigenvalue (g.e.v.) s.t. $\forall u \in \mathcal{D}_+(H_\omega) = \{\psi \in \mathcal{D}(H_\omega) \cap \mathcal{H}_+ | H_\omega \psi \in \mathcal{H}_+\}$,*

$$(\phi, H_\omega u)_\pm = E(\phi, u)_\pm,$$

where $(\cdot,\cdot)_\pm : \mathcal{H}_- \times \mathcal{H}_+ \to \mathbb{C}$ is the continuous extension of original inner product on $\mathcal{H} \times \mathcal{H}$, by density of $\mathcal{H}$ in $\mathcal{H}_-$. Let's denote the set of g.e.f. w.r.t. g.e.v. E for H_ω by $\Theta_\omega(E)$. Let $\tilde{\Theta}_\omega(E) = \Theta_\omega(E) \cup \{0\}$.

If g.e.f. $\phi \in \mathcal{H}$, then by density of $\mathcal{H} \in \mathcal{H}_-$, ϕ is an eigenfunction.

3.4 Preliminary Lemmas

Below are the preliminary lemmas needed for the proof of the main theorem. The first lemma is deterministic and describes the stability of good boxes.

Lemma 3.4.1. *Assume ω, E_0, L_0, x are fixed and $\forall L \geq L_0$, $\Lambda_L(x)$ is (ω, E_0, m_0, s)-good. Then $\forall m < m_0$, $\exists L_m$, s.t. $\forall L \geq L_m$, $\forall E$ satisfying $|E - E_0| \leq e^{-m_0 L}$, we have $\Lambda_L(x)$ is (ω, E, m, s)-good.*

Proof. Recall the resolvent identity:

$$G_{\omega,E,\Lambda_L(x)} - G_{\omega,E_0,\Lambda_L(x)} = (E_0 - E) G_{\omega,E,\Lambda_L(x)} G_{\omega,E_0,\Lambda_L(x)}.$$

Thus,

$$\begin{aligned} ||G_{\omega,E,\Lambda_L(x)}|| &\leq ||G_{\omega,E_0,\Lambda_L(x)}|| + |E_0 - E| ||G_{\omega,E,\Lambda_L(x)}|| ||G_{\omega,E_0,\Lambda_L(x)}|| \\ &\leq Ce^{L^{1-s}} + Ce^{-m_0 L + L^{1-s}} ||G_{\omega,E,\Lambda_L(x)}||, \end{aligned}$$

and so,

$$||G_{\omega,E,\Lambda_L(x)}|| \leq \frac{Ce^{L^{1-s}}}{1 - Ce^{-mL+L^{1-s}}} \leq 2Ce^{L^{1-s}}.$$

Also, if $m < m_0$,

$$\begin{aligned} |G_{\omega,E,\Lambda_L(x)}(a,b)| &\leq |G_{\omega,E_0,\Lambda_L(x)}(a,b)| + |E - E_0||G_{\omega,E,\Lambda_L(x)}(a,b)||G_{\omega,E_0,\Lambda_L(x)}(a,b)| \\ &\leq e^{-m_0|a-b|} + e^{-m_0 L} e^{L^{1-s}+L^{1-s}} \\ &\leq e^{-m|a-b|} \end{aligned}$$

when L is large enough. Denote the threshold by L_m. □

The following lemma provides the connection between good loops and the quantity W. In particular, we obtain a connection between the definition of a good box and W.

Lemma 3.4.2. *Let $l > 12$. For fixed ω, if there is a (l, E_0, m_0, s)-good loop $\mathcal{A}$ in $\Lambda_{L_2,L_1}(x_0)$, then $\exists C_d = \frac{\sqrt{d}}{4}$, s.t. $\forall |E - E_0| \leq e^{-m_0 l}$, $E \in \mathcal{I}$, $\forall m < m_0$, we have*

$$dist(E, \sigma(H_{\Lambda_{L_2}}))W_{\omega,x}(E) \leq CL_2^{d+\nu} 4^d e^{-\frac{m\ell}{3}}.$$

Proof. Let $M = ||V||_{\ell^\infty} + \sup_{E\in\mathcal{I}} |E|$. Let $l > 12$ so that $\frac{l}{2} - 2 > \frac{l}{3}$. Note that

$$\text{dist}(E, \sigma(H_{\Lambda_2})) = ||(H_{\Lambda_{L_2}} - E)^{-1}||^{-1} = \inf_{\psi\in\ell^2} \frac{||(H_{\Lambda_{L_2}} - E)\psi||}{||\psi||}.$$

Recall $\mathcal{A}$ is a closed loop in $\mathcal{G}$. So it"circles" a region $A \subset \mathbb{R}^d$. Let $\chi_A : \mathbb{Z}^d \to \{0, 1\}$ be the characteristic function of A on $\mathbb{Z}^d$.

Let ϕ_0 be a g.e.f. of H_ω w.r.t. E_0, i.e. $(H_\omega - E_0)\phi_0 = 0$. Take $\psi = \chi_A\phi_0$, we found that $(H_{\Lambda_{L_2}} - E_0)\psi(x)$ will be 0 at most points $x \in \Lambda_{L_2}$ except for those near $\mathcal{A}$.

$$\begin{aligned}
\operatorname{dist}(E, \sigma(H_{\Lambda_2})) &\leq \frac{||(H_{\Lambda_{L_2}} - E_0 + E_0 - E)\chi_A\phi_0||}{||\chi_A\phi_0||} \\
&\leq \frac{\sum\limits_{\operatorname{dist}(x,\mathcal{A})\leq 1, x\in\mathbb{Z}^d} (2^d+1)2^{d+1}M \max\limits_{|y-x|\leq 2, y\in\mathbb{Z}^d} |\phi(y)|}{|\phi(x_0)|} \\
&\leq \frac{C(L_2^d - L_1^d)4^d M e^{-\frac{m_0 l}{3}} l^{d-1} \max\limits_{y\in\Lambda_{L_2,L_1}} |\phi(y)|}{|\phi(x_0)|} \\
&\leq \frac{CM4^d L_2^d e^{-\frac{ml}{3}} \max\limits_{y\in\Lambda_{L_2,L_1}} |\phi(y)|}{|\phi(x_0)|}
\end{aligned}.$$

Notice $\max\limits_{y\in\Lambda_{L_2,L_1}} |\phi(y)| = \max\limits_{y\in\Lambda_{L_2,L_1}} \langle y-x_0\rangle^\nu \frac{|\phi(y)|}{\langle y-x_0\rangle^\nu} \leq \langle L_2\rangle^\nu ||T_\nu^{-1}\phi||$. Thus, by the definition of $W_{\omega,x}(E)$, we get

$$\operatorname{dist}(E, \sigma(H_{\Lambda_{L_2}}))W_{\omega,x}(E) \leq CL_2^{d+\nu} 4^d e^{-\frac{m\ell}{3}}.$$

□

Let $\mathcal{L}_c$ be the coarse lattice $\mathcal{N}_c = \frac{3l}{5}\mathbb{Z}^d$ with sides $\mathcal{S} = \{(x,y) \in \mathcal{N}\times\mathcal{N} : |x-y|_1 = \frac{3l}{5}\}$. Let $\mathcal{L}_f$ be the fine lattice $\mathcal{N}_f = \mathbb{Z}^d$ with sides $\mathcal{S}_f = \{(x,y) \in \mathcal{N}\times\mathcal{N} : |x-y|_1 = 1\}$. In the following context, we will always choose $l \in 3\mathbb{Z}^d$ so that $\mathcal{L}_c$ is a sub-lattice of $\mathcal{L}_f$.

Definition 3.6 (Good node and loop)**.** *Fix ω, for each $x \in \mathcal{N}$, we say*

1. *$x \in \mathcal{N}$ is a (ω, E, m, s)-good (-bad) node if $\Lambda_l(x)$ is a (ω, E, m, s)-good (-bad) box.*

2. *$\mathcal{A} \subset \mathcal{S}$ is a (l, E, m, s)-good loop (shell) if it is a closed loop (shell) in $\mathcal{S}$ where each node $x \in \mathcal{A}$ is a good node.*

3. *$\mathcal{P} \subset \mathcal{S}$ is a bad path if it is a non-self-intersecting path of graph $\mathcal{S}$ and each node is a bad node.*

Remark 3.3.

We say a (l, E, m, s)-good shell $\mathcal{A}$ is totally inside a subset $S \subset \mathbb{R}^d$ if $\bigcup_{x \in \mathcal{A}} \Lambda_{l+2}(x) \subset S$.

Let $\mathcal{Y}^{(E)}_{x_0,l,L_1,L_2}$ denote the event

$$\{\omega : \text{an } (\omega, l, E, m, s)\text{-good loop exists totally inside } \Lambda_{L_2,L_1}(x_0)\}.$$

Since the probability of each node being good is quite large by Theorem 3.1. We expect relatively large probability for having good loops as well. The next lemma quantifies the probability estimates of this event.

Lemma 3.4.3. *Assume E is fixed, and the scale l is (E, m, s, p)-good. We have*

$$\mathbb{P}\left\{\mathcal{Y}^{(E)}_{x_0,l,L_1,L_2}\right\} \geq 1 - 2d(\frac{L_1 + 3l}{l})^{d-1}(2^d)^{\frac{L_2-L_1-l}{l}} l^{-pd\frac{L_2-L_1-l}{(3^d-1)l}}.$$

In particular, if $l = \sqrt{L}$, $L_1 = \frac{L}{2}$, $L_2 = L$, when L is large enough, then

$$\mathbb{P}\left\{\mathcal{Y}^{(E)}_{x_0,l,\frac{L}{2},L}\right\} \geq 1 - L^{-c_{d,p}\sqrt{L}}.$$

Remark 3.4.

Notice that $\mathcal{Y}^{(E)}_{x_0,l,L_1,L_2}$ only depends on $\Omega_{\Lambda_{L_2,L_1}(x_0)}$.

Proof. Fix $E \in \mathcal{I}$. First notice that

$$(\mathcal{Y}^{(E)}_{x_0,l,L_1,L_2})^c = \{\omega : \text{there is no good loop totally inside } \Lambda_{L_2,L_1}\}$$
$$= \{\omega : \text{there is a bad path escaping from } \partial\Lambda^+_{L_1+l+2} \text{ to } \partial\Lambda^-_{L_2-l-2}\}$$

Notice that each such bad path must contain at least $N := \frac{L_2-L_1-2l-4}{6l/5} + 1$ many bad nodes starting from $\partial\Lambda^+_{L_1+l+2}$, which means it should contain $\frac{N}{(3^d-1)l}$-many independent bad nodes. And the number of all possible paths like this is less than $2d(\frac{L_1+l+2}{3l/5})^{d-1}(2^d)^N$. So we get

$$\mathbb{P}\{(\mathcal{Y}^E_{x_0,l,\frac{L}{2},\frac{L}{2}})^c\} \leq 2d(\frac{L_1+l+2}{3l/5}+1)^{d-1}(2^d)^N l^{-pdN}$$
$$= 2d(\frac{L_1+l+2}{3l/5}+1)^{d-1}(2^d)^{\frac{L_2-L_1-2l-4}{6l/5}+1} l^{-pd(\frac{L_2-L_1-2l-4}{(3^d-1)l}+1)}$$

The second inequality follows from the first one by plugging in and let L be large enough. $\square$

3.5 Multiscale to Localization

In order to prove Theorem 3.3, we need two intermediary results: Theorem 3.6 and Theorem 3.7.

3.5.1 The first spectral reduction

Theorem 3.6. *Given $b \geq 1$, there exists a constant $K_{d,p,b} \geq 1$ s.t. for any $K \geq K_{d,p,b}$, for large enough L, for any $x_0 \in \mathbb{Z}^d$, there is an event $\mathcal{Q}_{L,x_0}$, with*

1. *$\mathcal{Q}_{x_0,L} \in \mathcal{F}_{\Lambda_L(x_0)}$ and $\mathbb{P}\{\mathcal{Q}_{L,x_0}\} \geq 1 - (\frac{L}{K})^{-5d}$*

2. *for any $\omega \in \mathcal{Q}_{x_0,L}$, given $E \in \mathcal{I}$ s.t. if there exists a g.e.f. ϕ with $\phi(x_0) \neq 0$, then for L large enough*

$$dist\left(E, \sigma^{(\mathcal{I})}(H_{\omega,\Lambda_L(x_0)})\right) \leq e^{-\hat{m}\frac{L}{K}}.$$

Proof. The strategy here is two-fold:

1. Construct $\mathcal{Q}_{x_0,L}$ by layers.

2. Estimate the probability of the event $\mathcal{Q}_{x_0,L}$ occurring.

Given L_0, we define $l_0 = \sqrt{L_0}$, and $l_k = l_{k-1}^{1+\eta}$, for $k = 1, 2, \cdots, \hat{n}$, where $(1+\eta)^{\hat{n}} = 2$, so $l_{\hat{n}} = l_0^2 = L_0$. Let $L_k = L_{k-1} + 2Jl_k$ where J is a large constant to be determined later. Then we have

$$L_{\hat{n}} = L_0 + 2J\sum_{k=1}^{\hat{n}} l_k \leq (1 + 2J\hat{n})L_0.$$

Now we use an inductive construction to form the set $\mathcal{Q}_{x,L}$.

1. Given $m_0 > 0$. For the initial layer Λ_{L_0}, we pick $\mathcal{Y}^{E_{0,i}}_{l_0,\sqrt{L_0},L_0}$ where $E_{0,i}$ are energies s.t. the union of $[E_{0,i} - e^{-m_0 l_0}, E_{0,i} + e^{-m_0 l_0}]$ covers $\mathcal{I}$. We need to choose $\frac{|\mathcal{I}|}{2e^{-m_0 l_0}} = O(e^{\sqrt{L_0}})$ many of them. Let $Y_0 = \bigcap_i \mathcal{Y}^{E_{0,i}}_{l_0,\sqrt{L_0},L_0}$, then we have

$$\mathbb{P}\{Y_0\} \geq 1 - Ce^{\sqrt{L_0}}L_0^{-c_{d,p}\sqrt{L_0}} \geq 1 - Ce^{-\sqrt{L_0}}.$$

   Recall that $\mathcal{Y}^{(E)}_{l,L_1,L_2}$ only depends on $\Omega_{\Lambda_{L_2,L_1}}$. In particular, $\mathcal{Y}^{E_{0,i}}_{l_0,\sqrt{L_0},L_0}$ only depends on $\Omega_{\Lambda_{L_0}}$.

2. For the remaining events, we use an inductive scheme. If $\omega_{\Lambda_{L_{k-1}}} \in \Omega_{\Lambda_{L_{k-1}}}$ is given, we can consider all eigenvalues $E_{k,j} = E_{k,j}(\omega_{\Lambda_{L_{k-1}}})$ of $H_{\Lambda_{L_{k-1}},\omega_{\Lambda_{L_{k-1}}}}$. For each such $E_{k,j}$, we can then consider $\mathcal{Y}^{E_{k,j}}_{L_{k-1},l_k,L_k}$. Notice the dependence here is only on $\omega \in \Omega_{\Lambda_{L_k,L_{k-1}}}$ so there is no conflict with the previous $\omega_{\Lambda_{L_{k-1}}}$ and the induction is well-defined.

   Let $Y_k(\omega_{\Lambda_{L_{k-1}}}) = \bigcap_j \mathcal{Y}^{E_{k,j}}_{L_{k-1},l_k,L_k}(\omega_{\Lambda_{L_{k-1}}})$, where we pick $J \geq J_{p,d}$ (note that J depends on $c_{d,p}$). In this situation, we have

$$\mathbb{P}\{Y_k\} \geq 1 - L_{k-1}^{d+d-1} 3^{Jd} l_k^{-c_{d,p}2J} \geq 1 - L_0^{-6d}.$$

Having obtained the box $\Lambda_{L_{\hat{n}}}$ and we choose the event depending on $\Omega_{\Lambda_{L_{\hat{n}}}}$ to be

$$\mathcal{Q}_{x_0,L_{\hat{n}}} = \bigcap_{k=0}^{\hat{n}} Y_k(\omega_{k-1})$$

and we have

$$\mathbb{P}\{\mathcal{Q}_{x_0,L_{\hat{n}}}\} \geq 1 - L_0^{-5d}.$$

Thus, to obtain $\mathcal{Q}_{x_0,L}$, we choose an L_0 s.t. $L = L_{\hat{n}} \leq (1 + 2J_{d,p}\hat{n})L_0$. Then for any $K \geq K_{d,p} = 1 + 2J_{d,p}\hat{n}$, we have

$$\mathcal{Q}_{x_0,L} \geq 1 - L_0^{-5d} \geq 1 - (\frac{L}{K})^{-5d}.$$

We now need to verify that for $\omega \in \mathcal{Q}_{x_0,L}$, $E \in \mathcal{I}$ with some g.e.f. $\phi(x_0) \neq 0$, we have

$$\operatorname{dist}\left(E, \sigma^{(\mathcal{I})}(H_{\omega,\Lambda_L(x_0)})\right) \leq e^{-\hat{m}\frac{L}{K}}.$$

Notice, by lemma 3.2 applied to $\mathcal{Y}^{E_{0,i}}_{\sqrt{L_0},l_0,L_0}$, $E \in \mathcal{I}$ implies there exists an $E_{0,i}$ s.t.

$|E - E_{0,i}| \leq e^{-m_0 l_0}$. So, we have,

$$\text{dist}\left(E, \sigma^{(\mathcal{I})}(H_{\omega,\Lambda_{L_0}(x_0)})\right) \leq e^{-m_1 l_0},$$

where we choose $m_1 < m_0$. Thus, there exists an $E_{1,j} \in \sigma(H_{\omega,\Lambda_{L_0}(x_0)})$ s.t. $|E - E_{1,j}| \leq e^{-m_1 l_0} \leq e^{-m_1 l_1}$. We now apply lemma 3.2 to $\mathcal{Y}^{E_{1,j}}_{L_0,l_1,L_1}$ and repeat this process $\hat{n}$ times to obtain:

$$\text{dist}\left(E, \sigma^{(\mathcal{I})}(H_{\omega,\Lambda_L(x_0)})\right) \leq e^{-\hat{m}\frac{L}{K}}.$$

□

3.5.2 Second spectral reduction

Definition 3.7 (reduced spectrum). *The reduced spectrum of H_ω in $\Lambda_L(x_0)$, in the energy interval $\mathcal{I}$ is given by*

$$\sigma^{(\mathcal{I},red)}(H_{\omega,\Lambda_L(x_0)}) :=$$
$$\left\{E \in \sigma^{(\mathcal{I})}(H_{\omega,\Lambda_L(x_0)}) : \ \textit{dist}\ \left(E, \sigma^{(\mathcal{I})}(H_{\omega,\Lambda_{L_n}(x_0)}\right) \leq 2e^{-\frac{\hat{m}}{K}L_n}, n = 1, \cdots, n_1\right\}$$

where $L_n = L^{\rho^n}$ for $n = 0, 1, 2, 3, \cdots, N$, and $\rho^N = \beta$.

Theorem 3.7 (second spectral reduction). *Let $b \geq 1$, Given large enough L, for each $x \in \mathbb{Z}^d$ there exists an event χ_{L,x_0}, with*

$$\chi_{L,x_0} \in \mathcal{F}_{\Lambda_L(x_0)} \quad \textit{and} \quad \mathbb{P}\{\chi_{L,x_0}\} \geq 1 - L^{-b\beta d},$$

s.t. $\forall \omega \in \chi_{L,x_0}$

1. If $E \in \mathcal{I}$ satisfies

$$W_{\omega,x_0}(E) > e^{-\hat{m}\sqrt{\frac{L^\beta}{K}}} \quad \textit{and} \quad \textit{dist}\ (E, \mathbb{R} \setminus \mathcal{I}) > 2^{-\hat{m}\sqrt{\frac{L^\beta}{K}}} \tag{3.3}$$

then

$$\textit{dist}(E, \sigma^{(\mathcal{I},red)}(H_{\omega,\Lambda_L(x_0)})) \leq e^{-\hat{m}/K} \tag{3.4}$$

2. and we have

$$\#\sigma^{(\mathcal{I},red)}(H_{\omega,\Lambda_L(x_0)}) \leq CL^{(n_1+1)\beta d} \tag{3.5}$$

To obtain (3.4) from (3.3), one needs $\tilde{\mathcal{Q}}_{x_0,L} = \bigcap_{n=0}^{N} \mathcal{Q}_{x_0,L_n}$. In this case, by Theorem 3.6 and the definition of the reduced spectrum, the desired results follow.

Proving (3.5) requires sufficiently more work.

First notice that, compared with the typical estimates on the number of eigenvalues of $H_{\omega,\Lambda_L(x_0)}$, i.e. $\#\sigma(H_{\omega,\Lambda_L(x_0)}) \leq CL^d$, we want much tighter bound. Because $(n_1+1)\beta > 0$ could be very, very small. The reduced spectrum nomenclature stems from the fact that the number of elements in it is largely reduced (but there are still enough to obtain the required deterministic estimate). To achieve these goals, we introduce the notion of a "notsobad set" which helps control the number of close eigenvalues for $\Lambda_L(x_0)$ and $\Lambda'_L(x_0)$.

Let $L' < L$, $x_0 \in \mathbb{R}$, and consider $\Lambda_{L,L'}$ the annulus centered at x_0(we omit x_0 from the notation for the time being). Let $L_n = L^{\rho^n}$, for $n = 0, 1, 2, \cdots, n_1$. Let $\mathcal{R}_n = \{\Lambda_{L_n}(r)\}_{r \in R_n}$ be the standard L_n-covering of $\Lambda_{L,L'}$. Given $K_2 \in \mathbb{N}$ (where K_2 will be chosen later), we say

Definition 3.8. *The annulus $\Lambda_{L,L'}$ is (ω, E, K_2)-notsobad if there are at most K_2 points in R_{n_1}, denoted by $r_i, 1 \leq i \leq K_2$, s.t. $\forall x \in \Lambda_{L,L'} \setminus \Theta$, where $\Theta = \bigcup_{r_i} \Lambda_{3L_{n_1}}(r_i)$,* *there exists a (ω, E, m, s)-good box $\Lambda_{L_{n_x}}(r) \in \mathcal{R}_{n_x}$ s.t. $x \in \Lambda_{L_{n_x}}(r)$ for some $n_x \in \{1, 2, \dots, n_1\}$. And an event $\mathcal{N}$ is $(\Lambda_{L,L'}, E, K_2)$-notsobad if $\mathcal{N} \in \mathcal{F}_{\Lambda_{L,L'}}$, and $\Lambda_{L,L'}$ is (ω, E, K_2)-notsobad for all $\omega \in \mathcal{N}$.*

Remark 3.5. Θ is called the singular set and the above definition captures the fact that outside of the singular set, each point is good in at least one level L_n, $n \in \{1, 2, \dots, n_1\}$.

The following lemma gives a probability estimates on such a set.

Lemma 3.5.1 (Probability estimates). *If $K_2 \geq \hat{K}_2 = \hat{K}_2(d, p, b)$ and $L \geq \hat{L} = \hat{L}(d, p, b, K_2)$, then for all $E \in \mathcal{I}$, there exists a $(\Lambda L, L', E, K_2)$-notsobad event $\mathcal{N}^{(E)}_{\Lambda_{L,L'}}$ with*

$$\mathbb{P}\{\mathcal{N}_{\Lambda^{(E)}_{L,L'}}\} > 1 - L^{-5bd}$$

We can now define the set we need to satisfy (3.5) by:

$$\mathcal{N}_{\Lambda_{L,L'}} = \bigcap_{E \in \sigma(H_{\omega,\Lambda_{L'}})} \mathcal{N}^{(E)}_{\Lambda_{L,L'}} \in \mathcal{F}_{\Lambda_L}$$

$$\mathcal{N}_{L,x_0} = \bigcap_{n=1}^{n_1} \mathcal{N}_{\Lambda_{L_{n-1},L_n(x_0)}}$$

Thus, $\mathbb{P}\{\mathcal{N}_{L,x_0}\} > 1 - Cn_1 L_{n_1-1}^{-4bd} \geq 1 - Cn_1 L^{-4b\beta d/\rho}$ by Lemma 3.5.1.

We also have:

Lemma 3.5.2 (Deterministic nice property). *If $\omega \in \mathcal{N}_{L,x_0}$, then*

$$\#\sigma(\mathcal{I}, red)(H_{\omega,\Lambda_L}) \leq CL^{(n_1+1)\beta d}$$

.

Proof. First notice by definition that

$$\begin{aligned}\#\sigma(\mathcal{I}, red)(H_{\omega,\Lambda_L}) &\leq \#\{\{E_n\}_{n=0}^{n_1} : E_n \in \sigma(H_{\omega,\Lambda_{L_n}}) \text{ \& } |E_i - E_j| \leq 2e^{-\frac{\hat{m}}{K} L_{\max i,j}}\} \\ &:= \#D_0^{n_1}\end{aligned}$$

We can count the RHS by layers inductively. We start with the layer L_{n_1} and omit x_0 and ω for convenience so,

$$\#D_{n_1}^{n_1} = \#\sigma(H_{\omega,\Lambda_{L_{n_1}}}) \leq C(L_{n_1})^d.$$

Given $\{E_n\}_k^{n_1} \in D_k^{n_1}$, we compute $\#\{E : \text{if } E_{k-1} = E, \text{ then } \{E_n\}_{k-1}^{n_1} \in D_{k-1}^{n_1}\}$. Denote the previous set by B_{k-1}. Since $\omega \in \mathcal{N}_{\Lambda_{L_{n-1},L_n}}$ for any n, Λ_{L_{n-1},L_n} is an $(\omega, L_{n-1}, L_n, E_n)$-notsobad set. Let Θ_n be the corresponding singular set and set $\Theta_k^{n_1} = \bigcup_{n=k}^{n_1} \Theta_n \cup \Lambda_{L_{n_1}}$. Then we have:

$$|\Theta_k^{n_1}| \leq L_{n_1}^d + \sum_{n=k}^{n_1} K_2(3(L_{k-1})_{n_1})^d = L^{\beta d} + (n_1 - k + 1)K_2 3^d L^{\rho^{k-1}\beta d} \leq CL^{\beta d}$$

If $E \in B_{k-1}$, then $\forall x,\ x \in \lambda_{L_{k-1}} \setminus \Theta_k^{n_1}$. So there is $n_x \in \{k, k+1, \ldots, n_1\}$, s.t. $x \in \Lambda_{L_{n_x-1},L_{n_x}} \setminus \Theta_{n_x}$. and there exists a (ω, E_{n_x}, m, s)-good box $\Lambda_{(L_{n_x-1})_j}$ containing x for some $j \in 1, 2, \ldots, n_1$, where $(L_{n_x-1})_j = L^{\rho^{n_x+j-1}}$. Since

$$|E - E_{n_x}| \leq e^{-\frac{\hat{m}}{K} L_{n_x}} \leq e^{-\frac{\hat{m}}{K} L^{\rho^{n_x}}} \leq e^{-\frac{\hat{m}}{K} (L_{n_x})_j}$$

, $\Lambda_{(L_{n_x-1})_j}$ is also (ω, E, m, s)-good by Lemma 3.4.1.

Let ϕ_E be the normalized eigenfunction of E on $H_{\omega,\Lambda_{L_{k-1}}}$, Then

$$|\phi_E(x)| \leq e^{-m' L^{\rho^{n_x+j-1}}} \leq e^{-m' L^{\rho^{2n_1-1}}}$$

So we have

$$\sum_{x\in\Theta_k^{n_1}} |\phi_E(x)|^2 = 1 - \sum_{x\in\Lambda_{L_{k-1}}\backslash\Theta_k^{n_1}} |\phi_E(x)|^2 \geq 1 - CL^{\beta d}e^{-m'L^{\rho^{2n_1-1}}} \geq 1/2$$

when L is large enough.

$$\#B_{k-1}\sum_{x\in\Theta_k^{n_1}} |\phi_E(x)|^2 \leq tr\{P_{\Theta_k^{n_1}}P_{\mathcal{I}}(H_{\omega,\Lambda_{L_{k-1}}})\} \leq C|\Theta_k^{n_1} \leq CL^{\beta d}$$

Thus $\#B_{k-1} \leq 2CL^{\beta d}$ and using the inductive estimate from layer L_{n_1} to layer L_1, we have

$$\#D_0^{n_1} \leq CL_{n_1}(L^{\beta d})^{n_1} \leq CL^{(n_1+1)\beta d}$$

□

proof of Theorem 3.7. $\chi_{L,x_0} = \tilde{Q}_{L,x_0} \cup \mathcal{N}_{L,x_0}$ provides the desired event. □

proof of Theorem 3.3. For fixed E, let $\mathcal{M}_{L,x_0}^{(E)} = \{E : W_{\omega,x_0,L}(E) \leq e^{-ML}\}$, then $\mathbb{P}(\mathcal{M}_{L,x_0}^{(E)}) \geq 1 - C_{p,d}L^{-pd}$. The goal here is to extend the estimate uniformly for $E \in \mathcal{I}$ while maintaining the probability estimates. We achieve this by a judicious choice of E_n and then estimating the remaining terms by through E_n. We set

$$\mathcal{M}_{L,x_0} = \bigcap_{E\in\sigma^{\mathcal{I},red}(H_{\omega,\Lambda_L(x_0)})} \mathcal{M}_{L,x_0}^{(E)}$$

and we have $\mathbb{P}(\mathcal{M}_{L,x_0}) \geq 1 - CL^{-(p-(n_1+1)\beta)d}$.

Also by our assumptions, $W_{\omega,x_0} \geq e^{-ML^{\beta/2}} \geq e^{-\hat{m}\sqrt{\frac{L^\beta}{K}}}$, whenever $K \geq 900$, so we are free to choose $K = 900$ for Theorem 3.6. Having done this, if we take $b = 1 + \frac{1}{\beta}(p - (n_1+1)\beta)$, and take $\mathcal{U}_{L,x_0} = \chi_{L,x_0} \cup \mathcal{M}_{L,x_0}$, then $\mathbb{P}\{\mathcal{U}_{L,x_0}\} \geq 1 - L^{-\tilde{p}d}$.

Now if $\omega \in \mathcal{U}_{L,x_0}$, by Theorem 3.7, we have

$$\text{dist}(E, \sigma^{(\mathcal{I},red)}(H_{\omega,\Lambda_L(x_0)})) \leq e^{-\frac{\hat{m}}{K}L}$$

i.e. there exists $E_0 \in \sigma^{(\mathcal{I},red)}(H_{\omega,\Lambda_L(x_0)})$ s.t. $|E - E_0| \leq e^{-\frac{\hat{m}}{K}L}$. Since craig $W_{\omega,x_0,L}(E_0) \leq e^{-\frac{\hat{m}}{K}L}$, by the stability of $W_{\omega,x_0,L}$, we have $W_{\omega,x_0,L}(E) \leq e^{-\frac{\hat{m}}{K}L}$.

□

3.6 Dynamical Localization in Expectation

In this section, we extract localization results from Theorem 3.3.

proof of Theorem 3.4. By (3.1), we have

$$\begin{aligned}
\|\chi_{x,L} f(H_\omega) P_\omega(I)\delta_x\|_1 &\leq \int_I |f(E)| \|\chi_{x,L} Q_\omega(E)\delta_x\|_1 d\mu_\omega(E) \\
&\leq \int_I |f(E)| \|\chi_{x,L} Q_\omega(E)\|_2 \|\delta_x Q_\omega(E)\|_2 d\mu_\omega(E) \\
&\leq \int_I |f(E)| |W_\omega(x;E) W_\omega(x;E)| \mu_{\omega,x}(\{E\}) d\mu_\omega(E) \\
&\leq \mu_\omega(I) ||f||_{L^\infty(I,d\mu_\omega)} \sup_{E\in I} |W_\omega(x;E) W_\omega(x;E)|
\end{aligned}$$

where by Theorem 3.3, we have

$$\begin{aligned}
\mathbb{E}\left\{\|W_{\omega,x_0}(E) W_{\omega,x_0,L}(E)\|^s_{L^\infty(I,d\mu_\omega(E))}\right\} &\leq C e^{-\frac{s}{2}L^\mu} \mathbb{P}\{\mathcal{U}^c_{x_0,L}\} + C2^{s\nu} L^{s\nu} \mathbb{P}\{\mathcal{U}_{x_0,L}\} \\
&\leq C e^{-\frac{s}{2}L^\mu} + C2^{s\nu} L^{s\nu} L^{\tilde{p}d} \\
&\leq C L^{(s\nu - \tilde{p}d)}
\end{aligned}$$

Thus

$$\mathbb{E}\left\{\|\chi_{x,L} f(H_\omega) P_\omega(I)\delta_x\|_1^s\right\} \le C||f||_{L^\infty(I,d\mu_\omega)} L^{-(\tilde{p}d-s\nu)}$$

By taking $L = 2^k$ above and summing it up, we get

$$\begin{aligned}\mathbb{E}\left\{\sup_t \left\|\langle X-x\rangle^{bd} e^{-itH_\omega} P_\omega(I)\delta_x\right\|^s\right\} &\le C\sum_k 2^{\frac{sbd}{2}+(k+1)sbd-k(\tilde{p}d-s\nu)} \\ &\le C\sum_k \left(2^{sbd-\tilde{p}d+s\nu}\right)^k < \infty.\end{aligned}$$

where $s < \frac{\tilde{p}d}{bd+\nu}$. □

Part II

Relatively flat bands of TBG

Chapter 4

Magnetic response of Twisted bilayer graphene

4.1 Introduction

The introduction is already given in Section 0.2, we only recall the BM model and outline below for the reader's convenience:

The BM model is an effective 4×4 matrix-valued Hamiltonian $\begin{pmatrix} H_D^\theta & T^\theta(x) \\ (T^\theta(x))^* & \tilde{H}_D^{-\theta} \end{pmatrix}$, $x \in \mathbb{R}^2$, composed of two twisted-Dirac-operators $H_D^\theta, H_D^{-\theta}$ representing two isolated graphene sheets [84] respectively, and a tunneling potential term

$$T^\theta(x) = \begin{pmatrix} \alpha_0 V(x/\lambda_\theta) & \alpha_1 \overline{U}(-x/\lambda_\theta) \\ \alpha_1 U(-x/\lambda_\theta) & \alpha_0 V(x/\lambda_\theta) \end{pmatrix}$$

where the diagonal and off-diagonal terms represent two different types of interlayer

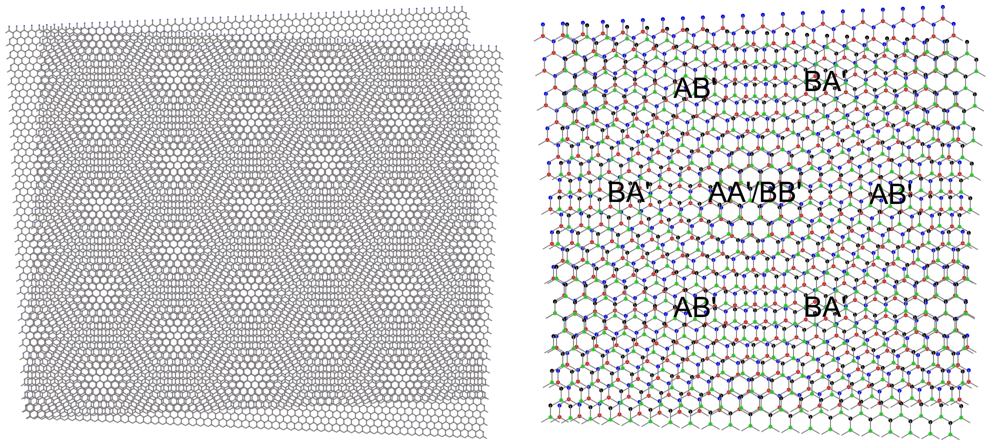


Figure 4.1: Left: Visible moiré pattern at $\theta = 5°$. Right: Single moiré hexagon, with (A=red, B=blue) and (A'=green, B'=black) denote vertices of two sheets of graphene respectively.

tunneling potentials. In fact, when two layers of graphene are twisted at an angle θ, a macroscopic honeycomb structure of scale λ_θ, called the moiré pattern, is formed (by a purely geometrical superposition of two sheets of graphene; see Fig.4.1). Then the two different types of interlayer tunnelings (see Fig.4.1) are respectively:

1. the chiral tunnelings $U(x/\lambda_\theta)$ and $\overline{U}(-x/\lambda_\theta)$ localized near the vertices of each moiré hexagon, with tunneling strength α_1 and a stacking similar to AB' and BA'-stacking;

2. the anti-chiral tunneling $V(x/\lambda_\theta)$, localized near the centers of moiré hexagon, with a tunneling strength α_0 and a stacking similar to AA'/BB'-stacking.

Here A and B label the equivalence classes of vertices on the honeycomb lattice and atoms on the lower lattice are indicated by a prime, cf. Figure 4.1. We refer to the BM model as the *chiral* or *anti-chiral* model in the limit of purely chiral ($\alpha_0 = 0$) or anti-chiral ($\alpha_1 = 0$) tunneling interaction, respectively.

Outline and all results. We summarize all our main results with an outline of the paper below:

- In Section 4.2, we introduce the BM model with external magnetic field for TBG.
- In Section 4.3, we proved that
  - periodic magnetic fields do not affect the presence of flat bands in Theorem 4.1.
  - flat bands are persisted under rational magnetic flux in Theorem 4.2, 4.3.
  - lots of quasimodes are located close to, and squeezing towards the zero energy level in Theorem 4.4.
- In Section 4.4, we discuss general properties of the DOS including
- In Section 4.5, we derive asymptotic formulae for the DOS:
  - of the chiral model: Theorem 4.5;
  - of the anti-chiral model: Theorem4.6;
  - is termwise-deferentiable w.r.t. B: Prop 4.5.9).
- In Section 4.6, we discuss physical applications of our semiclassical formulae.
- The article also contains two technical appendices to which some of the computations and auxiliary results for the derivation of the DOS are outsourced.

4.2 Introduction of magnetic BM model

We start by introducing relevant notation.

Notation. *Throughout this article we identify* $\mathbb{R}^2 \simeq \mathbb{C}$ *by* $x = (x_1, x_2) \simeq z = x_1 + ix_2$. *We denote by* L *the Lebesgue measure on* $\mathbb{R}^2 \sim \mathbb{C}$. *For functions of complex variables* $f(z, \bar{z})$ *we often just write* $f(z)$. *We write* $f = \mathcal{O}_\alpha(g)_H$ *if there is a constant* C_α *such that* $\|f\|_H \le C_\alpha g$. *In particular,* $f = \mathcal{O}(h^\infty)_H$ *means that for any* N *there exists* C_N *such that* $\|f\|_H \le C_N h^N$. *We also use the short notations* $\langle x \rangle := \sqrt{1 + |x|^2}$, $B_r(x) = \{y : |y - x| \le r\}$.

We introduce the symbol class $S(\mathbb{R}^{2n}; \mathscr{H}) := \Big\{ p \in C^\infty(\mathbb{R}^{2n} \times \mathbb{R}_{>0}) : \exists h_0,\ \text{for all } \gamma \in \mathbb{N}^2, \exists c_\gamma > 0 \quad \text{s.t. for all } (x, \xi) \in \mathbb{R}^{2n} \text{ for all } h \in (0, h_0) : |D^\gamma_{(x,\xi)} p(x, \xi, h)| \le c_\gamma \Big\}$. *In addition, let* $S^k_\delta(\mathbb{R}^2_{x,\xi})$ *denote the class of symbols* $a \in C^\infty(\mathbb{R}^{2n} \times \mathbb{R}_{>0})$ *such that*

$$|\partial^\alpha_x \partial^\beta_\xi a(x, \xi; h)| \le C_{\alpha,\beta} h^{-k-\delta(\alpha+\beta)}, \quad \text{for all } \alpha, \beta > 0.$$

We denote standard partial derivatives by ∂_{x_i} *and accordingly* $D_{x_i} := -i\partial_{x_i}$. *The principal symbol of a semiclassical operator* $a(x, hD_x)$ *is denoted by* $\sigma_0(a(x, hD_x))$. *We say a symbol* a *has an asymptotic expansion in* S^k_δ, $a \sim \sum_{j=0}^\infty a_j$, *if* $a \in S^k_\delta$ *and there is a sequence of* $a_j \in S^{k_j}_\delta$ *s.t.* $k_j \to -\infty$ *as* $j \to \infty$ *and* $a - \sum_{j=0}^N a_j \in S^{k_{N+1}}_\delta$. *When* k *or* $\delta = 0$, *we omit the respective sub and superscript. The spectrum of a linear operator* T *is denoted by* $\operatorname{Spec}(T)$. *We also introduce rotated Pauli matrices* $\sigma^\theta_k = e^{-i\frac{\theta}{4}\sigma_3} \sigma_k e^{i\frac{\theta}{4}\sigma_3}$, *for* $k = 1, 2$.

4.2.1 Moiré lattices and TBG

We recall from the introducting that by twisting two honeycomb lattices against each other, the emerging moiré honeycomb pattern exhibits different scales λ_θ at different twisting angles θ.[1] Thus it is easier to characterize such macroscopic honeycomb

[1]In fact, $\lambda_\theta = \frac{C\sqrt{3}}{2\sin(|\theta|/2)}$ when $|\theta| < \pi/6$ by [70, 33].

structures using a "unit-size honeycomb lattice" of side length $\frac{4\pi}{\sqrt{3}}$:

Let $\omega = \exp(\frac{2\pi i}{3})$, $\zeta_1 = 4\pi i\omega$, $\zeta_2 = 4\pi i\omega^2$. The "unit-size honeycomb lattice" is invariant under translations along a triangular lattice $\Gamma = \zeta_1\mathbb{Z} \oplus \zeta_2\mathbb{Z}$. We denote its unit cell, dual lattice, and the Brillouin zone of the dual lattice by $E = \mathbb{C}/\Gamma$, $\Gamma^* = \eta_1\mathbb{Z} \oplus \eta_2\mathbb{Z}$, and $E^* = \mathbb{C}/\Gamma^*$, where $\eta_1 = \frac{\omega^2}{\sqrt{3}}$ and $\eta_2 = -\frac{\omega}{\sqrt{3}}$. We also define the corresponding terms for a scaled honeycomb lattice of scale λ by $\Gamma_\lambda = \lambda\zeta_1\mathbb{Z} \oplus \lambda\zeta_2\mathbb{Z}$, $E_\lambda = \mathbb{C}/\Gamma_\lambda$, $\Gamma^*_\lambda = \lambda^{-1}\eta_1\mathbb{Z} \oplus \lambda^{-1}\eta_2\mathbb{Z}$ and $E^*_\lambda = \mathbb{C}/\Gamma^*_\lambda$. Let $T_1, T_2 \in \mathcal{L}(L^2(\mathbb{C}))$ be the standard translation operators $(T_i u)(x) := u(x - \zeta_i)$.

4.2.2 Chiral and anti-chiral tunnelings

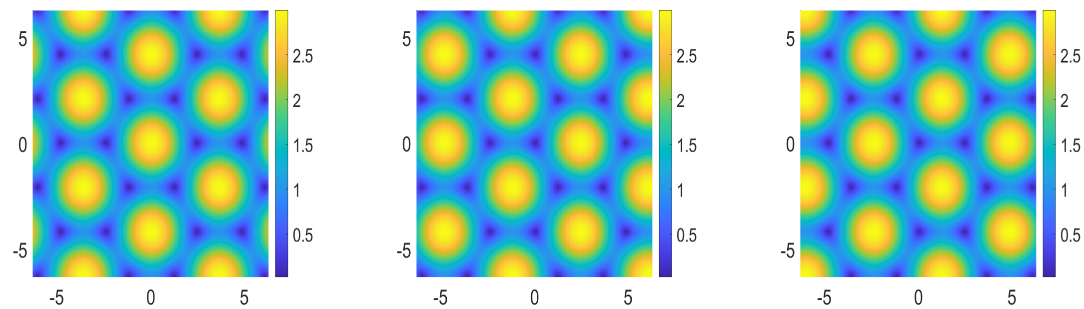
The chiral and anti-chiral tunneling potentials, V and U, are smooth "unit-size" periodic functions (cf. [12]) satisfying for $\mathbf{a}_j = \frac{4}{3}\pi i\omega^j$ with $j = 0, 1, 2$ the following symmetries

$$V(z + \mathbf{a}_j) = \bar{\omega}V(z), \quad V(\omega z) = V(z), \quad \overline{V(z)} = V(-z), \quad V(\bar{z}) = V(-z),$$
$$U(z + \mathbf{a}_j) = \bar{\omega}U(z), \quad U(\omega z) = \omega U(z), \quad \overline{U(\bar{z})} = U(z).$$

In particular, since $\zeta_1 = 3\mathbf{a}_1$, $\zeta_2 = 3\mathbf{a}_2$, we have $V(z+\zeta_j) = V(z)$ and $U(z+\zeta_j) = U(z)$ for $j = 1, 2$. Thus $V(z)$, $U(z)$, $U_-(z) := U(-z)$ are periodic on Γ. The tunneling potentials on the physical moiré scale are then $V(z/\lambda_\theta), U(z/\lambda_\theta), \overline{U_-}(z/\lambda_\theta)$.

4.2.3 Magnetic BM model with Adiabatic scaling

To introduce the BM model with magnetic field we start with the physical or adiabatic scaling. Since we will immediately change to a semiclassical scaling, we denote all



(a) tunneling potential $|V|^2$ for AA′/BB′-coupling. (b) tunneling potential $|U|^2$ for AB′-coupling. (c) tunneling potential $|U_-|^2$ for BA′-coupling.

Figure 4.2: The tunneling potentials for different coupling types on unit-size honeycomb lattice.

objects with a "$\sim$" in this paragraph. Let $\tilde{A}(\tilde{z}) = (\tilde{A}_1(\tilde{z}), \tilde{A}_2(\tilde{z}), 0) \in C^\infty(\mathbb{C};\mathbb{R}^3)$ be the magnetic vector potential of a magnetic field perpendicular to the TBG. The tunneling potentials, U and V, defined on the "unit-size honeycomb lattice" are then rescaled to the physical moiré-size by rescaling coordinates by λ_θ. Thus the magnetic BM model is $\tilde{\mathscr{H}}^\theta : D(\tilde{\mathscr{H}}^\theta) \subset L^2(\mathbb{C};\mathbb{C}^4) \to L^2(\mathbb{C};\mathbb{C}^4)$

$$\tilde{\mathscr{H}}^\theta := \tilde{\mathscr{H}}_0^\theta + \tilde{\mathscr{V}} := \begin{pmatrix} \tilde{H}_D^\theta & 0 \\ 0 & \tilde{H}_D^{-\theta} \end{pmatrix} + \begin{pmatrix} 0 & \tilde{T}^\theta \\ (\tilde{T}^\theta)^* & 0 \end{pmatrix}$$

with $\tilde{H}_D^\theta = \sum_{i=1}^2 \sigma_i^\theta (D_{\tilde{x}_i} - \tilde{A}_i(\tilde{z}))$ and $\tilde{T}^\theta(\tilde{z}) = \begin{pmatrix} \tilde{\alpha}_0 V(\tilde{z}/\lambda_\theta) & \tilde{\alpha}_1 \overline{U_-}(\tilde{z}/\lambda_\theta) \\ \tilde{\alpha}_1 U(\tilde{z}/\lambda_\theta) & \tilde{\alpha}_0 V(\tilde{z}/\lambda_\theta) \end{pmatrix}$, where λ_θ, U and V are given above and $\tilde{\alpha}_i$ represent the tunneling strength, $i = 1, 2$.

4.2.4 Magnetic BM model with Semiclassical Scaling

We shall now rescale the Hamiltonian in the previous paragraph to "unit-size" and multiply the Hamiltonian by λ_θ to work in another more convenient scaling called the *semiclassical scaling*: Let $z = \tilde{z}/\lambda_\theta$, $\alpha_i = \lambda_\theta \tilde{\alpha}_i$, $A_i(z) = \lambda_\theta \tilde{A}_i(\lambda_\theta z)$ (overall represented by a unitary operator U), we consider

$$\mathscr{H}^\theta(z) := \lambda_\theta (U \tilde{\mathscr{H}}^\theta U^{-1})(z) = \begin{pmatrix} H_D^\theta & 0 \\ 0 & H_D^{-\theta} \end{pmatrix} + \begin{pmatrix} 0 & T(z) \\ T(z)^* & 0 \end{pmatrix} =: \mathscr{H}_0^\theta + \mathscr{V}(z), \quad (4.1)$$

where $H_D^\theta = \sum_{i=1}^2 \sigma_i^\theta (D_{x_i} - A_i(z))$, or equivalently, $H_D^\theta = e^{-i\frac{\theta}{4}\sigma_3} H_D e^{i\frac{\theta}{4}\sigma_3}$ where

$$H_D = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \text{ with } \begin{cases} a = 2D_z - \overline{A(z)} \\ a^* = 2D_{\overline{z}} - A(z) \end{cases}, \ T(x) = \begin{pmatrix} \alpha_0 V(z) & \alpha_1 \overline{U_-}(z) \\ \alpha_1 U(z) & \alpha_0 V(z) \end{pmatrix}. \quad (4.2)$$

We denote the *chiral model* by $\mathscr{H}_{\mathrm{c}}^\theta = \mathscr{H}^\theta|_{\alpha_0=0}$ and the *anti-chiral model* by $\mathscr{H}_{\mathrm{ac}}^\theta = \mathscr{H}^\theta|_{\alpha_1=0}$.

Remark 4.1 (Strong magnetic field). Here $\alpha_i = \alpha_i(\theta)$ depends on θ, but we shall not emphasize this dependence further. Instead, we observe that small twisting angles naturally correspond, for constant magnetic fields, to the limiting regimes $\alpha \gtrsim 1$ and $B \gg 1$. This provides the basis of our study of large magnetic fields which we coin the *semiclassical scaling*.

Remark 4.2 (Periodic magnetic potentials). When considering periodic magnetic potentials $A \in C^\infty(\mathbb{C}/\Gamma')$, where Γ' is commensurable with respect to the moiré lattice Γ, it suffices to consider a larger lattice Γ'' with respect to which both the magnetic potentials and tunnelling potentials are periodic, i.e. $\Gamma'' \subset n_1\Gamma \cap n_2\Gamma'$ for some $n \in \mathbb{N}^2$. Therefore, we shall restrict us, when discussing periodic magnetic potentials, to potentials that are just periodic with respect to Γ.

Bloch-Floquet theory for magnetic BM. We now recall the standard Bloch-Floquet theory when the magnetic potential is composed of a linear part $A_{\mathrm{con}}(z) =$

$-\frac{B}{2}zi$ and a periodic part $A_{\text{per}} \in C^\infty(E;\mathbb{C})$, which corresponds to a constant magnetic field B and a zero-flux magnetic field respectively, and the electric potential is periodic $\mathscr{V} \in C^\infty(E;\mathbb{C})$.

Lemma 4.2.1. *Let $\lambda \in \mathbb{Z} \setminus \{0\}$, $B = \frac{\mu}{8\pi\lambda^2 \operatorname{Im}(\omega)}$ for some $\mu \in \mathbb{Z}$. Let $\Gamma_\lambda = \gamma_1\mathbb{Z} \times \gamma_2\mathbb{Z}$ with $\gamma_1 = 4\pi\omega i\lambda$, $\gamma_2 = 4\pi\omega^2 i\lambda$. Define the magnetic translations for all $m \in \mathbb{Z}^2$ and $\gamma \in \Gamma_\lambda$ by*

$$\mathbb{T}_{m_1\gamma_1+m_2\gamma_2} := e^{im_1m_2\mu\pi} T_{m_1\gamma_1+m_2\gamma_2}, \text{ where } T_\gamma u(z) = \exp\left(\frac{iB}{2}\operatorname{Im}(\bar{\gamma}z)\right) u(z+\gamma). \tag{4.3}$$

There is a unitary map $\mathcal{U}_B : L^2(\mathbb{C};\mathbb{C}^n) \to \mathcal{H}_B$ defined by

$$\mathcal{U}_B u(\mathbf{k}, z) = \sum_{\gamma\in\Gamma_\lambda} e^{-i\operatorname{Re}((\gamma+z)\bar{\mathbf{k}})} \mathbb{T}_\gamma u(z), \qquad \text{for all } z, \mathbf{k} \in \mathbb{C}, \text{ where} \tag{4.4}$$

$$\mathcal{H}_B = \left\{ v \in L^2(\mathbb{C}^2;\mathbb{C}^n) : \mathbb{T}_\gamma v_{\mathbf{k}} = v_{\mathbf{k}}, v_{\mathbf{k}+\mathbf{k}'} = e^{-i\operatorname{Re}(\bullet\overline{\mathbf{k}'})} v_{\mathbf{k}}, \text{ for all } \gamma \in \Gamma_\lambda, \mathbf{k}' \in \Gamma_\lambda^* \right\},$$

such that

$$\mathcal{U}_B \mathscr{H}^\theta(z, D_z)\mathcal{U}_B^{-1} = \int_{E_\lambda^*}^{\oplus} \mathscr{H}_{\mathbf{k}}^\theta(z, D_z) d\mathbf{k} \text{ where } \mathscr{H}_{\mathbf{k}}^\theta(z, D_z) = \mathscr{H}^\theta(z, D_z + \tfrac{\bar{\mathbf{k}}}{2}). \tag{4.5}$$

where each $\mathscr{H}_{\mathbf{k}}^\theta$ is defined on $L^2_B(E_\lambda) := \{f \in L^2_{\text{loc}}(\mathbb{C}) : \mathbb{T}_\gamma f = f, \text{ for all } \gamma \in \Gamma_\lambda\}$.

Proof. Recall by (4.1), $a = 2D_z - \bar{A}_{\text{con}}(z) - \bar{A}_{\text{per}}(z)$. By direct computation, we get

$$\begin{cases} [T_\gamma, 2D_z - \bar{A}_{\text{con}}] = [T_\gamma, 2D_{\bar{z}} + A_{\text{con}}] = [T_\gamma, A_{\text{per}}] = [T_\gamma, \mathscr{V}] = 0, & \text{for all } \gamma \in \Gamma_\lambda, \\ T_\alpha T_\beta = \exp\left(\frac{iB}{2}\operatorname{Im}(\bar{\beta}\alpha)\right) T_{\alpha+\beta} = \exp\left(iB\operatorname{Im}(\bar{\beta}\alpha)\right) T_\beta T_\alpha, & \text{for all } \alpha, \beta \in \Gamma_\lambda. \end{cases}$$

In particular, for B, λ as specified in the Lemma, we have $\frac{iB}{2}\operatorname{Im}(\bar{\gamma}_2\gamma_1) = -ik_2\pi$. Then

$T_{\gamma_1}T_{\gamma_2} = (-1)^{k_2}T_{\gamma_1+\gamma_2}$. Thus, with $\mathbb{T}_\gamma$ defined as in (4.3), we have

$$\begin{cases} \mathbb{T}_\alpha \mathbb{T}_\beta = \mathbb{T}_{\alpha+\beta} = \mathbb{T}_\beta \mathbb{T}_\alpha, & \text{for all } \alpha, \beta \in \Gamma_B, \\ [\mathbb{T}_\gamma, a] = [\mathbb{T}_\gamma, a^*] = [\mathbb{T}_\gamma, \mathscr{V}] = [\mathbb{T}_\gamma, \mathscr{H}^\theta] = 0, & \text{for all } \gamma \in \Gamma_B. \end{cases} \tag{4.6}$$

Thus $\{\mathbb{T}_\gamma\}_{\gamma\in\Gamma_B}$ forms an abelian group of transformations commuting with $\mathscr{H}^\theta$. Furthermore, one can check by (4.4) and (4.6) that

$$\begin{cases} \mathbb{T}_\gamma U_B u(\mathbf{k}, z) = U_B u(\mathbf{k}, z), & \text{for all } \gamma \in \Gamma_B, \\ U_B u(\mathbf{k}+\mathbf{k}', z) = e^{-i\operatorname{Re}(\mathbf{k}'\bar{z})} U_B u(\mathbf{k}, z), & \text{for all } \mathbf{k}' \in \Gamma_B^*, z \in \mathbb{C}. \end{cases} .$$

Finally, from $[\mathbb{T}_\gamma, a] = 0$, $e^{-i\operatorname{Re}((\gamma+z)\bar{\mathbf{k}})} a e^{i\operatorname{Re}((\gamma+z)\bar{\mathbf{k}})} = a + \mathbf{k}$ and $\mathcal{U}_B \mathscr{V} \mathcal{U}_B^{-1} = \mathscr{V}$, we conclude (4.5).

□

Remark 4.3 (Floquet transformed operators). In this article, we shall use the convention that for operators S that commute with translations $\{\mathbb{T}_\gamma\}_{\gamma\in\Gamma_\lambda}$, as defined in Lemma 4.2.1 we introduce the family of operators $\int_{E_\lambda^*}^{\oplus} S_{\mathbf{k}} \, d\mathbf{k} := \mathcal{U}_B S \mathcal{U}_B^*$, where $S_{\mathbf{k}}(z, D_z) = S(z, D_z + \frac{\bar{\mathbf{k}}}{2})$.

Proof. Assume $\psi_{0,\mathbf{k}}(z) = e^{-S} f_{\mathbf{k}}(\bar{z}) e^{-\frac{\bar{z}^2 B}{4}} e^{-i\operatorname{Re}(z\bar{\mathbf{k}})}$ for $S(z) = \frac{|z|^2 B}{4}$ and some analytic function $f_{\mathbf{k}}(z)$. By direct computation, one observes that $a_{\mathbf{k}}\psi_{0,\mathbf{k}} = 0$. Thus we only need to find an $f_{\mathbf{k}}$ such that $\psi_{0,\mathbf{k}} \in L^2_B(E_\lambda)$, i.e. $\mathbb{T}_{\gamma_j}\psi_{0,\mathbf{k}} = \psi_{0,\mathbf{k}}$, $j = 1, 2$. Let $\gamma_0 = -\gamma_1 - \gamma_2$. By linearity, it is enough to show $\mathbb{T}_{\gamma_j}\psi_{0,\mathbf{k}} = \psi_{0,\mathbf{k}}$ for $j = 0, 2$. It is more convenient to work with these two base vectors. Now applying e^S to both sides and noticing that $\tilde{\mathbb{T}}_{\gamma_j} := e^S \mathbb{T}_{\gamma_j} e^{-S} = \exp\left(-\frac{B}{4}\gamma_j(2\bar{z}+\bar{\gamma}_j) + \gamma_j\partial_z + \bar{\gamma}_j\partial_{\bar{z}}\right)$, we get

$$\begin{aligned}
&\tilde{\mathbb{T}}_{\gamma_j}\left(f_{\mathbf{k}}(\bar{z})e^{-\frac{\bar{z}^2 B}{4}}e^{-i\,\mathrm{Re}(z\bar{\mathbf{k}})}\right) = f_{\mathbf{k}}(\bar{z})e^{-\frac{\bar{z}^2 B}{4}}e^{-i\,\mathrm{Re}(z\bar{\mathbf{k}})}, \\
\Rightarrow \quad & \frac{f_{\mathbf{k}}(\bar{z}+\bar{\gamma}_i)}{f_{\mathbf{k}}(\bar{z})} = \exp\left(\frac{B}{4}(\gamma_j+\bar{\gamma}_j)(2\bar{z}+\bar{\gamma}_j) + \frac{i}{2}(\gamma_j\bar{\mathbf{k}}+\bar{\gamma}_j\mathbf{k})\right).
\end{aligned}$$

In particular, let $w = \bar{z}$, for $\gamma_0 = 4\pi\lambda i$ and $\gamma_2 = 4\pi\omega^2\lambda i$, we have

$$\begin{aligned}
&\frac{f_{\mathbf{k}}(w-4\pi\lambda i)}{f_{\mathbf{k}}(w)} = e^{4\pi\lambda\,\mathrm{Im}(\mathbf{k})i}, \\
&\frac{f_{\mathbf{k}}(w-4\pi\lambda\omega i)}{f_{\mathbf{k}}(w)} = \exp\left[2\pi\lambda B\,\mathrm{Im}\,\omega(2w-4\pi\lambda\omega i) + 2\pi\lambda(\omega\mathbf{k}-\bar{\omega}\bar{\mathbf{k}})\right].
\end{aligned} \tag{4.7}$$

General functions that satisfy such boundary conditions are $f_{\mathbf{k}}(z) = e^{\theta z}\vartheta_{\frac{1}{2},\frac{1}{2}}(\frac{z}{-4\pi\lambda i} - z_{\mathbf{k}}|\omega)$ where θ and $z_{\mathbf{k}}$ are to be determined to satisfy (4.7) and $\vartheta_{\frac{1}{2},\frac{1}{2}}(z|\tau)$ is a Jacobi theta function. Using the properties of Jacobi theta functions below

$$\begin{aligned}
&\vartheta_{a,b}(z|\tau) := \sum_{n\in\mathbb{Z}} \exp(\pi i(a+n)^2\tau + 2\pi i(n+a)(z+b)), \quad \mathrm{Im}\,\tau > 0, \\
&\vartheta_{a,b}(z+1|\tau) = e^{2\pi i a}\vartheta_{a,b}(z|\tau), \quad \vartheta_{a,b}(z+\tau|\tau) = e^{-2\pi i(z+b)-\pi i\tau}\vartheta_{a,b}(z|\tau),
\end{aligned}$$

we see

$$\begin{aligned}
&\frac{f_{\mathbf{k}}(z-4\pi\lambda i)}{f_{\mathbf{k}}(z)} = \exp(\pi i(1-4\theta\lambda)), \\
&\frac{f_{\mathbf{k}}(z-4\pi\lambda\omega i)}{f_{\mathbf{k}}(z)} = \exp(-4\pi\lambda\omega\theta i + \frac{z}{2\lambda} + 2\pi z_{\mathbf{k}} i - \pi i - \pi\omega i)
\end{aligned} \tag{4.8}$$

Comparing (4.7) with (4.8), and use $8\pi\lambda^2 B\,\mathrm{Im}(\omega^2) = k_2 = 1$, we find $\theta = \mathrm{Im}\,\mathbf{k} + \frac{1}{4\lambda}$ and $z_{\mathbf{k}} = 2\lambda\omega\mathbf{k} - i\lambda\omega\mathbf{k} - i\lambda\bar{\omega}\bar{\mathbf{k}} + \frac{1}{2} + \frac{\omega}{2}$.

Then since $[a_{\mathbf{k}}, a^*_{\mathbf{k}}] = 2B$, we can inductively define a sequence of $\psi_{n,\mathbf{k}}$ such that

$$a^*_{\mathbf{k}}\psi_{n,\mathbf{k}} = \sqrt{2B(n+1)}\psi_{n+1,\mathbf{k}} \quad \text{and} \quad a_{\mathbf{k}}\psi_{n,\mathbf{k}} = \sqrt{2Bn}\psi_{n-1,\mathbf{k}}.$$

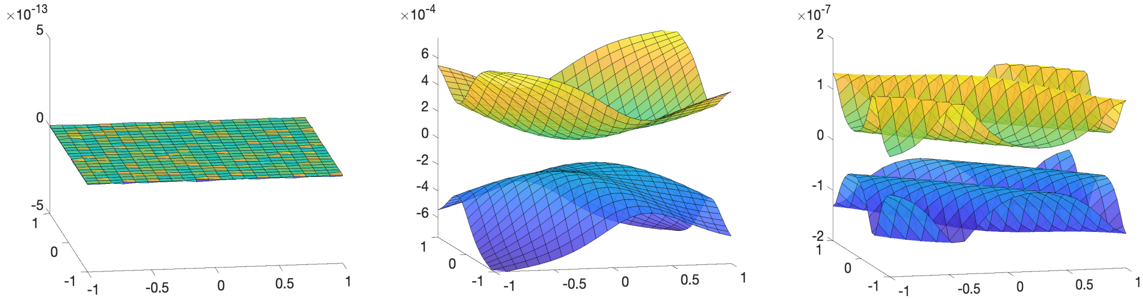


Figure 4.3: Constant magnetic field: On the left, flat bands for chiral model ($\alpha_1 = 1$); in the middle ($\theta = 0$) and on the right ($\theta = \pi$) non-flat bands for anti-chiral model, ($\alpha_0 = 1$).

By (4.2), one can check for $n \geq 0$ we have $H_D \begin{pmatrix} \psi_{n-1,\mathbf{k}} \\ \pm\psi_{n,\mathbf{k}} \end{pmatrix} = \pm\sqrt{2nB} \begin{pmatrix} \psi_{n-1,\mathbf{k}} \\ \pm\psi_{n,\mathbf{k}} \end{pmatrix}$. □

4.3 Spectral properties

In this section, we provide a basic spectral analysis of the magnetic BM model. We start by reducing the chiral and anti-chiral Hamiltonians to an off-diagonal form.

4.3.1 Spectral properties of chiral and anti-chiral model

The chiral model is described by the Hamiltonian (4.1) for $\alpha_0 = 0$, which after conjugation by $\mathscr{U} = \operatorname{diag}(e^{i\theta/4}, e^{-i\theta\sigma_3/4}, e^{i\theta/4})$, $\mathscr{H}_{\mathrm{c}} = \mathscr{U}\mathscr{H}^\theta\mathscr{U}$, reads

$$\mathscr{H}_{\mathrm{c}} = \begin{pmatrix} 0 & (\mathcal{D}_c)^* \\ \mathcal{D}_c & 0 \end{pmatrix} \text{ with } \mathcal{D}_c = \begin{pmatrix} 2D_{\bar{z}} - A_1(z) - iA_2(z) & \alpha_1 U(z) \\ \alpha_1 U_-(z) & 2D_{\bar{z}} - A_1(z) - iA_2(z) \end{pmatrix} \quad (4.9)$$

Instead, when setting $\alpha_1 = 0$ and conjugating by a unitary $\mathcal{V}$, with $\lambda = e^{i\frac{\pi}{4}}$, we obtain the *anti-chiral model* which is described by the Hamiltonian

$$\mathscr{H}^\theta_{\mathrm{ac}} := \mathcal{V}\mathscr{H}^\theta\mathcal{V} = \begin{pmatrix} 0 & (\mathcal{D}^\theta_{\mathrm{ac}})^* \\ \mathcal{D}^\theta_{\mathrm{ac}} & 0 \end{pmatrix} \text{ with}$$

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_1 & \mathcal{V}_2 \\ \mathcal{V}_2 & \mathcal{V}_1 \end{pmatrix} \text{ for } \mathcal{V}_1 = \begin{pmatrix} i\lambda & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & -\bar{\lambda} \end{pmatrix}, \tag{4.10}$$

$$\mathcal{D}^\theta_{\mathrm{ac}} = \begin{pmatrix} \alpha_0 V(z) & e^{i\theta/2}(2D_{\bar{z}} - (A_1(z) + iA_2(z))) \\ e^{i\theta/2}(2D_z - (A_1(z) - iA_2(z))) & \alpha_0\overline{V(z)} \end{pmatrix}.$$

The off-diagonal structore implies that for both the chiral and anti-chiral model with magnetic field, the spectrum is symmetric with respect to zero. In particular, let $U := (\sigma_3 \otimes \mathrm{id}_{\mathbb{C}^2})$ then it follows that $U\mathscr{H}_c U = -\mathscr{H}_c$ and $U\mathscr{H}^\theta_{\mathrm{ac}}U = -\mathscr{H}^\theta_{\mathrm{ac}}$.

We start by studying the existence of flat bands in magnetic fields that are periodic with respect to the moiré lattice. Consider the Hamiltonian $\mathscr{H}^\theta$ introduced in (4.1). We shall use the Floquet operators $\mathscr{H}^\theta_{\mathbf{k}}$ as introduced in Lemma 4.2.1 for quasi-momenta $\mathbf{k} \in \mathbb{C}$ acting on the fundamental cell $\mathbb{C}/\Gamma$ with periodic boundary conditions. We then introduce the parameter set, of flat bands at energy zero, for the chiral Hamiltonian

$$\mathcal{A}_{\mathrm{c}} := \left\{ \alpha_1 \in \mathbb{C}; 0 \in \bigcap_{\mathbf{k}\in\mathbb{C}} \mathrm{Spec}_{L^2(\mathbb{C}/\Gamma)}\left(\mathscr{H}^\theta_{c,\mathbf{k}}(\alpha_1)\right) \right\}$$

and denote the analogous set of α_0 for the anti-chiral model by $\mathcal{A}_{\mathrm{ac}}$. Our first theorem shows that in the chiral Hamiltonian, periodic magnetic fields do not affect the presence of flat bands as characterized in [79, 6] and shown to exist in [6, 87].

Theorem 4.1 (Magic angles–Periodic magnetic fields)**.** *Consider the BM model with* Γ*-periodic magnetic potentials* $A \in C^\infty(E; \mathbb{R}^2)$*:*

<u>For chiral Hamiltonian:</u> The magic angles are independent of the magnetic potential,

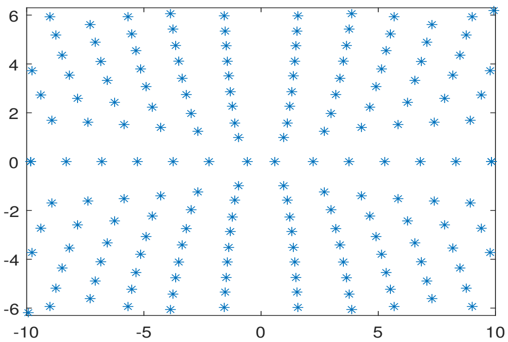


Figure 4.4: Spectrum of $T_{\mathbf{k}}$ for $\mathbf{k} \notin \Gamma^*$ with periodic magnetic field. The *TKV19* α_1 do not depend on the magnetic field strength.

i.e. $\alpha \in \mathcal{A}_c$ *for* $A = 0$ *if and only if* $\alpha \in \mathcal{A}_c$ *for non-zero* $A \in C^\infty(E;\mathbb{R}^2)$. *In particular, we have the upper bound* $|\mathcal{A}_c \cap B_R(0)| = \mathcal{O}(R^2)$.

For anti-chiral Hamiltonian: The anti-chiral Hamiltonian, with magnetic potentials as above, does not possess any flat bands at zero, i.e. $\mathcal{A}_{\mathrm{ac}} = \emptyset$.

We split the proof of Theorem 4.1 on the existence/absence of flat bands into two parts, separating the statement about the chiral Hamiltonian from the statement about the anti-chiral Hamiltonian. We start with a discussion of the chiral Hamiltonian.

Proof of Theo. 4.1, Chiral part. For the chiral Hamiltonian (4.9), $\alpha_0 = 0$, it suffices to analyze the nullspaces of the off-diagonal operators. Without loss of generality, we can study the nullspace of $\mathcal{D}_{\mathbf{k},c}(\alpha_1)$ where $0 \in \operatorname{Spec}(\mathcal{D}_{\mathbf{k},c})(\alpha_1) \Leftrightarrow \alpha_1^{-1} \in \operatorname{Spec}(T_{\mathbf{k}})$ with *Birman-Schwinger operator* $T_{\mathbf{k}} = (2D_{\bar z} - (A_1(z) + iA_2(z)) + \mathbf{k})^{-1} \begin{pmatrix} 0 & U(z) \\ U_-(z) & 0 \end{pmatrix}$ for $\mathbf{k} \notin \Gamma^*$. For any zero mode $\chi_{\mathbf{k}} \in L^2(E)$ to $D_{\mathbf{k},c}(\alpha_1, B = 0)$ it follows that $\psi_{\mathbf{k}} = \chi_{\mathbf{k}} \cdot \psi_0 \in L^2(E)$, with ψ_0 as in (E.3) solves

$$\mathcal{D}_{\mathbf{k},c}(\alpha_1)\psi_{\mathbf{k}} = \psi_0 \underbrace{\mathcal{D}_{\mathbf{k},c}(\alpha_1, B = 0)\chi_{\mathbf{k}}}_{=0} + \chi_{\mathbf{k}} \underbrace{(2D_{\bar z} - (A_1(z) + iA_2(z)))\psi_0(z)}_{=0} = 0.$$

This shows that $\mathscr{H}_c(B=0)$ possesses a flat band if and only if $\mathscr{H}_c(B)$ possesses one for B a Γ-periodic magnetic field. That magic angles α_1 of the chiral Hamiltonian can then be characterized by reciprocals of eigenvalues of $T_{\mathbf{k}}$ with $A=0$ follows then from [6, Theo. 2]. We now utilize the compactness of $T_{\mathbf{k}}$, with $A=0$, to give an upper bound on the number of magic angles.

Indeed, let $z = x_1 + ix_2 = 2i\omega(y_1 + i\omega y_2)$, we shall here consider $D_{\bar{z}}$ and $\mathcal{V}$ in new coordinates (y_1, y_2). Thus, decomposing for $A_N := \Pi_N T_{\mathbf{k}}$, with $\Pi_N : L^2(\mathbb{R}^2/2\pi\mathbb{Z}^2;\mathbb{C}^2) \to \ell^2(\mathbb{Z}^2_{2N+1};\mathbb{C}^2)$ such that $\Pi_N\left(\sum_{n\in\mathbb{Z}^2} a_{\mathbf{n}} e^{i\langle y,\mathbf{n}\rangle}\right) = \{a_{(n_1,n_2)}\}_{|n_j|\le N}, a_{\mathbf{n}} \in \mathbb{C}^2$ and $B_N := T_{\mathbf{k}} - A_N$ we can estimate specializing to $\mathbf{k} = 1/2$

$$\begin{aligned}
\|A_N\|_1 &\le \|\Pi_N(D_{\bar{z}} - \mathbf{k})^{-1}\|_1 \|\mathcal{V}\| \\
&\le \sqrt{3}\|U\|_\infty \sum_{|m|_\infty \le N} \frac{1}{\left|\left(m_1+\frac{1}{2}\right)^2 + \left(m_1+\frac{1}{2}\right)\left(m_2+\frac{1}{2}\right) + \left(m_2+\frac{1}{2}\right)^2\right|^{1/2}} \\
&= \sqrt{3}\|U\|_\infty \Bigg(\sum_{|m|_\infty \le 2} \left| \left(m_1+\tfrac{1}{2}\right)^2 + \left(m_1+\tfrac{1}{2}\right)\left(m_2+\tfrac{1}{2}\right) + \left(m_2+\tfrac{1}{2}\right)^2 \right|^{-1/2} \\
&\quad + \sum_{2<|m|_\infty \le N} \left| \left(m_1+\tfrac{1}{2}\right)^2 + \left(m_1+\tfrac{1}{2}\right)\left(m_2+\tfrac{1}{2}\right) + \left(m_2+\tfrac{1}{2}\right)^2 \right|^{-1/2} \Bigg) \\
&\le \sqrt{3}\|U\|_\infty \left(17 + \int_{1/2}^{\sqrt{2}N} \int_0^{2\pi} \frac{1}{\sqrt{1+\frac{1}{2}\sin(2\varphi)}}\, d\varphi \right) \\
&= \sqrt{3}\|U\|_\infty \left(17 + (\sqrt{2}N - \tfrac{1}{2})7 \right) \le C_2 N.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
\|B_N\| &\le \sup_{|m|_\infty > N} \left| \left(m_1+\tfrac{1}{2}\right)^2 + \left(m_1+\tfrac{1}{2}\right)\left(m_2+\tfrac{1}{2}\right) + \left(m_2+\tfrac{1}{2}\right)^2 \right|^{-1/2} \|U\|_\infty \\
&\le \frac{\|U\|_\infty}{(N-\frac{1}{2})} \le \frac{C_1}{N}.
\end{aligned}$$

Thus, for $|\alpha_1| \le R$ we take N, large enough, such that $\|\alpha_1 B_N\| \le \frac{RC_1}{N} < 1/2$. Thus, we may pick $N = \lceil RC_1/2 \rceil$. Hence, we can write $1-\alpha_1 T_{\mathbf{k}} = (1-\alpha_1 B_N)(1-(1-\alpha_1 B_N)^{-1}\alpha_1 A_N)$ and the magic α_1's are the zeros of $f(z) = \det(1-(1-\alpha_1 B_N)^{-1}\alpha_1 A_N)$. Using the standard bound for Fredholm determinants, we have $|f(z)| \le e^{2R\|A_1\|_1} \le e^{2RC_2N} \le e^{R^2C_1C_2}$. Hence, as $f(0)=1$, Jensen's formula implies that the number n of zeros of f for $\alpha \in B_0(R)$ is bounded by $n(R) \le \log(2)^{-1}\Big(4R^2C_1C_2\Big)$. □

We now continue by showing that the anti-chiral Hamiltonian does not possess flat bands at energy zero.

Proof of Theo.4.1, Anti-Chiral part. By (4.10), we have to study the invertibility of

$$\mathcal{K}_{\mathbf{k}}(\alpha_0) := \begin{pmatrix} \lambda & Q_{\mathbf{k}}(\alpha_0) \\ Q_{\mathbf{k}}(\alpha_0)^* & \lambda \end{pmatrix} \text{ where } Q_{\mathbf{k}}(\alpha_0) := \begin{pmatrix} \alpha_0 e^{-i\theta/2}V(z) & \mathscr{D}_z(\bar{\mathbf{k}}, A) \\ \mathscr{D}_{\bar{z}}(\mathbf{k}, A) & \alpha_0 e^{-i\theta/2}\overline{V(z)} \end{pmatrix}$$

and we introduced

$$\mathscr{D}_z(\bar{\mathbf{k}}, A) = 2D_z + \bar{\mathbf{k}} - (A_1(z) - iA_2(z)) \text{ and } \mathscr{D}_{\bar{z}}(\mathbf{k}, A) = 2D_{\bar{z}} + \mathbf{k} - (A_1(z) + iA_2(z)).$$

We shall omit the α_0 dependence and set $\theta = 0$ to simplify notation. The formal inverse of $\mathcal{K}_{\mathbf{k}}$ is given by $\mathcal{K}_{\mathbf{k}}^{-1} = \begin{pmatrix} \lambda(\lambda^2 - Q_{\mathbf{k}}Q_{\mathbf{k}}^*)^{-1} & -Q_{\mathbf{k}}(\lambda^2 - Q_{\mathbf{k}}^*Q_{\mathbf{k}})^{-1} \\ -(\lambda^2 - Q_{\mathbf{k}}^*Q_{\mathbf{k}})^{-1}Q_{\mathbf{k}}^* & \lambda(\lambda^2 - Q_{\mathbf{k}}^*Q_{\mathbf{k}})^{-1} \end{pmatrix}$. The operator $\mathbb{R} \ni \mathbf{k}_1 \mapsto \mathcal{K}_{\mathbf{k}}$ for $\mathbf{k}_2$ fixed, and $\lambda \in \mathbb{R}$ is a self-adjoint holomorphic family with compact resolvent on $L^2(\mathbb{C}/\Gamma)$. A flat band would imply that $\mathcal{K}_{\mathbf{k}}(\alpha_0)$ is not invertible for any $\mathbf{k} \in \mathbb{C}$. To simplify the analysis, we write

$$Q_{\mathbf{k}} = \underbrace{\sum_{j=1}^{2}(D_j + \mathbf{k}_j - A_j(x))\sigma_j}_{=:H_{\mathrm{D},\mathbf{k}}} + \mathcal{V} \text{ and } Q_{\mathbf{k}}^* = \underbrace{\sum_{j=1}^{2}(D_j + \mathbf{k}_j - A_j(x))\sigma_j}_{=:H_{\mathrm{D},\mathbf{k}}} + \overline{\mathcal{V}},$$

where $\mathcal{V} = \operatorname{diag}(V, \overline{V})$. Recall also the Pauli operator $H_{\mathrm{P},\mathbf{k}}$ given as

$$H_{\mathrm{P},\mathbf{k}} = (H_{\mathrm{D},\mathbf{k}})^2 = \big((D_1 + \mathbf{k}_1 - A_1(x))^2 + (D_2 + \mathbf{k}_2 - A_2(x))^2\big) - (\partial_1 A_2 - \partial_2 A_1)(x)\sigma_3.$$

In this setting, we have that both $\mathbf{k}_1, \mathbf{k}_2$ are real. Thus, we have for $S(\lambda) = H_{\mathrm{P},\mathbf{k}} - \lambda^2$

$$\begin{aligned} Q_{\mathbf{k}}^* Q_{\mathbf{k}} - \lambda^2 &= (1 + \underbrace{([Q_{\mathbf{k}}^*, \mathcal{V}] + \mathcal{V} Q_{\mathbf{k}}^* + \mathcal{V}^* Q_{\mathbf{k}} + \mathcal{V}\mathcal{V}^*) S(\lambda)^{-1}}_{=:W_1(\lambda)}) S(\lambda) \\ Q_{\mathbf{k}} Q_{\mathbf{k}}^* - \lambda^2 &= (1 + \underbrace{([Q_{\mathbf{k}}, \mathcal{V}^*] + \mathcal{V}^* Q_{\mathbf{k}} + \mathcal{V} Q_{\mathbf{k}}^* + \mathcal{V}\mathcal{V}^*) S(\lambda)^{-1}}_{=:W_2(\lambda)}) S(\lambda). \end{aligned}$$

We now complexify the real part of $\mathbf{k}$, which is $\mathbf{k}_1$, and choose $\mathbf{k} = \mathbf{k}_1 + i\mathbf{k}_2$ with $\mathbf{k}_1 := (\mu + iy)$, where $\mu, y, \mathbf{k}_2 \in \mathbb{R}$. Since $\sigma_0(Q_{\mathbf{k}})$ is the Dirac operator and the Pauli operator its square, we find by self-adjointness that

$$\|S(\lambda)^{-1}\|, \|Q_{\mathbf{k}} S(\lambda)^{-1}\|, \|Q_{\mathbf{k}}^* S(\lambda)^{-1}\| = \mathcal{O}(|y|^{-1}).$$

Assuming that there exists a flat band to $\mathcal{K}_{\mathbf{k}}$, it follows that

$$-1 \in \operatorname{Spec}(W_1(\lambda)), \operatorname{Spec}(W_2(\lambda))$$

in a complex neighbourhood of $\mathbf{k}_1 \in \mathbb{R}$ by Rellich's theorem. Then [54, Thm 1.9] implies that for all $\mathbf{k}_1 \in \mathbb{C}$ we have $-1 \in \operatorname{Spec}(W_1(\lambda)), \operatorname{Spec}(W_2(\lambda))$. But this is impossible, by the estimates on $\|S(\lambda)^{-1}\|$ for $|y|$ large enough. □

After discussing periodic magnetic potentials in such detail, we shall now study the effect of constant magnetic fields.

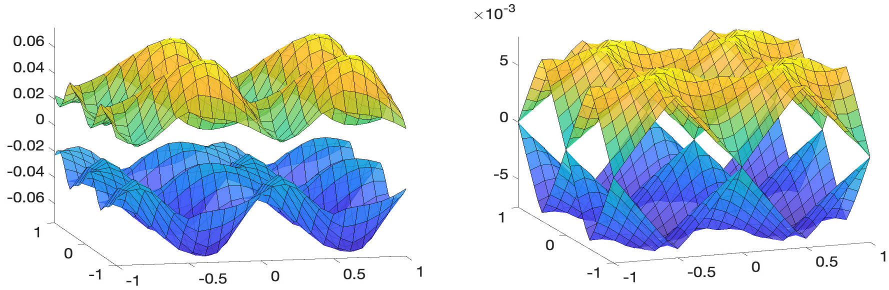


Figure 4.5: Periodic magnetic field $A_1(z) = 2\sqrt{3}\cos(\operatorname{Im}(z))$: On the left, the lowest bands of the anti-chiral Hamiltonian, $\alpha_0 = 1$, where 0 is not protected, under periodic magnetic perturbations, on the right the lowest bands of the chiral Hamiltonian, $\alpha_1 = 1$, where 0 is protected.

The zero energy level of the relativistic Dirac operator with zero potential and non-zero constant magnetic field is a flat band with respect to any lattice for which the magnetic flux $\Phi := \int_E B(z)\, dz > 0$ through the fundamental domain satisfies $\Phi \in 2\pi\mathbb{Q}$. We will argue next that the flat band persists for the chiral Hamiltonian where the fundamental domain is determined by the moiré lattice. For general magnetic fields, the concept of bands does not apply. Instead, since a flat band for a Floquet operator, corresponds to an eigenvalue of infinite multiplicity of the original operator, one should study the presence of eigenvalues of infinite multiplicity. Then, we have the following result that we split up into one statement on flat bands and one on eigenvalues of infinite multiplicity

Theorem 4.2 (Bands)**.** *Let A be a magnetic potential associated with a constant field $B > 0$ such that the magnetic flux Φ through any moiré cell $\mathbb{C}/\Gamma$ satisfies $\Phi \in 2\pi\mathbb{Q}$, then the chiral Hamiltonian possesses a flat band at zero energy. This flat band persists when adding a periodic magnetic potential $A_{\text{per}} \in C^\infty(E)$.*

Theorem 4.3 (Eigenvalues)**.** *Let $\alpha_1 \in \mathbb{C}$ be such that the chiral non-magnetic Hamiltonian $\mathscr{H}_c(B = 0)$ possesses a flat band, i.e. a magic angle. When adding to $\mathscr{H}_c(B = 0)$*

any magnetic field $B \in L^\infty_{\text{comp}}$ with flux $\lfloor \Phi/2\pi \rfloor \geq 1$[2] *or any periodic magnetic potential $A_{\text{per}} \in C^\infty(E)$, the operator $\mathscr{H}_c$ has an eigenvalue of infinite multiplicity. If α_1 is not magic, and $B \in L^\infty_{\text{comp}}$ as above, then $\mathscr{H}_c$ possesses an eigenvalue of multiplicity $\lfloor \Phi/2\pi \rfloor$ at zero. In particular, for non-zero constant magnetic fields the chiral Hamiltonian possesses an eigenvalue of infinite multiplicity at zero for any $\alpha_1 \in \mathbb{R}$.*

Proof. To see that 0 is in the spectrum of the Hamiltonian of the chiral Hamiltonian, we use that we can multiply any $\psi = (\psi_1, \psi_2) \in \mathbb{C}^4$ such that $\mathscr{H}_c(B = 0)\psi = 0$ and define the new function $\chi = (\varphi_1\psi_1, \varphi_1\psi_2)$ which then satisfies $\mathscr{H}_c(B)\chi = 0$.

By the Aharonov-Casher effect [26, Sec.6.4], there are precisely $\lfloor \Phi/(2\pi) \rfloor$ linearly independent square-integrable zero modes. Multiplying this with the Floquet-periodic zero modes of the chiral model, which exist for all $\alpha_1 \in \mathbb{C}$ gives the claim for the magnetic fields of compact support. When α_1 is magic, the

Turning to constant magnetic fields with flux $\Phi \in 2\pi\mathbb{Q}$, through a fundamental domain E for some n, then by adding potentials $A_{\text{per}} \in C^\infty(E)$, there is by Proposition E.0.2 a $\varphi_{\mathbf{k}}$ such that $(a_k + A_{\text{per}})\varphi_{\mathbf{k}} = 0$ and the existence of a flat band at zero follows. If the fields are not commensurable, the same argument shows the existence of an eigenvalue of infinite multiplicity.

The stability under perturbations by periodic magnetic potentials, follows directly from the existence of periodic $\psi_0 \neq 0$ such that, $(2D_{\bar{z}} + A_{\text{per}})\psi_0 = 0$. □

[2] We let $\lfloor y \rfloor$ be the largest integer *strictly* less than y.

4.3.2 Hörmander condition and exponential localization of bands

In this section, we study the exponential squeezing of bands for periodic magnetic fields and small angles. In particular, we shall see that in the chiral model, there will be at least $\sim 1/\theta$ many bands in an exponentially (in θ) small neighbourhood around zero. We conclude this property by studying the existence of localized quasi-modes in phase space. Phrased differently, for small twisting angles any angle *wants to be magic*. We shall prove this for the chiral model and then show that in the anti-chiral model such quasi-modes do not exist. In the case of the non-magnetic BM Hamiltonian, this has been established in [6, 7].

4.3.3 Exponential squeezing in chiral model

The chiral model possess in general a lot quasi-modes located close to the zero energy level. Indeed, since $h = 1/B$ is our semiclassical parameter, the principal symbol of $h\mathcal{D}_c^\theta$, with $\mathcal{D}_c^\theta$ as in (4.9), is just in semiclassical Weyl quantization $p(z,\zeta) := \sigma_0(hD_c^\theta)(z,\zeta) = 2\bar{\zeta} - A(z)$. The existence of localized modes will depend on the vanishing/non-vanishing of the bracket

$$\{p, \bar{p}\}(z) = 2(\partial_{\bar{z}}\overline{A(z)} - \partial_z A(z)) = 4iB(z). \tag{4.11}$$

We observe that with our quantization, the principal symbol and consequently the Poisson bracket are independent of the potentials. To see the effect of the potentials, one may look at the non-equivalent tight-binding limit

$$\mathcal{D}^{\theta}_{c,TB} = \begin{pmatrix} 2\theta D_{\bar z} - (A_1(z) + iA_2(z)) & U(z) \\ U_-(z) & 2\theta D_{\bar z} - (A_1(z) + iA_2(z)) \end{pmatrix}. \tag{4.12}$$

The semiclassical principal symbol of $\mathcal{D}^{\theta}_{c,TB}$ is given by

$$\sigma_0(\mathcal{D}^{\theta}_{c,TB})(z,\zeta) = \begin{pmatrix} 2\bar\zeta - (A_1(z) + iA_2(z)) & U(z) \\ U_-(z) & 2\bar\zeta - (A_1(z) + iA_2(z)) \end{pmatrix}.$$

The determinant of the principal symbol of $\mathcal{D}^{\theta}_{c,TB}$ and its conjugate symbol is given for $W(z) := U(z)U_-(z)$ by $q(z,\zeta) := \left(2\bar\zeta - (A_1(z) + iA_2(z))\right)^2 - W(z)$.

We then have the following existence of quasimodes result, which for the semiclassical scaling in $\mathcal{D}^{\theta}_c$ follows from the Poisson-bracket (4.11) along the lines as presented for $\mathcal{D}^{\theta}_{c,TB}$ below.

Proposition 4.3.1. *There exists an open set $\Omega \subset \mathbb{C}$ and a constant c such that for any $\mathbf{k} \in \mathbb{C}$ and $z_0 \in \Omega$, there exists a family $\theta \mapsto \mathbf{u}_\theta \in C^\infty(\mathbb{C}/\Gamma; \mathbb{C}^2)$ such that for $0 < \theta < \theta_0$,*

$$|(\mathcal{D}^{\theta}_{c,TB} - \theta\mathbf{k})\mathbf{u}_\theta(z)| \le e^{-c/\theta}, \quad \|\mathbf{u}_\theta\|_{L^2} = 1, \quad |\mathbf{u}_\theta(z)| \le e^{-c|z-z_0|^2/\theta}. \tag{4.13}$$

Proof. Since the Poisson-bracket in complex coordinates reads

$$\{q_1, q_2\} = \partial_\zeta q_1 \partial_z q_2 + \partial_{\bar\zeta} q_1 \partial_{\bar z} q_2 - \partial_z q_1 \partial_\zeta q_2 - \partial_{\bar z} q_1 \partial_{\bar\zeta} q_2,$$

we find that under the constraint that $q=\bar q=0$ at some point (z,ζ)

$$\{q,\bar q\}(z,\zeta)=(\partial_{\bar\zeta}q\partial_{\bar z}\bar q-\partial_z q\partial_\zeta\bar q)(z,\zeta)=8i|W(z)|B(z)-8i\operatorname{Im}(\partial_z W(z)\overline{W(z)}^{1/2}).$$

We then have that using that $U(z)=0$

$$W(z)=-z^2(\partial_z U(0))^2(1+\mathcal{O}(|z|))\text{ and }\partial_z W(z)=-2z(\partial_z U(0))^2(1+\mathcal{O}(|z|)),$$

which shows that

$$\overline{W(z)}^{1/2}\partial_z W(z)=2i|z|^2|\partial_z U(0)|^2\partial_z U(0)(1+\mathcal{O}(|z|)).$$

This implies the following expansion of the Poisson bracket at zero

$$\{q,\bar q\}=8i|z|^2|\partial_z U(0)|^2(B(z)-2\operatorname{Re}(\partial_z U(0))+\mathcal{O}(|z|)).$$

The result then follows from a real-analytic version [29, Theorem 1.2] of Hörmander's local solvability condition: For a differential operator $Q=\sum_{|\alpha|\le m}a_\alpha(x,\theta)(\theta D)^\alpha$ with real-analytic maps $x\mapsto a_\alpha(x,\theta)$ near some x_0, we let $q(x,\xi)$ be the semiclassical principal symbol of Q. If for phase space coordinates (x_0,ξ_0) we have $q(x_0,\xi_0)=0,\ \ \{q,\bar q\}(x_0,\xi_0)\neq 0$, then there exists a family $v_\theta\in C_c^\infty(\Omega)$, Ω a neighbourhood of x_0, such that for some $c>0$

$$|(\theta D)_x^\alpha Q v_\theta(x)|\le C_\alpha e^{-c/\theta},\quad \|v_\theta\|_{L^2}=1,\quad |(\theta\partial_x)^\alpha v_\theta(x)|\le C_\alpha e^{-c|x-x_0|^2/\theta}.$$

□

We then have the following result exhibiting the exponential squeezing of bands:

Theorem 4.4 (Exponential squeezing of bands)**.** *Consider the semiclassical scaling of the chiral Hamiltonian with magnetic potential $A \in C^\infty(E)$ inducing a non-zero magnetic field or consider the chiral Hamiltonian with tight-binding scaling* (4.12) *and arbitrary magnetic potential $A \in C^\infty(E)$. For the Floquet-transformed operator, the spectrum is a union of bands*

$$\operatorname{Spec}_{L^2(E)}(\mathscr{H}_{\mathbf{k}}^\theta) = \{E_j(\mathbf{k},\theta)\}_{j\in\mathbb{Z}}, E_j(\mathbf{k},\theta) \le E_{j+1}(\mathbf{k},\theta), \mathbf{k} \in \mathbb{C},$$

where $E_0(\mathbf{k},\theta) = \min_j |E_j(\mathbf{k},\theta)|$. Then there exist constants $c_0, c_1, c_2 > 0$ and $\theta_0 > 0$ such that for all $\mathbf{k} \in \mathbb{C}$ and $\theta \in (0,\theta_0)$,

$$|E_j(\mathbf{k},\theta)| \le c_0 e^{-c_1/\theta}, |j| \le c_2\theta^{-1}.$$

Proof. By using the above proposition and [6, Prop. 4.2], we can use the proof of [6, Theo. 5] to deduce the result. □

4.3.4 Anti-chiral model

To see that the conclusion of Theorem 4.4 does not hold for the anti-chiral Hamiltonian, we proceed as follows, and shall again restrict us to the slightly more technical tight-binding scaling. Consider in (4.10) for small $\theta > 0$ the operator, with $b_{\mathbf{k}} := 2\theta D_z - (A_1 - iA_2) - \theta\overline{\mathbf{k}}$,

$$\mathcal{D}_{\mathrm{ac},TB}^\theta = \begin{pmatrix} V & b_{\mathbf{k}} \\ b_{\mathbf{k}}^* & \overline{V} \end{pmatrix}.$$

Then, the existence of a zero mode is equivalent, for $z \in \Omega := \{w \in \mathbb{C}; V(w, \bar{w}) \neq 0\}$, to a zero mode of the operator

$$P(\theta)v_\theta(z) = \left(V(z)b_{\mathbf{k}}^*(V(z)^{-1}b_{\mathbf{k}} - |V(z)|^2\right) v_\theta(z).$$

We then find

Proposition 4.3.2. *If $u(\theta)$ is smooth on a bounded domain with*

$$\mathrm{WF}_h(u(\theta)) = \{(z_0, \zeta_0)\} \in T^*\Omega,$$

then it follows that $\|u(\theta)\| \leq \frac{C}{\theta}\|P(\theta)u(\theta)\|, \theta \downarrow 0$. Thus, there do not exist any quasi-modes $P(\theta)u(\theta) = \mathcal{O}(\theta^\infty)$.

Proof. Since $\sigma_0(p)$ is real-valued, the condition $\partial p \neq 0$ on $\{p = 0\}$ precisely means that p is of real principal type which implies the result by [89, Theo. 12.4]. □

The principal symbol of $P(\theta)$ is given by $\sigma_0(P(\theta))(z, \bar{z}, \zeta, \bar{\zeta}) = |2\zeta - (A_1 - iA_2)|^2 - \alpha_0^2|V(z, \bar{z})|^2$. This is of real principal type since real valued and $\sigma_0(P(\theta)) = 0$ implies $\partial_\zeta \sigma_0(P(\theta)) = 4(2\bar{\zeta} - (A_1 + iA_2)) \neq 0$ assuming $V(z, \bar{z}) \neq 0$ for all $z \in \mathbb{C}$. By the proposition above, $\|P(\theta)u(\theta)\|$ is bounded below by function of order θ. In particular, (4.13) does not hold.

4.4 Density of states

In this section we study general properties of the density of states and study the possible values the density of states takes for the Hamiltonian of TBG.

4.4.1 General properties

In this subsection, we assume that the magnetic potential of the Hamiltonian is of the form $A = A_{\text{per}} + A_{\text{con}}$ where $A_{\text{per}} \in C^\infty(E)$ and A_{con} is the vector potential of a constant magnetic field of strength B. Let $f \in C_c(\mathbb{R})$ then we define the *regularized trace*

$$\tilde{\text{Tr}}(f(\mathscr{H}^\theta)) = \lim_{r\to\infty} \frac{\text{Tr}(\mathbb{1}_{B_R}\, f(\mathscr{H}^\theta)\, \mathbb{1}_{B_R})}{|B_R|}$$

where $\mathbb{1}_{B_R}$ is the indication function of the square centered at 0 of side length $2R$. By Riesz's theorem, there exists the so-called *density of states (DOS)* measure ρ satisfying

$$\tilde{\text{Tr}}(f(\mathscr{H}^\theta)) = \int_{\mathbb{R}} f(t)\ d\rho(t). \tag{4.14}$$

We start by showing the existence and smoothness of the DOS.

Lemma 4.4.1. *For $f \in C_c^\infty(\mathbb{R})$ the regularized trace of $f(\mathscr{H}^\theta)$ exists, satisfies*

$$\tilde{\text{Tr}}(f(\mathscr{H}^\theta)) = \frac{1}{|E|}\, \text{Tr}_{L^2(E)}(f(\mathscr{H}^\theta)) = \frac{1}{|E|} \int_E f(\mathscr{H}^\theta)(x,x)\ dx,$$

and depends smoothly on $B \in \mathbb{R}$ and $\theta \in \mathbb{R} \setminus \{0\}$, with Schwartz kernel $f(\mathscr{H}^\theta)(x,y)$ of $f(\mathscr{H}^\theta)$.

Proof. Let $N_r, N_R \subset \Gamma$ be $N_r := \{\zeta \in \Gamma : \zeta + E \subset B_R\}$ and $N_R := \{\zeta \in \Gamma : \zeta + E \subset B_R \neq \emptyset\}$. Then

$$S_r := \bigcup_{\zeta \in N_r} E + \zeta \subset B_R \subset \bigcup_{\zeta \in N_R} E + \zeta =: S_R.$$

Thus for nonnegative f,

$$\frac{1}{|S_R|}\, \text{Tr}(\mathbb{1}_{S_r}\, f(\mathscr{H}^\theta)) \leq \frac{1}{|B_R|}\, \text{Tr}(\mathbb{1}_{B_R}\, f(\mathscr{H}^\theta)) \leq \frac{1}{|S_r|}\, \text{Tr}(\mathbb{1}_{S_R}\, f(\mathscr{H}^\theta)). \tag{4.15}$$

Furthermore, by definition, we see that for some $C, C' > 0$, for all R,

$$\#(N_R \setminus N_r) \leq CR, \quad \text{and} \quad |S_R \setminus S_r| \leq C'R. \tag{4.16}$$

Recall that $\mathbb{T}_\zeta$, defined in Lemma 4.2.1, satisfy $[\mathbb{T}_\zeta, \mathscr{H}^\theta] = 0$, therefore $[\mathbb{T}_\zeta, f(\mathscr{H}^\theta)] = 0$. Furthermore, since $\mathbb{T}_\zeta \, \mathbb{1}_{E+\zeta} \, \mathbb{T}_{-\zeta} = \mathbb{1}_E$, thus $\operatorname{Tr}(\mathbb{1}_{E+\zeta} \, f(\mathscr{H}^\theta)) = \operatorname{Tr}(\mathbb{1}_E \, f(\mathscr{H}^\theta))$. Hence,

$$\operatorname{Tr}(\mathbb{1}_{S_r} \, f(\mathscr{H}^\theta)) = \sum_{\zeta \in N_r} \operatorname{Tr}(\mathbb{1}_{E+\zeta} \, f(\mathscr{H}^\theta)) = (\# N_r) \operatorname{Tr}(\mathbb{1}_E \, f(\mathscr{H}^\theta))$$

and similarly $\operatorname{Tr}(\mathbb{1}_{S_R} \, f(\mathscr{H}^\theta)) = (\# N_R) \operatorname{Tr}(\mathbb{1}_E \, f(\mathscr{H}^\theta))$. Inserting this into (4.15), taking $R \to \infty$ we get by using (4.16) that

$$\widetilde{\operatorname{Tr}}(f(\mathscr{H}^\theta)) = \frac{1}{|E|} \operatorname{Tr}_{L^2(E)}(f(\mathscr{H}^\theta)).$$

To conclude the smooth dependence on θ and B, it suffices to adapt the arguments starting at [13, p.251].

□

In the next Proposition, we show that the integrated density of states of the twisted bilayer graphene Hamiltonian is stable under small perturbations of the magnetic field that do not close any spectral gaps.

Proposition 4.4.2. *Let the magnetic vector potential* $A = A_{\text{con}} + A_{\text{per}}$ *be the sum of a linear potential associated with a constant field* B_0 *and* $A_{\text{per}} \in C^\infty(E)$*. Assuming* $t_0, t_1 \notin \operatorname{Spec}(\mathscr{H}^\theta)$*, there exists a neighbourhood* $\mathcal{B} \subset \mathbb{R}$*, open, connected, with* $B_0 \in \mathcal{B}$ *as well as* $m = (m_1, m_2) \in \mathbb{Z}^2$ *such that for any perturbation of the constant magnetic* *field* $B \in \mathcal{B}$*,* $t_0, t_1 \notin \mathrm{Spec}(\mathscr{H}^\theta)$ *the DOS satisfies*

$$\rho((t_0,t_1)) = \frac{1}{|E|}\left(m_1 \frac{B|E|}{2\pi} + m_2\right).$$

Proof. By density, we may assume that $B_0|E| = 2\pi\frac{p}{q} \in 2\pi\mathbb{Q}$. This implies by choosing $\lambda = q$ that $B_0|E_\lambda| \in 2\pi\mathbb{Z}$. Let $\lambda_{n,\mathbf{k}}$ be the n-th Bloch band of $\mathscr{H}^\theta_{\mathbf{k}}$ for $n \in \mathbb{Z}$ on $\mathbf{k} \in E^*_\lambda$. The spectrum of $\mathscr{H}^\theta$ has band structure and is given by $\mathrm{Spec}(\mathscr{H}^\theta) = \cup_n J_n$ where $J_n = \bigcup\limits_{\mathbf{k}\in E^*_\lambda} \lambda_{n,\mathbf{k}}$. Let $t_0, t_1 \notin \mathrm{Spec}(\mathscr{H}^\theta)$. We call $\mathcal{I}$ the set of bands fully contained in (t_0,t_1). In terms of $\mathbf{k} \mapsto u_{n,\mathbf{k}}$ given by the eigenvectors associated with $\lambda_{n,\mathbf{k}}$ spectral projection of $\mathscr{H}^\theta_{\mathbf{k}}$ is given by

$$\begin{aligned}
&\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta_{\mathbf{k}})v_{\mathbf{k}}(x) = \int_{E_\lambda} \mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta_{\mathbf{k}})(x,y)v_{\mathbf{k}}(y)dy \text{ with}\\
&\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta_{\mathbf{k}})(x,y) := \sum_{j\in\mathcal{I}} u_{j,\mathbf{k}}(x)\overline{u_{j,\mathbf{k}}(y)}.
\end{aligned}$$

So the spectral projection $\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta) = \mathcal{U}_{B_0}^{-1}\int^{\oplus}_{E^*_\lambda} \mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta_{\mathbf{k}})\frac{d\mathbf{k}}{|E^*_\lambda|}\mathcal{U}_{B_0}$ of $\mathscr{H}^\theta$ is

$$\begin{aligned}
&\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)u(x) = \int_{\mathbb{R}} \mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)(x,y)u(y)dy \text{ with}\\
&\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)(x,y) = \int_{E^*_\lambda} \mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta_{\mathbf{k}})(x,y)\frac{d\mathbf{k}}{|E^*_\lambda|}.
\end{aligned}$$

Since $t_0, t_1 \notin \mathrm{Spec}(\mathscr{H}^\theta)$ and let $N := |\mathcal{I}|$, then by Lemma 4.4.1

$$\rho((t_0,t_1)) := \int_{E_\lambda} \mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)(x,x)\ \frac{dx}{|E_\lambda|} = \int_{E^*_\lambda}\sum_{j\in\mathcal{I}} 1\ \frac{d\mathbf{k}}{4\pi^2} = \frac{N}{|E_\lambda|},$$

thus $\tilde{\mathrm{Tr}}(\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)) = \frac{N}{|E_\lambda|}$. If $f \in C^\infty_c(\mathbb{R})$, such that $f(x) = 1$ for $x \in \mathrm{conv}\bigcup_n J_n$[3]

[3]conv is the convex hull

and $f(x)=0$ for $x \in \mathrm{Spec}(\mathscr{H}^\theta) \setminus \mathrm{conv} \bigcup_n J_n$, then

$$\rho((t_0,t_1)) = \int_{\mathbb{R}} f(t)\rho(dt) = \frac{N}{|E_\lambda|}.$$

Recall that $B_0|E| = \frac{B_0|E_\lambda|}{q} = 2\pi\frac{p}{q} \in 2\pi\mathbb{Q}$. We then introduce a new lattice $\tilde{\Gamma} \subset \Gamma$ generated by $\widetilde{\zeta_1} = \zeta_1$ and $\widetilde{\zeta_2} = q\zeta_2$. Then $B_0|\mathbb{C}/\tilde{\Gamma}| \in 2\pi\mathbb{Z}$ and $|\Gamma/\tilde{\Gamma}| = q$. As before, if $t_0, t_1 \notin \mathrm{Spec}(\mathscr{H}^\theta)$, then

$$\phi(B_0) := |E|\tilde{\mathrm{Tr}}(\mathbb{1}_{(t_0,t_1)}(\mathscr{H}^\theta)) = |E|\rho((t_0,t_1)) = |E|\int_{\mathbb{R}} f(t)d\rho(t) \in \frac{1}{q}\mathbb{Z} \subset \frac{B_0|E|}{2\pi}\mathbb{Z} + \mathbb{Z}$$

where the last inclusion follows since p, q are coprime i.e. there exist $c, d \in \mathbb{Z}$ such that $cp + dq = 1$. Note that if $z_0 \in \mathbb{R} \setminus \mathrm{Spec}(\mathscr{H}^\theta)$, then there exists $\varepsilon > 0$ such that $z \notin \mathrm{Spec}(\mathscr{H}^\theta)$ for all $|z - z_0|$ and small perturbations of the constant field $|B - B_0| < \varepsilon$ and $\phi(B)$ is locally a smooth function of the constant field B by Lemma 4.4.1, so there exists $B_0 \in \mathcal{B} \subset \mathbb{R}$ open, connected and $m \in \mathbb{Z}^2$ such that for $B \in \mathcal{B}$,

$$\rho((t_0,t_1)) = \int_{\mathbb{R}} f(t)d\rho(t) = \frac{1}{|E|}\left(m_1\frac{B|E|}{2\pi} + m_2\right).$$

□

4.5 Semiclassical expansion of Density of states

In this section, we provide explicit asymptotic expansions of the regularized trace in the semiclassical limit $B \gg 1$ for constant magnetic field in the spirit of Remark 4.1 for the chiral and anti-chiral model respectively. We also comment on the differentiability of the DOS at the end of this section in preparation for applications in the next section.

We consider (4.1) with fixed θ and constant magnetic field B:

$$\mathscr{H}^\theta = \mathscr{H}_0^\theta + \mathscr{V}(x) = \begin{pmatrix} H_D^\theta & 0 \\ 0 & H_D^{-\theta} \end{pmatrix} + \begin{pmatrix} 0 & T(x) \\ T^*(x) & 0 \end{pmatrix}. \tag{4.17}$$

Notice that the spectrum of $\mathscr{H}_0^\theta$ is composed of *Landau levels* $\lambda_{n,B} := \operatorname{sgn}(n)\sqrt{2|n|B}$ (see Lemma 4.5.2) which will be perturbed by the tunnelling potential $\mathscr{V}$ (see Remark 4.5). To simplify the notation, we therefore introduce the *Landau bands* $\Lambda_{n,B,\mathscr{V}} := (\lambda_{n-1,B} + \|\mathscr{V}\|_\infty, \lambda_{n+1,B} - \|\mathscr{V}\|_\infty)$ for $n \in \mathbb{Z}$, in which the spectrum of $\mathscr{H}^\theta$ is contained around the n-th Landau level $\lambda_{n,B}$, cf. Remark 4.6.

Now we state the main result of this section which is the asymptotic expansion of the DOS. We start with the chiral model:

Theorem 4.5 (Chiral model)**.** *Let $\lambda_{n,B} = \operatorname{sgn}(n)\sqrt{2|n|B}$. For a fixed $n \in \mathbb{Z}$, for $\varepsilon > 0$ small enough, for all $f \in C_c^K(\Lambda_{n,B,\mathscr{V}})$ with $K \geq \frac{6}{\varepsilon} - 2$, we have*

$$\widetilde{\operatorname{Tr}}(f(\mathscr{H}_c)) = \left[\frac{B}{\pi} f(\lambda_{n,B}) + \frac{|n|}{2\pi} \operatorname{Ave}(\mathfrak{U}) f''(\lambda_{n,B})\right] + \mathcal{O}_{n,K,f,\mathscr{V}}(B^{-\frac{1}{2}+\varepsilon}) \tag{4.18}$$

with

$$\begin{aligned} \mathfrak{U}(\eta) &= \frac{\alpha_1^2}{8}\left[\alpha_1^2(|U_-(\eta)|^2 - |U(\eta)|^2)^2 + 4|\partial_{\bar\eta}\overline{U_-(\eta)} - \partial_\eta U(\eta)|^2\right], \\ \operatorname{Ave}(g) &= \frac{1}{|E|}\int_E g(\eta) L(d\eta), \\ \eta &= x_2 + i\xi_2, \text{ and } \mathcal{O}_{n,K,f,\mathscr{V}} = \mathcal{O}_n(\|\mathscr{V}\|_\infty \|f\|_{C^K}) \end{aligned}$$

Furthermore, fix $N \in \mathbb{N}^+$ and consider $2N+1$ Landau bands with $n \in \{-N, .., N\}$, then for all $\varepsilon > 0$ small enough, for any $f \in C_c^K([\lambda_{-(N+1),B} + \|\mathscr{V}\|_\infty, \lambda_{N+1,B} - \|\mathscr{V}\|_\infty])$

with $K \geq \frac{6}{\varepsilon} - 2$*, we have*

$$\tilde{\operatorname{Tr}}(f(\mathscr{H}_{\mathrm{c}})) = \sum_{n=-N}^{N} \left[\frac{B}{\pi} f(\lambda_{n,B}) + \frac{|n|}{2\pi} \operatorname{Ave}(\mathfrak{U}) f''(\lambda_{n,B}) \right] + \mathcal{O}_{(N),K,f,\mathscr{V}}(B^{-\frac{1}{2}+\varepsilon})$$

where $\mathcal{O}_{(N),K,f,\mathscr{V}} := \sum_{n=-N}^{N} \mathcal{O}_{n,K,f,\mathscr{V}}$.

Our proof also shows that all higher order terms, which in general have complicated expressions, in the expansion of $\tilde{\operatorname{Tr}}(f(\mathscr{H}_{\mathrm{c}}))$ are of the form $f^{(k)}(\lambda_{n,B})$ (see (4.49)), which is different from the anti-chiral case studied below.

Next, we consider the anti-chiral model, where the sub-leading correction in the regularized trace is already of order $\sqrt{B}$. Since the dominant sub-leading correction in the anti-chiral case is one order higher than in the chiral case, we only state the correction up to order $\sqrt{B}$.

Theorem 4.6 (Anti-chiral model)**.** *Under the same assumption as in Theorem 4.5, we have for all* $\varepsilon > 0$ *small enough,* $f \in C_c^K(\Lambda_{n,B,\mathscr{V}})$ *with* $K \geq \frac{3}{\varepsilon} - 1$

$$\tilde{\operatorname{Tr}}(f(\mathscr{H}_{ac}^{\theta})) = \frac{B}{2\pi} t_{n,0}(f) - \frac{\sqrt{B}}{2\pi} t_{n,1}(f) + \mathcal{O}_{n,K,f,\mathscr{V}}(B^{\varepsilon}), \tag{4.19}$$

where $\mathcal{O}_{n,K,f,\mathscr{V}} = \mathcal{O}_n(\|\mathscr{V}\|_\infty \|f\|_{C^K})$, $\operatorname{Ave}(g) = \frac{1}{|E|}\int_E g(\eta) dL(\eta)$,

$$t_{n,0}(f) = \operatorname{Ave}\left(f(\lambda_{n,B} + c_n) + f(\lambda_{n,B} - c_n)\right),$$
$$t_{n,1}(f) = \operatorname{Ave}\left(s_n^2 f'(\lambda_{n,B} + c_n) + s_n^2 f'(\lambda_{n,B} - c_n)\right),$$
$$s_n(\eta;\theta) = \begin{cases} \alpha_0 \sin(\frac{\theta}{2})|V(\eta)|, \\ \alpha_0 |V(\eta)|, \end{cases} \quad c_n(\eta;\theta) = \begin{cases} \alpha_0 \cos(\frac{\theta}{2})|V(\eta)|, & n \neq 0, \\ \alpha_0 |V(\eta)|, & n = 0. \end{cases}.$$

Furthermore, fix $N \in \mathbb{N}^+$ and consider $2N+1$ Landau bands with $n \in \{-N, .., N\}$. For any $\varepsilon > 0$, $f \in C_c^K([\lambda_{-N-1,B} + \|\mathscr{V}\|_\infty, \lambda_{N+1} - \|\mathscr{V}\|_\infty])$ with $K \geq \frac{3}{\varepsilon} - 1$, we have

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}_{\mathrm{ac}}^\theta)) = \sum_{n=-N}^{N} \left[\frac{B}{2\pi} t_{n,0}(f) + \frac{\sqrt{B}}{2\pi} t_{n,1}(f) \right] + \mathcal{O}_{(N),f,K,\mathscr{V}}(B^\varepsilon)$$

where $\mathcal{O}_{(N),K,f,\mathscr{V}} := \sum_{n=-N}^{N} \mathcal{O}_{n,K,f,\mathscr{V}}$.

For the rest of this section, we shall temporarily stop using the identification $x = (x_1, x_2) \simeq z = x_1 + ix_2$ and start with some preparations to prove the two results, before. Let $\Sigma_i^\theta = \mathrm{diag}(\sigma_i^\theta, \sigma_i^{-\theta})$. We can rewrite (4.17) as $\mathscr{H}_0^\theta = \Sigma_1^\theta D_{x_1} + \Sigma_2^\theta (D_{x_2} + Bx_1)$. We will only use $x = (x_1, x_2)$ to denote the position, while z is used as another unrelated arbitrary complex number in the resolvent $(\mathscr{H}^\theta - z)^{-1}$.

Quantizations. Let $x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. For a symbol $a(x, \xi) \in S(\mathbb{R}^4_{x,\xi})$, we define the (h_1, h_2)-Weyl quantization $a^W(x, h_1 D_{x_1}, h_2 D_{x_2}) : L^2(\mathbb{R}^2_x) \to L^2(\mathbb{R}^2_x)$ as

$$(a^W(x, h_1 D_{x_1}, h_2 D_{x_2})u)(x) = \frac{1}{2\pi} \int e^{\frac{i}{h_1}(x_1 - y_1)\xi_1 + \frac{i}{h_2}(x_2 - y_2)\xi_2} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \tag{4.20}$$

In this section, we shall employ two different quantizations: in Subsection 4.5.1 to 4.5.2, we use the $(h_1, h_2) = (1, 1)$-Weyl quantization. Starting from Subsection 4.5.3, we use the (x_2, hD_{x_2})-Weyl quantization of the operator-valued symbol which is related to the $(h_1, h_2) = (1, h)$-Weyl quantization (see Subsection 4.5.3 for more details). Occasionally, we denote $a^W(x, h_1 D_{x_1}, h_2 D_{x_2})$ by a^W for convenience when there is no ambiguity of (h_1, h_2).

4.5.1 First Reduction: Symplectic reduction

In this subsection, we first apply a symplectic reduction to $\mathscr{H}^\theta$, then provide a spectral description of $\mathscr{H}_0^\theta$ and $\mathscr{H}^\theta$. In the end, we introduce the Helffer-Sjöstrand formula for our study of the regularized trace $\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$.

Symplectic Reduction. Let $(h_1, h_2) = (1,1)$ for this subsection. Then the operator $\mathscr{H}_0^\theta$ and $\mathscr{V}$, when viewed as a $(1,1)$-Weyl quantization, have symbols $\mathscr{H}_0^\theta(x,\xi) = \Sigma_1^\theta \xi_1 + B\Sigma_2^\theta(\xi_2 + x_1)$ and $\mathscr{V}(x)$ respectively. The following lemma provide the symplectic reduction of $\mathscr{H}^\theta$:

Lemma 4.5.1. *Let $h = 1/B$. Then there is a unitary operator $\mathscr{U}$, symbols $\mathscr{G}_0^\theta(x,\xi) = \Sigma_1^\theta \xi_1 + \Sigma_2^\theta x_1$ and $\mathscr{W}(x,\xi) = \mathscr{V}(x_2 + h^{1/2}x_1, h\xi_2 - h^{1/2}\xi_1)$, s.t.*

$$\mathscr{U}\mathscr{H}_0^\theta(x, D_x)\mathscr{U}^{-1} = \sqrt{B}\mathscr{G}_0^\theta(x, D_x), \tag{4.21}$$

$$\mathscr{U}\mathscr{V}(x)\mathscr{U}^{-1} = \mathscr{W}^W(x, D_x). \tag{4.22}$$

Remark 4.4. Notice that $\mathscr{G}_0^\theta(x,\xi)$ does not depend on (x_2, ξ_2), thus the $(1,1)$-Weyl-quantization is $\mathscr{G}_0^\theta(x, D_x) = (\Sigma_1^\theta D_{x_1} + \Sigma_2^\theta x_1) \otimes \mathbb{1}_{L^2(\mathbb{R}_{x_2})}$, where $\mathbb{1}_{L^2(\mathbb{R}_{x_2})}$ is the identity map on $L^2(\mathbb{R}_{x_2})$.

Remark 4.5. It follows that $\mathscr{U}\mathscr{H}^\theta\mathscr{U}^{-1} = \sqrt{B}(\mathscr{G}_0^\theta + \sqrt{h}\mathscr{W}^W)$. When $B \to \infty$, we can interpret $\mathscr{G}^\theta := \mathscr{G}_0^\theta + \sqrt{h}\mathscr{W}^W$ as a small perturbation of $\mathscr{G}_0^\theta$.

Proof. Recall that a symplectic transformation $(y,\eta) = \kappa(x,\xi)$ applying to a symbol $a(x,\xi) = a \circ \kappa^{-1}(y,\eta) \in S(\mathbb{R}^4)$, implies the existence of a unitary operator $U_\kappa : L^2(\mathbb{R}_x^2) \to L^2(\mathbb{R}_y^2)$ s.t.

$$U_\kappa a^W(x, D_x) U_\kappa^{-1} = (a \circ \kappa^{-1})^W(y, D_y). \tag{4.23}$$

By applying the following three symplectic transformations to $\mathscr{H}^\theta(x,\xi)$:

$$\kappa_1(x,\xi) = (x_1, \xi_2, \xi_1, -x_2), \ \ \kappa_2(x,\xi) = \left(x_1 + \tfrac{x_2}{B}, x_2, \xi_1, \xi_2 - \tfrac{x_1}{B}\right),$$
$$\kappa_3(x,\xi) = \left(\sqrt{B}x_1, -\tfrac{x_2}{B}, \tfrac{\xi_1}{\sqrt{B}}, -B\xi_2\right),$$

we find

$$\begin{cases} \mathscr{H}_0^\theta \circ \kappa_1^{-1} \circ \kappa_2^{-1} \circ \kappa_3^{-1}(x,\xi) = \sqrt{B}(\Sigma_1^\theta \xi_1 + \Sigma_2^\theta x_1), \\ \mathscr{V} \circ \kappa_1^{-1} \circ \kappa_2^{-1} \circ \kappa_3^{-1}(x,\xi) = \mathscr{V}(x_2 + h^{\frac{1}{2}}x_1, h\xi_2 + h^{\frac{1}{2}}\xi_1). \end{cases} \tag{4.24}$$

By (4.23) and (4.24), the unitary operator $U_\kappa := U_{\kappa_3} \circ U_{\kappa_2} \circ U_{\kappa_1}$ has then the desired properties. □

Spectral Properties. As mentioned in Remark 4.5, we study the spectral properties of $\mathscr{G}^\theta$ and $\mathscr{H}^\theta$ by viewing them as perturbations of $\mathscr{G}_0^\theta$ and $\mathscr{H}_0^\theta$. Therefore, we start with $\mathscr{G}_0^\theta$ and $\mathscr{H}_0^\theta$:

Lemma 4.5.2. *The spectral decompositions of $\mathscr{G}_0^\theta$ and $\mathscr{H}_0^\theta$ are given by*

$$\mathrm{Spec}(\mathscr{G}_0^\theta) = \{\lambda_n := \mathrm{sgn}(n)\sqrt{2|n|} : n \in \mathbb{Z}\} \text{ with eigenspace } N_n^\theta,$$
$$\mathrm{Spec}(\mathscr{H}_0^\theta) = \{\lambda_{n,B} := \mathrm{sgn}(n)\sqrt{2|n|B} : n \in \mathbb{Z}\} \text{ with eigenspace } \mathscr{U} N_n^\theta,$$

where

$$N_n^\theta = \mathrm{span}\left\{ \begin{pmatrix} x \mapsto u_n^\theta(x_1)s_1(x_2) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \mapsto u_n^{-\theta}(x_1)s_2(x_2) \end{pmatrix} : \forall s_1, s_2 \in L^2(\mathbb{R}_{x_2}) \right\}.$$

Here $u_n^\theta = e^{-\frac{i\theta}{4}\sigma_3} u_n e^{\frac{i\theta}{4}\sigma_3}$, $u_n = C_n \begin{pmatrix} \mathrm{sgn}(n) r_{|n|-1} \\ i r_{|n|} \end{pmatrix}$, $C_n = \begin{cases} \frac{1}{\sqrt{2}}, & n \in \mathbb{Z}\setminus\{0\} \\ 1, & n = 0 \end{cases}$, *as well as* $r_{-1} = 0$, $r_m = C_m'(D_{x_1} + ix_1)^m e^{-\frac{x_1^2}{2}}$ *where* C_m' *is constant s.t.* $\|r_m\|_{L^2(\mathbb{R}_{x_1})} = 1$ *for*

$m \in \mathbb{N}$.

Proof. The main observation here is for $G_D := \sigma_1 D_{x_1} + \sigma_2 x_1 = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$ where $a = D_{x_1} - ix_1$, we have $[a, a^*] = 2$. Thus a and a^* form a pair of annihilator and creator. By the standard argument for the ladder operators, there is a sequence of normalized $r_m(x_1) = C'_m \cdot (a^*)^m e^{-\frac{x_1^2}{2}} = C'_m (D_{x_1} + ix_1)^m e^{-\frac{x_1^2}{2}}$, for $m \geq 0$ s.t. $ar_m = \sqrt{2m} r_{m-1}$ and $a^* r_m = \sqrt{2(m+1)} r_{m+1}$. Then one can check by computation and (4.21) that u_n^θ, N_n^θ and $\mathscr{U} N_n^\theta$ defined above are eigenvectors and eigenspace of $\mathscr{G}_0^\theta$, $\mathscr{G}_0^\theta$ and $\mathscr{H}_0^\theta$ w.r.t. eigenvalue λ_n, λ_n and $\lambda_{n,B}$, for all $n \in \mathbb{Z}$. □

Remark 4.6. Since $\mathscr{H}^\theta = \mathscr{H}_0^\theta + \mathscr{V}$, thus

$$\operatorname{Spec}(\mathscr{H}^\theta) \subset \overline{B_{\|\mathscr{V}\|_\infty}\left(\operatorname{Spec}(\mathscr{H}_0^\theta)\right)} = \bigcup_n \overline{B_{\|\mathscr{V}\|_\infty}(\lambda_{n,B})}.$$

Fix n, since $\mathscr{V}$ is bounded, when B is large enough, $\left\{\overline{B_{\|\mathscr{V}\|_\infty}(\lambda_{j,B})}\right\}_{|j-n|\leq 1}$ are disjoint. Since the DOS measure ρ is supported on the spectrum, by (4.14), the regularized trace $\widetilde{\operatorname{Tr}}(f(\mathscr{H}^\theta))$ is not affected by modifying f within the spectral gap $(\lambda_{k-1,B} + \|\mathscr{V}\|_\infty, \lambda_{k,B} - \|\mathscr{V}\|_\infty)$, i.e. $\widetilde{\operatorname{Tr}}((\chi_{\Lambda_{k,B,\mathscr{V}}} f)(\mathscr{H}^\theta)) = \widetilde{\operatorname{Tr}}((\chi_{B_{\|\mathscr{V}\|_\infty}(\lambda_{k,B})} f)(\mathscr{H}^\theta))$, for any $k \in \mathbb{Z}$. Thus we will start with f supported on a fixed $\Lambda_{n,B,\mathscr{V}}$ to avoid the influence of bands nearby and then consider the general case of f supported on a fixed number of bands (see Theorem 4.5, 4.6 and their proofs in Subsection 4.5.5).

Remark 4.7. Both $\lambda_{n,B}$ and λ_n are called Landau levels of $\mathscr{H}_0^\theta$ and $\mathscr{G}_0^\theta$ respectively. To study the corresponding operators near the Landau levels, we denote $\mathscr{H}_n^\theta := \mathscr{H}^\theta - \lambda_{n,B}$, $\mathscr{H}_{0,n}^\theta := \mathscr{H}_0^\theta - \lambda_{n,B}$, $\mathscr{G}_n^\theta := \mathscr{G}^\theta - \lambda_n$ and $\mathscr{G}_{0,n}^\theta := \mathscr{G}_0^\theta - \lambda_n$.

Helffer-Sjöstrand formula and regularized traces. We proceed by recalling the Helffer-Sjöstrand formula. Let $K \in \mathbb{N}$. Given $f \in C_c^{K+1}(\mathbb{R})$, we can always find $\tilde{f}$, a order-K quasi-analytic extension of f, by which we mean a function $\tilde{f} \in C_c^{K+1}(\mathbb{C})$, such that

$$\tilde{f}|_{\mathbb{R}} = f, \text{ and } |\partial_{\bar{z}} f| \leq C\|f\|_{C^{K+1}} |\operatorname{Im} z|^K, \text{ for some } C > 0. \tag{4.25}$$

The concrete construction can be found in [4, Sec. 4.1] or [30, Theorem 8.1], where we can also choose $\tilde{f}$ s.t. $\operatorname{supp}(\tilde{f}) \supset \operatorname{supp}(f)$ is arbitrarily close to $\operatorname{supp}(f)$. We omit the proof which can be found in the quoted references.

Lemma 4.5.3 (Helffer-Sjöstrand formula)**.** *Let H be a self-adjoint operator on a Hilbert space. Let $f \in C_c^{K+1}(\mathbb{R})$ and $\tilde{f}$ be its order-K quasi-analytic extension, then*

$$f(H) = \frac{1}{2\pi i} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}(z)(z - H)^{-1} dz \wedge d\bar{z}. \tag{4.26}$$

In particular, for $f \in C_c^{K+1}(\Lambda_{n,B,\mathscr{V}})$, define $f_0(x) = f(x + \lambda_{n,B})$ a function localized around zero. By Remark 4.7, (4.21) and (4.26), we have

$$\begin{aligned} \mathscr{U} f(\mathscr{H}^\theta)\mathscr{U}^{-1} = \mathscr{U} f_0(\mathscr{H}_n^\theta)\mathscr{U} &= -\frac{i}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0(z)(z - \mathscr{U}\mathscr{H}_n^\theta \mathscr{U})^{-1} \, dz \wedge d\bar{z} \\ &= \frac{i\sqrt{h}}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0(z)(\mathscr{G}_n^\theta - \sqrt{h}z)^{-1} \, dz \wedge d\bar{z}. \end{aligned} \tag{4.27}$$

Thus to study $f(\mathscr{H}^\theta)$, it is enough to study the resolvent $(\mathscr{G}_n^\theta - \sqrt{h}z)^{-1}$.

4.5.2 Second reduction: Grushin problem

In this subsection, we apply the Schur complement formula twice for operators $\mathscr{G}_{0,n}^\theta$ and $\mathscr{G}_n^\theta$ to characterize $(\mathscr{G}_n^\theta - \sqrt{h}z)^{-1}$ using the effective Hamiltonian. In our context, the Schur complement formula is also called a Grushin problem and we shall use that terminology in the sequel. See [77] for more information on Grushin problem.

Unperturbed Grushin problem. To set up our Grushin problem, we introduce the space $B^k_{x_1} := B^k(\mathbb{R}_{x_1};\mathbb{C}^4) := (1 + D^2_{x_1} + x_1^2)^{-k/2} L^2(\mathbb{R}_{x_1};\mathbb{C}^4)$. Then

$$\mathscr{G}^\theta_{0,n}, \mathscr{G}^\theta_n : \; B^{k+1}_{x_1} \otimes L^2(\mathbb{R}_{x_2};\mathbb{C}) \to B^k_{x_1} \otimes L^2(\mathbb{R}_{x_2};\mathbb{C}) \subset L^2(\mathbb{R}^2_x;\mathbb{C}^4)$$

are bounded. Define $R^+_n = R^+_n(\theta) : B^k_{x_1} \otimes L^2(\mathbb{R}_{x_2};\mathbb{C}) \to L^2(\mathbb{R}_{x_2};\mathbb{C}^2)$ and $R^-_n = R^-_n(\theta)$: $L^2(\mathbb{R}_{x_2};\mathbb{C}^2) \to B^k_{x_1} \otimes L^2(\mathbb{R}_{x_2};\mathbb{C})$ by

$$(R^+_n t)(x_2) = \int_{\mathbb{R}} K^\theta_n(x_1)^* t(x_1, x_2)\, dx_1 \text{ and } R^-_n(s)(x) = K^\theta_n(x_1) s(x_2) \tag{4.28}$$

with

$$K^\theta_n(x_1) = \begin{pmatrix} u^\theta_n(x_1) & 0 \\ 0 & u^{-\theta}_n(x_1) \end{pmatrix}_{4\times 2}. \tag{4.29}$$

Then $(R^+_n)^* = R^-_n$.

First, we consider the Grushin problem for the unperturbed operator $\mathscr{G}^\theta_{0,n} - \sqrt{h}z$:

Lemma 4.5.4 (Unperturbed Grushin)**.** *Fix $n \in \mathbb{Z}$. Let R^+_n and R^-_n be defined as* (4.28)*. Let*

$$\mathcal{P}_{0,n} = \mathcal{P}_{0,n}(z; h, \theta) := \begin{pmatrix} \mathscr{G}^\theta_{0,n} - \sqrt{h}z & R^-_n \\ R^+_n & 0 \end{pmatrix}.$$

Then $\mathcal{P}_{0,n}$ is invertible iff $\sqrt{h}z \notin \{\lambda_m - \lambda_n : m \neq n\}$, and the inverse is

$$\mathcal{E}_{0,n} := (\mathcal{P}_{0,n})^{-1} =: \begin{pmatrix} E_{0,n} & E_{0,n,+} \\ E_{0,n,-} & E_{0,n,\pm} \end{pmatrix} \tag{4.30}$$

where $E_{0,n,+} = R_n^-$, $E_{0,n,-} = R_n^+$, $E_{0,n,\pm}(z;h) = \sqrt{h}z\,\mathbb{1}_{\mathbb{C}^{2\times 2}}$ *and*

$$E_{0,n}^{\theta}(z;h) = \sum_{m\neq n} \frac{K_m^{\theta}(K_m^{\theta})^*}{\lambda_m - \lambda_n - \sqrt{h}z} = \sum_{m\neq n} \frac{\begin{pmatrix} u_m^{\theta}(u_m^{\theta})^* & 0 \\ 0 & u_m^{-\theta}(u_m^{-\theta})^* \end{pmatrix}}{\lambda_m - \lambda_n - \sqrt{h}z} =: \begin{pmatrix} e_{0,n}^{\theta} & 0 \\ 0 & e_{0,n}^{-\theta} \end{pmatrix} \tag{4.31}$$

with $\lambda_n = \operatorname{sgn}(n)\sqrt{2|n|}, n \in \mathbb{Z}$. *Furthermore, we have*

$$E_{0,n,-}(\mathscr{G}_{0,n}^{\theta} - \sqrt{h}z)E_{0,n,+} = -E_{0,n,\pm} \quad \text{and}$$
$$(\mathscr{G}_{0,n}^{\theta} - \sqrt{h}z)^{-1} = E_{0,n} - E_{0,n,+}(E_{0,n,\pm})^{-1}E_{0,n,-}.$$

Remark 4.8. One can verify that $E_{0,n}^{\theta}$ maps N_n^{θ} to 0 and N_m^{θ} to $\frac{N_m^{\theta}}{\lambda_m - \lambda_n - \sqrt{h}z}$ if $m \neq n$.

Perturbed Grushin problem. Next, we consider the perturbed Grushin problem for $\mathscr{G}_n^{\theta} - \sqrt{h}z$.

Lemma 4.5.5 (Perturbed Grushin)**.** *Let* $R_n^{\pm}$, $\mathscr{W}^W$ *be defined as* (4.28), (4.22). *Let*

$$\mathcal{P}_n = \mathcal{P}_n(z;h,\theta) := \begin{pmatrix} \mathscr{G}_n^{\theta} - \sqrt{h}z & R_n^- \\ R_n^+ & 0 \end{pmatrix} = \mathcal{P}_{0,n} + \sqrt{h}\mathcal{W}$$

where $\mathcal{W} := \operatorname{diag}(\mathscr{W}_{4\times 4}^W, 0_{2\times 2})$. *Fix* $n \in \mathbb{Z}$, *there exist* $h_0 = \min\left\{\frac{1}{2\|\mathscr{W}\|_\infty}, \frac{\lambda_{|n|+1} - \lambda_{|n|}}{4\|\mathscr{W}\|_\infty}\right\}$, *s.t. for all* $h \in [0, h_0)$, $\mathcal{P}_n$ *is invertible with inverse*

$$\mathcal{E}_n := (\mathcal{P}_n)^{-1} =: \begin{pmatrix} E_n & E_{n,+} \\ E_{n,-} & E_{n,\pm} \end{pmatrix} \tag{4.32}$$

which is analytic in $|z| \leq 2\|\mathscr{W}\|_\infty$. $E_{n,\pm}(z) : L^2(\mathbb{R}_{x_2}; \mathbb{C}^2) \to L^2(\mathbb{R}_{x_2}; \mathbb{C}^2)$ *is called the* *effective Hamiltonian and satisfy*

$$E_{n,\pm}(z) = \sqrt{h}\left(z - R_n^+ \mathscr{W}^W (\mathbb{1} + \sqrt{h} E_{0,n} \mathscr{W}^W)^{-1} R_n^-\right) =: \sqrt{h}(z - Z^W). \tag{4.33}$$

In addition, we have

$$E_{n,-}(\mathscr{G}_n^\theta - \sqrt{h}z)E_{n,+} = -E_{n,\pm} \Rightarrow \sqrt{h}E_{n,-}E_{n,+} = \partial_z E_{n,\pm}, \tag{4.34}$$

$$(\mathscr{G}_n^\theta - \sqrt{h}z)^{-1} = E_n - E_{n,+}E_{n,\pm}^{-1}E_{n,-}. \tag{4.35}$$

Proof. Let h_0 be defined as above. When $h \in [0, h_0)$, $|z| < 2\|\mathscr{W}\|_\infty$, we have

$$\begin{cases} \sqrt{h}z \notin \{\lambda_m - \lambda_n : m \neq n\} & \Rightarrow \mathcal{P}_{0,n} \text{ is invertible with } \|\mathcal{P}_{0,n}\| \geq 1. \\ \sqrt{h}\|\mathcal{W}\|_\infty \leq \frac{1}{2} & \Rightarrow \mathcal{P}_n = \mathcal{P}_{0,n} + \sqrt{h}\mathcal{W} \text{ is invertible with inverse } \mathcal{E}_n. \\ |\sqrt{h}z| \leq \frac{\lambda_{|n|+1} - \lambda_{|n|}}{2} & \Rightarrow E_{0,n}(z), \mathcal{E}_{0,n}(z) \text{ are analytic by (4.31), (4.30).} \end{cases}$$

Furthermore,

$$\mathcal{E}_n := \mathcal{P}_n^{-1} = (I + \sqrt{h}\mathcal{P}_{0,n}^{-1}\mathcal{W})^{-1}\mathcal{P}_{0,n}^{-1} = \sum_{j=0}^{\infty}(-1)^j h^{j/2} \left(\mathcal{E}_{0,n}\mathcal{W}\right)^j \mathcal{E}_{0,n}.$$

In particular, we get from the $(2,2)$-block of $\mathcal{P}_n^{-1}$ that

$$\begin{aligned} E_{n,\pm}(z) &= E_{0,n,\pm}(z) + \sum_{j=1}^{\infty}(-1)^j h^{j/2} E_{0,n,-}\mathscr{W}^W (E_{0,n}\mathscr{W}^W)^{j-1}E_{0,n,+} \\ &= \sqrt{h}z - \sqrt{h}R_n^+\mathscr{W}^W(\mathbb{1} + \sqrt{h}E_{0,n}\mathscr{W}^W)^{-1}R_n^-. \end{aligned}$$

In fact, by direct computation, one get that $E_{0,n}$, $E_{n,+}$ and $E_{n,-}$ can all be represented by entries of $\mathcal{E}_{0,n}$ which we proved are analytic, thus $\mathcal{E}_n(z)$ is also analytic.

In the end, (4.34) and (4.35) follows from $\mathcal{E}_n\mathcal{P}_n\mathcal{E}_n = \mathcal{P}_n$ and the diagonalization on

$\mathcal{P}_n$. $\square$

4.5.3 Properties of effective Hamiltonian

In this subsection, we proceed with our study of $E_{n,\pm}(z)$, $E_{n,\pm}^{-1}(z)$ and $\partial_z E_{n,\pm} \circ E_{n,\pm}^{-1}$, with their symbols denoted by $E_{n,\pm}(x_2,\xi_2;z,h)$, $E_{n,\pm}^{-1}(x_2,\xi_2;z,h)$ and $r_n(x_2,\xi_2;z,h) := \partial_z E_{n,\pm} \# E_{n,\pm}^{-1}(x_2,\xi_2;z,h)$. Apart from analyzing boundedness and asymptotic expansions of symbols, we are especially interested in understanding the z-dependence and z vs. h competition of the symbols.

Before starting to analyze these properties, we introduce a key concept of this section: the operator-valued symbol and its quantization.

Operator-valued symbol. Let $b^w(x_2,\xi_2;x_1,D_{x_1}) \in S\left(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B^{k+1}_{x_1}; B^k_{x_1})\right)$, which we shall call an operator(-in-(x_1,D_{x_1}))-valued symbol (in (x_2,ξ_2)), then its (x_2,hD_{x_2})-Weyl quantization is defined as $b^W(x_2,hD_{x_2};x_1,D_{x_1}) : L^2(\mathbb{R}_{x_2}; B^{k+1}_{x_1}) \to L^2(\mathbb{R}_{x_2}; B^k_{x_1})$ such that

$$\begin{aligned}&\left(b^W(x_2,hD_{x_2};x_1,D_{x_1})u\right)(x_2)\\ &= \int e^{\frac{i(x_2-y_2)\xi_2}{h}} \left(b^w\left(\frac{x_2+y_2}{2},\xi_2;x_1,D_{x_1}\right)u\right)(x_1;\xi_1)\frac{dy_2 d\xi_2}{2\pi h}.\end{aligned}$$

In particular, if we have a symbol $a \in S(\mathbb{R}^4_{x,\xi})$, and we view (x_2,ξ_2) as parameters and consider the (x_1,D_{x_1})-Weyl quantization of it, we get $a^w(x,D_{x_1},\xi_2)$ which is an operator-valued symbol in (x_2,ξ_2) (the superscript w represent the (x_1,D_{x_1})-Weyl quantization). If we do a further (x_2,hD_{x_2})-Weyl quantization of $a^w(x,D_{x_1},\xi_2)$, then we get the $(1,h)$-Weyl quantization defined in (4.20).

Remark 4.9. For the rest of this section, given an operator, e.g. $\mathscr{G}^\theta_0$, $E_{n,\pm}$ and $\mathscr{W}^W$ in

(4.21), (4.32) and (4.22), instead of viewing them as the $(1,h)$-Weyl quantization of the scalar-valued symbol in $S(\mathbb{R}^4_{x,\xi})$, we will view them as the (x_2, hD_{x_2})-Weyl quantization of the operator-valued symbol in $S\left(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B^{k_1}_{x_1}; B^{k_2}_{x_1})\right)$, for appropriate $k_1, k_2 \in \mathbb{Z}$.

In particular, since $\mathscr{G}_0^\theta$ only depends on (x_1, D_{x_1}), $E_{n,\pm}$ only depeneds on (x_2, hD_{x_2}), $\mathscr{W}^W(x, D_x)$ is the $(1,h)$-Weyl quantization of the symbol $\mathscr{V}(x_2 + \sqrt{h}x_1, h\xi_2 - \sqrt{h}\xi_1)$, we see that the operator-valued symbol of $\mathscr{G}_0^\theta$, $E_{n,\pm}$ and $\mathscr{W}^W$ are respectively

$$\Sigma_1^\theta x_1 + \Sigma_2^\theta D_{x_1},\ E_{n,\pm}(x_2, \xi_2; z, h),\ \text{and}\ \tilde{\mathscr{V}}^w(x, D_{x_1}, \xi_2) := \mathscr{V}^w(x_2 + \sqrt{h}x_1, \xi_2 - \sqrt{h}D_{x_1})$$

where $\tilde{f}(x,\xi) = f(x_2 + \sqrt{h}x_1, \xi_2 - \sqrt{h}\xi_1)$. And since now

$$\mathscr{U}\mathscr{V}(x)\mathscr{U}^{-1} = \mathscr{W}^W(x, D_x) = \tilde{\mathscr{V}}^W(x, D_{x_1}, hD_{x_2}), \tag{4.36}$$

we will use $\tilde{\mathscr{V}}^W$ to replace $\mathscr{W}^W$ in Lemma 4.5.1 and 4.5.5. Finally, we mention that the proof of Lemma 4.5.1 implies in general

$$\mathscr{U} f(x)\mathscr{U}^{-1} = \tilde{f}^W(x, D_{x_1}, hD_{x_2}). \tag{4.37}$$

Boundedness with z dependence. We now study the boundedness of the operator-valued symbol $E_{n,\pm}$, $E_{n,\pm}^{-1}$ and r_n as well as the z dependence of them.

Notice that since $E_{n,\pm}$ only depends on (x_2, hD_{x_2}), when viewed as a (x_2, hD_{x_2})-Weyl quantization, its operator-in-(x_1, D_{x_1})-valued symbol coincides with its $\mathbb{C}_{2\times 2}$-valued symbol. For convenience, we write $S^k_\delta(\mathbb{R}^2_{x_2,\xi_2}; \mathbb{C}_{2\times 2})$ as S^k_δ and omit the "0" in δ and k.

Lemma 4.5.6 (Boundedness)**.** *Let h_0, $E_{n,\pm}$ be as in Lemma 4.5.5. Then for all $h \in [0, h_0)$, we have the symbol of $E_{n,\pm}$, $E_{n,\pm}(x_2, \xi_2; z, h)$, belongs to $S^{-\frac{1}{2}}$ uniformly in*

$|z| \le 2\|\mathscr{V}\|_\infty$, i.e. for any $\alpha, \beta > 0$, there is $C_{\alpha,\beta,n} = C_{\alpha,\beta,n}(\|\mathscr{V}\|_\infty)$, s.t.

$$\sup_{(x_2,\xi_2)\in\mathbb{R}^2} \|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta E_{n,\pm}(x_2,\xi_2;z,h)\|_{\mathbb{C}_{2\times 2}} \le C_{\alpha,\beta,n}\sqrt{h}, \quad \textit{for all } |z| \le 2\|\mathscr{V}\|_\infty.$$

Furthermore, if $|\operatorname{Im} z| \neq 0$, then we also have that for all $h \in [0, h_0)$, $|z| \le 2\|\mathscr{V}\|_\infty$, $\alpha, \beta > 0$,

$$\|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta E_{n,\pm}^{-1}(x_2,\xi_2;z,h)\|_{\mathbb{C}_{2\times 2}} \le C_{\alpha,\beta,n} \max\left(1, \frac{h^{3/2}}{|\operatorname{Im} z|^3}\right) h^{-\frac{1}{2}} |\operatorname{Im} z|^{-(|\alpha|+|\beta|)-1}, \tag{4.38}$$

$$\|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta r_n(x_2,\xi_2;z,h)\|_{\mathbb{C}_{2\times 2}} \le C_{\alpha,\beta,n} \max\left(1, \frac{h^{3/2}}{|\operatorname{Im} z|^3}\right) |\operatorname{Im} z|^{-(|\alpha|+|\beta|)-1}. \tag{4.39}$$

In particular, if $0 < \delta < \frac{1}{2}$ and $|\operatorname{Im} z| \ge h^\delta$, then $E_{n,\pm}^{-1} \in S_\delta^{\frac{1}{2}+\delta}$ and $r_n \in S_\delta^\delta$.

Proof. When $h \in [0, h_0)$, $|z| \le 2\|\mathscr{V}\|_\infty$, $E_{n,\pm}$ is a ΨDO because $\mathcal{P}_n$ is. In fact, we can check term by term that the operator-valued symbol

$$\mathcal{P}_n(x, D_{x_1}, \xi_2) \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B^{k+1}_{x_1} \times \mathbb{C}^2; B^k_{x_1} \times \mathbb{C}^2)).$$

By invertibility and Beal's lemma,

$$\mathcal{E}_n(x_2,\xi_2;z,h) \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B^{k+1}_{x_1} \times \mathbb{C}^2; B^k_{x_1} \times \mathbb{C}^2)).$$

In particular, we have

$$\begin{cases} R_n^+ \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B^k_{x_1}; \mathbb{C}^2)),\ R_n^- \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(\mathbb{C}^2; B^{k+1}_{x_1})), \\ E_{0,n} \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B^k_{x_1}; B^{k+1}_{x_1})),\ \tilde{\mathscr{V}}^w \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B^{k+1}_{x_1}; B^k_{x_1})). \end{cases} \tag{4.40}$$

Furthermore, by (4.31), when $|\sqrt{h}z| \le \frac{\lambda_{|n|+1}-\lambda_{|n|}}{2}$, $E_{0,n}$ is uniformly bounded. Thus

$E_{n,\pm}, \partial_z E_{n,\pm} \in S^{-\frac{1}{2}}$ uniformly.

Then we consider $E_{n,\pm}^{-1}$ and r_n. Let l_1, l_2, $\cdots$ be linear forms on $\mathbb{R}^2_{x_2,\xi_2}$. Let $L_j = l_j(x_2, hD_{x_2})$. Since $E_{n,\pm} \circ E_{n,\pm}^{-1} = I$, we get

$$\operatorname{ad}_{L_j} E_{n,\pm}^{-1} = -E_{n,\pm}^{-1} \circ \operatorname{ad}_{L_j} E_{n,\pm} \circ E_{n,\pm}^{-1},$$

where $\operatorname{ad}_{L_j} A = [L_j, A]$. Since $\operatorname{ad}_{L_j}(A \circ B) = (\operatorname{ad}_{L_j} A) \circ B + A \circ \operatorname{ad}_{L_j} B$, thus

$$\operatorname{ad}_{L_j}(\partial_z E_{n,\pm} \circ E_{n,\pm}^{-1}) = -\partial_z E_{n,\pm} \circ E_{n,\pm}^{-1} \circ \operatorname{ad}_{L_j} E_{n,\pm} \circ E_{n,\pm}^{-1} + \operatorname{ad}_{L_j} \partial_z E_{n,\pm} \circ E_{n,\pm}^{-1}.$$

By (4.35), $\|\sqrt{h} E_{n,\pm}^{-1}\|_{\mathbb{C}_{2\times 2}} = \mathcal{O}(|\operatorname{Im} z|^{-1})$. Recall that $E_{n,\pm}$, $\partial_z E_{n,\pm} \in S^{-\frac{1}{2}}$, thus

$$\|\operatorname{ad}_{L_j}(\sqrt{h} E_{n,\pm}^{-1})\|_{\mathbb{C}_{2\times 2}} = \mathcal{O}\left(\frac{h}{|\operatorname{Im} z|^2}\right) \text{ and } \|\operatorname{ad}_{L_j}(\partial_z E_{n,\pm} \circ E_{n,\pm}^{-1})\|_{\mathbb{C}_{2\times 2}} = \mathcal{O}\left(\frac{h}{|\operatorname{Im} z|^2}\right)$$

By induction,

$$\begin{aligned} &\|\operatorname{ad}_{L_1} \circ \cdots \circ \operatorname{ad}_{L_N}(\sqrt{h} E_{n,\pm}^{-1})\|_{\mathbb{C}_{2\times 2}} = \mathcal{O}\left(\tfrac{h^N}{|\operatorname{Im} z|^{N+1}}\right) \\ &\|\operatorname{ad}_{L_1} \circ \cdots \circ \operatorname{ad}_{L_N}(\partial_z E_{n,\pm} \circ E_{n,\pm}^{-1})\|_{\mathbb{C}_{2\times 2}} = \mathcal{O}\left(\tfrac{h^N}{|\operatorname{Im} z|^{N+1}}\right). \end{aligned}$$

By a parametrized version of Beal's lemma, [30, Prop. 8.4], we get (4.38) and (4.39).

□

Asymptotic Expansion with z dependence. We proceed by discussing the asymptotic expansion of $E_{n,\pm}$, $E_{n,\pm}^{-1}$ and r_n. Again, we are concerned with z-dependence of each term in the asymptotic expansions. In order to focus on the main points, we outsource further details concerning the asymptotic expansion of $E_{n,\pm}$ and $E_{n,\pm}^{-1}$, c.f. Prop. C.0.1, and its proof in the Appendix C, and present a shorter version here that only summarizes the results that we eventually need in the sequel.

Lemma 4.5.7 (Asymptotic expansion)**.** *Let h_0, $E_{n,\pm}$ be as in Lemma 4.5.5, $0<\delta<1/2$. If $h\in[0,h_0)$, $|z|\le 2\|\mathscr{V}\|_\infty$, $|\operatorname{Im} z|\ge h^\delta$, then $r_n(x_2,\xi_2;z,h)=\partial_z E_{n,\pm}\#E_{n,\pm}^{-1}$ has an asymptotic expansion in S^δ_δ:*

$$r_n(x_2,\xi_2;z,h)\sim\sum_{j=0}^{\infty}h^{\frac{j}{2}}r_{n,j}(x_2,\xi_2;z),\ \text{with } h^{\frac{j}{2}}r_{n,j}\in S_\delta^{(j+1)\delta-\frac{j}{2}}. \tag{4.41}$$

More specifically, there are $d_{n,j,k,l}(x_2,\xi_2;z)$, $e_{n,j,k,\alpha}(x_2,\xi_2)\in S$ s.t.

$$r_{n,j}=\sum_{k=0}^{j}(z-z_{n,0})^{-1}\prod_{l=0}^{k}\left[d_{n,j,k,l}(x_2,\xi_2;z)(z-z_{n,0})^{-1}\right], \tag{4.42}$$

with $\prod_{l=0}^{k}d_{n,j,k,l}(x_2,\xi_2;z)=\sum_{\alpha=0}^{j+k-2}z^\alpha e_{n,j,k,\alpha}(x_2,\xi_2)$ and $z_{n,0}$ given in Prop. C.0.2. Let $R_{n,J}:=r_n-\sum_{j=0}^{J-1}h^{\frac{j}{2}}r_{n,j}$, then $R_{n,J}\in S_\delta^{(J+1)\delta-\frac{J}{2}}$, i.e. for all $\alpha,\beta>0$, there is $C'_{\alpha,\beta,n}$ s.t.

$$\sup_{(x_2,\xi_2)\in\mathbb{R}^2}|\partial^\alpha_{x_2}\partial^\beta_{\xi_2}R_{n,J}|\le C'_{\alpha,\beta,n}h^{\frac{J}{2}-(J+1)\delta-\delta(|\alpha|+|\beta|)}. \tag{4.43}$$

Furthermore, for the expansion of $\operatorname{Tr}_{\mathbb{C}^2}(r_n)$, we have for $\eta=x_2+i\xi_2$,

$$\begin{aligned}\text{Chiral }\ \mathscr{H}_{\mathrm{c},n}(J=3):\operatorname{Tr}_{\mathbb{C}^2}(r_{\mathrm{c},n,0}+h^{\frac12}r_{\mathrm{c},n,1}+hr_{\mathrm{c},n,2})&=\frac{2}{z}+0+\frac{\lambda_n^2}{z^3}\mathfrak{U}(\eta)h,\\ \text{Anti-Chiral }\ \mathscr{H}^\theta_{\mathrm{ac},n}(J=2):\operatorname{Tr}_{\mathbb{C}^2}(r_{\mathrm{ac},n,0}+h^{\frac12}r_{\mathrm{ac},n,1})&=\frac{2z}{z^2-c_n^2}+\frac{2s_n^2(z^2+c_n^2)}{(z^2-c_n^2)^2}\sqrt{h},\end{aligned} \tag{4.44}$$

where $\mathfrak{U}(\eta)=\frac{\alpha_1^2}{8}\left[\alpha_1^2(|U_-(\eta)|^2-|U(\eta)|^2)^2+4|\partial_{\bar\eta}\overline{U_-(\eta)}-\partial_\eta U(\eta)|^2\right]$, $\partial_\eta=\frac12(\partial_{x_2}-i\partial_{\xi_2})$,

$$s_n(\eta)=\begin{cases}\alpha_0\sin(\frac{\theta}{2})|V(\eta)| & n\ne0\\ \alpha_0|V(\eta)| & n=0,\end{cases}\quad\text{and } c_n(\eta)=\begin{cases}\alpha_0\cos(\frac{\theta}{2})|V(\eta)| & n\ne0\\ \alpha_0|V(\eta)| & n=0.\end{cases}$$

Remark 4.10. Notice by Prop. C.0.2, $z_{n,0} = 0$ for the chiral model. Thus we have $r_{c,n,j} = \sum_{k=0}^{2(j-1)} z^{k-j-1} f_{n,j,k}(x_2, \xi_2)$ for appropriate $f_{n,j,k} \in S$ when $j \geq 1$.

4.5.4 Trace formula

Now we are ready to characterize $\tilde{\mathrm{Tr}} f(\mathscr{H}^\theta))$ using $E_{n,\pm}$ and still use the operator-valued symbol and (x_2, hD_{x_2})-quantization in this subsection.

Lemma 4.5.8. *Let $E_{n,\pm}$ be as in Lemma 4.5.5. Let $f \in C_c^{K+1}(\Lambda_{n,B,\mathscr{V}})$ and $f_0(x) = f(x + \lambda_{n,B})$ be as in* (4.27)*. Then the regularized trace $\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$ satisfies*

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta)) = -\frac{i}{4\pi^2 h|E|} \int_{\mathbb{C}} \int_E \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(r_n(x_2, \xi_2; z, h)) \; dx_2 \; d\xi_2 \; dz \wedge d\bar{z}, \tag{4.45}$$

Lemmas needed for the following proof are outsourced to Appendix D.

Proof. By (4.27), (4.35), and the analyticity of $E_n(z)$ when $h \in [0, h_0)$, $|z| \leq 2\|\mathscr{V}\|_\infty$,

$$\mathscr{U} f(\mathscr{H}^\theta)\mathscr{U}^{-1} = -\frac{i\sqrt{h}}{2\pi} \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0(E_{n,+} E_{n,\pm}^{-1} E_{n,-})(z) \; dz \wedge d\bar{z}.$$

Thus we have

$$
\begin{aligned}
\tilde{\operatorname{Tr}} f(\mathscr{H}^\theta) &= \lim_{R\to\infty} \frac{1}{4R^2} \operatorname{Tr}_1 \left(\mathbb{1}_R\, f(\mathscr{H}^\theta)\, \mathbb{1}_R \right) = \lim_{R\to\infty} \frac{1}{4R^2} \operatorname{Tr}_1 (\tilde{\mathbb{1}}_R^W \mathscr{U} f(\mathscr{H}^\theta) \mathscr{U}^{-1} \tilde{\mathbb{1}}_R^W) \\
&= \lim_{R\to\infty} -\frac{i\sqrt{h}}{8\pi R^2} \operatorname{Tr}_1 \left(\int_{\mathbb{C}} \partial_{\bar z} \tilde f_0 (\tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-} \tilde{\mathbb{1}}_R^W)\; dz\wedge d\bar z \right) \\
&= \lim_{R\to\infty} -\frac{i\sqrt{h}}{8\pi R^2} \int_{\mathbb{C}} \partial_{\bar z} \tilde f_0 \operatorname{Tr}_1 \left(\tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-} \tilde{\mathbb{1}}_R^W \right)\; dz\wedge d\bar z \\
&= \lim_{R\to\infty} -\frac{i\sqrt{h}}{8\pi R^2} \int_{\mathbb{C}} \partial_{\bar z} \tilde f_0 \operatorname{Tr}_2 \left(\bar{\mathbb{1}}_R^W E_{n,-} E_{n,+} E_{n,\pm}^{-1} \bar{\mathbb{1}}_R^W \right)\; dz\wedge d\bar z \\
&= \lim_{R\to\infty} -\frac{i}{8\pi R^2} \int_{\mathbb{C}} \partial_{\bar z} \tilde f_0 \operatorname{Tr}_2 \left(\bar{\mathbb{1}}_R^W \partial_z E_{n,\pm} E_{n,\pm}^{-1} \bar{\mathbb{1}}_R^W \right)\; dz\wedge d\bar z \\
&= \lim_{R\to\infty} -\frac{i}{16\pi^2 h R^2} \int_{\mathbb{C}} \partial_{\bar z} \tilde f_0 \int_{\mathbb{R}^2} \operatorname{Tr}_{\mathbb{C}^2} \left(\bar{\mathbb{1}}_R \# \partial_z E_{n,\pm} \# E_{n,\pm}^{-1} \# \bar{\mathbb{1}}_R \right)\; dx_2\; d\xi_2\; dz\wedge d\bar z \\
&= -\frac{i}{4\pi^2 h |E|} \int_{\mathbb{C}} \int_E \partial_{\bar z} \tilde f_0 \operatorname{Tr}_{\mathbb{C}^2} \left(\partial_z E_{n,\pm} \# E_{n,\pm}^{-1} \right)\; dx_2\; d\xi_2\; dz\wedge d\bar z
\end{aligned}
$$

where $\mathscr{U}\, \mathbb{1}_R\, \mathscr{U}^{-1} =: \tilde{\mathbb{1}}_R^W$ follows from (4.37). And $\bar{\mathbb{1}}_R^W = \bar{\mathbb{1}}_R^W(x_2, hD_{x_2})$ where $\bar{\mathbb{1}}_R(x_2,\xi_2)$ coincides with $\mathbb{1}_R(x_1,x_2)$ but is viewed as a function of phase space variables (x_2,ξ_2) rather than x. In addition, $\operatorname{Tr}_1 = \operatorname{Tr}_{L^2(\mathbb{R}^2_x;\mathbb{C}^4)}$, $\operatorname{Tr}_2 = \operatorname{Tr}_{L^2(\mathbb{R}_{x_2};\mathbb{C}^2)}$.

The second line follows from the Helffer-Sjöstrand formula in Lemma 4.5.3. The third line follows from Lemma D.0.3, where we proved $\mathbb{1}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-} \mathbb{1}_R^W$ is trace class. The fourth line follows directly from Lemma D.0.4. The fifth line follows from (4.35). The sixth line follows from

$$
\operatorname{Tr}_{\mathcal{L}(L^2(\mathbb{R}_{x_2};H_1);L^2(\mathbb{R}_{x_2};H_2))}(a^W(x_2,hD_{x_2})) = \frac{1}{2\pi h} \int_{\mathbb{R}^2_{x_2,\xi_2}} \operatorname{Tr}_{\mathcal{L}(H_1,H_2)}(a(x_2,\xi_2))dx_2 d\xi_2.
$$

The seventh line follows from periodicity of $\mathscr{V}$ and thus periodicity of $\partial_z E_{n,\pm} \# E_{n,\pm}^{-1}$, which follows immediately by looking at the asymptotic expansions. $\square$

4.5.5 Proof of main results

Now we can prove our main Theorems 4.5 and 4.6:

Proof of Theo. 4.5, 4.6. Let $0 < \delta < 1/2$. Assume $f \in C_c^{N+1}(\Lambda_{n,B,\mathscr{V}})$. Let $f_0(x) := f(x + \lambda_{n,B})$ which is supported on a nbhd of 0. Recall by Lemma 4.5.8, we need to compute

$$\tilde{\mathrm{Tr}}(f(\mathscr{H}_n^\theta)) = -\frac{i}{4\pi^2 h|E|}\int_E \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(r_n(x_2, \xi_2; z, h)) \; dz \wedge d\bar{z} \; dx_2 \; d\xi_2. \qquad (4.46)$$

We can rewrite the integral

$$\begin{aligned}\left[\int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(r_n) dz \wedge d\bar{z}\right](x_2, \xi_2; h) =& \int_{\mathbb{C}} \partial_{\bar{z}} \tilde{f}_0 \sum_{j=0}^{J-1} h^{\frac{j}{2}} \, \mathrm{Tr}_{\mathbb{C}^2}(r_{n,j}) dz \wedge d\bar{z} \\ &+ \int_{|\operatorname{Im} z| \ge h^\delta} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(R_{n,J}) dz \wedge d\bar{z} \\ &+ \int_{|\operatorname{Im} z| \le h^\delta} \partial_{\bar{z}} \tilde{f}_0 \, \mathrm{Tr}_{\mathbb{C}^2}(R_{n,J}) dz \wedge d\bar{z} \\ :=& A_1 + A_2 + A_3.\end{aligned}$$

Notice that by Remark 4.6, we only need to consider f_0 supported at $|z| \le \|\mathscr{V}\|$, for which we can pick $\tilde{f}_0$ s.t. $\tilde{f}_0$ is supported inside $|z| \le 2\|\mathscr{V}\|_\infty$ for the integral. As in Lemma 4.5.7, we take $J = 3$ in the chiral case and $J = 2$ in the anti-chiral case.

First of all, we compute A_1 by (4.44) and the general version of Cauchy's integral formula, see [49, (3.1.11)]: Let X be an open subset of $\mathbb{C}$. Let $g \in C_c^1(X)$, then

$$2\pi i g^{(n)}(\zeta) = \int_X \partial_{\bar{z}} g(z) \frac{n!}{(z-\zeta)^{n+1}} dz \wedge d\bar{z}. \qquad (4.47)$$

In particular, take X to be an small open neighborhood of $\mathrm{supp}(\tilde{f}_0)$. By , (4.47) and the definition of f_0, we have

$$A_{1,\mathrm{c}} = \int_{\mathbb{C}} \partial_{\bar z} \tilde f_0 \left[\frac{2}{z} + \frac{\lambda_n^2}{z^3}\mathfrak{U}(\eta)h\right] dz \wedge d\bar z = 2\pi i \left[2f(\lambda_{n,B}) + \frac{\lambda_n^2}{2}\mathfrak{U}(\eta) f''(\lambda_{n,B})h\right],$$

$$\begin{aligned} A_{1,\mathrm{ac}} &= \int_{\mathbb{C}} \partial_{\bar z} \tilde f_0 \operatorname{Tr}_{\mathbb{C}^2}\left[\frac{1}{z-c_n} + \frac{1}{z+c_n} + \frac{s_n^2\sqrt{h}}{(z-c_n)^2} + \frac{s_n^2\sqrt{h}}{(z+c_n)^2}\right] dz \wedge d\bar z \\ &= 2\pi i \left[f(\lambda_{n,B}+c_n) + f(\lambda_{n,B}-c_n) + f'(\lambda_{n,B}+c_n)s_n^2\sqrt{h} + f'(\lambda_{n,B}-c_n)s_n^2\sqrt{h}\right]. \end{aligned}$$

For A_2, by (4.43) and $|z| \le 2\|\mathscr{V}\|_\infty$, when $|\operatorname{Im} z| \ge h^\delta$, there are C_n, C'_n s.t.

$$|A_2| \le \int_{|\operatorname{Im} z| \ge h^\delta} |\partial_{\bar z}\tilde f_0| C_n h^{\frac{J}{2}-(J+1)\delta} 2L(dz) \le C'_n \|f\|_{C^{K+1}} \|\mathscr{V}\|_\infty h^{\frac{J}{2}-(J+1)\delta}.$$

Finally, by (4.25), (4.39), (4.42), $0 < \delta < 1/2$ and $|z| \le 2\|\mathscr{V}\|_\infty$, we have for some $C_{n,j}, C''_n$

$$\begin{aligned} |A_3| &\le \int_{|\operatorname{Im} z| \le h^\delta} |\partial_{\bar z}\tilde f_0| \left[|\operatorname{Tr}_{\mathbb{C}^2}(r_n)| + \sum_{j=0}^{J-1}\left|\operatorname{Tr}_{\mathbb{C}^2}(h^{\frac{j}{2}} r_{n,j})\right|\right] dz \wedge d\bar z \\ &\le \int_{|\operatorname{Im} z| \le h^\delta} \|f\|_{C^{K+1}} |\operatorname{Im} z|^K \left[\max\left(\frac{1}{|\operatorname{Im} z|}, \frac{h^{\frac{3}{2}}}{|\operatorname{Im} z|^4}\right) + \sum_{j=0}^{J-1} \frac{C_{n,j} h^{\frac{j}{2}}}{|\operatorname{Im} z|^{j+1}}\right] dz \wedge d\bar z \\ &\le 2C''_n \|f\|_{C^{K+1}} \|\mathscr{V}\|_\infty \left[\max\left(h^{(K-1)\delta}, h^{(K-4)\delta+\frac{3}{2}}\right) + \sum_{j=0}^{J-1} h^{\frac{j}{2}+(K-j-1)\delta}\right] dz \wedge d\bar z \\ &\le C''_n \|f\|_{C^{K+1}} \|\mathscr{V}\|_\infty h^{(K-1)\delta}, \end{aligned}$$

Define $C_{n,K,f,V} = \max(C_n, C'_n, C''_n)\|\mathscr{V}\|_\infty \|f\|_{C^{K+1}}$. We see

$$|A_{2,\mathrm{c}}| \le C_{n,K,f,\mathscr{V}} h^{\frac{3}{2}-4\delta}, \quad |A_{2,\mathrm{ac}}| \le C_{n,K,f,\mathscr{V}} h^{1-3\delta}, \quad |A_3| \le C_{n,K,f,\mathscr{V}} h^{(K-1)\delta}.$$

Combine the estimates of A_1, A_2, A_3, and plug them into (4.46), we have

$$\begin{aligned}\tilde{\mathrm{Tr}} f(\mathscr{H}_{\mathrm{c}}) &= \frac{1}{\pi h} f(\lambda_{n,B}) + \frac{|n|}{2\pi} f''(\lambda_{n,B})\mathfrak{U}(\eta) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{\frac{1}{2}-4\delta} + h^{(K-1)\delta-1}),\\ \tilde{\mathrm{Tr}} f(\mathscr{H}^{\theta}_{\mathrm{ac}}) &= \frac{1}{2\pi h} t_{n,0}(f) + \frac{1}{2\pi\sqrt{h}} t_{n,1}(f) + \mathcal{O}(h^{-3\delta} + h^{(K-1)\delta-1})\end{aligned} \tag{4.48}$$

where $t_{n,0}(f) = \mathrm{Ave}[f(\lambda_{n,B} - c_n) + f(\lambda_{n,B} + c_n)]$, $t_{n,1}(f) = \mathrm{Ave}[s_n^2 f(\lambda_{n,B} - c_n) + s_n^2 f(\lambda_{n,B} + c_n)]$, and $\mathrm{Ave}(g) = \frac{1}{|E|}\int_E g(\eta) d\eta$. Thus we proved (4.18) and (4.19).

In general, fix $N \in \mathbb{N}^+$ and we consider $2N+1$ Landau levels centered at 0. Let B be large enough s.t. $\left\{\overline{B_{\|\mathscr{V}\|_\infty}(\lambda_{n,B})}\right\}_{n=-N}^{N}$ do not intersect. For any $f \in C_c^{K+1}(\lambda_{-(N+1),B} + \|\mathscr{V}\|_\infty, \lambda_{N+1,B} - \|\mathscr{V}\|_\infty)$, by Remark 4.6, values of f on the gap do not contribute to $\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$, thus we can apply the partition of unity of f on $\{\Lambda_{n,B,\mathscr{V}}\}_{n=-N}^{N}$, i.e. find f_n s.t. $f = \sum_{n=-N}^{N} f_n$ and $\operatorname{supp} f_n \subset \Lambda_{n,B,\mathscr{V}}$. Then we can apply (4.18) and (4.19) to each f_n and take the sum. That gives us the rest of the Theorem 4.5 and 4.6.

Furthermore, as mentioned in Remark 4.10, $z_{n,0} = 0$ in the chiral case, thus each term in the expansion of $r_{n,\mathrm{c}}$ is of the form $r_{n,j,\mathrm{c}} = \sum_{k=0}^{k-j-1} z^{k-j-1} f_{n,j,k}(x_2, \xi_2)$. Now assume f is smooth enough, then for any $J \in \mathbb{N}$, by (4.47), we can see that

$$A_{1,\mathrm{c}} = \sum_{j=0}^{J-1} h^{j/2} \sum_{k=0}^{k-j-1} F_{n,j,k}(\eta) f^{(j-k)}(\lambda_{n,B}), \quad \text{for some } F_{n,j,k}(\eta) \in S. \tag{4.49}$$

Thus for the chiral case, every term in the asymptotic expansion of $\tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$ only depends on derivatives $f^{(k)}$ at $\lambda_{n,B}$.

□

4.5.6 Differentiability

Finally, we comment on the differentiability of the regularized trace with respect to the magnetic field. That $h \mapsto \tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$ is a differentiable function follows already from Lemma 4.4.1. However, what does not follow from Lemma 4.4.1 is that the asymptotic expansion itself in Theorems 4.5 and 4.6 is differentiable. The following Proposition, which uses the same notation as Theorems 4.5 and 4.6 shows that term-wise differentiation yields the right asymptotic expansion:

Proposition 4.5.9 (Differentiability). *Under the same assumption of $\lambda_{n,B}$, ε, as in Theorem 4.5, we have that $B \mapsto \tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$ is differentiable. For all ε, $f \in C^K(\Lambda_{n,B,\mathscr{V}})$, that $K > \frac{6}{\varepsilon} - 2$, then for $\mathcal{O}_{n,K,f,\mathscr{V}} = \mathcal{O}_n(\|\mathscr{V}\|_\infty \|f\|_{C^K})$, we have: For the chiral model $\mathscr{H}^\theta = \mathscr{H}_{\mathrm{c}}$,*

$$\begin{aligned}\partial_B \tilde{\mathrm{Tr}}(f(\mathscr{H}_{\mathrm{c}})) = &\frac{\sqrt{2|n|B}}{2\pi} f'(\lambda_{n,B}) + \frac{f(\lambda_{n,B})}{\pi} + \\ &\frac{(2|n|)^{\frac{3}{2}}}{8\pi\sqrt{B}} \operatorname{Ave}(\mathfrak{U}) f'''(\lambda_{n,B}) + \mathcal{O}_{n,K,f,\mathscr{V}}(B^{-1+\varepsilon})\end{aligned} \tag{4.50}$$

For the anti-chiral model $\mathscr{H}^\theta = \mathscr{H}^\theta_{\mathrm{ac}}$,

$$\partial_B \tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta_{\mathrm{ac}})) = \frac{\sqrt{2|n|B}}{4\pi} t_{n,0}(f') + \frac{1}{4\pi}\Big(2t_{n,0}(f) + \sqrt{2|n|}\, t_{n,1}(f')\Big) + \mathcal{O}_{n,K,f,\mathscr{V}}(B^{-\frac{1}{2}+3\delta}) \tag{4.51}$$

In particular, when $n = 0$, we get a better estimate for the chiral and anti-chiral case respectively:

$$\begin{aligned}\partial_B \tilde{\mathrm{Tr}}(f(\mathscr{H}_{\mathrm{c}})) &= \frac{1}{\pi} f(0) + \mathcal{O}_{0,K,f,\mathscr{V}}(B^{-\frac{3}{2}+4\delta}) \\ \partial_B \tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta_{\mathrm{ac}})) &= \frac{1}{2\pi} t_{0,0}(f) + \frac{3}{4\pi\sqrt{B}} t_{0,1}(f) + \mathcal{O}_{0,K,f,\mathscr{V}}(B^{-1+3\delta})\end{aligned} \tag{4.52}$$

where $t_{n,0}(f)$, $t_{n,1}(f)$, $\mathfrak{U}$, s_n and c_n are the same as in Theorem 4.5, 4.6.

To prove this proposition, we will need to prove two auxiliary Lemmas 4.5.10 and 4.5.11 discussing properties of $\partial_h E_{n,\pm}$, $\partial_h E_{n,\pm}^{-1}$ and $\partial_h r_n$, which are similar to the two properties needed for $E_{n,\pm}$, $E_{n,\pm}^{-1}$ and r_n previously in 4.5.6 and 4.5.7. The rest of the proof is similar to Sec. 4.5.5. We start with some preparations: To discuss the differentiability of asymptotic expansions, we define $\#_h^M$ for $a(x,\xi;h), b(x,\xi;h) \in S(\mathbb{R}^2_{x,\xi})$ by

$$\begin{aligned} a\#_h^M b &= \left[e^{\frac{ih}{2}\sigma(D_x,D_\xi;D_y,D_\eta)} \left(\frac{i}{2}\sigma(D_x,D_\xi;D_y,D_\eta) \right)^M \right] (a(x,\xi,h)b(y,\eta,h)) \,|_{\substack{x=y\\ \xi=\eta}} \\ &= \sum_{|\alpha|=|\beta|=M} C_{\alpha,\beta} (\partial_{x,\xi}^\alpha a) \# (\partial_{y,\eta}^\beta b), \end{aligned} \tag{4.53}$$

where $\sigma(x,\xi;y,\eta) = \langle \xi, y\rangle - \langle x,\eta\rangle$. Then we see that,

$$\partial_h^M (a\#b) = a\#_h^M b \; + \sum_{\substack{i+j+k=M\\ j\neq M}} C_{i,j,k} \; (\partial_h^i a) \; \#_h^j \; (\partial_h^k b). \tag{4.54}$$

The following result is derived for general $M \in \mathbb{N}$ but we will, for simplicity, only consider the $M = 1$ case later:

Lemma 4.5.10 (Boundedness)**.** *Let h_0, $E_{n,\pm}$ be as in Lemma 4.5.5. The symbol $E_{n,\pm}(x_2,\xi_2;z,h)$ is smooth in h when $h < h_0$ and for any $M \in \mathbb{N}$, $\partial_h^M E_{n,\pm} \in S^{M-\frac{1}{2}}$ uniformly in $|z| \leq 2\|\mathscr{V}\|_\infty$, i.e. for any multi-index α, β, there is $C_{\alpha,\beta,n} = C_{\alpha,\beta,n}(\|\mathscr{V}\|_\infty)$ s.t.*

$$\|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta \partial_h^M E_{n,\pm}(x_2,\xi_2;z,h)\|_{\mathbb{C}_{2\times 2}} \leq C_{\alpha,\beta,n}\sqrt{h}, \quad \textit{for all } |z| \leq 2\|\mathscr{V}\|_\infty.$$

If $|\operatorname{Im} z| \neq 0$*,* $M > 0$*, then* $\partial_h^M E_{n,\pm}^{-1}$ *and* $\partial_h^M r_n$ *satisfy*

$$\|\partial_{x_2}^{\alpha}\partial_{\xi_2}^{\beta}\partial_h^M E_{n,\pm}^{-1}(x_2,\xi_2;z,h)\|_{\mathbb{C}_{2\times 2}} \leq C_{\alpha,\beta,n} \max\left(1, \frac{h^{\frac{3}{2}}}{|\operatorname{Im} z|^3}\right) h^{-\frac{1+2M}{2}} |\operatorname{Im} z|^{-2M-|\alpha|-|\beta|}, \tag{4.55}$$

$$\|\partial_{x_2}^{\alpha}\partial_{\xi_2}^{\beta}\partial_h^M r_n(x_2,\xi_2;z,h)\|_{\mathbb{C}_{2\times 2}} \leq C_{\alpha,\beta,n} \max\left(1, \frac{h^{\frac{3}{2}}}{|\operatorname{Im} z|^3}\right) h^{-M} |\operatorname{Im} z|^{-2M-|\alpha|-|\beta|}. \tag{4.56}$$

In particular, when $0 < \delta < 1/2$ *and* $|\operatorname{Im} z| \geq h^{\delta}$*, we have* $\partial_h^M E_{n,\pm}^{-1} \in S_{\delta}^{M(2\delta+1)+\frac{1}{2}}$ *and* $\partial_h^M r_n \in S_{\delta}^{M(2\delta+1)}$.

Proof. Let $\mathcal{P}_n$ be as in Lemma 4.5.5, by 4.40, $\mathscr{G}^{\theta} - \sqrt{h}z$, $R_n^{\pm} \in S(\mathbb{R}^2_{x_2,\xi_2})$. Furthermore, since $\mathscr{G}^{\theta} = \mathscr{G}_0^{\theta} + \sqrt{h}\tilde{\mathscr{V}}^w$, by direct computation, we see $\partial_h^M(\mathscr{G}^{\theta} - \sqrt{h}z) \in S^{M-\frac{1}{2}}$ while $\partial_h^M R_n^{\pm} = 0$, for $M > 0$.

Then consider $\mathcal{E}_n = \mathcal{P}_n^{-1}$. First of all, by the proof of Lemma 4.5.6, we have

$$\mathcal{E}_n(x, D_{x_1}, \xi_2) \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B^k_{x_1} \times \mathbb{C}^2; B^{k+1}_{x_1} \times \mathbb{C}^2)).$$

By differentiating $\mathcal{E}_n = \mathcal{E}_n \# \mathcal{P}_n \# \mathcal{E}_n$ w.r.t. h and using (4.53) and (4.54), we have

$$\partial_h \mathcal{E}_n = -\mathcal{E}_n \# \partial_h \mathcal{P}_n \# \mathcal{E}_n + \sum_{|\alpha|=|\beta|=1} C_{\alpha,\beta}\left(\partial^{\alpha}_{x_2,\xi_2}\mathcal{E}_n \# \partial^{\beta}_{x_2,\xi_2}\mathcal{P}_n \# \mathcal{E}_n\right). \tag{4.57}$$

Since $\partial_h \mathcal{P}_n \in S^{\frac{1}{2}}$, thus $\partial_h \mathcal{E}_n \in S^{\frac{1}{2}}$ above. By differentiating (4.57) w.r.t. h and using (4.53) and (4.54), we see that $\partial_h^2 \mathcal{E}_n \in S^{\frac{3}{2}}$. An iterative argument shows that $\partial_h^M \mathcal{E}_n \in S^{M-\frac{1}{2}}$. In particular, $\partial_h^M E_{n,\pm} \in S^{M-\frac{1}{2}}$. Furthermore, by differentiating $E_{n,\pm}^{-1} = E_{n,\pm}^{-1} \# E_{n,\pm} \# E_{n,\pm}^{-1}$ w.r.t. h and using (4.54) and (4.53), we have

$$\partial_h E_{n,\pm}^{-1} = -E_{n,\pm}^{-1} \# \partial_h E_{n,\pm} \# E_{n,\pm}^{-1} - \sum_{|\alpha|=|\beta|=1} C_{\alpha,\beta}\partial^{\alpha}_{x_2,\xi_2} E_{n,\pm}^{-1} \# \partial^{\beta}_{x_2,\xi_2} E_{n,\pm} \# E_{n,\pm}^{-1}. \tag{4.58}$$

When $|\operatorname{Im} z| \geq h^\delta$, by (4.39) and [89, Theorem 4.23(ii)], we see that

$$\|\partial_h E_{n,\pm}^{-1}\| = \mathcal{O}(h^{-\frac{3}{2}}|\operatorname{Im} z|^{-2}) + \mathcal{O}(h^{-\frac{1}{2}}|\operatorname{Im} z|^{-3}) = \mathcal{O}(h^{-\frac{3}{2}}|\operatorname{Im} z|^{-2}).$$

Furthermore, since $[D_{x_j}, A^W] = (D_{x_j}A)^W$ and $-[x_j, A^W] = (hD_{\xi_j}A)^W$, we see that

$$\|\operatorname{ad}_{L_{j_1}} \circ \cdots \operatorname{ad}_{L_{j_N}} (\partial_h E_{n,\pm}^{-1})^W\| = \mathcal{O}\left(\frac{h^{-\frac{3}{2}}}{|\operatorname{Im} z|^2} \frac{h^N}{|\operatorname{Im} z|^N}\right).$$

By [30, Prop. 8.4], we get

$$\|\partial_{x_2}^\alpha \partial_{\xi_2}^\beta \partial_h E_{n,\pm}^{-1}(x_2, \xi_2; z, h)\|_{\mathbb{C}_{2\times 2}} \leq C_{\alpha,\beta} \max\left(1, \frac{h^{\frac{3}{2}}}{|\operatorname{Im} z|^3}\right) h^{-\frac{3}{2}}|\operatorname{Im} z|^{-2-|\alpha|-|\beta|}. \quad (4.59)$$

Iterating this process by taking ∂_h of (4.58), expanding it and using (4.53), (4.54) and (4.59), we see that every time we differentiate, we derive an extra order of $1/(h|\operatorname{Im} z|^2)$. Thus we obtain (4.55) for $M > 0$. Then

$$\partial_h r_n = \partial_h \partial_z E_{n,\pm} \# E_{n,\pm}^{-1} + \partial_z E_{n,\pm} \# \partial_h E_{n,\pm}^{-1} + \sum_{|\alpha|=|\beta|=1} C_{\alpha,\beta} (\partial_{x_2,\xi_2}^\alpha \partial_z E_{n,\pm}) \# (\partial_{x_2,\xi_2}^\beta E_{n,\pm}^{-1}).$$

By (4.39), (4.55) and [89, Theorem 4.23(ii)], we see that $\|\partial_h r_n^W\| = \mathcal{O}(h^{-1}|\operatorname{Im} z|^{-2})$. By the same argument as for $E_{n,\pm}^{-1}$, we get (4.56). □

We shall now focus on $M = 1$, for simplicity, and study the asymptotic expansion of $\partial_h r_n$.

Lemma 4.5.11 (Asymptotic expansion). *Let $0 < \delta < 1/2$ and $|\operatorname{Im} z| \geq h^\delta$, then $\partial_h r_n$ has an asymptotic expansion in $S_\delta^{1+2\delta}$:*

$$\partial_h r_n \sim \sum_{j=1}^{\infty} \frac{j}{2} h^{\frac{j}{2}-1} r_{n,j} =: \sum_{j=1}^{\infty} h^{\frac{j}{2}-1} q_{n,j}, \ \textit{where } r_{n,j} \textit{ are given in Lemma 4.5.7.}$$

Then $h^{\frac{j}{2}-1}q_{n,j} \in S_\delta^{(j+1)\delta+1-\frac{j}{2}}$. *Let* $Q_{n,J} := \partial_h r_n - \sum_{j=1}^{J-1} h^{\frac{j}{2}-1}q_{n,j} \in S_\delta^{(J+1)\delta+1-\frac{J}{2}}$, *i.e., for all* $\alpha, \beta > 0$, *there is* $C''_{\alpha,\beta,n}$ *s.t.*

$$\sup_{(x_2,\xi_2)\in\mathbb{R}^2} |\partial^\alpha_{x_2}\partial^\beta_{\xi_2} Q_{n,J}| \le C''_{\alpha,\beta,n} h^{\frac{J}{2}-1-(J+1)\delta-\delta(|\alpha|+|\beta|)}. \tag{4.60}$$

Furthermore, for the expansion of $\mathrm{Tr}_{\mathbb{C}^2}(\partial_h r_n)$, *we have for* $\eta = x_2 + i\xi_2$,

$$\begin{aligned} \text{Chiral } \mathscr{H}_{\mathrm{c},n}(J=3) : \mathrm{Tr}_{\mathbb{C}^2}(h^{-\frac{1}{2}}q_{n,1} + q_{n,2}) &= \frac{\lambda_n^2}{z^3}\mathfrak{U}(\eta), \\ \text{Anti-Chiral } \mathscr{H}^\theta_{\mathrm{ac},n}(J=2) : \mathrm{Tr}_{\mathbb{C}^2}(h^{-\frac{1}{2}}q_{n,1}) &= \frac{s_n^2(z^2+c_n^2)}{(z^2-c_n^2)^2\sqrt{h}}. \end{aligned} \tag{4.61}$$

We will prove that the termwise differentiation of the asymptotic expansion of r_n in (4.41) is indeed an asymptotic expansion of $\partial_h r_n$ in $S_\delta^{2\delta+1}$.

Proof. Let $g = \sqrt{h}$ and consider $r_n \sim \sum_{j=0}^\infty g^j r_{n,j}$. By Borel's theorem (c.f. [89, Theorem 4.15]) (c.f. [49, Theorem 1.2.6]), we see that for such $r_{n,j} \in C^\infty(\mathbb{R}^2_{x_2,\xi_2})$, there is $\tilde{r}_n \in C^\infty(\mathbb{R}^+_g \times \mathbb{R}_{x_2,\xi_2})$ s.t. $\tilde{r}_n = \sum_{j=0}^\infty g^j r_{n,j}$. Thus

$$\partial_g \tilde{r}_n = \sum_{j=1}^\infty j g^{j-1} r_{n,j}. \tag{4.62}$$

On the other hand, by uniqueness in Borel's theorem, we see that $\tilde{r}_n - r_n = \mathcal{O}(h^\infty)$. Thus $\partial_g\tilde{r}_n - \partial_g r_n = \mathcal{O}(g^\infty)$. Thus (4.62) is also an asymptotic expansion of $\partial_g r_n$. Furthermore, since $\partial_h r_n = \frac{1}{2\sqrt{h}}\partial_g r_n$, thus we proved $\partial_h r_n$ has the following asymptotic expansion in $S_\delta^{1+2\delta}$:

$$\partial_h r_n \sim \frac{1}{2\sqrt{h}} \sum_{j=1}^\infty j h^{\frac{j-1}{2}} r_{n,j} = \sum_{j=1}^\infty \frac{j}{2} h^{\frac{j}{2}-1} r_{n,j}.$$

The rest of the Lemma follows from Lemma 4.5.7. □

Proof of Prop. 4.5.9 . Recall that $f_0(z) = f(z + \sqrt{2|n|/h})$ also depends on h. By differentiating (4.45) w.r.t. h, we get

$$\begin{aligned}
\partial_h \tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta_{c,n})) &= \frac{i}{4\pi^2 h^2 |E|} \int_{\mathbb{C}} \int_E \partial_{\bar z} \tilde f_0(z) \,\mathrm{Tr}_{\mathbb{C}^2}(r_n)\; dx_2\; d\xi_2\; dz \wedge d\bar z \\
&+ \frac{i\sqrt{2|n|/h}}{8\pi^2 h^2 |E|} \int_{\mathbb{C}} \int_E \partial_{\bar z} \tilde f'{}_0(z + \sqrt{2n/h}) \,\mathrm{Tr}_{\mathbb{C}^2}(r_n)\; dx_2\; d\xi_2\; dz \wedge d\bar z \\
&- \frac{i}{4\pi^2 h |E|} \int_{\mathbb{C}} \int_E \partial_{\bar z} \tilde f_0(z) \,\mathrm{Tr}_{\mathbb{C}^2}(\partial_h r_n)\; dx_2\; d\xi_2\; dz \wedge d\bar z := -B_1 - B_2 - B_3.
\end{aligned}$$

where the asymptotic expansion of $B_1 = \frac{1}{h} \tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta))$ and $B_2 = \sqrt{\frac{|n|}{2h^3}} \tilde{\mathrm{Tr}}(f'(\mathscr{H}^\theta))$ are known by (4.48). While B_3 can be computed by splitting the integral as in Subsection 4.5.5:

$$\begin{aligned}
\left[\int_{\mathbb{C}} \partial_{\bar z} \tilde f_0 \,\mathrm{Tr}_{\mathbb{C}^2}(\partial_h r_n) dz \wedge d\bar z\right](x_2, \xi_2; h) &= \int_{\mathbb{C}} \partial_{\bar z} \tilde f_0 \sum_{j=1}^{J-1} h^{\frac{j}{2}-1} \,\mathrm{Tr}_{\mathbb{C}^2}(q_{n,j}) dz \wedge d\bar z \\
&+ \int_{|\operatorname{Im} z| \ge h^\delta} \partial_{\bar z} \tilde f_0 \,\mathrm{Tr}_{\mathbb{C}^2}(Q_{n,J}) dz \wedge d\bar z \\
&+ \int_{|\operatorname{Im} z| \le h^\delta} \partial_{\bar z} \tilde f_0 \,\mathrm{Tr}_{\mathbb{C}^2}(Q_{n,J}) dz \wedge d\bar z \\
&:= A'_1 + A'_2 + A'_3,
\end{aligned}$$

and we imitate the estimates of A_1, A_2, A_3 in the Subsection 4.5.5 with $\partial_h r_n$ instead of r_n, and we use Lemma 4.5.10 and 4.5.11 instead of Lemma 4.5.6 and 4.5.7. In short, we need (4.47) and (4.61) for A'_1, (4.60) for A'_2, (4.56) and Lemma 4.5.11 for A'_3 and we derive that

$$\begin{cases}
A'_{1,\mathrm{c}} = \pi i f''(\lambda_{n,B}) \lambda_n^2 \mathfrak{U}(\eta), \quad A'_{1,\mathrm{ac}} = \frac{\pi i}{\sqrt{h}} \left(s_n^2 f'(\lambda_{n,B} - c_n) + s_n^2 f'(\lambda_{n,B} + c_n)\right), \\
|A'_2| \le C_{n,K,f,\mathscr{V}} h^{\frac{J}{2} - 1 - (J+1)\delta}, |A'_3| \le C_{n,K,f,\mathscr{V}} h^{(K-2)\delta - 1},
\end{cases}$$

from which we can find B_3. And we summarize B_1, B_2, B_3 below:

For the chiral model where $J = 3$, we have

$$
\begin{aligned}
B_{1,\mathrm{c}} &= \frac{1}{\pi h^2} f(\lambda_{n,B}) + \frac{|n|}{2\pi h} \operatorname{Ave}(\mathfrak{U}) f''(\lambda_{n,B}) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-\frac{1}{2}-4\delta} + h^{(K-1)\delta-2}), \\
B_{2,\mathrm{c}} &= \frac{\sqrt{2|n|}}{2\pi h^{\frac{5}{2}}} f'(\lambda_{n,B}) + \frac{(2|n|)^{\frac{3}{2}}}{8\pi h^{\frac{3}{2}}} \operatorname{Ave}(\mathfrak{U}) f'''(\lambda_{n,B}) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-1-4\delta} + h^{(K-1)\delta-\frac{5}{2}}) \\
B_{3,\mathrm{c}} &= \frac{|n|}{2\pi h} f''(\lambda_{n,B}) \operatorname{Ave}(\mathfrak{U}) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-\frac{1}{2}-4\delta} + h^{(K-2)\delta-2}).
\end{aligned}
$$

When $n \neq 0$ and $K > \frac{3}{2\delta} - 3$, we have

$$
\begin{aligned}
\partial_h \tilde{\operatorname{Tr}}(f(\mathscr{H}_{\mathrm{c},n})) = &-\frac{\sqrt{2|n|}}{2\pi h^{\frac{5}{2}}} f'(\lambda_{n,B}) - \frac{1}{\pi h^2} f(\lambda_{n,B}) - \frac{(2|n|)^{\frac{3}{2}}}{8\pi h^{\frac{3}{2}}} \operatorname{Ave}(\mathfrak{U}) f'''(\lambda_{n,B}) \\
&+ \mathcal{O}_{n,K,f,\mathscr{V}} h^{-1-4\delta}.
\end{aligned}
$$

When $n = 0$ and $K > \frac{3}{2\delta} - 3$, since $B_2 = 0$, we get a better estimate:

$$
\partial_h \tilde{\operatorname{Tr}}(f(\mathscr{H}_{\mathrm{c},0})) = -\frac{1}{\pi h^2} f(0) - O_{0,K,f,\mathscr{V}} h^{-\frac{1}{2}-4\delta}.
$$

For the anti-chiral model where $J = 2$, we have

$$
\begin{aligned}
B_{1,\mathrm{ac}} &= \frac{1}{2\pi h^2} t_{n,0}(f) + \frac{1}{2\pi h^{\frac{3}{2}}} t_{n,1}(f) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-1-3\delta} + h^{(K-1)\delta-2}), \\
B_{2,\mathrm{ac}} &= \frac{\sqrt{2|n|}}{4\pi h^{\frac{5}{2}}} t_{n,0}(f') + \frac{\sqrt{2|n|}}{4\pi h^2} t_{n,1}(f') + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-\frac{3}{2}-3\delta} + h^{(K-1)\delta-\frac{5}{2}}), \\
B_{3,\mathrm{ac}} &= \frac{1}{4\pi h^{\frac{3}{2}}} t_{n,1}(f) + \mathcal{O}_{n,K,f,\mathscr{V}}(h^{-1-3\delta} + h^{(K-2)\delta-2}).
\end{aligned}
$$

Thus when $n \neq 0$ and $K > \frac{1}{\delta} - 2$, we have

$$
\begin{aligned}
\partial_h \tilde{\operatorname{Tr}}(f(\mathscr{H}^{\theta}_{\mathrm{ac}})) = &-\frac{\sqrt{2|n|}}{4\pi h^{\frac{5}{2}}} t_{n,0}(f') - \frac{1}{4\pi h^2} \left(2 t_{n,0}(f) + \sqrt{2|n|}\, t_{n,1}(f')\right) \\
&- \mathcal{O}_{n,K,f,\mathscr{V}} h^{-\frac{3}{2}-3\delta}.
\end{aligned}
$$

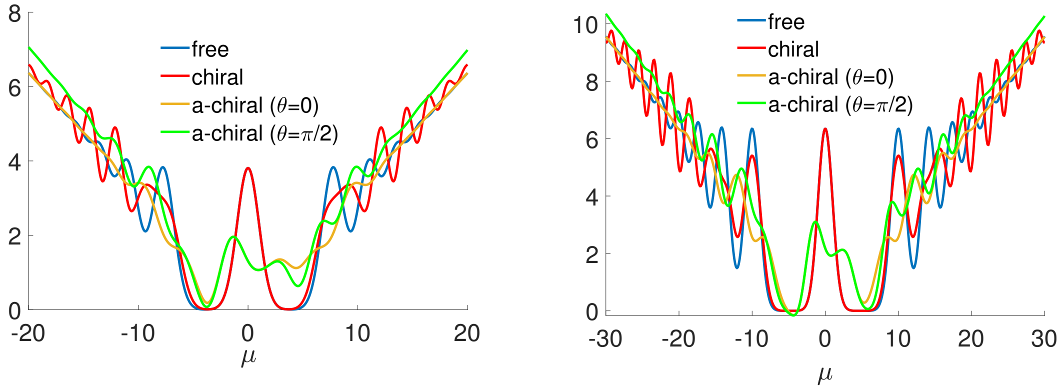


Figure 4.6: SdH oscillations: Smoothed out DOS $\rho(f_\mu)$ with $f_\mu(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}/\sqrt{2\pi}\sigma$ illustrating the oscillatory features. On the left, $B = 30$ and on the right $B = 50$ for $\sigma = 1$.

If $n = 0$ and $K > \frac{1}{\delta} - 2$, since $B_2 = 0$, we get a better estimate:

$$\partial_h \tilde{\mathrm{Tr}}(f(\mathscr{H}^\theta_{\mathrm{ac}})) = -\frac{1}{2\pi h^2} t_{0,0}(f) - \frac{3}{4\pi h^{\frac{3}{2}}} t_{0,1}(f) + O_{0,K,f,\mathscr{V}} h^{-1-3\delta}.$$

Recall $h = \frac{1}{B}$. By $\partial_B = -\frac{1}{B^2}\partial_h$, we get the results (4.50), (4.51) and (4.52). $\square$

4.6 Magnetic response quantities

This section discusses applications of the regularized trace expansions derived in the previous section, cf. Theorems 4.5 and 4.6 as well as Proposition 4.5.9. They form the rigorous foundation of our analysis in this section and we shall focus on qualitative features rather here, instead.

Our main contribution on magnetic response properties of TBG is a careful analysis of the oscillatory behaviour of the DOS. While this effect can be easily explained using the Poisson summation formula, we shall illustrate this phenomenon, by considering a

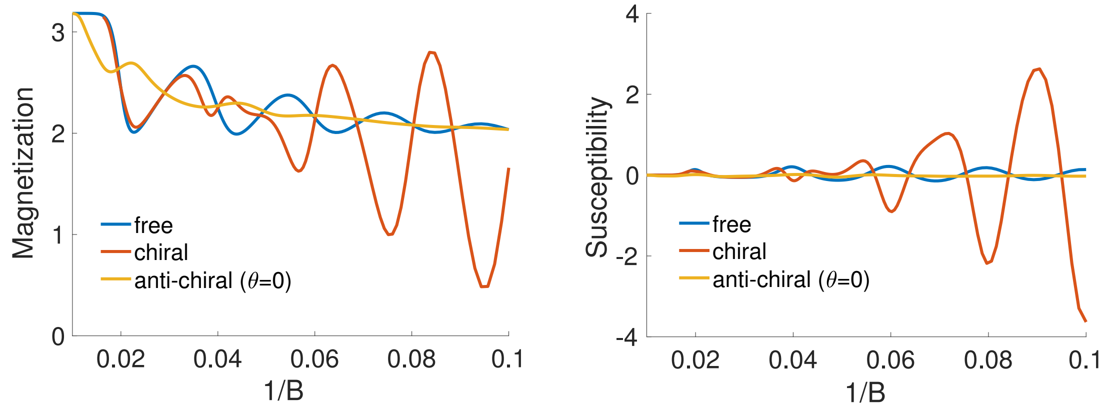


Figure 4.7: Magnetization and susceptibility for $\beta = 4$, $\alpha_i = 3/5$, and chemical potentials $\mu = 5$ (left) and $\mu = 10$ (right).

Gaussian density $f_\mu(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}/\sqrt{2\pi}\sigma$ and analyze the Shubnikov–de Haas (SdH) oscillations in a smoothed-out version of the DOS $\mu \mapsto \rho(f_\mu)$ in Figure 4.6 for $\sigma = 1$ using the asymptotic formulae of Theorems 4.5 and 4.6. As a general rule from our study, we find that the AB/BA interaction leads to an enhancement of this oscillatory behaviour compared to the non-interacting case, while the AA'/BB' interaction damps oscillations. The smoothing effect of the AA'/BB' interaction is due to a splitting and broadening of the highly degenerate Landau levels. This splitting has also consequences for the Quantum Hall effect, see Fig. 4.12. We also study the de Haas–van Alphen (dHvA) effect in TBG, see Fig. 4.7 and 4.10 for which we find a similar phenomenon.

We study magnetic response quantities by thoroughly analyzing the following cases:

- The free or non-interacting case, corresponds to two non-interacting sheets of graphene modeled by the direct sum of two magnetic Dirac operators, see also [9, 8] for similar results in a quantum graph model and [71] for a thorough analysis of the magnetic Dirac operator, directly.
- The chiral case, which corresponds to pure AB/BA interaction.

- The anti-chiral case, which corresponds to pure AA'/BB' interaction.

For our analysis of the de Haas-van Alphen effect, we shall employ a cut-off function $\eta_N \in C_c^\infty(\mathbb{R})$ that is one on the interval $[0, \sqrt{2BN}]$ and smoothly decays to zero outside of that interval, enclosing precisely $N+1$ Landau levels and η_N^{sym} which is equal to one on $[-\sqrt{2BN}, \sqrt{2BN}]$. The choice of cut-off function mainly plays the role of a reference frame. In particular, for the study of magnetic oscillations it seems more natural to consider η_N instead of η_N^{sym} as the former cut-off function singles out the effect of individual Landau levels moving past a fixed chemical potential μ. We shall employ the leading order terms for the regularized trace in this section, as specified in Theorems 4.5 and 4.6 and Proposition 4.5.9. For this reason, we write functionals $\rho(f)$, where $f \in C^\infty(\mathbb{R})$, as $\rho(f) \sim g$, to indicate that g are the first terms in the asymptotic expansion of $\rho(f)$ and analogously for derivatives of $\rho(f)$ with respect to the magnetic field.

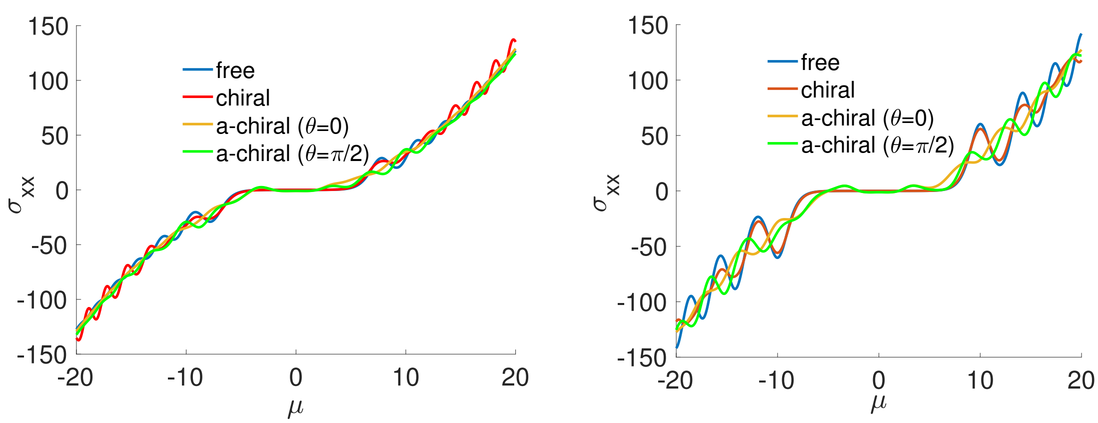


Figure 4.8: Smoothed out longitudinal conductivity $\sigma_{xx} \propto -\rho(\lambda n'_\beta(\lambda - \mu))$ with n_β, the Fermi-Dirac distribution, showing Shubnikov-de Haas oscillations. On the left, $B = 30$ and on the right $B = 50$ for $\beta = 1.5$. with $\alpha_i = \frac{3}{5}$.

4.6.1 Shubnikov-de Haas oscillations

We shall start by discussing *Shubnikov - de Haas (SdH)* oscillations in the density of states. A common method of measuring SdH oscillations is by measuring longitudinal conductivity and resistivity, see also [83, 78]. In the following, let $\sigma \in \mathbb{R}^{2\times 2}$ be the conductivity matrix, such that the current density $j = \sigma E$, where E is an external electric field, then the resistivity matrix is just $\rho = \sigma^{-1}$. Hence, we shall focus on conductivities in the sequel.

The SdH oscillations are most strongly pronounced at low temperatures in the regime of strong magnetic fields and describe oscillations in the longitudinal conductivity σ_{xx} of the material.

The expression for the longitudinal conductivity goes back to Ando et al [3] who derived the following relation, see also [39],

$$\sigma_{xx}(\beta, \mu, B) = -\int_0^\infty n'_\beta(\lambda - \mu)\lambda \eta_N^{\text{sym}}(\lambda)\ d\rho(\lambda),$$

where $n_\beta(x) = \frac{1}{e^{\beta x}+1}$ is the Fermi-Dirac statistics. In the free case, i.e. without any tunnelling potential, the oscillations happen precisely at the relativistic Landau levels. For the chiral model, oscillations caused by higher Landau levels are enhanced compared to the free case, whereas oscillations in the anti-chiral case are much more smoothed out.

The oscillatory behaviour of the longitudinal conductivity is visible both as a function of chemical potential, for a fixed magnetic field strength, as shown in Fig. 4.8 as well as function of inverse magnetic field in Fig. 4.9 for fixed chemical potential.

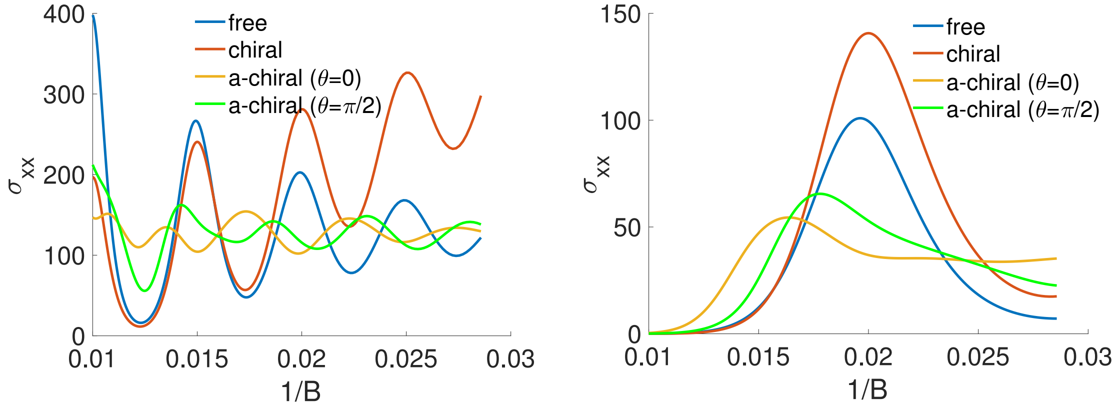


Figure 4.9: Smoothed out longitudinal conductivity $\sigma_{xx} \propto -\rho(\lambda n'_\beta(\lambda - \mu))$ with n_β, the Fermi-Dirac distribution, showing Shubnikov-de Haas oscillations. On the left, $B = 30$ and on the right $B = 50$, both for $\beta = 2.5$. with $\alpha_i = 0.35$.

4.6.2 De Haas-van Alphen oscillations

In 1930, de Haas and van Alphen who discovered that both the magnetization and the magnetic susceptibility of metals show an oscillatory profile as a function of B^{-1}. This effect is called the de Haas-van Alphen (dHvA) effect. Even in the simpler case of graphene, both the experimental as well as theoretical foundations of that effect are not yet well-understood [64, 57, 71]. One problem in understanding the dHvA effect [71], lies in the dependence of the chemical potential on the external magnetic field. To simplify mathematical analysis, it is more convenient to work in the grand-canonical ensemble, which is also discussed in [23, 71, 60]. The comparison with the canonical ensemble is made in this subsection as well.

The grand thermodynamic potential for a DOS measure ρ, at inverse temperature β, and field-independent chemical potential μ is defined as

$$\Omega_\beta(\mu, B) := (f_\beta * (\eta_N \rho))(\mu),$$

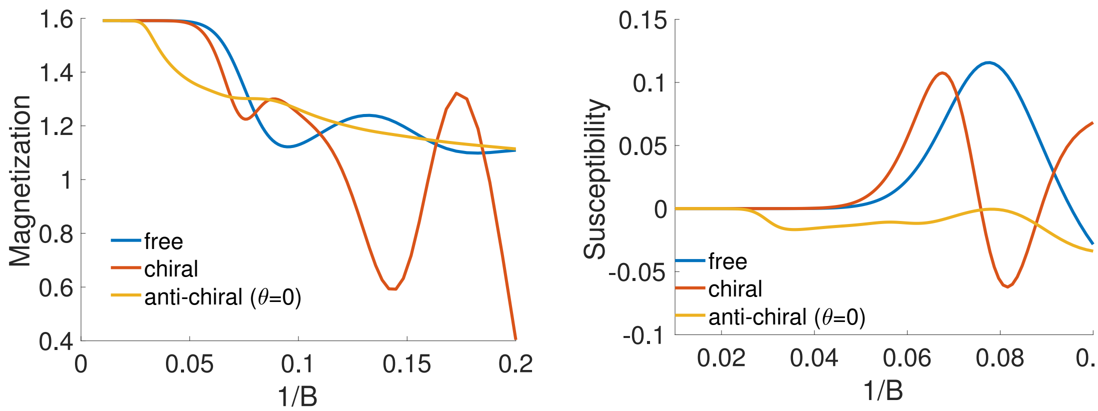


Figure 4.10: Magnetization and susceptibility for $\beta = 4$, $\alpha_i = 3/5$, and chemical potential $\mu = 5$.

where $f_\beta(x) := -\beta^{-1}\log(e^{\beta x}+1)$. The magnetization M and susceptibility χ are then in the grand-canonical ensemble defined as

$$M(\beta,\mu,B) = -\frac{\partial \Omega_\beta(\mu,B)}{\partial B} \text{ and } \chi(\beta,\mu,B) = \frac{\partial M_\beta(\mu,B)}{\partial B}.$$

The susceptibility describes the response of a material to an external magnetic field. When $\chi > 0$ the material is paramagnetic, when $\chi < 0$ diamagnetic, and strongly enhanced $\chi \gg 1$ for ferromagnets.

While the approximation of computing the magnetization in the grand canonical ensemble is common, one should strictly speaking compute it in the canonical ensemble, instead.

In this case, the charge density ϱ given by the Fermi-Dirac statistics, with $n_\beta(x) := \frac{1}{e^{\beta x}+1}$, according to

$$\varrho = -\frac{\Omega_\beta(\mu,B)}{\partial \mu} = \rho(n_\beta(\cdot - \mu))$$

is fixed and the chemical potential becomes a function of ρ and B.

To see that this uniquely defines μ as a function of ϱ and B large enough, it is sufficient to observe that

$$\mu \mapsto \sum_{n\in\mathbb{Z}} (\eta_N n_\mu)(\lambda_n \sqrt{B})$$

is a monotonically increasing function. The Helmholtz free energy is then given as

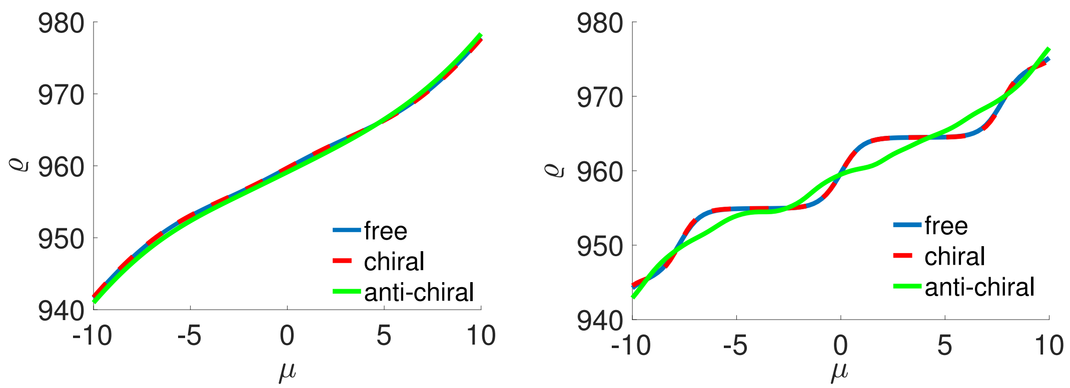


Figure 4.11: Charge density with respect to chemical potential. Magnetic field $B = 30$ for $\beta = 1/2$ and $\beta = 2$. We consider 100 Landau levels around zero and an anti-chiral model with $\theta = 0$.

$$F_\beta(\varrho, B) = \Omega_\beta(\mu(\varrho, B), B) + \mu(\rho, B)\varrho$$

with the magnetization given as the derivative $M(\beta, \varrho, B) = -\frac{\partial F_\beta(\varrho,B)}{\partial B}$. Hence, the magnetization in the canonical ensemble is also given by

$$M(\beta, \varrho, B) = -\frac{\partial \Omega_\beta(\mu, B)}{\partial B}\Big|_{\mu=\mu(\varrho,B)},$$

where the difference to the grand-canonical ensemble lies in the B-dependent chemical potential. The dHvA oscillations are shown in Figures 4.7 and 4.10, with the AB'/BA' interaction leading to enhanced oscillations and the AA'/BB' interaction damping the oscillations, compared to the non-interacting case.

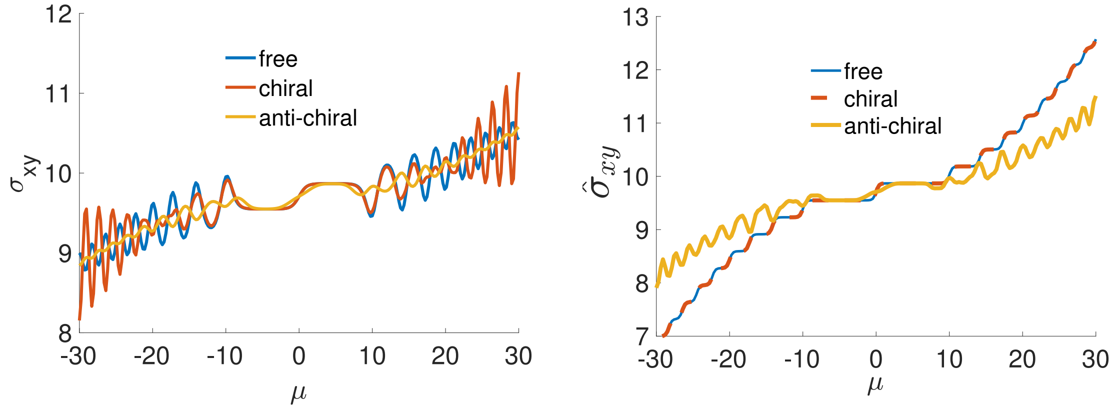


Figure 4.12: Full quantum Hall conductivity (4.56) on the left with $\beta = 2$, $B = 40$ and on the right the high temperature conductivity (4.57) with $\beta = 5$,$B = 50$.

4.6.3 Quantum Hall effect

The (transversal) quantum Hall conductivity σ_{xy} is, by the Streda formula [66, (16)], for a Fermi energy μ given by

$$\sigma_{xy}(\beta,\mu,B) = \sum_{n=-N}^{N} \frac{\partial \rho(\eta_N n_\beta(\bullet - \mu))}{\partial B}.$$

In case of the chiral Hamiltonian, the Gibbs factor $\gamma_{\beta,n}(\mu) = e^{\beta(\lambda_n\sqrt{B}-\mu)}$ allows us to write

$$\begin{aligned}&\sigma_{xy,c}(\beta,\mu,B) = \\ &(1+o(1))\left(\sum_{n=-N}^{N} \frac{n_\beta(\lambda_n\sqrt{B}-\mu)}{\pi}\left(1-\frac{\beta\lambda_n\sqrt{B}}{2}\gamma_{\beta,n}(\mu)n_\beta(\lambda_n\sqrt{B}-\mu)\right)\right. \\ &\left.+\sum_{n=-N}^{N} -\frac{\lambda_n|\lambda_n|^2\beta^3\operatorname{Ave}(\mathfrak{U})}{4\pi\sqrt{B}} n_\beta^4(\lambda_n\sqrt{B}-\mu)\left(\gamma_{\beta,n}(\mu)-4\gamma_{\beta,n}(\mu)^2+\gamma_{\beta,n}(\mu)^3\right)\right)\end{aligned}$$

At very low temperatures, and μ well between two Landau levels, the contribution of the derivative of the Landau levels with respect to B can be discarded.

We then obtain the high-temperature limiting expression

$$\hat{\sigma}_{xy,c}(\beta,\mu,B) := \sum_{n=-N}^{N} \frac{\eta_\beta(\lambda_{n,B}-\mu)}{\pi} \xrightarrow[\beta\to\infty]{} |\{n; |\lambda_{n,B}| \le \mu\}|$$

as $n_\beta(\lambda_{n,B}-\mu) \to 1 - H(\lambda_n\sqrt{B}-\mu)$ for $\beta \uparrow \infty$, where H is the Heaviside function.

This expression reveals the well-known staircase profile of the Hall conductivity which can already be concluded in this model in the $\beta \to \infty$ limit from Proposition 4.4.2.

For the AA'/BB' interaction, the situation is rather different. Due to the broadening and splitting of the Landau levels, the staircase profile is less pronounced at non-zero temperature. Setting $\hat{\sigma}_{xy,ac}(\beta,\mu,B) := t_{n,0}(n_\beta(\bullet-\mu)) - \frac{t_{n,1}(n_\beta(\bullet-\mu))}{2\sqrt{B}}$, where in the limit $\beta \to \infty$, the second term vanishes, for μ away from the spectrum as n'_β is a δ_0 approximating sequence such that also in case of the AA'/BB' interaction $\lim_{\beta\to\infty} \hat{\sigma}_{xy}(\beta,\mu,B) = |\{n; |\lambda_{n,B}| \le \mu\}|$.

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Appendix A

Tridiagonality

This appendix set up Lemma A.0.1 needed in Chapter 2.

Lemma A.0.1. *The matrix $S_z = (z(\mathcal{L})^* - \mathcal{M})$ is tridiagonal. We omit z and denote the (i,j)-entry by $A_{i,j}$ for convenience, when $a \leq i, j \leq b$. Then we have*

$$A_{j,j} = \begin{cases} z\alpha_j + \alpha_{j-1}, & j \text{ even}, \\ -z\overline{\alpha_{j-1}} - \overline{\alpha_j}, & j \text{ odd}, \end{cases}, \quad A_{j+1,j} = A_{j,j+1} = \begin{cases} z\rho_j, & j \text{ even}, \\ -\rho_j, & j \text{ odd}. \end{cases}$$

Remark A.1. If we modify the extended CMV matrix at $a-1$ and b by β and γ, then the corresponding matrix $A^{\beta,\gamma}_{[a,b],z}$ is the restriction of S_z on $[a,b]$ but with $\alpha_{a-1} = \beta$ and $\alpha_b = \gamma$. Fix an interval $[a,b]$. We denote $S^{\beta,\gamma}_z$ by the infinite matrix S_z with $\alpha_{a-1} = \beta$ and $\alpha_b = \gamma$.

Lemma A.0.2. *Let* Ψ *solve* $\mathcal{E}\Psi = z\Psi$*, then for* $a < n < b$*,*

$$\Psi(n) = -G^{\beta,\gamma}_{[a,b],z}(n,a)\begin{cases}\Psi(a)(z\bar{\beta} - z\bar{\alpha}_{a-1}) + \Psi(a-1)z\rho_{a-1}, & a \text{ odd}\\ \Psi(a)(\alpha_{a-1} - \beta) - \Psi(a-1)\rho_{a-1}, & a \text{ even}\end{cases}$$

$$- G^{\beta,\gamma}_{[a,b],z}(n,b)\begin{cases}\Psi(b)(-\bar{\alpha}_b + \bar{\gamma}) - \Psi(b+1)\rho_b, & b \text{ odd,}\\ \Psi(b)(z\alpha_b - z\gamma) + \Psi(b+1)z\rho_b, & b \text{ even}\end{cases}$$

Proof.

$$\begin{aligned}
&P_{[a,b]}S_z\Psi = 0\\
\Rightarrow &P_{[a,b]}S^{\beta,\gamma}_z\Psi + P_{[a,b]}(A - A^{\beta,\gamma})\Psi = 0\\
\Rightarrow &P_{[a,b]}S^{\beta,\gamma}_z(P_{[a,b]}\Psi + P_{[a,b]^c}\Psi) + P_{[a,b]}(S_z - S^{\beta,\gamma}_z)\Psi = 0\\
\Rightarrow &A^{\beta,\gamma}_{[a,b],z}\Psi = -P_{[a,b]}(S_z - S^{\beta,\gamma}_z)\Psi - P_{[a,b]}S^{\beta,\gamma}_z P_{[a,b]^c}\Psi
\end{aligned}$$

There are two terms on the right hand side. One is always 0:

$$P_{[a,b]}S^{\beta,\gamma}_z P_{[a,b]^c}\Psi =$$

$$\begin{bmatrix} \ddots & \ddots & & & \\ A^{\beta,\gamma}_{a,a-1} & A^{\beta,\gamma}_{a,a} & A^{\beta,\gamma}_{a,a+1} & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ \cdots & 0 & A^{\beta,\gamma}_{b,b-1} & A^{\beta,\gamma}_{b,b} & A^{\beta,\gamma}_{b,b+1} \\ & & & \ddots & \ddots \end{bmatrix}\begin{bmatrix}\Psi(a-1)\\ 0\\ \vdots\\ 0\\ \Psi(b+1)\end{bmatrix} = \begin{bmatrix}0\\0\\ \vdots\\0\\0\end{bmatrix}\begin{matrix}\text{row } a-1\\ \text{row } a\\ \vdots\\ \text{row } b\\ \text{row } b+1\end{matrix},$$

where we use Remark A.1 and Lemma A.0.1 to get $A^{\beta,\gamma}_{a,a-1} = A^{\beta,\gamma}_{b,b+1} = 0$.

The other has two non-zero terms:

$$P_{[a,b]}(S_z - S_z^{\beta,\gamma})\Psi =$$

$$\begin{bmatrix} \ddots & \ddots & & & \\ A^-_{a,a-1} & A^-_{a,a} & A^-_{a,a+1} & 0 & \cdots \\ & \ddots & \ddots & \ddots & \\ \cdots & 0 & A^-_{b,b-1} & A^-_{b,b} & A^-_{b,b+1} \\ & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \Psi(a-1) \\ \Psi(a) \\ \vdots \\ \Psi(b) \\ \Psi(b+1) \end{bmatrix} = \begin{bmatrix} \vdots \\ \Psi(a-1)A^-_{a,a-1} + \Psi(a)A^-_{a,a} \\ \cdots \\ \Psi(b)A^-_{b,b} + \Psi(b+1)A^-_{b,b+1} \\ \vdots \end{bmatrix}$$

where $A^-_{x,y} = A_{x,y} - A^{\beta,\gamma}_{x,y}$.

Now

$$\begin{aligned}\Psi(n) = &- G^{\beta,\gamma}_{[a,b],z}(n,b)\left(\Psi(b)A^-_{b,b} + \Psi(b+1)A^-_{b,b+1}\right) \\ &- G^{\beta,\gamma}_{[a,b],z}\left(\Psi(a-1)A^-_{a,a-1} + \Psi(a)A^-_{a,a}\right)\end{aligned}$$

where $A^-_{x,y}$ is derived from Lemma A.0.1 and Remark A.1

$$A^-_{a,a-1} = \begin{cases} z\rho_{a-1}, & a \text{ odd}, \\ -\rho_{a-1}, & a \text{ even}. \end{cases} \qquad A^-_{a,a} = \begin{cases} -z\bar{\alpha}_{a-1} + z\bar{\beta}, & a \text{ odd}, \\ \alpha_{a-1} - \beta, & a \text{ even}. \end{cases}$$

$$A^-_{b,b+1} = \begin{cases} -\rho_b, b \text{ odd}, \\ z\rho_b, b \text{ even}. \end{cases} \qquad A^-_{b,b} = \begin{cases} -\bar{\alpha}_b + \bar{\gamma}, & b \text{ odd}, \\ z\alpha_b - z\gamma, & b \text{ even}. \end{cases} .$$

That proves the result. □

Appendix B

Corrections

In this appendix, we provide corrections for some of the issues from [58], [74], [18], [73] as mentioned in Chapter 2. We first provide the correct results in their notations and then, for the reader's convenience, we rewrite them in our notation when there is a correspondence. Finally, we give either a short proof or a reference for those citations in [58] which are invalid now.

B.1 Corrections for [64]

1. Formula (3.6) in Lemma 3.3 should be $\mathcal{C} = \mathcal{E}_{-1,\cdot}^{[0,\infty)}$. Or in our notation, $\mathcal{C} = \mathcal{E}^{-1,\cdot}_{[0,+\infty]}$. It follows from the definition, see Remark 2.3.

2. Formula (3.14) in Lemma 3.6 should be $\Phi_n(z) = \Phi^{[0,n-1]}_{-1,\cdot}(z)$. Or in our notation, $\Phi_n(z) = \mathcal{P}^{-1,\cdot}_{[0,n-1]}(z)$. See [73, Theorem 5.3] for a proof.

3. Formula (3.16) and (3.17) in Lemma 3.7 should be

$$\Phi_n^{\beta}(z) = \Phi_{-\bar{\beta},\cdot}^{[0,n-1]}(z) \quad \text{and} \quad \Phi_n^{\beta}(z;\gamma) = \Phi_{-\bar{\beta},\gamma}^{[0,n-1]}(z) \tag{B.1}$$

where $\Phi_n^{\beta}(z;\gamma)$ means first replacing α_{n-1} by γ, then multiplying every $\alpha_k, 0 \le k < n-1$ and γ by β (instead of the reversed order). In our notation there is no direct correspondence, but if we denote $X_{[a,b]}^{\beta,\gamma}(\zeta)$ to be $X_{[a,b]}^{\beta,\gamma}$ with all coefficients $\alpha_a, \cdots, \alpha_{b-1}, \gamma$ being multiplied by ζ, where X can be $\mathcal{C}, \mathcal{E}, \mathcal{P}, \cdots$, then

$$\mathcal{P}_{[0,n-1]}^{-1,\cdot}(\beta) = \mathcal{P}_{[0,n-1]}^{-\bar{\beta},\cdot} \quad \text{and} \quad \mathcal{P}_{[0,n-1]}^{-1,\gamma}(\beta) = \mathcal{P}_{[0,n-1]}^{-\bar{\beta},\gamma}.$$

See [74, Theorem 4.2.9] for a proof. See also [73, Theorem 5.6] for a clear restatement but with a typo: If D is a diagonal matrix with elements $1, \lambda^{-1}, 1, \lambda^{-1}, \cdots$, and $\mathcal{M}_\lambda$ differs from $\mathcal{M}$ by having λ in the $(0,0)$-position instead of 1, then $D\mathcal{C}(\lambda\alpha)D^{-1} = \mathcal{L}(\{\alpha_n\})\mathcal{M}_{\bar{\lambda}}(\{\alpha_n\})$.

4. Formula (3.18) in Prop. 3.8 should be

$$|G_{\beta,\gamma}^{[a,b]}(z;k,l)| = \frac{1}{\rho_l}\left|\frac{\phi_{\beta,\cdot}^{[a,k-1]}(z)\phi_{\cdot,\gamma}^{[l+1,b]}(z)}{\phi_{\beta,\gamma}^{[a,b]}(z)}\right|.$$

In our notation, the equality is given in (2.8). Notice that we have no extra parameters $\frac{1}{\rho_l}$ because our definition of $P_{[a,b],\omega,z}^{\beta,\gamma}$ is different from the corresponding definition of $\phi_{\beta,\gamma}^{[a,b]}(z)$. This result follows by direct computation using Cramer's rule.

5. Formula (3.22) in Lemma 3.10 should be

$$T_{[a,b]}(z) = \frac{1}{2}\begin{pmatrix} \phi^{[a,b]}_{-1,\cdot}(z) + \phi^{[a,b]}_{1,\cdot}(z) & \phi^{[a,b]}_{-1,\cdot}(z) - \phi^{[a,b]}_{1,\cdot}(z) \\ (\phi^{[a,b]}_{-1,\cdot})^*(z) - (\phi^{[a,b]}_{1,\cdot})^*(z) & (\phi^{[a,b]}_{-1,\cdot})^*(z) + (\phi^{[a,b]}_{1,\cdot})^*(z) \end{pmatrix}.$$

where we used a different formula for $T_{[a,b]}$, i.e. (2.9). For a proof of the correct form, see [74, (3.2.17), (3.2.27)].

6. Formula (3.23), (3.24) in Cor. 3.11 should be

$$\begin{pmatrix} \phi^{[a,b]}_{\beta,\cdot}(z) \\ -\beta(\phi^{[a,b]}_{\beta,\cdot})^*(z) \end{pmatrix} = T^{[a,b]}(z)\begin{pmatrix} 1 \\ -\beta \end{pmatrix}$$

and

$$\phi^{[a,b]}_{\beta,\gamma}(z) = \frac{1}{\rho_b}\left\langle \begin{pmatrix} z \\ \beta\bar{\gamma} \end{pmatrix}, T_{[a,b-1]}\begin{pmatrix} 1 \\ -\beta \end{pmatrix} \right\rangle.$$

Note in the proof of Cor. 3.11, they used (3.2.26) in [74], which has a typo and the correct form should be

$$\begin{pmatrix} \phi^{\lambda}_{n+1} \\ \bar{\lambda}(\phi^{\lambda}_{n+1})^* \end{pmatrix} = T_n(z)\begin{pmatrix} 1 \\ \bar{\lambda} \end{pmatrix}. \tag{B.2}$$

Proof. The first equality follows from (B.1) and (B.2). The second equality follows from the first equality and

$$\psi^{[a,b]}_{\beta,\gamma} = \Phi^{-\bar{\beta}}_n(z;\gamma) = \frac{1}{\rho_b}(\Phi^{[a,b-1]}_{\beta,\cdot} z + \beta\bar{\gamma}(\phi^{[a,b-1]}_{\beta,\cdot})^*).$$

□

B.2 Corrections for [19]

1. Equation (7.4) should be

$$\phi^{\beta,\gamma}_{\omega,[a,b]}(z) = \frac{1}{\rho_b}\left\langle \begin{pmatrix} z \\ \beta\bar{\gamma} \end{pmatrix}, S^z_{b-a}(T^a\omega)\begin{pmatrix} 1 \\ -\beta \end{pmatrix} \right\rangle.$$

   We believe this can be used then to derive (2.12) by [81, Theorem 5]. Then one could complete the proof of double elimination in [18].

2. The second to the last equation of Page 39 should be

$$|G^{\tau_1,\tau_2}_{\omega,\Lambda}(j,k;z)| = \left|\frac{\phi^{\tau_1,\cdot}_{\omega,[a,j-1]}(z)\phi^{\cdot,\tau_2}_{\omega,[k+1,b]}(z)}{\rho^{\tau_1,\tau_2}_{\omega,[a,b]}(z)}\right| \prod_{i=j}^{k-1}\rho_i.$$

Appendix C

Asymptotic expansion

In this appendix, we shall prove Prop. C.0.1 which, in particular, includes the proof of Lemma 4.5.7. The quantization is as in Subsection 4.5.3.

Proposition C.0.1. *Let h_0, $E_{n,\pm}$ be as in Lemma 4.5.5. For $h \in [0, h_0)$, $|z| \leq 2\|\mathscr{W}\|_\infty$, we have*

1. *The symbol $\frac{1}{\sqrt{h}} E_{n,\pm}$ has an asymptotic expansion in S: There are $a_{n,j,k} \in S$ s.t.*

$$\frac{1}{\sqrt{h}} E_{n,\pm}(x_2, \xi_2; z, h) \sim \sum_{j=0}^{\infty} h^{\frac{j}{2}} E_{n,j}(x_2, \xi_2; z) \text{ with } E_{n,j} = \sum_{k=0}^{j-1} a_{n,j,k}(x_2, \xi_2) z^k, j \geq 1. \tag{C.1}$$

   In particular, $E_{n,0} = z - z_{n,0}$, $E_{n,1} = -z_{n,1}$, $E_{n,2} = -z_{n,2}$, where $z_{n,j}$ are given explicitly in Lemma C.0.2.

2. *Let $0 < \delta < 1/2$, if $|\operatorname{Im} z| \geq h^\delta$, then $\sqrt{h} E_{n,\pm}^{-1}$ has an asymptotic expansions in S_δ^δ: There are $b_{n,j,k,l}, c_{n,j,k,\alpha} \in S$ s.t. if $\prod_{l=0}^{k} b_{n,j,k,l}(x_2, \xi_2; z) = \sum_{\alpha=0}^{j+k-2} z^\alpha c_{n,j,k,\alpha}(x_2, \xi_2)$, we*

have

$$\sqrt{h}E_{n,\pm}^{-1} \sim \sum_{j=0}^{\infty} h^{\frac{j}{2}} F_{n,j}(x_2,\xi_2;z), \ \textit{with}$$
$$F_{n,j} = \sum_{k=0}^{j} (z-z_{n,0})^{-1} \prod_{l=0}^{k} \left(b_{n,j,k,l}(x_2,\xi_2;z)(z-z_{n,0})^{-1} \right). \tag{C.2}$$

Thus $h^{\frac{j}{2}} F_{n,j} \in S^{j(\delta-\frac{1}{2})+\delta}$. *In particular, we have*

$$F_{n,0} = (z-z_{n,0})^{-1}, \quad F_{n,1} = F_{n,0} z_{n,1} F_{n,0},$$
$$F_{n,2} = F_{n,0} \left(z_{n,1} F_{n,1} + z_{n,2} F_{n,0} - \frac{\{F_{n,0}, z-z_{n,0}\}}{2i} \right), \tag{C.3}$$

where $\{\cdot,\cdot\}$ *is the Poisson bracket.*

3. *Let* $0 < \delta < 1/2$, *if* $|\operatorname{Im} z| \geq h^{\delta}$, *then* r_n *has an asymptotic expansions in* S^{δ}_{δ}: *There are* $d_{n,j,k,l}(x_2,\xi_2;z), e_{n,j,k,\alpha}(x_2,\xi_2) \in S$, *s.t. if* $\prod_{l=0}^{k} d_{n,j,k,l}(x_2,\xi_2;z) = \sum_{\alpha=0}^{j+k-2} z^{\alpha} e_{n,j,k,\alpha}(x_2,\xi_2)$, *we have*

$$r_n(x_2,\xi_2;z,h) \sim \sum_{j=0}^{\infty} h^{\frac{j}{2}} r_{n,j}(x_2,\xi_2;z,h), \textit{with}$$
$$r_{n,j} = \sum_{k=0}^{j} (z-z_{n,0})^{-1} \prod_{l=0}^{k} \left(d_{n,j,k,l}(x_2,\xi_2;z)(z-z_{n,0})^{-1} \right).$$

Thus $h^{\frac{j}{2}} r_{n,j} \in S_0^{(J+1)\delta-\frac{J}{2}}$. *In particular,*

$$r_{n,0} = F_{n,0}, \quad r_{n,1} = F_{n,1}, \quad r_{n,2} = F_{n,2} - (\partial_z z_{n,2}) F_{n,0}.$$

4. In particular, denote $\eta = x_2 + i\xi_2$, then the leading terms of $\mathrm{Tr}_{\mathbb{C}^2}(r_n)$ *are:*

$$\text{Chiral } \mathscr{H}_{\mathrm{c},n} : \mathrm{Tr}_{\mathbb{C}^2}(r_{\mathrm{c},n,0} + h^{\frac{1}{2}} r_{\mathrm{c},n,1} + h r_{\mathrm{c},n,2}) = \frac{2}{z} + 0 + \frac{\lambda_n^2}{z^3}\mathfrak{U}(\eta)h,$$
$$\text{Anti-Chiral } \mathscr{H}^{\theta}_{\mathrm{ac},n} : \mathrm{Tr}_{\mathbb{C}^2}(r_{\mathrm{ac},n,0} + h^{\frac{1}{2}} r_{\mathrm{ac},n,1}) = \frac{2z}{z^2 - c_n^2} + \frac{2s_n^2(z^2 + c_n^2)}{(z^2 - c_n^2)^2}\sqrt{h},$$

where $\mathfrak{U}(\eta) = \frac{\alpha_1^2}{8}\left[\alpha_1^2(|U_-(\eta)|^2 - |U(\eta)|^2)^2 + 4|\partial_{\bar\eta}\overline{U_-(\eta)} - \partial_\eta U(\eta)|^2\right]$, $\partial_\eta = \frac{1}{2}(\partial_{x_2} - i\partial_{\xi_2})$, $s_n(\eta) = \begin{cases} \alpha_0 \sin(\frac{\theta}{2})|V(\eta)| & n \neq 0 \\ \alpha_0 |V(\eta)| & n = 0, \end{cases}$ *and* $c_n(\eta) = \begin{cases} \alpha_0 \cos(\frac{\theta}{2})|V(\eta)| & n \neq 0 \\ \alpha_0 |V(\eta)| & n = 0. \end{cases}$

We will prove Proposition C.0.1 in the rest of this appendix in two steps: First, we compute explicitly the leading terms (three terms for the chiral model, two for anti-chiral model) in the expansion of $Z_n(x_2, \xi_2; z, h)$, the symbol of Z_n^W, where $E_{n,\pm} = \sqrt{h}(z - Z_n^W)$ by (4.33). Then, we derive in general the z dependence for each term in the expansion of $E_{n,\pm}$, from which we build up both the legitimacy of the existence of asymptotic expansions of $E_{n,\pm}^{-1}$ and r_n, and the z dependence for each terms in the expansions.

Explicit leading terms. Recall that by (4.33) and (4.36), $E_{n,\pm} = \sqrt{h}(z - Z_n^W)$ with

$$\begin{aligned} Z_n^W(x_2, hD_{x_2}; h) &= R_n^+ \tilde{\mathscr{V}}^W (I + \sqrt{h} E^\theta_{0,n} \tilde{\mathscr{V}}^W)^{-1} R_n^- \\ &= \sum_{k=0}^\infty h^{\frac{k}{2}} (-1)^k R_n^+ \tilde{\mathscr{V}}^W (E^\theta_{0,n}\tilde{\mathscr{V}}^W)^k R_n^- =: \sum_{k=0}^\infty h^{\frac{k}{2}} Q^W_{n,k}(x_2, hD_{x_2}; h), \end{aligned} \tag{C.4}$$

where $R_n^\pm$, $E^\theta_{0,n}$, $\tilde{\mathscr{V}}^W$ are given in (4.28), (4.31) and (4.36). Then we can express the asymptotic expansion of $Z_n(x_2, \xi_2)$ in terms of $Q_{n,k}(x_2, \xi_2)$:

Proposition C.0.2. *Let* $Q^W_{n,k}(x_2, hD_{x_2}; h) = (-1)^k R_n^+ \tilde{\mathscr{V}}^W (E^\theta_{0,n}\tilde{\mathscr{V}}^W)^k R_n^-$. *Then sym-*

bols $Q_{n,0}$, $Q_{n,1}$, $Q_{n,2}$ have the following asymptotic expansions

$$Q_{n,0}(x_2,\xi_2;h) = Q^{(0)}_{n,0}(x_2,\xi_2) + \sqrt{h}Q^{(1)}_{n,0}(x_2,\xi_2) + hQ^{(2)}_{n,0}(x_2,\xi_2) + \mathcal{O}_S(h^{\frac{3}{2}}),$$

$$Q_{n,1}(x_2,\xi_2;h) = Q^{(0)}_{n,1}(x_2,\xi_2) + \sqrt{h}Q^{(1)}_{n,1}(x_2,\xi_2) + \mathcal{O}_S(h),$$

$$Q_{n,2}(x_2,\xi_2;h) = Q^{(0)}_{n,2}(x_2,\xi_2) + \mathcal{O}_S(\sqrt{h}).$$

In the chiral model, for $\eta = x_2 + i\xi_2$, $D_\eta = \frac{1}{2}(D_{x_2} - iD_{\xi_2})$,

$$Q^{(0)}_{\mathrm{c},n,0} = Q^{(2)}_{\mathrm{c},n,0} = Q^{(0)}_{\mathrm{c},n,2} = 0, \quad Q^{(0)}_{\mathrm{c},n,1} = -\frac{\alpha_1^2\lambda_n}{4}\left[|U|^2 - |U_-|^2\right]\sigma_3,$$

$$Q^{(1)}_{\mathrm{c},n,0} = \frac{\lambda_n\alpha_1}{2}\begin{pmatrix} 0 & D_\eta U - D_{\bar\eta}\overline{U_-} \\ D_\eta U_- - D_{\bar\eta}\overline{U} & 0 \end{pmatrix},$$

$$Q^{(1)}_{\mathrm{c},n,1} = \begin{cases} -\frac{\alpha_1^2 z}{4}\left[2|n|(|U|^2 + |U_-|^2)\,\mathbb{1}_{2\times 2} + (|U|^2 - |U_-|^2)\sigma_3\right] & n \neq 0, \\ -\frac{\alpha_1^2 z}{2}\begin{pmatrix} |U|^2 & 0 \\ 0 & |U_-|^2 \end{pmatrix} & n = 0. \end{cases}$$

While in the anti-chiral model, when $\mathscr{H}^\theta = \mathscr{H}_{\mathrm{ac}}$, we have, when $n = 0$,

$$Q^{(1)}_{\mathrm{ac},0,0} = Q^{(0)}_{\mathrm{ac},0,1} = Q^{(1)}_{\mathrm{ac},0,1} = Q^{(0)}_{\mathrm{ac},0,2} = 0,$$

$$Q^{(0)}_{\mathrm{ac},0,0} = \alpha_0\begin{pmatrix} 0 & e^{-\frac{\theta}{2}i}V \\ e^{\frac{\theta}{2}}V^* & 0 \end{pmatrix}, Q^{(2)}_{\mathrm{ac},0,0} = \frac{\alpha_0}{4}\begin{pmatrix} 0 & e^{-\frac{\theta}{2}i}\Delta_{x_2,\xi_2}V \\ e^{\frac{\theta}{2}i}\Delta_{x_2,\xi_2}\bar{V} & \end{pmatrix},$$

when $n \neq 0$,

$$Q^{(1)}_{\mathrm{ac},n,0} = 0,\ Q^{(0)}_{\mathrm{ac},n,1} = \frac{\alpha_0^2|V|^2\sin^2(\frac{\theta}{2})}{2\lambda_n}\,\mathbb{1}_{2\times 2},\ Q^{(1)}_{\mathrm{ac},n,1} = -\frac{z\alpha_0^2|V|^2\sin^2(\frac{\theta}{2})}{4\lambda_n^2}\,\mathbb{1}_{2\times 2},$$

$$Q^{(0)}_{\mathrm{ac},n,0} = \alpha_0\cos(\tfrac{\theta}{2})\begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}, Q^{(0)}_{\mathrm{ac},n,2} = -\frac{\alpha_0^3|V|^2\sin^2(\frac{\theta}{2})\cos(\frac{\theta}{2})}{4\lambda_n^2}\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix},$$

$$Q^{(2)}_{\mathrm{ac},n,0} = \frac{\alpha_0}{4}\left(2|n|\cos(\tfrac{\theta}{2}) - i\sigma_3\sin(\tfrac{\theta}{2})\right)\begin{pmatrix} 0 & \Delta_{x_2,\xi_2}V \\ \Delta_{x_2,\xi_2}\bar{V} & 0 \end{pmatrix}.$$

In particular, Z_n *has an asymptotic expansion* $Z_n \sim \sum_{k=0}^{\infty} h^{\frac{k}{2}} z_{n,k}$ *in* S *with*

$$z_{n,0} = Q^{(0)}_{n,0},\ z_{n,1} = Q^{(0)}_{n,1} + Q^{(1)}_{n,0},\ z_{n,2} = Q^{(0)}_{n,2} + Q^{(1)}_{n,1} + Q^{(2)}_{n,0}.$$

Proof. Notice that $Q_{n,k}(x_2,\xi_2) = (-1)^k\int_{\mathbb{R}_{x_1}}(K_n^\theta(x_1))^*\tilde{\mathscr{V}}^w\#(E^\theta_{0,n}\tilde{\mathscr{V}}^w)^{\#k}K_n^\theta(x_1)dx_1$. Recall that by (4.29), (4.1), and (4.31), we have

$$K_n^\theta = \begin{pmatrix} u_n^\theta & 0 \\ 0 & u_n^{-\theta} \end{pmatrix},\ \mathscr{V} = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix},\ T = \begin{pmatrix} \alpha_0 V & \alpha_1\overline{U_-} \\ \alpha_1 U & \alpha_0 V \end{pmatrix},\ E^\theta_{0,n} = \begin{pmatrix} e^\theta_{0,n} & 0 \\ 0 & e^{-\theta}_{0,n} \end{pmatrix}.$$

Thus, inserting the above expressions into the definition of $Q_{n,k}$, we find for its symbol

$$Q_{n,k} = \int \begin{pmatrix} u_n^{\theta\,*} & 0 \\ 0 & u_n^{-\theta *} \end{pmatrix}\begin{pmatrix} 0 & \tilde{T}^w \\ (\tilde{T}^w)^* & 0 \end{pmatrix}\left(\begin{pmatrix} e^\theta_{0,n} & 0 \\ 0 & e^{-\theta}_{0,n} \end{pmatrix}\begin{pmatrix} 0 & \tilde{T}^w \\ (\tilde{T}^w)^* & 0 \end{pmatrix}\right)^k\begin{pmatrix} u_n^{\theta\,*} & 0 \\ 0 & u_n^{-\theta} \end{pmatrix}\frac{dx_1}{(-1)^k}$$

where $\tilde{T}^w = T^w(x_2 + h^{\frac{1}{2}}x_1, \xi_2 - h^{\frac{1}{2}}D_{x_1})$. In particular,

$$Q_{n,0} = \begin{pmatrix} 0 & \int (u_n^\theta)^* \tilde{T}^w u_n^{-\theta} dx_1 \\ \int (u_n^{-\theta})^* (\tilde{T}^w)^* u_n^\theta dx_1 & \end{pmatrix},$$
$$Q_{n,1} = \begin{pmatrix} -\int (u_n^\theta)^* \tilde{T}^w e_{0,n}^{-\theta} (\tilde{T}^w)^* u_n^\theta dx_1 & 0 \\ 0 & -\int (u_n^{-\theta})^* (\tilde{T}^w)^* e_{0,n}^\theta \tilde{T}^w u_n^{-\theta} dx_1 \end{pmatrix}, \text{ and}$$
$$Q_{n,2} = \begin{pmatrix} 0 & \int (u_n^\theta)^* \tilde{T}^w e_{0,n}^{-\theta} (\tilde{T}^w)^* e_{0,n}^\theta \tilde{T}^w u_n^{-\theta} dx_1 \\ \int (u_n^{-\theta})^* (\tilde{T}^w)^* e_{0,n}^\theta \tilde{T}^w e_{0,n}^{-\theta} (\tilde{T}^w)^* u_n^\theta dx_1 & 0 \end{pmatrix}. \tag{C.5}$$

Notice that since both $\tilde{T}^w$ and $e_{0,n}^\theta$ depend on h, we need to further expand them in order to obtain asymptotic expansions of $Q_{n,k}$. Thus the proof of Proposition C.0.2 rests now on the following two lemmas.

Lemma C.0.3 (Expansion of $\tilde{T}^w$ and $e_{0,n}^\theta$)**.**

1. *Let $T \in C_b^\infty(\mathbb{R}_x^2)$. Recall that $\tilde{T}(x,\xi) := T(x_2 + h^{\frac{1}{2}}x_1, \xi_2 - h^{\frac{1}{2}}\xi_1) \in S(\mathbb{R}_{x,\xi}^4)$. Then*

$$\tilde{T}^w(x, D_{x_1}, \xi_2) = T(x_2,\xi_2) + \sqrt{h}\langle \nabla_{x_2,\xi_2} T(x_2,\xi_2), (x_1, -D_{x_1})\rangle$$
$$+ \frac{h}{2}\langle (x_1, -D_{x_1}), \operatorname{Hess} T(x_2,\xi_2)(x_1, -D_{x_1})^T\rangle + \mathcal{O}_{S(\mathbb{R}^2_{x_2,\xi_2};\mathcal{L}(B^3_{x_1};B^0_{x_1}))}(h^{\frac{3}{2}})$$

2. *Let $e_{0,n}^\theta$ be as in (4.31). Then $e_{0,n}^\theta(x, D_{x_1}, \xi_2)$ has asymptotic expansion $e_{0,n}^\theta \sim \sum_{k=0}^{\infty} h^{\frac{k}{2}} \sigma_k(e_{0,n}^\theta)$ where $\sigma_k(e_{0,n}^\theta) = \sum_{m \neq n} \frac{z^k u_m^\theta (u_m^\theta)^*}{(\lambda_m - \lambda_n)^{k+1}}$.*

Lemma C.0.4 (Projections)**.** *Let $S_n^\theta = \operatorname{span}\{u_n^\theta, u_{-n}^\theta\}$ with $S_n := S_n^0$. The following properties hold:*

1. *We have* $S_n^\theta = S_n^{-\theta}$, *in particular* $u_n^\theta = \cos\left(\frac{\theta}{2}\right) u_n^{-\theta} + i\sin\left(\frac{\theta}{2}\right) u_{-n}^{-\theta}$.

2. *Let* $M = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$ *then* $Mu_n \in S_{n-1} \cup S_{n+1}$, *for any* $n \geq 0$. *More specifically for* $\theta = 0$

$$\begin{aligned} Mu_{\pm n} &= \frac{\alpha i}{2}(u_{n+1} - u_{-(n+1)}) \mp \frac{\beta i}{2}(u_{n-1} + u_{-(n-1)}), \text{ for } n \geq 2 \\ Mu_{\pm 1} &= \frac{\alpha i}{2}(u_2 - u_{-2}) \pm \frac{\beta}{\sqrt{2}} u_0, \text{ and } Mu_0 = \frac{\alpha}{\sqrt{2}}(u_1 - u_{-1}). \end{aligned}$$

3. *We have* $x_1 u_n^\theta \in S_{n-1}^\theta \cup S_{n+1}^\theta$, $D_{x_1} u_n^\theta \in S_{n-1}^\theta \cup S_{n+1}^\theta$. *More specifically*

$$\begin{aligned} x_1 u_{\pm n}^\theta = \frac{\sqrt{2}}{4}[&u_{n-1}^\theta(\sqrt{n} \pm + \sqrt{n-1}) + u_{-(n-1)}^\theta(\sqrt{n} \mp \sqrt{n-1}) \\ &+ u_{n+1}^\theta(\sqrt{n+1} + \sqrt{n}) \pm u_{-(n+1)}^\theta(\sqrt{n+1} \mp \sqrt{n})], \text{ for } |n| \geq 2 \\ x_1 u_{\pm 1}^\theta = \frac{i}{2}u_0^\theta &+ \frac{\sqrt{2}}{4}[u_2^\theta(\sqrt{2} \pm \sqrt{1}) + u_{-2}^\theta(\sqrt{2} \mp \sqrt{1})] \text{ and } x_1 u_0^\theta = \frac{\sqrt{2} i}{4}(u_1^\theta + u_{-1}^\theta). \end{aligned}$$

Proof. We omit the proof of this Lemma here as it follows from straightforward but lengthy basis expansions and the simple observation that $\langle u_m^{-\theta}, u_n^\theta \rangle = \cos\left(\frac{\theta}{2}\right)\delta_{m,n} + i\sin\left(\frac{\theta}{2}\right)\delta_{m,-n}$. □

From the preceding Lemmas C.0.3 and C.0.4, we can compute the asymptotic expansion of each term of $Q_{n,k}$ in (C.5) and therefore prove Prop.C.0.2.

For the $(1,2)$-entry of $Q_{n,0}$, by Lemma C.0.3, we have

$$\begin{aligned} \int (u_n^\theta)^* \tilde{T}^w u_n^{-\theta} dx_1 &= \int (u_n^\theta)^* T u_n^{-\theta} dx_1 + \sqrt{h} \int (u_n^\theta)^* \langle \nabla_{x_2,\xi_2} T, (x_1, -D_{x_1}) \rangle u_n^{-\theta} dx_1 \\ &\quad + \frac{h}{2} \int (u_n^\theta)^* \langle (x_1, -D_{x_1}), \operatorname{Hess} T(x_2,\xi_2)(x_1, -D_{x_1})^T \rangle u_n^{-\theta} dx_1 \\ &=: t_{n,0}^{(0)} + \sqrt{h} t_{n,0}^{(1)} + h t_{n,0}^{(2)} + \mathcal{O}_{S(\mathbb{R}^2_{x_2,\xi_2}; \mathbb{C}_{2\times 2})}(h^{\frac{3}{2}}). \end{aligned}$$

Specializing now to the chiral case, in which case $\theta = 0$, we choose

$$T(x_2,\xi_2) = \begin{pmatrix} 0 & \alpha_1 \overline{U(x_2,\xi_2)} \\ \alpha_1 U_-(x_2,\xi_2) & 0 \end{pmatrix}$$

where in the chiral case, by Lemmas C.0.3 and C.0.4, we see that

$$t^{(0)}_{\mathrm{c},n,0} = 0,\ t^{(1)}_{\mathrm{c},n,0} = \frac{\lambda_n \alpha_1 i}{2}(\partial_{\bar w}\overline{U_-} - \partial_w U),\ \text{and}\ t^{(2)}_{\mathrm{c},n,0} = 0,$$

while in the anti-chiral case, choosing $T(x_2,\xi_2) = \alpha_0 V(x_2,\xi_2)\,\mathrm{id}_{\mathbb{C}_{2\times 2}}$

$$t^{(0)}_{\mathrm{ac},n,0} = \begin{cases} \alpha_0 \cos(\frac{\theta}{2}) V & n \neq 0, \\ \alpha_0 e^{-\frac{\theta}{2} i} V & n = 0, \end{cases},\quad t^{(1)}_{\mathrm{ac},n,0} = 0,\ \text{and}$$

$$t^{(2)}_{\mathrm{ac},n,0} = \begin{cases} \frac{\alpha_0}{4}(2|n|\cos(\frac{\theta}{2}) - i\sigma_3 \sin(\frac{\theta}{2}))\Delta_{x_2,\xi_2} V, & n \neq 0 \\ \frac{\alpha_0}{4} e^{-i\frac{\theta}{2}} \Delta_{x_2,\xi_2} V & n = 0. \end{cases}$$

Due to the conjugacy relation $\int (u_n^\theta)^* (\tilde{T}^w)^* u_n^{-\theta} dx_1 = (\int (u_n^{-\theta})^* \tilde{T}^w u_n^\theta dx_1)^*$, the expansion of $Q^\theta_{n,0}$ follows by (C.5).

Similarly for the $(1,1)$-entry $Q^\theta_{n,1}$, denote

$$-\int (u_n^\theta)^* \tilde{T}^w e^{-\theta}_{0,n} (\tilde{T}^w)^* u_n^\theta dx_1 =: t^{(0)}_{n,1} + t^{(1)}_{n,1}\sqrt{h} + \mathcal{O}_{S(\mathbb{R}^2_{x_2,\xi_2};\mathbb{C}_{2\times 2})}(h)$$

where, using Lemma 1, in the chiral case,

$$t^{(0)}_{\mathrm{c},n,1} = -\frac{\alpha_1^2\lambda_n}{4}(|U|^2 - |U_-|^2) \text{ and}$$

$$t^{(1)}_{\mathrm{c},n,1} = \begin{cases} -\frac{\alpha_1^2 z}{4}[2|n|(|U|^2 + |U_-|^2) + (|U|^2 - |U_-|^2)], & n \neq 0 \\ -\frac{\alpha_1^2 z}{2}|U|^2, & n = 0 \end{cases}$$

and in the anti-chiral case

$$t^{(0)}_{\mathrm{ac},n,1} = \begin{cases} \frac{\alpha_0^2|V|^2\sin^2(\frac{\theta}{2})}{2\lambda_n}, & n \neq 0 \\ 0, & n = 0 \end{cases} \text{ and } t^{(1)}_{\mathrm{ac},n,1} = \begin{cases} -\frac{\alpha_0^2|V|^2\sin^2(\frac{\theta}{2})z}{4\lambda_n^2}, & n \neq 0 \\ 0, & n = 0. \end{cases}$$

In a similar fashion, the $(2,2)$-entry of $Q_{n,1}$, defined in (C.5), can be obtained by precisely the same computations after only replacing θ by $-\theta$ and T^* by T, i.e. U switching with U_- and using V^* instead of V. Thus the asymptotic expansion of $Q^\theta_{n,1}$ follows.

Similarly for $Q^\theta_{n,2}$ we restrict us to the $(1,2)$ entry in (C.5). Then, we denote

$$\int (u_n^\theta)^* \tilde{T}^w e^{-\theta}_{0,n} (\tilde{T}^w)^* e^\theta_{0,n} \tilde{T}^w u_n^{-\theta} dx_1 =: t^{(0)}_{n,2} + \mathcal{O}_{S(\mathbb{R}^2_{x_2,\xi_2};\mathbb{C}_{2\times 2})}(\sqrt{h}).$$

It follows then by Lemma 1, that in the chiral model, $t^{(0)}_{\mathrm{c},n,2} = 0$ while in the anti-chiral model, $t^{(0)}_{n,2} = -\frac{\alpha_0^3|V|^2\sin^2(\frac{\theta}{2})\cos(\frac{\theta}{2})}{4\lambda_n^2}V$. By the conjugacy relation

$$\int (u_n^{-\theta})^* (\tilde{T}^w)^* e^\theta_{0,n} \tilde{T}^w e^{-\theta}_{0,n} (\tilde{T}^w)^* u_n^\theta dx_1 = [\int (u_n^\theta)^* \tilde{T}^w e^{-\theta}_{0,n} (\tilde{T}^w)^* e^\theta_{0,n} \tilde{T}^w u_n^{-\theta} dx_1]^*,$$

this also yields directly the expansion of $Q^\theta_{\mathrm{ac},n,2}$. □

Existence, derivation and z-dependence. Now we prove the rest of Prop. C.0.1, which includes the existence and derivation of asymptotic expansion of $E_{n,\pm}^{-1}$ and r_n and the z dependence of each terms in the expansions of $E_{n,\pm}$, $E_{n,\pm}^{-1}$ and r_n.

Proof of Prop. C.0.1. By (C.4) and Prop. C.0.2, $E_{n,\pm} = \sqrt{h}(z - Z_n)$, and Z_n has an asymptotic expansion in S. Thus, $\frac{1}{\sqrt{h}}E_{n,\pm}$ also has an asymptotic expansion in S: $\frac{1}{\sqrt{h}}E_{n,\pm} \sim \sum_j h^{\frac{j}{2}} E_{n,j}$ with $E_{n,j} \in S$. To exhibit the z-dependence, we notice that only $E_{0,n}$ depends on z in (C.4). Thus, by (4.40), we have

$$\begin{aligned}
Z_n^W &= R_n^+ \tilde{\mathscr{V}}^W (\mathbb{1} + \sqrt{h} E_{0,n} \tilde{\mathscr{V}}^W)^{-1} R_n^- = R_n^+ \tilde{\mathscr{V}}^W R_n^- + \sum_{\alpha=1}^{\infty} R_n^+ \tilde{\mathscr{V}}^W (\sqrt{h} E_{0,n} \tilde{\mathscr{V}}^W)^{\alpha} R_n^- \\
&= R_n^+ \tilde{\mathscr{V}}^W R_n^- + \sum_{\alpha=1}^{\infty} h^{\frac{\alpha}{2}} R_n^+ \tilde{\mathscr{V}}^W \left[\sum_{m \neq n} \frac{K_m^{\theta} (K_m^{\theta})^*}{\lambda_m - \lambda_n} \sum_{\beta=0}^{\infty} \left(\frac{\sqrt{h} z}{\lambda_m - \lambda_n} \right)^{\beta} \right]^{\alpha} R_n^- \\
&= R_n^+ \tilde{\mathscr{V}}^W R_n^- + \sum_{\alpha=1}^{\infty} \sum_{\gamma=0}^{\infty} h^{\frac{\alpha+\gamma}{2}} z^{\gamma} A_{n,\alpha,\gamma}^W (x_2, hD_{x_2}) \\
&= R_n^+ \tilde{\mathscr{V}}^W R_n^- - \sum_{j=1}^{\infty} h^{\frac{j}{2}} \left(\sum_{k=0}^{j-1} z^k a_{n,j,k}^W (x_2, hD_{x_2}) \right)
\end{aligned}$$

for some appropriate $A_{n,\alpha,\gamma}(x_2, \xi_2) \in S$ and $a_{n,j,k}(x_2, \xi_2) \in S$. Thus we proved part (1).

For $\sqrt{h} E_{n,\pm}^{-1}$, first of all, by a parametrix construction using the formal expansion of the sharp product

$$a \widetilde{\#} b \sim \sum_k \frac{1}{k!} \left(\left(\frac{ih}{2} \sigma(D_{x_2}, D_{\xi_2}; D_y, D_{\eta}) \right)^k (a(x_2, \xi_2) b(y, \eta)) \right) \Bigg|_{x_2 = y,\ \xi_2 = \eta}, \tag{C.6}$$

we can formally derive (C.2) and (C.3). More specifically, there is a formal expansion of $\sqrt{h} E_{n,\pm}^{-1}$, which is denoted by $\sqrt{h} F_n \sim \sum_j h^{\frac{j}{2}} F_{n,j}$, s.t. $\frac{1}{\sqrt{h}} E_{n,\pm} \widetilde{\#} \sqrt{h} F_n = \mathbb{1}_{2\times 2}$. Denote

$\sigma(D_{x_2}, D_{\xi_2}; D_y, D_\eta)$ in (C.6) by σ, we can solve for $F_{n,j}$ by considering

$$
\begin{aligned}
\mathbb{1}_{2\times 2} = E_{n,\pm}\widetilde{\#}F_n^{-1} &\sim \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} h^{\frac{\alpha+\beta}{2}} E_{n,\alpha}\widetilde{\#}F_{n,\beta} \\
&= \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} h^{\frac{\alpha+\beta}{2}} \sum_{\gamma=0}^{\infty} h^{\gamma} \left(\left(\frac{i\sigma}{2}\right)^{\gamma} (E_{n,\alpha}(x_2,\xi_2)F_{n,\beta}(y,\eta)) \right)\Big|_{x_2=y,\xi_2=\eta} \\
&= \sum_{j=0}^{\infty}\sum_{\beta=0}^{j}\sum_{\alpha=0}^{j-\beta} h^{\frac{j}{2}} \left(\left(\frac{i\sigma}{2}\right)^{\frac{j-\alpha-\beta}{2}} (E_{n,\alpha}(x_2,\xi_2)F_{n,\beta}(y,\eta)) \right)\Big|_{x_2=y,\xi_2=\eta}.
\end{aligned}
$$

Then we compare the parameter of the term of $h^{\frac{j}{2}}$ on both sides and get

$$
-E_{n,0}F_{n,j} = \sum_{\beta=0}^{j-1}\sum_{\alpha=0}^{j-\beta} \left(\left(\frac{i\sigma}{2}\right)^{\frac{j-\alpha-\beta}{2}} (E_{n,\alpha}(x_2,\xi_2)F_{n,\beta}(y,\eta)) \right)\Big|_{x_2=y,\xi_2=\eta},
$$

from which we can solve for $F_{n,j}$. Furthermore, by (C.1) and $E_{n,0} = z - z_{n,0}$, we can check inductively that for $j \geq 0$, there are $b_{n,j,k,l}$, $c_{n,j,k}$ s.t.

$$
\begin{aligned}
&F_{n,j} = \sum_{k=0}^{j} (z - z_{n,0})^{-1} \prod_{l=0}^{k} \left(b_{n,j,k,l}(x_2,\xi_2;z)(z-z_{n,0})^{-1} \right), \\
&\text{with } \prod_{l=0}^{k} b_{n,j,k,l}(x_2,\xi_2;z) = \sum_{\alpha=0}^{j+k-2} z^{\alpha} c_{n,j,k}(x_2,\xi_2), \text{ for appropriate } c_{n,j,k} \in S.
\end{aligned}
$$

Notice that $\widetilde{\#}$ differs from the actual sharp product $\#$:

$$
a\#b = e^{\frac{ih}{2}\sigma(D_{x_2},D_{\xi_2};D_y,D_\eta)} \left(a(x_2,\xi_2)b(y,\eta)\right)|_{x_2=y,\ \xi_2=\eta}. \tag{C.7}
$$

Now we claim that this formal expansion for $\sqrt{h}F_n$ is legitimate as an asymptotic expansion in S_δ^δ and in fact, it is exactly the asymptotic expansion of $\sqrt{h}E_{n,\pm}$ when $|z| \leq 2\|\mathscr{V}\|_\infty$ and $|\operatorname{Im} z| \geq h^\delta$. In fact, $\sqrt{h}(E_{n,\pm}^{-1} - F_n) \in S^{-\infty}$.

In fact, since $|z|$ is bounded and $|\operatorname{Im} z| \geq h^\delta$ and $F_{n,j}$ is a rational function in z, thus $h^{\frac{j}{2}} F_{n,j} \in S_\delta^{j(\delta-\frac{1}{2})+\delta}$. Since $j(\delta - \frac{1}{2}) + \delta \to -\infty$, (C.2) is not only a formal expansion but is indeed an asymptotic expansion of F_n in the symbol class S_δ^δ.

Furthermore, comparing (C.6) with (C.7), we see that $F_n \# E_{n,\pm} = 1 - R_n$ with $R_n \in S^{-\infty}$. By Beal's lemma, there is $\tilde{R}_n \in S^{-\infty}$ s.t. $(1 - R_n^W)^{-1} = 1 - \tilde{R}_n^W$. Thus $\sqrt{h} E_{n,\pm}^{-1} = F \#(1 - \tilde{R}_n^W) \in S_\delta^\delta$ and have exactly the same asymptotic expansion as F_n in (C.2) since $\tilde{R}_n \in S_\delta^{-\infty}$. Thus part (2) is proved.

It follows that $r_n := \partial_z E_{n,\pm} \# E_{n,\pm}^{-1}$ is also well-defined with an asymptotic expansion in S_δ^δ. Since

$$
\begin{aligned}
r_n &\sim \sum_{\alpha=0}^{\infty} h^{\frac{\alpha}{2}} \partial_z E_{n,\alpha} \# \sum_{\beta=0}^{\infty} h^{\frac{\beta}{2}} F_{n,j} \\
&= \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} h^{\frac{\alpha+\beta}{2}} \sum_{\gamma=0}^{\infty} h^{\gamma} \left(\left(\frac{i\sigma}{2} \right)^{\gamma} (E_{n,\alpha}(x_2, \xi_2; z) F_{n,\beta}(y, \eta; z)) \right) \Big|_{x_2=y, \xi_2=\eta} \\
&= \sum_{j=0}^{\infty} \sum_{\alpha=0}^{j} \sum_{\beta=0}^{j-\alpha} h^{\frac{j}{2}} r_{n,j,\alpha,\beta} \left(\left(\frac{i\sigma}{2} \right)^{\frac{j-\alpha-\beta}{2}} (E_{n,\alpha}(x_2, \xi_2; z) F_{n,\beta}(y, \eta; z)) \right) \Bigg|_{x_2=y, \xi_2=\eta} .
\end{aligned}
$$

Combining it with part (1) and (2) and the fact that σ is linear in D_{x_2}, D_{ξ_2}, we get part (3). Part (4) follows directly from part (1), (2), (3) with Prop. C.0.2. □

Appendix D

For the proof of Lemma 4.5.8

In this appendix, we provide several lemmas that together complete the proof of Lemma 4.5.8. We start with a proposition that expresses the Hilbert-Schmidt norm of the quantization in terms of its operator-valued symbol.

Proposition D.0.1. *Let $\mathscr{H}_1$, $\mathscr{H}_2$ be two Hilbert spaces. Let $P : \mathbb{R}^2 \to \mathcal{L}(\mathscr{H}_1; \mathscr{H}_2)$ be an operator-valued symbol in the symbol class $S(\mathbb{R}^2_{y,\eta}; \mathcal{L}(\mathscr{H}_1; \mathscr{H}_2))$. Furthermore, let $\|\cdot\|_{\mathrm{HS}}$ denote the Hilbert-Schmidt norm of maps $\mathscr{H}_1$ to $\mathscr{H}_2$ or $L^2(\mathbb{R}_y; \mathscr{H}_1)$ to $L^2(\mathbb{R}_y; \mathscr{H}_2)$. Then*

$$\|P^W(y, hD_y)\|^2_{\mathrm{HS}} = \frac{1}{2\pi h} \int_{\mathbb{R}^2} \|P(y,\eta)\|^2_{\mathrm{HS}}\, dy\, d\eta.$$

In particular, if $\mathscr{H}_1 = \mathscr{H}_2 = \mathbb{R}$, for the scalar-valued symbol P, we have

$$\|P^W(y, hD_y)\|^2_{\mathrm{HS}} = \frac{\|P(y,\eta)\|^2_{L^2(\mathbb{R}^2;\mathbb{R})}}{2\pi h}. \tag{D.1}$$

The next Lemma allows us to interchange the order of trace and integration.

Lemma D.0.2. *Let $E_{n,-}$, $E_{n,+}$ be as in (4.32). Let $\tilde{\mathbb{1}}^W_R$, $\bar{\mathbb{1}}^W_R$ be as in the proof of*

Lemma 4.5.7. Then, there exists a constant $C > 0$ such that

$$\|\bar{\mathbb{1}}_R^W E_{n,-}\|_{\mathrm{HS}(L^2(\mathbb{R}_x^2),L^2(\mathbb{R}_{x_2}))} \leq Ch^{-1/2}R \text{ and } \|E_{n,-}\tilde{\mathbb{1}}_R^W\|_{\mathrm{HS}(L^2(\mathbb{R}_x^2),L^2(\mathbb{R}_{x_2}))} \leq Ch^{-1/2}R.$$

Proof. The first equation follows from (D.1). For the second equation, we first recall that

Claim D.1. *If $a \in S(\mathbb{R}^{2n}; \mathcal{L}(X,Y); m_1)$, $b \in S(\mathbb{R}^{2n}; \mathrm{HS}(Y,Z); m_2)$ and $m_1 m_2(x,\xi) \in L^2(\mathbb{R}^{2n}_{x,\xi})$, where m_1, m_2 are order functions, then*

$$b\#a \in S(\mathbb{R}^{2n}; \mathrm{HS}(X,Z); m_1 m_2) \text{ and } (b\#a)^W = b^W a^W \in \mathrm{HS}(L^2(\mathbb{R}^n_x; X); L^2(\mathbb{R}^n_x; Y)).$$

Similar to Lemma 1 in [86], we can show that

Claim D.2. *For any k' s.t. $1 < k'$, we have*

1. *$E_{n,-}(x_2,\xi_2) \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathcal{L}(B^{-k'}_{x_1}; \mathbb{C}^2))$,*
2. *$\tilde{\mathbb{1}}_R^w(x, D_{x_1}, \xi_2) \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathrm{HS}(L^2_{x_1}; B^{-k'}_{x_1}); m)$, where $m(x_2,\xi_2) = (1 + (|(x_2,\xi_2)| - R)_+)^{-k'}$ is the order function.*

Then it follows that, by Claim D.1, we have $E_{n,-}\#\tilde{\mathbb{1}}_R^w \in S(\mathbb{R}^2_{x_2,\xi_2}; \mathrm{HS}(L^2_{x_1}); m)$, i.e.

$$\|E_{n,-}\#\tilde{\mathbb{1}}_R^w(x_2,\xi_2)\|_{\mathrm{HS}(L^2_{x_1})} \leq m(x_2,\xi_2) = (1 + (|(x_2,\xi_2)| - R)_+)^{-k'}.$$

Thus by Prop. D.0.1, since for all $k > 0$,

$$\int_{\mathbb{R}^2} [1 + (|(x_2,\xi_2)| - R)_+]^{-2k} dx d\xi = \pi R^2 + \mathcal{O}(R^{\max(1,-2k+2)}) = \mathcal{O}(R^2),$$

we get $\|E_{n,-}\tilde{\mathbb{1}}_R^W\|_{\mathrm{HS}(L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1};\mathbb{C}^4));L^2(\mathbb{R}_{x_2};\mathbb{C}^2))} \leq Ch^{-1/2}R$ and the Lemma is proved.

□

Lemma D.0.3. *Let $E_{n,-}, E_{n,+}, E_{n,\pm}$ be as in* (4.32)*. For* $\operatorname{Im} z \neq 0$*, both operators*

$$\tilde{\mathbb{1}}_R^W E_{n,+}E_{n,\pm}^{-1}E_{n,-}\tilde{\mathbb{1}}_R^W \text{ and } \bar{\mathbb{1}}_R^W E_{n,-}E_{n,+}E_{n,\pm}^{-1}\bar{\mathbb{1}}_R^W$$

are trace class as bounded linear operators $\mathcal{L}(L^2(\mathbb{R}_{x_2};L^2(\mathbb{R}_{x_1};\mathbb{C}^4)))$ and $\mathcal{L}(L^2(\mathbb{R}_{x_2};\mathbb{C}^2))$, respectively.

Proof. By Lemma D.0.2, the fact that $\tilde{\mathbb{1}}_R^W E_{n,+}$ is the adjoint of $E_{n,-}\tilde{\mathbb{1}}_R^W$ and boundedness of $E_{n,\pm}$ from (4.35), we have

$$\mathrm{Tr}_1(\tilde{\mathbb{1}}_R^W E_{n,+}E_{n,\pm}^{-1}E_{n,-}\tilde{\mathbb{1}}_R^W) \leq \frac{CR^2}{h^{\frac{3}{2}}|\operatorname{Im} z|} \text{ and } \mathrm{Tr}_2(\bar{\mathbb{1}}_R^W E_{n,-}E_{n,+}E_{n,\pm}^{-1}\bar{\mathbb{1}}_R^W) \leq \frac{CR^2}{h^{\frac{3}{2}}|\operatorname{Im} z|}.$$

□

The second proposition allows us to change the position of $E_{n,-}$ in the averaging and limiting process in the proof of Lemma 4.5.8.

Lemma D.0.4. *Let $E_{n,-}, E_{n,+}, E_{n,\pm}$ be as in* (4.32)*, then*

$$\mathrm{Tr}_{L^2(\mathbb{R}_x^2;\mathbb{C}^4)}(\mathbb{1}_R^W\, E_{n,+}E_{n,\pm}^{-1}E_{n,-}\,\mathbb{1}_R^W) - \mathrm{Tr}_{L^2(\mathbb{R}_{x_2};\mathbb{C}^2)}(\bar{\mathbb{1}}_R^W E_{n,-}E_{n,+}E_{n,\pm}^{-1}\bar{\mathbb{1}}_R^W) \leq \frac{CR^{\frac{3}{2}}}{h|\operatorname{Im} z|}.$$

Proof. Since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ when AB and BA are both of trace class.

$$\begin{aligned}
&\mathrm{Tr}_1(\tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1} E_{n,-} \tilde{\mathbb{1}}_R^W) - \mathrm{Tr}_2(\bar{\mathbb{1}}_R^W E_{n,-} E_{n,+} E_{n,\pm}^{-1} \bar{\mathbb{1}}_R^W) \\
&= \mathrm{Tr}_2(E_{n,-} (\tilde{\mathbb{1}}_R^W)^2 E_{n,+} E_{n,\pm}^{-1}) - \mathrm{Tr}_2((\bar{\mathbb{1}}_R^W)^2 E_{n,-} E_{n,+} E_{n,\pm}^{-1}) \\
&= \mathrm{Tr}_2 \left[(E_{n,-} \tilde{\mathbb{1}}_R^W - \bar{\mathbb{1}}_R^W E_{n,-}) \tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1} \right] + \mathrm{Tr}_2 \left[\bar{\mathbb{1}}_R^W (E_{n,-} \tilde{\mathbb{1}}_R^W - \bar{\mathbb{1}}_R^W E_{n,-}) E_{n,+} E_{n,\pm}^{-1} \right] \\
&:= \mathrm{Tr}_2 \left[[E_{n,-}, \mathbb{1}_R]_w \tilde{\mathbb{1}}_R^W E_{n,+} E_{n,\pm}^{-1} \right] + \mathrm{Tr}_2 \left[\bar{\mathbb{1}}_R^W [E_{n,-}, \mathbb{1}_R]_w E_{n,+} E_{n,\pm}^{-1} \right] \\
&:= \mathrm{Tr}_2(A_1) + \mathrm{Tr}_2(A_2)
\end{aligned}$$

where $[E_{n,-}, \mathbb{1}_R]_W := E_{n,-} \tilde{\mathbb{1}}_R^W - \bar{\mathbb{1}}_R^W E_{n,-}$. Then the following claim completes the proof.

Claim D.3. *For* $\operatorname{Im} z \neq 0$, A_1, A_2 *are trace class operators and there is a* $C > 0$ *such that*

$$\mathrm{Tr}_2(A_1), \mathrm{Tr}_2(A_2) \leq Ch^{-1} |\operatorname{Im} z|^{-3/2} R^{3/2}.$$

□

Proof of Claim D.3. From Lemma D.0.2, we already know

$$\|[E_{n,-}, \mathbb{1}_R]_W\|_{\mathrm{HS}^{\mathrm{W}}} \leq Ch^{-1/2} R,$$

where $\mathrm{HS}^{\mathrm{W}} = \mathrm{HS}(L^2(\mathbb{R}_{x_2}; L^2(\mathbb{R}_{x_1}; \mathbb{C}^4)); L^2(\mathbb{R}_{x_2}; \mathbb{C}^2))$. We will improve the upper bound from $Ch^{-1/2} R$ to $Ch^{-1/2} R^{1/2}$.

Let $\bar{\chi}_R^c = 1 - \bar{\chi}_R$, $\tilde{\mathbb{1}}_R^c = 1 - \tilde{\mathbb{1}}_R$. First notice that from the proof of Lemma D.0.2, and replacing $\bar{\chi}_R$ by $\bar{\chi}_R^c$, we have

$$\|[E_{n,-}, \mathbb{1}_R]_w(x_2, \xi_2)\|_{\mathrm{HS}} \leq \tfrac{C_k}{[1+(R-|(x_2,\xi_2)|)_+]^k}, \ \|[E_{n,-}, \mathbb{1}_R^c]_w(x_2, \xi_2)\|_{\mathrm{HS}} \leq \tfrac{C_k}{[1+(|(x_2,\xi_2)|-R)_+]^k}$$

where $[E_{n,-}, \mathbb{1}_R]_w(x_2,\xi_2) = E_{n,-}\#\tilde{\mathbb{1}}_R^w - \bar{\mathbb{1}}_R\#E_{n,-}$ is the symbol in (x_2,ξ_2) of $[E_{n,-}, \mathbb{1}_R]_W$ and $\mathrm{HS} = \mathrm{HS}(L^2(\mathbb{R}_{x_1};\mathbb{C}^4);\mathbb{C}^2)$. Since $[E_{n,-}, \mathbb{1}_R]_w = -[E_{n,-}, \mathbb{1}_R^c]_w$, we have

$$\|[E_{n,-}, \mathbb{1}_R]_w(x_2,\xi_2)\|_{\mathrm{HS}} \leq C_k[1 + ||(x_2,\xi_2)| - R|]^{-k}.$$

Thus by Prop. D.0.1 and a strightfoward computation of the following integral

$$\int_{\mathbb{R}^2_{x_2,\xi_2}} [1 + ||(x_2,\xi_2)| - R|]^{-2k} dx_2 d\xi_2 = \frac{1}{(2k-2)(2k-1)} + \frac{R}{2k-1} = \mathcal{O}(R),$$

we find that $\|[E_{n,-}, \mathbb{1}_R]_W\|_{\mathrm{HS^w}} \leq Ch^{-1/2}R^{1/2}$. Since $\tilde{\mathbb{1}}_R^W E_{n,+}$ is the adjoint of $E_{n,-}\tilde{\mathbb{1}}_R^W$, this yields that

$$\mathrm{Tr}(A_1) \leq Ch^{-3/2}R^{3/2}, \quad \mathrm{Tr}(A_2) \leq Ch^{-3/2}R^{3/2}.$$

□

In next Lemma, we summarize the averaging property of the periodic symbols to reduce the regularized trace to a fundamental cell.

Lemma D.0.5. *Let* $E_{n,-}, E_{n,+}, E_{n,\pm}$, $\bar{\mathbb{1}}_R$ *be as in* (4.32). *Then*

$$\begin{aligned}&\lim_{R\to\infty}\frac{1}{4R^2}\int_{\mathbb{R}^2} \mathrm{Tr}_{\mathbb{C}^2}(\bar{\mathbb{1}}_R\#\partial_z E_{n,\pm}\#E_{n,\pm}^{-1}\#\bar{\mathbb{1}}_R)\ dx_2\ d\xi_2\\ =&\frac{1}{|E|}\int_E \partial_{\bar z}\tilde f\ \mathrm{Tr}_{\mathbb{C}^2}(\partial_z E_{n,\pm}\#E_{n,\pm}^{-1})\ dx_2\ d\xi_2.\end{aligned}$$

The proof of this Lemma can be found in [86, Prop.3].

Appendix E

Magnetic Bloch function on torus

In this appendix, we construct Bloch functions associated to the magnetic Dirac operator H_D defined in (4.1) following [45], for a constant magnetic field B under the symmetric gauge $A(z) = -\frac{B}{2}iz$.

We now consider the Bloch-Floquet transformed magnetic Dirac operator H_D with $a_{\mathbf{k}} := a + \mathbf{k} = 2D_z - \overline{A(z)}$, $H_{D,\mathbf{k}} := \begin{pmatrix} 0 & a_{\mathbf{k}} \\ a_{\mathbf{k}}^* & 0 \end{pmatrix}$.

Proposition E.0.1. *For any $\mathbf{k} \in E_\lambda^*$, there is a sequence of Bloch function $\psi_{n,\mathbf{k}} \in L_B^2(E_\lambda)$, such that*

$$a_{\mathbf{k}}\psi_{0,\mathbf{k}} = 0, \quad a_{\mathbf{k}}\psi_{n,\mathbf{k}} = \sqrt{n}\psi_{n-1,\mathbf{k}}, \quad a_{\mathbf{k}}^*\psi_{n,\mathbf{k}} = \sqrt{n+1}\psi_{n+1,\mathbf{k}}.$$

The zero mode function $\psi_{0,\mathbf{k}}(z)$ is given by

$$\psi_{0,\mathbf{k}}(z) = C_0 e^{-S(z)} f_{\mathbf{k}}(\bar{z}) e^{\frac{B\bar{z}^2}{4}} e^{-i\,\mathrm{Re}(z\bar{\mathbf{k}})} \text{ with } f_{\mathbf{k}}(z) = e^{(\mathrm{Im}\,\mathbf{k} + \frac{1}{4\lambda})z} \vartheta_{\frac{1}{2},\frac{1}{2}}\left(\frac{zi}{4\pi\lambda} - z_{\mathbf{k}} \Big| \omega\right),$$

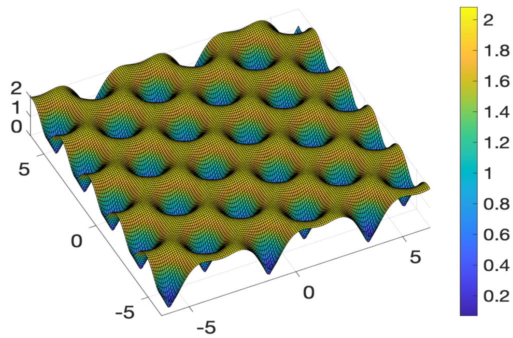

Figure E.1: Periodic wavefunction at zeroth order Landau level.

where $S(z) = \frac{|z|^2 B}{4}$, ϑ *is a Jacobi theta function and* $z_{\mathbf{k}} = 2\lambda\omega\mathbf{k} - i\lambda\omega\mathbf{k} - i\lambda\bar{\omega}\bar{\mathbf{k}} + \frac{1}{2} + \frac{\omega}{2}$. *Let* $\psi_{-1,\mathbf{k}} := 0$. *In particular,* $H_{D,\mathbf{k}}$ *has eigenfunction* $u_{n,\mathbf{k}}$ *with eigenvalues* $\lambda_{n,\mathbf{k}}$ *for*

$$u_{n,\mathbf{k}} = C'_n \begin{pmatrix} \operatorname{sgn}(n)\psi_{|n|-1,\mathbf{k}} \\ \psi_{|n|,\mathbf{k}} \end{pmatrix}, \; C'_n = \begin{cases} \frac{1}{\sqrt{2}} & n \neq 0 \\ 1 & n = 0 \end{cases}, \; \lambda_{n,\mathbf{k}} = \operatorname{sgn}(n)\sqrt{2|n|B}, \; \forall n \in \mathbb{N}.$$

Any Γ-periodic function f is, for some suitable coefficients (f_n), of the form $f(z) = \sum_{n \in \mathbb{Z}^2} f_n e^{2\pi i \mathbf{n}\cdot\mathbf{x}}$ where $\mathbf{n}\cdot\mathbf{x} = n_1 x_1 + n_2 x_2$ with

$$x_1 = \frac{i(\omega z - \bar{\omega}\bar{z})}{\sqrt{3}\lambda} \text{ and } x_2 = \frac{i(\omega\bar{z} - \bar{\omega}z)}{\sqrt{3}\lambda}. \tag{E.1}$$

We now aim to construct Bloch functions in the presence of a constant magnetic field in addition to a periodic magnetic potential, for which the vector potential is also periodic, with Fourier expansion following (E.1)

$$A_{\text{per}}(x_1(\lambda\omega) + x_2(\lambda\omega^2)) = \sum_{\mathbf{n} \in \mathbb{Z}^2 \setminus \{0\}} A_{\mathbf{n}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} \in L^2(\mathbb{C}/\Gamma). \tag{E.2}$$

Proposition E.0.2. *Let* $\Gamma = \lambda(\mathbb{Z} \times \omega\mathbb{Z})$ *where* $\lambda = \sqrt{\frac{2\pi}{B\,\operatorname{Im}(\omega)}}$. *The zero mode Bloch* *function* $(\psi_{0,A,\mathbf{k}},0)$ *to a Floquet transformed magnetic Dirac operator*

$$H_{D,\mathrm{per}} = \begin{pmatrix} 0 & (a_{\mathbf{k}}^* + A_{\mathrm{per}}^*) \\ (a_{\mathbf{k}} + A_{\mathrm{per}}) & 0 \end{pmatrix},$$

with both a constant magnetic field B_0 *and a periodic magnetic field with vector potential* (E.2)*, where* $A_{\mathrm{per}} \in L^2(\mathbb{C}/\Gamma)$*, is characterised by* $(a_{\mathbf{k}} + A_{\mathrm{per}})\psi_{0,A,\mathbf{k}} = 0$*. For each* $\mathbf{k} \in \mathbb{C}$*, there exists a unique solution*

$$\psi_{0,A,\mathbf{k}}(z) = \psi_{0,\mathbf{k}}(z) \exp\left[-\frac{\sqrt{3}\lambda i}{4\pi} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{A_n e^{2\pi i n \cdot x}}{n_1\omega^2 - n_2\omega} \right] \tag{E.3}$$

for $z = \lambda\omega(x_1 + \omega x_2)$ *where we use (E.1). Similarly,* $(-2i\partial_z + A_{\mathrm{per}}(z))\widetilde{\psi}_0(z) = 0$ *has a periodic solution* $\widetilde{\psi}_0(z) = \exp\left[-\frac{\sqrt{3}\lambda i}{4\pi} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{A_n e^{2\pi i n \cdot x}}{n_2\omega^2 - n_1\omega} \right]$.

Proof. Notice that for arbitrary ϕ, we have $e^{-\phi} a_{\mathbf{k}} e^{\phi} = a_{\mathbf{k}} - 2i\partial_{\bar{z}}\phi(z)$. If we can find ϕ s.t. $-2i\partial_{\bar{z}}\phi(z) = A_{\mathrm{per}}$, then $\psi_{0,A,\mathbf{k}} = e^{-\phi}\psi_{0,\mathbf{k}}$ satisfies $(a_{\mathbf{k}} + A_{\mathrm{per}})\psi_{0,A,\mathbf{k}} = 0$. From (E.1) we find

$$\partial_{\bar{z}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} = \frac{2\pi}{\sqrt{3}\lambda}(n_1\omega^2 - n_2\omega)e^{2\pi i \mathbf{n}\cdot\mathbf{x}} \text{ and } \partial_{\bar{z}} e^{2\pi i \mathbf{n}\cdot\mathbf{x}} = \frac{2\pi}{\sqrt{3}\lambda}(n_2\omega^2 - n_1\omega)e^{2\pi i \mathbf{n}\cdot\mathbf{x}}$$

we see that $\phi(z) = \frac{\sqrt{3}\lambda i}{4\pi} \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{A_n}{n_1\omega^2 - n_2\omega} e^{2\pi i n \cdot x}$ is a solution. ☐