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COISOTROPIC SYMPLECTIC TOPOLOGY AND PERIODIC ORBITS IN SYMPLECTIC DYNAMICS

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Abstract

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The main theme of this thesis is the interaction between symplectic topology and Hamiltonian and symplectic dynamics.

The first problem considered in this thesis concerns symplectic topology of coisotropic submanifolds. We revisit the definition of the coisotropic Maslov index and prove a Maslov index rigidity result for stable coisotropic submanifolds in a broad class of ambient symplectic manifolds. Furthermore, we establish a nearby existence theorem for the same class of ambient manifolds. The main tools used to achieve these goals are Hamiltonian Floer homology and Kerman's "pinned" action selector.

The existence of periodic orbits of symplectomorphisms lies at the center of the second problem we consider. We are interested in a variant of the Conley conjecture which asserts the existence of infinitely many periodic orbits of a symplectomorphism if it has a fixed point which is *unnecessary* in some sense. More specifically, we show that, for a certain class of closed monotone symplectic manifolds, any symplectomorphism isotopic to the identity with a hyperbolic fixed point must necessarily have infinitely many periodic orbits as long as the symplectomorphism satisfies some constraints on the flux. The main tool used to prove this result is Floer homology for symplectomorphisms, i.e. the Floer-Novikov homology.

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Chapter 1

Introduction

The subject of this thesis is symplectic topology with particular focus on the topology of coisotropic submanifolds and on periodic orbits of symplectomorphisms.

Periodic orbits are among the most fundamental objects in dynamics. The question about the existence of such orbits in Hamiltonian dynamics, where there is no dissipation of energy, initially arose in the study of classical systems in celestial mechanics. Classical mechanics is concerned with the laws describing the motion of bodies under the action of a system of forces. Classically very little was known about periodic orbits of Hamiltonian systems or fixed points of symplectomorphisms until in the 1920s Birkhoff proved *Poincare's last geometrical theorem* which asserts that an area preserving twist map of the annulus must have at least two different fixed points.

Modern symplectic topology has its historical origin in classical Hamiltonian mechanics on cotangent bundles. In this physical setting, a manifold X can be thought of as the possible positions of particles in the physical system and the symplectic manifold T^*X, the cotangent bundle of X, is the phase space: all the possible positions and momenta. A function defined on the phase space T^*X is then the Hamiltonian.

Coisotropic submanifolds play an important role in symplectic geometry: they describe systems with symmetries and provide a method to generate new symplectic spaces (symplectic reduction) and they appear in homological mirror symmetry.

Two important classes of examples of coisotropic submanifolds are Lagrangian submanifolds, which have half the dimension of the ambient manifold, and hypersurfaces, which have codimension one. The dimension of a coisotropic submanifold ranges between these two cases. These submanifolds also carry a natural foliation, called the characteristic foliation, whose leaves have dimension equal to the codimension of the submanifold. One of the topics of this thesis is the existence of special closed curves *in* this foliation which can be viewed as a generalization of the existence of closed characteristics on a hypersurface in $\mathbb{R}^{2n}$: if the parametrization of the periodic solutions is neglected, the later problem aims for closed characteristics of a distinguished line bundle over a hypersurface in a symplectic manifold, the characteristic line bundle. Weinstein conjectured in [Wei79] that a hypersurface of a certain type always carries a closed characteristic and Viterbo established it in [Vit87]. Other results within this framework include the existence of periodic solutions of certain Hamiltonian systems on prescribed energy levels; for instance, in [Rab79] and [HZ11].

In this setting, our first main result in this thesis is on the Maslov index and symplectic area rigidity for coisotropic submanifolds in a broad class of ambient symplectic manifolds. In [Zil09] and [Gin11], the Maslov index is defined for loops in coisotropic submanifolds which are tangent to the characteristic foliation of the coisotropic submanifold. The Maslov index of such a loop, $x\colon S^1 \to M$, is the (Conley-Zehnder) mean index Δ of a symplectic path which is a lift of the holonomy along the loop to the pull-back bundle x^*TM. Although such a lift is not unique, the coisotropic Maslov index μ is a well-defined real valued index.

With this definition of the coisotropic Maslov index, we prove (cf. [Bat12]) a result on the Maslov class rigidity. More specifically, given a closed displaceable stable coisotropic submanifold, we show that there exists a non-trivial loop lying in the submanifold with Maslov index bounded below by 1 and above by $2n + 1 - k$, where $2n$ is the dimension of the symplectic manifold and k the codimension of the coisotropic submanifold (see Theorem 2.1.3). Moreover, the result gives bounds on the symplectic area bounded by the loop; this area is positive and bounded above by the displacement energy of the coisotropic submanifold. This result was proved by Ginzburg in [Gin11] for ambient symplectic manifolds which are symplectically aspherical. The case where the characteristic foliation is a fibration is also considered in [Zil09]. We extend the result in [Gin11] to certain rational manifolds which need not be symplectically aspherical. In the *spherical* case, the obtained loop may be trivial with non-trivial capping. Hence, in our theorem we state conditions on the ambient manifold for which this loop is non-trivial and has the referred bounds on the Maslov index and on the symplectic area.

The Maslov class rigidity for Lagrangian submanifolds was originally studied by Viterbo in [Vit90] for the Lagrangian torus and by Polterovich in [Pol91a, Pol91b], for instance, for monotone Lagrangian submanifolds. These results show that the Maslov class satisfies certain restrictions. Namely, the minimal Maslov number lies between 1 and $n+1$. Audin was the first to conjecture (as far as we know) that the minimal Maslov number is 2 for the Lagrangian torus; cf. [Aud88]. Fukaya proved this conjecture in [Fuk06]. There are two methods to prove this type of results. One approach, introduced by Gromov in [Gro85], uses holomorphic curves. This approach is the one used, for instance, by Audin and Polterovich (see also [ALP94]). A different approach relies on Hamiltonian Floer homology and is found, for instance, in the work of Viterbo, Kerman and Şirikçi; see also [Ker09, KŞ10].

The proof of our result follows the method used by Ginzburg in [Gin11] which is based on the second approach mentioned above together with the stability condition and certain lower bounds on the energy estimated by Bolle in [Bol96, Bol98]. The proof also relies on a suitable action selector introduced in [Ker09, KŞ10].

The second part of our theorem, which gives bounds on the symplectic area bounded by the loop, complements and partially generalizes numerous rigidity results for the Liouville class. Among these are Liouville class rigidity results for Lagrangian submanifolds (see e.g. [Che96, Che98, Gro85, Oh97, Pol93]), for stable coisotropic submanfiolds (see e.g. [Gin07, Ker08, Ush11]) and for hypersurfaces of restricted contact type (see e.g. [Sch06]).

Furthermore, we prove a theorem of dense or nearby existence (Theorem 2.1.4) which guarantees the existence of periodic orbits for a dense set of energy levels. This result is established in [Gin07] for symplectically aspherical manifolds and, as mentioned there, it can be viewed as a generalization of the existence of closed characteristics on stable hypersurfaces in $\mathbb{R}^{2n}$ established in [HZ11]. We state this nearby existence theorem for a broader class of rational symplectic manifolds.

Our theorems hold for a large class of manifolds including negative monotone manifolds (see definition in Section 3.1). Standard examples are surfaces of genus greater than two and the hypersurface in $\mathbb{CP}^n$ defined by $z_0^m + \ldots + z_n^m = 0$ where $m > n+1$.

The second main result of our investigation concerns the geometric behavior of symplectomorphisms. Namely, we are interested in the existence and *number* of their periodic points. There are many interesting results for Hamiltonian diffeomorphisms.

Floer, in [Flo89], introduced a new approach, now called Floer theory, to establish the Arnold conjecture giving a positive lower bound on the number of fixed points of such diffeomorphisms. However, the main focus of our work is on symplectomorphisms which need not be generated by Hamiltonians. There are also variations of Floer theory applicable in this case introduced by Dostoglou and Salamon in [DS93] and Lê and Ono in [LO95, Ono95], but, in contrast with the Hamiltonian case, one cannot expect Floer homology to immediately yield the existence of fixed points. For example, a rotation of the two-torus is a symplectomorphism without fixed points.

Although here we are interested in symplectomorphisms which do not necessarily arise from Hamiltonians, our result (cf. [Bat]) can be viewed in the context of what is often referred to the Conley conjecture ([Con84]) which claims the existence of infinitely many periodic orbits (of a Hamiltonian diffeomorphism). The conjecture was shown to be true for symplectic manifolds with $c_1|_{\pi_2(M)} = 0$ and also for negative monotone manifolds; see [CGG11, GG09, Hei12] and also [FH03, Gin10, GG12a, Hin09, LC06, SZ92]. The main difference between the Conley conjecture and our result is that in the Conley conjecture the existence of periodic orbits is unconditional whereas in our result the symplectomorphism is required to have one contractible (hyperbolic) periodic orbit. Without loss of generality, when a periodic orbit is contractible, we may assume it is a fixed point. Hence, for the sake of simplicity, from now on we consider the hyperbolic periodic orbit γ to be constant; see beginning of Section 5.2.2 for more details.

Due to this assumption on the existence of a periodic orbit of a specific type, our result fits more accurately under what Gürel describes in [Gür12b, Gür12a] as the generalized HZ-conjecture; see also [GG12b]. This variant of the Conley conjecture claims that a Hamiltonian diffeomorphism with *more than necessary* fixed points has infinitely many periodic points where *more than necessary* is interpreted as the lower bound on the number of fixed points provided by some form of the Arnold conjecture. For $\mathbb{CP}^n$, the expected threshold is $n + 1$. The HZ-conjecture was originally stated (as far as we know) in this form by Hofer and Zehnder in [HZ11, p.263] and was motivated by the results of Gamboudo and Le Calvez in [GLC99] and Franks in [Fra88] (see also [Fra92, Fra96]) where they prove that an area preserving diffeomorphism of S^2 having at least three fixed points admits automatically infinitely many periodic points; see also [BH11, CKR$^+$12, Ker12] for symplectic topological proofs. In a broader context, it appears that the presence of a fixed point that is *unnecessary* from a homological or geometrical perspective is already sufficient to force the existence of infinitely many periodic points. In fact, our theorem (see Theorem 2.2.1 and cf. [Bat]) asserts that, for a certain class of symplectic manifolds, a symplectomorphism (isotopic to the identity) with a hyperbolic fixed point must admit infinitely many periodic points (as long as it satisfies some condition on its flux). The theorem is a symplectic analogue of a result proved in [GG12b] for Hamiltonian diffeomorphisms. There are few results directly supporting the conjecture for dimension greater than two: in addition to [GG12b], a "local version" of the conjecture is considered in [GG12b] and the conjecture is presented for non-contractible orbits in [Gür12b].

Some examples of manifolds that meet the requirements of Theorem 2.2.1 and admit symplectomorphisms (which are not Hamiltonian diffeomorphisms) with periodic points are the product of complex projective spaces with tori $\mathbb{CP}^n \times \mathbb{T}^{2m}$ (with $m + 2 \leq n$) and the product of complex Grassmannians with tori $Gr(2, N) \times \mathbb{T}^2$.

Chapter 2

Main results

2.1 Rigidity of the coisotropic Maslov index and the nearby existence theorem

In this section we state and discuss our result on the Maslov index and symplectic area rigidity for coisotropic submanifolds (cf. [Bat12]). In order to state the theorem, we must first briefly describe the Maslov index for loops in coisotropic submanifolds (for a more detailed description of this index we refer the reader to Section 3.4.1).

In the Lagrangian setting, the Maslov index gives an explicit isomorphism, $\mu\colon \pi_1(\Lambda(n)) \to \mathbb{Z}$, between the fundamental group of the Lagrangian Grassmannian and $\mathbb{Z}$ (cf. [Arn67]) and the Maslov class is given by the generator of

$$H^1(\Lambda(n), \mathbb{Z}) \cong \pi_1(\Lambda(n)) \cong \mathbb{Z}.$$

Then, the Maslov index of a loop $x\colon S^1 \to L$ in a closed Lagrangian submanifold, $\mu(x) \in \mathbb{Z}$, is obtained using the above index together with a trivialization of $T_{x(t)}L$.

For closed coisotropic submanifolds, the Maslov index is defined ([Gin11, Zil09]) for loops tangent to the characteristic foliation as the mean (Conley-Zehnder) index of a symplectic path obtained from a lift of the holonomy along the loop to the pull back x^*TM. This is a well-defined real valued index, $\mu(x, u) \in \mathbb{R}$ (see Example 2.1.2), and it generalizes the usual Lagrangian Maslov index. More specifically, let (M^{2n}, ω) be a symplectic manifold and N^{2n-k} a closed coisotropic submanifold of M of codimension k. Then $(T_pN)^\omega \subseteq T_pN$ for each $p \in N$ and, denoting by ω_N the restriction of ω to N, we note that the distribution $TN^\omega := \ker \omega_N$ on N is integrable. By the Frobenius theorem, there is a foliation $\mathcal{F}$ (the *characteristic foliation*) on N whose tangent spaces are given by TN^ω, i.e. $T\mathcal{F} = \ker \omega_N$, and the rank of this foliation is k. Consider a capped loop $\bar{x} = (x, u)$ tangent to $T\mathcal{F}$ and the holonomy along x

$$H_t \colon T^\perp \mathcal{F}_{x(0)} \to T^\perp \mathcal{F}_{x(t)}.$$

There is a symplectic vector bundle decomposition of the restriction of TM to N:

$$TM\big|_N = (T\mathcal{F} \oplus T^\perp N) \oplus T^\perp \mathcal{F}$$

where we identify the normal bundle $T^\perp \mathcal{F}$ to $\mathcal{F}$ in N with $TN/T\mathcal{F}$ and the normal bundle $T^\perp N$ to N in M with TM/TN. Lift the holonomy along x to x^*TM. The capping u gives rise to a symplectic trivialization of x^*TM, unique up to homotopy, and hence this lift can be viewed as a symplectic path

$$\Psi \colon [0, 1] \to \mathrm{Sp}(2n).$$

Following [Zil09] (see also [Gin11]) we adopt

Definition 2.1.1. The *coisotropic Maslov index* is defined (up to a sign) as the mean index of this path, i.e.

$$\mu(x, u) := -\Delta(\Psi).$$

This Maslov index is independent of the lift of the holonomy along x. However, in general, it depends on the trivialization arising from the capping u (see Section 3.4.1 for the definitions of the indices). The proof that this Maslov index is well-defined can be found in [Zil09]. In Section 3.4.1, for the sake of completeness, we give a direct proof of this fact.

Example 2.1.2. Consider the Hamiltonian defined in $(\mathbb{C}^n, \omega_0)$ by

$$H(z) := 1/2 \sum_{l=1}^{n} \lambda_l |z_l|^2$$

with $\lambda_l \in \mathbb{R}^+$ (where ω_0 is the standard symplectic form). The ellipsoid defined as the regular level set $H^{-1}(\{1\})$ is a hypersurface (and hence a coisotropic submanifold) of $\mathbb{C}^n$. For each $j = 1, \ldots, n$, the loop parameterized by

$$\gamma_j(t) := (0, \ldots, 0, z_j(t), 0, \ldots, 0)$$

where

$$z_j(t) = e^{-i\lambda_j t} z_j$$

(with $|z_j|^2 = 2/\lambda_j$ and $t \in [0, 2\pi/\lambda_j]$) is a periodic orbit of the Hamiltonian system of H lying in $H^{-1}(\{1\})$. A calculation shows that the Maslov index of the loop (γ_j, u_j) is given by

$$\mu(\gamma_j, u_j) = -\Delta(\gamma_j, u_j) = \frac{2}{\lambda_j} \sum_{l=1}^{n} \lambda_l$$

where u_j is some capping of γ_j. In this case, the index is independent of the capping we use.

To compute $\mu(\gamma_j, u_j)$, we use $\Psi_t = d(\varphi_H^t)_{\gamma(0)}$ the linearized flow along γ. The foliation $\mathcal{F}$ is formed by the integral curves of φ_H^t. See Section 3.2.1 for the description of the Maslov index when the loop is a periodic orbit of a Hamiltonian.

With this definition of the coisotropic Maslov index, we prove (in Section 5.1.1) the following result on the Maslov class rigidity.

Theorem 2.1.3. *Let (M^{2n}, ω) be a rational closed symplectic manifold, $N^{2n-k} \subset M^{2n}$ a closed stable displaceable coisotropic submanifold of M and $\mathcal{F}$ its characteristic foliation.*

Assume that one of the following conditions is satisfied

- *M is negative monotone,*
- *$e(N) < h_0$, where $e(N)$ is the displacement energy of N and h_0 is the rationality constant of M,*
- *$2n + 1 < 2\mathcal{N}$, where $\mathcal{N}$ is the minimal Chern number of M.*

Then, for all $\varepsilon > 0$, there exists a capped loop $\bar{\gamma} = (\gamma, v)$ such that γ is a non-trivial loop tangent to $\mathcal{F}$ and

$$\begin{aligned} 1 \leq \quad & \mu(\bar{\gamma}) && \leq 2n + 1 - k, \\ 0 < \quad & Area(\bar{\gamma}) && \leq e(N) + \varepsilon, \end{aligned}$$

where $Area(\bar{\gamma}) := \int_v \omega$.

The condition that M is closed can be replaced in the theorem by geometrically bounded and wide. Recall that a symplectic manifold is said to be *wide* if it admits an arbitrarily large, compactly supported, autonomous Hamiltonian whose Hamiltonian flow has no non-trivial contractible periodic orbits of period less than or equal to one; see [Gür08] for more details. The proof of the theorem in this case is essentially the same as when M is closed.

The requirements that N is displaceable and stable are essential. For instance, a closed manifold N viewed as the zero section of its cotangent bundle T^*N is not displaceable (cf. [Gro85]) and the Maslov index of a loop in N is always trivial since $\pi_2(T^*N, N) = 0$. Moreover, the assumption that N is stable cannot be entirely omitted: there exist Hamiltonian systems having no periodic orbits on a compact energy level which arise as counterexamples to the Seifert conjecture; cf. [Gin99, GG03].

Furthermore, as a corollary of the previous main result we obtain (cf. [Bat12]) a theorem of dense or nearby existence, that is, a theorem which guarantees the existence of periodic orbits for a dense set of energy levels. This result is presented in [Gin07] for symplectically aspherical manifolds. Here, we state this nearby existence theorem for a broader class of rational symplectic manifolds. The structure of our proof is essentially the same as in the referred paper and the necessary changes are contained in the proof of the theorem in Section 5.1.2.

Let M be a closed rational symplectic manifold and consider a map $\overrightarrow{F} = (F_1, \dots, F_k)\colon M \to \mathbb{R}^k$ whose components F_j are Poisson-commuting Hamiltonians, i.e. $\{F_i, F_j\} = 0$ for $i \neq j$ and satisfy $dF_1 \wedge \ldots \wedge dF_k \neq 0$ in N_0 where $N_a := \overrightarrow{F}^{-1}(\{a\})$, for $a \in \mathbb{R}^k$, and N_0 is a displaceable coisotropic submanifold of M with codimension k. Assume that one of the following conditions is satisfied

- M is negative monotone,
- $e(N_0) < h_0$,
- $2n + 1 < 2\mathcal{N}$.

Then we have the following nearby existence result.

Theorem 2.1.4. *For a dense set of regular values $a \in \mathbb{R}^k$ near the origin, the level set N_a carries a closed curve x (with capping u in M) tangent to the characteristic* *foliation $\mathcal{F}_a$ on N_a.*

2.2 Hyperbolic points and periodic orbits of symplectomorphisms

In this section we state and discuss our result on the existence of infinitely many periodic orbits of symplectomorphisms. More specifically, we proved (cf. [Bat]) the existence of infinitely many periodic orbits of symplectomorphisms isotopic to the identity as long as they admit at least one hyperbolic periodic orbit and satisfy some constraints on the flux.

Consider a symplectomorphism ϕ in the identity component of the group of symplectomorphisms of (M, ω). The flux homomorphism (see definition in Section 3.3) associates with ϕ a cohomology class $[\theta]$ in $H^1(M, \mathbb{R})$. We say that ϕ has *rational flux* if the group formed by the integrals of θ over the loops in M is discrete, that is,

$$\langle [\theta], \pi_1(M) \rangle = h_1 \mathbb{Z}$$

for some $h_1 \in \mathbb{R}$.

Then we have the following result on the periodic orbits of such symplectomorphisms.

Theorem 2.2.1. *Let M^{2n} be strictly monotone (i.e. M is monotone and $c_1|_{\pi_2(M)} \neq 0$ and $[\omega]|_{\pi_2(M)} \neq 0$). Assume that*

- *$\mathcal{N} \geq n/2 + 1$ (where $\mathcal{N}$ is the minimal Chern number) and*

$$\beta * \alpha = \mathfrak{q}[M] \quad \text{in } HQ_*(M) = H_*(M) \otimes \Lambda \tag{2.2.1}$$

for some ordinary homology classes α, $\beta \in H_*(M)$, with $\deg(\alpha)$, $\deg(\beta) < 2n$.

Then any symplectomorphism in $Symp_0(M, \omega)$ with

- *a contractible hyperbolic periodic orbit γ and*
- *rational flux where $h_1 = (p/r)h_0$ (where h_0 is the rationality constant of M)*

has infinitely many periodic orbits.

Here p and r are coprime positive integers and $\mathfrak{q}$ is the element of the Novikov ring defined as in Section 4.3.1.

The assumption on the existence of a hyperbolic periodic orbit γ is extremely important. A significant feature of hyperbolic orbits is the fact that the energy of (*Floer*) trajectories approaching an iteration of γ and crossing its fixed neighborhood cannot be *small*, i.e. is bounded away from zero by a constant independent of the order of iteration (see Section 5.2.1). The main tool used to prove our result is filtered Floer-Novikov homology (see Section 4.2.2). Our assumption that the flux and the action *grow together* plays an important role in the proof.

The following proposition (proved in Section 5.2.3) leads to examples of symplectomorphisms which meet the requirements of the main theorem.

Proposition 2.2.2. *Given a symplectic isotopy ϕ_t of (M, ω) with $\phi_0 = id$ and a loop γ in M, there exists a Hamiltonian deformation, ψ_t, of ϕ_t such that γ is a hyperbolic one-periodic orbit of ψ_1.*

In particular, the flux of $\{\phi_t\}$ is equal to the flux of $\{\psi_t\}$.

Example 2.2.3. Consider $M = \mathbb{CP}^m \times \mathbb{T}^{2n}$ (where $m + 2 \leq n$) with the standard symplectic form. Recall that, in this case, the symplectic area of $\mathbb{T}^{2n}$ is one, $c_1(\mathbb{T}^{2n}) = 0$ and that, under the normalization $\omega_{\text{FS}}[g_0] = m + 1$ (where ω_{FS} is the Fubini-Studi form on $\mathbb{CP}^m$ and g_0 is the generator of $H_2(\mathbb{CP}^m, \mathbb{R})$), $\mathbb{CP}^m$ is a monotone symplectic manifold. Denote the rationality constant of M by h_0. Consider a symplectomorphism ϕ given by the shift of $\theta \in H^1(M, \mathbb{R})$ with $h_1 = (p/r)h_0$. The symplectomorphism ϕ has no fixed points. According to the previous proposition, there exists a Hamiltonian deformation ψ_t of a symplectic path ϕ_t connecting the identity to ϕ such that ψ_1 has a hyperbolic fixed point.

Remark 2.2.4. The hyperbolicity condition is required so that the orbit has the important feature mentioned above and which is also described in Section 5.2.1. Hence, more *generally*, a symplectomorphism with a periodic orbit having the property in Theorem 5.2.1 (and satisfying the requirement on its flux) admits infinitely many periodic orbits.

Remark 2.2.5. Hypothetically the requirements on the minimal Chern number and the interdependence of the flux and the (spherical) rationality of the manifold can possibly be relaxed or even eliminated. However, the homological assumption (2.2.1) is crucial in the proof.

The proof of the theorem goes by contradiction and, if a symplectomorphism admits finitely many fixed points, it admits an iteration k for which the *action* of all the k-periodic orbits are in a small neighborhood of $\lambda_0\mathbb{Z}$. Using the feature of the hyperbolic orbit, the fact that quantum homology acts on the (filtered) Floer-Novikov homology and condition (2.2.1), we obtain a k-periodic orbit with action outside the small neighborhood of $\lambda_0\mathbb{Z}$. Our assumption that the flux and the action *grow together* plays an important role in the proof.

Chapter 3

Some facts from symplectic geometry

In this chapter we introduce the notation used throughout the thesis and recall some facts about symplectic manifolds and symplectomorphisms considered in the main theorems.

We are interested in certain structures and properties of symplectic manifolds which are described in Section 3.1.

Our first result, on the rigidity of the coisotropic Maslov index, concerns the existence of periodic orbits of a Hamiltonian on coisotropic submanifolds tangent to its characteristic foliation. The periodic orbits of Hamiltonian systems can be characterized as critical points of a functional on the space of capped loops. In Section 3.2, we describe this action functional and the mean and Conley-Zehnder indices. In Section 3.4, we recall the definition of the Maslov index for coisotropic submanifolds and give some properties of this index for the so called *stable* coisotropic submanifolds.

The periodic orbits of symplectomorphisms can also be viewed as critical points of some functional but in this case on the space of *tailed*-capped loops. In Section 3.3, we describe this action functional and the mean index.

3.1 Symplectic manifolds

Our main results are for closed rational symplectic manifolds. In this section we describe and discuss the properties of such manifolds needed in the proofs. For more details see [MS95].

Recall that (M, ω) is *closed* if it is compact with no boundary and is said to be *(spherically) rational* if the group

$$\langle [\omega], \pi_2(M) \rangle \subset \mathbb{R}$$

formed by the integrals of ω over the spheres in M is discrete, that is,

$$\langle [\omega], \pi_2(M) \rangle = h_0 \mathbb{Z}$$

where $h_0 \geq 0$. When $\langle [\omega], \pi_2(M) \rangle = 0$ we set $h_0 = \infty$. The constant h_0 is called the *rationality constant* and it is the infimum over the symplectic areas of all nonconstant spheres in M with positive area. More explicitly,

$$h_0 := \inf_{A \in \pi_2(M)} \big\{ \langle \omega, A \rangle \ | \ \langle \omega, A \rangle > 0 \big\}.$$

Consider an almost complex structure J on M compatible with ω, i.e. such that $\langle \xi, \eta \rangle := \omega(\xi, J\eta)$ is a Riemannian metric on M. For every symplectic manifold (M, ω), the space of compatible almost complex structures is non-empty and contractible. Similarly, a Riemannian metric g is compatible with ω if it is of the form $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ for some almost complex structure J.

Denote by $c_1 := c_1(M, J) \in H^2(M, \mathbb{Z})$ the first Chern class of M. The *minimal Chern number* of a symplectic manifold (M, ω) is the integer $\mathcal{N}$ which generates the discrete group $\langle c_1, \pi_2(M) \rangle \subset \mathbb{R}$ formed by the integrals of c_1 over the spheres in M, i.e.

$$\langle c_1, \pi_2(M) \rangle = \mathcal{N} \mathbb{Z}$$

where $\mathcal{N} \in \mathbb{Z}^+$. When $\langle c_1, \pi_2(M) \rangle = 0$, we set $\mathcal{N} = \infty$. The constant $\mathcal{N}$ is given explicitly by

$$\mathcal{N} := \inf_{A \in \pi_2(M)} \big\{ \langle c_1, A \rangle \ | \ \langle c_1, A \rangle > 0 \big\}.$$

A symplectic manifold (M, ω) is called *monotone* (*negative monotone*) if the cohomology classes c_1 and $[\omega]$ satisfy the condition

$$c_1|_{\pi_2(M)} = \lambda \, [\omega]|_{\pi_2(M)}$$

for some non-negative (respectively, negative) constant $\lambda \in \mathbb{R}$.

The manifold (M, ω) is called *symplectically aspherical* if

$$c_1|_{\pi_2(M)} = 0 = [\omega]|_{\pi_2(M)}.$$

Notice that a symplectically aspherical manifold is monotone and a monotone (or negative monotone) manifold is rational.

3.2 Hamiltonian diffeomorphisms

All the Hamiltonians H on M are assumed to be one-periodic in time, namely,

$$H\colon S^1 \times M \to \mathbb{R},$$

where $S^1 = \mathbb{R}/\mathbb{Z}$, and we set $H_t = H(t, \cdot)$ for $t \in S^1$. The Hamiltonian vector field X_H of H is defined by $\iota_{X_H}\omega = -dH$. The time-one map of the flow of the Hamiltonian vector field X_H is called a *Hamiltonian diffeomorphism* and denoted by φ_H.

The composition $\varphi_H^t \circ \varphi_K^t$ of two Hamiltonian flows is again Hamiltonian and it is generated by $K\#H$ where

$$(K\#H)_t := K_t + H_t \circ (\varphi_K^t)^{-1}. \tag{3.2.1}$$

Remark 3.2.1. In general $K\#H$ is not a one-periodic Hamiltonian. However, $K\#H$ is one-periodic if $H_0 = 0 = H_1$. This condition can be met by reparametrizing the Hamiltonian as a function of time without changing the time-one map. Thus, in Section 5.1, we will usually treat $K\#H$ as a one-periodic Hamiltonian.

When two Hamiltonians K and H are one-periodic, we denote by $K\natural H$ the two periodic Hamiltonian equal to K_t for $t \in [0, 1]$ and H_{t-1} for $t \in [1, 2]$ (where we assume $K_1 = H_0$ and $H_1 = K_0$). Then define the k-periodic Hamiltonian $H^{\natural k} := H\natural \ldots \natural H$ (k times) the natural way.

The *Hofer norm* of a one-periodic Hamiltonian H is defined by

$$||H|| := \int_0^1 (\max_M H_t - \min_M H_t)dt.$$

The Hamiltonian diffeomorphism φ_H is said to *displace* a subset U of M if

$$\varphi_H(U) \cap U = \emptyset.$$

When such a map exists, we call U *displaceable* and define the *displacement energy of U* to be

$$e(U) := \inf\left\{||H||\colon \varphi_H \text{ displaces } U\right\}$$

where $||\cdot||$ is the Hofer norm.

3.2.1 Capped loops, the Hamiltonian action functional

Let $x\colon S^1 \to M$ be a contractible loop with capping $u\colon D^2 \to M$, i.e. $u|_{\partial D^2} = x$. Two cappings u and v of x are called *equivalent* if the integrals of ω and of c_1 over the sphere obtained by attaching u to v are both equal to zero. For instance, when M is symplectically aspherical, all cappings of x are equivalent. A *capped closed curve* $\bar{x}$ is, by definition, a closed curve x equipped with an equivalence class of cappings.

The *action functional* of a one-periodic Hamiltonian H on a capped closed curve $\bar{x} = (x, u)$ is defined by

$$\mathcal{A}_H(\bar{x}) := -\int_u \omega + \int_{S^1} H_t(x(t))dt.$$

The space of capped closed curves is a covering space of the space of contractible loops and the critical points of the action functional are exactly the capped one-periodic orbits of the Hamiltonian vector field X_H. The *action spectrum* $\mathcal{S}(H)$ of H is the set of critical values of the action.

A (capped) periodic orbit $\bar{x}$ of H is said to be *non-degenerate* if the linearized return map

$$d\varphi_H\colon T_{x(0)}M \to T_{x(0)}M$$

has no eigenvalues equal to one. Note that capping has no effect on degeneracy or non-degeneracy of $\bar{x}$.

Using a trivialization of x^*TM arising from the capping of $\bar{x}$, the linearized flow along x

$$d\varphi_H^t\colon T_{x(0)}M \to T_{x(t)}M$$

can be viewed as a symplectic path $\Phi\colon [0,1] \to \mathrm{Sp}(2n)$. The mean index of $\bar{x}$ is defined by $\Delta(\bar{x}) := \Delta(\Phi)$; see Definition 3.4.3. When we need to emphasize the role of H, we write $\Delta_H(\bar{x})$. A list of properties of the mean index can be found in Section 3.4.1. In general, the mean index and the action depend on the equivalence class of the capping

u of the loop x. More concretely, let A be a 2-sphere and denote by $\bar{x}\#A$ the recapping of $\bar{x}$ by attaching A. Then we have

$$\Delta(\bar{x}\#A) = \Delta(\bar{x}) - 2\langle c_1, A\rangle \quad \text{and} \quad \mathcal{A}_H(\bar{x}\#A) = \mathcal{A}_H(\bar{x}) - \int_A \omega.$$

Consider a non-degenerate path $\Phi\colon [0,1] \to \mathrm{Sp}(2n)$, i.e. such that $\Phi(1)$ has no eigenvalues equal to one. We denote by $\mu_{\mathrm{CZ}}(\Phi)$ the *Conley-Zehnder index* of Φ. For a non-degenerate capped closed orbit $\bar{x} = (x, u)$, its Conley-Zehnder index is given by the Conley-Zehnder index of the symplectic path Φ obtained from the linearized flow $d\varphi_H^t$ and a trivialization arising from the capping u. Up to a sign, it is defined as in [Sal99, SZ92] and we use the normalization such that $\mu_{\mathrm{CZ}}(\bar{x}) = n$ when $\bar{x}$ is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian; cf. [GG09].

We have the following relation between the Conley-Zehnder and mean indices for non-degenerate paths and orbits; cf. [SZ92]:

$$|\Delta(\Phi) - \mu_{\mathrm{CZ}}(\Phi)| < n \quad \text{and hence} \quad |\Delta(\bar{x}) - \mu_{\mathrm{CZ}}(\bar{x})| < n. \tag{3.2.2}$$

3.3 Symplectomorphisms

In this section, we recall some properties of symplectomorphisms following [LO95].

We denote by $\mathrm{Symp}(M,\omega)$ the symplectomorphism group of (M,ω) and by $\mathrm{Symp}_0(M,\omega)$ the identity component in $\mathrm{Symp}(M,\omega)$.

Let $\phi \in \mathrm{Symp}_0(M,\omega)$ and consider ϕ_t a symplectic path connecting the identity to ϕ, i.e. $\phi_0 = id$ and $\phi_1 = \phi$. A vector field X_t is defined by:

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t. \tag{3.3.1}$$

Recall that the *flux homomorphism* is defined on the universal cover of $\mathrm{Symp}_0(M,\omega)$ as follows:

$$\begin{aligned} \widetilde{\mathrm{Flux}} : \widetilde{\mathrm{Symp}_0}(M,\omega) &\to H^1(M,\mathbb{R}) \\ \widetilde{\phi} &\mapsto \left[\int \omega(X_t,\cdot)dt\right]. \end{aligned} \tag{3.3.2}$$

Let θ be a closed one-form which represents the cohomology class $\widetilde{\mathrm{Flux}}(\widetilde{\phi})$. Throughout this work, we assume that the group formed by the integrals of θ over the loops in M is discrete, that is,

$$\langle [\theta], \pi_1(M) \rangle = h_1 \mathbb{Z} \tag{3.3.3}$$

for some $h_1 \in \mathbb{R}$. When the group $\langle [\theta], \pi_1(M) \rangle = 0$ we set $h_1 = \infty$. The symplectomorphism ϕ is said to have *rational flux* when (3.3.3) holds.

A symplectomorphism ϕ is called *exact* if it is the time-one map of a Hamiltonian vector field (see Section 3.2 for the definition).

3.3.1 Tailed-capped loops and the action functional

Let $x\colon S^1 \to M$ be a contractible loop, where $S^1 = \mathbb{R}/\mathbb{Z}$, with capping $u\colon D^2 \to M$, i.e. $u|_{\partial D^2} = x$. Two cappings v and w of x are called *equivalent* if the integrals of ω and of c_1 over the sphere obtained by attaching v to w are both equal to zero. Denote by $\mathcal{L}M$ the space of (capped) loops in M.

The integration of the form θ defined by (3.3.2) along loops gives a homomorphism

$$\begin{aligned} I_\theta\colon \pi_1(M) &\to \mathbb{R} \\ \gamma &\mapsto \int_\gamma \theta. \end{aligned}$$

Consider the covering $\pi\colon \widetilde{M} \to M$ associated with $\ker I_\theta \subset \pi_1(M)$, i.e. the deck transformation group of the cover $\pi : \widetilde{M} \to M$ is isomorphic to

$$\frac{\pi_1(M)}{\ker I_\theta}. \tag{3.3.4}$$

Denote by $ev\colon \mathcal{L}M \to M$ the evaluation map $x \mapsto x(0)$ and by $\widetilde{ev}$ the corresponding map on $\mathcal{L}\widetilde{M}$. Consider the covering space of $\mathcal{L}M$ associated to the homomorphisms

$$\begin{aligned} I_{c_1}: \pi_2(M) &\to \mathbb{R} \\ A &\mapsto 2\int_A c_1 =: 2\langle c_1, A \rangle \end{aligned}$$

and

$$\begin{aligned} I_\omega\colon \pi_2(M) &\to \mathbb{R} \\ A &\mapsto -\int_A \omega =: -\langle \omega, A \rangle \end{aligned}$$

which we denote by $\widetilde{\mathcal{L}}M$. The deck transformation group of the obtained covering space is isomorphic to the quotient group

$$\frac{\pi_2(M)}{\ker I_{c_1} \cap \ker I_\omega}.$$

This construction gives rise to a covering space of $\mathcal{L}M$, which we denote by $\widetilde{\mathcal{L}}\widetilde{M}$, so that the following diagram commutes:

$$\begin{array}{ccccc} \widetilde{\mathcal{L}}\widetilde{M} & \xrightarrow{\widetilde{j}} & \mathcal{L}\widetilde{M} & \xrightarrow{\widetilde{ev}} & \widetilde{M} \\ \downarrow{\scriptstyle \widetilde{\Pi}} & & \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \pi} \\ \widetilde{\mathcal{L}}M & \xrightarrow{j} & \mathcal{L}M & \xrightarrow{ev} & M \end{array} \tag{3.3.5}$$

where j is the projection from $\widetilde{\mathcal{L}}M$ to $\mathcal{L}M$ and $\widetilde{j}$ the corresponding projection from $\widetilde{\mathcal{L}}\widetilde{M}$ to $\mathcal{L}\widetilde{M}$. The deck transformation group of $\widetilde{\mathcal{L}}\widetilde{M} \to \mathcal{L}M$ is the direct sum

$$(\pi_1(M)/\ker I_\theta) \bigoplus \pi_2(M)/(\ker I_{c_1} \cap \ker I_\omega).$$

An element of the covering space $\widetilde{\mathcal{L}}\widetilde{M}$ is represented by an equivalence class of pairs $(\widetilde{x}, \widetilde{v})$ where

i) $\widetilde{x}$ is a loop in $\widetilde{M}$,

ii) $\widetilde{v}$ is a disc in $\widetilde{M}$ bounding $\widetilde{x}$ and

iii) $(\widetilde{x}, \widetilde{v})$ is equivalent to $(\widetilde{y}, \widetilde{w})$ if $\widetilde{x} = \widetilde{y}$ and

$$I_{c_1}(v\#(-w)) = 0 = I_\omega(v\#(-w))$$

where $v = \pi(\widetilde{v})$ and $w = \pi(\widetilde{w})$.

An element of $\widetilde{\mathcal{L}}\widetilde{M}$ can be viewed as a capped loop with a *tail* attached to it in M (see Figure 3.1). Consider an element $[(\widetilde{x}, \widetilde{v})] \in \widetilde{\mathcal{L}}\widetilde{M}$ and a capped loop (x, v) in M such that $\pi(\widetilde{x}) = x$ and $\pi(\widetilde{v}) = v$. Fix a point $p_0 \in M$ and consider a path in M, t, connecting p_0 to $x(0)$. We say that two objects $\widehat{\widetilde{x}} := (x, v, t)$ and $\widehat{\widetilde{y}} := (y, w, t')$ are equivalent if

i) $x = y$,

ii) $I_{c_1}(v\#(-w)) = 0 = I_\omega(v\#(-w))$ and

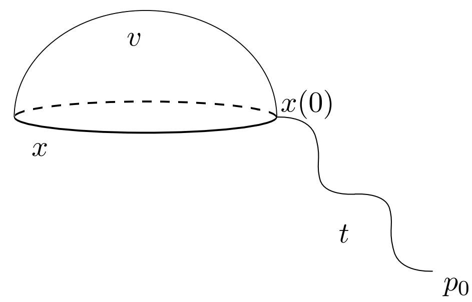

Figure 3.1: Tailed-capped loop

iii) $I_\theta(t\#t') = 0$ where $t\#t'$ is the concatenation of the paths t and t'.

The equivalence class $[\widehat{x}]$ (in M) corresponds to the equivalence class $[(\widetilde{x}, \widetilde{v})]$ (in $\widetilde{M}$).

Let $\phi \in \mathrm{Symp}_0(M)$, ϕ_t a symplectic path connecting $\phi_0 = id$ to $\phi_1 = \phi$ and X_t the vector field associated with ϕ_t (as in (3.3.1)).

Lê and Ono proved in [LO95, Lemma 2.1] that we can deform $\{\phi_t\}$ through symplectic isotopies (keeping the end points fixed) so that the cohomology classes $[\omega(X'_t, \cdot)]$ and $\widetilde{\mathrm{Flux}}(\widetilde{\phi}) = [\theta]$ are the same (where X'_t is the vector field associated with the deformed symplectic isotopies ϕ'_t).

Lemma 3.3.1 (Deformation Lemma)**.** *For $\widetilde{\phi} \in \widetilde{Symp_0}(M, \omega)$, there exist*

i) a smooth path ϕ_t in $Symp_0(M, \omega)$ with $\phi_0 = id$, $\phi_1 = \phi$ and $\phi_{t+1} = \phi_t \circ \phi_1$ and

ii) a Hamiltonian $H_t : M \to M$ with $H_{t+1} = H_t$

such that

$$-\omega(X_t, \cdot) = \theta + dH_t =: \theta_t.$$

The fixed points of $\phi = \phi_1$ are in one-to-one correspondence with one-periodic solutions of the differential equation

$$\dot{x}(t) = X_{\theta_t}(t, x(t)) \tag{3.3.6}$$

where X_{θ_t} is defined by $-\omega(X_{\theta_t}, \cdot) = \theta_t$.

The set of one-periodic solutions of (3.3.6) is denoted by $\mathcal{P}(\theta_t)$ and coincides with the zero set of the closed one-form defined on the loop space of M, $\mathcal{L}M$, by

$$\alpha_{\{\phi_t\}}(x, \xi) = \int_0^1 \omega(\dot{z}, \xi) + \theta_t(z(t))(\xi) dt \tag{3.3.7}$$

where $x \in \mathcal{L}M$ and $\xi \in T_x\mathcal{L}M$ (i.e. ξ is a tangent vector field along the loop x).

By the Deformation Lemma 3.3.1, there exists a periodic Hamiltonian

$$\widetilde{H}\colon S^1 \times \widetilde{M} \to \mathbb{R} \quad \text{such that} \quad d\widetilde{H}_t = \pi^*\theta_t \quad (t \in S^1) \tag{3.3.8}$$

where $\theta_t := -\omega(X_t, \cdot)$. The time-dependent Hamiltonian flow on $\widetilde{M}$ generated by $\widetilde{H_t}$ is the pull back of the original symplectic flow on M. In particular, the set of contractible periodic solutions of the Hamiltonian system $\mathcal{P}(\widetilde{H})$ is the set $\pi^{-1}(\mathcal{P}(\theta_t))$ and $\widetilde{\mathcal{P}}(\widetilde{H}) := \widetilde{j}^{-1}(\mathcal{P}(\widetilde{H}))$ is the critical set of the functional:

$$\mathcal{A}_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]) = -\int_D v^*\omega + \int_0^1 \widetilde{H}(t, \widetilde{x}(t))dt \tag{3.3.9}$$

where $\pi(\widetilde{v}) = v$ (recall j is given by (3.3.5)). The action functional is homogeneous with respect to iterations

$$\mathcal{A}_{\widetilde{H}^{\natural k}}([\widetilde{x}, \widetilde{v}]^k) = k\mathcal{A}_{\widetilde{H}}([\widetilde{x}, \widetilde{v}])$$

where $[\widetilde{x}, \widetilde{v}]^k$ is the k-th iteration of $[\widetilde{x}, \widetilde{v}]$ and depends on the equivalence class of the capping $\widetilde{v}$ of the loop $\widetilde{x}$:

$$\mathcal{A}_{\widetilde{H}}((\widetilde{x}, \widetilde{v})\#A) = \mathcal{A}_{\widetilde{H}}(\widetilde{x}, \widetilde{v}) + I_\omega(A)$$

where $A \in \pi_2(M)$.

3.3.2 Mean index and the augmented action

A one-periodic orbit x of $\phi \in \mathrm{Symp}(M, \omega)$ (i.e. a periodic solution of (3.3.6)) is said to be *non-degenerate* if

$$d\phi_{x(0)} : T_{x(0)}M \to T_{x(0)}M$$

has no eigenvalues equal to one and x is called hyperbolic if none of the eigenvalues has absolute value equal to one. Observe that a hyperbolic periodic orbit is non-degenerate. We say that ϕ (or H when $\phi = \varphi_H$) is non-degenerate if all its one-periodic orbits are non-degenerate.

Let (x, v) be a capped periodic orbit of ϕ. Using a trivialization of x^*TM arising from the capping v, the linearized flow along x

$$d\phi_t\colon T_{x(0)}M \to T_{x(t)}M$$

can be viewed as a symplectic path $\Phi\colon [0,1] \to \mathrm{Sp}(2n)$. The mean index of (x, v) is defined by $\Delta_\phi(x, v) := \Delta_\phi(\Phi)$; see [SZ92].

Recall that the time-dependent Hamiltonian flow on $\widetilde{M}$ generated by $\widetilde{H}$ (obtained in Lemma 3.3.1) is the pull back of the original symplectic flow on M, hence a periodic orbit $\widetilde{x} \in \widetilde{\mathcal{P}}(\widetilde{H})$ is non-degenerate if and only if $\pi(\widetilde{x})$ is non-degenerate as a periodic orbit of ϕ and it is hyperbolic if and only if $\pi(\widetilde{x})$ is hyperbolic as a periodic orbit of ϕ. Moreover,

$$\Delta_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]) = \Delta_\phi((x, v))$$

and it has the following properties:

$$\Delta_{\widetilde{H}^{\natural k}}([\widetilde{x}, \widetilde{v}]^k) = k\Delta_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]),$$

and

$$\Delta_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]\#A) = \Delta_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]) - I_{c_1}(A)$$

where $A \in \pi_2(M)$. The *augmented action* is defined by

$$\widetilde{\mathcal{A}}_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]) := \mathcal{A}_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]) - \frac{\lambda}{2}\Delta_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]). \tag{3.3.10}$$

Notice that the augmented action is independent of the capping $\widetilde{v}$ and it is homogeneous with respect to iteration, i.e.

$$\widetilde{\mathcal{A}}_{\widetilde{H}^{\natural k}}([\widetilde{x}, \widetilde{v}]^k) = k\widetilde{\mathcal{A}}_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]).$$

3.4 Coisotropic submanifolds and the Maslov index

We are particularly interested in closed coisotropic submanifolds. Recall, from Section 2.1, that a N^{2n-k} is a coisotropic submanifold if $(T_pN)^\omega \subseteq T_pN$ for each $p \in N$ and that these submanifolds admit a foliation $\mathcal{F}$ defined by $T\mathcal{F} = TN^\omega$. In this section, we recall the definition of the Maslov index in this setting and discuss some properties of this index for stable coisotropic submanifolds.

3.4.1 Definition of the coisotropic the Maslov index

The objective of this section is to revisit the definition of the coisotropic Maslov index and give a direct proof of the fact that it is well defined. As mentioned in the introduction, similar notions of index are originally considered in [Gin11, Zil09].

First, we define the Maslov index of a loop of coisotropic subspaces of $(\mathbb{R}^{2n}, \omega_0)$ where $\omega_0 := dx \wedge dy$ and (x, y) are the coordinates in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Then, we define the Maslov index of a capped loop lying in a coisotropic submanifold and tangent to the characteristic foliation of the coisotropic submanifold. We start by recalling the definition of the mean index given in [SZ92]. For its construction, we need a collection of mappings given by the following theorem:

Theorem 3.4.1 ([SZ92]). *There is a unique collection of continuous mappings*

$$\rho\colon \operatorname{Sp}(V, \omega) \to S^1$$

(one for every symplectic vector space V) satisfying the following conditions:

- *Naturality: If $T\colon (V_1, \omega_1) \to (V_2, \omega_2)$ is a symplectic isomorphism (that is, $T^*\omega_2 = \omega_1$), then*

$$\rho(T\varphi T^{-1}) = \rho(\varphi)$$

for $\varphi \in \operatorname{Sp}(V_1, \omega_1)$.

- *Product: If $(V, \omega) = (V_1 \times V_2, \omega_1 \times \omega_2)$, then*

$$\rho(\varphi) = \rho(\varphi_1)\rho(\varphi_2)$$

for $\varphi \in \operatorname{Sp}(V, \omega)$ of the form $\varphi(z_1, z_2) = (\varphi_1 z_1, \varphi_2 z_2)$ where $\varphi_i \in \operatorname{Sp}(V_i, \omega_i)$.

- *Determinant: If $\varphi \in \operatorname{Sp}(2n) \cap O(2n) \simeq U(n)$ is of the form*

$$\varphi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix},$$

then

$$\rho(\varphi) = \det(X + iY)$$

- *Normalization: If φ has no eigenvalues on the unit circle, then*

$$\rho(\varphi) = \pm 1$$

Remark 3.4.2. The map $\rho\colon \operatorname{Sp}(2n) \to S^1$ is given explicitly by

$$\rho(\varphi) := (-1)^{m_0} \prod_{\lambda \in \sigma(\varphi) \cap S^1 \setminus \{-1,1\}} \lambda^{m_+(\lambda)}$$

where $\sigma(\varphi)$ is the set of eigenvalues of φ, m_0 is given by

$$m_0 := \#\big\{\{\lambda, \lambda^{-1}\} \mid \lambda \in \sigma(\varphi) \cap \mathbb{R}^-\big\}$$

and $m_+(\lambda)$ is some multiplicity assigned to an eigenvalue $\lambda \in S^1\backslash\{-1, 1\}$; see page 1316 in [SZ92] for the details of the definition of m_+.

Notice that only the eigenvalues of φ on the unit circle and on the negative real axis contribute to $\rho(\varphi)$.

Then, the definition of the mean index of a path $\Psi\colon [0,1] \to \mathrm{Sp}(2n)$ is given by:

Definition 3.4.3 (*Mean Index*; [SZ92])**.** Let $\Psi\colon [0,1] \to \mathrm{Sp}(2n)$ be a path of symplectic matrices. Then choose a function $\alpha\colon [0,1] \to \mathbb{R}$ such that $\rho(\Psi_t) = e^{\pi i \alpha(t)}$. The *Mean index* of the path Ψ is defined by

$$\Delta(\Psi) := \alpha(1) - \alpha(0)$$

The mean index Δ has the following properties:

1. Homotopy Invariance: $\Delta(\Psi)$ is an invariant of homotopy of Ψ with fixed end points

2. Concatenation: Δ is additive with respect to concatenation of paths:

$$\Delta(\Psi) = \Delta(\Psi|_{[0,a]}) + \Delta(\Psi|_{[a,1]})$$

   where $0 < a < 1$

3. Loop: $\Delta(\varphi\Psi) = \Delta(\varphi) + \Delta(\varphi_0\Psi)$ if either φ or Ψ is a loop

4. Naturality: $\Delta(T\Psi T^{-1}) = \Delta(\Psi)$ where $T\colon (V_1, \omega_1) \to (V_2, \omega_2)$ is a symplectic isomorphism and $\Psi \in \mathrm{Sp}(V_1, \omega_1)$

5. Product: $\Delta(\Psi) = \Delta(\Psi_1) + \Delta(\Psi_2)$ where $\Psi \in \mathrm{Sp}(V = V_1 \times V_2, \omega = \omega_1 \times \omega_2)$ is given by $\Psi(z_1, z_2) = (\Psi_1 z_1, \Psi_2 z_2)$ where $\Psi_i \in \mathrm{Sp}(V_i, \omega_i)$.

The Maslov index of a loop of coisotropic subspaces is given (up to a sign) as the mean index of a certain path of symplectic matrices.

Definition 3.4.4 (*Maslov Index for Coisotropic Subspaces*)**.** Consider

$$\mathcal{C} = (\mathcal{C}_t)_{t\in[0,1]}$$

an oriented loop of coisotropic subspaces of $(\mathbb{R}^{2n}, \omega_0)$ and

$$H_t\colon \mathcal{C}_0/\mathcal{C}_0^{\omega_0} \to \mathcal{C}_t/\mathcal{C}_t^{\omega_0}$$

a path of symplectic linear maps. Recall that a loop $\mathcal{C}$ is oriented if one can orient the space $\mathcal{C}_t$ (continuous in t) so that $\mathcal{C}_0$ and $\mathcal{C}_1$ have the same orientation. Pick a path

$$\Psi\colon [0,1] \to \mathrm{Sp}(2n) \quad \text{satisfying} \quad \Psi_0 = Id,\ \Psi_t(\mathcal{C}_0) = \mathcal{C}_t \text{ and } \Psi_t\Big|_{C_0/C_0^{\omega}} = H_t \qquad (3.4.1)$$

and define the real valued index $\mu\colon \mathfrak{C} \to \mathbb{R}$ by

$$\mu(\mathcal{C}, H) := -\Delta(\Psi),$$

where $\mathfrak{C}$ is the set of loops of coisotropic subspaces of $(\mathbb{R}^{2n}, \omega_0)$.

If the loop $\mathcal{C}$ is not oriented, we define the Maslov index $\mu(\mathcal{C}, H)$ as half of the Maslov index of the loop obtained by traversing the initial loop twice.

Proposition 3.4.5. *The Maslov index given in Definition 3.4.4 is well defined.*

Proof. We prove this proposition in three steps by considering the following cases:

1. The loop $\mathcal{C}$ is constant with $\mathcal{C}_t = L_0$ a fixed Lagrangian subspace of $(\mathbb{R}^{2n}, \omega_0)$.

2. The loop $\mathcal{C}$ is constant with $\mathcal{C}_t = \mathcal{C}_0$ a fixed coisotropic subspace of $(\mathbb{R}^{2n}, \omega_0)$.

3. General case: $\mathcal{C}$ is a loop of coisotropic subspaces of $(\mathbb{R}^{2n}, \omega_0)$.

Step 1: Assume, without loss of generality, that $\mathcal{C}$ is the constant *horizontal loop* $L_0 := \{(x, y) \in \mathbb{R}^{2n} \mid y = 0\}$. Then consider $\Psi\colon [0,1] \to \mathrm{Sp}(2n)$ as in (3.4.1) and notice that since $\mathcal{C}_t = L_0$ is Lagrangian, $H \equiv 0$. For $t \in [0,1]$, we have that Ψ_t fixes the lagrangian L_0 if and only if it is of the form

$$\begin{pmatrix} A_t & B_t \\ 0 & A_t^{-T} \end{pmatrix} \text{ where } B_t^T A_t^T = A_t^{-1} B_t.$$

This path is homotopic to the concatenation of two symplectic paths of the form:

$$\Psi_t^{'} = \begin{pmatrix} \tilde{A}_t & 0 \\ 0 & \tilde{A}_t^{-T} \end{pmatrix} \quad \text{and} \quad \Psi_t^{''} = \begin{pmatrix} \tilde{A}_1 & \tilde{B}_t \\ 0 & \tilde{A}_1^{-T} \end{pmatrix}$$

where we essentially first travel along Ψ_t with $B_t = 0$ and then, when we reach

$$\begin{pmatrix} A_1 & B_0 = 0 \\ 0 & A_1^{-T} \end{pmatrix},$$

we build up B_t from 0 to B_1.

Since $\Psi_t^{''}$ has constant eigenvalues, $\Delta(\Psi^{''}) = 0$. Hence, by property (2), the mean index of Ψ is equal to the mean index $\Psi^{'}$.

Suppose that $\tilde{A}_t$ is diagonalizable, i.e., it can be written in the form

$$\tilde{A}_t = P_t \underbrace{\begin{pmatrix} (A_1)_t & & 0 \\ & \ddots & \\ 0 & & (A_n)_t \end{pmatrix}}_{=:\, D_t} (P_t)^{-1} \tag{3.4.2}$$

where $P_t \in \mathrm{Sp}(2n)$ and each block $(A_j)_t$ corresponds to an eigenvalue $(\lambda_j)_t$ of $\tilde{A}_t$. Then, in this case,

$$\begin{pmatrix} \tilde{A}_t & 0 \\ 0 & \tilde{A}_t^{-T} \end{pmatrix} = \begin{pmatrix} P_t & 0 \\ 0 & P_t^{-T} \end{pmatrix} \underbrace{\begin{pmatrix} D_t & 0 \\ 0 & D_t^{-T} \end{pmatrix}}_{=:\, \Gamma_t} \begin{pmatrix} P_t & 0 \\ 0 & P_t^{-T} \end{pmatrix}^{-1}$$

and, by the naturality property of the map ρ, we have $\rho(\Psi_t^{'}) = \rho(\Gamma_t)$ for all $t \in [0,1]$.

Claim 3.4.6. *For all $t \in [0,1]$, we have $\rho(\Gamma_t) = 1$.*

Proof. For the sake of simplicity, we will drop, for now, the subscript t in the notation. By Remark 3.4.2, we have

$$\begin{aligned} \rho(\Gamma) &:= (-1)^{m_0} \prod_{\lambda \in \sigma(\Gamma) \cap S^1 \setminus \{-1,1\}} \lambda^{m_+(\lambda)} \\ &= (-1)^{m_0} \prod_{\substack{\lambda \in \sigma(\Gamma) \cap S^1 \setminus \{-1,1\} \\ \mathrm{Im}\lambda > 0}} \lambda^{m_+(\lambda)} \, \bar{\lambda}^{m_+(\bar{\lambda})} \\ &= (-1)^{m_0} \prod_{\substack{\lambda \in \sigma(\Gamma) \cap S^1 \setminus \{-1,1\} \\ \mathrm{Im}\lambda > 0}} \lambda^{m_+(\lambda) - m_+(\bar{\lambda})} \end{aligned} \tag{3.4.3}$$

where $\sigma(\Gamma)$ is the spectrum of Γ. Recall that only the eigenvalues of Γ on the unit circle and on the negative real axis contribute to $\rho(\Gamma)$. Regarding the eigenvalues on S^1, it can be proved, directly from the definition of m_+, that $m_+(\lambda) = m_+(\overline{\lambda})$. Hence, using the notation with the subscript t, we obtain by (3.4.3) that $\rho(\Gamma_t) = (-1)^{(m_0)_t}$, for each $t \in [0,1]$, where

$$(m_0)_t := \#\big\{\{\lambda_t, \lambda_t^{-1}\} \in \sigma(\Gamma_t) \colon \lambda_t \in \mathbb{R}^-\big\} = \#\big\{\lambda_t \in \sigma(D_t) \colon \lambda_t \in \mathbb{R}^-\big\}.$$

The last equality follows from the fact that λ_t is an eigenvalue of D_t if and only if λ_t and λ_t^{-1} are eigenvalues of Γ_t. Since D_t is continuous in t and $\det(D_t) \neq 0$, the signs of $\det(D_0)$ and $\det(D_1)$ are the same. The determinant of D_t is given by

$$\det(D_t) = \prod_{\lambda_t \in \mathbb{R}^-} \lambda_t \underbrace{\prod_{\lambda_t \in \mathbb{R}^+} \lambda_t}_{>0} \underbrace{\prod_{\lambda_t \in \mathbb{C}\setminus\mathbb{R}} \lambda_t}_{>0}$$

where the products run over $\lambda_t \in \sigma(D_t)$. Then the sign of $\det(D_t)$ is determined by the number (mod 2) of the real negative eigenvalues of D_t and we have $(-1)^{(m_0)_0} = (-1)^{(m_0)_t}$ for all $t \in [0,1]$. Since, by (3.4.1) $D_0 = Id$ the result follows immediately. □

Hence, we have proved that, under the assumption (3.4.2), $\rho(\Psi_t^{'}) = 1$ for a fixed $t \in [0,1]$. Since the set of diagonalizable matrices is dense in the set of matrices, the result holds for a "general" $\Psi_t^{'}$. It follows that $\Delta(\Psi^{'}) = 0$ and hence we have $\Delta(\Psi) = 0$.

Step 2: Consider $\Psi\colon [0,1] \to \mathrm{Sp}(2n)$ as in (3.4.1) and the symplectic decomposition of $\mathbb{R}^{2n}$:

$$\mathbb{R}^{2n} = (\mathbb{R}^{2n}/\mathcal{C}_0 \oplus \mathcal{C}_0^{\omega_0}) \oplus \mathcal{C}_0/\mathcal{C}_0^{\omega_0}. \tag{3.4.4}$$

Since $\Psi_t \in \mathrm{Sp}(2n)$, $\Psi_t(V) = V$ and $\Psi_t(\mathcal{C}_0/\mathcal{C}_0^{\omega_0}) = \mathcal{C}_0/\mathcal{C}_0^{\omega_0}$, the path Ψ_t has the form

$$\begin{bmatrix} (\Psi_t)|_V & 0 \\ 0 & H_t \end{bmatrix}$$

with respect to decomposition (3.4.4), where $V := \mathbb{R}^{2n}/\mathcal{C}_0 \oplus \mathcal{C}_0^{\omega_0}$. By property (5) of the mean index,

$$\Delta(\Psi) = \Delta(\Psi|_V) + \Delta(H).$$

Since V is symplectic and $\mathcal{C}_0^{\omega_0}$ is Lagrangian in V, we have by step 1 that $\Delta(\Psi|_V) = 0$ and hence $\Delta(\Psi) = \Delta(H)$. Therefore, the mean index $\Delta(\Psi)$ only depends on the mean index of H and the result is proved for case (2).

Step 3: Let $\Psi\colon [0,1] \to \mathrm{Sp}(2n)$ be a path as in (3.4.1) and consider a loop $\Phi\colon [0,1] \to \mathrm{Sp}(2n)$ which depends only on $\mathcal{C}$ and satisfies $\Phi_t(\mathcal{C}_0) = \mathcal{C}_t$. Recall that $\mathcal{C}$ is an orientable loop and hence we may consider such a loop Φ. Define the path $\tilde{\Psi}\colon [0,1] \to \mathrm{Sp}(2n)$ by $\tilde{\Psi}_t := \Phi_t^{-1}\Psi_t$ which satisfies $\tilde{\Psi}_t(\mathcal{C}_0) = \mathcal{C}_t$ for all $t \in [0,1]$. By step 2, $\Delta(\tilde{\Psi}) = \Delta(\tilde{H})$, where $\tilde{H}_t\colon \mathcal{C}_0/\mathcal{C}_0^{\omega_0} \to \mathcal{C}_t/\mathcal{C}_t^{\omega_0}$ is given by

$$\tilde{H}_t = \Phi_t^{-1}\big|_{(\mathcal{C}_t/\mathcal{C}_t^{\omega_0})}\Psi_t\big|_{(\mathcal{C}_0/\mathcal{C}_0^{\omega_0})} = \Phi_t^{-1}\big|_{(\mathcal{C}_t/\mathcal{C}_t^{\omega_0})}H_t.$$

Since Φ is a loop, then by property (3) of the mean index we have $\Delta(\tilde{\Psi}) = \Delta(\Phi^{-1}\Psi) = \Delta(\Phi^{-1}) + \Delta(\Psi)$ and $\Delta(\tilde{H}) = \Delta\big(\Phi^{-1}\big|_{(\mathcal{C}_0/\mathcal{C}_0^{\omega})}\big) + \Delta(H)$. Hence

$$\Delta(\Psi) = \Delta(\tilde{H}) - \Delta(\Phi^{-1})$$

which only depends on H and on Φ. Since Φ_t only depends on $\mathcal{C}_t$, $\Delta(\Psi)$ only depends on H and $\mathcal{C}$. Therefore, the Maslov index $\mu(\mathcal{C}, H) := -\Delta(\Psi)$ depends only on the loop $\mathcal{C} = (\mathcal{C}_t)$ and the linear map H and not on the choice of the path Ψ as long as it satisfies the properties in (3.4.1). □

We, now, define the Maslov index of a capped loop lying in a coisotropic submanifold and tangent to the characteristic foliation of the coisotropic submanifold.

Let (M,ω) be a symplectic manifold, N^{2n-k} a coisotropic submanifold of (M,ω) and $\mathcal{F}$ its characteristic foliation. Consider $x\colon S^1 \to N$ a loop in N tangent to $\mathcal{F}$ and $u\colon D^2 \to M$ a capping of the loop x in M. We have the symplectic vector bundle decomposition

$$TW\big|_N = (TM/TN \oplus T\mathcal{F}) \oplus TN/T\mathcal{F}.$$

Assume $x^*T\mathcal{F}$ is orientable and hence trivial. Denote by ξ a trivialization of $x^*T\mathcal{F}$:

$$x^*T\mathcal{F} \overset{\xi}{\cong} S^1 \times T_{x(0)}\mathcal{F}.$$

Moreover, we have the following isomorphism

$$TM/TN \cong T^*\mathcal{F},$$

and hence $\xi \oplus \xi^*$ can be viewed as a family of symplectic maps

$$\Xi_t\colon TM/TN_{x(0)} \oplus T_{x(0)}\mathcal{F} \to TM/TN_{x(t)} \oplus T_{x(t)}\mathcal{F}.$$

Denote by $H_t\colon (TN/T\mathcal{F})_{x(0)} \to (TN/T\mathcal{F})_{x(t)}$ the holonomy along x. The capping u gives rise to a symplectic trivialization, unique up to homotopy, of x^*TM. Using such a trivialization, the map $\Xi_t \oplus H_t$ can be viewed as a path

$$\Psi\colon [0,1] \to \mathrm{Sp}(2n)$$

which, up to some identifications, satisfies

$$\Psi_0 = Id, \quad \Psi_t(T_{x(0)}N) = T_{x(t)}N \text{ and } \Psi_t\big|_{(TN/T\mathcal{F})_{x(0)}} = H_t. \tag{3.4.5}$$

Definition 3.4.7 (*Maslov Index of a Capped Loop*)**.** The *Maslov index* of (x,u) is defined by

$$\mu(x,u) := -\Delta(\Psi).$$

If $x^*T\mathcal{F}$ is not orientable, we define $\mu(x,u)$ as $\mu(x^2,u^2)/2$ where (x^2,u^2) is the double cover of (x,u).

Remark 3.4.8. By Proposition 3.4.5, $\mu(x,u)$ is independent of the trivialization ξ. However it may depend on the capping u. We give some properties of the coisotropic Maslov index (cf. [Gin11]):

- Homotopy Invariance: $\mu(x,u)$ is invariant under a homotopy of x in a leaf of $\mathcal{F}$.
- Recapping: $\mu(x,u\#A) = \mu(x,u) + 2\langle c_1, A\rangle$ where $u\#A$ is the notation for the recapping of (x,u) by a 2-sphere A.
- Homogeneity: $\mu(x^k,u^k) = k\mu(x,u)$ where (x^k,u^k) is the k-fold cover of (x,u).

3.4.2 Stable coisotropic submanifolds

In this section, we give the definition and some properties of stable coisotropic submanifolds. This class of coisotropic submanifolds was introduced in [Bol96, Bol98] and is defined as follows.

The submanifold N is said to be *stable* if there exist k one-forms $\alpha_1,\dots,\alpha_k$ on N such that

$$\mathrm{Ker}\, d\alpha_i \supset \mathrm{Ker}\, \omega_N \quad \text{for all } i = 1,\dots,k$$

and

$$\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \omega_N^{n-k} \neq 0 \quad \text{on N.}$$

Notice that this condition is rather restrictive. For instance, a stable Lagrangian submanifold is necessarily a torus and a stable coisotropic submanifold is automatically orientable. Thus, examples of stable coisotropic submanifolds include Lagrangian tori and also contact hypersurfaces. Moreover, the stability condition is closed under products. For more details, we refer the reader to [Bol96, Bol98, Gin07, Ush11].

As a consequence of the Weinstein symplectic neighborhood theorem, we obtain tubular neighborhoods of stable coisotropic submanifolds:

Proposition 3.4.9 ([Bol96, Bol98]). *Let N^{2n-k} be a closed stable coisotropic submanifold of (M^{2n}, ω). Then, for $r > 0$ sufficiently small, there exists a neighborhood of N in M which is symplectomorphic to*

$$U_r = \left\{(q, p) \in N \times \mathbb{R}^k \colon |p| < r\right\}$$

equipped with the symplectic form

$$\omega = \omega_N + \sum_{j=1}^{k} d(p_j \alpha_j)$$

where $p = (p_1, \ldots, p_k)$ are the coordinates in $\mathbb{R}^k$ and $|p|$ is the Euclidean norm of p.

Thus, such a neighborhood is foliated by a family of coisotropic submanifolds $N_p = N \times \{p\}$ with $p \in B_r^k := \left\{p \in \mathbb{R}^k \colon |p| < r\right\}$ and a leaf of the characteristic foliation on N_p projects onto a leaf of the characteristic foliation on N.

Furthermore, we have

Proposition 3.4.10 ([Bol96, Bol98, Gin07]). *Let N^{2n-k} be a stable coisotropic submanifold of (M^{2n}, ω). Then*

- *the leaf-wise metric $(\alpha_1)^2 + \ldots + (\alpha_k)^2$ on $\mathcal{F}$ is leaf-wise flat;*
- *the Hamiltonian flow of $\rho = (p_1^2 + \ldots + p_k^2)/2$ is the leaf-wise geodesic flow of this metric.*

3.4.3 Stable coisotropic submanifolds and the Maslov index

Consider $\bar{x} = (x, u)$ a non-trivial (capped) periodic orbit of the Hamiltonian flow of ρ. Then, as a consequence of Proposition 3.4.5, we obtain that the mean index

$\Delta_\rho(\bar{x})$ of a periodic orbit $\bar{x}$ of a leaf-wise geodesic flow on N is equal to, up to a sign, the coisotropic Maslov index of the projection of $\bar{x}$ on N. More precisely,

$$\mu(\pi(x), \hat{u}) = -\Delta_\rho(x, u) \tag{3.4.6}$$

where $\hat{u}$ is the capping of the orbit $\pi(x)$ given by the capping u of x together with the cylinder obtained from the projection of x on N; see Figure 3.2.

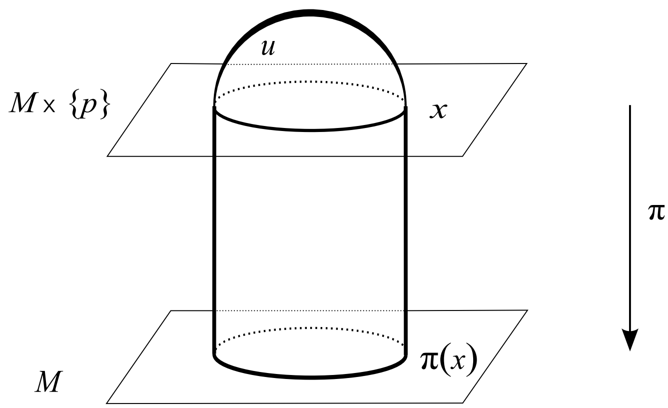

Figure 3.2: Capping $\hat{u}$.

The following result establishes bounds on the Conley-Zehnder index of a small non-degenerate perturbation of a capped periodic orbit (x, u) of ρ which goes beyond (3.2.2). (Here as above N is stable.)

Proposition 3.4.11 ([Gin11]). *Let ρ' be a small perturbation of the Hamiltonian ρ defined in Proposition 3.4.10 and x' a non-degenerate periodic orbit of ρ' (with a capping u') close to a non-trivial periodic orbit x of ρ (with a capping u). Then*

$$\Delta_\rho(x, u) - n \leq \mu_{\mathrm{CZ}}((x, u)') \leq \Delta_\rho(x, u) + (n - k)$$

where $(x, u)' := (x', u')$.

Chapter 4

Floer homology

Symplectic topology offers a powerful tool for finding periodic orbits, Floer homology. The Conley-Zehder index mentioned in Section 3.2 is used for the grading of (Hamiltonian) Floer homology which is the main tool used to prove our first main result (Theorem 2.1.3). This version of Floer theory is described in Section 4.1 (for more details see [Sal99]).

The main tool used to prove the result on periodic orbits of symplectomorphisms is a variation of (Hamiltonian) Floer homology. In Section 4.2, we discuss Floer homology for symplectomorphisms, that is, Floer-Novikov homology. The quantum homology (defined in Section 4.3.1) acts on the Floer-Novikov homology. This action is described in Section 4.3.2 and needed in the statement of our second main result (Theorem 2.2.1).

4.1 Floer homology for Hamiltonians

4.1.1 Definition of Floer homology

Let us recall the definition of the Floer homology for a non-degenerate Hamiltonian H. The Floer chain groups are generated by the capped one-periodic orbits of H and graded by the Conley-Zehnder index. The boundary operator is defined by counting solutions of the Floer equation

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = -\nabla H_t(u)$$

with finite energy. Floer trajectories for a non-degenerate Hamiltonian H with finite energy converge to periodic orbits $\bar{x}$ and $\bar{y}$ as $s \to \pm\infty$ and satisfy

$$E(u) = \mathcal{A}_H(\bar{x}) - \mathcal{A}_H(\bar{y}) = \int_{-\infty}^{\infty} \int_{S^1} \left|\left|\frac{\partial u}{\partial s}\right|\right|^2 dtds.$$

The boundary operator counts Floer trajectories converging to periodic orbits y and x as $s \to \pm\infty$ and satisfying the condition [(capping of $\bar{x}$)$\#u$] = [capping of $\bar{y}$].

This construction extends by continuity from non-degenerate Hamiltonians to all Hamiltonians; see [Sal99, SZ92] for more details.

Remark 4.1.1. The total Floer homology is independent of the Hamiltonian and, up to a shift of the grading and the effect of recapping, is isomorphic to the homology of M. More precisely, we have

$$HF_*(H) \cong H_{*+n}(M) \otimes \Lambda$$

as graded Λ-modules; see, for instance, [GG12a, MS12] and references therein for details on the definition of the Novikov ring Λ or Section 4.3.1 for the description of Λ in the case where M is strictly monotone. In particular, the fundamental class $[M]$ can be viewed as an element of $HF_n(H)$.

Remark 4.1.2. To ensure that the Floer differential is defined, we either assume M to be weakly monotone (see e.g. [HS95, MS12, Ono95, Sal99]) or utilize the machinery of virtual cycles (see e.g. [FO99, FOOO09, LT98]). In our main result, one of the possible conditions on M is negative monotonicity. In this case, M^{2n} is weakly monotone if and only if $\mathcal{N} \geq n-2$, where $\mathcal{N}$ is the minimal Chern number.

4.1.2 Filtered Floer homology

Let us recall the definition of the filtered Floer homology for a non-degenerate Hamiltonian H (see e.g. [GG09] and references therein). The (total) chain Floer complex $CF_*(H) =: CF_*^{(-\infty,\infty)}(H)$ admits a filtration by $\mathbb{R}$. For each $b \in (-\infty, \infty]$ outside $\mathcal{S}(H)$, the chain complex $CF_*^{(-\infty,b)}(H)$ is generated by the capped one-periodic orbits of H with action $\mathcal{A}_H$ less than b. For $-\infty \leq a < b \leq \infty$ outside $\mathcal{S}(H)$, set

$$CF_*^{(a,b)}(H) := CF_*^{(-\infty,b)}(H)/CF_*^{(-\infty,a)}(H).$$

The boundary operator $\partial\colon CF_*(H) \to CF_{*-1}(H)$ descends to $CF_*^{(a,b)}(H)$ and hence the *filtered Floer homology* $HF_*^{(a,b)}(H)$ is defined.

This construction also extends by continuity to all Hamiltonians. For an arbitrary (one-periodic in time) Hamiltonian H on M, set

$$HF_*^{(a,b)}(H) := HF_*^{(a,b)}(\tilde{H}) \tag{4.1.1}$$

where $\tilde{H}$ is a non-degenerate perturbation of H and $-\infty \leq a < b \leq \infty$ are outside $\mathcal{S}(H)$.

When $a < b < c$, we have $CF_*^{(b,c)}(H) = CF_*^{(a,c)}(H)/CF_*^{(a,b)}(H)$ and thus obtain the long exact sequence

$$\ldots \to HF_*^{(a,b)}(H) \to HF_*^{(a,c)}(H) \to HF_*^{(b,c)}(H) \to HF_{*-1}^{(a,b)}(H) \to \ldots. \tag{4.1.2}$$

4.1.3 Homotopy maps

By definition, a *homotopy* of Hamiltonians on M is a family of (one-periodic in time) Hamiltonians H^s smoothly parameterized by $s \in \mathbb{R}$ and such that $H^s \equiv H^0$ when s is near $-\infty$ and $H^s \equiv H^1$ when s is near ∞; see [Gin07] and references therein for the definitions, properties and proofs.

Set

$$E := \int_{-\infty}^{\infty} \int_{S^1} \max_M \partial_s H_t^s dtds.$$

For every $C \geq E$, the homotopy induces a map of the filtered Floer homology, which we denote by $\Psi_{H^0 H^1}$, shifting the action filtration by C:

$$\Psi_{H^0 H^1} \colon HF_*^{(a,b)}(H^0) \to HF_*^{(a+C,b+C)}(H^1). \tag{4.1.3}$$

Example 4.1.3. Let H^s be an *increasing linear homotopy* from H^0 and H^1, i.e.

$$H^s = (1 - f(s))H^0 + f(s)H^1$$

where $f \colon \mathbb{R} \to [0,1]$ is a monotone increasing compactly supported function equal to zero near $-\infty$ and equal to one near ∞. Since

$$E \leq \int_{S^1} \max_M (H^1 - H^0)\, dt, \tag{4.1.4}$$

we have the homomorphism $\Psi_{H^0 H^1}$ for every $C \geq \displaystyle\int_{S^1} \max_M (H^1 - H^0)\, dt$.

Furthermore, we have the following continuity property for filtered homology: let (a^s, b^s) be a family (smooth in s) of non-empty intervals such that a^s and b^s are outside $\mathcal{S}(H^s)$ for some homotopy H^s and such that (a^s, b^s) is equal to (a^0, b^0) when s is near $-\infty$ and equal to (a^1, b^1) when s is near ∞. Then there exists an isomorphism of homology

$$HF^{(a_0,b_0)}(H^0) \xrightarrow{\cong} HF^{(a_1,b_1)}(H^1). \tag{4.1.5}$$

When the interval is fixed and the homotopy is monotone decreasing, the isomorphism (4.1.5) is in fact $\Psi_{H^0H^1}$ which in general is not the case.

4.1.4 Kerman's "pinned" action selector

One important tool used in the proof of Theorem 2.1.3 is an action selector defined for "pinned" Hamiltonians. This tool was first introduced in [Ker09, KŞ10] for a class of Hamiltonians and manifolds which are somewhat different from those we work with. However, the definition of the action selector is essentially the same. In this section, we describe this action selector and a *special orbit* associated with it.

Let M be a rational symplectic manifold and U an open neighborhood of the coisotropic submanifold N of M. Consider $K\colon M \to \mathbb{R}$ a compactly supported autonomous Hamiltonian such that the neighborhood U contains the support of K, $\operatorname{supp} K$, and U is displaced by a Hamiltonian H. We may assume H is non-negative with minimum value equal to zero. Suppose that K is constant on N where it attains its maximum value $\max K =: L$, the maximum value L is greater than $||H||$ and that K is strictly decreasing and C^2-close to L on a small neighborhood of N.

Consider the quotient map $j_K\colon HF_n(K) \to HF_n^{(L-\delta,L+\delta)}(K)$ and define the element $[\max_K] \in HF_n^{(L-\delta,L+\delta)}(K)$ as

$$[\max_K] := j_K([M])$$

where the fundamental class $[M]$ is seen as an element of $HF_n(K)$; recall Remark 4.1.1.

Definition 4.1.4 (*"Pinned" Action Selector*)**.** For $\delta > 0$ small and $\alpha > L+\delta$, consider the inclusion map

$$\iota_\alpha\colon HF_n^{(L-\delta,\, L+\delta)}(K) \hookrightarrow HF_n^{(L-\delta,\, \alpha)}(K).$$

Define

$$c(K) := \inf_{\delta>0} \inf \left\{\alpha > L+\delta \colon \iota_\alpha([\max_K]) = 0\right\}.$$

We have $c(K) \in \mathcal{S}(K)$ and $c(K) = \mathcal{A}_K(\bar{x})$ for some capped orbit $\bar{x}$ which is called a *special one-periodic orbit.*

Claim 4.1.5. *There exists $\mathcal{C} \in HF_{n+1}^{(L+\delta,\,\infty)}(K)$ such that $\partial\mathcal{C} = [\max_K]$ where*

$$\partial\colon HF_{n+1}^{(L+\delta,\,L+\delta+||H||)}(K) \to HF_n^{(L-\delta,\,L+\delta)}(K)$$

is the connecting differential in the long exact sequence (4.1.2) (with $a = L-\delta,\ b = L+\delta$ and $c = L+\delta+||H||$).

Proof. For $\delta > 0$ sufficiently small, namely such that $L - \delta > ||H||$, consider the following commutative diagram:

$$\begin{array}{ccc}
 & & \mathrm{HF}_{n+1}^{(L+\delta,\,L+\delta+||H||)}(K) \\
 & & \downarrow \partial \\
 & & \mathrm{HF}_n^{(L-\delta,\,L+\delta)}(K) \\
 & & \downarrow \iota \\
\mathrm{HF}_n^{(L-\delta-||H||,\,L+\delta)}(K) & \xrightarrow{\Psi\circ\Phi} & \mathrm{HF}_n^{(L-\delta,\,L+\delta+||H||)}(K) \\
\Phi \downarrow & \nearrow \Psi & \\
\mathrm{HF}_n^{(L-\delta,\,L+\delta+||H||)}(K\#H) & \xrightarrow{\Theta} & \mathrm{HF}_n^{(L-\delta,\,L+\delta+||H||)}(H)
\end{array}$$

where ι is the inclusion and ∂ is the *connecting differential* in the long exact sequence (4.1.2) (with $a = L-\delta,\ b = L+\delta$ and $c = L+\delta+||H||$). The maps Φ and Ψ are induced by monotone homotopies between K and $K\#H$: the map Φ is induced by the linear monotone increasing homotopy from K to $K\#H$ (recall that $H \geq 0$) where, in Example 4.1.3, $C = ||H||$; the map Ψ is induced by the linear monotone decreasing homotopy from $K\#H$ to K where, in (4.1.3), $C = 0$.

Since φ_H displaces $\operatorname{supp} K$, the one-periodic orbits of $K\#H$ are exactly the one-periodic orbits of H and moreover $\mathcal{S}(K\#H) = \mathcal{S}(H)$; see [HZ11]. Then the map Θ is an isomorphism induced by a linear monotone homotopy between $K\#H$ and H due to the continuity property (4.1.5) of filtered homology.

Note that the vertical part of the diagram, which consists of the maps ∂ and ι, is part of a long exact sequence as in (4.1.2).

Consider the projection

$$j_H\colon HF(H) \to HF^{(L-\delta, L+\delta+||H||)}(H).$$

and the image

$$j_H([M]) \in HF^{(L-\delta, L+\delta+||H||)}(H)$$

of the class $[M] \in HF_n(H)$. Since

$$L - \delta > ||H||,$$

we have

$$0 = j_H([M]) \in \mathrm{HF}_n^{(L-\delta,\, L+\delta+||H||)}(H).$$

(This last equality is proved similarly to Lipschitz continuity of the action selector with respect to the Hofer norm.) Hence

$$\mathrm{HF}_n^{(L-\delta,\, L+\delta+||H||)}(K) \ni \Psi \circ \Theta^{-1} \circ j_H([M]) = \iota([\max_K]) = 0$$

where the first equality follows from the fact that $j_H([M])$ is equal to the image $\Theta \circ \Phi \circ j([M])$ of the class $[M]$ seen as an element of $HF_n(K)$ and the map j is the projection

$$j\colon HF_n(K) \to HF_n^{(L-\delta-||H||, L+\delta)}(K).$$

Then

$$0 = [\max_K] \in \mathrm{HF}_n^{(L-\delta,\, L+\delta)}(K)$$

and, since ι and ∂ are part of a long exact sequence, it follows that there exists $\mathcal{C} \in \mathrm{HF}_{n+1}^{(L+\delta, L+\delta+||H||)}(K)$ such that

$$\partial\mathcal{C} = [\max_K] \in \mathrm{HF}_n^{(L-\delta,\, L+\delta)}(K).$$

□

Consider a small non-degenerate perturbation $K'\colon S^1 \times M \to M$ of K with $\max K' = L$ attained at a point $p \in N$ (which does not depend on the perturbation K') and such that

$$HF_j^{(a_0,a_1)}(K) := HF_j^{(a_0,a_1)}(K') \tag{4.1.6}$$

with $a_0, a_1 \notin \mathcal{S}(K)$, $\mathcal{S}(K')$; recall definition (4.1.1).

Consider the class $[\max_{K'}] := j_{K'}([M]) \in HF_n^{(L-\delta, L+\delta)}(K')$ and define

$$c(K') := \inf_{\delta>0} \inf \big\{\alpha > L+\delta \colon \iota_\alpha([\max_{K'}]) = 0\big\}.$$

where $\iota_\alpha \colon HF_n^{(L-\delta,\, L+\delta)}(K') \hookrightarrow HF_n^{(L-\delta,\, \alpha)}(K')$ is the inclusion map. We have $c(K') \to c(K)$ as $K' \to K$ and $c(K') = \mathcal{A}_{K'}(\bar{x}')$ for some capped orbit $\bar{x}'$. A special one-periodic orbit $\bar{x}'$ for K' is obtained explicitly the following way: by (4.1.6) and Claim 4.1.5, we obtain a class $[\bar{c}'] \in HF_{n+1}^{(L+\delta,\, \infty)}(K')$ such that $\partial[\bar{c}'] = [\max_{K'}]$. Within each chain $\bar{c}'$ pick a capped orbit with the largest action and then among the resulting capped orbits choose a capped orbit $\bar{x}'$ with the least action. Moreover, we have $\mu_{\mathrm{CZ}}(\bar{x}') = n+1$.

Remark 4.1.6. The orbit $\bar{x}'$ does not have to be connected with the constant orbit (γ_p, u_p) by a Floer downward trajectory (where γ_p is the constant loop p and u_p is its trivial capping). However, there exists a capped orbit $\bar{y}'$ with this property and such that

$$L \leq \mathcal{A}_K(\bar{y}') \leq \mathcal{A}_K(\bar{x}').$$

The orbit $\bar{y}'$ is given explicitly by the following construction: take all chains $\bar{c}'$ such that $\partial[\bar{c}'] = [\max'_K]$. Within each chain consider a capped orbit connected to (γ_p, u_p) with the least action and among these orbits consider one with the least action, $\bar{y}'$.

For a Hamiltonian K as above, consider a sequence (K_j) such that K_j is as K' above and $K_j \to K$ as $j \to \infty$. By the Arzela-Ascoli theorem, there exists a subsequence of special one-periodic orbits $\bar{x}_j$ which converges to an orbit $\bar{x}$ of K which is called a *special one-periodic orbit* of K. Recall that $c(K_j) \to c(K)$ as $j \to \infty$ and $\mu_{\mathrm{CZ}}(\bar{x}_j) = n+1$.

The following results give upper and lower bounds for the action of a special one-periodic orbit.

Lemma 4.1.7. *For a special one-periodic orbit $\bar{x}$ of K, we have the following action upper bound:*

$$\mathcal{A}_K(\bar{x}) \leq L + ||H||. \tag{4.1.7}$$

Proof. Since $\iota([\max_K]) = 0$ (proved in Claim 4.1.5), $c(K) \leq L + ||H||$. By the definition of Kerman's "pinned" action selector, we have $c(K) \geq L$. Then the result follows immediately from the fact that $\bar{x}$ is a carrier of the action selector c. □

Lemma 4.1.8. *A capped loop $\bar{x}$ as in Lemma 4.1.7 satisfies*

$$\mathcal{A}_K(\bar{x}) - L \geq \epsilon \tag{4.1.8}$$

where $\epsilon > 0$ is independent of K.

Proof. Consider a sequence (K_j) as above. Let u_j be a Floer downward trajectory connecting the orbit $\bar{y}_j$ defined in Remark 4.1.6 and the constant orbit (γ_p, u_p). If $E(u_j)$ is below h_0, then we may apply a similar argument to that in lemmas 6.2 and 6.4 in [Gin07] which draws heavily from [Bol96, Bol98] and we obtain

$$d < E(u_j) = \mathcal{A}_{K_j}(\bar{y}_j) - \mathcal{A}_{K_j}(\bar{\gamma}_p)$$

where $d > 0$ is independent of K_j. Define

$$\epsilon := \max\{h_0, d\} > 0.$$

Then $E(u_j) = \mathcal{A}_{K_j}(\bar{y}_j) - \mathcal{A}_{K_j}(\bar{\gamma}_p) \geq \epsilon$ and, since $\mathcal{A}_{K_j}(\bar{y}_j) \leq \mathcal{A}_{K_j}(\bar{x}_j)$, it follows that

$$\mathcal{A}_{K_j}(\bar{x}_j) - L \geq \epsilon. \tag{4.1.9}$$

Then take (4.1.9) to the limit when $j \to \infty$ and we obtain the desired result

$$\mathcal{A}_K(\bar{x}) - L \geq \epsilon.$$

□

4.2 Floer homology for symplectomorphisms

In this section, we recall the construction of the Floer homology for symplectomorphisms following [LO95] (references therein and [Ono06]).

4.2.1 Definition of Floer-Novikov homology

Let ϕ be a symplectomorphism (isotopic to the identity) defined on a strictly monotone manifold M, consider an almost complex structure J on M and fix an almost complex structure $\widetilde{J}$ on $\widetilde{M}$ corresponding to J. Consider the Hamiltonian $\widetilde{H}$ associated with ϕ as in (3.3.8) and recall that we denote by $\mathcal{P}(\widetilde{H})$ the set of contractible periodic orbits of the Hamiltonian system associated with $\widetilde{H}$ and by $\widetilde{\mathcal{P}}(\widetilde{H}) := \widetilde{j}^{-1}(\mathcal{P}(\widetilde{H}))$ (see Section 3.3.1). The maps $\widetilde{u}\colon \mathbb{R} \times S^1 \to \widetilde{M}$ which satisfy the equation

$$\partial_s \widetilde{u} + \widetilde{J}(\widetilde{u})(\partial_t \widetilde{u} - X_{\widetilde{H}}(\widetilde{u})) = 0 \tag{4.2.1}$$

with boundary conditions

$$\lim_{s \to \pm\infty} \widetilde{u}(s,t) = [\widetilde{x}^{\pm}, \widetilde{v}^{\pm}] \tag{4.2.2}$$

and

$$[\widetilde{x}^{-}, \widetilde{v}^{-} \# \widetilde{u}] = [\widetilde{x}^{+}, \widetilde{v}^{+}] \tag{4.2.3}$$

form the space of connecting orbits on $\widetilde{\mathcal{L}}\widetilde{M}$. The energy of a connecting orbit in this space is given by

$$E(\widetilde{u}) = \int_{-\infty}^{\infty} \int_0^1 |\partial_s \widetilde{u}|^2 dtds = \mathcal{A}_{\widetilde{H}}([\widetilde{x}^{-}, \widetilde{v}^{-}]) - \mathcal{A}_{\widetilde{H}}([\widetilde{x}^{+}, \widetilde{v}^{+}]) \tag{4.2.4}$$

Denote by $\mathcal{M}([\widetilde{x}^{-}, \widetilde{v}^{-}], [\widetilde{x}^{+}, \widetilde{v}^{+}]) := \mathcal{M}([\widetilde{x}^{-}, \widetilde{v}^{-}], [\widetilde{x}^{+}, \widetilde{v}^{+}]; \widetilde{H}, J)$ the space of solutions of (4.2.1), (4.2.2) and (4.2.3) with $[\widetilde{x}^{\pm}, \widetilde{v}^{\pm}] \in \widetilde{\mathcal{P}}(\widetilde{H})$.

The Conley-Zehnder index μ_{CZ} of a non-degenerate periodic solution $[\widetilde{x}, \widetilde{v}] \in \widetilde{\mathcal{L}}\widetilde{M}$ satisfies

$$0 \neq |\Delta_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]) - \mu_{\mathrm{CZ}}([\widetilde{x}, \widetilde{v}])| < n. \tag{4.2.5}$$

and is Γ_1-invariant (where Γ_1 is $\pi_1(M)/\ker I_\theta$), i.e. $\mu_{\mathrm{CZ}}([\widetilde{x}, \widetilde{v}]) = \mu_{\mathrm{CZ}}(a \cdot [\widetilde{x}, \widetilde{v}])$ for any $a \in \Gamma_1$. This index satisfies the following identities:

i) $\mu_{\mathrm{CZ}}([\widetilde{x}, \widetilde{v} \# A]) = \mu_{\mathrm{CZ}}([\widetilde{x}, \widetilde{v}]) + I_{c_1}(A)$

ii) $\dim \mathcal{M}([\widetilde{x}^{-}, \widetilde{v}^{-}], [\widetilde{x}^{+}, \widetilde{v}^{+}]) = \mu_{\mathrm{CZ}}([\widetilde{x}^{-}, \widetilde{v}^{-} \# A]) - \mu_{\mathrm{CZ}}([\widetilde{x}^{+}, \widetilde{v}^{+} \# A])$

for $A \in \pi_2(M)$.

Denote by $\widetilde{\mathcal{P}}_k(\widetilde{H})$ the subset of $\widetilde{\mathcal{P}}(\widetilde{H})$ of periodic solutions with $\mu_{\mathrm{CZ}}([\widetilde{x}, \widetilde{v}]) = k$. Consider the chain complex whose k-th chain group $C_k(\widetilde{H})$ consists of all formal sums

$$\sum \xi_{[\widetilde{x}, \widetilde{v}]} \cdot [\widetilde{x}, \widetilde{v}]$$

with $[\widetilde{x}, \widetilde{v}] \in \widetilde{\mathcal{P}}_k(\widetilde{H})$, $\xi_{[\widetilde{x}, \widetilde{v}]} \in \mathbb{Z}_2$ and such that, for all $c \in \mathbb{R}$,

$$\#\big\{[\widetilde{x}, \widetilde{v}] \mid \xi_{[\widetilde{x}, \widetilde{v}]} \neq 0,\ \mathcal{A}_{\widetilde{H}}([\widetilde{x}, \widetilde{v}]) > c\big\} < \infty.$$

For a generator $[\widetilde{x}, \widetilde{v}]$ in $C_k(\widetilde{H})$, the boundary operator ∂_k is defined as follows

$$\partial_k([\widetilde{x}, \widetilde{v}]) = \sum_{\mu_{\mathrm{CZ}}([\widetilde{y}, \widetilde{w}])=k-1} n_2([\widetilde{x}, \widetilde{v}], [\widetilde{y}, \widetilde{w}])[\widetilde{y}, \widetilde{w}]$$

where $n_2([\widetilde{x}, \widetilde{v}], [\widetilde{y}, \widetilde{w}]) \in \mathbb{Z}_2$ is the modulo-2 reduction of the number of elements in the quotient space $\mathcal{M}([\widetilde{x}, \widetilde{v}], [\widetilde{y}, \widetilde{w}])/\mathbb{R}$. The boundary operator ∂ satisfies $\partial^2 = 0$ and we have the homology groups

$$HFN_k(\theta_t) = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}. \tag{4.2.6}$$

Moreover, this homology is invariant under exact deformations (see Theorem 4.3 in [LO95]).

4.2.2 Filtered Floer-Novikov homology

The (total) chain Floer complex $C_*(\widetilde{H}) =: C_*^{(-\infty,\infty)}(\widetilde{H})$ admits a filtration by $\mathbb{R}$. Define $\mathcal{S}(\widetilde{H})$ the set of values of the functional $\mathcal{A}_{\widetilde{H}}$ (defined in (3.3.9)) which is called the *action spectrum*. For each $b \in (-\infty, \infty]$ outside $\mathcal{S}(\widetilde{H})$, the chain complex $C_*^{(-\infty,b)}(\widetilde{H})$ is generated by equivalence classes of capped loops $[(\widetilde{x}, \widetilde{v})]$ with action $\mathcal{A}_{\widetilde{H}}$ less than b. For $-\infty \leq a < b \leq \infty$ outside $\mathcal{S}(\widetilde{H})$, set

$$C_*^{(a,b)}(\widetilde{H}) := C_*^{(-\infty,b)}(\widetilde{H})/C_*^{(-\infty,a)}(\widetilde{H}).$$

The boundary operator $\partial\colon C_*(\widetilde{H}) \to C_{*-1}(\widetilde{H})$ descends to $C_*^{(a,b)}(\widetilde{H})$ and hence the *filtered Floer-Novikov homology* $HFN_*^{(a,b)}(\theta)$ is well defined.

This construction also extends by continuity to all symplectomorphisms in $\mathrm{Symp}_0(M, \omega)$. For an arbitrary $\phi \in \mathrm{Symp}_0(M, \omega)$, set

$$HFN_*^{(a,b)}(\theta) := HFN_*^{(a,b)}(\theta') \tag{4.2.7}$$

where $[\theta'] := \widetilde{\mathrm{Flux}}(\widetilde{\phi}')$ with ϕ' a non-degenerate perturbation of ϕ and $-\infty \leq a < b \leq \infty$ outside the closure of the action spectrum of a Hamiltonian corresponding to ϕ obtained as in (3.3.8). Observe that since the symplectic manifold (M, ω) and the flux are rational (in the sense of Section 3.1) the action spectrum is nowhere dense (see e.g. [HZ11, Sch00]) and hence we may assume a and b are just outside the action spectrum of the referred Hamiltonian. This definition does not depend on the perturbation.

4.3 Quantum homology

The quantum homology of M, $HQ_*(M)$, is an algebra over the Novikov ring, Λ. In this section we recall their definitions; see (Chapter 11 in) [MS12] for more details. Here we follow (Section 2.2 in) [GG12b].

4.3.1 Novikov ring and quantum homology

In the case where M is strictly monotone, the *Novikov ring* Λ is the group algebra of a group Γ over $\mathbb{Z}_2$, $\mathbb{Z}_2[\Gamma]$. The group Γ is the quotient of $\pi_2(M)$ by the equivalence relation $\sim$ where $A \sim B$ if $I_{c_1}(A) = I_{c_1}(B)$, or equivalently, if $I_\omega(A) = I_\omega(B)$, i.e.

$$\Gamma = \pi_2(M)/\ker I_\omega = \pi_2(M)/\ker I_{c_1}.$$

An element in Λ is a formal finite linear combination,

$$\sum \alpha_A e^A,$$

where $\alpha_A \in \mathbb{Z}_2$. We set the degree of e^A, for $A \in \Gamma$, as $I_{c_1}(A)$ which grades the ring Λ. We have $\Gamma \simeq \mathbb{Z}$ and denote by A_0 the generator of Γ with $I_{c_1}(A_0) = -2N$. Then $\mathfrak{q} := e^{A_0} \in \Lambda$ has degree $-2N$ and the Novikov ring is the ring of Laurent polynomials $\mathbb{Z}_2[\mathfrak{q}^{-1}, \mathfrak{q}]$.

The *quantum homology* of M is defined by

$$HQ_*(M) = H_*(M) \otimes \Lambda$$

(where Λ is the Novikov ring) where the degree of the generator $\alpha \otimes e^A$ is $\deg(\alpha)+ I_{c_1}(A)$ ($\alpha \in H_*(M)$, $A \in \Gamma$,). The product structure is given by the *quantum product*:

$$\alpha * \beta = \sum_{A\in\Gamma} (\alpha * \beta)_A e^A \tag{4.3.1}$$

where $(\alpha * \beta)_A \in H_*(M)$ is defined via some Gromov-Witten invariants of M and has degree $\deg(\alpha) + \deg(\beta) - 2n - I_{c_1}(A)$. Thus

$$\deg(\alpha * \beta) = \deg(\alpha) + \deg(\beta) - 2n.$$

When $A = 0$, $(\alpha * \beta)_0 = \alpha \cap \beta$, where $\cap$ stands for the intersection product of ordinary homology classes.

Recall that in (4.3.1) it suffices to restrict the summation to the negative cone $I_\omega(A) \leq 0$ and, under our assumptions on M, we can write

$$\alpha * \beta = \alpha \cap \beta + \sum_{k>0} (\alpha * \beta)_k \mathfrak{q}^k,$$

where $\deg((\alpha * \beta)_k) = \deg(\alpha) + \deg(\beta) - 2n + 2Nk$ and the sum is finite.

The product $*$ is a Λ-linear, associative, graded-commutative product on $HQ_*(M)$. The fundamental class $[M]$ is the unit in the algebra $HQ_*(M)$. Thus $a\alpha = (a[M]) * \alpha$, where $a \in \Lambda$ and $\alpha \in H_*(M)$ is canonically embedded in $HQ_*(M)$.

The map I_ω extends to $HQ_*(M)$ as

$$I_\omega(\alpha) = \max\big\{I_\omega(A) \mid \alpha_A \neq 0\big\} = \max\big\{-h_0 k \mid \alpha_k \neq 0\big\}$$

where $\alpha = \sum \alpha_A e^A = \sum \alpha_k \mathfrak{q}^k$. We have

$$I_\omega(\alpha + \beta) \leq \big\{I_\omega(\alpha), I_\omega(\beta)\big\} \tag{4.3.2}$$

and

$$I_\omega(\alpha * \beta) \leq I_\omega(\alpha) + I_\omega(\beta).$$

4.3.2 Quantum product action

We describe an action of the quantum homology on the filtered Floer-Novikov homology. We follow [GG12b, Section 2.3] for the Floer-Novikov setting; see [LO96, Section 3] for more details. Let $[\sigma] \in H_*(M)$. Denote by $\mathcal{M}([\widetilde{x}, \widetilde{v}], [\widetilde{y}, \widetilde{w}]; \sigma)$ the moduli space of solutions $\widetilde{u}$ of (4.2.1), (4.2.2) and (4.2.3) with $[\widetilde{x}, \widetilde{v}], [\widetilde{y}, \widetilde{w}] \in \widetilde{\mathcal{P}}(\widetilde{H})$ and such that $u(0,0) \in \sigma$ where σ is a generic cycle representing $[\sigma]$ and $\pi \circ \widetilde{u} = u$.

Then the dimension of this moduli space is given by

$$\dim \mathcal{M}([\widetilde{x}, \widetilde{v}], [\widetilde{y}, \widetilde{w}]; \sigma) = \mu_{\mathrm{CZ}}([\widetilde{x}, \widetilde{v}]) - \mu_{\mathrm{CZ}}([\widetilde{y}, \widetilde{w}]) - \operatorname{codim}(\sigma).$$

and let $m([\widetilde{x}, \widetilde{v}], [\widetilde{y}, \widetilde{w}]; \sigma) \in \mathbb{Z}_2$ be the parity of $\#\mathcal{M}([\widetilde{x}, \widetilde{v}], [\widetilde{y}, \widetilde{w}]; \sigma)$ when this moduli space is zero-dimensional and zero otherwise.

For any c, $c' \notin S(\widetilde{H})$, there is a map

$$\Phi_\sigma : C_*^{(c,c')}(\widetilde{H}) \to C_{*-\operatorname{codim}(\sigma)}^{(c,c')}(\widetilde{H})$$

induced by

$$\Phi_\sigma([\widetilde{x}, \widetilde{v}]) = \sum_{[\widetilde{y},\widetilde{w}]} m([\widetilde{x}, \widetilde{v}], [\widetilde{y}, \widetilde{w}]; \sigma)[\widetilde{y}, \widetilde{w}].$$

This map commutes with the Floer-Novikov differential ∂ and descends (independently of the choice of the cycle representing $[\sigma]$) to a map

$$\Phi_{[\sigma]} : HFN_*^{(c,c')}(\theta) \to HFN_{*-\mathrm{codim}(\sigma)}^{(c,c')}(\theta).$$

The action of the class $\alpha = \mathfrak{q}^\nu[\sigma] \in HQ_*(M)$

$$\Phi_\alpha : HFN_*^{(c,c')}(\theta) \to HFN_{*-2n+\deg(\alpha)}^{(c,c')+I_\omega(\alpha)}(\theta). \tag{4.3.3}$$

is induced by the map

$$\Phi_{\mathfrak{q}^\nu\sigma}([\widetilde{x}, \widetilde{v}]) := \sum_{[\widetilde{y},\widetilde{w}]} m(\mathfrak{q}^\nu[\widetilde{x}, \widetilde{v}], [\widetilde{y}, \widetilde{w}]; \sigma)[\widetilde{y}, \widetilde{w}]$$

where $\mathfrak{q}$ is as in Section 4.3.1 and $\mathfrak{q}^\nu[\widetilde{x}, \widetilde{v}]$ is $[\widetilde{x}, \widetilde{w}] \in \widetilde{\mathcal{P}}\widetilde{H}$ where $w = \pi \circ \widetilde{w}$ is obtained by recapping $v = \pi \circ \widetilde{v}$ the following way $w = v\#(\nu A_0)$ (where A_0 is the generator of the group Γ defined in Section 4.3.1).

By linearity over Λ, the map Φ_α can be extended with $\alpha \in HQ_*(M)$ so that (4.3.3) holds.

The maps Φ_α also give an action of the quantum homology on the filtered Floer-Novikov homology. We have the following properties:

$$\Phi_{[M]} = id$$

and

$$\Phi_\beta \Phi_\alpha = \Phi_{\beta * \alpha}. \tag{4.3.4}$$

Remark 4.3.1. Observe that in the multiplicative property (4.3.4) the maps on the two sides of the identity have, in general, different target spaces. For any interval (a, b), consider the following diagram:

$$\begin{array}{ccccc} HFN_*^{(c,c')}(\theta) & \xrightarrow{\Phi_\alpha} & HFN_{*-2n+\deg(\alpha)}^{(c,c')+I_\omega(\alpha)}(\theta) & \xrightarrow{\Phi_\beta} & HFN_{*-4n+\deg(\alpha)+\deg(\beta)}^{(c,c')+I_\omega(\alpha)+I_\omega(\beta)}(\theta) \\ & \searrow^{\Phi_{\beta*\alpha}} & & & \downarrow \\ & & HFN_{*-2n+\deg(\beta*\alpha)}^{(c,c')+I_\omega(\beta*\alpha)}(\theta) & \longrightarrow & HFN_{*-2n+\deg(\beta*\alpha)}^{(a,b)}(\theta) \end{array} \tag{4.3.5}$$

where $a \geq c + I_\omega(\alpha) + I_\omega(\beta)$ and $b \geq c' + I_\omega(\alpha) + I_\omega(\beta)$. Then the identity (4.3.4) should be understood as that the diagram (4.3.5) commutes.

Chapter 5

Proofs of the main results

5.1 Rigidity of the coisotropic Maslov index

In this section we focus on the rigidity result of the Maslov index for coisotropic submaniofolds. The proof of Theorem 2.1.3 is presented in Section 5.1.1. As a corollary of the main theorem we obtain the nearby existence theorem (Theorem 2.1.4) which is proved in Section 5.1.2.

5.1.1 Proof of Theorem 2.1.3

Fix R such that $U_R = N \times B^k_R$ is defined by Proposition 3.4.9. Consider $\varepsilon > 0$ small and $0 < r < R/2$. Assume U_r is displaced by some Hamiltonian H and consider $L > e(U_r)$. Let $K_{L,r,\varepsilon} \colon [0, R] \to \mathbb{R}$ be a smooth decreasing map such that

- $K_{L,r,\varepsilon} \geq 0$
- $K_{L,r,\varepsilon}(0) = L$
- $K_{L,r,\varepsilon}$ is strictly decreasing and C^2-close to L on $[0, \varepsilon]$
- $K_{L,r,\varepsilon}$ is concave on $[\varepsilon, 2\varepsilon]$
- $K_{L,r,\varepsilon}$ is linear decreasing from $L - \varepsilon$ to ε on $[2\varepsilon, r - \varepsilon]$
- $K_{L,r,\varepsilon}$ is convex on $[r - \varepsilon, r]$
- $K_{L,r,\varepsilon} \equiv 0$ on $[r, R]$.

We also denote by $K_{L,r,\varepsilon}$ the Hamiltonian

$$K_{L,r,\varepsilon}\colon M \to \mathbb{R}$$

defined by $K_{L,r,\varepsilon}(|p|)$ on U_R and equal to zero outside U_R.

Fix r and consider the family of functions $K_{L,\varepsilon}$ depending smoothly on the parameters L and ε. These Hamiltonians have the same properties as the Hamiltonian K in the previous subsection.

The key to the proof, as in [Gin11], is the following result which gives the location of a sequence of special one-periodic orbits $\bar{x}_i$.

Lemma 5.1.1 ([Gin11])**.** *There exists $L > e(U_R)$ and a sequence $\varepsilon_i \to 0$ such that a special one-periodic orbit of K_{L,ε_i} $\bar{x}_i$ satisfies*

$$|p(x_i)| \in [\varepsilon_i, 2\varepsilon_i]$$

where $p = (p_1, \ldots, p_k)$ are the coordinates introduced in Proposition 3.4.9.

Remark 5.1.2. In [Gin11], the result of Lemma 5.1.1 is proved for a class of Hamiltonians which is slightly different from the one we work with. However the above lemma holds for the same reasons as the result in the referred paper.

Consider L and the sequences ε_i and $\bar{x}_i$ as in Lemma 5.1.1. By Proposition 3.4.10, if we reparametrize $\bar{x}_i$ and reverse its orientation, then $\bar{x}_i$ can be viewed as a periodic orbit $\bar{x}_i^-$ of ρ. Since the slopes of the Hamiltonians K_{L,ε_i} are bounded from above (for instance, by $2L/r$), then (by the Arzela-Ascoli theorem) we define

$$\bar{\gamma} \colon = \text{limit of (a subsequence of) } (\pi(x_i^-), \widehat{u_i}^-).$$

where $\mu(\pi(x_i^-), \widehat{u_i}^-) = -\Delta_\rho(x_i^-, u_i^-)$ by (3.4.6). Then, by (3.2.2),

$$-n \le \mu_{\mathrm{CZ}}((x_i^-, u_i^-)') - \Delta(x_i^-, u_i^-) \le n$$

and hence

$$\begin{array}{rcll} -n \le \mu(\pi(x_i^-), u_i^-) & + & \mu_{\mathrm{CZ}}((x_i^-, u_i^-)') & \le n \\ & & \| & \\ & & -\,\mu_{\mathrm{CZ}}((x_i, u_i)') & = -(n+1) \end{array}$$

where the first equality uses the fact that x_i is in the region where K_{L,ε_i} is concave, i.e., where $|p(x_i)| \in [\varepsilon_i, 2\varepsilon_i]$ and we obtain the following bounds for the Maslov index of $(\pi(x_i^-), \widehat{u_i}^-)$:

$$1 \leq \mu(\pi(x_i^-), u_i^-) \leq 2n + 1. \tag{5.1.1}$$

Considering the limit (of a subsequence) of (5.1.1), we have

$$1 \leq \mu(\bar{\gamma}) \leq 2n + 1. \tag{5.1.2}$$

By Proposition 3.4.9, we obtain

$$\begin{aligned} A_{K_{L,\varepsilon_i}}(\bar{x}_i) &= K_{L,\varepsilon_i}(\bar{x}_i) - \int_{u_i} \omega \\ &= K_{L,\varepsilon_i}(\bar{x}_i) - \int_{\hat{u}_i} \omega - |p(x_i)| l(\pi(x_i)) \end{aligned} \tag{5.1.3}$$

where $\hat{u}_i$ is constructed as in Section 3.4.3; see Figure 3.2.

Moreover, by (4.1.7), (4.1.8) and (5.1.3), we have

$$0 < \epsilon \leq K_{L,\varepsilon_i}(x_i^-) - \int_{\hat{u}_i^-} \omega - |p(x_i^-)| l(\pi(x_i^-)) - L \leq e(U_r). \tag{5.1.4}$$

The limit (of a subsequence) of $-\int_{\hat{u}_i^-} \omega$ is Area$(\bar{\gamma})$ since the (sub)sequence of the symplectic areas C^0-converges and the norm of the derivative of $\hat{u}_i$ is uniformly bounded. Since $|p(x_i^-)| \in [\varepsilon_i, 2\varepsilon_i]$, $K_{L,\varepsilon_i}(x_i^-) \in [\varepsilon_i, L - \varepsilon_i]$ and the sequence $l(\pi(x_i^-))$ is bounded (since the slope of K_{L,ε_i} is bounded), then, taking the limit (of a subsequence) of (5.1.4), we obtain

$$0 < \epsilon \leq \text{Area}(\bar{\gamma}) \leq e(U_r). \tag{5.1.5}$$

Recall that ϵ is independent of ε_i. Then, taking $r > 0$ sufficiently small, we have

$$0 < \text{Area}(\bar{\gamma}) \leq e(N) + \varepsilon.$$

Hence, we have the desired bounds for the area of $\bar{\gamma}$. To obtain the Maslov index bounds as presented in the theorem (which go beyond (5.1.2)), we will first prove that the orbit γ is non-trivial. Assume the contrary, that is, that γ is a trivial orbit. Then, by (5.1.5), the capping v of γ must be non-trivial. Recall that we have one of the following conditions:

- W is negative monotone,
- $e(N) < h_0$,
- $2n+1 < 2\mathcal{N}$.

Suppose that W is negative monotone. Then, $\langle c_1, v\rangle$ and $\text{Area}(\bar{\gamma})$ have opposite signs. However, by (5.1.2) and (5.1.5), they are both positive and we obtain a contradiction. If $e(N) < h_0$ or $2n+1 < 2\mathcal{N}$, we obtain contradictions by the definition of the rationality constant h_0 and (5.1.5) or by the definition of the minimal Chern number $\mathcal{N}$ and (5.1.2), respectively. Therefore, γ is a non-trivial orbit. Furthermore, there exists a (sub)sequence of non-trivial orbits x_i as in Lemma 5.1.1 which converges to γ. Then, by Proposition 3.4.11, we have

$$\begin{array}{rcl} -\mu(\pi(x_i^-), u_i^-) - n \leq & \mu_{\text{CZ}}((x_i^-, u_i^-)') & \leq -\mu(\pi(x_i^-), \widehat{u_i}^-) + n - k \\ & \| & \\ & -\mu_{\text{CZ}}((x_i, u_i)') & = -(n+1) \end{array}$$

where the first equality uses the fact that x_i is in the region where K_{L,ε_i} is concave, i.e., where $|p(x_i)| \in [\varepsilon_i, 2\varepsilon_i]$. Then

$$1 \leq \mu(\pi(x_i^-), \widehat{u_i}^-) \leq 2n+1-k$$

and considering the limit (of a subsequence) we obtain the desired bounds for the Maslov index of $\bar{\gamma}$:

$$1 \leq \mu(\bar{\gamma}) \leq 2n+1-k.$$

5.1.2 Proof of the nearby existence theorem

We prove the existence of an orbit (with the required properties) in a level N_a arbitrarily close to N_0 and the wanted result follows immediately. Consider $K := f(F_1, \ldots, F_k)$ where $f\colon \mathbb{R}^k \to \mathbb{R}$ is a bump function supported in a small neighborhood of the origin in $\mathbb{R}^k$ and such that the maximum value of f is large enough. Since the support of f is small, we may assume that the support of K is displaceable and all $a \in \operatorname{supp} f$ are regular values of $\overrightarrow{F}$. Hence the coisotropic submanifolds N_a are compact and close to N_0 when $a \in \mathbb{R}^k$ is near the origin. By lemmas 4.1.7 and 4.1.8, there exists a capped one-periodic orbit of K (in some regular level N_a) such that

$$\max K < A_K(\bar{x}) \leq \max K + ||H|| \tag{5.1.6}$$

where H displaces $\operatorname{supp} K$. The capped orbit $\bar{x}$ can be approximated by non-degenerate capped orbits with Conley-Zehnder index equal to $n+1$ and hence, by (3.2.2), we obtain

$$1 \leq \Delta(\bar{x}) \leq 2n + 1.$$

Since one of the three conditions mentioned in the statement of the theorem is satisfied, the orbit x is non-trivial. Indeed, assume that x is a trivial orbit. Then (5.1.6) is equivalent to

$$0 < \operatorname{Area}(\bar{x}) \leq e(N).$$

Then using the area and (mean) index bounds on $\bar{x}$ and assuming one of the above three conditions, we obtain a contradiction (following the same reasoning as in Section 5.1.1).

Furthermore, since the Hamiltonian K Poisson-commutes with all F_j, the orbit x is tangent to the characteristic foliation $\mathcal{F}_a$ on N_a.

5.2 Hyperbolic points and periodic orbits of symplectomorphisms

In this section we focus on the result on hyperbolic fixed points and periodic orbits of symplectomorphisms. The proof of the main theorem of this section (Theorem 2.2.1) relies on an important feature of hyperbolic fixed points of symplectomorphisms which is described in Section 5.2.1. The proof of Theorem 2.2.1 is then presented in Section 5.2.2. In Section 5.2.3, we prove Proposition 2.2.2.

5.2.1 Ball-crossing energy theorem

Here, we describe the key property of hyperbolic periodic orbits which supports the proof of the main theorem (see [GG12b, Section 3] for more details including the proof of the Ball-crossing energy Theorem).

Let ϕ be a symplectomorphism (isotopic to the identity) on a symplectic manifold (M, ω) and fix a one-periodic in time almost complex structure J compatible with ω. We consider solutions $\widetilde{u} : \Sigma \to \widetilde{M}$ of the equation (4.2.1) where $\Sigma \subset \mathbb{R} \times S^1_k$ is a closed domain (i.e. a closed subset with non-empty interior). By definition, the energy of $\widetilde{u}$ is

$$E(\widetilde{u}) := \int_{\Sigma} ||\partial_s \widetilde{u}||_{\widetilde{M}} \, dtds$$

where $||\partial_s \widetilde{u}||_{\widetilde{M}}$ is $||\partial_s(\pi \circ \widetilde{u})||$ where $||\cdot||$ stands for the norm with respect to $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$.

Let γ be a hyperbolic one-periodic solution of (3.3.6) in M and $\widetilde{\gamma}$ a lift of γ to $\widetilde{M}$, i.e. $\widetilde{\gamma} \in \mathcal{P}(\widetilde{H})$ hyperbolic. Recall the definition of the covering space $\widetilde{M}$ in Section 3.3.1 and of the Hamiltonian $\widetilde{H}$ associated with ϕ in (3.3.8).

We say that $\widetilde{u}$ is asymptotic to $\widetilde{\gamma}^k$ as $s \to \infty$ if Σ contains a cylinder $[s_0, \infty) \times S^1_k$ and $\widetilde{u}(s,t) \to \widetilde{\gamma}^k(t)$ C^∞-uniformly in t as $s \to \infty$.

Let U be a (sufficiently small) neighborhood of γ with smooth boundary and define $\widetilde{U} := \pi^{-1}(U)$.

Theorem 5.2.1 (Ball-Crossing Energy Theorem; [GG12b])**.** *There exists a constant $c_\infty > 0$ (independent of k and Σ) such that for any solution $\widetilde{u}$ of the equation (4.2.1), with $\widetilde{u}(\partial\Sigma) \subset \partial\widetilde{U}$ and $\partial\Sigma \neq \emptyset$, which is asymptotic to $\widetilde{\gamma}^k$ as $s \to \infty$, we have*

$$E(\widetilde{u}) > c_\infty. \tag{5.2.1}$$

Moreover, the constant c_∞ can be chosen to make (5.2.1) hold for all k-periodic almost complex structures (varying in k) C^∞-close to $\widetilde{J}$ uniformly on $\mathbb{R} \times \widetilde{U}$.

5.2.2 Proof of Theorem 2.2.1

As mentioned in the introduction, to simplify the setting, we may assume that γ is a constant orbit. This is due to the fact that there exists a one periodic loop of Hamiltonian diffeomorphisms ψ^t defined on a neighborhood of γ such that $\psi^t(\gamma(0)) = \gamma(t)$. We may think of $\gamma(t) \equiv \gamma(0)$ as a fixed point of $(\psi^t)^{-1} \circ \phi_t$ (see Section 5.1 in [Gin10] for more details). Furthermore, attach a capping w to γ and fix a lift, $\widehat{\gamma} := [\widetilde{\gamma}, \widetilde{w}] \in \widetilde{\mathcal{L}}\widetilde{M}$, so that

$$\Delta_{\widetilde{H}}(\widehat{\gamma}) = 0 = \mathcal{A}_{\widetilde{H}}(\widehat{\gamma}). \tag{5.2.2}$$

Remark 5.2.2.

1. To ensure condition (5.2.2), we may have to consider an iteration of ϕ which we continue denoting by ϕ: in fact, by passing if necessary to an iteration, we can guarantee that the mean index of γ with respect to any capping is divisable by $2\mathcal{N}$. Then there exists a capping such that the mean index is zero and finally by adding a constant to the obtained associated Hamiltonian we can assume the action is also zero.

2. Observe that since γ is hyperbolic, the mean index $\Delta_{\widetilde{H}}(\widehat{\gamma})$ is equal to the Conley-Zehnder index $\mu_{\text{CZ}}(\widehat{\gamma})$ and hence $\mu_{\text{CZ}}(\widehat{\gamma}) = 0$.

Arguing by contradiction, assume that ϕ has finitely many periodic orbits. Consider an iteration of ϕ, still denoted by ϕ, so that p is sufficiently large, namely,

$$(2p-3)h_0 - \lambda(n+1) > 0 \tag{5.2.3}$$

where λ is the monotonicity constant of M. The r-th iteration ϕ^r (where r is defined in Theorem 2.2.1) has finitely many periodic orbits and we denote them by $x_1, \ldots, x_m$.

Remark 5.2.3.

1. Observe that $\widetilde{\text{Flux}}(\widetilde{\phi^r}) = r\widetilde{\text{Flux}}(\widetilde{\phi})$.

2. The periodic orbit γ^r of ϕ^r is hyperbolic and we keep the notation γ for this orbit and ϕ for the iteration ϕ^r.

Fix a one-periodic in time almost complex structure J' (and denote by $\widetilde{J}'$ the corresponding almost complex structure on $\widetilde{M}$). Let U be a neighborhood of γ such that no periodic orbit of ϕ except γ intersects U. By Theorem 5.2.1, there exists a constant $c_\infty > 0$ such that, for all k, all non-trivial k-periodic solution of (4.2.1) asymptotic to $\widetilde{\gamma}^k$ as $s \to \infty$ has energy greater than c_∞.

For each $i = 1, \ldots, m$, attach a capping v_i to the loop x_i, fix a lift $[(\widetilde{x}_i, \widetilde{v}_i)] =: \widehat{x}_i \in \widetilde{\mathcal{L}M}$ and define

$a_i \in S^1_{h_0}$ by $A_{\widetilde{H}}(\widehat{x}_i) \bmod h_0$,

$\overline{a_i} \in S^1_{ph_0}$ by $\widetilde{A}_{\widetilde{H}}(\widehat{x}_i) \bmod ph_0$.

Remark 5.2.4. Observe that a_i and $\overline{a_i}$ are independent of the initially attached capping and fixed lift. (The second follows from Remark 5.2.3 (1).)

Take $\epsilon,\ \delta > 0$ small, namely,

$$2(\epsilon + \delta) < \lambda \text{ and } \epsilon < c_\infty. \tag{5.2.4}$$

Then, by Kronecker's Theorem, there exists k (large) such that for all $i = 1, \ldots, m$

$$||ka_i||_{h_0} < \epsilon \quad \text{and} \quad ||k\overline{a_i}||_{ph_0} < \delta.$$

Here $||a||_h \in [0, h/2]$ stands for the distance from $a \in S^1_h = \mathbb{R}/h\mathbb{Z}$ to 0. Observe that k depends on ϵ (and δ), hence on c_∞ and ultimately on the neighborhood U.

Consider a non-degenerate perturbation ϕ' of ϕ^k such that (4.2.7) holds and the Hamiltonian $\widetilde{K}$ associated to ϕ' (in the sense of (3.3.8)) satisfies the following properties:

- $\widetilde{K}$ is k-periodic and C^2-close to $\widetilde{H}^{\natural k}$,
- $\widetilde{K}$ coincides with $\widetilde{H}^{\natural k}$ on the neighborhood U and
- $\widetilde{K}$ is non-degenerate.

If ϕ^k is non-degenerate, we can take $\phi' = \phi^k$. Then, by Remark 5.2.4 and assuming $\delta < h_0$, there exists k (large) such that for all $\widehat{x}$ k-periodic solution of $\widetilde{K}$

$$\left\|\mathcal{A}^{h_0}_{\widetilde{K}}(\widehat{x})\right\|_{h_0} < \epsilon \tag{5.2.5}$$

and

$$\text{either } \left|\widetilde{\mathcal{A}}_{\widetilde{K}}(\widehat{x})\right| < \delta \text{ or } \left|\widetilde{\mathcal{A}}_{\widetilde{K}}(\widehat{x})\right| > (p-1)h_0 \tag{5.2.6}$$

where $\mathcal{A}^{h_0}_{\widetilde{K}}(\widehat{x})$ stands for $\mathcal{A}_{\widetilde{K}}(\widehat{x})$ mod h_0.

For any k-periodic almost complex structure $\widetilde{J}$ sufficiently close to (the k-periodic extension of) $\widetilde{J}'$, all non-trivial k-periodic solutions of the equation (4.2.1) for the pair $(\phi', \widetilde{J})$ asymptotic to $\widetilde{\gamma}^k$ as $s \to \infty$ have energy greater than c_∞.

Lemma 5.2.5. [GG12b, Lemma 4.1] *Let $\tau := (p-1)h_0 - \frac{\lambda}{2}(n+1)$. The orbit $\widehat{\gamma}^k$ is not connected by a solution of (4.2.1) to any $\widehat{x} \in \widetilde{P}(\widetilde{K})$ with relative index ± 1 with action in $(-\tau, \tau)$.*

In particular, $\widehat{\gamma}^k$ is closed in $C^{(-\tau,\tau)}_*(\widetilde{K})$ and $0 \neq [\widehat{\gamma}^k] \in HFN^{(-\tau,\tau)}_*(\theta')$. Moreover, $\widehat{\gamma}^k$ must enter every cycle representing its homology class $[\widehat{\gamma}^k]$ in $HFN^{(-\tau,\tau)}_*(\theta')$.

Proof. Assume the orbit $\widehat{\gamma}^k$ is connected, by a solution $\widetilde{u}$ of (4.2.1), to some $\widehat{x} \in \widetilde{P}(\widetilde{K})$ with index $\mu_{\mathrm{CZ}}(\widehat{x}) = \pm 1$ with action in $(-\tau, \tau)$.

Consider the first case in (5.2.6), i.e. $\left|\widetilde{\mathcal{A}}_{\widetilde{K}}(\widehat{x})\right| < \delta$: since

i) $\left\|\mathcal{A}^{h_0}_{\widetilde{K}}(\widehat{x})\right\|_{h_0} < \epsilon$ (by (5.2.5)),

ii) $E(\widetilde{u}) > c_\infty > \epsilon$ (by Theorem 5.2.1 and (5.2.4)) and

iii) $\mathcal{A}_{\widetilde{K}}(\widehat{\gamma}^k) = 0$ (by (5.2.2)),

we have

$$\left|\mathcal{A}_{\widetilde{K}}(\widehat{x})\right| > h_0 - \epsilon.$$

Then, by the definition of augmented action (3.3.10) and since

i) $\left|\widetilde{\mathcal{A}}_{\widetilde{K}}(\widehat{x})\right| < \delta$ and

ii) $2(\epsilon + \delta) < \lambda$ (by 5.2.4),

we have

$$\left|\Delta_{\widetilde{K}}(\widehat{x})\right| > \frac{2}{\lambda}(h_0 - \epsilon - \delta) = 2\mathcal{N} - \frac{2(\epsilon+\delta)}{\lambda} > 2\mathcal{N} - 1.$$

Thus, by (4.2.5),

$$\left|\mu_{\mathrm{CZ}}(\widehat{x})\right| > 2\mathcal{N} - 1 - n \geq n + 2 - 1 - n = 1$$

where the second inequality follows from the requirement that $\mathcal{N} \geq n/2 + 1$. We obtained a contradiction since $\mu_{\mathrm{CZ}}(\widehat{x}) = \pm 1$.

Consider now the second case in (5.2.6), i.e. $\left|\widetilde{\mathcal{A}}_{\widetilde{K}}(\widehat{x})\right| > (p-1)h_0$: by the definition of augmented action (3.3.10), we obtain

$$\left|\mathcal{A}_{\widetilde{K}}(\widehat{x})\right| > (p-1)h_0 - \frac{\lambda}{2}\left|\Delta_{\widetilde{K}}(\widehat{x})\right| > (p-1)h_0 - \frac{\lambda}{2}(n+1) =: \tau$$

where the second inequality follows from the fact that $\left|\Delta_{\widetilde{K}}(\widehat{x})\right| < n+1$ (which holds since $\mu_{\mathrm{CZ}}(\widehat{x}) = \pm 1$ and by (4.2.5)). Hence the action of $\widehat{x}$ is outside the interval $(-\tau, \tau)$ and we obtained a contradiction. □

The previous lemma also holds for $\mathfrak{q}\widehat{\gamma}^k$ with the shifted range of actions $(-\tau, \tau) - h_0$. For an interval (a, b) contained in the intersection of the action intervals $(-\tau, \tau)$ and $(-\tau, \tau) - h_0$, Lemma 5.2.5 holds for both *tailed*-capped orbits $\widehat{\gamma}^k$ and $\mathfrak{q}\widehat{\gamma}^k$ and the interval (a, b).

Remark 5.2.6. Observe that such an interval (a, b) exists since $-\tau < \tau - h_0$ due to our initial assumption on p, namely, $(2p-3)h_0 - \lambda(n+1) > 0$ (5.2.3).

For the sake of completeness, we state the result in the following lemma.

Lemma 5.2.7. *The orbits* $\widehat{\gamma}^k$ *and* $\mathfrak{q}\widehat{\gamma}^k$ *are not connected by a solution of (4.2.1) to any* $\widehat{x} \in \widetilde{P}(\widetilde{K})$ *with relative index* ± 1 *with action in*

$$(a, b) \subset (-\tau, \tau) \cap (-\tau - h_0, \tau - h_0).$$

In particular, $\widehat{\gamma}^k$ *and* $\mathfrak{q}\widehat{\gamma}^k$ *are closed in* $C_*^{(a,b)}(\widetilde{K})$ *and* $[\widehat{\gamma}^k] \neq 0 \neq [\mathfrak{q}\widehat{\gamma}^k] \in HFN_*^{(a,b)}(\theta')$*. Moreover, the orbits* $\widehat{\gamma}^k$ *and* $\mathfrak{q}\widehat{\gamma}^k$ *must enter every cycle representing their homology classes, respectively* $[\widehat{\gamma}^k]$ *and* $\mathfrak{q}[\widehat{\gamma}^k]$*, in* $HFN_*^{(a,b)}(\theta')$*.*

Recall that, by (5.2.5), all periodic orbits of ϕ' have action values in the ϵ-neighborhood of $h_0\mathbb{Z}$. With the following lemma we obtain a contradiction and the main theorem follows.

Lemma 5.2.8. *The symplectomorphism* ϕ' *has a periodic orbit with action outside the* ϵ*-neighborhood of* $h_0\mathbb{Z}$*.*

Proof. For ordinary homology classes $\alpha,\ \beta \in H_*(M)$ with $\deg(\alpha),\ \deg(\beta) < 2n$ as in the statement of Theorem 2.2.1, consider $\Phi_{\beta*\alpha}([\widehat{\gamma}^k])$ as an element of the group $HFN_*^{(a,b)}(\theta')$ with $(a, b) = (-\tau, \tau)$. Since $\beta * \alpha = \mathfrak{q}[M]$, then by (4.3.4) and (4.3.5) we have

$$\Phi_\beta \Phi_\alpha([\widehat{\gamma}^k]) = \Phi_{\beta*\alpha}([\widehat{\gamma}^k]) = \Phi_{\mathfrak{q}[M]}([\widehat{\gamma}^k]) = \mathfrak{q}\Phi_{[M]}([\widehat{\gamma}^k]) = \mathfrak{q}[\widehat{\gamma}^k].$$

Take σ and η generic cycles representing the ordinary homology classes α and β, respectively. The chain $\Phi_\eta\Phi_\sigma(\widehat{\gamma}^k)$ represents the homology class $\mathfrak{q}[\widehat{\gamma}^k]$ and hence the orbit $\mathfrak{q}\widehat{\gamma}^k$ enters the chain $\Phi_\eta\Phi_\sigma(\widehat{\gamma}^k)$ (by Lemma 5.2.7). Hence, (see Figure 5.1) there exists an *orbit* $\widehat{y}$ in the chain $\Phi_\sigma(\widehat{\gamma}^k)$ which is connected to both $\widehat{\gamma}^k$ and $\mathfrak{q}\widehat{\gamma}^k$ by trajectories which are solutions of (4.2.1). By the Ball-Crossing Energy Theorem 5.2.1, (5.2.4) and

i) $\mathcal{A}_{\widetilde{K}}(\widehat{\gamma}^k) = 0$

ii) $\mathcal{A}_{\widetilde{K}}(\mathfrak{q}\widehat{\gamma}^k) = -h_0,$

we obtain

$$-\epsilon > \mathcal{A}_{\widetilde{K}}(\widehat{y}) > -h_0 + \epsilon.$$

□

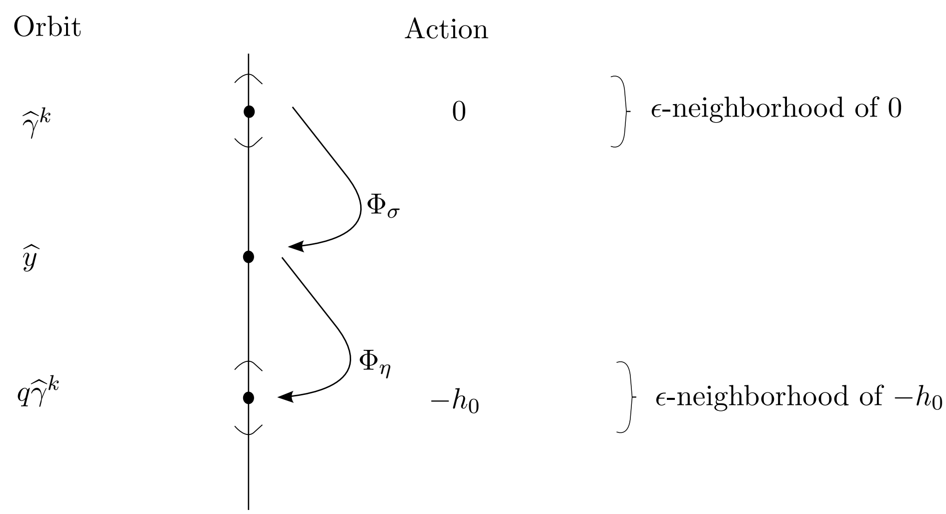

Figure 5.1: The ϵ-neighborhood of $h_0\mathbb{Z}$

5.2.3 Proof of Proposition 2.2.2

We will prove this proposition in four steps. In the first three we assume γ is a constant loop and in the fourth step we consider the general case.

Step 1: Assume that γ is a constant loop, i.e. $\gamma(t) \equiv p$. Then there exists a Hamiltonian $H\colon S^1 \times M \to M$ such that $\phi_t(p) = \varphi_H^t(p)$ where φ_H^t is the Hamiltonian flow associated with H. (Observe that dH is given by $\frac{d}{dt}\phi_t(p) = X_{H_t}(\phi_t(p))$.) The point p is a fixed point of the composition $(\varphi_H^t)^{-1} \circ \phi_t$. Notice that the flux of this composition is equal to the flux of ϕ_t since $(\varphi_H^t)^{-1}$ is the flow of some Hamiltonian usually denoted by H^{inv}. We keep the notation ϕ_t for this composition.

Step 2: There exists a Hamiltonian $H^{'}\colon S^1 \times M \to M$ such that $\phi_t = \varphi_{H'}^t$ near p since $\theta_t = dH_t^{'}$ near p for some $H^{'}$. The composition $(\varphi_{H'}^t)^{-1} \circ \phi_t \equiv id$ near p. (Observe that the flux of $\{\phi_t\}$ is equal to the flux of $\{(\varphi_{H'}^t)^{-1} \circ \phi_t\}$.) Again, keep the notation ϕ_t for this composition.

Step 3: Consider a Hamiltonian $K\colon S^1 \times M \to M$ such that p is a hyperbolic fixed point of φ_K. We have obtained an isotopy $\varphi_K^t \circ \phi_t$ (with the same flux as $\{\phi_t\}$) such that $\gamma(t) \equiv p$ is a hyperbolic fixed point of the symplectomorphism $\varphi_K \circ \phi_1$. We continue denoting the composition $\varphi_K^t \circ \phi_t$ by ϕ_t.

Step 4: Consider now the general case where $\gamma(t)$ is a loop and denote $\gamma(0)$ by p. Applying steps 1 through 3 to the point p, we obtain a symplectic path ϕ_t such that p

is a fixed point of ϕ_1. There exists a loop of Hamiltonian diffeomorphisms η_t such that $\eta_t(p) = \gamma(t)$ (see e.g. [Gin10, Section 5.1] for more details). Then γ is a hyperbolic periodic orbit of the time-one map of the composition $\eta_t \circ \phi_t$. □

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