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REPRESENTATIONS OF RANDOM FLOW*

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ABSTRACT

A representation theorem for two dimensional homogeneous random fields is established, and used to analyze the scope of the Wiener-Hermite expansion method, the significance of Onsager's conjecture, and the assumption of universal equilibrium. It is further used to derive a numerical method for the study of random flow. Generalizations to three dimensional flow are also presented, but remain mostly conjectural.

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INTRODUCTION

A number of ingenious statistical theories of turbulence have become available in recent years (see, e.g., Saffman (1968), Orszag (1970)); unfortunately, most of these theories rely on mathematical assumptions whose physical significance is unclear, their mutual relationships are not evident, and they do not readily lead to algorithms for use in practical applications. In the present paper, we shall attempt to overcome these difficulties. Most of the work will be applicable only to the admittedly simplified problem of two-dimensional incompressible flow.

The main tool in the analysis is a representation of a two dimensional homogeneous isotropic vorticity field as a running average of circular vortices. The validity of this representation is first established, and then used to analyze the scope of the Wiener-Hermite expansion method (Meecham and Jeng (1968)), the significance of Onsager's conjecture (Onsager (1949)), and the meaning of universal equilibrium (Batchelor (1960)). It is further used to obtain a numerical method (Chorin (1973)) and to establish a theory of the inertial range (Chorin (1970)). Generalizations to three dimensional flow are presented, but remain mostly conjectural.

We shall consider flow fields $\underline{u} = (u_1, u_2, u_3)$ satisfying the Navier-Stokes equations for an incompressible fluid,

(1a) $$\partial_t \underline{u} + (\underline{u} \cdot \underline{\nabla})\, \underline{u} = \text{grad } p + \frac{1}{R} \Delta \underline{u},$$

(1b) $$\text{div } \underline{u} = 0,$$

where t is the time, $\underline{\nabla}$ is the gradient operator, p is the pressure, $\Delta \equiv \underline{\nabla}^2$, and R is the Reynolds number, which is assumed to be large. The flow fills out the whole (x_1, x_2, x_3) space ((x_1, x_2) space if there are two dimensions); the initial data are random; the randomness presumably hides our ignorance of the precise nature of the data. The vorticity vector, which we shall use extensively, is

$$\underline{\xi} = \text{curl } \underline{u} \ ;$$

in the case of two dimensional flow $\underline{\xi}$ has a single component denoted by ξ.

REPRESENTATION OF A TWO DIMENSIONAL HOMOGENEOUS FIELD AS A RUNNING AVERAGE

Let $\xi(x_1, x_2)$ be a homogenous random field in the two dimensional (x_1, x_2) space. (For definitions and analysis, see Gelfand and Vilenkin (1964), Doob (1953)). The field ξ has a spectral representation

$$\xi(\underline{x}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp(2\pi i \, \underline{k} \cdot \underline{x}) \, Z(d\underline{k}) \ ,$$

where $\underline{x} = (x_1, x_2)$, $\underline{k} = (k_1, k_2)$, and $Z(\underline{k})$ is a random measure. The correlation function $B(\underline{r})$ of ξ is

$$B(\underline{r}) = E[\xi(\underline{x} + \underline{r}) \, \xi(\underline{x})] \ ,$$

where $E[\cdot]$ denotes an expected value and $B(\underline{r})$ is independent of $\underline{x}$ by definition. We have

$$B(\underline{r}) = \int\int \exp(2\pi i \, \underline{r} \cdot \underline{k}) \, dF(\underline{k}) \ ,$$

where

$$dF(\underline{k}) = E[|Z(d\,\underline{k})|^2] .$$

Assume that

(2) $$\iint |B(\underline{r})|\, d\underline{r} < \infty .$$

$F(\underline{k})$ is then absolutely continuous and $B(\underline{r})$ is an ordinary Fourier transform,

$$B(\underline{r}) = \iint \exp(2\pi i\, \underline{k}\cdot\underline{r})\, f(\underline{k})\, d\underline{k} ,$$

where

$$f(\underline{k}) = \frac{\partial^2 F}{\partial k_1 \partial k_2} .$$

If (and only if) F is absolutely continuous, the field ξ is a field of moving averages, i.e., it has a representation of the form

(3) $$\xi(\underline{x}) = \iint f^*(\underline{x}-\underline{s})\, d\eta(\underline{s}) ,$$

where $\underline{s} = (s_1, s_2)$, f^* is an ordinary function of its argument, the process η has orthogonal increments and satisfies

$$E[|d\eta|^2] = ds_1\, ds_2 \equiv d\underline{s} ;$$

furthermore,

(4) $$B(\underline{r}) = \iint |f(k)|^2 \exp(2\pi i\, \underline{r}\cdot\underline{k})\, d\underline{k} .$$

In intuitive terms, if ξ is a homogeneous field satisfying the condition (2), it is a superposition, with random coefficients, of translates of a single function f;* f^* determines the spectrum of the field. The proof is a simple generalization to two dimensions of the argument given in Doob (1953), page 532. Formally,

$$\xi(\underline{x}) = \iint \exp(2\pi i\, \underline{x}\cdot\underline{k})\, f(\underline{k})\, d\eta^*(\underline{k}) ,$$

where $d\eta^*(\underline{k})$ is a field with orthogonal increments and

$$E[\,|d\eta^*(\underline{k})|^2] = dk_1\, dk_2 \equiv d\underline{k}\,.$$

Symbolically,

$$\xi(x, y) = \int\int \exp(2\pi i\, \underline{x}\cdot\underline{k})\, f(\underline{k})\, \eta^{*'}(\underline{k})\, d\underline{k}\,, \tag{4}$$

where $\eta^{*'}$ is the (generalized) derivative of η^* with respect to k_1 and k_2. Let $\eta(\underline{k})$ be the Fourier transform of $\eta^*(\underline{k})$, i.e., symbolically

$$\eta(\underline{k}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp(2\pi i\, \underline{k}\,\underline{s})\, \eta^*(\underline{s})\, d\underline{s}\,,$$

then Parseval's identity applied to (4) yields

$$\xi(x, y) = \int\int f^*(\underline{s})\, \eta'(\underline{x}+\underline{s})\, d\underline{s} = \int\int f^*(\underline{s}-\underline{x})\, d\xi(\underline{s}).$$

The formal operations can be justified by proper definition of the Fourier transform (see Doob (1953)). Furthermore,

$$E[\xi(\underline{x}+\underline{r})\;\; \xi(\underline{x})] = \int\int f^*(\underline{s}-\underline{r})\, \overline{f^*(s)}\, d\underline{s}\,,$$

$$= \int\int |f(\underline{s})|^2 \exp(2\pi i\, \underline{r}\cdot\underline{s})\, d\underline{s}\,,$$

thus establishing equation (3). Finally, if ξ is isotropic as well as homogeneous, $B(\underline{r}) = B(r)$, $r = |\underline{r}|$, and thus $f(\underline{s}) = f(s)$, $s = |\underline{s}|$.

Our purpose is to identify the field ξ with the vorticity field of a two dimensional incompressible flow. The preceding argument shows that if ξ is a homogeneous isotropic field, it is in fact a superposition of circular vortices. The task at hand is to find out from the equations of motion whether such vortices do in fact arise and thus whether vorticity can in fact be regarded as a homogeneous isotropic field. This task will be undertaken in the next section. It is worthwhile to note that formula

(2) provides the means for constructing a Gaussian homogeneous and isotropic field admitting a spectrum with an inertial range; this possibility is in itself of some significance (see the discussion at end of Hopf (1952)).

THE ONSAGER CONJECTURE

Onsager (1949) studied the behavior of a system of point vortices in the plane, and, on the basis of an argument drawn from statistical mechanics, conjectured that vortices of the same sign will tend to cluster around the strong ones, and furthermore, that the larger compound vortices formed will be the only conspicuous features of the motion. An argument based on the behavior of a system of point vortices fails to take into account the conservation of vorticity per unit area in incompressible hydrodynamics (Batchelor (1967)); if this conservation property is taken into account, one can conjecture that in a system of vortices of finite area, vortices of the same sign will order themselves around the strongest ones to produce vortices of ever increasing total strength and total area. The resulting vortices would be such that the vorticity per unit area would be highest in absolute value at their center.

Substantial numerical evidence in support of this conjecture is available. In particular, a number of problems in plasma physics are formally identical to problems in two dimensional vortex motion, and their solutions exhibit the process of vortex consolidation (see Hockney (1970)). A number of calculations with the Navier-Stokes equations also display the process of vortex formation and growth, and attendant energy transfer from small to large "eddies" (see Lilly (1969)).

A further argument in favor of this conjecture is the following: consider a vorticity patch in which the vorticity per unit area increases as one approaches a center. If a subpatch is introduced, whose vorticity density is larger than the ambient density, it will migrate outward. The reasons for this behavior are obvious (see fig. 1). If the subpatch P has a vorticity higher than the ambient positive vorticity, it will move the strong vorticity in A outward, and the weak vorticity B inward, as shown on the figure. The displaced vorticity will induce a motion of P towards the center. A similar argument explains the outward migration of weak vorticity; the occurrence of these phenomena is readily verified numerically. Circularly stratified patches of vorticity are thus stable, and it is reasonable to believe that they will be formed whatever the initial vorticity distribution may be. This argument is unaffected by the presence of a small viscosity (large Reynolds number R), since in the absence of boundaries the effect of a small viscosity is small (Ebin and Marsden (1970)).

The Onsager conjecture relates the properties of homogeneous isotropic random fields to properties of the Navier-Stokes equation. One can visualize the flows as follows: given arbitrary initial data, vortices will form in them; these vortices will get organized into larger vortices, etc. One can identify these vortices with the vortices occurring in formula (2). Clearly the vortices cannot be individually though of as circular, in particular since neighboring vortices will deform each other; but if one is interested in phenomena of a scale large compared to the scale of the vortices, this may not matter, just like

Wiener paths idealize brownian motion on time scales large compared to the time intervals between collisions independently of exact collision times. Only inasmuch as this idealization is valid can a random solution of the Navier-Stokes equations be thought of as being homogeneous and isotropic.

One can use the preceding argument to determine the form of the energy spectrum of two dimensional random flow in the limit as the viscosity tends to zero and the wave number tends to infinity (in this order). As the viscosity tends to zero, the solutions of the Navier-Stokes equations tends to solution of the corresponding inviscid equations. The inviscid (Euler) equations conserve the integral of ξ^2 (Batchelor (1967)); this means in particular that $\int k^2 E(k)\,dk$ is bounded, where $k = |\underline{k}|$, and $E(k) = |f(k)|^2$ is the energy spectrum, (Batchelor 1960)). The distribution of values of ξ depends on the initial data (since ξ per unit area is conserved), but if a broad range of values of ξ is introduced initially, each individual vortex will have a vorticity distribution encompassing a wide range of values of ξ; ξ must increase towards the center slower than $1/r$, where r is the distance from the center of the vortex. It is, however, reasonable to assume that one comes arbitrarily closely to this distribution in each individual vortex, and thus $E(k) \sim k^{-3+\epsilon}$, ϵ small and positive; in each individual vortex, $\xi \sim r^{-1+\epsilon'}$, $\epsilon' > 0$. Such a spectrum is indistinguishable from the k^{-3} spectrum derived by Leith (1968) by a different argument. (See also Chorin (1969, 1970)).

In summary, a random homogeneous isotropic vorticity field must be a superposition of circular vortices; and there is every reason to think that an arbitrary solution of the Navier-Stokes equations at large R is in fact such a superposition.

UNIVERSAL EQUILIBRIUM THEORY

Before making use of the argument of the preceding sections we would like to derive the representation of the vorticity field by a more heuristic argument, which will shed some additional light on its significance. This argument was previously presented in Chorin (1969, 1970).

The occurrence, in turbulent flow, of an energy spectrum E(k) which has for large k a universal form, is often explained by the universal equilibrium theory, well summarized in Batchelor (1960): The range of wave numbers k which contain most of the energy ("the energy containing eddies") can be regarded as a definite group, with characteristic velocity $u = \sqrt{E[\int u^2 dx]}$ and characteristic length $\ell = k_{en}^{-1}$, where k_{en} is a typical wave number in the group. The characteristic time of these eddies is ℓ/u, and the time scale of their decay is $u/|du/dt|$; these times are experimentally found to be comparable, and thus this range of k's has no feature resembling an equilibrium. It is, however, assumed that for large k the eddies have a characteristic time small in comparison with the scale of the over-all decay, and thus may be associated with degrees of freedom in approximate statistical equilibrium. Let $\hat{\underline{u}}(\underline{k})$ be the (formal) Fourier transform of $\underline{u}(x)$, let $\underline{k}_{eq}$ be a wave number typical of the equilibrium range, with magnitude $k_{eq} = |\underline{k}_{eq}|$, and let $\hat{u}_{eq}$ be a typical amplitude of $\hat{\underline{u}}(\underline{k})$, $k = |\underline{k}|$ on the equilibrium range, for example,

$\hat{u}_{eq} = |\underline{\hat{u}}(\underline{k}_{eq})|$. Write

(5) $$K = k_{eq}/k_{en}, \quad U = \hat{u}_{eq}/u.$$

The characteristic time of $\hat{u}_{eq}$ is $(k_{eq}\,\hat{u}_{eq})^{-1} = (KU)^{-1}(k_{en}u)^{-1}$, and the assumption of universal equilibrium reads

(6) $$(KU)^{-1}\left|\frac{du}{dt}\right| << k_{en}\,u^2.$$

The quantity u in (6) is the result of an averaging operation; it is reasonable, and consistent with experience in classical statistical mechanics, to assume that if (6) holds on the average on an ensemble it holds for most systems in that ensemble, i.e., for most flows there exists a range of k's satisfied by k_{eq} such that

(7) $$(KU)^{-1}\left|\frac{du}{dt}\right| << k_{en},u^2 = \sqrt{\int u^2\,dx}\ .$$

If this stronger condition is satisfied, condition (6) will be satisfied à fortiori.

Consider a solution of the Navier-Stokes equations (1); take its (formal) Fourier transform $\underline{\hat{u}}(\underline{k})$ (there is a substantial difficulty in finding a proper definition here; we shall overlook this difficulty in the present heuristic argument). $\underline{\hat{u}}(k)=(\hat{u}_1, \hat{u}_2)$ satisfies the following equation:

(8) $$\partial_t\,\hat{u}_\alpha = ik_\beta\,P_{\alpha\gamma}\,Q_{\delta\gamma} - \frac{1}{R}\,k^2\,\hat{u}_\alpha, \qquad k = |\underline{k}|,$$

where

$$Q_{\delta\gamma} = \int\int \hat{u}_\beta (\underline{k} - \underline{k}') \, \hat{u}_\gamma (\underline{k}') \, dk'$$

$$P_{\alpha\gamma} = \delta_{\alpha\gamma} - \frac{k_\alpha k_\gamma}{k^2} , \qquad (\delta_{\alpha\gamma} \text{ the Kronecker delta})$$

and the summation convention is in use. $\hat{\underline{u}}(\underline{k})$ is the complex conjugate of $\hat{\underline{u}}(-\underline{k})$, and the pressure has been eliminated through the use of the equation of continuity

$$k_\alpha \hat{u}_\alpha = 0 ,$$

giving rise, in the well known manner, to the projection $P_{\alpha\gamma}$. We now study the behavior of the solutions in the inertial range, i.e., in the limit as $R \to \infty$ and $k \to \infty$ (in that order). As $R \to \infty$, the solutions of the Navier-Stokes equations which vanish outside a compact domain tend to the solutions of the Euler equations strongly, as well as on L_1 (Ebin and Marsden (1970)); the respective Fourier transforms then tend to each other uniformly on every bounded region in $\underline{k}$ space and the limit $R \to \infty$ may be studied by setting $1/R = 0$. We assume that this is still the case here, when the region in which $\underline{u} \neq 0$ is not specified. Under these conditions, the last term in equation (8) can be dropped. To study the second limit, $k \to \infty$, we perform the scaling

$$\underline{u}^* = \underline{u}/U$$
$$k^* = \underline{k}/K ,$$

with U, K defined above. Substitution into (8) leads to

$$\text{(a)} \qquad (KU)^{-1} \, \partial_t \hat{u}_\alpha^* = ik_\beta^* (\delta_{\alpha\gamma} - \frac{k_\alpha^* k_\gamma^*}{k^{*2}}) \, Q_{\beta\gamma}^* ,$$

If k is in the inertial range, the right hand side of (a) is of order $k_{en} u^2$, the left hand side of order $(KU)^{-1} |du/dt|$; by (6), we have

$$(10) \qquad \lim_{k \to \infty} ik_\beta \left(\delta_{\alpha\gamma} - \frac{k_\alpha^* k_\gamma^*}{k^{*2}} \right) Q_{\beta\gamma}^* = 0.$$

Consider first the limiting equation

$$ik_\beta \left(\delta_{\alpha\gamma} - \frac{k_\alpha^* k_\gamma^*}{k^2} \right) Q_{\beta\gamma}^* = 0.$$

This is merely the Fourier transform of the steady (time independent) Euler equations; it is satisfied by the Fourier transform of any circular vortex, in particular by any vortex of the form

$$(11) \qquad \hat{u}_1^* = - ik_2/k^\beta, \quad \hat{u}_2^* = ik_1/k^\beta, \qquad \beta \text{ constant.}$$

It is readily seen, e.g., by application of Dirichlet's lemma (see, e.g., Carslaw (1950)), that an arbitrary superposition of translates of vortices of the form (11),

$$\sum_j C_j(t)\, \underline{u}^* e^{i\underline{a}_j(t)\cdot \underline{k}}$$

where C_j, $\underline{a}_j = (a_{j1}, a_{j2})$ are functions of t, will satisfy the weaker condition (10). We have thus rederived the fact that the large frequency spectrum of the flow is a superposition of the contributions due to a collection of circular vortices, each one of which is in fact a solution of the time-independent equations. The conservation of vorticity can then be used to show that

$$E(k) \sim k^{-3+\epsilon}, \quad \epsilon > 0,$$

in a manner similar to the one employed at the end of the preceding section. We have thus both used and justified the universal equilibrium hypothesis.

EVOLUTION OF THE FLOW

So far we have established a representation of a random vorticity field for a fixed time t. We now turn to the crucial problem of determining the evolution of the field when each of its sample flows evolves according to the Navier-Stokes equations (1). We first present a plausible but incorrect argument which dutifully leads to disaster. The reasons for this procedure are, first, that analysis of the wrong argument will point the way to the right one, and secondly, that some of the negative conclusions will also apply to the Wiener-Hermite expansion method, which has recently been the object of lively controversy.

The incorrect argument runs as follows: First, derive equations describing the motion of vorticity. This is (Batchelor (1967))

$$\begin{aligned} \partial_t \xi + (\underline{u} \cdot \underline{\nabla}) \xi &= \frac{1}{R} \Delta \xi \\ \Delta \psi &= -\xi, \\ u_1 = -\partial_2 \psi, \quad u_2 &= \partial_1 \psi \end{aligned} \tag{12}$$

where ψ is a stream function. Substitute the representation (3) into this set of equations, multiply by $d\eta(s)$, and examine the expected values of the result. Since the increments $d\eta(s)$ are orthogonal, we obtain an equation for a single circular vortex with $\xi(\underline{x}) = f^*(\underline{x}-\underline{s})$. The term $(\underline{u}\cdot\underline{\nabla})\xi$ vanishes (since a circular vortex is a solution of the stationary Euler equation), and we obtain the following result:

$$\partial_t f^* = \frac{1}{R}\,\Delta f^*\,,$$

i.e., since f^* defines the spectrum E(k), there is no change in E except that due to viscous decay - surely a fallacious conclusion. The error lies, of course, in the fact that the representation (3) is valid only to a fixed time t. The homogeneous field η is not time invariant, and differentiation of (3) with respect to t must take this fact into account. This is best understood in conjunction with Onsager's conjecture. The energy transfer between "eddies" and the evolution of the correlation function occur because vortices group themselves into larger vortices; thus, the effect of the nonlinear terms is to induce new correlations between the flow at different points and to destroy the orthogonal character of the increments $d\eta$. On a scale large with respect to the vortex size, the evolving flow can be described at each time t as a running average with respect to a field with orthogonal increments, but each time t with a new f and a new η.

This difficulty also arises the Wiener-Hermite expansion method (see, e.g., Meecham and Jeng (1968)), where it was substantially analyzed by Crow and Canavan (1970). In the Wiener-Hermite expansion method, the fields are expanded in an infinite series of running averages with respect to Hermite functionals of white noise, where white noise is defined as the generalized derivative of a field with independent Gaussian increments. The occurrence of an infinite series is the price one pays for expressing the fields in terms of Gaussian fields only. If the field is exactly Gaussian, the Wiener-Hermite expansion has a single term, which is identical to (3). (In two dimensional incompressible flow, a vorticity field which is initially Gaussian will remain Gaussian for all time.) In practical applications, the Wiener-Hermite expansion is truncated after a few terms, on the plausible ground that the observed turbulence is nearly Gaussian. Relations between the coefficients in the expansion are obtained by substitution into the Navier-Stokes equations and use of the statistical orthogonality of the various terms, but no provision is made for change in the white noise fields. The resulting equations are thus incorrect, and in particular do not provide a description of energy transfer between scales. Illustrative sample calculations were given by Crow and Canavan.

One way to solve the problem is to construct a few sample flows by approximating the integral (3). In particular, if one assumes that the flow has had some time to evolve, one can set

$$\xi = \sum_i \eta_i f^*(\underline{x} - \underline{x}_1) \tag{13}$$

where $f^* \sim |\underline{x} - \underline{x}_i|^{-1}$ for small $|\underline{x} - \underline{x}_i|$, x_i is a center of a vortex, and the η_i are random numbers. Then allow the flows to evolve according to equations (12). If the f^* have small support, one can assume that the induced velocity field varies little over that support and thus the motion of each one of the vortices in (13) is described by the motion of its center $\underline{x}_i$. This leads to a system of ordinary differential equations for the $\underline{x}_i$. Such a method was in fact derived in Chorin (1973) and shown to be applicable to problems in which the flow is not homogeneous and where boundaries are present.

THREE DIMENSIONAL FLOW

It would be of great interest to generalize the preceding results to the case of three dimensional flow. There are, however, major differences between the two and three dimensional cases. In particular, in three dimensions the vorticity is a three dimensional solenoidal vector, and the argument which leads to the representation (3) fails. There appears to be no possibility of existence for a three dimensional homogeneous isotropic random solenoidal vector field. I conjecture that one has only intermittent homogeneity and isotropy, i.e., if $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$, and if we define the correlation tensor by

$$B_{ij}(\underline{x}+\underline{r}) = E[\xi_i(\underline{x})\, \xi_j(\underline{x}+\underline{r})],$$

then if $|\underline{r}_1| = |\underline{r}_2| = r$, for every ϵ and $\underline{x}$ there would exist an $\underline{X}$ such that

$$|B_{ij}(\underline{x}+\underline{r}_1) - B_{ij}(\underline{x}+\underline{X}+\underline{r}_2)| \leq \epsilon\ .$$

It is furthermore reasonable to assume by analogy that the vorticity field is a sum of vortex tubes. Such tubes can be stretched, and it is therefore no longer true that if the flow is initially Gaussian it will remain Gaussian for all time. The velocity field is subject to two constraints: it is incompressible, and its energy is non-increasing. I conjecture that these constraints result in the following Hölder inequality

(14) $$|\underline{u}(\underline{x}+\underline{r}) - \underline{u}(x)| \leq \text{constant} \quad |\underline{r}|^{\beta}, \quad \beta > 1/3,$$

independently of the amount of stretching. As already noted by Onsager (1949), such an inequality, which essentially restricts the possible structures of the vortex cores, leads to a Kolmogoroff spectrum $E(k) \sim k^{-5/3}$. An interesting counterexample was given by Marsden et al. (to appear), but it is presumably unstable and thus cannot occur except for special initial data. The arguments in favor of the conjecture above are the following:

(i) Not only does (14) lead to a Kolmogoroff law, but it also provides a clear physical picture in which the large "eddies" (which cause the stretching) are independent statistically from the small eddies (which make up the cores); this is assumed in Kolmogoroff's derivation.

(ii) As explained in Marsden, Ebin and Fischer (to appear), the exponent in Kolmogoroff's laws is intimately tied to the possibility of proof of existence of solution for the Navier-Stokes equations. One can thus conjecture that all existing flows must satisfy (14).

(iii) Crow (1970) made the remarkable discovery that there apparently exists a universal cut-off coefficient for the numerical evaluation of the self induction of vortex lines, suggesting the existence of a universal vortex structure.

(iv) Finally, the assumption of universal equilibrium leads to an equation similar to (10), which can be satisfied only by vortex tubes whose radius of curvature is outside the initial range, with the whole contribution to the energy spectrum in the inertial range coming from the core.

An effort to confirm this conjecture by numerical means is presently in progress.

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FIGURE CAPTION

Fig. 1. Stability of a Circularly Stratified Vorticity Field.

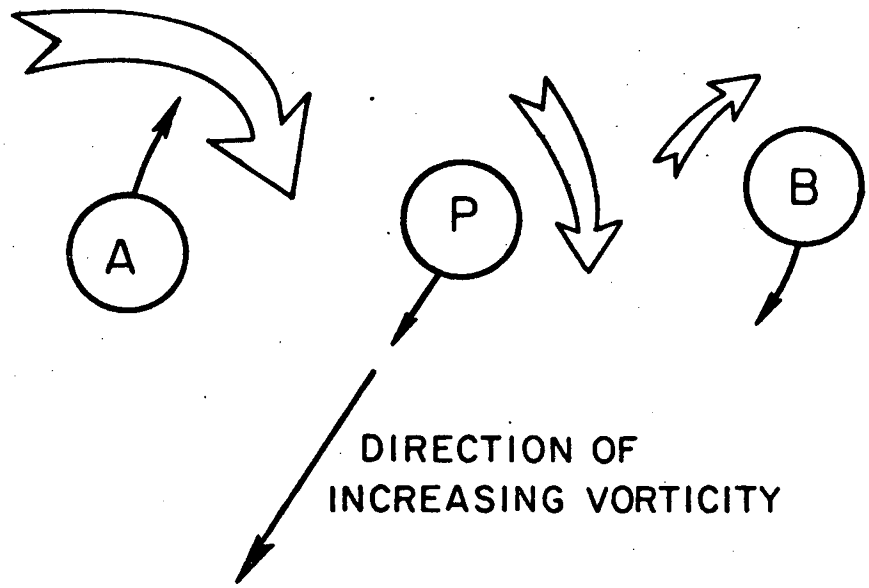

Fig. 1

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