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Role of Wilson Lines in 3D Quantum Gravity

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Physics

by

Ashwin Dushyantha Hegde

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ABSTRACT OF THE DISSERTATION

Role of Wilson Lines in 3D Quantum Gravity

by

Ashwin Dushyantha Hegde

Doctor of Philosophy in Physics

University of California, Los Angeles, 2019

Professor Per J. Kraus, Chair

Gravity in three dimensions and $\text{AdS}_3/\text{CFT}_2$ is an interesting playground to study some of the issues of quantum gravity. Using the Chern-Simons description of 3D gravity, we construct gravitational Wilson lines in asymptotiacally AdS spacetimes. In this thesis, we study the role of gravitational Wilson lines and find the correct dictionary to the dual CFT.

On the CFT side, networks of Wilson lines correspond to semi-classical limits of conformal blocks. We establish some of the dictionary in Chapter 1. By explicit computation, we verify these for the cases of pure gravity and particular instances of gravity with higher spin fields.

In chapter 2 we develop the quantization of this Wilson line. We compute the gravitational self energy of a particle in AdS_3 to first non-trivial order using a single boundary to boundary Wilson line. In the CFT, this reproduces the $1/c$ correction to the two point function of the corresponding primary operator.

Various renormalization ambiguities arise at higher order. We study these in Chapter 3 and extend the previous computation to order $1/c^3$, finding agreement with CFT results. Renormalization ambiguities are fixed by imposing conformal invariance.

The dissertation of Ashwin Dushyantha Hegde is approved.

Peter Petersen

Eric D'Hoker

Per J. Kraus, Committee Chair

University of California, Los Angeles

2019

To my parents

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M. Besken, E. D'Hoker, A. Hegde, P.Kraus,

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CHAPTER 1

Holographic Conformal Blocks from Interacting Wilson Lines

We present a simple prescription for computing conformal blocks and correlation functions holographically in AdS_3 in terms of Wilson lines merging at a bulk vertex. This is shown to reproduce global conformal blocks and heavy-light Virasoro blocks. In the case of higher spin theories the space of vertices is in one-to-one correspondence with the space of $\mathcal{W}_N$ conformal blocks, and we show how the latter are obtained by explicit computations.

1.1 Introduction

This paper continues a program aimed at determining the AdS gravity description of conformal blocks. For previous work see $[5, 13, 17, 29, 45, 46, 56, 58-60]$. The conformal block decomposition of correlation functions, combined with the constraints of unitarity and crossing symmetry, is a powerful nonperturbative framework in which to study strongly interacting conformal field theories $[35, 72, 74, 76]$. It has also proven to be very effective in elucidating the AdS/CFT correspondence, in particular the emergence of local physics in the bulk $[36, 40, 44, 45, 54, 55, 64, 68]$.

To push this program forward it is very useful to have in hand bulk AdS representations of conformal blocks. In $[59]$ it was shown that global conformal blocks with external scalar operators have a simple bulk representation in terms of "geodesic Witten diagrams". This refers to a tree level exchange Witten diagram with a pair of cubic vertices, except that the vertices are not integrated over all of AdS, but only over geodesics connecting the boundary points hosting the external operators. This result leads to a strikingly simple procedure for

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expanding the full Witten diagram in conformal blocks.

In the case of AdS_3/CFT_2 the story is especially rich since the global conformal algebra is enhanced to an infinite dimensional algbebra, namely Virasoro or something larger, such as a $\mathcal{W}$-algebra. Here one focusses on the regime of large central charge, since this is the regime where the bulk becomes classical. In [5, 13, 45, 58, 60] it was shown that heavy-light Virasoro blocks (defined by scaling some operator dimensions with c, while keeping others fixed) are reproduced by geodesic Witten diagram operators, now not in pure AdS_3 but in a new geometry produced by backreaction from the heavy operators.

Conformal blocks for $\mathcal{W}$-algebras are relevant to the recent interest in higher spin AdS/CFT dualities. In particular, Gaberdiel and Gopakumar [47] proposed to consider the minimal model cosets

$$\frac{SU(N)_k \oplus SU(N)_1}{SU(N)_{k+1}} \tag{1.1.1}$$

in the 't Hooft limit $k, N \to \infty$ with $\lambda = N/(N+k)$ fixed. This was argued to be holographically dual to the higher spin theory of Prokushkin and Vasiliev [71]. The theory in the 't Hooft limit has left and right moving $\mathcal{W}_\infty(\lambda)$ algebras [48, 49]. These are nonlinear algebras with an infinite tower of conserved currents. It is then of interest to know the corresponding conformal blocks, but these are rather challenging to obtain directly on account of the complexity of the algebra.

At fixed N the algebras are $\mathcal{W}_N$, with conserved currents of spins $s = 2, \ldots N$. One of the main results of this paper is to provide a very simple bulk prescription for the conformal blocks of these algebras in the large c limit. Furthermore, this can be used as a backdoor approach for obtaining (some of) the $\mathcal{W}_\infty(\lambda)$ blocks, as this can be achieved by the analytic continuation $N \to -\lambda$; see [21, 56] for examples of this approach. We also note that upon setting $N = 2$ the conformal blocks are those of the Virasoro algebra.

The setup we use can be motivated as follows. We note that the central charge of the coset theory is

$$c = (N - 1)\left(1 - \frac{N(N+1)}{(N+k)(N+k+1)}\right) \tag{1.1.2}$$

To take $c \to \infty$ at fixed N we can take the limit $k \to -N - 1$, dubbed the "semiclassical

limit" in [70]. The negative value of k results in a non-unitary theory, manifested for example by negative dimension primaries in the spectrum. As a result, this limit does not provide a healthy example of the AdS/CFT correspondence in Lorentzian signature (see also [69] for related discussion). However, as noted above it does act as a useful stepping stone for obtaining results in the unitary 't Hooft limit via analytic continuation in N. It is also of interest — perhaps as a warmup example — as a very explicit and tractable setup where many details of AdS/CFT an be worked out.

For example, all coset primaries in this limit can be identified in the bulk, at least below the black hole threshold. The bulk description is in terms of SL(N)$\times\widetilde{SL(N)}$ Chern-Simons theory coupled to matter. Coset primaries are labelled by a pair of SL(N) highest weights, (Λ_+, Λ_-). These are highest weights of finite dimensional representations of SL(N). Primaries of the form $(0, \Lambda_-)$ have scaling dimension $\Delta \sim c$; they are "heavy" operators, and are described in the bulk by flat SL(N)$\times\widetilde{SL(N)}$ connections [23]. On the other hand primaries of the form $(\Lambda_+, 0)$ have $\Delta \sim O(1)$; these light operators are described by perturbative matter in the bulk. The general (Λ_+, Λ_-) is then described by light matter fields propagating in the heavy classical background [57, 70].

The main result of this work is a simple and usable expression for computing correlators of these operators, significantly extending previous work. Let us first consider the case of n light operators. The correlator is described by n bulk-to-boundary propagators meeting at an n-point vertex, according to the following rules. Each light operator corresponds to a representation of SL(N)$\times\widetilde{SL(N)}$ with highest weight state $|hw\rangle_i|\widetilde{hw}\rangle_i$, $i = 1, \ldots n$. We then attach a Wilson line to each such state[1], emanating from the associated boundary point x_i to a point in the bulk, $Pe^{\int_{x_i}^{x_b} A}Pe^{\int_{x_i}^{x_b} \tilde{A}}$. Since the connections are flat, the choice of path does not matter. The bulk vertex located at x_b is defined by choosing a singlet state $|S\rangle$ in the tensor product of representations corresponding to the boundary operators. In general, there are many choices for such singlet states, and as we discuss below these are in one-to-one correspondence with conformal blocks, as can be seen by taking the tensor product of pairs

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[1] Wilson lines first made an appearance in these theories in the context of entanglement entropy [8, 30], and have appeared more recently as a probe of black hole solutions [24].

of operators, and then combining terms in the product into singlets. With these ingredients in hand, the correlator is[2]

$$G_S(z_i, \bar{z}_i) = \langle S| \prod_{i=1}^n Pe^{\int_{x_i}^{x_b} A}|hw\rangle_i Pe^{\int_{x_i}^{x_b} \tilde{A}}|\widetilde{hw}\rangle_i \qquad (1.1.3)$$

The correlator is independent of the choice of x_b, as seen by noting that changing x_b just introduces a group element that acts on the singlet state as the identity. To include heavy primaries $(0, \Lambda_-)$ we still use (1.1.3) but now with $(A, \tilde{A})$ taken to be the flat connection representing the heavy background; this is especially simple in the case of two heavy operators in conjugate representations, which is all that we consider in this paper, while more generally one needs to solve a nontrivial monodromy problem [29]. The general (Λ_+, Λ_-) primary is included by taking the location of a light $(\Lambda_+, 0)$ primary to coincide with the insertion point of the heavy $(0, \Lambda_-)$ primary.

Our master formula (1.1.3) reduces the problem of computing correlators to computing SL(N) matrix elements. We will verify that we correctly reproduce various known results for four-point functions. First, it's easy to see that we reproduce all previous results [29,45,56,57] for vacuum blocks. Setting $N = 2$ and taking all operators to be light we obtain the well known formula for global conformal blocks. Taking two operators to be heavy we correctly reproduce heavy-light Virasoro blocks. For $N = 3$ with four light operators we obtain the result for $\mathcal{W}_3$ blocks found in [38]. Allowing N to be arbitrary and taking light operators in the fundamental and anti-fundamental representations we reproduce previous results derived using the Coulomb gas formalism [67]. In all these cases, the primaries we consider have negative scaling dimension, due to the underlying non-unitarity. However, it is easy to analytically continue to positive dimensions and obtain results in the unitary regime.

In our construction, each choice of singlet state yields a correlator. As we already mentioned, there is a natural basis for such singlet states that gives a one-to-one correspondence with conformal blocks. The general correlator is then a general sum over products of left and right moving conformal blocks. Of course, any particular theory will lead to particular

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[2]An equivalent formula was proposed and studied in the $N = 2$ context in the recent paper [17], which appeared while this work was in progress.

coefficients in this sum. For example, this would be the case if we had derived (1.1.3) starting from, say, a Lagrangian. In principle, it should be possible to start from the equations of Prokushkin and Vasiliev and derive the precise correlators that reproduce those of the coset theory, and it would be very interesting to do so.

Apart from a relation to any particular CFT, what the Wilson line approach does is allow one to compute conformal blocks for operators in degenerate representations of the chiral algebra. For example, in the $N = 2$ Virasoro case the dimensions of degenerate primaries are given by the famous Kac formula, $h = h_{r,s}(c)$. As $c \to \infty$,

$$h_{1,s}(c) = -\frac{s-1}{2} + O(1/c) , \quad h_{r,1}(c) = -\frac{r^2-1}{24}c + O(c^0) . \tag{1.1.4}$$

Light operators of dimension $h_{1,s}$ will be seen to be described by Wilson lines in the spin $j = (s-1)/2$ representation of SL(2), while heavy operators of dimension $h_{r,1}$ correspond to flat connections whose holonomy around the boundary has winding number r. Since minimal models are built up out of degenerate representations, we can use Wilson lines and flat connections to compute correlators in these theories.

1.2 Correlation functions: general formulation

In this section we motivate and present our general expression for correlators and conformal blocks, and illustrate with a few simple examples.

1.2.1 Preliminaries

We will be dealing with the group SL(N)$\times\widetilde{SL(N)}$. The generators of the principally embedded SL(2) are denoted as[3] T_i, $i = -1, 0, 1$ and obey $[T_i, T_j] = (i-j)T_{i+j}$. We similarly introduce $\tilde{T}_i$ generators for $\widetilde{SL(2)}$.

Each primary $\mathcal{O}_i$ will be associated with a finite dimensional representation $(\mathcal{R}_i, \tilde{\mathcal{R}}_i)$ of SL(N)$\times\widetilde{SL(N)}$. We denote the highest weight state in this representation as $|hw\rangle_i|\widetilde{hw}\rangle_i$,

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[3]These are typically denoted as L_i, but we reserve L_i for SL(2) matrices in the N dimensional defining representation of SL(N).

where the notion of highest weight is determined by maximizing the eigenvalues of T_0 and $\tilde{T}_0$. The scaling dimensions of these operators $(h_i, \tilde{h}_i)$ are determined by the highest weights:

$$T_0|hw\rangle_i = -h_i|hw\rangle_i , \quad \tilde{T}_0|\widetilde{hw}\rangle_i = -\tilde{h}_i|\widetilde{hw}\rangle_i . \tag{1.2.1}$$

The connections for SL(N) and $\widetilde{SL(N)}$ are denoted A and $\tilde{A}$ respectively. AdS$_3$ with planar boundary is described by

$$A = e^\rho T_1 dz + T_0 d\rho , \quad \tilde{A} = e^\rho \tilde{T}_1 d\bar{z} - \tilde{T}_0 d\rho \tag{1.2.2}$$

As is standard, a gauge transformation can be performed to effectively remove all reference to the radial coordinate ρ, so that we work with $a = T_1 dz$ and $\tilde{a} = \tilde{T}_1 d\bar{z}$. More general backgrounds are obtained by replacing the generators T_1 and $\tilde{T}_1$ by other group generators, and we describe these later as needed. More details can be found in any number of references; e.g. [9, 20]

1.2.2 Correlators

We start out by considering the correlation function of n primary operators on the plane

$$G(x_i) = \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle . \tag{1.2.3}$$

An n-point correlator is built out of n bulk-to-boundary propagators meeting at a bulk vertex located at the point $(\rho_b, z_b, \bar{z}_b)$. Since results will not depend on the choice of ρ_b we suppress it throughout. Neither will results depend on the choice of $(z_b, \bar{z}_b)$, but intermediate computations simplify for certain choices, so dependence on these quantities will be retained. The bulk-to-boundary propagator emanating from boundary point $(z_i, \bar{z}_i)$ is

$$Pe^{\int_{x_i}^{x_b} a}|hw\rangle_i e^{\int_{x_i}^{x_b} \tilde{a}}|\widetilde{hw}\rangle_i = e^{z_{bi}T_1^{(i)}}|hw\rangle_i e^{\bar{z}_{bi}\tilde{T}_1^{(i)}}|\widetilde{hw}\rangle_i . \tag{1.2.4}$$

where $z_{bi} = z_b - z_i$ and $\bar{z}_i = \bar{z}_b - \bar{z}_i$. Note that (1.2.4) is a state in the representation $(\mathcal{R}_i, \tilde{\mathcal{R}}_i)$.

The bulk vertex is defined by choosing a singlet state in the tensor product $(\mathcal{R}_1, \tilde{\mathcal{R}}_1) \otimes \dots \otimes (\mathcal{R}_n, \tilde{\mathcal{R}}_n)$. As discussed below, a particular basis for such singlet states corresponds to a basis of conformal blocks in which to expand the correlation function. Certain linear combinations

of these basis states can then be used to construct a correlation function obeying crossing symmetry. Given a choice of singlet state $|S\rangle$, the corresponding correlator is given by the matrix element

$$G_S(z_i, \overline{z}_i) = \langle S| \prod_{i=1}^{n} e^{z_{bi} T_1^{(i)}} |hw\rangle_i e^{\overline{z}_{bi} \tilde{T}_1^{(i)}} |\widetilde{hw}\rangle_i \ . \tag{1.2.5}$$

We show below that this object transforms correctly under the global conformal group.

It is natural to adopt a basis of singlet states which factorize as $|S\rangle = |s\rangle|\tilde{s}\rangle$. The general correlation function is then a sum of holomorphically factorized terms,

$$G(x_i) = \sum_{s\tilde{s}} A_{s\tilde{s}} w_s(z_i) \tilde{w}_{\tilde{s}}(\overline{z}_i) \ , \tag{1.2.6}$$

with

$$w_s(z_i) = \langle s| \prod_{i=1}^{n} e^{z_{bi} T_1^{(i)}} |hw\rangle_i \ , \quad \text{and} \quad \tilde{w}_{\tilde{s}}(\overline{z}_i) = \langle \tilde{s}| \prod_{i=1}^{n} e^{\overline{z}_{bi} \tilde{T}_1^{(i)}} |\widetilde{hw}\rangle_i. \tag{1.2.7}$$

Once we have computed $w_s(z_i)$ the corresponding result for $\tilde{w}_{\tilde{s}}(\overline{z}_i)$ follows by making obvious replacements.

We now note a few key properties satisfied by $w_s(z_i)$. First, we establish that the expresssion in (1.2.7) is independent of the choice of bulk point z_b. Suppose that instead of z_b we place the vertex at $z_{b'}$; this gives back the same result:

$$w'_s(z_i) = \langle s| \prod_{i=1}^{n} e^{z_{b'i} T_1^{(i)}} |hw\rangle_i = \langle s| \prod_{i=1}^{n} e^{z_{b'b} T_1^{(i)}} \prod_{i=1}^{n} e^{z_{bi} T_1^{(i)}} |hw\rangle_i = w_s(z_i) \ , \tag{1.2.8}$$

where we used the fact that $\langle s|$ is a singlet, and hence invariant under the action of the group element $\prod_{i=1}^{n} e^{z_{b'b} T_1^{(i)}}$.

A similar argument explains why we do not have to consider any additional "exchange" type diagrams in addition to the "contact" diagram defined above. An exchange diagram would have bulk vertices connected by bulk-to-bulk propagators. But since the location of bulk vertices is arbitrary, we can always choose to move them all to a single point, in which case the bulk-to-bulk propagators are absent, and we simply recover a contact diagram. The completeness of contact diagrams will be corroborated by the fact that these will be seen to yield a complete set of conformal blocks, out of which any correlator can be assembled.

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We next establish that $w_s(z_i)$ transforms as it should under conformal transformations, namely

$$w_s(z_i') = \left[\prod_{i=1}^{n} \left(\frac{\partial z_i'}{\partial z_i} \right)^{-h_i} \right] w_s(z_i) \ , \quad z_i' = \frac{az_i + b}{cz_i + d} \ . \tag{1.2.9}$$

We do this by applying a gauge transformation that acts as $z_i \to z_i'$. The details are given in appendix 1.9.

While our main focus will be on 4-point functions, let us first illustrate by considering the computation of 2-point and 3-point functions. Given (1.2.9), the dependence on z is guaranteed to come out correctly in these cases, but verifying this is a useful warmup.

For the 2-point function, in order to construct a singlet state we need that the representations $\mathcal{R}_1$ and $\mathcal{R}_2$ be conjugates of each other. In particular, this implies the familiar fact that the 2-point function vanishes unless the two operators have the same scaling dimension. We use the freedom to choose z_b arbitrarily to set $z_b = z_2$, which yields

$$w_s(z_1, z_2) = \langle s | e^{-z_{12} T_1^{(1)}} | hw \rangle_1 | hw \rangle_2 \ . \tag{1.2.10}$$

The singlet state is $|s\rangle = |-hw\rangle_1 |hw\rangle_2 + \dots$. The omitted terms contain states other than $|hw\rangle_2$, but it's clear from (1.2.10) that these won't contribute, and so

$$w_s(z_1, z_2) = \langle -hw | e^{-z_{12} T_1^{(1)}} | hw \rangle_1 = \frac{C}{(z_{12})^{2h}} \ , \tag{1.2.11}$$

for some constant C. To arrive at (1.2.11) we just used that the highest weight has T_0 eigenvalue $-h$, together with the fact that T_1 lowers the weight by one unit, to note that the only contribution comes from picking out the $-2h$ power from the expansion of the exponential. The result (1.2.11) is of course the one dictated by conformal invariance.

We now turn to the three point function. For this to be nonzero we need that $\mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \mathcal{R}_3$ contains a singlet. Although $\mathcal{R}_i$ are representations of SL(N) with highest weights $-h_i$, for the purposes of this computation we can take them to be representations of SL(2) of spin $j_i = -h_i$, and the singlet to be the SL(2) singlet built out of these three representations. The reason is that in (1.2.7) we are acting with SL(2) group elements on the highest weight states, and these can only yield states in the same SL(2) representation. That is, terms in

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the SL(N) singlet containing SL(2) spins different from j_i yield no contribution. With this in mind, the singlet is given by the Wigner 3j symbol as

$$|s\rangle = \sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle . \tag{1.2.12}$$

Using our freedom to choose the location of the bulk vertex, we take $z_b = z_1$, and note that this implies that only the term $m_1 = j_1$ in the sum contributes. The three point function is

$$
\begin{aligned}
&w_s(z_1, z_2, z_3) \\
&= \sum_{m_1, m_2, m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \langle j_1 m_1 | e^{z_{b1} T_1^{(1)}} | j_1 j_1 \rangle \langle j_2 m_2 | e^{z_{b2} T_1^{(2)}} | j_2 j_2 \rangle \langle j_3 m_3 | e^{z_{b3} T_1^{(3)}} | j_3 j_3 \rangle .
\end{aligned}
$$

$$\tag{1.2.13}$$

The sum can be evaluated using the known expression for the Wigner 3j symbol. Alternatively, we can work in terms of tensors. The latter approach generalizes more readily to our four-point computations, and in appendix 1.10 we show that this yields

$$w_s(z_1, z_2, z_3) = \frac{C(j_1, j_2, j_3)}{z_{12}^{h_1 + h_2 - h_3} z_{13}^{h_1 + h_3 - h_2} z_{23}^{h_2 + h_3 - h_1}} , \tag{1.2.14}$$

where $C(j_1, j_2, j_3)$ is nonzero provided the product of the three representations contains a singlet. Again, this is the standard result dictated by conformal invariance.

1.3 Four-point functions

This paper focuses mainly on the study of four-point functions of primary operators on the plane.

$$G(x_i) = \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle . \tag{1.3.1}$$

As in the previous section, each primary corresponds to the highest weight state of an irreducible representation of $\mathrm{SL(N)} \times \widetilde{SL(N)}$ that we denote $(\mathcal{R}_i, \tilde{\mathcal{R}}_i)$. In the following subsections we review the conformal block decomposition of four-point functions, we explain the construction of conformal blocks through the assembly of singlets, and we discuss restrictions due to crossing symmetry.

1.3.1 Conformal block decomposition

We now quickly review the conformal block decomposition of four-point correlators on the plane. The correlator is expressed as a sum of conformal partial waves (CPWs), each of which corresponds to inserting a projector onto a single representation of the relevant symmetry algebra,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x)P_P\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) = C_{12}^P C_{34}^P W_P(x_i) \ . \tag{1.3.2}$$

The projection operator P_P projects onto the space of states in a representation labelled by the primary operator $\mathcal{O}_P$. Pulling out the OPE coefficients renders $W_P(x_i)$ an object that is completely determined by symmetry, and in terms of which the full correlator is expanded as

$$G(x_i) = \sum_P C_{12}^P C_{34}^P W_P(x_i) \ . \tag{1.3.3}$$

Since the symmetry algebra factorizes into commuting left and right moving algebras, the same is true of the CPWs,

$$W_P(x_i) = w_p(z_i)\tilde{w}_{\tilde{p}}(\overline{z}_i) \ . \tag{1.3.4}$$

Invariance under the global conformal group allows us to reduce the dependence to

$$w_p(z_i) = \left(\frac{z_{24}}{z_{14}}\right)^{h_{12}} \left(\frac{z_{14}}{z_{13}}\right)^{h_{34}} \left(\frac{z_{34}}{z_{13}}\right)^{h_1+h_2} \frac{g_p(z)}{z_{24}^{h_1+h_2} z_{34}^{h_3+h_4}} \ , \tag{1.3.5}$$

where $h_{ij} \equiv h_i - h_j$, $z_{ij} \equiv z_i - z_j$, and z is the conformally invariant cross ratio

$$z = \frac{z_{12} z_{34}}{z_{13} z_{24}} \ . \tag{1.3.6}$$

The analogous result holds for $\tilde{w}_{\tilde{p}}(\overline{z}_i)$ upon making the obvious substitutions. We note that $g_p(z)$ depends on the quantum numbers of the primary operators appearing in the correlation function as well as those of the exchanged primary.

Another way to express the above is to use conformal invariance to set $x_1 = z$, $x_2 = 0$,

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$x_3 = \infty$ and $x_4 = 1$. We then have [4]

$$\langle \mathcal{O}_1(z,\overline{z})\mathcal{O}_2(0,0)P_P\mathcal{O}_3(\infty,\infty)\mathcal{O}_4(1,1)\rangle$$
$$= C_{12}^P C_{34}^P \left[(1-z)^{h_{34}-h_{12}} g_p(z) \right] \left[(1-\overline{z})^{\tilde{h}_{34}-\tilde{h}_{12}} \tilde{g}_{\tilde{p}}(\overline{z}) \right] \tag{1.3.7}$$

where $\mathcal{O}_3(\infty,\infty) = \lim_{x_3 \to \infty} z^{2h_3}\overline{z}^{2\tilde{h}_3}\mathcal{O}_3(x_3)$ inside the correlator. The form of $g_p(z)$ depends on what symmetry algebra is controlling the conformal block decomposition. Explicit formulas will be given below.

1.3.2 Conformal blocks from singlets

In this subsection we describe how to holographically construct conformal blocks which can be combined to give crossing symmetric four-point functions of primary operators. We will focus on the holomorphic part of a conformal block denoted $g_p(z)$. This implies that we will ignore the representations $\tilde{\mathcal{R}}_i$ and deal only with the construction of singlets in the tensor product $\otimes_i \mathcal{R}_i$.

Following the discussion in section 1.2 we consider four representations $\mathcal{R}_i$ of SL(N) and separate the operators into two pairs (12) and (34). These give rise to the tensor products

$$\mathcal{R}_1 \otimes \mathcal{R}_2 = \oplus_a \mathcal{R}_a^{(12)}, \qquad\qquad \mathcal{R}_3 \otimes \mathcal{R}_4 = \oplus_a \mathcal{R}_a^{(34)}. \tag{1.3.8}$$

Picking complex conjugate representations from the two sums we can construct singlets. We choose a representation $\mathcal{R}_a^{(12)} = \mathcal{R}_p$ in the first sum, its conjugate $\mathcal{R}_a^{(34)} = \overline{\mathcal{R}}_p$ in the second sum and denote by $|s_p^{12,34}\rangle$ the singlet in $\mathcal{R}_p \otimes \overline{\mathcal{R}}_p$. Each singlet defines a conformal block when used in (1.2.7) which we adapt here to the case in consideration

$$w_p(z_i) = \langle s_p^{12,34} | \prod_{i=1}^{4} e^{z_{bi}T_1^{(i)}} |hw\rangle_i . \tag{1.3.9}$$

Figure 1.1 shows a picture of this object.

Once we have obtained the blocks, the four point function can be constructed as

$$G(x_i) = \langle \mathcal{O}_1(z_1,\overline{z}_1)\dots\mathcal{O}_4(z_4,\overline{z}_4)\rangle = \sum_{p,\tilde{p}} A_{p\tilde{p}}^{12,34} w_p(z_i)\tilde{w}_{\tilde{p}}(\overline{z}_i) \tag{1.3.10}$$

---

[4]The prefactor in (1.3.5) was chosen such that the w_p reduce to g_p for pairwise identical operators at these distinguished positions. In the sections to follow we will assume that the prefactor has been chosen so.
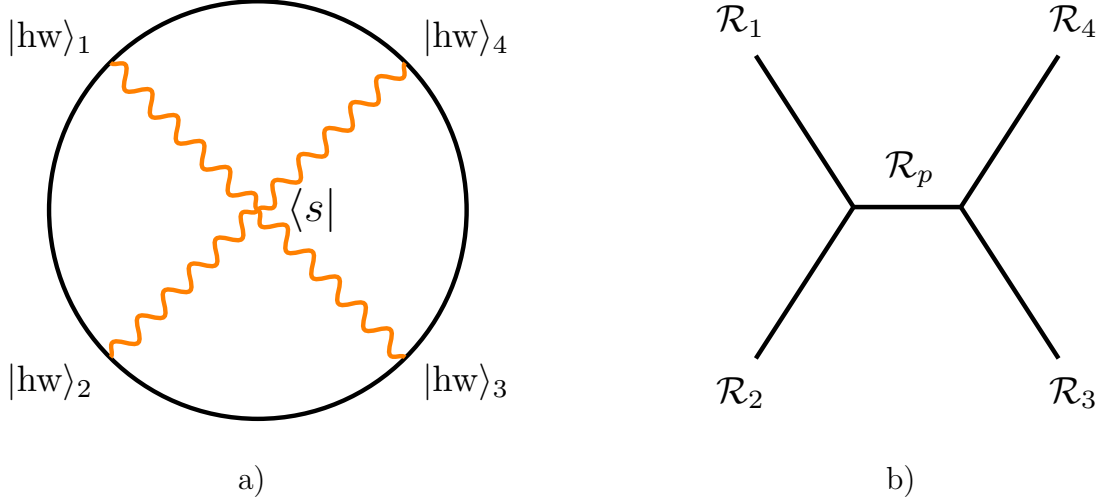
Figure 1.1: a) Holographic calculation of a conformal block. Four bulk-to-boundary propagators consisting of Wilson lines in some representation $\mathcal{R}_i$ meet at a common bulk point where a singlet state is assembled. b) Construction of the singlet state $|s_p^{12,34}\rangle$. A representation $\mathcal{R}_p$ is chosen from the tensor product $\mathcal{R}_1 \otimes \mathcal{R}_2$, while its conjugate $\overline{\mathcal{R}}_p$ is chosen from the tensor product $\mathcal{R}_3 \otimes \mathcal{R}_4$. The singlet state is the one appearing in $\mathcal{R}_p \otimes \overline{\mathcal{R}}_p$.

where $A_{p\tilde{p}}^{12,34}$ are in principle unknown constants related to the OPE coefficients as $A_{p\tilde{p}}^{12,34} = C_{p\tilde{p}}^{12} C_{p\tilde{p}}^{34}$. Alternatively, denoting the tensor product basis elements by $|S_{p\tilde{p}}^{12,34}\rangle \equiv |s_p^{12,34}\rangle |\tilde{s}_{\tilde{p}}^{12,34}\rangle$, we can define the singlet

$$|S\rangle = \sum_{p,\tilde{p}} A_{p\tilde{p}}^{12,34} |S_{p\tilde{p}}^{12,34}\rangle \tag{1.3.11}$$

and then write the four point function as

$$G(z_i, \overline{z}_i) = \langle S| \prod_{i=1}^{4} e^{z_{bi} T_1^{(i)}} |hw\rangle_i e^{\overline{z}_{bi} \tilde{T}_1^{(i)}} |\widetilde{hw}\rangle_i . \tag{1.3.12}$$

1.3.3 Crossing symmetry

In the above we expanded in the $(12)(34)$ channel and wrote the corresponding basis of singlets as $\{|S_{p\tilde{p}}^{12,34}\rangle\}$, but we can expand in other channels as well, for example $(14)(32)$. The corresponding basis of singlets will differ from the previous one and we denote it $\{|S_{p'\tilde{p}'}^{14,32}\rangle\}$.

We can expand the singlet (1.3.11) in the new basis

$$|S\rangle = \sum_{p,\tilde{p}} A^{12,34}_{p\tilde{p}} |S^{12,34}_{p\tilde{p}}\rangle = \sum_{p',\tilde{p}'} A^{14,32}_{p'\tilde{p}'} |S^{14,32}_{p'\tilde{p}'}\rangle. \tag{1.3.13}$$

The bases appearing in (1.3.13) are complete and given coefficients $A^{12,34}_{p\tilde{p}}$ we can find coefficients $A^{14,32}_{p'\tilde{p}'}$ such that (1.3.13) is obeyed. Crossing symmetry in the case that all $\mathcal{R}_i$ are distinct relates OPE coefficients in one channel to those of another. The set of operators that appears in each channel has already been fixed by the rules above.

The situation changes if two of the operators carry the same representation; for example suppose $\mathcal{R}_2 = \mathcal{R}_4$. Then $G(x_i)$ should be invariant under $x_2 \leftrightarrow x_4$. Looking at (1.3.12), this implies that $|S\rangle$ should be invariant under interchanging the states associated with $\mathcal{R}_2$ and $\mathcal{R}_4$. This crossing symmetry condition imposes a constraint on the OPE coefficients. To see this we study the holomorphic singlet states $|s^{12,34}_p\rangle$.

The change of basis associated with $x_2 \leftrightarrow x_4$ is given by

$$|s^{12,34}_p\rangle = \sum_{p'} O_{pp'} |s^{14,32}_{p'}\rangle \tag{1.3.14}$$

for some orthogonal matrix $O_{pp'}$ which we call the exchange matrix. We then have

$$\begin{aligned} |S\rangle &= \sum_{p,\tilde{p}} A^{12,34}_{p\tilde{p}} |s^{12,34}_p\rangle |\tilde{s}^{12,34}_{\tilde{p}}\rangle \\ &= \sum_{p,\tilde{p}} (O^{-1} A^{12,34} O)_{p\tilde{p}} |s^{14,32}_p\rangle |\tilde{s}^{14,32}_{\tilde{p}}\rangle \ , \end{aligned} \tag{1.3.15}$$

which implies $A^{14,32} = O^{-1} A^{12,34} O$. This constraint on the OPE coefficients will play a role in section 1.6 when we build four-point functions as sums over SL(N) conformal blocks.

1.4 General SL(2) result

We turn now to the evaluation of conformal blocks for the case of SL(2) representations. Each operator is associated with the highest weight state of a finite dimensional representation of SL(2). The Young tableaux for the representations $\mathcal{R}_i$ consist of a single row whose length

is the Dynkin label λ.

$$\mathcal{R}_i = \underbrace{\boxed{\,|\,\,|\cdots|\,\,|\,}}_{\lambda_i} = \{\lambda_i\} \ . \tag{1.4.1}$$

The Dynkin label is related to the spin of the representation as $\lambda_i = 2j_i$. The conformal dimension associated to the highest weight state $|\mathrm{hw}\rangle_i$ is given by $h_i = -\lambda_i/2 = -j_i$. The negative value of h is a manifestation of the non-unitary nature of the theory in which the primaries lie in finite dimensional representations of SL(2). This will not pose any obstacle towards verifying precise and detailed agreement between bulk and boundary observables in the limit of large central charge.

In this section we examine the calculation of a holographic conformal block whose external primary operators are highest weight states of representations $\mathcal{R}_i$ with Dynkin labels λ_i placed at the points z_i on the plane. Likewise, the exchanged primary is associated to a representation $\mathcal{R}_p$ with Dynkin label λ_p. As explained above (1.2.7), the object we need to evaluate reads

$$w_s(z_i) = \langle s| \prod_i e^{z_{bi} T_1^{(i)}} |\mathrm{hw}\rangle_i \ , \tag{1.4.2}$$

where $|s\rangle$ is the singlet state corresponding to the exchange of the representation $\mathcal{R}_p$. Figure 1.1 shows an intuitive picture of the setup. We will implement the following strategy. First, we will construct the states of the representation $\mathcal{R}_p$ out of the states of $\mathcal{R}_1$ and $\mathcal{R}_2$. Likewise, we will obtain the states of $\overline{\mathcal{R}}_p$ out of those of $\mathcal{R}_3$ and $\mathcal{R}_4$. The singlet $|s\rangle$ is built by contracting all the SL(2) indices of the states in $\mathcal{R}_p$ with those of $\overline{\mathcal{R}}_p$ using the Levi-Civita symbol, which is an invariant tensor. To make the calculation easier, we will perform certain tricks involving gauge invariance. First, we will exploit conformal invariance to move three of the external primaries to $z_1 = \infty$, $z_2 = 1$, and $z_3 = 0$. After this, the configuration of external primaries reads

$$\begin{aligned} z_1 = \infty &: \quad \mathcal{R}_1 = \{\lambda_1\} \ , \qquad z_2 = 1 \quad : \quad \mathcal{R}_2 = \{\lambda_2\} \ , \\ z_3 = 0 &: \quad \mathcal{R}_3 = \{\lambda_3\} \ , \qquad z_4 = z \quad : \quad \mathcal{R}_4 = \{\lambda_4\} \ . \end{aligned} \tag{1.4.3}$$

Before attempting to write the singlet state $|s\rangle$, it is useful to notice that the Wilson line operator coming from infinity projects the highest weight state $|\mathrm{hw}\rangle_1$ to the lowest weight

state

$$\lim_{z_1 \to \infty} z_1^{2h_1} e^{z_{b\infty} T_1^{(1)}} |\mathrm{hw}\rangle_1 \propto |-\mathrm{hw}\rangle_1 \ . \tag{1.4.4}$$

This observation simplifies the calculation of the singlet greatly, as we now need to focus only on the terms in $|s\rangle$ that are lowest weight for the primary $\mathcal{O}_1$. A further simplification of the calculation consists in choosing the bulk point where the Wilson lines meet to be at $z_b = 0$. This gauge choice immediately implies that the Wilson line operator coming from the boundary point $z_3 = 0$ corresponds to the identity, and so it projects the highest weight state to itself.

$$\lim_{z_b \to 0} e^{z_{b0} T_1^{(3)}} |\mathrm{hw}\rangle_3 = |\mathrm{hw}\rangle_3 \ . \tag{1.4.5}$$

As a consequence the only terms in $|s\rangle$ contributing to $w_s(z_i)$ are highest weight for $\mathcal{O}_3$ and lowest weight for $\mathcal{O}_1$. Instead of writing down $\langle s|$ we will compute $w_s(z_i)$ directly by replacing the states $|e_j\rangle_i$ by the objects $q_j^{(i)} \equiv \langle e_j| e^{z_{bi} T_1^{(i)}} |e_1\rangle_i$, where $|e_j\rangle_i$ are the states in the defining representation of SL(2) and the subscript i refers to the representation $\mathcal{R}_i$ (see appendices 1.10 and 1.11.1). We start with the following expressions for the Wilson line matrix elements involving states of the boundary representations

$$
\begin{aligned}
\langle (\mathcal{R}_1)_{i_1 \ldots i_{\lambda_1}} | e^{z_{b1} T_1^{(1)}} |\mathrm{hw}\rangle_1 &= \delta_{i_1}^2 \ldots \delta_{i_{\lambda_1}}^2 \ , \\
\langle (\mathcal{R}_2)_{i_1 \ldots i_{\lambda_2}} | e^{z_{b2} T_1^{(2)}} |\mathrm{hw}\rangle_2 &= q_{(i_1}^{(2)} \ldots q_{i_{\lambda_2})}^{(2)} \ , \\
\langle (\mathcal{R}_3)_{i_1 \ldots i_{\lambda_3}} | e^{z_{b3} T_1^{(3)}} |\mathrm{hw}\rangle_3 &= \delta_{i_1}^1 \ldots \delta_{i_{\lambda_3}}^1 \ , \\
\langle (\mathcal{R}_4)_{i_1 \ldots i_{\lambda_4}} | e^{z_{b4} T_1^{(4)}} |\mathrm{hw}\rangle_4 &= q_{(i_1}^{(4)} \ldots q_{i_{\lambda_4})}^{(4)} \ ,
\end{aligned}
\tag{1.4.6}
$$

where we have projected the states of $\mathcal{R}_1$ to their lowest weight, and the states of $\mathcal{R}_3$ to their highest weight. We now build the representation $\mathcal{R}_p$ out of the states in the first pair. This representation must consist of λ_p symmetric indices. There are a total of $\lambda_1 + \lambda_2$ indices and each contraction with the Levi-Civita symbol subtracts two indices. It follows that $(\lambda_1 + \lambda_2 - \lambda_p)/2$ contractions are needed. The result reads

$$\langle (\mathcal{R}_p)_{i_1 \ldots i_{\lambda_p}} | e^{z_{b1} T_1^1} |\mathrm{hw}\rangle_1 e^{z_{b2} T_1^2} |\mathrm{hw}\rangle_2 = (q_1^{(2)})^{\frac{\lambda_1 + \lambda_2 - \lambda_p}{2}} \delta_{(i_1}^2 \ldots \delta_{i_{\frac{\lambda_p + \lambda_1 - \lambda_2}{2}}}^2 q_{i_{\frac{\lambda_p + \lambda_1 - \lambda_2}{2}+1}}^{(2)} \ldots q_{i_{\lambda_p})}^{(2)} \ . \tag{1.4.7}$$

The same logic follows for the construction of the states in $\overline{\mathcal{R}}_p$. In this case, there will be $(\lambda_3 + \lambda_4 - \lambda_p)/2$ contractions with the Levi-Civita symbol

$$\langle (\overline{\mathcal{R}}_p)_{i_1 \dots i_{\lambda_p}} | e^{z_{b3} T_1^3} | \mathrm{hw} \rangle_3 e^{z_{b4} T_1^4} | \mathrm{hw} \rangle_4 = (q_2^{(4)})^{\frac{\lambda_3 + \lambda_4 - \lambda_p}{2}} \delta^1_{(i_1} \dots \delta^1_{i_{\frac{\lambda_p + \lambda_3 - \lambda_4}{2}}} q^{(4)}_{i_{\frac{\lambda_p + \lambda_3 - \lambda_4}{2} + 1}} \dots q^{(4)}_{i_{\lambda_p})} \, . \tag{1.4.8}$$

Finally, the singlet is obtained by contracting all the indices of (1.4.7) with the indices of (1.4.8) using Levi-Civita symbols:

$$g_s(z) = (q_1^{(2)})^{\frac{\lambda_1 + \lambda_2 - \lambda_p}{2}} (q_2^{(4)})^{\frac{\lambda_3 + \lambda_4 - \lambda_p}{2}} \epsilon^{i_1 j_1} \dots \epsilon^{i_{\lambda_p} j_{\lambda_p}}$$

$$\times \delta^2_{(i_1} \dots \delta^2_{i_{\frac{\lambda_p + \lambda_1 - \lambda_2}{2}}} q^{(2)}_{i_{\frac{\lambda_p + \lambda_1 - \lambda_2}{2} + 1}} \dots q^{(2)}_{i_{\lambda_p})} \times \delta^1_{(j_1} \dots \delta^1_{j_{\frac{\lambda_p + \lambda_3 - \lambda_4}{2}}} q^{(4)}_{j_{\frac{\lambda_p + \lambda_3 - \lambda_4}{2} + 1}} \dots q^{(4)}_{j_{\lambda_p})} \, . \tag{1.4.9}$$

The last step of the calculation is to evaluate the object (1.4.9). The strategy is the following: we first classify the different symmetric permutations that give rise to inequivalent contributions to $g_s(z)$. We will then sum over all permutation classes, taking into account their contribution and multiplicity.

In order to classify the different permutations, let us define "red" indices as the indices appearing in the objects δ^1_j. We also define "green" indices as the indices appearing in $q_j^{(4)}$. Each permutation will contribute differently depending of how many red and green indices appear in the delta functions δ^2_i (Box 1) and the objects $q_i^{(2)}$ (Box 2). We then define our permutation class as those with k red indices in Box 1. This also implies there will be $\frac{\lambda_p + \lambda_1 - \lambda_2}{2} - k$ green indices in Box 1, $\frac{\lambda_p + \lambda_3 - \lambda_4}{2} - k$ red indices in Box 2, and $\frac{\lambda_4 - \lambda_3 + \lambda_2 - \lambda_1}{2} + k$ green indices in Box 2. Each permutation of this class will contribute to the block as follows

$$g_s^{(k)}(z) = (q_1^{(2)})^{\frac{\lambda_1 + \lambda_2 - \lambda_p}{2}} (q_2^{(4)})^{\frac{\lambda_3 + \lambda_4 - \lambda_p}{2}}$$

$$\times (\epsilon^{i_R j_R} \delta^2_{i_R} \delta^1_{j_R})^k (\epsilon^{i_G j_G} \delta^2_{i_G} q^{(4)}_{j_G})^{\frac{\lambda_p + \lambda_3 - \lambda_4}{2} - k} (\epsilon^{i_R j_R} q^{(2)}_{i_R} \delta^1_{j_R})^{\frac{\lambda_p + \lambda_3 - \lambda_4}{2} - k} (\epsilon^{i_G j_G} q^{(2)}_{i_G} q^{(4)}_{j_G})^{\frac{\lambda_4 - \lambda_3 + \lambda_2 - \lambda_1}{2} + k}$$

$$= z^{\frac{\lambda_3 + \lambda_4 - \lambda_p}{2}} (1 - z)^{\frac{\lambda_4 - \lambda_3 + \lambda_2 - \lambda_1}{2} + k} \, . \tag{1.4.10}$$

The multiplicity of each class consists of choosing k red indices out of a total of $\frac{\lambda_p + \lambda_3 - \lambda_4}{2}$, choosing $\frac{\lambda_p + \lambda_1 - \lambda_2}{2} - k$ green indices out of a total of $\frac{\lambda_p + \lambda_4 - \lambda_3}{2}$, and ordering the indices of each box. We then have

$$C^{(k)} = \binom{\frac{\lambda_p + \lambda_3 - \lambda_4}{2}}{k} \binom{\frac{\lambda_p + \lambda_4 - \lambda_3}{2}}{\frac{\lambda_p + \lambda_1 - \lambda_2}{2} - k} \Gamma\left(\frac{\lambda_p + \lambda_1 - \lambda_2}{2} + 1\right) \Gamma\left(\frac{\lambda_p + \lambda_4 - \lambda_3}{2} + 1\right) \, . \tag{1.4.11}$$

We are now ready to sum over permutation classes. This consists of a sum over k. The result reads

$$g_s(z) = \sum_{k=0}^{\frac{\lambda_p + \lambda_3 - \lambda_4}{2}} C^{(k)} g_s^{(k)}(z) = z^{\frac{\lambda_3 + \lambda_4 - \lambda_p}{2}} {}_2F_1\left(-\frac{\lambda_p + \lambda_2 - \lambda_1}{2}, -\frac{\lambda_p + \lambda_4 - \lambda_3}{2}; -\lambda_p; z\right) .$$

(1.4.12)

This result can be written in a more suggestive way by replacing $\lambda_i \to -2h_i$

$$g_s(z) = z^{-h_3 - h_4 + h_p} {}_2F_1\left(h_p + h_{21}, h_p + h_{43}; 2h_p; z\right) .$$

(1.4.13)

This is the standard result for the chiral half of the global conformal block [34]. This result was also obtained in [17].

1.5 SL(3) Result

After the warmup with SL(2), we can now move on to the more difficult task of computing SL(3) blocks. Our goal here is to compute conformal blocks of $\mathcal{W}_3$ in the large central charge limit with the operator dimensions and charges kept fixed as $c \to \infty$. The $\mathcal{W}_3$ algebra reduces to SL(3) in the large central charge limit. Our strategy as before will be to compute blocks in finite dimensional representations of SL(3) and then continue the result to more general representations. Finite dimensional irreducible representations of SL(3) are labelled by two integers (the Dynkin labels) λ_1 and λ_2. Alternatively, they can be written as symmetric traceless tensors with λ_1 lower and λ_2 upper indices where the lower and upper indices denote states in the defining representation and its conjugate respectively (see appendix 1.11 for details). Our main goal in this section is to reproduce the result for $\mathcal{W}_3$ conformal blocks obtained in [38].

In terms of SL(3) tensors, constructing the singlet amounts to contracting all lower and upper indices. It turns out to be computationally more tractable if we consider two of the representations to have only upper (or only lower) indices i.e. the tensor product $(\lambda_1, \lambda_2) \otimes (0, \mu) \otimes (0, \mu') \otimes (\lambda_1', \lambda_2')$. Let the exchanged representation be $\mathcal{R}_p = (x, y)$. Below

we list the Young tableaux associated to these representations

$$\mathcal{R}_1 \quad \boxed{\begin{array}{|c|c|}\hline \lambda_2 & \lambda_1 \\\hline \lambda_2 \\\cline{1-1}\end{array}} \quad z_1 = 0 \;, \quad \mathcal{R}_2 \quad \boxed{\begin{array}{|c|}\hline \mu \\\hline \mu \\\hline\end{array}} \quad z_2 = z \;,$$

$$\mathcal{R}_3 \quad \boxed{\begin{array}{|c|}\hline \mu' \\\hline \mu' \\\hline\end{array}} \quad z_3 = 1 \;, \quad \mathcal{R}_4 \quad \boxed{\begin{array}{|c|c|}\hline \lambda_2' & \lambda_1' \\\hline \lambda_2' \\\cline{1-1}\end{array}} \quad z_4 = \infty \;, \tag{1.5.1}$$

$$\mathcal{R}_p \quad \boxed{\begin{array}{|c|c|}\hline y & x \\\hline y \\\cline{1-1}\end{array}} \quad z_p = z_b$$

To avoid cluttering, in the above Young tableau we have used $\boxed{\lambda}$ to denote a row of λ boxes. To evaluate the conformal block in the $(12)(34)$ channel, we first construct the tensor products $\mathcal{R}_1 \otimes \mathcal{R}_2$ and $\mathcal{R}_3 \otimes \mathcal{R}_4$ in terms of $SL(3)$ tensors. The singlet is then obtained by contracting all indices between the tensors coming from the two tensor products[5]. To be a little more explicit, the representation $\mathcal{R}_p$ in the tensor product $\mathcal{R}_1 \otimes \mathcal{R}_2$ can be written as

$$M_{i_1 \cdots i_x}^{j_1 \cdots j_y} = (P_{i_1 \cdots i_x}^{j_1 \cdots j_y})_{b_1 \cdots b_{\lambda_2} c_1 \cdots c_\mu}^{a_1 \cdots a_{\lambda_1}} |e_{a_1} \dots e_{a_{\lambda_1}} \bar{e}^{b_1} \dots \bar{e}^{b_{\lambda_2}} \bar{e}^{c_1} \dots \bar{e}^{c_\mu}\rangle \tag{1.5.2}$$

where the indices a and b denote states of the representation $\mathcal{R}_1$ and c of $\mathcal{R}_2$. As a consequence all the a, b and c indices are symmetrized and any contraction between a and b vanishes. The tensor P projects onto the representation (x, y) and as we explain below must be built out of δ_k^l's and ϵ_{klm}'s. Note that for the new tensor M to be irreducible, it must be completely symmetric and traceless. The tensor N for the representation $\overline{\mathcal{R}}_p$ can be constructed out of the tensor product $\mathcal{R}_3 \otimes \mathcal{R}_4$ in a similar manner.

$$N_{j_1 \cdots j_y}^{i_1 \cdots i_x} = (P_{j_1 \cdots j_y}^{i_1 \cdots i_x})_{g_1 \cdots g_{\lambda_2'} h_1 \cdots h_{\mu'}}^{f_1 \cdots f_{\lambda_1'}} |\bar{e}^{h_1} \dots \bar{e}^{h_{\mu'}} e_{f_1} \dots e_{f_{\lambda_1'}} \bar{e}^{g_1} \dots \bar{e}^{g_{\lambda_2'}}\rangle \tag{1.5.3}$$

where the indices f and g denote states of $\mathcal{R}_4$ and h of $\mathcal{R}_3$. In the full tensor product the singlet state is then obtained as

$$|s\rangle = M_{i_1 \cdots i_x}^{j_1 \cdots j_y} N_{j_1 \cdots j_y}^{i_1 \cdots i_x} \tag{1.5.4}$$

---

[5]For a singlet to exist, the two irreps coming from the two tensor products must be conjugate to each other. Since conjugating irreps of $SL(3)$ is equivalent to switching the Dynkin labels, the singlet exists only if the number of upper indices on the first tensor is equal to the number of lower indices on the second and vice versa.

Now let's study the kinds of representations that can appear in (1.5.2). We start out with λ_1 lower indices and $\lambda_2 + \mu$ upper indices. The operations we can perform that are invariant under SL(3) are contraction with the invariant tensors δ_k^l, ϵ_{klm} and ϵ^{klm}. Taking the symmetry properties of a, b and c into account, we are allowed to do one of two things: contract indices a and c using δ_c^a, or convert indices b and c into a lower index using ϵ_{ibc}. If we perform d contractions using δ's and e conversions using ϵ's, a simple counting of indices requires the relation

$$(x, y) = (\lambda_1 - d + e, \lambda_2 + \mu - d - 2e) \tag{1.5.5}$$

We still need to make the tensor symmetric and traceless. The procedure for making a symmetric tensor traceless is described in appendix 1.12. We will deal with this later as it doesn't change the relation in (1.5.5). Performing similar operations on the $\mathcal{R}_3 \otimes \mathcal{R}_4$ tensor product with d' contractions and e' conversions, we obtain

$$(y, x) = (\lambda_1' - d' + e', \lambda_2' + \mu' - d' - 2e') \tag{1.5.6}$$

The projectors in (1.5.2) and (1.5.3) without the tracelessness constraint imposed now look like

$$
(P_{i_1 \cdots i_x}^{j_1 \cdots j_y})_{b_1 \cdots b_{\lambda_2} c_1 \cdots c_\mu}^{a_1 \cdots a_{\lambda_1}} = \delta_{i_1}^{a_1} \cdots \delta_{i_{l_1}}^{a_{l_1}} \delta_{b_1}^{j_1} \cdots \delta_{b_{l_3}}^{j_{l_3}} \delta_{c_1}^{j_{l_3+1}} \cdots \delta_{c_{l_4}}^{j_y} \epsilon_{i_{l_1+1} b_{l_3+1} c_{l_4+1}} \cdots \epsilon_{i_x b_{\lambda_2} c_{l_4+l_2}}
$$
$$
\times \delta_{c_{l_4+l_2+1}}^{a_{l_1+1}} \cdots \delta_{c_\mu}^{a_{\lambda_1}}
$$
$$
(P_{j_1 \cdots j_y}^{i_1 \cdots i_x})_{g_1 \cdots g_{\lambda_2'} h_1 \cdots h_{\mu'}}^{f_1 \cdots f_{\lambda_1'}} = \delta_{j_1}^{f_1} \cdots \delta_{j_{n_4}}^{f_{n_4}} \delta_{g_1}^{i_1} \cdots \delta_{g_{n_1}}^{i_{n_1}} \delta_{h_1}^{i_{n_1+1}} \cdots \delta_{h_{n_2}}^{i_x} \epsilon_{j_{n_4+1} g_{n_1+1} h_{n_2+1}} \cdots \epsilon_{j_y g_{n_1+n_5} h_{n_2+n_5}}
$$
$$
\times \delta_{h_{n_2+n_5+1}}^{f_{n_4+1}} \cdots \delta_{h_{\mu'}}^{f_{\lambda_1'}}
$$

$$\tag{1.5.7}$$

where the i's and j's are to be completely symmetrized. We have made the following definitions for notational convenience

$$
l_1 = \lambda_1 - d \ , \quad l_2 = e \ , \quad l_3 = \lambda_2 - e \ , \quad l_4 = \mu - d - e
$$
$$
n_1 = \lambda_2' - e' \ , \quad n_2 = \mu' - d' - e' \ , \quad n_4 = \lambda_1' - d' \ , \quad n_5 = e' \ ,
$$

$$\tag{1.5.8}$$

In simple terms, l_1 is the number of i indices that appear in δ_i^a, the rest of them (l_2 in number) being in ϵ_{ibc}. Similarly, n_1 is the number of j indices that appear in δ_b^j, the rest

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of them (n_2 in number) being in δ_c^j and so on. Recall that our final goal is to calculate the Wilson line

$$w_s(z_k) = \langle s| \prod_{k=1}^{4} e^{z_{bk} T_1^{(k)}} |hw\rangle_k \tag{1.5.9}$$

To make direct comparison with the results of [38], we choose the operator positions – $z_1 = 0, z_2 = z, z_3 = 1, z_4 = \infty$ where z_k denotes the position associated to the representation $\mathcal{R}_k$. To this end we define

$$q_a^{(k)} = \langle e_a| e^{z_{bk} T_1^{(k)}} |e_1\rangle_k \ , \ \ \bar{q}_{(k)}^b = \langle \bar{e}^b| e^{z_{bk} T_1^{(k)}} |\bar{e}^3\rangle_k \tag{1.5.10}$$

using which we can directly write out the matrix elements rather than the states appearing in the singlet $|s\rangle$. The contributions from the $\mathcal{R}_1 \otimes \mathcal{R}_2$ tensor product can then be written as

$$(M_{0z})_{i_1 \cdots i_x}^{j_1 \cdots j_y} = \left((M^\dagger)_{i_1 \cdots i_x}^{j_1 \cdots j_y} \right) e^{z_{b1} T_1^{(1)}} \otimes e^{z_{b2} T_1^{(2)}} |hw\rangle_1 |hw\rangle_2$$

$$= q_{(i_1}^{(1)} \cdots q_{i l_1}^{(1)} \tilde{q}_{i l_1 + 1} \cdots \tilde{q}_{i_x)} \bar{q}_{(1)}^{(j_1} \cdots \bar{q}_{(1)}^{j_{l_3}} \bar{q}_{(2)}^{j_{l_3}+1} \cdots \bar{q}_{(2)}^{j_y)} \tag{1.5.11}$$

$$\times \delta_{c l_4 + l_2 + 1}^{a l_1 + 1} \cdots \delta_{c_\mu}^{a \lambda_1} q_{a l_1 + 1}^{(1)} \bar{q}_{(2)}^{c l_4 + l_2 + 1} \cdots q_{a \lambda_1}^{(1)} \bar{q}_{(2)}^{c_\mu}$$

with $\tilde{q}_i \equiv \epsilon_{ibc} \bar{q}_{(1)}^b \bar{q}_{(2)}^c$. In the tensor M_{0z}, it is clear that there are two types of lower indices: ones that appear on $q^{(1)}$ and ones on $\tilde{q}$. There are two types of upper indices too: ones on $\bar{q}_{(1)}$ and ones on $\bar{q}_{(2)}$. As in the SL(2) case, we refer to these different types of indices by colors. The indices on $q^{(1)}$ we call red, $\tilde{q}$ blue, $\bar{q}_{(1)}$ green and $\bar{q}_{(2)}$ yellow. In this language, the l_i defined in (1.5.8) are just the number of indices of each color. There are further simplifications once we fix the positions of the bulk and boundary points. We use our freedom of choosing the bulk point to set $z_b = 0$ such that the Wilson line projects out the highest weight state of $\mathcal{R}_1$. This forces all red and green indices to be highest weight indices i.e. $q_1^{(1)}$ and $\bar{q}_{(1)}^3$ respectively.

A similar story plays out for the $\mathcal{R}_3 \otimes \mathcal{R}_4$ tensor product

$$(N_{1\infty})_{j_1 \cdots j_y}^{i_1 \cdots i_x} = q_{j_1}^{(4)} \cdots q_{j n_4}^{(4)} \tilde{q}_{j n_4 + 1}' \cdots \tilde{q}_{j_y}' \bar{q}_{(4)}^{i_1} \cdots \bar{q}_{(4)}^{i_{n_1}} \bar{q}_{(3)}^{i_{n_1}+1} \cdots \bar{q}_{(3)}^{i_x}$$

$$\times \delta_{h n_2 + n_5 + 1}^{f n_4 + 1} \cdots \delta_{h_{\mu'}}^{f \lambda_1'} q_{f n_4 + 1}^{(4)} \bar{q}_{(3)}^{h n_2 + n_5 + 1} \cdots q_{f \lambda_1'}^{(4)} \bar{q}_{(3)}^{h_{\mu'}} \tag{1.5.12}$$

with $\tilde{q}_j' = \epsilon_{jgh} \bar{q}_{(4)}^g \bar{q}_{(3)}^h$. Again mimicking the SL(2) computations, we refer to the indices of $N_{1\infty}$ as boxes. We call the indices on $\bar{q}_{(4)}$ and $\bar{q}_{(3)}$ box 1 and box 2 respectively. Saving box

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3 for a different purpose, we call the indices on $q^{(4)}$ and $\tilde{q}'$ box 4 and box 5 respectively. The n_i of (1.5.8) count the number of boxes of each type. As discussed before in (1.4.4), setting $z_4 \to \infty$ projects out the lowest weight state from the singlet in the Wilson line. In other words all box 1 and box 4 indices are forced to be lowest weight indices i.e. $\bar{q}^1_{(4)}$ and $q^{(4)}_3$ respectively.

The explicit form of the matrices L_1 and L_{-1} (see appendix 1.11) in the defining representation gives

$$
\begin{aligned}
q^{(k)}_a &= \langle e_a | e^{-z_k L_1} | e_1 \rangle_k = \delta^1_a + \sqrt{2} z_k \delta^2_a + z^2_k \delta^3_a \\
\bar{q}^a_{(k)} &= \langle \bar{e}^a | e^{-z_k L_{-1}} | \bar{e}^3 \rangle_k = \delta^a_3 - \sqrt{2} z_k \delta^a_2 + z^2_k \delta^a_1
\end{aligned}
\tag{1.5.13}
$$

Using $z_1 = 0$, $z_2 = z$ and the fact that all the $q^{(1)}$ indices are highest weight indices we have $\delta^a_c q^{(1)}_a \bar{q}^c_{(2)} = \delta^1_c q^{(1)}_1 \bar{q}^c_{(2)} = z^2$. Similarly, all $q^{(4)}$ indices are lowest weight giving $\delta^f_h q^{(4)}_f \bar{q}^h_{(3)} = \delta^3_h q^{(4)}_3 \bar{q}^h_{(3)} = 1$.

Next , let us deal with the issue of making the exchanged tensor traceless. As discussed in appendix 1.12, we first subtract all possible traces of the tensor. Then we subtract out traces of the new terms added and so on until we run out of traces. The result from (1.12.10) is

$$
(\tilde{N}_{1\infty})^{i_1 \cdots i_x}_{j_1 \cdots j_y} = \sum_{n=0}^{min(x,y)} C_n \delta^{(i_1}_{(j_1} \cdots \delta^{i_n}_{j_n} (N_{1\infty})^{i_{n+1} \cdots i_x) k_1 \cdots k_n}_{j_{n+1} \cdots j_y) k_1 \cdots k_n}
\tag{1.5.14}
$$

where the C_n are read off from (1.12.10). In doing this we have introduced new types of upper and lower indices – the ones appearing on δ^i_j. We call the upper index box 3, and lower box 6. A caveat here is that the trace of some indices vanishes like the ones coming from the representation (λ'_1, λ'_2). In other words some terms in the symmetrization in (1.5.14) vanish depending on what indices are being traced out. Note that in the absence of this constraint all terms in the symmetrization would contribute in exactly the same manner. To account for the constraint we simply assume that all possible traces are allowed but then correct by multiplying by the fraction of terms that would survive in the symmetrization. From (1.5.12), we see $\bar{q}_{(4)} \cdot \tilde{q}' = 0 = \bar{q}_{(3)} \cdot \tilde{q}' = \bar{q}_{(4)} \cdot q^{(4)}$ allowing us to trace out only box 2 and box 4. The fraction of terms for a given value of n is then found as follows : choose n indices

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from box 2 and box 4 to trace out, multiply by the number of permutations that preserve this structure and divide by the total number of terms. This gives an additional factor to add onto (1.5.14)

$$C'_n = \binom{n_2}{n}\binom{n_4}{n}\frac{\Gamma(x-n+1)\Gamma(y-n+1)\Gamma(n+1)^2}{\Gamma(x+1)\Gamma(y+1)}$$
$$= \frac{\Gamma(n_2+1)\Gamma(n_4+1)\Gamma(x-n+1)\Gamma(y-n+1)}{\Gamma(n_4-n+1)\Gamma(n_2-n+1)\Gamma(x+1)\Gamma(y+1)} \tag{1.5.15}$$

We now have the two objects $\tilde{M}_{0z}$ and $\tilde{N}_{1\infty}$ and the only thing left to do is to contract the indices between them in order to assemble the singlet. Note that since we are contracting all indices, it is sufficient to make just one of them symmetric and traceless. Putting all of this together, we have

$$g_s(z) = (M_{0z})^{j_1\cdots j_y}_{i_1\cdots i_x}(\tilde{N}_{1\infty})^{i_1\cdots i_x}_{j_1\cdots j_y}$$
$$= q^{(1)}_{i_1}\cdots q^{(1)}_{i_{l_1}}\tilde{q}_{i_{l_1+1}}\cdots\tilde{q}_{i_x}\bar{q}^{j_1}_{(1)}\cdots\bar{q}^{j_{l_3}}_{(1)}\bar{q}^{j_{l_3+1}}_{(2)}\cdots\bar{q}^{j_y}_{(2)} \times z^{2d}$$
$$\times \sum_{n=0}^{min(x,y)} C_n C'_n \delta^{(i_1}_{(j_1}\cdots\delta^{i_n}_{j_n}q^{(4)}_{j_{n+1}}\cdots q^{(4)}_{j_{n_4}}\tilde{q}'_{j_{n_4+1}}\cdots\tilde{q}'_{j_y}\bar{q}^{i_{n+1}}_{(4)}\cdots\bar{q}^{i_{n_1+n+1}}_{(4)}\bar{q}^{i_{n_1+n+2}}_{(3)}\cdots\bar{q}^{i_x}_{(3)} \times 1$$

$$(1.5.16)$$

As for the SL(2) case, keeping track of the permutations is a combinatorial problem; we need to find different ways to color boxes 1, 2 and 3 red or blue and boxes 4, 5 and 6 green or yellow. The details are relegated to appendix 1.13. Ignoring all factors that are independent of z and the integer n, we obtain

$$g_s(z) = z^{2d}\sum_{n=0}^{\infty}\frac{z^{2n}}{n!}\frac{(-n_2)_n(-l_4)_n(-l_1)_n(-n_4)_n}{(-x)_n(-y)_n(-x-y-1)_n} \tag{1.5.17}$$
$$\times {}_2F_1(-l_2, n-n_2; n-x; z){}_2F_1(-n_5, n-l_4; n-y; z)$$

The representations we consider here are of the same form as the ones in [38]. The r_i and s_i there are defined to be the negative of the Dynkin labels : $r_1 = -\lambda_1$, $s_1 = -\lambda_2$ and so on. Using this we find the following map to the definitions in equation (2.65) of [38] : $n_5 \to -\alpha$, $l_2 \to -\beta$, $n_2 \to -\gamma$ and $\lambda_4 \to -\delta$. With these relations our result in (1.5.17) agrees with their CFT calculation of the $\mathcal{W}_3$ blocks in the large c limit.

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1.6 An SL(N) example

We now consider an example at arbitrary N, but with simple representations so as to keep the computation tractable. In particular, we will study the four-point function of two primaries in the fundamental (defining) representation of SL(N), and two primaries in the anti-fundamental representation of SL(N),

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \langle \phi_+(x_1)\overline{\phi}_+(x_2)\phi_+(x_3)\overline{\phi}_+(x_4)\rangle \ . \tag{1.6.1}$$

Using conformal invariance and identifying the conformal cross ratios

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}} \ , \quad \overline{z} = \frac{\overline{z}_{12}\overline{z}_{34}}{\overline{z}_{13}\overline{z}_{24}} \tag{1.6.2}$$

this reduces to

$$G_{\phi_+\overline{\phi}_+\phi_+\overline{\phi}_+} = \langle \phi_+(\infty)\overline{\phi}_+(1,1)\phi_+(z,\overline{z})\overline{\phi}_+(0,0)\rangle \ , \tag{1.6.3}$$

where $\mathcal{O}_1 = \mathcal{O}_2 = \phi_+$ and $\mathcal{O}_3 = \mathcal{O}_4 = \overline{\phi}_+$ are primaries corresponding to the highest weight states of the following representations

$$\phi_+ \ : \ \mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_+ = \left(\Box, 0\right) \qquad \text{and} \qquad \overline{\phi}_+ \ : \ \overline{\mathcal{R}}_3 = \overline{\mathcal{R}}_4 = \overline{\mathcal{R}}_+ = \left(\overline{\Box}, 0\right) \ . \tag{1.6.4}$$

We denote by $|hw\rangle_i$ the highest weight state of $\mathcal{R}_i$, and by $|\overline{hw}\rangle_i$ the highest weight state of $\overline{\mathcal{R}}_i$.

The holographic calculation of the blocks corresponding to this four point function follows the logic of section 1.3. We first construct the matrix elements of the Wilson lines acting on the boundary states. We then build the states corresponding to the exchanged representations, and we end the calculation by assembling the singlet. We will work in the channel where the pair $\phi_+(\infty)\overline{\phi}_+(1)$ exchanges states with the pair $\phi_+(z)\overline{\phi}_+(0)$. In order to see what representations can be exchanged, we decompose the tensor product of the representations of ϕ_+ and $\overline{\phi}_+$

$$\Box \otimes \overline{\Box} = \mathbf{1} \oplus \mathbf{Adj} \ , \qquad \text{where} \qquad \mathbf{Adj} = \boxed{\vdots} \ . \tag{1.6.5}$$

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The adjoint representation is conjugate to itself, so there are two different blocks we can construct. One of them corresponds to the exchange of the identity representation, the other corresponds to the exchange of the adjoint representation. We construct each block in a separate subsection.

1.6.1 Exchange of 1

We start by building the matrix elements of the bulk-to-boundary Wilson lines acting on the highest weight states at the boundary.

$$
\begin{aligned}
\langle (\mathcal{R}_1)_j | e^{z_{b1} L_1} | \mathrm{hw} \rangle_1 &= q_j^{(1)} \ , \\
\langle (\mathcal{R}_2)^k | e^{z_{b2} L_1} | \overline{\mathrm{hw}} \rangle_2 &= \bar{q}_{(2)}^k \ , \\
\langle (\mathcal{R}_3)_j | e^{z_{b3} L_1} | \mathrm{hw} \rangle_3 &= q_j^{(3)} \ , \\
\langle (\mathcal{R}_4)^k | e^{z_{b4} L_1} | \overline{\mathrm{hw}} \rangle_4 &= \bar{q}_{(4)}^k \ ,
\end{aligned}
\tag{1.6.6}
$$

with $q_j^{(i)} = \langle e_j | e^{z_{bi} T_1^{(i)}} | \mathrm{hw} \rangle_i$ and $\bar{q}_{(i)}^k = \langle e^k | e^{z_{bi} T_1^{(i)}} | \overline{\mathrm{hw}} \rangle_i$. The next step is to build the trivial representation out of each pair. We do this by contracting indices with the invariant tensor δ_k^j

$$
\begin{aligned}
\langle (\mathbf{1}) | e^{z_{b1} L_1} | \mathrm{hw} \rangle_1 e^{z_{b2} L_1} | \overline{\mathrm{hw}} \rangle_2 &= q_j^{(1)} \bar{q}_{(2)}^k \delta_k^j \ , \\
\langle (\overline{\mathbf{1}}) | e^{z_{b3} L_1} | \mathrm{hw} \rangle_3 e^{z_{b4} L_1} | \overline{\mathrm{hw}} \rangle_4 &= q_j^{(3)} \bar{q}_{(4)}^k \delta_k^j \ .
\end{aligned}
\tag{1.6.7}
$$

The last step is to assemble the singlet out of $\mathbf{1}$ and $\overline{\mathbf{1}}$. No contractions with any tensor are needed

$$
w_{\mathbf{1}}(z_i) = \langle s | e^{z_{b1} L_1} | \mathrm{hw} \rangle_1 e^{z_{b2} L_1} | \overline{\mathrm{hw}} \rangle_2 | e^{z_{b3} L_1} | \mathrm{hw} \rangle_3 e^{z_{b4} L_1} | \overline{\mathrm{hw}} \rangle_4 = \frac{1}{N} (q_j^{(1)} \bar{q}_{(2)}^k \delta_k^j)(q_{j'}^{(3)} \bar{q}_{(4)}^{k'} \delta_{k'}^{j'})
\tag{1.6.8}
$$

where we normalized the singlet. Using the explicit form of $q_j^{(i)}$ and $\bar{q}_{(i)}^j$ we obtain

$$
g_{\mathbf{1}}(z) = \frac{1}{N} z^{N-1} \ .
\tag{1.6.9}
$$

1.6.2 Exchange of Adj

We proceed in the same fashion as in the previous subsection. We start with the expressions for the matrix elements of the bulk-to-boundary Wilson lines acting on the highest weight

states at the boundary. These are written in (1.6.6). The next step is to build the adjoint representation using the matrix elements of the first pair, the same can be done for the second pair. For this we need an object with one index down (index in the fundamental representation), and one index up (index in the anti-fundamental representation). For the representation to be irreducible we also must impose a tracelessness condition. The answer reads

$$
\begin{aligned}
(M_{12})^k_{\ j} &\equiv \langle (\mathbf{Adj})^k_{\ j}|e^{z_{b1}L_1}|\mathrm{hw}\rangle_1 e^{z_{b2}L_1}|\overline{\mathrm{hw}}\rangle_2 = q^{(1)}_j \bar{q}^k_{(2)} - \frac{1}{N}\delta^k_j q^{(1)}_i \bar{q}^i_{(2)} \ , \\
(M_{34})^k_{\ j} &\equiv \langle (\overline{\mathbf{Adj}})^k_{\ j}|e^{z_{b3}L_1}|\mathrm{hw}\rangle_3 e^{z_{b4}L_1}|\overline{\mathrm{hw}}\rangle_4 = q^{(3)}_j \bar{q}^k_{(4)} - \frac{1}{N}\delta^k_j q^{(3)}_i \bar{q}^i_{(4)} \ ,
\end{aligned}
\tag{1.6.10}
$$

where the second term in each expression ensures tracelessness. The singlet can now be built by contracting all indices of $(M_{12})^k_{\ j}$ with all indices of $(M_{34})^k_{\ j}$ using Kronecker delta functions

$$
\begin{aligned}
w_{\mathbf{Adj}}(z_i) &= \langle s|e^{z_{b1}L_1}|\mathrm{hw}\rangle_1 e^{z_{b2}L_1}|\overline{\mathrm{hw}}\rangle_2 e^{z_{b3}L_1}|\mathrm{hw}\rangle_3 e^{z_{b4}L_1}|\overline{\mathrm{hw}}\rangle_4 \\
&= \frac{1}{\sqrt{N^2-1}}\delta^j_k \delta^{j'}_{k'}(M_{12})^{k'}_{\ j}(M_{34})^k_{\ j'} \\
&= \frac{1}{\sqrt{N^2-1}}\left(q^{(1)}_j \bar{q}^k_{(2)} q^{(3)}_k \bar{q}^j_{(4)} - \frac{1}{N}q^{(1)}_j \bar{q}^j_{(2)} q^{(3)}_k \bar{q}^k_{(4)}\right) \ .
\end{aligned}
\tag{1.6.11}
$$

Using the explicit form of $q^{(i)}_j$ and $\bar{q}^j_{(i)}$ we obtain

$$
g_{\mathbf{Adj}}(z) = \frac{1}{\sqrt{N^2-1}}\left((z-1)^{N-1} - \frac{1}{N}z^{N-1}\right) \ .
\tag{1.6.12}
$$

1.6.3   The four-point function $G_{\phi_+ \bar{\phi}_+ \phi_+ \bar{\phi}_+}$

As explained below (1.2.6), we have only computed the holomorphic conformal blocks. In order to obtain the four point function we need to sum over the products of holomorphic conformal blocks $g_s(z)$ and anti-holomorphic conformal blocks $\tilde{g}_{\tilde{s}}(\bar{z})$. We then write

$$
G_{\phi_+ \bar{\phi}_+ \phi_+ \bar{\phi}_+} = \sum_{p,\tilde{p}=\mathbf{1},\mathbf{Adj}} A_{p\tilde{p}} g_p(z) \tilde{g}_{\tilde{p}}(\bar{z}) = g^T(z) A g(\bar{z}) \ ,
\tag{1.6.13}
$$

where we introduced the matrix A and the vectors

$$
g(z) = \begin{pmatrix} g_{\mathbf{1}}(z) \\ g_{\mathbf{Adj}}(z) \end{pmatrix} \ , \quad g(\bar{z}) = \begin{pmatrix} g_{\mathbf{1}}(\bar{z}) \\ g_{\mathbf{Adj}}(\bar{z}) \end{pmatrix}.
\tag{1.6.14}
$$

The correlator written explicitly in (1.6.3) is invariant under the exchange $x_2 \leftrightarrow x_4$. This translates to a constraint on the matrix A in our construction. To see this we first observe that the vector $g(z)$ transforms under the exchange as

$$g(z) = \begin{pmatrix} g_{\mathbf{1}}(z) \\ g_{\mathbf{Adj}}(z) \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{N} & \frac{\sqrt{N^2-1}}{N} \\ \frac{\sqrt{N^2-1}}{N} & -\frac{1}{N} \end{pmatrix} \begin{pmatrix} g_{\mathbf{1}}(z) \\ g_{\mathbf{Adj}}(z) \end{pmatrix} \equiv Og(z) \qquad (1.6.15)$$

where O is the orthogonal exchange matrix defined in subsection 1.3.3 and $g(\overline{z})$ transforms similarly. This means that the correlator transforms as

$$g^T(z)Ag(\overline{z}) \rightarrow g^T(z)O^T AOg(\overline{z}). \qquad (1.6.16)$$

Demanding invariance of the correlator amounts to the constraint $O^T AO = A$. Any linear combination of the identity matrix and the exchange matrix O satisfies this equation and will lead to a crossing symmetric correlation function when plugged in (1.6.13). We continue to compute these crossing symmetric building blocks. With $A = I$ we get

$$G_I(z_i, \overline{z}_i) = g_{\mathbf{1}}(z)g_{\mathbf{1}}(\overline{z}) + g_{\mathbf{Adj}}(z)g_{\mathbf{Adj}}(\overline{z})$$
$$= \frac{1}{N^2-1}\left[\left(|z|^2\right)^{N-1} + \left(|z-1|^2\right)^{N-1} - \frac{1}{N}((z-1)\overline{z})^{N-1} - \frac{1}{N}(z(\overline{z}-1))^{N-1} \right].$$
$$(1.6.17)$$

And with $A = O$ we get

$$G_O(z_i, \overline{z}_i) = \frac{1}{N}\left(g_{\mathbf{1}}(z)g_{\mathbf{1}}(\overline{z}) - g_{\mathbf{Adj}}(z)g_{\mathbf{Adj}}(\overline{z}) \right) + \frac{\sqrt{N^2-1}}{N}\left(g_{\mathbf{1}}(z)g_{\mathbf{Adj}}(\overline{z}) + g_{\mathbf{Adj}}(z)g_{\mathbf{1}}(\overline{z}) \right)$$
$$= \frac{1}{N^2-1}\left[(z(\overline{z}-1))^{N-1} + ((z-1)\overline{z})^{N-1} - \frac{1}{N}\left(|z|^2\right)^{N-1} - \frac{1}{N}\left(|z-1|^2\right)^{N-1} \right].$$
$$(1.6.18)$$

One can see in (1.6.1)-(1.6.3) that the exchange $x_2 \leftrightarrow x_4$ corresponds to $(z, \overline{z}) \rightarrow (1-z, 1-\overline{z})$ and (1.6.17) and (1.6.18) are indeed invariant under this transformation. A specific linear combination of G_I and G_O gives

$$G(z_i, \overline{z}_i) = \left(|z|^2\right)^{N-1} + \left(|z-1|^2\right)^{N-1}. \qquad (1.6.19)$$

This is the semiclassical limit of the result computed in [67] using the Coulomb gas formalism. Another linear combination of interest is the following

$$G(z_i, \overline{z}_i) = \left(|z|^2\right)^{N-1} + \left(|z-1|^2\right)^{N-1} + ((z-1)\overline{z})^{N-1} + (z(\overline{z}-1))^{N-1}. \qquad (1.6.20)$$

For $N = -1$, this expression reduces to the following correlator of free complex bosons

$$G(z_i, \overline{z}_i) = \langle \partial\phi\overline{\partial}\overline{\phi}(x_1)\partial\overline{\phi}\overline{\partial}\phi(x_2)\partial\phi\overline{\partial}\overline{\phi}(x_3)\partial\overline{\phi}\overline{\partial}\phi(x_4)\rangle$$

$$= (z_{12}\overline{z}_{12})^{-2}(z_{34}\overline{z}_{34})^{-2} + (z_{14}\overline{z}_{14})^{-2}(z_{23}\overline{z}_{23})^{-2} \qquad (1.6.21)$$

$$+ (z_{12}\overline{z}_{14})^{-2}(z_{34}\overline{z}_{23})^{-2} + (z_{14}\overline{z}_{12})^{-2}(z_{23}\overline{z}_{34})^{-2}$$

after implementing coordinates as in (1.6.2) and (1.6.3).

1.7 Heavy-light Virasoro blocks

We now show how to use our approach to obtain Virasoro blocks in the heavy-light limit. This refers to a limit in which we take $c \to \infty$ while scaling operator dimensions in a specific way. In particular, we consider a four-point function of two light operators and two heavy operators, $\langle O_{L_1}O_{L_2}O_{H_1}O_{H_2}\rangle$. Light operators have scaling dimensions $h_{1,2}$ that are held fixed in the limit, while heavy operator dimensions $H_{1,2}$ scale like c, while their difference $H_{12} = H_1 - H_2$ is held fixed. Further, the exchanged primary is taken to be light, with its scaling dimension h_p held fixed.

Rather than working on the z-plane, in this section it will be more convenient to work on the cylinder, $z = e^{iw}$, with $w = \phi + i\tau$. Of course, the conformal blocks in the two cases are simply related by a conformal transformation. We will further use conformal invariance to place the heavy operators in the far past and future, and one of the light operators at $w = 0$. With these comments in mind, the heavy-light Virosoro blocks on the cylinder are [46]

$$\langle O_{L_1}(w, \overline{w})O_{L_2}(0,0)\, P_p\, O_{H_1}(\tau = -\infty)O_{H_2}(\tau = \infty)\rangle = \mathcal{F}(h_i, h_p; w)\overline{\mathcal{F}}(\tilde{h}_i, \tilde{h}_p; \overline{w})$$

$$(1.7.1)$$

with

$$\mathcal{F}(h_i, h_p; w) = \left(\sin\frac{\alpha w}{2}\right)^{-2h_{L_1}}\left(1 - e^{i\alpha w}\right)^{h_p + h_{12}}{}_2F_1\left(h_p + h_{12}, h_p - \frac{H_{12}}{\alpha}, 2h_p; 1 - e^{i\alpha w}\right).$$

$$(1.7.2)$$

Here

$$\alpha = \sqrt{1 - \frac{24h_{H_1}}{c}}\ . \qquad (1.7.3)$$

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Setting $\alpha = 1$ yields the result for the global block. We then note that the heavy-light Virasoro block is obtained from the global block by the replacements

$$w \to \alpha w , \quad H_{12} = \frac{H_{12}}{\alpha} . \tag{1.7.4}$$

We now show how this result comes out in our approach.

As shown in previous work, the relevant bulk geometry is a conical defect spacetime whose energy matches the dimension of the heavy operators. The corresponding connection is

$$a = (L_1 + \frac{\alpha^2}{4} L_{-1}) dw \tag{1.7.5}$$

We now write

$$e^{(T_1 + \frac{\alpha^2}{4} T_{-1})w} = e^{c_1(w)T_1} [c_0(w)]^{2T_0} e^{c_{-1}(w)T_{-1}} \tag{1.7.6}$$

with

$$c_1(w) = \frac{2}{\alpha} \tan \frac{\alpha w}{2} , \quad c_0(w) = \cos \frac{\alpha w}{2} , \quad c_{-1}(w) = \frac{\alpha}{2} \tan \frac{\alpha w}{2} \tag{1.7.7}$$

obtained by matching the two sides in the two-dimensional rep of SL(2).

The conformal block is given by[6]

$$\mathfrak{w}_s(w_i) = \sum_{\{m_i\}} S_{m_1,m_2,m_3,m_4} \prod_{i=1}^{4} \langle j_i m_i | e^{c_1(w_{bi})T_1} [c_0(w_{bi})]^{2T_0} | j_i j_i \rangle \tag{1.7.8}$$

where we have written the singlet state as $\langle s | = \sum_{\{m_i\}} S_{m_1,m_2,m_3,m_4} \prod_{i=1}^{4} \langle j_i m_i |$. We will think of the first two spins as representing the light operators, so $h_1 = -j_1$ and $h_2 = -j_2$. The insertion point of last two spins will be taken to $\tau = \pm\infty$, since this is where the heavy operators are inserted. The heavy operators correspond to the background connection with the contribution of the spins added on top. Below we will see that $H_{12} = -\alpha(j_3 - j_4)$.

We use conformal invariance to set

$$w_1 \to w , \quad w_2 \to 0 , \quad w_3 \to -i\infty , \quad w_4 \to +i\infty , \quad w_b \to 0 \tag{1.7.9}$$

---

[6]To avoid confusion with the cylinder coordinates w_i, we use $\mathfrak{w}_s$ to denote the conformal blocks in this section.

The functions behave as

$$c_1(w_{b2}) \sim 0 , \quad c_0(w_{b2}) \sim 1$$
$$c_1(w_{b3}) \sim \frac{2i}{\alpha} , \quad c_0(w_{b3}) \sim \frac{1}{2}e^{\frac{i\alpha w_3}{2}} \to \infty$$
$$c_1(w_{b4}) \sim -\frac{2i}{\alpha} , \quad c_0(w_{b4}) \sim \frac{1}{2}e^{-\frac{i\alpha w_4}{2}} \to \infty \tag{1.7.10}$$

The first limit picks out $m_2 = j_2$ from the sum. After stripping off the $w_{3,4}$ dependent factors (which are absorbed into the definition of the operators at $\tau = \pm\infty$) we are left with

$$\mathfrak{w}_s(w_i) = (\cos\frac{\alpha w}{2})^{2j_1} \sum_{m_1,m_3,m_4} S_{m_1,j_2,m_3,m_4} \langle j_1 m_1 | e^{-\frac{2}{\alpha}\tan\frac{\alpha w}{2}T_1} | j_1 j_1 \rangle$$
$$\langle j_3 m_3 | e^{\frac{2i}{\alpha}T_1} | j_3 j_3 \rangle \langle j_4 m_4 | e^{-\frac{2i}{\alpha}T_1} | j_4 j_4 \rangle \tag{1.7.11}$$

Now, starting from the $\alpha = 1$ case we obtain (1.7.11) by the replacements

$$w \to \alpha w , \quad T_1 \to \frac{1}{\alpha}T_1 \tag{1.7.12}$$

We first establish that the rescaling of T_1 has no effect other than contributing an overall multiplicative constant. This is because upon expanding the exponentials only a fixed overall power of T_1 contributes, since $m_1 + j_2 + m_3 + m_4 = 0$ by the singlet condition. We simply pick up one power of α for each power of T_1, which as noted above just yields a fixed overall constant which we ignore.

Besides the rescaling of w, we also need to account for the rescaling of H_{12} in (1.7.4). At $\alpha = 1$ we have only light operators and we would write $H_{12} = -(j_3 - j_4)$. For general α we can read off the contribution to the scaling dimension from $j_{3,4}$ from the $w_{3,4}$ dependent prefactor that we stripped off. From the behavior of the functions $c_0(w_{b3})$ and $c_0(w_{b4})$ we see that this factor is $e^{\frac{i\alpha j_3 w_3}{2}}e^{-\frac{i\alpha j_3 w_4}{2}}$. This tells us that it is $\alpha j_{3,4}$ that contributes to the scaling dimensions, and so $-(j_3 - j_4) = \frac{H_{12}}{\alpha}$. This accounts for the rescaling of H_{12}.

Altogether, we see that if we have established the correct result for the global conformal block, as we have indeed done in section 1.4, then agreement for the heavy-light block follows. This completes the argument.

1.8 Discussion

We close with a few comments. The main result of this work is formula (1.1.3), yielding large c correlators and conformal blocks of $\mathcal{W}_N$ theories. We showed by explicit computation how the choice of light external operators yields global blocks, recovering known results in a new way that is well adapted to holographic considerations. We can equally well obtain heavy-light blocks, as was demonstrated in the $N = 2$ case where we obtained heavy-light Virasoro blocks. Similarly, heavy-light blocks for $\mathcal{W}_N$ can be obtained through more work, if desired. In all these cases, all our results directly pertain to the case where operator dimensions are negative; however, after the result has been obtained one can analytically continue to positive dimensions. Of course, this requires some knowledge of the analytic structure as a function of operator dimension. This is usually no obstacle: for example, one knows that each term in the series expansion of a conformal block in the cross ratio is a rational function of operator dimensions, rendering analytic continuation trivial. Similarly, one can analytically continue in N to obtain blocks of $\mathcal{W}_\infty(\lambda)$.

Looking ahead, it would be very interesting to obtain (1.1.3) directly from the equations of Prokushkin and Vasiliev. At present, we only know how to do this in the case of two light operators, corresponding to computing a two-point function in a heavy background. Starting from the Prokushkin-Vasiliev equations, it is well known (e.g. [10]) how to linearize in the matter field to obtain a description of a free scalar interacting with Chern-Simons gauge fields, and how the computation of two-point functions leads to a special case of (1.1.3). However, the system of equations becomes much more complicated when matter self-interactions are included, and they have so far not been put into a usable form. We also note that the case of two light operators includes all existing computations of entanglement entropy in higher spin theories, which correspond to two-point functions of operators with quantum numbers chosen to match those of twist operators [8, 25, 30, 56].

Results obtained here pertain to the large c limit, which corresponds to the classical limit in the bulk. On the CFT side one can work out $1/c$ corrections [41], and it is interesting to ask how these might arise in the bulk as quantum corrections. For example, one might

entertain computing loop diagrams in the bulk via Wilson lines. However, the most obvious way of defining such diagrams does not lead to anything new when we recall that gauge invariance implies that the location of bulk vertices can be moved without changing the result. The same argument that said that tree level exchange diagrams can be reduced to contact diagrams by merging vertices also tells us that such loop diagrams can be reduced to tree level contact diagrams. Apparently some new ingredient is needed to compute quantum corrections.

Appendices

1.9 Conformal invariance of correlators

Here we show that our correlation functions transform properly under global conformal transformations, as in (1.2.9). We start from our general expression for an n-point function

$$w_s(z_i) = \langle s| \prod_{i=1}^n e^{z_{bi} T_1^{(i)}} |hw\rangle_i \ . \tag{1.9.1}$$

Under a gauge transformation of the connection

$$a \to LaL^{-1} + LdL^{-1} \tag{1.9.2}$$

a Wilson line transforms as

$$Pe^{\int_x^y a} \to L(y)Pe^{\int_x^y a}L^{-1}(x) \tag{1.9.3}$$

An arbitrary $SL(2)$ transformation can be written as

$$L(z) = e^{c_{-1}T_{-1}} \ e^{2\log c_0 T_0} \ e^{c_1 T_1} \tag{1.9.4}$$

where the c_i are functions of z. Starting with the connection corresponding to pure AdS in Poincaré coordinates , $a = T_1 dz$, a gauge transformation by $L(z)$ gives

$$a' = \left[\frac{1-c_1'}{c_0^2}T_1 - \frac{2(c_{-1} + c_0 c_0' - c_{-1} c_1')}{c_0^2}T_0 - \frac{c_0^2 c_{-1}' - 2c_0 c_0' c_{-1} - c_{-1}^2 + c_{-1}^2 c_1'}{c_0^2}T_{-1}\right] dz \tag{1.9.5}$$

To verify this one can first work out the result in the 2×2 matrix representation of $SL(2)$ and then use the fact that the group multiplication is independent of the representation. We demand that $a' \propto T_1$, so that the coefficients of T_0 and T_{-1} vanish. It will prove sufficient to take

$$c_1(z) = 0 , \quad c_0(z) = cz + d , \quad c_{-1}(z) = -c c_0(z) \tag{1.9.6}$$

corresponding to the new connection

$$a' = \frac{T_1 dz}{(cz + d)^2} = T_1 dz' \tag{1.9.7}$$

where

$$z' = \frac{az + b}{cz + d} , \quad ad - bc = 1 . \tag{1.9.8}$$

Returning to (1.9.1) we write

$$
\begin{aligned}
w_s(z_i) &= \langle s | \prod_{i=1}^{n} L^{-1}(z_b) L(z_b) e^{z_{bi} T_1^{(i)}} L^{-1}(z_i) L(z_i) | hw \rangle_i \\
&= \langle s | \prod_{i=1}^{n} e^{z'_{bi} T_1^{(i)}} L(z_i) | hw \rangle_i
\end{aligned}
\tag{1.9.9}
$$

We further have

$$
\begin{aligned}
L(z_i) | hw \rangle_i &= e^{c_{-1} T_{-1}} \, e^{2 \log c_0 T_0} | hw \rangle_i \\
&= e^{-2 h_i \log c_0} | hw \rangle_i \\
&= (cz_i + d)^{-2 h_i} | hw \rangle_i
\end{aligned}
\tag{1.9.10}
$$

which yields

$$w_s(z_i) = \left[\prod_{i=1}^{n} (cz_i + d)^{-2 h_i} \right] w_s(z'_i) . \tag{1.9.11}$$

This is equivalent to (1.2.9).

1.10   Computation of three-point function

In this appendix we give the details for deriving (1.2.14). We work with a description of SL(2) representations based on symmetric tensors, or equivalently Young tableau with a

single row. We start with Young tableaux with one single row of length $\lambda = 2j$ for a spin j representation. In tensor notation the states of this representation are $A^{\alpha_1\ldots\alpha_\lambda} |e_{\alpha_1}\ldots e_{\alpha_\lambda}\rangle$ where A is a symmetric tensor, and $|e_1\rangle$ and $|e_2\rangle$ are the spin up and spin down states of the spin half representation of SL(2), respectively. In other words, $|e_1\rangle$ and $|e_2\rangle$ are states in the fundamental representation of SL(2). The highest weight state is $|e_1\ldots e_1\rangle$. Wilson lines emanating from the boundary points z_1, z_2 and z_3 carry Dynkin labels λ_1, λ_2 and λ_3, respectively, and we take $\lambda_1 \geq \lambda_2$ without loss of generality. The tensor product of the first two representations decomposes as

$$\boxed{\lambda_1} \otimes \boxed{\lambda_2} = \sum_{\lambda = |\lambda_1 - \lambda_2|}^{\lambda_1 + \lambda_2} \boxed{\lambda} \tag{1.10.1}$$

where representations of label $\lambda \in \{|\lambda_1 - \lambda_2|, \ldots, \lambda_1 + \lambda_2\}$ appear. If λ_3 lies in this interval we can build a singlet out of the three representations. Once we have the singlet we need to evaluate the bulk-to-boundary Wilson lines. These act independently on each state of the fundamental representation so it is convenient to first evaluate matrix elements on these factors and then assemble the singlet, which will then lead directly to the three point function. We denote by $q_\alpha^{(i)}$ the following matrix element of the Wilson line

$$q_\alpha^{(i)} = \langle e_\alpha | \, e^{z_{bi} T_1^{(i)}} \, |e_1\rangle_i = \delta_\alpha^1 - z_{bi}\delta_\alpha^2. \tag{1.10.2}$$

We now exploit gauge invariance to set $z_1 = z_b$. After this we see that $q_\alpha^{(1)} = \delta_\alpha^1$, which simplifies the calculation. We now define the tensor $(M_i)_{j_1\ldots j_{\lambda_i}} = \langle (\mathcal{R}_i)_{j_1\ldots j_{\lambda_i}} | e^{z_{bi} L_1} | \mathrm{hw}\rangle_i$ representing the matrix element of the Wilson line for any state in the representation $\mathcal{R}_i$.

$$
\begin{aligned}
z_1 \quad &: \quad (M_1)_{\alpha_1\ldots\alpha_{\lambda_1}} = \delta_{\alpha_1}^1 \ldots \delta_{\alpha_{\lambda_1}}^1 \\
z_2 \quad &: \quad (M_2)_{\beta_1\ldots\beta_{\lambda_2}} = q_{(\beta_1}^{(2)} \ldots q_{\beta_{\lambda_2})}^{(2)} \\
z_3 \quad &: \quad (M_3)_{\rho_1\ldots\rho_{\lambda_3}} = q_{(\rho_1}^{(3)} \ldots q_{\rho_{\lambda_3})}^{(3)}
\end{aligned}
\tag{1.10.3}
$$

We now build a tensor of λ_3 symmetric indices out of M_1 and M_2.

$$(M_{12})_{\gamma_1\ldots\gamma_{\lambda_3}} = \epsilon^{\alpha_1\beta_1} \ldots \epsilon^{\alpha_{\frac{\lambda_1+\lambda_2-\lambda_3}{2}} \beta_{\frac{\lambda_1+\lambda_2-\lambda_3}{2}}} (M_1)_{\alpha_1\ldots\alpha_{\frac{\lambda_1+\lambda_2-\lambda_3}{2}} (\gamma_1\ldots\gamma_{\frac{\lambda_1+\lambda_3-\lambda_2}{2}}}$$

$$(M_2)_{\gamma_{\frac{\lambda_1+\lambda_3-\lambda_2}{2}+1}\ldots\gamma_{\lambda_3})\beta_1\ldots\beta_{\frac{\lambda_1+\lambda_2-\lambda_3}{2}}} \tag{1.10.4}$$

$$= (q_2^{(2)})^{\frac{\lambda_1+\lambda_2-\lambda_3}{2}} \delta_{(\gamma_1}^1 \ldots \delta_{\gamma_{\frac{\lambda_1+\lambda_3-\lambda_2}{2}}}^1 q_{\gamma_{\frac{\lambda_1+\lambda_3-\lambda_2}{2}+1}}^{(2)} \ldots q_{\gamma_{\lambda_3})}^{(2)}$$

where we have contracted indices with the invariant tensor $\epsilon^{\alpha\beta}$. Finally we bring M_{12} and M_3 together and construct the singlet

$$
\begin{aligned}
w_s(z_1, z_2, z_3) &= \epsilon^{\alpha_1\beta_1} \ldots \epsilon^{\alpha_{\lambda_3}\beta_{\lambda_3}} (M_3)_{\alpha_1\ldots\alpha_{\lambda_3}} (M_{12})_{\beta_1\ldots\beta_{\lambda_3}} \\
&= (q_2^{(2)})^{\frac{\lambda_1+\lambda_2-\lambda_3}{2}} \epsilon^{\alpha_1\beta_1} \ldots \epsilon^{\alpha_{\lambda_3}\beta_{\lambda_3}} q_{\alpha_1}^{(3)} \ldots q_{\alpha_{\lambda_3}}^{(3)} \delta_{\beta_1}^1 \ldots \delta_{\beta_{\frac{\lambda_1+\lambda_3-\lambda_2}{2}}}^1 q_{\beta_{\frac{\lambda_1+\lambda_3-\lambda_2}{2}+1}}^{(2)} \ldots q_{\beta_{\lambda_3}}^{(2)}
\end{aligned}
$$

$$(1.10.5)$$

where we have made use of the symmetric structure of M_3 and M_{12}, and discarded constant factors. Using now the explicit form of $q_\alpha^{(i)}$ from (1.10.2) we obtain

$$
w_s(z_1, z_2, z_3) = z_{12}^{\frac{\lambda_1+\lambda_2-\lambda_3}{2}} z_{13}^{\frac{\lambda_1+\lambda_3-\lambda_2}{2}} z_{23}^{\frac{\lambda_2+\lambda_3-\lambda_1}{2}}
$$

$$(1.10.6)$$

This yields the result (1.2.14) upon using $h_i = -\lambda_i/2$.

1.11 SL(N) Conventions and Facts

1.11.1 Conventions

We use the same conventions as in [23]. All the Wilson lines that appear in this paper are valued in the SL(2) subgroup of SL(N). The matrices we then need are for the generators of SL(2) which in the N dimensional defining representation are

$$
L_1 = - \begin{pmatrix}
0 & & \cdots & & & & 0 \\
\sqrt{N-1} & 0 & & & & \cdots & \\
0 & \sqrt{2(N-2)} & 0 & & & & \\
\vdots & & \ddots & \ddots & & & \vdots \\
& & & \sqrt{i(N-i)} & 0 & & \\
& & & & \ddots & \ddots & \\
0 & & \cdots & & & \sqrt{N-1} & 0
\end{pmatrix}
$$

$$(1.11.1)$$

$$
L_0 = \mathrm{diag}\left(\frac{N-1}{2}, \frac{N-3}{2}, \ldots, \frac{N-2i+1}{2}, \ldots, -\frac{N-1}{2}\right)
$$

$$
L_{-1} = -(L_1)^\dagger
$$

Using these matrices, we find for the defining representation

$$\langle -\text{hw}|e^{zL_1}|\text{hw}\rangle = \frac{z^{N-1}}{(N-1)!} \langle -\text{hw}|(L_1)^{N-1}|\text{hw}\rangle$$
$$= (-z)^{N-1}$$

(1.11.2)

1.11.2 Irreducible Tensors

We denote states of the defining representation of SL(N) by an n-dimensional lower indexed vector $|e_i\rangle$, $i = 1, \ldots, N$. It is natural then to denote states of the conjugate representation by upper indexed objects $|\bar{e}^i\rangle$, such that the invariant tensors are given by δ_i^j, $\epsilon_{i_1 \ldots i_N}$ and $\epsilon^{j_1 \ldots j_N}$. Their invariance follows from the fact that the matrices of SL(N) have unit determinant.

This characterization is useful for the SL(3) calculations of section 1.5. The invariant tensors are now δ_i^j, ϵ_{ijk} and ϵ^{ijk}. Consider a tensor with arbitrary number of lower and upper indices. First focus on a pair of lower indices. The part that is antisymmetric in these two indices can be converted into a single upper index using an ϵ^{ijk}. Next we do the same with pairs of upper indices. We can keep doing this until we have a tensor that has completely symmetric upper and lower indices. We can also contract an upper and a lower index using δ_i^j to give a lower rank tensor. Thus an irreducible tensor of SL(3) should be completely traceless and symmetric in upper and lower indices.

We can construct a symmetric traceless tensor with m lower and n upper indices, $T_{i_1 \ldots i_m}^{j_1 \ldots j_n}$, by taking a tensor product of m copies of the defining and n copies of its conjugate representation, symmetrizing and subtracting out traces. Since the traces are all lower rank tensors, we have

(1.11.3)

where the . . . on the right denote Young tableau with boxes $< m + 2n$. So we conclude $T_{i_1 \ldots i_m}^{j_1 \ldots j_n} \sim (m, n)$. Conjugation of a representation simply conjugates each factor in the tensor product above which is equivalent to exchanging upper and lower indices or $\overline{(m, n)} = (n, m)$.

1.12 Removing Traces of Symmetric Tensors

Consider a tensor with x lower indices and y upper indices, $A_{i_1 \cdots i_x}^{j_1 \cdots j_y}$, where the upper and lower indices are completely symmetrized. For the purposes of having an irreducible representation of SL(3) we also need this tensor to be traceless. Since all the indices are symmetric we can consider a particular trace, say $\delta_{j_1}^{i_1}$, all other traces being equivalent. The trace of the tensor $A_{i_1 \cdots i_x}^{j_1 \cdots j_y}$ gives a term with one upper and one lower index contracted — a single trace expression. To make this tensor traceless, we need to subtract out single trace expressions with appropriate coefficients while maintaining the symmetry of the indices. Trace of the single trace terms we just added to our tensor gives new double trace expressions. We then subtract those double trace terms and keep on going until we run out of indices to contract. In general, the expression for the traceless tensor looks like

$$\tilde{A}_{i_1 \cdots i_x}^{j_1 \cdots j_y} = A_{i_1 \cdots i_x}^{j_1 \cdots j_y} + \sum_{n=1}^{min(x,y)} C_n \delta_{(i_1}^{(j_1} \cdots \delta_{i_n}^{j_n} A_{i_{n+1} \cdots i_x) k_1 \cdots k_n}^{j_{n+1} \cdots j_y) k_1 \cdots k_n} \tag{1.12.1}$$

where the parentheses denote symmetrization. Our goal is then to fix the coefficients $\tilde{C}_n$. First note that since both i and j are symmetrized, a lot of the terms have the same tensor structure. For example, $\delta_{i_1}^{j_1} \delta_{i_2}^{j_2}$ is the same as $\delta_{i_2}^{j_2} \delta_{i_1}^{j_1}$ (but different from $\delta_{i_1}^{j_2} \delta_{i_2}^{j_1}$). To account for these degeneracies (given n), fix the indices that appear on the tensor A. There are $(x-n)!(y-n)!$ terms which are the same, coming from permutations of the $(x-n)$ lower and $(y-n)$ upper indices on A. Further, we have a total of $(n!)^2$ terms coming from the permutations of the lower and upper indices on the Kronecker deltas but only $n!$ of them are distinct corresponding to keeping the sequence of lower indices fixed while permuting the upper indices. This gives an additional degeneracy factor of $n!$. We then redefine our constants

$$C_n = \frac{(-1)^n \tilde{C}_n}{(x-n)!(y-n)!n!} \tag{1.12.2}$$

such that each tensor structure appears with a factor of $(-1)^n \tilde{C}_n$ in the sum. Note that we have included a sign since the single trace terms cancel the zero trace terms, the double trace cancel the single trace terms and so on. We will use induction to determine $\tilde{C}_n$. Consider

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the term with n traces and $n+1$ traces respectively.

$$n: \frac{(-1)^n \tilde{C}_n}{(x-n)!(y-n)!n!} \delta^{(j_1}_{(i_1} \cdots \delta^{j_n}_{i_n} A^{j_{n+1}\cdots j_y)k_1\cdots k_n}_{i_{n+1}\cdots i_x)k_1\cdots k_n}$$

$$n+1: \frac{(-1)^{n+1}\tilde{C}_{n+1}}{(x-n-1)!(y-n-1)!(n+1)!} \delta^{(j_1}_{(i_1} \cdots \delta^{j_{n+1}}_{i_{n+1}} A^{j_{n+2}\cdots j_y)k_1\cdots k_{n+1}}_{i_{n+2}\cdots i_x)k_1\cdots k_{n+1}}$$

(1.12.3)

To facilitate counting, further restrict to a particular tensor structure after contracting with $\delta^{i_1}_{j_1}$, say $\delta^{j_2}_{i_2} \cdots \delta^{j_{n+1}}_{i_{n+1}} A^{j_{n+2}\cdots j_y k_1\cdots k_{n+1}}_{i_{n+2}\cdots i_x k_1\cdots k_{n+1}}$. This tensor structure can arise from the $n+1$ trace terms in one of 4 ways.

1. Both the indices i_1 and j_1 are among the Kronecker deltas and on the same Kronecker delta.

$$\delta^{j_1}_{i_1} \delta^{j_2}_{i_2} \cdots \delta^{j_{n+1}}_{i_{n+1}} A^{j_{n+2}\cdots j_y k_1\cdots k_{n+1}}_{i_{n+2}\cdots i_x k_1\cdots k_{n+1}}$$

(1.12.4)

There is exactly one such term after accounting for the degeneracies. Contracting with $\delta^{j_1}_{i_1}$ gives an additional factor of 3.

2. Both the indices i_1 and j_1 are among the Kronecker deltas but are on different Kronecker deltas.

$$\delta^{j_a}_{i_1} \delta^{j_2}_{i_2} \cdots \delta^{j_1}_{i_a} \cdots \delta^{j_{n+1}}_{i_{n+1}} A^{j_{n+2}\cdots j_y k_1\cdots k_{n+1}}_{i_{n+2}\cdots i_x k_1\cdots k_{n+1}}$$

(1.12.5)

where $2 \leq a \leq n+1$. There are n such terms and each gives a factor of 1.

3. i_1 is on a Kronecker delta but j_1 is on the tensor A.

$$\delta^{j_b}_{i_1} \delta^{j_2}_{i_2} \cdots \delta^{j_{n+1}}_{i_{n+1}} A^{j_{n+2}\cdots j_1\cdots j_y k_1\cdots k_{n+1}}_{i_{n+2}\cdots i_x k_1\cdots k_{n+1}}$$

(1.12.6)

where $n+2 \leq b \leq y$. There are $(y-n-1)$ such terms and each gives a factor of 1.

4. On a similar note, we can have j_1 on the Kronecker delta but i_1 on A.

$$\delta^{j_1}_{i_c} \delta^{j_2}_{i_2} \cdots \delta^{j_{n+1}}_{i_{n+1}} A^{j_{n+2}\cdots j_y k_1\cdots k_{n+1}}_{i_{n+2}\cdots i_1\cdots i_x k_1\cdots k_{n+1}}$$

(1.12.7)

where $n+2 \leq b \leq x$. There are $(x-n-1)$ such terms and each gives a factor of 1.

It is easily checked that no other possibility gives the right tensor structure. Combining all this we get a factor of $(x+y-n+1)$ accompanying the required tensor structure in the $n+1$ trace terms. Looking at the terms in (1.12.3) with n traces, the required tensor structure can appear only when both the i_1 and j_1 indices are on the tensor A and the Kronecker deltas are in the correct form. This term occurs exactly once after removing degeneracies. Hence, we get the recursion relation

$$\tilde{C}_{n+1} = \frac{\tilde{C}_n}{x + y - n + 1} \tag{1.12.8}$$

Note that we can think of the original tensor as the $n = 0$ term with $\tilde{C}_0 = 1$. The coefficients are then given by

$$\tilde{C}_n = \frac{1}{[x + y + 1]_n} \tag{1.12.9}$$

where $[a]_n$ is the descending Pochhammer symbol, $[a]_n = a(a - 1) \ldots (a - n + 1)$. Putting all of this together, the traceless tensor is given by

$$\tilde{A}^{j_1 \cdots j_y}_{i_1 \cdots i_x} = \sum_{n=0}^{min(x,y)} \frac{(-1)^n \Gamma(x + y - n + 2)}{\Gamma(x + y + 2)\Gamma(x - n + 1)\Gamma(y - n + 1)\Gamma(n + 1)} \delta^{(j_1}_{(i_1} \cdots \delta^{j_n}_{i_n} A^{j_{n+1} \cdots j_y)k_1 \cdots k_n}_{i_{n+1} \cdots i_x)k_1 \cdots k_n} \tag{1.12.10}$$

1.13 Details of SL(3) Calculation

In this appendix, we present some details of the SL(3) calculations of section 1.5. The singlet in terms of tensors of SL(3) was found in (1.5.16). All that is required now is to contract all the indices while keeping track of all the combinatorial factors and powers of z.

$$g_s(z) = z^{2d} \ q^{(1)}_{i_1} \cdots q^{(1)}_{i_{l_1}} \tilde{q}_{i_{l_1+1}} \cdots \tilde{q}_{i_x} \bar{q}^{j_1}_{(1)} \cdots \bar{q}^{j_{l_3}}_{(1)} \bar{q}^{j_{l_3+1}}_{(2)} \cdots \bar{q}^{j_y}_{(2)}$$
$$\times \sum_{n=0}^{min(x,y)} C_n C'_n \delta^{(i_1}_{(j_1} \cdots \delta^{i_n}_{j_n} q^{(4)}_{j_{n+1}} \cdots q^{(4)}_{j_{n_4}} \tilde{q}'_{j_{n_4+1}} \cdots \tilde{q}'_{j_y} \bar{q}^{i_{n+1}}_{(4)} \cdots \bar{q}^{i_{n_1+n+1}}_{(4)} \bar{q}^{i_{n_1+n+2}}_{(3)} \cdots \bar{q}^{i_x}_{(3)} \tag{1.13.1}$$

As mentioned before we refer to the indices on the first line by colors and the second line by boxes. The various labels we use for indices and the number of them are collected below

Label	Index on	Number
Red	$q^{(1)}$	$l_1 = \lambda_1 - d$
Blue	$\tilde{q}$	$l_2 = e$
Green	$\bar{q}_{(1)}$	$l_3 = \lambda_2 - e$
Yellow	$\bar{q}_{(2)}$	$l_4 = \mu - d - e$
Box 1	$\bar{q}_{(4)}$	$n_1 = \lambda'_2 - e'$
Box 2	$\bar{q}_{(3)}$	$n_2 = \mu' - d' - e'$
Box 3/6	δ	n
Box 4	$q^{(4)}$	$n_4 = \lambda'_1 - d'$
Box 5	$\tilde{q}'$	$n_5 = e'$

(1.13.2)

Each permutation will correspond to a particular way of coloring the boxes. Note that we are allowed to color boxes 1, 2 and 3 red or blue only and boxes 4, 5 and 6 green or yellow only. Taking this into account, the various contributions from different combinations are

Coloring	Contribution	Number
Box 1 Red	$q^{(1)} \cdot \bar{q}_{(4)}$	u
Box 2 Red	$q^{(1)} \cdot \bar{q}_{(3)}$	$l_1 - u - u'$
Box 3 Red	$q_j^{(1)}$	u'
Box 1 Blue	$\tilde{q} \cdot \bar{q}_{(4)}$	$n_1 - u$
Box 2 Blue	$\tilde{q} \cdot \bar{q}_{(3)}$	$l_2 - n_1 - n + u + u'$
Box 3 Blue	$\tilde{q}_j$	$n - u'$
Box 4 Green	$\bar{q}_{(1)} \cdot q^{(4)}$	v
Box 5 Green	$\bar{q}_{(1)} \cdot \tilde{q}'$	$l_3 - v - v'$
Box 6 Green	$\bar{q}_{(1)}^i$	v'
Box 4 Yellow	$\bar{q}_{(2)} \cdot q^{(4)}$	$n_4 - n - v$
Box 5 Yellow	$\bar{q}_{(2)} \cdot \tilde{q}'$	$l_4 - n_4 + v + v'$
Box 6 Yellow	$\bar{q}_{(2)}^i$	$n - v'$

(1.13.3)

Note that box 3 and box 6 must be contracted as they refer to lower and upper indices appearing on δ. Our definition $\tilde{q}_j = \epsilon_{jbc}\bar{q}^b_{(1)}\bar{q}^c_{(2)}$ automatically gives $\tilde{q} \cdot \bar{q}_{(1)} = 0 = \tilde{q} \cdot \bar{q}_{(2)}$. Since $\mathcal{R}_1$ is represented as a symmetric traceless tensor, we also have $q^{(1)} \cdot \bar{q}_{(1)} = 0$. We are then left with just one possible combination – color box 3 red and box 6 yellow giving a contribution of $q^{(1)} \cdot \bar{q}_{(2)}$. In the above table this means $u' = n$ and $v' = 0$.

Choosing the bulk point to coincide with $z_1 = 0$ and imposing $z_4 \to \infty$ constrains $q^{(1)}$, $\bar{q}_{(1)}$ to be highest weight ($q^{(1)}_1$ and $\bar{q}^3_{(1)}$) and $q^{(4)}$, $\bar{q}_{(4)}$ to be lowest weight ($q^{(4)}_3$ and $\bar{q}^1_{(4)}$). All the contributions can then be found by our knowledge of the matrix elements in the defining representation (1.5.13). For example we have

$$q^{(1)} \cdot \bar{q}_{(3)} = \bar{q}^1_{(3)} = 1$$
$$\tilde{q} \cdot \bar{q}_{(3)} = \epsilon_{i3c}\bar{q}^i_{(3)}\bar{q}^c_{(2)} = \sqrt{2}z(1 - z)$$

(1.13.4)

The next task is to find the combinatorial factors accompanying each combination and to sum them all up. As an example consider boxes of type 1 i.e. the first and fourth rows of table (1.13.3). We need to color u boxes red and the rest blue. First choose u red indices and $n_1 - u$ blue indices which can be done in $\binom{l_1}{u}\binom{l_2}{u}$ ways. The coloring of the n_1 boxes of type 1 can then be done in $\Gamma(n_1 + 1)$ ways. Proceeding in a similar manner with the rest of the boxes, we obtain

$$g_p(z) = z^{2d} \sum_{n=0}^{min(x,y)} C_n C'_n \sum_{u=0}^{n_1} \sum_{v=0}^{n_4-n} \binom{l_1}{u}\binom{l_2}{n_1-u}\Gamma(n_1+1)\binom{l_1-u}{n}\Gamma(n+1)\Gamma(n_2-n+1)$$
$$\times \binom{l_3}{v}\binom{l_4}{n_4-n-v}\Gamma(n_4-n+1)\binom{l_4-n_4+n+v}{n}\Gamma(n+1)\Gamma(n_5+1)$$
$$\times (\sqrt{2}z)^{n_1-u}(\sqrt{2}z(1-z))^{l_2-n_1+u}(-\sqrt{2})^{l_3-v}(-\sqrt{2}(1-z))^{l_4-n_4+v}z^{2n}$$
$$\sim z^{2d+l_2} \sum_{n=0}^{min(x,y)} C_n C'_n \frac{\Gamma(n_2-n+1)}{\Gamma(l_1-n+1)} z^{2n}(1-z)^{l_2-n_1+l_4-n_4}$$
$$\times \, _2F_1(-n_1, n-l_1; 1+l_2-n_1; 1-z)\,_2F_1(-l_3, n-n_4; 1+l_3-n_4; 1-z)$$

(1.13.5)

where the $\sim$ indicates that we have ignored factors that are independent of z and the summation variable n. We can put the hypergeometric functions into standard form using

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the identity

$$_2F_1(a, b; b - m; z) = \frac{(-1)^m (a)_m}{(1 - b)_m} (1 - z)^{-a-m} {_2F_1}(-m, b - a - m; 1 - a - m; 1 - z) , \quad m \in \mathbb{N}$$

(1.13.6)

where $(a)_m = a(a+1)\ldots(a+m-1)$ is the ascending Pochhammer symbol. We also use the following reflection formula for gamma functions

$$\frac{\Gamma(s - a + 1)}{\Gamma(s - b + 1)} = (-1)^{b-a} \frac{\Gamma(b - s)}{\Gamma(a - s)} , \quad a, b \in \mathbb{Z}, s \in \mathbb{C}$$

(1.13.7)

With $a = n$ and $b = 0$, we obtain

$$\Gamma(s - n + 1) = (-1)^n \frac{\Gamma(s + 1)\Gamma(-s)}{\Gamma(-s + n)}$$

$$\sim \frac{(-1)^n}{(-s)_n}$$

(1.13.8)

The only other ingredient required is the factor $C_n C'_n$ which is obtained from (1.12.10) and (1.5.15) to be

$$C_n C'_n \sim \frac{(-1)^n}{\Gamma(n + 1)} \frac{\Gamma(x + y - n + 2)}{\Gamma(n_4 - n + 1)\Gamma(n_2 - n + 1)}$$

(1.13.9)

We then put all the factors and identities together into (1.13.5) and after the dust settles, we have

$$g_p(z) \sim z^{2d+e} \sum_{n=0}^{min(x,y)} \frac{z^{2n}}{n!} \frac{(-n_2)_n(-l_4)_n(-l_1)_n(-n_4)_n}{(-x)_n(-y)_n(-x - y - 1)_n}$$

$$\times {_2F_1}(-l_2, n - n_2; n - x; z) {_2F_1}(-n_5, n - l_4; n - y; z)$$

(1.13.10)

Note that we have the relations $n_1 + n_2 = x = l_1 + l_2$ and $n_4 + n_5 = y = l_3 + l_4$ with all of the l's and n's being non-negative integers. We can then take the upper limit of the sum to be ∞ as all the extra terms in the sum vanish.

CHAPTER 2

Anomalous Dimensions from Quantum Wilson Lines

We study the self-energy of a gravitating point particle in AdS$_3$, and compare to operator dimensions in CFT$_2$. In particular, we compute the one and two loop diagram contributions to the expectation value of an open Wilson line in the SL(2,R)$\times$ SL(2,R) Chern-Simons formulation of AdS$_3$ gravity. This gives the two-point function of CFT primary operators to second order in a large c expansion, and hence yields the scaling dimension $h(j,c)$ as a function of the SL(2,R) spin j. Comparison to CFT is made in the context of constructing Virasoro representations starting from representations of SL(2,R) current algebra. Our Wilson line computations follow the framework advanced recently by Fitzpatrick et. al., which is based on earlier work by H. Verlinde. We encounter some renormalization scheme ambiguities at the two-loop level which we are not able to fully resolve, hampering a definitive comparison with CFT expressions at this order.

2.1   Introduction

In this paper we study the gravitational self-energy of a point particle in AdS$_3$, and in particular the relation between the energy of the particle when Newton's constant is vanishing or finite. Typically, the relation between these energies is not very interesting since it is cutoff dependent: the self-energy suffers from the classic UV divergence problem, necessitating a short distance cutoff, and there is no universal relation between the bare and renormalized energies. However, for a particle in AdS$_3$ the situation appears to be more favorable, as we now discuss.

The Hilbert space of a particle coupled to gravity in AdS$_3$ corresponds, via the AdS$_3$/CFT$_2$

duality, to a representation of the Virasoro algebra. The lowest allowed energy of the particle maps to the dimension of the primary operator which labels the representation, $E_0 = h + \overline{h} - \frac{c}{12}$. We will use the well-known fact [14], reviewed below, that representations of the Virasoro algebra can be obtained by starting from SL(2,R) current algebra and imposing constraints on the currents. Starting from an SL(2,R) primary of spin-j, one thereby obtains a Virasoro primary of dimension $h(j, c)$, which depends on j and the central charge c. The formula can be written as

$$h(j, c) = -j + \frac{m+1}{m} j(j+1) , \quad c = 1 - \frac{6}{m(m+1)} . \tag{2.1.1}$$

Recalling the Brown-Henneaux formula [19], $c = 3l/2G_N$, sending $G_N \to 0$ corresponds to $c \to \infty$, which can be accomplished by taking $m \to -1$. $h(j, c)$ admits an expansion in $1/c$,

$$h(j, c) = -j - \frac{6}{c} j(j+1) - \frac{78}{c^2} j(j+1) + \dots . \tag{2.1.2}$$

We aim to give the subleading terms an interpretation in terms of gravitational self-energy.[1]

The relation between the SL(2,R) current algebra and the Virasoro algebra has an analog on the AdS$_3$ side that is also well known; see [12] for a review. Starting from SL(2,R)$\times$ SL(2,R) Chern-Simons theory, which is equivalent [1, 78] (in perturbation theory) to three-dimensional Einstein gravity with a negative cosmological constant, imposing the boundary conditions that imply asymptotic AdS$_3$-ness has the effect of implementing the aforementioned reduction of the symmetry algebra. Our particle is described by a Wilson line in the spin-j representation of SL(2,R). An open Wilson line with endpoints on the AdS boundary computes a boundary two-point function, from which the dimension $h(j, c)$ can be deduced, and hence our task is to compute such a Wilson line perturbatively in $1/c$. Wilson lines in the context of AdS$_3$/CFT$_2$ duality first appeared in [8, 30] as a tool to compute entanglement entropy in higher spin theories.

Our setup is motivated by ongoing work [5, 17, 29, 41, 45, 46, 52, 58, 60] on the bulk interpretation of conformal blocks in two-dimensional CFTs, which is in turn aimed at gaining

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[1]In much of this paper we will take $2j$ to be a positive integer corresponding to a finite dimensional non-unitary representation of SL(2,R). This of course yields a negative "bare" energy. However, we stress that our analysis carries over immediately to j-values corresponding to positive energy unitary representations, as we discuss later.

insight into the emergence of local bulk physics — and its ultimate breakdown — starting from CFT. In particular, conformal blocks were given a bulk formulation in terms of particle worldlines in [4–7, 22, 26, 28, 43, 45, 46, 51, 58–60]. A Wilson line version of these constructions in the large c limit, with generalizations to higher spin theories, was given in [15, 17, 29, 56]. The fully quantum version incorporating $1/c$ corrections appears in [42]. We should also note that the main features of these Wilson line constructions already appeared long ago in [77], building on the famous connection between Chern-Simons theory and CFT developed in [79], albeit at a somewhat formal level that did not take into account such issues as UV divergences. This early work is reviewed in the modern AdS/CFT context in [42].
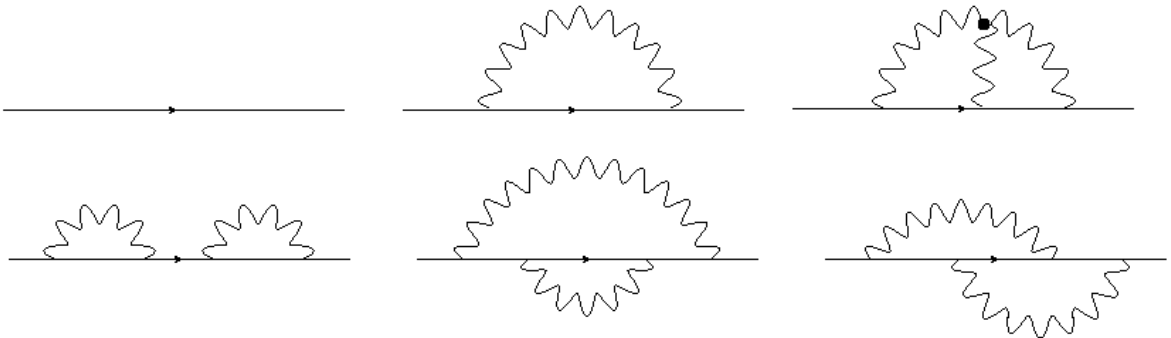
Figure 2.1: Wilson line diagrams to order $1/c^2$

We compute a Wilson line two-point function to the first two subleading orders in the $1/c$ expansion, corresponding to the diagrams shown in figure 2.1.[2] These diagrams are

Figure 2.2: Graviton self energy

UV divergent, as expected. The proper treatment of these divergences is not completely straightforward, as we are not starting from the standard framework of a local Lagrangian to which we can add counterterms, and this gives rise to some ambiguities. At order $1/c$

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[2]The graviton self-energy diagrams in figure 2.2 are implicitly taken into account, as will become clear.

simply removing power law divergences yields the first correction in (2.1.2). At order $1/c^2$ the two loop diagrams include contributions that can be unambiguously associated to the exponentiation of the order $1/c$ result, but ambiguity arises in trying to deduce the $1/c^2$ correction to $h(j,c)$, essentially due to the need to remove a divergent term of the same form as the finite term we are after. It seems likely that to resolve this ambiguity one needs to study in more detail how the Virasoro generators act in this setup and require that the symmetry is being implemented consistently.

2.2   CFT results

We begin by reviewing how imposing constraints on SL(2,R) current algebra representations yields representations of Virasoro [14]. The SL(2,R) current algebra at level k is

$$J^a(z)J^b(0) \sim \frac{(k/2)\eta^{ab}}{z^2} + \frac{i\epsilon^{ab}{}_c J^c(0)}{z} \tag{2.2.1}$$

Here $\eta^{ab} = (1,1,-1)$ and $\epsilon^{123} = 1$. We also define $J^{\pm} = J^1 \pm iJ^2$. The stress tensor is given via the Sugawara construction

$$T_{SL(2)} = \frac{1}{k-2}\eta_{ab}J^a J^b \tag{2.2.2}$$

Its modes obey a Virasoro algebra with central charge

$$c_{SL(2)} = \frac{3k}{k-2} \tag{2.2.3}$$

Current algebra primaries sit in representations of SL(2,R), as labelled by the quadratic Casimir $C_2 = \eta_{ab}J^a J^b$ and the J^3 eigenvalue. For ease of comparison with our later formulas it turns out to be convenient to focus on representations with J^3 bounded from below, and to define j as the negative of the smallest value of J^3 in the representation, so that the quadratic Casimir is $C_2 = -j(j+1)$. In this notation, taking $2j$ to be a positive integer yields a finite dimensional, non-unitary, representation of SL(2,R). The scaling dimension of a spin-j primary is

$$h_{SL(2)}[\Phi_j] = -\frac{j(j+1)}{k-2} \tag{2.2.4}$$

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The reduction to Virasoro proceeds by imposing the constraints $J^-(z) = k$ and $J^0(z) = 0$. For conformal invariance to be compatible with the J^- constraint the stress tensor needs to be modified so that $J^-(z)$ acquires vanishing scaling dimension. This is accomplished by adding to the stress tensor a term proportional to $\partial J^3(z)$. Also, ghosts are introduced so that the constraints can be implemented by a BRST construction. The full stress tensor is then

$$T = T_{SL(2)} + \partial J^3 + T_{gh} \tag{2.2.5}$$

with central charge

$$c = \frac{3k}{k-2} + 6k - 2 \tag{2.2.6}$$

with the -2 coming from the ghosts. The improvement term yields a contribution J^3 to the dimension of the original current algebra primaries, so the dimension of the Virasoro primary is

$$h[\Phi_j] = -j - \frac{j(j+1)}{k-2} \tag{2.2.7}$$

since $J^3 = -j$ yields the lowest dimension operator. It is convenient to write the central charge in the standard minimal model parametrization

$$c = 1 - \frac{6}{m(m+1)} \ , \quad k = \frac{m+2}{m+1} \ , \tag{2.2.8}$$

so that

$$h(j,c) \equiv h[\Phi_j] = -j + \frac{m+1}{m}j(j+1) \tag{2.2.9}$$

To put this in context, recall that the dimensions of the Kac degenerate representations are

$$h_{r,s} = \frac{\left(r(m+1) - sm\right)^2 - 1}{4m(m+1)} \ . \tag{2.2.10}$$

We have

$$h(j,c) = h_{r,s} \ , \quad r = 2j+1 \ , \ s = 1 \ . \tag{2.2.11}$$

Of interest to us is the large c limit obtained by taking $m \to -1$, which yields

$$h(j,c)] = -j - \frac{6j(j+1)}{c} - \frac{78j(j+1)}{c^2} + \dots \ . \tag{2.2.12}$$

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The alternative case $m \to 0$ is commented on below. As we have discussed, we expect the terms appearing in the expansion (2.2.12) to correspond, in the bulk, to perturbative gravitational self energy diagrams.

2.3   Bulk side: preliminary comments

The $1/c$ expansion on the CFT side maps to an expansion in the 3d Newton constant G, so we can hope to recover (2.2.12) by gravitational perturbation theory in AdS$_3$. The Brown-Henneaux formula $c = 3l/2G$ relates the expansions.[3]

Let us first give a heuristic explanation for the part of (2.2.12) which is due to classical self-energy. We consider a spinless point particle of mass $ml = 2h \gg 1$. In higher than three dimensions, as soon as gravity is turned on the particle would collapse into a black hole, but in three dimensions and for sufficiently light particles one instead gets a conical defect solution. In the absence of a cosmological constant, a particle of mass m yields a solution described by Minkowski space with a wedge of angle $\Delta\phi = 8\pi G m$ cut out [32]. Let us now think of placing this particle in AdS$_3$. We do so while keeping m fixed, meaning that we hold fixed the deficit angle computed by examining the geometry in the immediate neighborhood of the solution. Now, a conical defect solution in AdS$_3$ takes the form

$$ds^2 = -(r^2 - 8GMl^2)dt^2 + \frac{l^2 dr^2}{r^2 - 8GMl^2} + r^2 d\phi^2 \tag{2.3.1}$$

where $\phi \cong \phi + 2\pi$. Here M is the total energy measured at the asymptotic AdS boundary, with $M < 0$ for a conical defect. By rescaling coordinates, this metric can be written in standard form $ds^2 = -(r^2 + l^2)dt^2 + l^2 dr^2/(r^2 + l^2) + r^2 d\phi$ but with an angle $\Delta\phi = 2\pi(1 - \sqrt{-8GM})$ cut out. Equating our two expressions for $\Delta\phi$ yields the relation between the "bare" mass m and the physical energy M,

$$M = -\frac{1}{8G} + m - 2Gm^2 . \tag{2.3.2}$$

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[3]More precisely, we should recall that the Brown-Henneaux formula is a classical result in Einstein gravity. In the presence of higher derivative terms it is replaced by the Wald-like formula [65, 75] $c = \frac{l}{2G} g_{\mu\nu} \frac{\delta \mathcal{L}}{\delta R_{\mu\nu}}$.

Writing $Ml = -\frac{c}{12} + 2h$, $ml = -2j$, and using the Brown-Henneaux formula, this becomes

$$h = -j - \frac{6j^2}{c} \ . \tag{2.3.3}$$

The j^2/c contribution matches (2.2.12). To capture the j/c term we need to go beyond treating the particle as having a definite position and include the effect of its finite size quantum wavefunction, which is suppressed for $j \gg 1$. This effect is incorporated in the perturbative treatment given below.

Before turning to that analysis let us return to (2.2.9) and now expand around $m \to 0$,

$$h(j,c) = -\frac{j(j+1)}{6}c - j + \frac{13j(j+1)}{6} + \frac{6j(j+1)}{c} + \dots \ . \tag{2.3.4}$$

This result was given a nice bulk interpretation in [73]; to compare, set $j = (s-1)/2$ and write $L_0 = h(j,c) - \frac{c}{24} = -\frac{s^2 c}{24} + \frac{(13s+1)(s-1)}{24} + \dots$. These states correspond to classical solutions with conical excess angle $2\pi(s-1)$. The $O(c^0)$ contribution comes from quantizing the solutions using the method of coadjoint orbits. For s a positive integer these representations correspond to the degenerate $h_{1,s}$ representations of the Virasoro algebra, examined at large c.

2.4 Perturbative self-energy computation

2.4.1 Chern-Simons formulation, and correlators from Wilson lines

The Chern-Simons formulation of 3d gravity is perfectly adapted to our problem, since the above procedure of going from SL(2,R) current algebra to Virasoro has a precise counterpart in terms of imposing boundary conditions on the connection in SL(2,R)$\times$ SL(2,R) Chern-Simons theory. In the bulk, the Virasoro symmetry arises as the symmetry algebra preserving the asymptotic boundary conditions. We will not review the details of this, as it is well described in many references, e.g. [12]. We just note the following. AdS$_3$ in the form $ds^2 = d\rho^2 + e^{2\rho}dzd\bar{z}$ is represented by the connection $A = L_0 d\rho + e^{\rho}L_1 dz$, along with a similar expression for the second SL(2,R) factor which we henceforth suppress. Here L_n are standard SL(2,R) generators obeying $[L_m, L_n] = (m-n)L_{m+n}$. More generally, a metric

with boundary stress tensor $T(z)$ is represented by $A = L_0 d\rho + (L_1 + \frac{6}{c}T(z)e^{-\rho}L_{-1})dz$. The ρ dependence can be removed by a gauge transformation by $e^{\ln(\rho)L_0}$, allowing us to work with the reduced connection

$$a = \left(L_1 + \frac{6}{c}T(z)L_{-1}\right)dz \ . \tag{2.4.1}$$

Given a connection of the above form, the rule for computing correlators is extremely simple. More precisely, we focus here on the conformal blocks, and in particular just the holomorphic half of the conformal block. Each operator in the CFT corresponds to some spin-j representation of SL(2,R).

In the large c limit the rule for computing conformal blocks is as follows [15, 17]. We set $T(z) = 0$ corresponding to the vacuum state. Each primary operator is represented by its corresponding highest weight SL(2,R) state $|j_i j_i\rangle$. We then attach a Wilson line $W_{j_i}[z_i, z_b] = P e^{\int_{z_i}^{z_b} a}$ directed from the operator location to some arbitrary location z_b. At z_b there resides a singlet state $\langle S|$ in the tensor product of the representations of the primary operators. The large c conformal block is then simply

$$G(z_1, j_1; z_2, j_2; \ldots z_n, j_n) = \langle S| \prod_{i=1}^{n} W_{j_i}[z_i, z_b] |j_i j_i\rangle \ . \tag{2.4.2}$$

This expression satisfies two basic properties. First, it is independent of the choice of z_b, as moving z_b is easily seen to be realized by a gauge transformation, which acts trivially on the singlet state. Second, gauge invariance implies that it transforms as it should under conformal transformations. We also remark that there are in general multiple ways to construct singlet states out of the representations hosted by the primary operators, and this corresponds to the space of conformal blocks. A full fledged correlation function is constructed by combining holomorphic and anti-holomorphic conformal blocks in a manner compatible with crossing symmetry.

The above large c construction yields the global conformal blocks, in which exchanged operators fill out representations of the global conformal group SL(2,R). These conformal blocks can be viewed as the large c limit of Virasoro blocks, which are much richer objects. From the bulk point of view, the Virasoro blocks capture the effect of gravitational interactions, including both classical and quantum effects. Indeed, at finite c the Virasoro blocks

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in some sense contain non-perturbative quantum gravity effects [43, 66], and indeed this is the main motivation for trying to formulate them in bulk terms.

At finite c the same construction (2.4.2) applies, at least formally, except now we should integrate over all connections compatible with asymptotically AdS boundary conditions,

$$G(z_1, j_1; z_2, j_2; \ldots z_n, j_n) = \int DA_\mu e^{-S_{CS}(A)} \langle S | \prod_{i=1}^n W_{j_i}[z_i, z_b] | j_i j_i \rangle . \qquad (2.4.3)$$

Rather than performing the explicit path integral we can follow [42] and take the point of view that the effect is simply to produce correlation functions of the stress tensor appearing in (2.4.1). That is, we expand the path ordered exponentials in powers of $T(z)$, and then replace a string of $T(z)$ operators by the corresponding vacuum correlator, recalling that these are uniquely fixed by Virasoro symmetry. At a formal level this recipe is justified [77] on the grounds that the objects it produces satisfy the Virasoro Ward identities, and some explicit checks of the $1/c$ expansions applied to four-point blocks were carried out in [42].

We focus here on a two-point function since our goal is to compute scaling dimensions. To get a nonzero result the two representations appearing in (2.4.2) should be conjugates of each other, in order that their product contain a singlet. We then simplify by using the freedom to choose z_b to place z_b coincident with one of our operator insertions. The result is that the two-point function is

$$G_j(z_1, z_2) = \langle j, -j | W_j[z_1, z_2] | jj \rangle . \qquad (2.4.4)$$

As already mentioned, we are taking j to be a non-negative integer, so that we have a finite dimensional representation with states $|jm\rangle$, $m = -j, -j+1, \ldots j$, but this is essentially just for notational convenience. Using the prescription of [42], the same functional j dependence arises order by order in perturbation theory for the infinite dimensional representations. More explicitly, we have the following

$$G_j(z_1, z_2) = \langle j, -j | P e^{\int_{z_1}^{z_2} a(y) dy} | jj \rangle = \sum_{n=0}^\infty \int_{z_1}^{z_2} dy_n \int_{z_1}^{y_n} dy_{n-1} \ldots \int_{z_1}^{y_2} dy_1 \langle j, -j | a(y_n) \ldots a(y_1) | jj \rangle ,$$
$$(2.4.5)$$

with a given in (2.4.1) and where each string of stress tensors is replaced by its vacuum correlator.

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If the CFT operator has a definite scaling dimension the result should take the form

$$G_j(z_1, z_2) = C z_{21}^{-2h(j,c)} , \quad z_{ij} = z_i - z_j. \tag{2.4.6}$$

In the $1/c$ expansion we write

$$h(j, c) = \sum_{n=0}^{\infty} \frac{h_n(j)}{c^n} , \tag{2.4.7}$$

so that

$$G_j(z_1, z_2) = C z_{21}^{-2h_0(j)} \left(1 - \frac{2h_1(j)}{c} \ln z_{21} - \frac{2h_2(j)}{c^2} \ln z_{21} + \frac{2h_1(j)^2}{c^2} (\ln z_{21})^2 + \dots \right) . \tag{2.4.8}$$

The overall constant C will itself have a $1/c$ expansion. Based on our CFT discussion, we expect the results,

$$h_0(j) = -j , \quad h_1(j) = -6j(j+1) , \quad h_2(j) = -78j(j+1). \tag{2.4.9}$$

Our explicit computation of $G_j(z_1, z_2)$ will encounter UV divergences due to the collision of stress tensor insertions on the Wilson line. In the analogous computation of four-point conformal blocks in [42] a normal ordering prescription was adopted such that there were no contractions between any pair of stress tensors on the same Wilson line. That is of course not an option here, since we just have a single Wilson line and the entire result comes from such contractions.

2.5   Computation of the two-point function

2.5.1   Expansion in $T(z)$

We now perform a simple transformation so that we can expand the Wilson line in powers of $T(z)$ rather than $a(z)$. Starting from

$$W[z_1, z_2] = P e^{\int_{z_1}^{z_2} \left(L_1 + \frac{6}{c} T(y) L_{-1} \right) dy} \tag{2.5.1}$$

we define $V[z_1, z_2] = e^{-L_1 z_{21}} W[z_1, z_2]$ which obeys

$$\frac{d}{dz_2} V[z_1, z_2] = e^{-L_1 z_{21}} \frac{6}{c} T(z_2) L_{-1} e^{L_1 z_{21}} V[z_1, z_2] \tag{2.5.2}$$

$$= \frac{6}{c} (L_{-1} - 2 z_{21} L_0 + z_{21}^2 L_1) T(z_2) V[z_1, z_2] . \tag{2.5.3}$$

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Solving this by a path ordered exponential then yields

$$W[z_1, z_2] = e^{L_1 z_{21}} P e^{\frac{6}{c} \int_{z_1}^{z_2} (L_{-1} - 2(y-z_1)L_0 + (y-z_1)^2 L_1) T(y) dy} . \tag{2.5.4}$$

To implement the $1/c$ expansion we now just need to expand the second exponential factor. To streamline our expressions we now set

$$z_2 = z , \quad z_1 = 0 \tag{2.5.5}$$

so that $z_{21} = z$.

2.5.2   Order c^0

At leading order we have simply

$$G^{(0)}(z) = \langle j, -j | e^{L_1 z} | jj \rangle \sim z^{2j} , \tag{2.5.6}$$

so that $h_0(j) = -j$ as expected.

2.5.3   Order $1/c$

Since $\langle 0 | T(z) | 0 \rangle = 0$ the first nontrivial correction comes from expanding the second exponential factor in (2.5.4) to second order, yielding

$$G^{(1)}(z) = \frac{6^2}{c^2} \int_0^z dy_1 \int_0^{y_1} dy_2 \langle j, -j | e^{L_1 z} (L_{-1} - 2y_1 L_0 + y_1^2 L_1)(L_{-1} - 2y_2 L_0 + y_2^2 L_1) | jj \rangle \langle T(y_1) T(y_2) \rangle \tag{2.5.7}$$

The SL(2,R) matrix element is easily computed by the following strategy, which extends to more complicated higher order cases. Use the commutation relations to put the generators in the normal order $(L_1)^{n_1} (L_0)^{n_0} (L_{-1})^{n_{-1}}$. Using $L_{-1} | jj \rangle = 0$ and $L_0 | jj \rangle = j | jj \rangle$ we are left with only L_1 insertions, and only the power $(L_1)^{2j}$ has a nonzero matrix element. This gives

$$\langle j, -j | e^{L_1 z} (L_{-1} - 2y_1 L_0 + y_1^2 L_1)(L_{-1} - 2y_2 L_0 + y_2^2 L_1) | jj \rangle \tag{2.5.8}$$

$$= \langle j, -j | e^{L_1 z} | jj \rangle \frac{2jy_2(z - y_1)(2jy_1(z - y_2) - y_2(z - y_1))}{z^2} \tag{2.5.9}$$

As for the stress tensor correlator, we have the usual expression

$$\langle T(y_1) T(y_2) \rangle = \frac{c/2}{(y_1 - y_2)^4} . \tag{2.5.10}$$

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Note that c is the full central charge; this is why the self-energy diagrams of figure 2.2) is implicitly included. The integral in (2.5.7) diverges when $y_2 \to y_1$ and needs to be regulated. Our strategy will be as follows. In general, stress tensor correlators will be built out of products of factors of the form $1/(y_i - y_j)^2$, and we regulate these by making the replacement

$$\frac{1}{(y_i - y_j)^2} \to \frac{1}{(y_i - y_j)^2 + \epsilon^2} \ , \tag{2.5.11}$$

so in particular we now take

$$\langle T(y_1)T(y_2) \rangle = \frac{c/2}{\left((y_1 - y_2)^2 + \epsilon^2\right)^2} \ . \tag{2.5.12}$$

One way to motivate this is to express the stress tensor in terms of c free bosons, $T(z) = \sum_i \partial\phi_i(z)\partial\phi_i(z)$. Stress tensor correlators are then obtained by Wick's theorem. If we regulate the basic two-point function as $\langle \partial\phi(z)\partial\phi(0) \rangle = 1/(z^2 + \epsilon^2)$ then we recover the above procedure. The advantage of this regulator is that it is computationally tractable. On the other hand, introducing a nonzero ϵ of course breaks conformal invariance, and it is not immediately obvious how to subtract divergences such that conformal invariance is recovered as $\epsilon \to 0$.

We now compute

$$\begin{aligned}
G^{(1)}(z) &= \langle j, -j|e^{L_1 z}|jj\rangle \frac{36j}{c} \int_0^z dy_1 \int_0^{y_1} dy_2 \frac{y_2(z - y_1)\left(2jy_1(z - y_2) - y_2(z - y_1)\right)}{z^2\left((y_1 - y_2)^2 + \epsilon^2\right)^2} \\
&= \langle j, -j|e^{L_1 z}|jj\rangle \frac{36j}{c} \left[\frac{(2j - 1)\pi z^3}{120\epsilon^3} + \frac{z^2}{12\epsilon^2} - \frac{(j + 1)\pi z}{12\epsilon} + \frac{j + 1}{3}\ln\frac{z}{\epsilon} + \frac{2j - 1}{18} + O(\epsilon)\right]
\end{aligned} \tag{2.5.13}$$

We now perform a "minimal subtraction" and simply remove the divergent terms and then set $\epsilon = 0$, even though there is no clear relation at this stage to adding local counterterms to an underlying action. This gives

$$G^{(1)}(z) = \langle j, -j|e^{L_1 z}|jj\rangle \left[\frac{2j(2j - 1)}{c} + \frac{12j(j + 1)}{c}\ln z\right] \ . \tag{2.5.14}$$

Combining this with the order c^0 contribution, we have

$$G^{(0)}(z) + G^{(1)} = Cz^{2j}\left[1 + \frac{12j(j + 1)}{c}\ln z + O(\frac{1}{c^2})\right] \ , \tag{2.5.15}$$

from which we read off $h_1(j) = -6j(j + 1)$ in perfect agreement with (2.4.9).

2.5.4 Order $1/c^2$

At this order there are four contributing diagrams. One diagram comes from expanding the exponential in (2.5.4) to third order and using $\langle T(y_1)T(y_2)T(y_3)\rangle \sim c$. However we can also expand (2.5.4) to fourth order and use the fact that $\langle T(y_1)T(y_2)T(y_3)T(y_4)\rangle$ has order c^2 contributions, which can be thought of as the disconnected diagrams. There are three such disconnected diagrams. The four contributing diagrams are shown in figure 2.3.
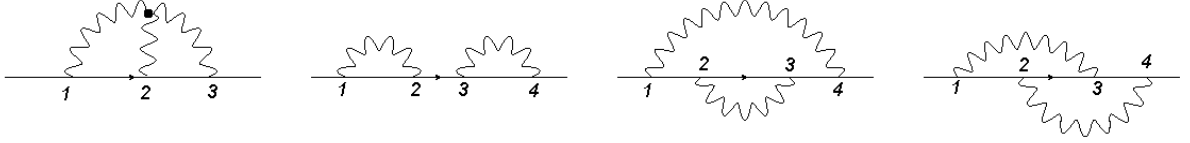
Figure 2.3: Diagrams contributing at order $1/c^2$, with stress tensor insertions on the Wilson line as indicated.

We evaluated these four diagrams using the approach described in the appendix. As in the above, we renormalize by dropping divergent terms.

2.5.4.1 $G_{123}^{(2)}(z)$

Here we use the regulated three-point function

$$\langle T(y_1)T(y_2)T(y_3)\rangle = \frac{c}{[(y_1 - y_2)^2 + \epsilon^2][(y_2 - y_3)^2 + \epsilon^2][(y_3 - y_1)^2 + \epsilon^2]} \tag{2.5.16}$$

The result is

$$\frac{G_{123}^{(2)}(z)}{\langle j, -j|e^{L_1 z}|jj\rangle} = \left[-\frac{168j(j+1) - 144j^3}{c^2}\ln z - \frac{144j(j+1)}{c^2}(\ln z)^2 \right] . \tag{2.5.17}$$

2.5.4.2 $G_{12;34}^{(2)}(z)$

We use

$$\langle T(y_1)T(y_2)T(y_3)T(y_4)\rangle\big|_{12;34} = \frac{c^2/4}{[(y_1 - y_2)^2 + \epsilon^2]^2[(y_3 - y_4)^2 + \epsilon^2]^2} , \tag{2.5.18}$$

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which yields

$$\frac{G^{(2)}_{12;34}(z)}{\langle j, -j | e^{L_1 z} | jj \rangle} = \left[\frac{1}{c^2} \left(\frac{72}{5} j - 264 j^2 + \frac{384}{5} j^3 + \frac{1776}{5} j^4 \right) \ln z + \frac{144 j^2 (j+1)^2}{c^2} (\ln z)^2 \right] .$$

(2.5.19)

2.5.4.3 $G^{(2)}_{14;23}(z)$

We use

$$\langle T(y_1) T(y_2) T(y_3) T(y_4) \rangle \big|_{14;23} = \frac{c^2/4}{[(y_1 - y_4)^2 + \epsilon^2]^2 [(y_2 - y_3)^2 + \epsilon^2]^2} ,$$

(2.5.20)

which yields

$$\frac{G^{(2)}_{14;23}(z)}{\langle j, -j | e^{L_1 z} | jj \rangle} = \left[\frac{1}{c^2} \left(-\frac{324}{5} j - \frac{492}{5} j^2 + \frac{1824}{5} j^3 + \frac{1992}{5} j^4 \right) \ln z + \frac{72 j^2 (j+1)^2}{c^2} (\ln z)^2 \right] .$$

(2.5.21)

2.5.4.4 $G^{(2)}_{13;24}(z)$

We use

$$\langle T(y_1) T(y_2) T(y_3) T(y_4) \rangle \big|_{13;24} = \frac{c^2/4}{[(y_1 - y_3)^2 + \epsilon^2]^2 [(y_2 - y_4)^2 + \epsilon^2]^2} ,$$

(2.5.22)

which yields

$$\frac{G^{(2)}_{13;24}(z)}{\langle j, -j | e^{L_1 z} | jj \rangle} = \left[\frac{1}{c^2} \left(\frac{396}{5} j + \frac{1908}{5} j^2 - \frac{2736}{5} j^3 - \frac{3528}{5} j^4 \right) \ln z + \frac{1}{c^2} \left(144 j - 288 j^3 - 144 j^4 \right) (\ln z)^2 \right] .$$

(2.5.23)

2.5.4.5 Complete result at order $1/c^2$

We now combine all of our results for the complete correlator up to this order. The result is

$$
\begin{aligned}
G(z) &= G^{(0)}(z) + G^{(1)}(z) + G^{(2)}(z) + \dots \\
&= C z^{2j} \left[1 + \frac{12 j (j+1)}{c} \ln z + \frac{24(3j - 29) j (j+1)}{5 c^2} \ln z + \frac{72 j^2 (j+1)^2}{c^2} (\ln z)^2 + O\left(\frac{1}{c^3} \right) \right]
\end{aligned}
$$

(2.5.24)

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Note that the $2j(2j-1)/c$ term in (2.5.14) contributed to this, since we have set the leading term in $[\ldots]$ to be 1 by absorbing the overall constant factor into C.

Comparing with expectations, we see that the $(\ln z)^2$ term is in agreement with (2.4.8), so that the result to this order takes the form of a single power of z. This is quite nontrivial from the diagrammatic point of view, as there are $(\ln z)^2$ contributions from all four of the $1/c^2$ diagrams which must all combine together to give the correct coefficient. On the other hand, the $\frac{1}{c^2}\ln z$ term does not have the expected coefficient $-2h_2(j)/c^2 = 156j(j+1)/c^2$.

We now make a few comments about this result. A feature that emerges at order $1/c^2$ but which is absent at order $1/c$ is the appearance of divergent terms of the form $\frac{1}{c^2\epsilon^n}\ln z$. If we take the general point of view that when removing a divergence we can also subtract a finite term with the same z dependence, then this renders the coefficient of the $\frac{1}{c^2}\ln z$ term ambiguous. By contrast, the absence of divergences of the form $\frac{1}{c\epsilon^n}\ln z$ and $\frac{1}{c^2\epsilon^n}(\ln z)^2$ suggests that the coefficients of the terms $\frac{1}{c}\ln z$ and $\frac{1}{c^2}(\ln z)^2$ are unambiguous, and indeed these coefficients precisely match expectations. Of course, what this emphasizes is the need for a more systematic renormalization approach. On the other hand, we again note the fact that our result to this order takes the form of a single power law in z, suggesting that conformal invariance is being respected by our procedure.

2.6 Discussion

We have computed the expectation value of an open Wilson line to order $1/c^2$. From this result we read off the scaling dimension of the corresponding primary operator and compared it to expectations from CFT considerations. This revealed partial agreement with CFT predictions as well as some unresolved issues. The order $1/c$ result was as expected, and furthermore we found that at order $1/c^2$ the result takes the form of a single power law, as dictated by conformal invariance. On the other hand, the order $1/c^2$ contribution to the scaling dimension is at odds with our expectations. More accurately, the result is ambiguous within the framework of our computation, as the desired coefficient of a $\frac{1}{c^2}\ln z$ term is "corrupted" by the presence of $\frac{1}{\epsilon^n c^2}\ln z$ UV divergences requiring renormalization. This

clearly points to the need for a more principled renormalization scheme.

There are of course other ways to regulate the stress tensor correlators. For example, instead of making the replacement in (2.5.11) we can implement a simple version of dimensional regularization. In particular, we can replace the exponent 2 in the denominator with $(2 - \epsilon)$, taking ϵ to be sufficiently positive so that the integrals converge, and then analytically continue the result to ϵ near 0. After a minimal subtraction of pole terms, the $1/c$ contribution we find is still in agreement with (2.4.9) but the $1/c^2$ contribution is not.

It is worth contrasting what we have found here with what one encounters in the computation of closed Wilson loops in ordinary Chern-Simons theory, which yield topological invariants [79]. The leading order contribution comes from a gluon exchanged between two points on the Wilson loop. This leads to an integral which is UV finite, but the result is not a topological invariant. To rectify this one needs to introduce a "framing", corresponding to displacing the worldines on which the two gluons are inserted. The result is a topological invariant that depends on the choice of framing [50, 79].

Our primary operators are labelled by an SL(2,R) spin j, which from the CFT side comes from constructing Virasoro representations by applying constraints to SL(2,R) current algebra representations. An SL(2,R) spin-j also naturally appears in the bulk, via the formulation of gravity in terms of SL(2,R) Chern-Simons theory, and it therefore seems meaningful to compare scaling dimensions in the two descriptions as a function of j and the central charge c. On the other hand, strictly from the Virasoro point of view, j is simply a label, so one might wonder if there is perhaps some c dependent relation between the j labels in the two descriptions. To address this we note that degenerate representations correspond to $2j$ being a positive integer, which precludes such a c dependent relation for such representations. This is to say that we certainly expect to be able to meaningfully compare the scaling dimensions of degenerate representations on the two sides as a function of c. Of course, these scaling dimensions are entirely fixed by Virasoro representation theory, but we do not want to use this, as the entire point here is to develop computational rules in the bulk that will apply more generally.

We have tried to extract scaling dimensions from two-point functions, but another approach is to adopt canonical quantization [37]. In particular, we can consider a single particle, associated to a spin-j representation of SL(2,R), coupled in a gauge invariant fashion to SL(2,R) Chern-Simons gauge fields. One should be able to realize the Virasoro generators on this Hilbert space, and demanding that the algebra is realized consistently may resolve the ambiguities associated with renormalizing UV divergences. We hope to report on this in the near future.

Appendices

2.7   Evaluation of integrals

We encounter nested integrals of the form

$$I(z) = \int_0^z dy_1 \int_0^{y_1} dy_2 \ldots \int_0^{y_{n-1}} dy_n \frac{P(z, y_i)}{\prod_{i<j}[(y_i - y_j)^2 + \epsilon^2]^{n_{ij}}} \ , \qquad (2.7.1)$$

where $P(z, y_i)$ is a polynomial and n_{ij} are non-negative integers.

We first rewrite this in terms of unconstrained integrals by introducing step functions,

$$\int_0^z dy_1 \int_0^{y_1} dy_2 \ldots \int_0^{y_{n-1}} dy_n \rightarrow \int d^n y \ \theta(z - y_1)\theta(y_1 - y_2) \ldots \theta(y_{n-1} - y_n)\theta(y_n) \ , \quad (2.7.2)$$

and use the Fourier representation

$$\theta(y) = \int \frac{dp}{2\pi i} \frac{e^{ipy}}{p - i\delta} \ , \quad \delta > 0 \ . \qquad (2.7.3)$$

We also write the denominator factors in momentum space using

$$\frac{1}{y^2 + \epsilon^2} = \frac{1}{2\epsilon} \int_{-\infty}^{\infty} dk e^{iky - |k|\epsilon} \ . \qquad (2.7.4)$$

The y-integrals can then be carried out, yielding n delta functions involving p and k. These delta functions soak up all but one of the p integrals, and the remaining p integral can be done by computing residues. This leaves some k-integrals, where the integrand is a sum of terms taking the form of exponentials time rational functions. Some of the denominator

factors can be removed by differentiating with respect to z, and the other by using relations like

$$\frac{1}{k_1 - k_2} = -\frac{i}{2} \int_{-\infty}^{\infty} du \ sgn(u) e^{i(k_1 - k_2)u} \ . \tag{2.7.5}$$

The k integrals are then carried out, followed by the u integrals. The result is then expanded for small ϵ, and we finally integrate to undo the earlier z differentiation. Due to the last step, this procedure will only determine the result up to a polynomial in z. However, if desired, this polynomial can easily be determined by directly studing the small z expansion of the original integral.

We present a representative example to make the procedure concrete,

$$I_3(z) = \int_0^z dy_1 \int_0^{y_1} dy_2 \int_0^{y_2} dy_3 \frac{1}{(y_1 - y_2)^2 + \epsilon^2} \frac{1}{(y_1 - y_3)^2 + \epsilon^2} \frac{1}{(y_2 - y_3)^2 + \epsilon^2} \ . \tag{2.7.6}$$

Proceeding as above, we have

$$
\begin{aligned}
I_3(z) \ &= \ \frac{1}{8\epsilon^3} \int \frac{d^4p\, d^3k}{(2\pi i)^4} \frac{e^{-|k_1|\epsilon - |k_2|\epsilon - |k_3|\epsilon}}{(p_1 - i\delta)(p_2 - i\delta)(p_3 - i\delta)(p_4 - i\delta)} \\
&\quad \times \int d^3y\, e^{ip_1(z - y_1) + ip_2 y_{12} + ip_3 y_{23} + ip_4 y_3 + ik_1 y_{12} + ik_2 y_{23} + ik_3 y_{31}} \\
&= \ \frac{i}{8\epsilon^3} \int \frac{dp_1\, d^3k}{2\pi i} \frac{e^{-|k_1|\epsilon - |k_2|\epsilon - |k_3|\epsilon}}{(p_1 - i\delta)^2(p_1 - k_1 + k_3 - i\delta)(p_1 - k_2 + k_3 - i\delta)} e^{ip_1 z} \\
&= \ \frac{i}{8\epsilon^3} \int d^3k\, e^{-|k_1|\epsilon - |k_2|\epsilon - |k_3|\epsilon} \left[\frac{e^{i(k_1 - k_3)z}}{(k_1 - k_3)^2(k_1 - k_2)} - \frac{e^{i(k_2 - k_3)z}}{(k_2 - k_3)^2(k_1 - k_2)} \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. + \frac{(k_1 k_2 + k_3^2 - k_1 k_3 - k_2 k_3)z + k_1 + k_2 - 2k_3}{(k_1 - k_3)^2(k_2 - k_3)^2} \right]
\end{aligned}
$$

$$\tag{2.7.7}$$

In getting to the final expression we performed the p_1 integral by residues, but discarded the contribution from the pole at $p_1 = i\delta$, since this will only contribute a degree 1 polynomial in z that will anyway be killed by the derivatives that we will apply in the next step. On the other hand, convergence of the k integrals in the above undifferentiated expression does require the presence of this polynomial part, as it is needed to render the integrand finite at the locations where the denominator factors vanish.

We now differentiate twice to get

$$
\begin{aligned}
\frac{\partial^2 I_3}{\partial z^2} &= -\frac{i}{8\epsilon^3} \int d^3k\, e^{-|k_1|\epsilon - |k_2|\epsilon - |k_3|\epsilon} e^{-ik_3 z} \frac{e^{ik_1 z} - e^{ik_2 z}}{k_1 - k_2} \\
&= -\frac{i}{4\epsilon^2} \frac{1}{z^2 + \epsilon^2} \int d^2k\, e^{-|k_1|\epsilon - |k_2|\epsilon} \frac{e^{ik_1 z} - e^{ik_2 z}}{k_1 - k_2}
\end{aligned}
\tag{2.7.8}
$$

Using (2.7.5) gives

$$
\begin{aligned}
\frac{\partial^2 I_3}{\partial z^2} &= -\frac{1}{8\epsilon^2} \frac{1}{z^2 + \epsilon^2} \int_{-\infty}^{\infty} du\ sgn(u) \int d^2k\, e^{i(k_1 - k_2)u} e^{-|k_1|\epsilon - |k_2|\epsilon} \left(e^{ik_1 z} - e^{ik_2 z} \right) \\
&= \frac{2z}{z^2 + \epsilon^2} \int_{-\infty}^{\infty} du\ sgn(u) \frac{u}{(u^2 + \epsilon^2)[(u+z)^2 + \epsilon^2][(u-z)^2 + \epsilon^2]} \\
&= 2 \frac{\tan^{-1}\left(\frac{z}{\epsilon}\right) + \frac{\epsilon}{z} \ln\left(1 + \frac{z^2}{\epsilon^2}\right)}{\epsilon(z^2 + \epsilon^2)(z^2 + 4\epsilon^2)} \\
&= \frac{\pi}{\epsilon z^4} + \frac{4}{z^5} \ln \frac{z}{\epsilon} - \frac{2}{z^5} + O(\epsilon)
\end{aligned}
\tag{2.7.9}
$$

and so we arrive at

$$
I_3(z) = \frac{\pi}{6\epsilon z^2} + \frac{1}{3z^3} \ln \frac{z}{\epsilon} + \frac{1}{36 z^3} + O(\epsilon) \ ,
\tag{2.7.10}
$$

where we fixed the integration constants by examining the original integral.

All of our integrals can be worked out this way. This somewhat circuitous procedure has the advantage that it can easily be automated.

CHAPTER 3

Renormalization of Gravitational Wilson Lines

We continue the study of the Wilson line representation of conformal blocks in two-dimensional conformal field theory; these have an alternative interpretation as gravitational Wilson lines in the context of the $\text{AdS}_3/\text{CFT}_2$ correspondence. The gravitational Wilson line involves a path-ordered exponential of the stress tensor, and its expectation value can be computed perturbatively in an expansion in inverse powers of the central charge c. The short-distance singularities which occur in the associated stress tensor correlators require systematic regularization and renormalization prescriptions, whose consistency with conformal Ward identities presents a subtle problem. The regularization used here combines dimensional regularization and analytic continuation. Representation theoretic arguments, based on $SL(2,\mathbb{R})$ current algebra, predict an exact result for the Wilson line anomalous dimension and, by building on previous work, we verify that the perturbative calculations using our regularization and renormalization prescriptions reproduce the exact result to order $1/c^3$ included. We also discuss a related, but somewhat simpler, Wilson line in Wess-Zumino-Witten models that yields current algebra conformal blocks, and we emphasize the distinction between Wilson lines constructed out of non-holomorphic and purely holomorphic currents.

3.1 Introduction

Wilson lines and Wilson loops are obtained by the path-ordered exponential integral of a connection respectively along an open interval and a closed contour. In gauge theory, the connection is the canonical gauge field and the resulting Wilson loop operator is a gauge-invariant observable with applications to elucidating the phases of gauge theory and beyond.

A different type of Wilson line operator has recently found use in two-dimensional conformal field theory; in this case the connection is a composite field involving the stress tensor of the CFT. What this object yields is a conformal block associated with a pair of primary operators, one at each endpoint of the Wilson line. Actually, the two types of Wilson lines just mentioned are closely related objects if viewed in the context of the AdS/CFT correspondence: the CFT Wilson line is the boundary image of a bulk Wilson line, and for this reason we often refer to it as a gravitational Wilson line, although it exists as an object in CFT independent of the AdS/CFT correspondence. In this paper we continue the study of these Wilson lines, focussing in particular on their status as well defined quantum mechanical operators. Their renormalization poses a rather subtle and nonstandard problem which we aim to understand better.

The general connection between Wilson lines in three dimensions and conformal field theory in two dimensions arose in [79], and the relation to the Virasoro algebra appeared in [77]. More recently, Wilson lines arose in the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, first as a tool for computing entanglement entropy in higher spin theories [8, 30], and then in the more general context of computing conformal blocks [15, 17]. Quantum aspects of these Wilson lines have been studied in [11, 16, 42, 61–63]. The related representation of CFT conformal blocks and OPE structures in terms of AdS appeared in [59] and in [28]. We also note that the notion of integrating the stress tensor over a contour arises in the context of the averaged null energy condition (proven in flat space in [39, 53]), and the related notion of a "length operator" discussed in [2] has connections to the Wilson line discussed here.

More motivation and details on the form of the Wilson line will be given in the next section, but for now it suffices to write,

$$W[z_2, z_1] = \langle j, -j| P \exp\left\{ \int_{z_1}^{z_2} dz \left(L_1 + \frac{6}{c} T(z) L_{-1} \right) \right\} |j, j\rangle. \qquad (3.1.1)$$

Except for the non-holomorphic Wilson line discussed in section 3.4.2 our formulas refer to a chiral half of a CFT and z denotes the corresponding holomorphic coordinate on the plane. The shape of the integration contour from z_1 to z_2 used to define the Wilson line is inconsequential, except when we introduce a regulator and break conformal invariance, and

then it is taken to be along the real line. L_0 and $L_{\pm 1}$ are generators of the Lie algebra of $SL(2, \mathbb{R})$, with the states $|j, \pm j\rangle$ being highest/lowest weight states of a spin j representation, corresponding to a primary of dimension $h = h(j, c)$, as will be discussed in more detail below. $T(z)$ is the stress tensor operator, such that $W[z_2, z_1]$ is supposed to represent the Virasoro vacuum OPE block corresponding to the bi-local $O(z_2)O(z_1)$, where $O(z)$ is a primary operator of dimension $h(j, c)$. That is, $W[z_2, z_1]$ captures all terms in the $O(z_2)O(z_1)$ OPE involving only stress tensors.

The Virasoro vacuum block is a rich object, capturing as it does the effect of an arbitrary number of stress tensors. Phrased in terms of AdS, it encodes the gravitational interaction [52]. The Wilson line provides an expression for the Virasoro vacuum block in a form admitting a convenient $1/c$ expansion, which in the bulk corresponds to an expansion in Newton's constant. Our goal here is to understand this perturbative expansion; once that is under control one can contemplate using the Wilson line to study non-perturbative effects as well.

The Wilson line as defined in (3.1.1) is a singular object due to the appearance of stress tensors at coincident points, and thus requires regularization and renormalization [16, 42, 61]. Here we adopt a type of dimensional regularization [61], in which the stress tensor is taken to have dimension $2 - \varepsilon$. Renormalization of the Wilson line then requires an overall multiplicative renormalization by a factor $N(\varepsilon)$, as well as a vertex renormalization factor $\alpha(\varepsilon)$ multiplying $T(z)$, where both $N(\varepsilon)$ and $\alpha(\varepsilon)$ depend on the regulator ε as well as on c and j. This regularization scheme breaks conformal invariance at intermediate stages, and from the point of view of diagrammatics it is highly nontrivial that conformal invariance is restored upon renormalization.

The most basic quantity to consider is the Wilson line expectation value itself; given what we have said, this should take the form of a conformal two-point function,

$$\langle W[z_2, z_1] \rangle \sim |z_2 - z_1|^{-2h(j,c)}. \tag{3.1.2}$$

At lowest order in the $1/c$ expansion one finds $h(j, c) = -j$, but this value receives corrections order by order in an expansion in powers of $1/c$. There is in fact an expectation for the

exact answer based on general conformal field theory considerations. The Wilson line, as we have defined it, is based on a representation of $SL(2, \mathbb{R})$ but once the stress tensors are included it describes an object in Virasoro representation theory. Hamiltonian reduction supplies a procedure for constructing a representation of the Virasoro algebra by imposing a constraint on a corresponding representation of $SL(2, \mathbb{R})$ current algebra. This procedure has an analog in bulk gravity, where the constraints are precisely those that correspond to imposing asymptotically AdS boundary conditions. The resulting relation between the $SL(2, \mathbb{R})$ spin j and the Virasoro dimension $h(j, c)$ is given by, (see e.g. [14]),

$$h(j, c) = -j + \frac{m+1}{m} j(j+1) \ , \qquad\qquad c = 1 - \frac{6}{m(m+1)} \ . \qquad (3.1.3)$$

Expanding $h(j, c)$ in powers of $1/c$ the first few contributions are given by

$$h(j, c) = -j - \frac{6}{c} j(j+1) - \frac{78}{c^2} j(j+1) - \frac{1230}{c^3} j(j+1) + \mathcal{O}(c^{-4}) \ . \qquad (3.1.4)$$

and provide a prediction for the perturbative expansion of the Wilson line expectation value. One of the main results of this paper is to verify, by explicit calculation, that the procedure of dimensional regularization and renormalization via the inclusion of the factors $N(\varepsilon)$ and $\alpha(\varepsilon)$, does indeed reproduce the dimension formula (3.1.4) to the order indicated, thereby extending previous results [16, 61].

It is also useful to give a bulk gravity perspective on the result (3.1.4) in terms of gravitational self-energy. If we take the classical point particle limit, $c, j \to \infty$, with j/c fixed we can write the result as $m = m_0 - \frac{2G}{\ell} m_0^2$. To obtain this we used the Brown-Henneaux formula $c = \frac{3\ell}{2G}$, the relation between the mass of a particle in AdS and the corresponding conformal dimension $m\ell = 2h$, and similarly $m_0 \ell = 2h_0 = -2j$. The relation between m and m_0 is the same as that obtained from considering the classical gravitational self-energy of a point particle in AdS [16]. The general formula (3.1.4) can thus be thought of as supplying quantum corrections to this result. This is interesting, because the gravitational self-energy is typically ill-defined, or rather sensitive to unknown UV physics, but the situation in three dimensions appears to be under better control.

3.1.1 Organization

We now summarize the remainder of this paper. In section 3.2 we review the logic behind the construction of the gravitational Wilson line. In section 3.3 we discuss the analog of the gravitational Wilson line for a level k current algebra with conserved current $J^a(z)$ given by the Wilson line $P \exp \frac{1}{k} \int J^a T^a$. Here T^a denote the generators of the relevant Lie algebra, and we denote by G the corresponding Lie group. This object yields the current algebra vacuum OPE block for a bi-local primary operator. Its evaluation poses a similar, but somewhat simpler, renormalization problem as compared to the stress tensor case. In this case the $1/k$ expansion should yield the standard expression for the scaling dimension h of a current algebra primary in terms of quadratic Casimirs, $h = C_2(r)/(2k + C_2(G))$. In order to better understand the origin of the Wilson line, we study an alternative construction starting from the WZW model. We consider the bi-local operator $g^{-1}(x_2)g(x_1)$ constructed from the basic WZW primary $g(x)$ which lives on the group manifold G. This can be written identically in terms of a Wilson line for a non-conserved current, $\mathcal{J}_\mu = -kg^{-1}\partial_\mu g$, and admits a relatively straightforward perturbative expansion using standard dimensional regularization (modulo subtleties associated with the appearance of epsilon tensors). What is not manifest in this approach is why this operator holomorphically factorizes.

In section 3.5 we turn to the gravitational Wilson line. We describe the systematics of the renormalization procedure and compute the expectation value of the Wilson line with zero and one additional stress tensor insertions through order $1/c^3$. Consistency of these two computations uniquely fixes all renormalization constants and yields an unambiguous answer for the anomalous dimension, which indeed reproduces the expansion (3.1.4). In section 3.6 we discuss an alternative regularization procedure. Rather than modifying the dimension of the stress tensor we adopt another method for softening the short distance singularities arising from collisions of stress tensors. This approach also involves a dimensionless regulator ε and *a priori* seems just as sensible as the prior scheme. However, our explicit computations reveal that conformal invariance is not recovered in this scheme. This serves to highlight the subtleties involved in renormalizing the Wilson line. We close the paper with some comments

in section 3.7. Various technical results appear in appendices.

3.2 The gravitational Wilson line operator

Consider a primary operator $O(z, \bar{z})$ in a two-dimensional CFT. As most of our considerations involve one chiral half of the CFT, we henceforth write $O(z)$. Under a conformal transformation, $z' = f(z)$, the bi-local operator $O(z_2)O(z_1)$ transforms as

$$O(z'_2)O(z'_1) = \left(f'(z_2)f'(z_1)\right)^{-h} O(z_2)O(z_1) , \qquad (3.2.1)$$

which identifies the scaling dimension h of O.

3.2.1 Wilson line covariant under global conformal transformations

We first discuss how to write down a Wilson line whose transformation is given by (3.2.1) under global conformal transformations, $f(z) = (az+b)/(cz+d)$, which describe an $SL(2, \mathbb{R})$ subgroup of the full Virasoro symmetry. To this end, let (L_{-1}, L_0, L_1) be $SL(2, \mathbb{R})$ generators obeying $[L_m, L_n] = (m - n)L_{m+n}$. We then consider the matrix element

$$W[z_2, z_1] = \langle h; out | P \exp \left\{ \int_{z_1}^{z_2} dz L_1 \right\} |h; in\rangle , \qquad (3.2.2)$$

for suitable in and out states to be defined momentarily.

To see how to implement the conformal transformation, consider the more general path ordered exponential $P \exp \int_{z_1}^{z_2} a(z)$, where the connection $a(z) = a_z(z)dz$ takes values in the Lie algebra of $SL(2, \mathbb{R})$. Under the action of an arbitrary group element $U(z) \in SL(2, \mathbb{R})$, the connection transforms by $U^{-1}(z)a(z)U(z) - U^{-1}(z)dU(z) = a_U(z)$ while the Wilson line transforms by

$$U^{-1}(z_2) P \exp \left\{ \int_{z_1}^{z_2} a(z) \right\} U(z_1) = P \exp \int_{z_1}^{z_2} a_U(z) . \qquad (3.2.3)$$

In the present case, $a(z) = L_1 dz$. The following transformation leaves $a(z)$ invariant, i.e. $a_U(z) = a(z)$, and hence represents a global conformal transformation

$$U(z) = e^{\lambda_1(z)L_1} e^{\lambda_0(z)L_0} e^{\lambda_{-1}(z)L_{-1}} \qquad (3.2.4)$$

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with

$$\lambda_1 = z - f(z) , \qquad \lambda_0(z) = -\ln(f'(z)) , \qquad \lambda_{-1}(z) = -\frac{f''(z)}{2f'(z)} , \qquad (3.2.5)$$

and $f(z) = (az + b)/(cz + d)$ as above. Together with (3.2.3) we then have

$$W[z_2, z_1] = \langle h; out | e^{-\lambda_{-1}(z_2)L_{-1}} e^{\ln[f'(z_2)]L_0} P \exp \left\{ \int_{z_1'}^{z_2'} dz L_1 \right\} e^{-\ln[f'(z_1)]L_0} e^{\lambda_{-1}(z_1)L_{-1}} | h; in \rangle \qquad (3.2.6)$$

again with $z' = f(z)$. We now observe that if the states are taken to obey

$$L_{-1}|h; in\rangle = 0 , \quad L_0|h; in\rangle = -h|h; in\rangle$$
$$L_1|h; out\rangle = 0 , \quad L_0|h; out\rangle = h|h; out\rangle , \qquad (3.2.7)$$

then we obtain the desired transformation law

$$W[z_2', z_1'] = \big(f'(z_2)f'(z_1)\big)^{-h} W[z_2, z_1] . \qquad (3.2.8)$$

It will be convenient to write $h = -j$, since if $2j$ is a non-negative integer the L_n can be taken to be a finite dimensional matrix representation of $SL(2, \mathbb{R})$. One can then carry out computations for such j and at the end set $j = -h$ for $h \geq 0$. This is just a computational shortcut, and the same results are obtained by working with representations with $h \geq 0$ throughout. A convenient representation for $h \geq 0$ is discussed in appendix 3.8.

With this in mind, our Wilson line is at this stage written as

$$W[z_2, z_1] = \langle j, -j | P \exp \left\{ \int_{z_1}^{z_2} dz L_1 \right\} | j, j \rangle \qquad (3.2.9)$$

with $L_0| \pm j\rangle = \pm| \pm j\rangle$ and $L_{\mp 1}| \pm j\rangle = 0$. The matrix element is readily evaluated using the fact that L_1 lowers the L_0 eigenvalue by one, and we have $W[z_2, z_1] \sim z^{2j} = z^{-2h}$. The Wilson line (3.2.9) thus gives the coefficient of the identity operator in the OPE expansion of the two primaries: $O(z_2)O(z_1) \sim W[z_2, z_1] + (\text{other operators})$.

The Wilson line (3.2.9) emerges naturally in the AdS/CFT correspondence when we describe gravity in the bulk in the Chern-Simons formulation. The AdS metric $ds^2 = d\rho^2 + e^{2\rho}dz d\bar{z}$ is represented by the pair of connections $A = e^\rho L_1 dz + L_0 d\rho$ and $\overline{A} = e^\rho L_{-1} d\bar{z} - L_0 d\rho$. See, e.g. [20]. The Wilson line in the Chern-Simons theory $W[z_2, z_1] =$

$\langle j, -j | P \exp \left\{ \int_{z_1}^{z_2} A \right\} | j, j \rangle$ reduces to the Wilson line (3.2.9) upon substituting for A with the reduced connection $a = L_1 dz$, as the ρ dependence can be gauged away.

This "global Wilson line" of (3.2.9) forms the basis of a convenient description of arbitrary global (i.e $SL(2, \mathbb{R})$) conformal blocks. Rather than a single Wilson line, one considers a network with trivalent vertices. Each vertex is represented by a singlet state in the tensor product of the three representations that enter the vertex. The space of conformal blocks is in one-to-one correspondence with the space of such networks; see [15, 17].

3.2.2 Wilson line covariant under local conformal transformations

The main point of the preceding subsection was to motivate the form of the Wilson line that incorporates the stress tensor. It should yield the Virasoro OPE block, which is to say that it should capture all contributions to the $O(z_2)O(z_1)$ OPE involving only stress tensors. One way to motivate the proposal is to repeat the analysis that led to (3.2.8) but now for an arbitrary local conformal transformation $z' = f(z)$. In this case, $a(z)$ cannot be left invariant, but must transform as follows

$$a_U(z) = \left(L_1 + \frac{6}{c} T(z) L_{-1} \right) dz \tag{3.2.10}$$

with $T(z)$ given in terms of $f(z)$ by

$$T(z) = \frac{c}{12} S_f(z) , \qquad S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 , \tag{3.2.11}$$

where $S_f(z)$ is the Schwarzian derivative. We then obtain

$$\langle j, -j | P \exp \left\{ \int_{z_1}^{z_2} dz \left(L_1 + \frac{6}{c} T(z) L_{-1} \right) \right\} | j, j \rangle = \frac{[f'(z_2) f'(z_1)]^h}{[f(z_2) - f(z_1)]^{2h}} , \tag{3.2.12}$$

where we again have $h = -j$. In this expression $T(z)$ is the classical function given in (3.2.11), not the stress tensor operator. However, this result naturally suggests an expression for the Virasoro vacuum OPE block as the gravitational Wilson line $W[z_2, z_1]$ given by,

$$W[z_2, z_1] \equiv \langle j, -j | P \exp \left\{ \int_{z_1}^{z_2} dz \left(L_1 + \frac{6}{c} T(z) L_{-1} \right) \right\} | j, j \rangle , \tag{3.2.13}$$

where now $T(z)$ is the stress tensor operator. In particular, suppose we take the expectation value of W in a CFT state with a classical stress tensor expectation value in the large c limit.

Such a stress tensor can be generated from $T(z) = 0$ by some conformal transformation $z' = f(z)$. The Wilson line expectation value should then be equal to the primary two-point function transformed by $f(z)$, and this is precisely what (3.2.12) says. At the level of correlation functions, the statement that $W[z_2, z_1]$ is the Virasoro vacuum block is the statement that it equals $O(z_2)O(z_1)$ inside any correlation function involving just stress tensors,

$$\langle O(z_2)O(z_1)T(z_3)\ldots T(z_n)\rangle = \langle W[z_2, z_1]T(z_3)\ldots T(z_n)\rangle . \tag{3.2.14}$$

See [42] for more discussion and tests of this proposal.

This Wilson line also arises naturally from the bulk Chern-Simons description. The most general asymptotically AdS$_3$ solution of Einstein's equations corresponds to the connections

$$
\begin{aligned}
A &= (e^\rho L_1 + \frac{6}{c}e^{-\rho}T(z)L_{-1})dz + L_0 d\rho \\
\overline{A} &= (e^\rho L_{-1} + \frac{6}{c}e^{-\rho}\overline{T}(\bar{z})L_1)dz - L_0 d\rho
\end{aligned}
\tag{3.2.15}
$$

where the holographic dictionary identifies $T(z)$ and $\overline{T}(\bar{z})$ as the components of the dual CFT stress tensor (e.g. [20]). The Wilson line therefore corresponds to $P\exp\int a$ where $a = (L_1 + \frac{6}{c}T(z)L_{-1})dz$ is the reduced connection. In the quantum theory we should integrate over all asymptotically AdS connections weighted by the Chern-Simons action. On general grounds, this should have the effect of replacing any string of stress tensors by their vacuum expectation value, and this is precisely what was meant above in saying that $T(z)$ appears in the Wilson line as an operator.

3.3   Current algebra Wilson lines in the WZW model

Just as the gravitational Wilson line defined in terms of the stress tensor encodes conformal blocks of the Virasoro algebra, we can define a Wilson line built out of a spin-1 current that encodes current algebra conformal blocks. The current algebra Wilson line is a somewhat simpler object, and we also have the useful Lagrangian realization of current algebra in terms of the WZW model. In this section we explore this current algebra Wilson line from

several complementary points of view. We first define a holomorphic Wilson line that is the direct spin-1 analog of our gravitational Wilson line and discuss its renormalization. We then turn to a non-holomorphic Wilson line, defined by a simple rewriting of a bi-local primary operator. Its renormalizaton proceeds somewhat differently, but we show that the anomalous dimensions of the two Wilson lines agree. We finally make some comments about the connection between these two constructions.

3.3.1 WZW model and current algebra

We first review some background material; see, e.g. [33]. The action of the WZW model is

$$S[g] = \frac{k}{4\pi} \int_{\Sigma} d^2x \sqrt{\gamma}\gamma^{\mu\nu} Tr'(\partial_\mu g^{-1}\partial_\nu g) + \frac{ik}{6\pi} \int_{\Gamma} Tr'(\omega)^3 \qquad (3.3.1)$$

where the theory lives on the Riemann surface Σ with local coordinates x^μ, metric $\gamma_{\mu\nu}$, and where $\gamma = det\,(\gamma_{\mu\nu})$. The surface Σ is the boundary of a three-manifold Γ. The field $g(x)$ takes values in a compact Lie group G, and the one-form $\omega = g^{-1}dg$ takes values in the Lie algebra $\mathcal{G}$ of G, in an arbitrary finite-dimensional irreducible representation r. Denoting the structure constants of $\mathcal{G}$ by f^{abc} and a basis of Hermitian generators of $\mathcal{G}$ in the representation r of $\mathcal{G}$ by T^a with $a,b,c = 1, \cdots, dim\,\mathcal{G}$, the structure relations are given by $[T^a, T^b] = \sum_c i f_{abc}T^c$, and we use the normalization for the trace in the representation r by $Tr'(T^aT^b) = \frac{1}{2}\delta^{ab}$. With these normalization conventions, the level k is quantized such that $2k$ is an integer. We denote by $C_2(r)$ the value of the quadratic Casimir operator $C_2 = \sum_a T^aT^a$ in representation r. For example, for $G = SU(N)$ and r the defining representation we have $C_2(r) = (N^2 - 1)/(2N)$, while in the adjoint representation $r = G$ we have $C_2(G) = N$.

Invariance of $S[g]$ under global transformations $g(x) \to g_L\,g(x)\,g_R^{-1}$ with $(g_L, g_R) \in G \times G$ implies the existence of two independent conserved currents, which take the form

$$J^\mu = -\frac{k}{2}(\gamma^{\mu\nu} - i\epsilon^{\mu\nu})g^{-1}\partial_\nu g \,, \qquad\qquad \bar{J}^\mu = -\frac{k}{2}(\gamma^{\mu\nu} + i\epsilon^{\mu\nu})\partial_\nu g\,g^{-1} \,. \qquad (3.3.2)$$

In a system of local complex coordinates $x^\mu = (z, \bar{z})$, the metric on Σ takes the form $\gamma_{\mu\nu}dx^\mu dx^\nu = dz d\bar{z}$ and the non-vanishing components of $\gamma_{\mu\nu}$ and of the anti-symmetric

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tensor $\epsilon_{\mu\nu}$ are given by $\gamma_{z\bar{z}} = \gamma_{\bar{z}z} = \frac{1}{2}$ and $\epsilon_{z\bar{z}} = -\epsilon_{\bar{z}z} = \frac{i}{2}$. In terms of these complex coordinates, the expressions for the currents simplify as follows

$$J_z = -k\,g^{-1}\partial_z g\ , \qquad\qquad\qquad J_{\bar{z}} = 0\ ,$$
$$\bar{J}_{\bar{z}} = -k\,\partial_{\bar{z}}g\,g^{-1}\ , \qquad\qquad\qquad \bar{J}_z = 0\ , \qquad\qquad (3.3.3)$$

and obey $\partial_{\bar{z}}J_z = \partial_z\bar{J}_{\bar{z}} = 0$. In view of these relations, J_z and $\bar{J}_{\bar{z}}$ are respectively referred to as the holomorphic and anti-holomorphic currents of the WZW theory, properties which will be reflected in the notation of their coordinate dependence $J_z(z)$ and $\bar{J}_{\bar{z}}(\bar{z})$. The holomorphic currents $J_z(z) = \sum_a J_z^a(z)T^a$ obey the OPE

$$J_z^a(z)J_z^b(0) \sim \frac{k\delta_{ab}}{z^2} + \sum_c if_{abc}\frac{J_z^c(0)}{z}\ , \qquad\qquad (3.3.4)$$

and similarly for the anti-holomorphic currents.

Primary operators of the WZW theory are group elements $g(x)$ taken in some representation r. They have conformal weight (h,h) where the dimension h is given by,

$$h = \frac{C_2(r)}{2k + C_2(G)}\ . \qquad\qquad (3.3.5)$$

The basic two-point function is $\langle g^{-1}(x_2)g(x_1)\rangle$, which is proportional to the identity matrix by virtue of the $G \times G$ global symmetry [1]. Expanded in powers of $1/k$ we have

$$\langle g^{-1}(x_2)g(x_1)\rangle \sim (x_{21}^2)^{-2h} \qquad\qquad (3.3.6)$$
$$\sim 1 - \frac{C_2(r)}{k}\ln(x_{21}^2) + \frac{C_2(r)^2}{2k^2}\big(\ln(x_{21}^2)\big)^2 + \frac{C_2(r)C_2(G)}{2k^2}\ln(x_{21}^2) + \mathcal{O}(k^{-3})$$

up to an overall multiplicative factor. Here we use the notation $x_{21}^\mu = x_2^\mu - x_1^\mu$.

In perturbation theory in powers of $1/k$ the scaling dimension h is extracted from the correlator by computing Feynman diagrams. The algebraic approach to the WZW model yields the full result (3.3.5), such that the perturbative series simply amounts to the shift $2k \to 2k + C_2(G)$. In diagrammatic terms it is not at all obvious how we just get this simple shift. However, agreement is expected, since we have good reason to believe that the path

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[1] In what follows we do not distinguish between a matrix proportional to the identity and one of its diagonal elements, including in the cases of the holomorphic and WZW Wilson lines.

integral and algebraic definitions of the WZW theory describe one and the same theory. Examples of perturbation theory computations in WZW include [18, 31]. However, we are not aware of any prior computation of the anomalous dimension of primary operators in perturbation theory.

3.3.2 Holomorphic Wilson line

Given the holomorphic current $J(z) \equiv J_z(z)$ a natural object to consider is the Wilson line operator $P \exp \frac{1}{k} \int_{z_1}^{z_2} dz\, J(z)$, where P denotes path ordering along the contour from z_1 to z_2. The basic claim is that, up to renormalization, this operator gives the current algebra vacuum OPE block. That is, consider the bi-local operator $g^{-1}(x_2) g(x_1)$, with g taken in some irreducible representation r. We can decompose the operator into irreducible representations of the current algebra. The Wilson line then gives the representation containing only holomorphic currents. An equivalent way of stating this is that the Wilson line should reproduce correlation functions with any number of holomorphic current insertions,

$$\langle g^{-1}(x_2) g(x_1) J^{a_3}(z_3) \dots J^{a_n}(z_n) \rangle = \mathcal{Z}(\bar{z}_1, \bar{z}_2) \langle P \exp \left\{ \frac{1}{k} \int_{z_1}^{z_2} dz\, J(z) \right\} J^{a_3}(z_3) \dots J^{a_n}(z_n) \rangle \quad (3.3.7)$$

where the factor $\mathcal{Z}(\bar{z}_1, \bar{z}_2)$ is independent of $z_3, \cdots, z_n$ and depends anti-holomorphically on x_1 and x_2 through $\bar{z}_1$ and $\bar{z}_2$ only. At lowest order in $1/k$ this is easy to establish using the OPE of the currents. At higher orders we encounter divergences requiring renormalization. In this section we wish to check this relation in a perturbative expansion in powers of $1/k$ in terms of suitably renormalized operators. Setting $z_1 = 0$ we consider the case of zero and one current insertions, and we wish to establish

$$\begin{aligned} W(z) \equiv \lim_{\varepsilon \to 0} \langle W_\varepsilon(z) \rangle &= z^{-2h}\,, \\ \lim_{\varepsilon \to 0} \langle J^a(x) W_\varepsilon(z) \rangle &= z^{-2h} \left(\frac{1}{x-z} - \frac{1}{x} \right) T^a \end{aligned} \quad (3.3.8)$$

where we introduced the notation $W_\varepsilon(z)$ for the regulated Wilson line operator and $W(z)$ for its renormalized vacuum expectation value.

Although the current algebra Wilson line can be understood on its own terms, it is usefully thought of as existing due to the well-known equivalence between Chern-Simons theory on

a three-dimensional manifold M and the WZW theory on the boundary of M [37, 79]. The natural observables in Chern-Simons theory are Wilson lines $P \exp \int A$, and in the present case we consider an open Wilson line with endpoints on the boundary. On account of the flatness of the connection, the precise shape of the Wilson line contour does not matter, only the location of its endpoints, hence the Wilson line is well-suited to represent the bi-local operator $g^{-1}(x_2)g(x_1)$ (or more precisely, its current OPE block).

To flesh this out a bit more, the boundary components of the Chern-Simons gauge field are mapped in the WZW model to the current and an external gauge field: $(A_z, A_{\bar{z}}) \leftrightarrow (J_z, A_{\bar{z}})$. In the Chern-Simons path integral we fix $A_{\bar{z}}$ on the boundary but allow A_z to fluctuate. Such a path integral is equal on the WZW side to a generating function for the current correlators, $\langle \exp \int d^2 z J_z^a A_{\bar{z}}^z \rangle$. This is established by relating the Chern-Simons equations of motion to the current algebra Ward identity. The same procedure can be carried out in the presence of a Wilson line. The Chern-Simons gauge field now gets a source due to the Wilson line, which maps on the WZW side to the Ward identity for the current in the presence of a primary operator inserted at each endpoint. This then leads to the equivalence (3.3.8) between current correlators computed in the presence of the bi-local $g^{-1}(x_2)g(x_1)$ and in the presence of the Wilson line $P \exp \frac{1}{k} \int_{z_1}^{z_2} dz J$. This discussion explains why such a relation exists, but it is purely formal, as it does not take into account UV divergences in the path integral. Here, we are trying to establish that the relation holds in the full quantum theory.

3.3.3 Lowest order calculations

The regulated holomorphic Wilson line operator takes the form

$$W_\varepsilon(z) = N(\varepsilon) P \exp \left\{ \frac{\alpha(\varepsilon)}{k} \int_0^z dy J(y) \right\}, \tag{3.3.9}$$

where ε is a UV regulator. Expanding out the exponential and taking the vacuum expectation value, we need to compute nested integrals of current correlators. All current correlators are obtained from the corresponding modification of the standard recursion relation, which is

determined by OPE and holomorphy considerations,

$$\langle J^a(y) J^{a_n}(y_n) \dots J^{a_1}(y_1)\rangle$$

$$= \sum_{i=1}^{n} \sum_{b} \frac{i f_{a a_i b}}{(y - y_i)^1} \langle J^{a_n}(y_n) \dots J^{a_{i+1}}(y_{i+1}) J^b(y_i) J^{a_{i-1}}(y_{i-1}) \dots J^{a_1}(y_1)\rangle$$

$$+ \sum_{i=1}^{n} \frac{k \delta_{a a_i}}{(y - y_i)^2} \langle J^{a_n}(y_n) \dots J^{a_{i+1}}(y_{i+1}) J^{a_{i-1}}(y_{i-1}) \dots J^{a_1}(y_{a_1})\rangle \qquad (3.3.10)$$

starting from $\langle \mathbb{1}\rangle = 1$ and $\langle J^a(y) = 0\rangle$. Singularities arise from collisions of pairs of currents, as in [61]. We implement a form of dimensional regularization in which we assign scaling dimension $1 - \varepsilon$ to the currents. For example, the regulated two-point function is

$$\langle J^a(y_1) J^b(y_2)\rangle = \frac{k \delta_{ab}}{(y_1 - y_2)^{2 - 2\varepsilon}} . \qquad (3.3.11)$$

Regulating correlators can be subtle since each term in the recursion relation (3.3.10) doesn't scale as the full correlator should. For example the three point function is obtained as

$$\langle J^a(y_1) J^b(y_2) J^c(y_3)\rangle = \sum_{d} \frac{i f_{abd}}{(y_1 - y_2)} \frac{k \delta_{dc}}{(y_2 - y_3)^2} + \sum_{d} \frac{i f_{acd}}{(y_1 - y_3)} \frac{k \delta_{bd}}{(y_2 - y_3)^2}$$

$$= \frac{i k f_{abc}}{(y_1 - y_2)(y_1 - y_3)(y_2 - y_3)} . \qquad (3.3.12)$$

Our prescription then is to first compute the correlator and write it in a form where scaling of each coordinate is manifest. Then simply replace every instance of $(y_i - y_j)$ with $(y_i - y_j)^{1-\varepsilon}$.

To illustrate the general procedure outlined above, we consider the Wilson line expectation value at order $1/k$,

$$\langle W_\varepsilon(z)\rangle = N(\varepsilon)\left[1 + \frac{\alpha(\varepsilon)^2}{k^2} \sum_{a,b} T^a T^b \int_0^z dy_1 \int_0^{y_1} dy_2 \langle J^a(y_1) J^b(y_2)\rangle + \dots\right]$$

$$= N(\varepsilon)\left[1 + \frac{\alpha(\varepsilon)^2}{k} C_2(r) \int_0^z dy_1 \int_0^{y_1} dy_2 \frac{1}{(y_1 - y_2)^{2 - 2\varepsilon}} + \dots\right]$$

$$= N(\varepsilon)\left[1 - \frac{\alpha(\varepsilon)^2}{k} C_2(r)\left(\frac{1}{2\varepsilon} + \ln z + 1 + \mathcal{O}(\varepsilon)\right) + \dots\right] \qquad (3.3.13)$$

At this order we can take $N(\varepsilon) = 1 + \frac{C_2(r)}{2k\varepsilon}$ and $\alpha(\varepsilon) = 1$. This gives the expected result

$$W(z) \sim z^{-2h} + \mathcal{O}\left(\frac{1}{k^2}\right) , \qquad h = \frac{C_2(r)}{2k} + \mathcal{O}\left(\frac{1}{k^2}\right) . \qquad (3.3.14)$$

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We can similarly verify the Ward identity at lowest order, which corresponds to expanding the exponential to first order. This gives

$$\begin{aligned}
\langle J^a(w) W_\varepsilon(z) \rangle &= T^a \int_0^z \frac{dy}{(y-w)^{2-2\varepsilon}} + \dots \\
&= T^a \left(\frac{1}{w-z} - \frac{1}{x} \right) + \dots
\end{aligned} \qquad (3.3.15)$$

which is the correct result at this order.

3.3.4 Higher order computations

We now make a few comments about the computation of the holomorphic Wilson line at higher orders in $1/k$. We will be brief here, as the most significant technical details will be discussed later in the context of the gravitational Wilson line.

Since the correlation function of n currents contains a maximal power k^p with $p = \lfloor \frac{n}{2} \rfloor$, to obtain the Wilson line at order $1/k^n$ we need to expand the exponential to order $2n$. The correlation function of up to $2n$ currents is obtained from the recursion relation (3.3.10). The nested integrals can be evaluated by the methods discussed below. Finiteness of the renormalized Wilson line as $\varepsilon \to 0$ only partially fixes the renormalization constants $N(\varepsilon)$ and $\alpha(\varepsilon)$ up to the given order in the $1/k$ expansion. The unfixed part of $N(\varepsilon)$ can be fixed by adopting a normalization convention, such as $\langle W_\varepsilon(1) \rangle = 1$. To fix α, which is needed to determine the scaling dimension, we need to demand that the Ward identity is satisfied. Rather than the general Ward identity (3.3.8), various integrals greatly simplify if we place the current at infinity, using the usual formula obtained from $z \to 1/z$: $J_\infty^a \equiv -\lim_{z\to\infty} z^2 J^a(z)$. So this amount to imposing

$$\lim_{\varepsilon\to 0} \langle J_\infty^a W_\varepsilon(z) \rangle = T^a z W(z) . \qquad (3.3.16)$$

We carry this out order by order in $1/k$, fixing the constants N and α up to that order as we go. These considerations completely fix the terms in the ε expansion that contribute to the finite parts of the correlators as $\varepsilon \to 0$. The program is in fact highly overconstrained, since just from counting terms there is no guarantee that constants N and α can be found

that satisfy these criteria. It is furthermore not guaranteed that the Wilson line correlator will be a pure power law. Nevertheless, explicit computations demonstrate that all these conditions are indeed satisfied, at least to third order in the $1/k$ expansion.

As an example, consider the Wilson line (3.3.9) expanded to order $1/k^2$. Focussing only on the term that involves three current insertions, we have

$$
\begin{aligned}
\langle W_\varepsilon(z) \rangle &\sim \frac{N\alpha^3}{k^3} \sum_{a,b,c} T^a T^b T^c \int_0^z dy_1 \int_0^{y_1} dy_2 \int_0^{y_2} dy_3 \, \langle J^a(y_1) J^b(y_2) J^c(y_3) \rangle \, , \qquad (3.3.17) \\
&= \frac{N\alpha^3}{k^2} \sum_{a,b,c} i f_{abc} T^a T^b T^c \int_0^z dy_1 \int_0^{y_1} dy_2 \int_0^{y_2} dy_3 \frac{1}{(y_1-y_2)^{1-\varepsilon}(y_1-y_3)^{1-\varepsilon}(y_2-y_3)^{1-\varepsilon}} \, .
\end{aligned}
$$

The integral is discussed in detail in the gravitational case and we will skip its derivation here. The Lie algebra factor multiplying the integral is easily computed as

$$
\begin{aligned}
i f_{abc} T^a T^b T^c &= \tfrac{1}{2} \sum_{a,b,c,d} i f_{abc} [T^a, T^b] T^d T^c \, , \\
&= -\tfrac{1}{2} \sum_{a,b,c,d} f_{abc} f_{abd} T^d T^c \, , \\
&= -\tfrac{1}{2} C_2(G) C_2(r) \, , \qquad\qquad (3.3.18)
\end{aligned}
$$

where we have used the anti-symmetry of the f_{abc} in the first line, the structure relations to obtain the second line, and the definitions of the quadratic Casimir values $C_2(G)$ and $C_2(r)$, respectively, in the adjoint representation and the representation r.

The Lie algebra factors for other diagrams can be computed in a similar manner. All the required integrals are simpler versions of the ones that appear in the gravitational case. We skip them here for brevity. Evaluating the Wilson line, we find the expected anomalous dimension to order $1/k^3$,

$$
h = \frac{C_2(r)}{2k} - \frac{C_2(r)C_2(G)}{4k^2} + \frac{C_2(r)C_2(G)^2}{8k^3} + \mathcal{O}(1/k^4) \qquad (3.3.19)
$$

which reproduces the expansion of the current algebra result in (3.3.5) to this order.

3.4 Non-holomorphic Wilson line from WZW

In this section we discuss the computation of primary two-point functions in WZW models in a manner that does not exhibit manifest holomorphic factorization. The virtue of this approach is that computations can be carried out using familiar dimensional regularization (modulo subtleties associated with epsilon tensors) with Feynman rules obtained from the WZW Lagrangian, and there is a simple relation between the bi-local primary operator and a Wilson line which holds even in the regulated theory. The drawback is the lack of manifest holomorphic factorization, which in turn makes computations more laborious than those in the previous section, although the results are mutually consistent.

3.4.1 Direct perturbative computation of $\langle g^{-1}(x_2)g(x_1)\rangle$

We proceed by computing $\langle g^{-1}(x_2)g(x_2)\rangle$ in perturbation theory, and then showing how this computation can be recast in terms of a non-holomorphic Wilson line. To carry out perturbation theory we parametrize the field $g(x)$ which takes values in the representation r of the group G in terms of the field $X(x)$ which takes values in the Lie algebra $\mathcal{G}$ of G [2]

$$g(x) = \exp\left\{\frac{i}{\sqrt{k}}X^a(x)T^a\right\} . \tag{3.4.1}$$

Expanding the exponential in powers of $k^{-\frac{1}{2}}$ and substituting into the WZW action yields

$$S[g] = \frac{1}{8\pi}\int d^2x\,\partial_\mu X^a\partial^\mu X^a + \frac{i}{24\pi k^{1/2}}\,f_{abc}\int d^2x\epsilon^{\mu\nu}X^a\partial_\mu X^b\partial_\nu X^c$$
$$-\frac{1}{24\pi k}\,K_{abcd}\int d^2x\,X^aX^b\partial^\mu X^c\partial_\mu X^d + \mathcal{O}(k^{-\frac{3}{2}}) \tag{3.4.2}$$

where the metric is taken to be $ds^2 = dx^\mu dx^\mu$, and the tensor K is given by

$$K_{abcd} = Tr'\big(T^aT^bT^cT^d - T^aT^cT^bT^d\big) = if_{bce}Tr'(T^aT^eT^d) . \tag{3.4.3}$$

We work in dimensional regularization, taking the spacetime dimensionality to be $d = 2 - \varepsilon$. The one subtlety is how to define quantities involving $\epsilon^{\mu\nu}$ in this scheme; this will be discussed

---

[2]In this section repeated Lie algebra indices are summed over.

below. The position space free field correlator in $d = 2 - \varepsilon$ dimensions is given by

$$\langle X^a(x) X^b(0) \rangle_{free} = \Delta(x)\, \delta^{ab} \tag{3.4.4}$$

where $\Delta(x)$ is the free-field propagator given by

$$\Delta(x) = \int \frac{d^d p}{(2\pi)^d} \frac{4\pi}{p^2} e^{ipx} = \frac{\Gamma(\frac{d}{2} - 1)}{(\pi x^2)^{\frac{d}{2} - 1}} = -\frac{2}{\varepsilon} - \ln(\pi x^2) - \gamma + \mathcal{O}(\varepsilon) \ . \tag{3.4.5}$$

and $x^2 = \gamma_{\mu\nu} x^\mu x^\nu$ the the d-dimensional norm of x^μ. As is familiar when using dimensional regularization, we are setting self-contractions to zero: $\langle X^a(0) X^b(0) \rangle_{free} = 0$. Renormalizing the two-point function of the primary field g to order $1/k$ by introducing a multiplicative renormalization factor $N(\varepsilon) = 1 + 2C_2(r)/(k\varepsilon) + \mathcal{O}(1/k^2)$, we find to this order

$$N(\varepsilon)\langle g^{-1}(x)g(0)\rangle = N(\varepsilon)\left[1 + \frac{C_2(r)}{k}\Delta(x) + \mathcal{O}(k^{-2})\right] = 1 - \frac{C_2(r)}{k}\ln(x^2) + \mathcal{O}(k^{-2}) \tag{3.4.6}$$
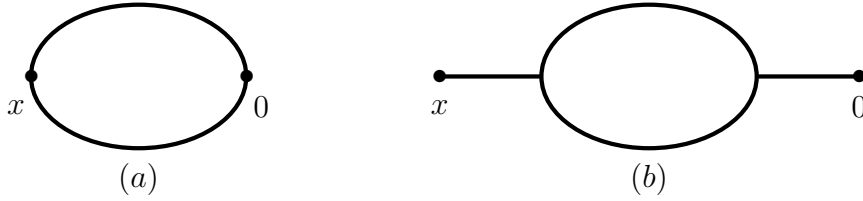
Figure 3.1: Feynman diagrams at order $1/k^2$.

At order $1/k^2$ we have the diagrams shown in Figure 3.1. Figure 3.1a comes from expanding each of the exponentials in $g^{-1}(x)$ and $g(0)$ to second order and taking Wick contractions. Figure 3.1b arises from bringing down two cubic interaction vertices. This yields

$$\begin{aligned}
\langle g^{-1}(x)g(0)\rangle_{1a} &= \frac{1}{2k^2}\frac{\Gamma(\frac{d}{2} - 1)^2}{\pi^{d-2}}\left(C_2(r)^2 - \frac{1}{4}C_2(r)C_2(G)\right)(x^2)^{2-d} \ . \\
\langle g^{-1}(x)g(0)\rangle_{1b} &= -\frac{1}{k^2}\frac{1}{2^d \pi^{d-\frac{5}{2}}}\frac{\Gamma(\frac{d}{2})\Gamma(d-2)}{(d-4)(d-2)\Gamma(\frac{d}{2} + \frac{1}{2})}C_2(r)C_2(G) \\
&\qquad \times \gamma_{\mu\alpha}\epsilon^{\mu\nu}\epsilon^{\alpha\beta}[2(2-d)x_\nu x_\beta + x^2 \gamma_{\nu\beta}](x^2)^{1-d} \ . \tag{3.4.7}
\end{aligned}$$

To proceed we need a rule for defining $\gamma_{\mu\alpha}\epsilon^{\mu\nu}\epsilon^{\alpha\beta}$ in d-dimensions. In $d = 2$ we have

$$\gamma_{\mu\alpha}\epsilon^{\mu\nu}\epsilon^{\alpha\beta} = \gamma^{\nu\beta} \ . \tag{3.4.8}$$

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One option is to adopt this rule in d dimensions. But there are alternative prescriptions as well. For instance, we could first use the $d = 2$ identity $\epsilon^{\mu\nu}\epsilon^{\alpha\beta} = \gamma^{\mu\alpha}\gamma^{\nu\beta} - \gamma^{\mu\beta}\gamma^{\nu\alpha}$ and then contract with $\gamma_{\mu\alpha}$ in d dimensions. This gives $\gamma_{\mu\alpha}\epsilon^{\mu\nu}\epsilon^{\alpha\beta} = (1 - \varepsilon)\gamma^{\nu\beta}$. More generally, we could multiply ε by any coefficient. These prescriptions differ in the sense that one can show that the value of the anomalous dimension depends on the value of this coefficient. However, conformal invariance singles out the rule (3.4.8). In particular, consider the current algebra Ward identity

$$\langle g^{-1}(x_2)g(x_1)J^a(x_3)\rangle \sim \left(\frac{1}{z_3 - z_2} - \frac{1}{z_3 - z_1}\right)\langle g^{-1}(x_2)g(x_1)\rangle T^a \ . \tag{3.4.9}$$

This Ward identity, together with the definition of the Sugawara stress tensor, is what fixes the conformal dimension of g in the algebraic approach to the WZW model. Evaluating both sides of (3.4.9) in $1/k$ perturbation theory we encounter, at order $1/k^2$, on the right hand side the same diagrams as above, including the ambiguity associated with the product of epsilon tensors. On the other hand, no epsilon tensors appear on the left hand side at this order, and hence there is no ambiguity. We then find that demanding (3.4.9) implies that we should adopt (3.4.8). In fact, it turns out that under this rule (3.4.9) holds for all d. This discussion of course raises the question as to the proper rule at higher loop orders, where higher powers of epsilon tensors will arise. There is a natural generalization of (3.4.9) in which one reduces all products of epsilon tensors directly in $d = 2$, but whether this is compatible with the Ward identity at higher orders in $1/k$ is an open question that we do not address here.

Returning to (3.4.7) we now have

$$\langle g^{-1}(x)g(0)\rangle_{1b} = \frac{C_2(r)C_2(G)}{k^2}\frac{\Gamma(\frac{d}{2} - 1)^2}{8\pi^{d-2}(d-1)}(x^2)^{2-d} \ , \tag{3.4.10}$$

where we used the Legendre duplication formula to simplify.

The computation of the renormalized correlator $N(\varepsilon)\langle g^{-1}(x)g(0)\rangle$ also receives a contribution from the $1/k$ term in $N(\varepsilon)$. However, we need not consider this as it has no bearing on the computation of the anomalous dimension, since the latter can be extracted from $x\partial_x \ln\langle g^{-1}(x)g(0)\rangle$.

Collecting all contributions through order $1/k^2$ we have

$$
\begin{aligned}
\langle g^{-1}(x)g(0)\rangle &= N(\varepsilon)\Big[1 + \frac{C_2(r)}{k}\frac{\Gamma(\frac{d}{2}-1)}{\pi^{d/2-1}}(x^2)^{1-\frac{d}{2}} + \frac{C_2(r)^2}{2k^2}\frac{\Gamma(\frac{d}{2}-1)^2}{\pi^{d-2}}(x^2)^{2-d} \\
&\quad - \frac{C_2(r)C_2(G)}{k^2}\frac{\Gamma(\frac{d}{2}-1)^2}{8\pi^{d-2}}\frac{d-2}{d-1}(x^2)^{2-d}\Big]
\end{aligned}
\tag{3.4.11}
$$

Since the scaling dimension h is identified via $\langle g^{-1}(x)g(0)\rangle \sim (x^2)^{-2h}$ we can extract it as

$$
h = -\frac{1}{4}\lim_{\varepsilon\to 0}x\partial_x \ln\langle g^{-1}(x)g(0)\rangle
\tag{3.4.12}
$$

Plugging in (3.4.11) we find

$$
h = \frac{C_2(r)}{2k} - \frac{C_2(r)C_2(G)}{4k^2} + \mathcal{O}(k^{-3})
\tag{3.4.13}
$$

in agreement with the expansion of (3.3.5) to this order.

3.4.2 Non-holomorphically factorized Wilson line

We can convert the bi-local primary operator considered above into a Wilson line type object by using the identity

$$
g^{-1}(x_2)g(x_1) = P\exp\Big\{-\int_{x_1}^{x_2} dy^\mu\, g^{-1}(y)\partial_\mu g(y)\Big\}\,.
\tag{3.4.14}
$$

This identity holds for any matrix-valued object $g(x)$. In particular, if we compute the expectation value of both sides we are guaranteed to get exact agreement even with a finite regulator in place. The computations of the previous section therefore establish that perturbation theory will yield $\big\langle P\exp\big\{-\int_{x_1}^{x_2} g^{-1}(y)\partial_\mu g(y)dy^\mu\big\}\big\rangle \sim (x_{21}^2)^{-2h}$; finiteness also requires the multiplicative renormalization factor $N(\varepsilon)$ that we will suppress.

We now write

$$
\langle g^{-1}(x_2)g(x_1)\rangle = \Big\langle P\exp\Big\{\frac{1}{k}\int_{x_1}^{x_2} dy^\mu\, \mathcal{J}_\mu(y)\Big\}\Big\rangle
\tag{3.4.15}
$$

where the "vector operator" $\mathcal{J}_\mu$ is defined as

$$
\mathcal{J}_\mu = -kg^{-1}\partial_\mu g\,.
\tag{3.4.16}
$$

This is not a conserved current, $\partial^\mu \mathcal{J}_\mu \neq 0$. Its components are related to those of the conserved currents J_μ and $\bar{J}_\mu$ as $\mathcal{J}_z = J_z$, $\mathcal{J}_{\bar{z}} = g^{-1}\bar{J}_{\bar{z}}g$. The computations we have performed so far establish that, as $\varepsilon \to 0$,

$$\left\langle P \exp\left\{\frac{1}{k}\int_{x_1}^{x_2} dy^\mu \, \mathcal{J}_\mu(y)\right\}\right\rangle = \left\langle P \exp\left\{\frac{\alpha}{k}\int_{z_1}^{z_2} dy \, J_z(y)\right\}\right\rangle\left\langle P \exp\left\{\frac{\alpha}{k}\int_{\bar{z}_1}^{\bar{z}_2} d\bar{y}\,\bar{J}_{\bar{z}}(y))\right\}\right\rangle \quad (3.4.17)$$

through at least $\mathcal{O}(1/k^2)$. We note that the chiral Wilson lines on the right hand side require vertex renormalization factors, while no such object is required on the left hand side, as follows from the identity (3.4.14). Roughly speaking, we may surmise that the α factors on the right compensate for the non-chiral correlators on the left.

To flesh this out a bit more, let us consider correlation functions involving the vector operator $\mathcal{J}_\mu$. To order k^0 we find the two-point functions

$$\begin{aligned}
\langle \mathcal{J}_z^a(x)\mathcal{J}_z^b(0)\rangle &= \frac{d}{2}\left(\frac{d}{2}-1\right)\frac{k}{z^2}\Delta(x)\delta^{ab}\,, \\
\langle \mathcal{J}_z^a(x)\mathcal{J}_{\bar{z}}^b(0)\rangle &= \left(\frac{d}{2}-1\right)^2\frac{k}{z\bar{z}}\Delta(x)\delta^{ab} + \frac{\left(\frac{d}{2}-1\right)^2}{2(d-1)}C_2(G)\frac{\Delta(x)^2}{z\bar{z}}\delta^{ab}\,, \\
\langle \mathcal{J}_{\bar{z}}^a(x)\mathcal{J}_{\bar{z}}^b(0)\rangle &= \frac{d}{2}\left(\frac{d}{2}-1\right)\frac{k}{\bar{z}^2}\Delta(x)\delta^{ab} + \left(\frac{d}{2}-1\right)C_2(G)\frac{\Delta(x)^2}{\bar{z}^2}\delta^{ab}\,, \quad (3.4.18)
\end{aligned}$$

where $\Delta(x)$ is the scalar propagator defined in (3.4.5). The fact that $\langle \mathcal{J}_z\mathcal{J}_z\rangle$ is uncorrected at order k^0 is consistent with the fact that this is the two-point function of the conserved current J_z, and hence is unrenormalized. The mixed correlator in the second line, $\langle \mathcal{J}_z\mathcal{J}_{\bar{z}}\rangle$ is finite as $\varepsilon \to 0$, and this contributes to the non-chiral Wilson line expectation value at order $1/k^2$. The correlator in the last line $\langle \mathcal{J}_{\bar{z}}\mathcal{J}_{\bar{z}}\rangle$ diverges as $\varepsilon \to 0$. We define the renormalized operator $\tilde{\mathcal{J}}_{\bar{z}}^a$,

$$\mathcal{J}_{\bar{z}}^a = \left(1 - \frac{C_2(G)}{k\varepsilon}\right)\tilde{\mathcal{J}}_{\bar{z}}^a\,. \quad (3.4.19)$$

After doing this and taking $\varepsilon \to 0$ we get

$$\begin{aligned}
\langle \mathcal{J}_z^a(x)\mathcal{J}_z^b(0)\rangle &= \frac{k}{z^2}\delta_{ab} + O(\varepsilon)\,, \\
\langle \mathcal{J}_z^a(x)\tilde{\mathcal{J}}_{\bar{z}}^b(0)\rangle &= \mathcal{O}(\varepsilon)\,, \\
\langle \tilde{\mathcal{J}}_{\bar{z}}^a(x)\tilde{\mathcal{J}}_{\bar{z}}^b(0)\rangle &= A\frac{k}{\bar{z}^2}(z\bar{z})^{-\frac{C_2(G)}{k}}\delta_{ab} + O(\varepsilon)\,, \quad (3.4.20)
\end{aligned}$$

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for some constant A. $\tilde{\mathcal{J}}_{\bar{z}}^a$ has acquired scaling dimension $(h, \overline{h}) = (1 + \frac{C_2(G)}{k}, \frac{C_2(G)}{k})$. Coming back to the Wilson line, even if we rewrite it in terms of the renormalized vector operator components $(\mathcal{J}_z, \tilde{\mathcal{J}}_{\bar{z}})$ it is not correct to omit the contributions from $\langle \mathcal{J}_z \tilde{\mathcal{J}}_{\bar{z}} \rangle$ even though this correlator vanishes as $\varepsilon \to 0$. This vanishing is compensated by $1/\varepsilon$ divergences, yielding a finite result. Thus, there is no manifest factorization.

3.4.3 Comments on holomorphic factorization

The conclusion of the above analysis is that the expectation value of the non-holomorphic Wilson line built out of $\mathcal{J}_\mu$ agrees with (the square of) the holomorphic Wilson line as the regulator is removed. The former thus exhibits factorization to the order we have considered, but this comes out from detailed computation rather than being manifest from the start. Here we add a few more comments regarding this state of affairs.

The classical WZW model exhibits holomorphic factorization in the following sense. The Euler-Lagrange equations are $\partial_{\bar{z}}(g^{-1}\partial_z g) = 0$. The general solution of this equation takes the factorized form $g(z, \bar{z}) = \overline{g}_L(\bar{z}) g_R(z)$, for arbitrary and independent (ignoring any reality conditions) matrices $\overline{g}_L(\bar{z})$ and $g_R(z)$. Formally, the quantum correlator of interest is then

$$\langle g^{-1}(z, \bar{z}) g(0) \rangle = \langle g_R^{-1}(z) \overline{g}_L^{-1}(\bar{z}) \overline{g}_L(0) g_R(0) \rangle . \tag{3.4.21}$$

We can try to argue for factorization from either the path integral or operator perspectives. In terms of the path integral, we can imagine independently integrating over g_L and g_R. Inside the g_L path integral $\overline{g}_L^{-1}(\bar{z}) \overline{g}_L(0)$ will be proportional to the unit matrix, and the correlator thence factorizes. Of course, this argument is little more than handwaving as it stands, since the fact that classical solutions take the factorized form does not imply that we can perform independent path integrals over the two factors. On the other hand, writing $g(z, \bar{z}) = \overline{g}_L(\bar{z}) g_R(z)$ makes more sense in the quantum theory if we work in the operator formalism. In this case, the outstanding issue is that although the oscillator modes can be uniquely associated to one of the two factors, the same is not true of the zero modes, which couples the two together.

We should also mention the argument by Witten [80] establishing the holomorphic fac-

torization of current correlators on arbitrary Riemann surfaces, which is formal in the sense of ignoring UV divergences and anomalies. Starting from the WZW action $S[g]$ one gauges the current J by coupling to an external gauge field A,

$$S[g, A] = S[g] + \frac{1}{2\pi} \int d^2z \, Tr A_{\bar{z}} g^{-1} \partial_z g - \frac{1}{4\pi} \int d^2z \, Tr A_{\bar{z}} A_z \ . \tag{3.4.22}$$

The path integral over g defines a wavefunction

$$\Psi(A) = \int Dg \, e^{-kS(g,A)} \tag{3.4.23}$$

which serves as a generating function for current correlators. The main result is to then show that the partition function, $Z(\Sigma)$, of the WZW model on the Riemann surface Σ is equal to the norm of the wavefunction, $Z(\Sigma) = |\Psi|^2$, where $|\Psi|^2 = \frac{1}{Vol(\hat{G})} \int DA \overline{\Psi(A)} \Psi(A)$. We might contemplate extending this to our context by cutting holes in the Riemann surface with prescribed holonomies to represent the primary operator insertions. Of course, one would still need to confront what for us is the main issue, namely making precise sense of these manipulations at the quantum level. We leave these questions for the future, and now return to the main case of interest, the gravitational Wilson line.

3.5   Renormalization of gravitational Wilson lines

In this section we shall regularize and renormalize the matrix elements of the gravitational Wilson line operator in two-dimensional conformal field theory in a perturbative expansion in inverse powers of the central charge c. We focus on the scaling dimension $h(j, c)$ of the Wilson line operator, whose exact expression is predicted from the twisted $SL(2, \mathbb{R})$ current algebra representations of spin j as discussed in the Introduction. Using the regularization and renormalization schemes developed here we shall calculate $h(j, c)$ up to order $1/c^3$ and find perfect agreement with the large c expansion to the same order of the exact expression (3.1.4), which we repeat here

$$h(j, c) = -j - \frac{6}{c} j(j+1) - \frac{78}{c^2} j(j+1) - \frac{1230}{c^3} j(j+1) + \mathcal{O}(c^{-4}). \tag{3.5.1}$$

As discussed in Section 3.2 the Wilson line is defined as a matrix element of

$$P \exp \int_0^z dy \left(L_1 + \frac{6}{c} T(y) L_{-1} \right). \tag{3.5.2}$$

The first step in implementing $1/c$ perturbation theory is to rewrite this in a manner analogous to what one does when passing to the interaction representation in quantum mechanical problems. In the present case this amounts to using the identity

$$P \exp \int_0^z dy \left(L_1 + \frac{6}{c} T(y) L_{-1} \right) = e^{zL_1} P \exp \int_0^z dy \left(\frac{6}{c} X(y) T(y) \right) \tag{3.5.3}$$

where $X(y)$ is given by,

$$X(y) = L_{-1} - 2yL_0 + y^2 L_1. \tag{3.5.4}$$

We shall consider matrix elements between states $|j, m\rangle$, with $2j+1 \in \mathbb{N}$ and $0 \leq j - m \leq 2j$, which are the tensor product of a spin j representation state of $SL(2, \mathbb{R})$ with the ground state of the two-dimensional conformal field theory. In the infinite c limit, the Wilson line operator reduces to e^{zL_1} whose matrix element $\langle j, -j| e^{zL_1} |j, j\rangle = z^{2j}$ gives the classical scaling dimension $-j$, in agreement with the leading term in (3.5.1).

For large but finite c we shall use perturbation theory in powers of $1/c$ to expand the Wilson line in terms of correlators which are polynomial in the stress tensor. Such correlators may be evaluated on the two-dimensional plane using the conformal Ward identities expressed, for example, in terms of the OPE of two stress tensors at points $w, z \in \mathbb{C}$,

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \mathcal{O}((z-w)^0). \tag{3.5.5}$$

The perturbative expansion of the matrix elements of the Wilson line operator is beset by short distance singularities resulting from the first term in (3.5.5), and require regularization. The use of a Pauli-Villars regulator in [16] correctly reproduced the $1/c$ term in (3.5.1) and the corresponding order $1/c^2$ term proportional to $(\ln z)^2$ in the expansion of the two point function, but gave a $1/c^2$ correction that disagrees with the corresponding term in (3.5.1). Dimensional regularization, and analytic continuation in $\varepsilon = 2 - d$ as applied to this problem in [61], is more successful, as we now discuss.

3.5.1 Dimensional regularization

No regulator of short distance singularities which preserves the infinite-dimensional conformal symmetry in two-dimensional space-time is known to exist. In fact most regulators will break the finite-dimensional conformal group and its dilation subgroup. However, dimensional regularization, in which the dimension of space-time is continued from two to $d = 2 - \varepsilon$ dimensions, preserves dilation symmetry in dimension d in each Feynman diagram contribution for all values of d where such diagrams are absolutely convergent. For this reason, dimensional regularization and analytic continuation in ε appears perhaps better-suited for regularizing correlators in scale invariant theories than other schemes. Unfortunately, the Ward identity (3.5.5), by which all correlators polynomial in the stress tensor can be computed on the two-dimensional plane, no longer holds and cannot be used to this end in $d \neq 2$.

Therefore, we need a concrete quantum field theory representation or model for the stress tensor which is valid for arbitrary dimension d and for arbitrary central charge c. Of course, upon proper renormalization, the Wilson line expectation values are expected to be independent of the model used to represent the CFT. To obtain an expansion for large c, we may take c to be an integer, without loss of generality. A simple model is then provided by the free field theory of c scalar fields ϕ^γ with $\gamma = 1, \cdots, c$ in d space-time dimensions. Parametrizing space-time $\mathbb{R}^d$ by coordinates $(z, \bar{z}, \vec{z})$ where $z, \bar{z}$ are the complex coordinates for $\mathbb{C}$ and $\vec{z} \in \mathbb{R}^{d-2}$, we readily evaluate the normalized two-point function of the field $\partial_z \phi^\gamma$,

$$\langle \partial_z \phi^\gamma(z) \partial_w \phi^{\gamma'}(w) \rangle = \frac{-V(d)\,\delta^{\gamma\gamma'}(\bar{z}-\bar{w})^2}{\left(|z-w|^2 + (\vec{z}-\vec{w})^2\right)^{\frac{d}{2}+1}} \ . \tag{3.5.6}$$

The normalization is given by $V(d) = \Gamma(\frac{d}{2}+1)/\pi^{\frac{d}{2}-1}$, but we shall soon see that its effect may absorbed by a renormalization, and therefore we shall set $V = 1$. For two points in the complex plane we have $\vec{z} = \vec{w} = 0$, and for two points on the real line the correlator in $d = 2 - \varepsilon$ dimensions simplifies to the following formula we shall use throughout,

$$\langle \partial_w \phi^\gamma(z) \partial_z \phi^{\gamma'}(w) \rangle = \frac{-\delta^{\gamma\gamma'}}{|z-w|^{2-\varepsilon}} \ . \tag{3.5.7}$$

In this model, the holomorphic stress tensor $T(z)$ for $z \in \mathbb{C}$ is defined as the T_{zz} component of the d-dimensional traceless stress tensor for the free field ϕ^γ, which is given by,

$$T(z) = -\frac{1}{2} \sum_{\gamma=1}^{c} : \partial_z \phi^\gamma(z) \partial_z \phi^\gamma(z) : \tag{3.5.8}$$

where the normal ordering symbol :: instructs us to omit all self-contractions in the calculation of correlators of $T(z)$. An equivalent definition in terms of the OPE of two fields $\partial_z \phi^\gamma$ may be given but will not be needed here.

Given the rules for calculating correlators in the free field theory model for the dimensionally regularized conformal field theory, it is straightforward to compute the correlator of the product of an arbitrary number of stress tensors, arranged at points y_i along the real line. Evidently, we have $\langle T(y) \rangle = 0$. The Feynman diagrams for a correlator $\langle T(y_1) \cdots T(y_n) \rangle$ for $n \geq 2$ may be distinguished by the number of connected one-loop sub-diagrams. Each sub-diagram may be labelled by a partition P into cycles of the set of points $\{y_1, \cdots, y_n\}$, with each cycle containing at least two points. Two partitions are equivalent if they are related by cyclic permutations and/or reversal of orientation of the points in each cycle, and under permutations of the cycles. This partitioning of a Feynman diagram into cycles is unique.

We shall denote a cycle of ordered points $y_{i_1}, \cdots, y_{i_\ell}$ by a square bracket $[i_1, \cdots, i_\ell]$ and the value of the corresponding one-loop diagram along this cycle by,

$$\begin{aligned}
\langle T^2 \rangle_{[i_1, i_2]} &= \frac{c/2}{|y_{i_1} - y_{i_2}|^{4-2\varepsilon}} , \\
\langle T^\ell \rangle_{[i_1, \cdots, i_\ell]} &= \frac{c}{|y_{i_1} - y_{i_2}|^{2-\varepsilon} |y_{i_2} - y_{i_3}|^{2-\varepsilon} \cdots |y_{i_\ell} - y_{i_1}|^{2-\varepsilon}} , \qquad \ell \geq 3 .
\end{aligned} \tag{3.5.9}$$

The y-dependence of $\langle T^\ell \rangle_{[i_1, \cdots, i_\ell]}$ is indicated through the indices $i_1, \cdots, i_\ell$ in the labeling of the cycle. The correlator is given by a sum over all possible inequivalent partitions $P = C_1 \cup C_2 \cup \cdots \cup C_p$ into p cycles, with $C_s \cap C_{s'} = \emptyset$ for $s' \neq s$, of the set $\{y_1, \cdots, y_n\}$,

$$\langle T(y_1) \cdots T(y_n) \rangle = \sum_P \langle T^n \rangle_P , \qquad\qquad \langle T^n \rangle_P = \prod_{s=1}^{p} \langle T^{\ell_s} \rangle_{C_s} . \tag{3.5.10}$$

The c-dependence of the contribution of P is given by c^p. For the calculation of the matrix elements of the Wilson line operator to order $1/c^3$, to be given in the next section, we shall

need the correlators for $n = 2, 3$ given in (3.5.9), as well as those for $n = 4$ with one and two cycles, for $n = 5$ with two cycles, and for $n = 6$ with three cycles, given as follows,

$$
\begin{aligned}
\langle T(y_1) \cdots T(y_4) \rangle &= \langle T^4 \rangle_{[12][34]} + \langle T^4 \rangle_{[13][24]} + \langle T^4 \rangle_{[14][23]} \\
&\quad + \langle T^4 \rangle_{[1234]} + \langle T^4 \rangle_{[1342]} + \langle T^4 \rangle_{[1324]}, \\
\langle T(y_1) \cdots T(y_5) \rangle &= \langle T^5 \rangle_{[12][345]} + 9 \text{ more partitions} + \mathcal{O}(c), \\
\langle T(y_1) \cdots T(y_6) \rangle &= \langle T^6 \rangle_{[12][34][56]} + 14 \text{ more partitions} + \mathcal{O}(c^2). \qquad (3.5.11)
\end{aligned}
$$

The contributions from each partition is given by (3.5.10) and $\langle T^4 \rangle_{[12][34]} = \langle T^2 \rangle_{[12]} \langle T^2 \rangle_{[34]}$, $\langle T^5 \rangle_{[12][345]} = \langle T^2 \rangle_{[12]} \langle T^3 \rangle_{[345]}$, $\langle T^6 \rangle_{[12][34][56]} = \langle T^2 \rangle_{[12]} \langle T^2 \rangle_{[34]} \langle T^2 \rangle_{[56]}$ and their permutations.

3.5.2 The regularized Wilson line matrix elements

We define the regularized matrix element of the Wilson line operator in dimension $d = 2 - \varepsilon$,

$$
W_\varepsilon(z) = N(\varepsilon) \langle j, -j | e^{z L_1} P \exp \left\{ \frac{6\alpha(\varepsilon)}{c} \int_0^z dy \, X(y) \, T(y) \right\} | j, j \rangle. \qquad (3.5.12)
$$

$X(y)$ was defined in (3.5.4) and the states $|j, m\rangle$ stand for the tensor product of the free field theory ground state and the spin j representation state of $SL(2, \mathbb{Z})$ of weight m. The multiplicative renormalization factor $N(\varepsilon)$ is required on general grounds for an exponential operator, while the factor $\alpha(\varepsilon)$ renormalizes the coupling to the stress tensor.

It will be shown below that the parameters $N(\varepsilon)$ and $\alpha(\varepsilon)$ may be chosen, order by order in powers of $1/c$, so as to cancel the poles in ε, and to define a renormalized matrix element whose scaling dimension is $h(j, c)$,

$$
W(z) = \lim_{\varepsilon \to 0} \langle W_\varepsilon(z) \rangle = z^{-2h(j,c)} \qquad z > 0 \qquad (3.5.13)
$$

up to order $1/c^3$ included. It will also be of interest to regularize and renormalize the matrix elements of the Wilson line operator multiplied by a single stress tensor $T(x)$ for $x \in \mathbb{R}$,

$$
T_x W_\varepsilon(z) = N(\varepsilon) \langle j, -j | T(x) |x|^{4-2\varepsilon} \, e^{z L_1} P \exp \left\{ \frac{6\alpha(\varepsilon)}{c} \int_0^z dy \, X(y) \, T(y) \right\} | j, j \rangle. \quad (3.5.14)
$$

By inspecting the scaling behavior of the correlators involving $T(x)$, it is clear that the expectation value $\langle T_x W_\varepsilon(z) \rangle$ tends to a finite limit as $x \to \infty$ and defines a matrix element

$\langle T_\infty W_\varepsilon(z)\rangle$ whose behavior is predicted from the dilation Ward identity,

$$\lim_{\varepsilon \to 0} \langle T_\infty W_\varepsilon(z)\rangle = h(j,c)\, z^2\, W(z). \tag{3.5.15}$$

We verify that the parameters $N(\varepsilon)$ and $\alpha(\varepsilon)$ required to renormalize W also renormalize $T_\infty W$, as may be expected on the basis of the dilation Ward identity in dimension $d = 2 - \varepsilon$.

3.5.3 Perturbative expansion

$\langle W_\varepsilon(z)\rangle$ may be evaluated by expanding the path ordered exponential in powers of α/c,

$$\langle W_\varepsilon(z)\rangle = z^{2j} N \sum_{n=0}^{\infty} \frac{(6\alpha)^n}{c^n} \int_0^z dy_n \cdots \int_0^{y_2} dy_1 F_n(z; y_n, \cdots, y_1)\langle T(y_n)\cdots T(y_1)\rangle \tag{3.5.16}$$

where we have suppressed the ε-dependence of N and α, which will be understood throughout. The $SL(2,\mathbb{R})$ group theory factor F_n is defined by,

$$z^{2j} F_n(z; y_n, \cdots, y_1) = \langle j, -j|e^{zL_1} X(y_n)\cdots X(y_1)|j, j\rangle. \tag{3.5.17}$$

A recursive formula for F_n is obtained in Appendix 3.9, while the calculations of the stress tensor correlators were given in the preceding section. To proceed further, it will be convenient to organize the calculation of $\langle W_\varepsilon(z)\rangle$ as follows,

$$\langle W_\varepsilon(z)\rangle = z^{2j} N \sum_{n=0}^{\infty} \alpha^n\, z^{n\varepsilon}\, W_{1\cdots n} \tag{3.5.18}$$

where $W_0 = 1$, $W_1 = 0$ and the contributions for $n \geq 2$ are given by,

$$W_{1\cdots n} = \frac{6^n}{c^n\, z^{n\varepsilon}} \int_0^z dy_n \cdots \int_0^{y_2} dy_1 F_n(z; y_n, \cdots, y_1)\langle T(y_n)\cdots T(y_1)\rangle. \tag{3.5.19}$$

The factors of $z^{n\varepsilon}$ have been inserted to make the coefficients $W_{1\cdots n}$ independent of z for any value of ε. To see this, we recall from Appendix 3.9 that the combination $z^n F_n(z; y_n, \cdots, y_1)$ is a homogeneous polynomial in $z, y_1, \cdots, y_n$ of total degree $2n$, while the correlator of n stress tensors is homogeneous in $y_1, \cdots, y_n$ of total degree $n(-2 + \varepsilon)$. Therefore $W_{1\cdots n}$ is homogeneous in $z, y_1, \cdots, y_n$ of total degree 0 and we may set $z = 1$ in the evaluation of $W_{1\cdots n}$ in (3.5.19) so that all z-dependence of $W_\varepsilon(z)$ resides in the coefficients $z^{n\varepsilon}$ in (3.5.16). The

expansion of $\langle T_\infty W_\varepsilon(z)\rangle$ proceeds analogously by replacing the correlator $\langle T(y_n)\cdots T(y_1)\rangle$ with $\langle T(x)|x|^{4-2\varepsilon}T(y_n)\cdots T(y_1)\rangle$ and then taking the $x\to\infty$ limit.

The coefficients $W_{1\cdots n}$ may be decomposed into a sum over inequivalent partitions P of the set of n points $\{y_1,\cdots,y_n\}$ by decomposing the correlator of n stress tensors in (3.5.19) into a sum over P using (3.5.10),

$$W_{1\cdots n} = \sum_P W_P, \qquad W_P = \frac{6^n}{c^n}\int_0^1 dy_n\cdots\int_0^{y_2} dy_1 F_n(1;y_n,\cdots,y_1)\langle T^n\rangle_P. \quad (3.5.20)$$

The expression for W_P may be simplified using the scaling and translation properties of $\langle T^n\rangle_P$ and the polynomial nature of the function $F_n(1;y_n,\cdots,y_1)$ to resolve the nested ordering of the integrals. We change variables from $(y_n,\cdots,y_1)$ to $(x_n,u,\alpha_{n-1},\cdots,\alpha_1)$,

$$\begin{aligned}
y_k &= x_n - u\alpha_{n-1} - u\alpha_{n-2}\cdots - u\alpha_k & 1\le k\le n-1,\\
y_n &= x_n & (3.5.21)
\end{aligned}$$

subject to $0\le u\le x_n\le 1$ and $0\le\alpha_i$ as well as $\alpha_{n-1}+\cdots\alpha_1 = 1$. Using the observation that the integration range of the variables u,x_n is independent of the integration range of the variables α_i, we rearrange the integrations as follows,

$$W_P = \frac{6^n}{2^{p_2}c^{n-p}}\int_0^1 d\alpha_{n-1}\cdots\int_0^1 d\alpha_1\,\delta\left(1-\sum_{k=1}^{n-1}\alpha_k\right)\frac{\mathcal{N}_n(\alpha_1,\cdots,\alpha_{n-1})}{\mathcal{D}_P(\alpha_1,\cdots,\alpha_{n-1})}. \quad (3.5.22)$$

The function $\mathcal{D}_P$ is given in terms of the contribution to the stress tensor correlator arising from the partition P and is given explicitly by,

$$\langle T^n\rangle_P = \frac{c^p}{2^{p_2}}\frac{u^{-2n+n\varepsilon}}{\mathcal{D}_P(\alpha_1,\cdots,\alpha_{n-1})} \quad (3.5.23)$$

where p is the total number of cycles in P and p_2 is the number of 2-cycles in P. The function $\mathcal{N}_n$ is defined as follows,

$$\mathcal{N}_n(\alpha_1,\cdots,\alpha_{n-1}) = \int_0^1 du\,u^{-n-2+n\varepsilon}\int_u^1 dx_n\,F_n(1;y_n,\cdots,y_1) \quad (3.5.24)$$

where $y_1,\cdots y_n$ are given in terms of $x_n,u,\alpha_1,\cdots\alpha_{n-1}$ by (3.5.21). Since $F_n(1;y_n,\cdots,y_1)$ is polynomial in y_i, the integral $\mathcal{N}_n$ is polynomial in α_i as well, with coefficients which are rational functions of ε. Finally, one of the α_k-integrals in (3.5.22) may be carried out by satisfying the δ-function, so that the number of non-trivial integrals left over is $n-2$.

3.5.4 Evaluation of $W_{1\cdots n}$

The details of the calculation of the functions W_P and their sum $W_{1\cdots n}$ are presented in Appendix 3.10. They include the list of the denominator functions $\mathcal{D}_P$ and the evaluations of some of the integrals over the parameters α_i, but we do not give the functions F_n or $\mathcal{N}_n$ whose length grows rapidly with n and were handled by MAPLE. The result may be summarized as follows. The contribution W_{12} is of order $1/c$ and is required up to order ε^2, the contribution W_{123} is of order $1/c^2$ while $W_{1234} = W_{1234}^{(2)} + W_{1234}^{(3)}$ has contributions of order $1/c^2$ and $1/c^3$ and both are required to order ε^0,

$$
\begin{aligned}
c\,W_{12} &= \frac{6j(j+1)}{\varepsilon} + j(10j+4) + \frac{j}{3}(74j+98)\varepsilon + \frac{j}{9}(418j+196)\varepsilon^2, \\
c^2\,W_{123} &= -\frac{96j(j+1)}{\varepsilon^2} + \frac{24j}{\varepsilon}(2j^2-9j-5) + 16\pi^2 j(j+1) + 6j(18j^2-143j-203), \\
c^2\,W_{1234}^{(2)} &= \frac{18}{\varepsilon^2}j(j+1)(j^2+j+2) + \frac{3}{\varepsilon}j(20j^3+16j^2+49j+29) \\
&\quad + 2j(99j^3+132j^2+436j+460) - 24j(j+1)\pi^2, \\
c^3\,W_{1234}^{(3)} &= \frac{1296}{\varepsilon^3}j(j+1) + \frac{648}{\varepsilon^2}j(-2j^2+5j+3) + \frac{216}{\varepsilon}j(2j^3-11j^2+89j+132) \\
&\quad + \frac{72}{5}j(4j^3+8j^2-39j-43)\pi^2.
\end{aligned}
\tag{3.5.25}
$$

Finally, the contributions W_{12345} and W_{123456} are required to order $1/\varepsilon$ and to order $1/c^3$, for the calculation of the dimension $h(j,c)$ to order $1/c^3$,

$$
\begin{aligned}
c^3\,W_{12345} &= -\frac{576}{5\varepsilon^3}j(j+1)(5j^2+5j+11) + \frac{96j}{5\varepsilon}j(1+j)(5j^2+5j+79)\pi^2 \\
&\quad + \frac{48j}{5\varepsilon^2}(30j^4-205j^3-152j^2-634j-387) \\
&\quad + \frac{4j}{5\varepsilon}(1410j^4-12341j^3-18640j^2-58776j-62077), \\
c^3\,W_{123456} &= \frac{36j}{\varepsilon^3}(j+1)(j^2+j+2)(j^2+j+4) - \frac{48j}{\varepsilon}(j+1)(3j^2+3j+13)\pi^2 \\
&\quad + \frac{6j}{\varepsilon^2}(30j^5+36j^4+201j^3+210j^2+361j+202) \\
&\quad + \frac{2j}{\varepsilon}(372j^5+468j^4+3873j^3+5967j^2+10100j+8684).
\end{aligned}
\tag{3.5.26}
$$

The calculation of $\langle T_\infty W_\varepsilon(z)\rangle$ is analogous. The results are given in the Appendix 3.12.3.

3.5.5 Renormalization of $W_\varepsilon(z)$ and $T_\infty W_\varepsilon(z)$ to order $1/c^3$

To order $1/c^3$, the regularized matrix element $\langle W_\varepsilon(z)\rangle$ of the Wilson line operator is given by (3.5.18), (3.5.25), and (3.5.26), as well as by the parameters N and α. We seek to determine N and α by requiring that $\langle W_\varepsilon(z)\rangle$ obey as renormalization conditions the scaling relation (3.5.13) to order $1/c^3$. By inspecting the expansion of $\langle W_\varepsilon(z)\rangle$ in terms of the coefficients $W_{1\cdots n}$ it is far from obvious that such a scaling relation can indeed be secured. However, once it has been, the parameter N is trivially fixed as follows,

$$\langle W_\varepsilon(1)\rangle = 1. \tag{3.5.27}$$

This leaves the parameter α at our disposal to enforce the scaling relation (3.5.13) by requiring that the function $\ln\langle W_\varepsilon(z)\rangle$ be linear in $\ln(z)$,

$$\ln\langle W_\varepsilon(z)\rangle = -2h(j,c)\ln z + \mathcal{O}(\varepsilon) \tag{3.5.28}$$

where $h(j,c)$ is to be determined in the process. By inspecting the relation between the order of expansion in powers of $1/c$ and the order of the pole in ε, we find that for order $1/c^m$ the maximal order is $1/\varepsilon^m$, thereby producing a polynomial in $\ln(z)$ of degree m in $\ln\langle W_\varepsilon(z)\rangle$, up to corrections of order $\mathcal{O}(\varepsilon)$. Therefore, to order $1/c$, the scaling condition is automatic, while to orders $1/c^2$ and $1/c^3$ the scaling condition imposes respectively two and three conditions. These conditions are satisfied by a function α given as follows,

$$\alpha = 1 + \frac{1}{c}\left(\frac{6}{\varepsilon} + 3 + \varepsilon a_1\right) + \frac{1}{c^2}\left(\frac{30}{\varepsilon^2} + \frac{55}{\varepsilon} + a_2 + \varepsilon a_3\right) + \mathcal{O}(c^{-3},\varepsilon^2). \tag{3.5.29}$$

The contributions proportional to a_1, a_2 and a_3 are not determined by the renormalization scaling conditions, and neither are higher order terms in $1/c$ or ε to this order in the expansion. The scaling dimension resulting from the renormalization of W is given by,

$$h_W(j,c) = -j - j(j+1)\left(\frac{6}{c} + \frac{78}{c^2} + \frac{60a_2 - 360a_1 + 2450 + 192\pi^2(3j^2 + 3j - 1)}{5c^3}\right) \tag{3.5.30}$$

up to contributions of order $1/c^4$ and ε. While the result for $h_W(j,c)$ to the orders $1/c$ and $1/c^2$ are uniquely determined by the renormalization procedure and precisely agree with the

predictions of $SL(2, \mathbb{R})$ current algebra in (3.5.1), the order $1/c^3$ is determined only once the particular combination $a_2 - 6a_1$ of the coefficients a_1 and a_2 is known.

The missing information may be obtained from the renormalization of the matrix element $T_\infty W_\varepsilon(z)$. Its detailed calculation is given in the Appendix. Using the same renormalization parameters N and α as we used for $W_\varepsilon(z)$, the prediction of the scaling dimension derived from $\langle T_\infty W_\varepsilon(z) \rangle$ is obtained via (3.5.15) and is given by,

$$h_{TW}(j, c) = -j - \frac{6j(j+1)}{c} - \frac{j}{c^2}\left(78j + \frac{49}{3} + \frac{16}{5}\pi^2\big(3j(j+1) - 1\big) - 6a_1 + a_2\right). \quad (3.5.31)$$

Matching the orders in $1/c^2$ gives the following result for the combination,

$$a_2 - 6a_1 = \frac{185}{3} - \frac{16\pi^2}{5}\big(3j(j+1) - 1\big) \quad (3.5.32)$$

which upon substitution in the $1/c^3$ term of $h_W(j, c)$ leads to perfect agreement with the predictions of (3.5.1) to order $1/c^3$.

We note that renormalization of the gravitational Wilson line matrix elements consistent with the conformal Ward identities has forced us to make the vertex renormalization parameter $\alpha(\varepsilon)$ dependent on j in the order $1/c^3$ contribution to the Wilson line, and to order $1/c^2$ in $\alpha(\varepsilon)$. This j-dependence of $\alpha(\varepsilon)$ is a new phenomenon that was absent at lower orders in $1/c$, and raises two issues. First, in terms of renormalization theory, it suggests that the gravitational Wilson line operator as originally defined cannot be renormalized at the operator level, since a dependence on the states governing its matrix elements enters. A slight modification of the original definition of the Wilson line can remedy this obstacle by promoting $\alpha(\varepsilon)$ itself to an operator which involves the quadratic Casimir of $SL(2, \mathbb{R})$. Second, to satisfy (3.5.32), we actually have a choice: setting $a_1 = 0$ we require a j-dependent renormalization at order $1/c^2$, while setting $a_2 = 0$ we can get away with a renormalization at order $1/c$ of an evanescent operator which, given its proportionality to ε, would vanish at the classical level as $\varepsilon \to 0$. The role of such evanescent operators remains to be understood.

3.6 Regularization scheme in two dimensions

Instead of "changing the theory" by extending the free field model for a conformal field theory with central charge c from two dimensions to $d = 2 - \varepsilon$ dimensions, we shall attempt in this section to keep conformal invariance intact in $d = 2$, and regularize and renormalize the operator W in this exactly conformal theory. As we shall show below, for the particular though natural regulator we choose, this attempt will ultimately fail.

3.6.1 A two-dimensional regulator for the Wilson line

We introduce a regulator, order by order in the $1/c$ expansion of the matrix elements of the Wilson line operator, in which the correlator of stress tensors $\langle T^n \rangle_{1 \cdots n}$ is regularized by,

$$\langle T^n \rangle_{1 \cdots n} = \langle T(y_1) \cdots T(y_n) \rangle \prod_{1 \leq i < j \leq n} |y_j - y_i|^\varepsilon \tag{3.6.1}$$

and the correlator $\langle T(y_1) \cdots T(y_n) \rangle$ is evaluated using the OPE for the stress tensor of (3.5.5) of a conformal field theory with central charge c, valid strictly in two dimensions. We have chosen the regulator to be symmetric under permutations of the points $y_1, \cdots, y_n$ just as the stress tensor correlator is, to be invariant under translations of the variables y_i, and to have good scaling behavior similar to, but different from, dimensional regularization. In the $\tilde{\alpha}/c$ expansion, and with the regularization defined above, the Wilson line correlator may be presented as a sum over contributions with a definite number of T-insertions,[3]

$$\langle \tilde{W}_\varepsilon(z) \rangle = z^{2j} \tilde{N} \sum_{n=0}^{\infty} \tilde{\alpha}^n \, z^{\frac{1}{2} n(n-1)\varepsilon} \, \tilde{W}_{1 \cdots n} \ . \tag{3.6.2}$$

The factors of $z^{\frac{1}{2} n(n-1)\varepsilon}$ have been extracted in order to make the coefficients $\tilde{W}_{1 \cdots n}$ independent of z, using arguments analogous to the ones used for $W_\varepsilon(z)$. The decomposition of the correlator into a sum over contributions arising from inequivalent partition cycles P proceeds as with dimensional regularization, and we have,

$$\tilde{W}_{1 \cdots n} = \sum_P \tilde{W}_P, \qquad \tilde{W}_P = \frac{6^n}{c^n} \int_0^1 dy_n \cdots \int_0^{y_2} dy_1 F_n(1; y_n, \cdots, y_1) \langle T^n \rangle_P \tag{3.6.3}$$

---

[3]Throughout this section, we shall use a tilde for the quantities defined with the regulator of (3.6.1) in order to distinguish them from those defined in the preceding section with dimensional regularization.

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where $\langle T^n \rangle_P$ is defined by (3.6.1) for the partition P.

Using the change of variables (3.5.21) we recast the expression for $\tilde{W}_P$ as follows,

$$\tilde{W}_P = \frac{6^n}{2^{p_2} c^{n-p}} \int_0^1 d\alpha_{n-1} \cdots \int_0^1 d\alpha_1 \, \delta \left(1 - \sum_{k=1}^{n-1} \alpha_k \right) \frac{\tilde{\mathcal{N}}_n(\alpha_1, \cdots, \alpha_{n-1})}{\tilde{\mathcal{D}}_P(\alpha_1, \cdots, \alpha_{n-1})} \qquad (3.6.4)$$

where $\tilde{\mathcal{D}}_P$ and $\tilde{\mathcal{N}}_P$ are defined by,

$$\langle T^n \rangle_P = \frac{c^p}{2^{p_2}} \frac{u^{-2n+\frac{1}{2}n(n-1)\varepsilon}}{\tilde{\mathcal{D}}_P(\alpha_1, \cdots, \alpha_{n-1})} \,,$$

$$\tilde{\mathcal{N}}_n(\alpha_1, \cdots, \alpha_{n-1}) = \int_0^1 du \, u^{-n-2+\frac{1}{2}n(n-1)\varepsilon} \int_u^1 dx_n \, F_n(1; y_n, y_{n-1}, \cdots, y_1) \quad (3.6.5)$$

with $\langle T^n \rangle_P$ given in (3.6.1), p and p_2 are respectively the total number of cycles and the number of two-cycles in P.

3.6.2 Calculation of the coefficients $\tilde{W}_{12}$, $\tilde{W}_{123}$ and $\tilde{W}_{1234}$

The coefficient $\tilde{W}_{12}$ coincides with the coefficient W_{12} computed in dimensional regularization after letting $2\varepsilon \to \varepsilon$, while $\tilde{W}_{123} = W_{123}$, and are given by,

$$c\, W_{12} = \frac{12j(j+1)}{\varepsilon} + j(10j+4) + \frac{j}{6}(74j+98)\varepsilon + \mathcal{O}(\varepsilon^2),$$

$$c^2\, W_{123} = -\frac{96j(j+1)}{\varepsilon^2} + \frac{24j}{\varepsilon}(2j^2 - 9j - 5) + \mathcal{O}(\varepsilon^0). \qquad (3.6.6)$$

To order $1/c^2$, the coefficient $\tilde{W}_{1234}$ receives contributions from the partitions [12][34], [13][24] and [14][23], whose denominator functions are given by,

$$\tilde{\mathcal{D}}_{[12][34]} = \alpha_1^{4-\varepsilon} \alpha_2^{-\varepsilon} \alpha_3^{4-\varepsilon} (\alpha_1 + \alpha_2)^{-\varepsilon} (\alpha_2 + \alpha_3)^{-\varepsilon},$$

$$\tilde{\mathcal{D}}_{[13][24]} = \alpha_1^{-\varepsilon} \alpha_2^{-\varepsilon} \alpha_3^{-\varepsilon} (\alpha_1 + \alpha_2)^{4-\varepsilon} (\alpha_2 + \alpha_3)^{4-\varepsilon},$$

$$\tilde{\mathcal{D}}_{[14][23]} = \alpha_1^{-\varepsilon} \alpha_2^{4-\varepsilon} \alpha_3^{-\varepsilon} (\alpha_1 + \alpha_2)^{-\varepsilon} (\alpha_2 + \alpha_3)^{-\varepsilon}. \qquad (3.6.7)$$

The function $\tilde{\mathcal{N}}_4(\alpha_1, \alpha_2, \alpha_3)$ is a polynomial in its variables, with coefficients which are rational functions of ε with simple poles. We satisfy the δ-function constraint by solving for

$\alpha_2 = 1 - \alpha_1 - \alpha_3$, and decompose the polynomial $\tilde{\mathcal{N}}_4$ in the following, equivalent ways,

$$
\begin{aligned}
\tilde{\mathcal{N}}_4(\alpha_1, 1 - \alpha_1 - \alpha_3, \alpha_3) &= \sum_{A,B=0}^{2} \mathcal{M}_{AB}^{(1)} \alpha_1^A \alpha_3^B = \sum_{A,B=0}^{2} \mathcal{M}_{AB}^{(2)} (1 - \alpha_1)^A (1 - \alpha_3)^B \\
&= \sum_{A=0}^{4} \sum_{B=0}^{2} \mathcal{M}_{AB}^{(3)} (1 - \alpha_1)^A (1 - \alpha_1 - \alpha_3)^B.
\end{aligned}
\tag{3.6.8}
$$

The expansion reduces the integrals to sums over basic families of integrals $\mathcal{Q}_\varepsilon^{(i)}$ for $i = 1, 2, 3$ given and evaluated in Appendix 3.12.4,

$$
\begin{aligned}
c^2 \tilde{W}_{[12][34]} &= \frac{6^4}{4} \sum_{A,B=0}^{2} \mathcal{M}_{AB}^{(1)} \mathcal{Q}_\varepsilon^{(1)}(A - 3, B - 3), \\
c^2 \tilde{W}_{[13][24]} &= \frac{6^4}{4} \sum_{A,B=0}^{2} \mathcal{M}_{AB}^{(2)} \mathcal{Q}_\varepsilon^{(2)}(A - 3, B - 3), \\
c^2 \tilde{W}_{[14][23]} &= \frac{6^4}{4} \sum_{A=0}^{4} \sum_{B=0}^{2} \mathcal{M}_{AB}^{(3)} \mathcal{Q}_\varepsilon^{(3)}(A + 1, B - 3).
\end{aligned}
\tag{3.6.9}
$$

The results are as follows,

$$
\begin{aligned}
c^2 \tilde{W}_{[12][34]} &= \frac{56}{\varepsilon^2} j^2(j+1)^2 + \frac{2}{15\varepsilon} j(j+1)(776j^2 - 1924j + 273), \\
c^2 \tilde{W}_{[13][24]} &= -\frac{16}{\varepsilon^2} j(j+1)(j^2 + j - 1) - \frac{2}{15\varepsilon} j(466j^3 + 1292j^2 - 21j - 487), \\
c^2 \tilde{W}_{[14][23]} &= \frac{8}{\varepsilon^2} j^2(j+1)^2 + \frac{4}{3\varepsilon} j(j+1)(29j^2 + 119j - 69)
\end{aligned}
\tag{3.6.10}
$$

giving a combined contribution of

$$
c^2 \tilde{W}_{1234} = \frac{16}{\varepsilon^2} j(j+1)(3j^2 + 3j + 1) + \frac{4j}{3\varepsilon}(60j^3 - 96j^2 - 113j + 7).
\tag{3.6.11}
$$

Expanding the parameter $\tilde{\alpha}$ in powers on $1/c$,

$$
\tilde{\alpha}(\varepsilon) = 1 + \frac{1}{c}\left(\frac{\tilde{a}_1}{\varepsilon} + \tilde{a}_2\right) + \mathcal{O}(c^{-2}),
\tag{3.6.12}
$$

setting $\langle \tilde{W}_\varepsilon(1) \rangle = 1$ and collecting all remaining contributions, we find,

$$
\begin{aligned}
\ln\langle \tilde{W}_\varepsilon(z) \rangle &= 2j \ln z + \frac{12}{c} j(j+1) \ln z + \frac{24}{c^2 \varepsilon} j(j+1)(6j^2 + 6j - 8 + \tilde{a}_1) \ln z \\
&\quad + \frac{4j}{c^2}(60j^3 - 240j^2 - 412j - 76 + 5j\tilde{a}_1 + 6j\tilde{a}_2 + 2\tilde{a}_1 + 6\tilde{a}_2) \ln z \\
&\quad + \frac{12}{c^2} j(j+1)(60j^2 + 60j - 12 + \tilde{a}_1)(\ln z)^2 + O(\varepsilon).
\end{aligned}
\tag{3.6.13}
$$

To obtain a finite result, we must cancel the pole in ε and thus set $\tilde{a}_1 = 8 - 6j(j+1)$. Having done so, the value of the coefficient of $(\ln z)^2$ becomes $24j(j+1)(27j^2 + 27j - 2)$ and no further adjustment of $\tilde{N}$ or $\tilde{\alpha}$ is available to cancel this obstruction to the scaling behavior of (3.5.13) for $\langle \tilde{W}_\varepsilon(z) \rangle$.

3.7   Discussion and Outlook

The main result of the paper is the computation of the expectation value of the gravitational Wilson line to order $1/c^3$. To deal with the short-distance singularities which arise in the integrations over stress tensor correlators, we have used a version of dimensional regularization to dimension $d = 2 - \varepsilon$ combined with a non-trivial analytic continuation in ε, and effectively treated the stress tensor as having dimension $d = 2 - \varepsilon$. Renormalization of the gravitational Wilson line matrix elements consistent with the conformal Ward identities was found to require, to order $1/c^3$ included, an overall multiplicative factor $N(\varepsilon)$ and a "vertex renormalization" factor $\alpha(\varepsilon)$. The multiplicative factor $N(\varepsilon)$ depends on ε and j in an expansion in powers of $1/c$. The vertex renormalization $\alpha(\varepsilon)$ is independent of j to orders $1/c$ and $1/c^2$ but requires dependence on j through its Casimir value $j(j+1)$ to order $1/c^3$. This result suggests that, to sufficiently high order in $1/c$, the renormalization of the Wilson line operator depends on the matrix element considered. Deepening the understanding of this dependence is left for future work.

From a purely diagrammatic point of view, the emergence of a bi-local conformal primary operator from the gravitational Wilson line matrix elements appears to be based on the magic of remarkable relations between contribution at different orders in $1/c$. For example, a simple fact about the anomalous dimension (3.1.4) is that it depends on j only through the $SL(2, \mathbb{R})$ Casimir eigenvalue $j(j+1)$. Yet each diagram by itself does produce higher powers of j which do not form a polynomial in $j(j+1)$. No regularization scheme appears to be known in which each contribution is polynomial in $j(j+1)$.

As a simpler example, we have computed the expectation values of Wilson line operators of holomorphic currents appearing in theories with level k current algebra symmetry to order

$1/k^3$. The computations are relatively simpler in this case but still retain a lot of the features of the gravitational case. We have also performed a more standard field theoretic perturbative calculation of the expectation value of a Wilson line for non-holomorphic currents using the WZW model to order $1/k^2$. The results of the two approaches are consistent; however the connection between the two calculations remains to be fully elucidated.

A promising approach towards a more geometrical understanding of the bi-local and conformal primary nature of gravitational Wilson lines is via Hamiltonian reduction, which produces Virasoro symmetry from $SL(2, \mathbb{R})$ current algebra symmetry (see [14] for details). The constraints we need to impose on the $SL(2, \mathbb{R})$ currents $J^a(z)$ are given by $J^-(z) = k$ and $J^0(z) = 0$. Under these constraints, the current algebra Wilson line reduces to the gravitational Wilson line (with central charge $c = 6k$)

$$J^a T^a \longrightarrow L_1 + \frac{6}{c} L_{-1} \ . \tag{3.7.1}$$

Further, it was shown in [3], that the geometric action can be obtained from the chiral WZW action by the same reduction. The geometric action is written in terms of the function $f(z)$ appearing in (3.2.11) and (3.2.12), and is the right object to compute stress tensor correlators. Note that the same reduction is done in the bulk Chern-Simons theory when we impose asymptotically AdS boundary conditions. As a consequence, at least formally, the expectation value of the gravitational Wilson line can be obtained by reduction of the $SL(2, \mathbb{R})$ current algebra Wilson line

$$\int \mathcal{D}g \, e^{-\mathrm{S_{WZW}}[g]} \, W(z) \longrightarrow \int \mathcal{D}f \, e^{-\mathrm{S_G}[f]} \, W(z) \ . \tag{3.7.2}$$

All this suggests that understanding the current algebra Wilson line might be sufficient to understand the gravitational case. However, the transformation from g to f in (3.7.2) remains formal. Addressing the subtle issues of regularization and renormalization of the transformation, and the emergence of conformal symmetry, are left for future work as well.

Recently, the connection between the geometric action and AdS_3 gravity was carefully studied in [27]. The authors used the geometric action and certain bi-local operators to calculate various quantities, such as the sphere and torus partition functions and corrections

to Virasoro blocks. The bi-local operators used in [27] are simply the Wilson line operators we consider (compare equation 6.9 there with (3.2.12) here). It would be interesting to see if their methods could be used to understand our problem better.

The advantage of our regulator over, for example, the Pauli-Villars type regulator used in [16] is that it is dimensionless. This greatly constrains the form of the divergences and allows a simple prescription to subtract divergences. Another natural dimensionless regulator was considered in section 3.6. Surprisingly, we found that it is not possible to restore conformal invariance in this case, as we take the regulator away. Understanding why dimensional regularization is superior might shed some light onto the renormalization problem.

By computing the Wilson line anchored on the boundary, we are computing the boundary to boundary scalar two point function in AdS_3 with graviton loop corrections (up to 3 loops). A conventional calculation would be quite complicated as we would have to use the bulk to bulk graviton propagator and involves integrating vertices over all of AdS. The Wilson line calculation is manifestly holomorphically factorized and needs only one integration per vertex. This is much simpler. It would be interesting to see if we could reduce the standard Witten diagram computation to the Wilson line one.

Ultimately, we are interested in finding a formalism that allows us to exploit Virasoro symmetry to understand non-perturbative gravity corrections in AdS_3. We believe that understanding the renormalized Wilson line better is a step towards this direction.

Appendices

3.8 $SL(2, \mathbb{R})$ representations as functions on unit disk

In the bulk of this paper we based the Wilson line on finite dimensional spin j representations of $SL(2, \mathbb{R})$. These representations are convenient to work with, but since they are non-unitary one must analytically continue in j at the end of any computation to obtain result valid for unitary representations. Alternatively, one can work directly with unitary

representations, for example by realizing $SL(2, \mathbb{R})$ in terms of functions of the complex variable u defined on the unit disk $D = \{u \in \mathbb{C}, |u| < 1\}$. We write

$$L_1 = \partial_u , \quad L_0 = u\partial_u + h , \quad L_{-1} = u^2 \partial_u + 2hu .\tag{3.8.1}$$

The $SL(2, \mathbb{R})$-invariant inner product between functions $f(u)$ and $g(u)$ is defined as an integral over the unit disk D,

$$\langle f|g \rangle = \int_D \frac{d^2 u}{(1 - u\bar{u})^{2-2h}} \overline{f(u)} g(u) .\tag{3.8.2}$$

This is defined to respect the relations $L_n^\dagger = L_{-n}$ between adjoint operators.

As reviewed in section 3.2, the Wilson line was built on $SL(2, \mathbb{R})$ states obeying

$$L_{-1}|h; in\rangle = 0 , \quad L_0|h; in\rangle = -h|h; in\rangle$$
$$L_1|h; out\rangle = 0 , \quad L_0|h; out\rangle = h|h; out\rangle ,\tag{3.8.3}$$

These states therefore correspond to the functions

$$|h_{in}\rangle \to u^{-2h} , \quad |h_{out}\rangle \to 1 .\tag{3.8.4}$$

The Wilson line is then given by

$$W[z_2, z_1] = \langle h; out| \exp\left\{ \int_{z_1}^{z_2} dz(L_1 + \frac{6}{c}T(z)L_{-1}) \right\} |h; in\rangle .\tag{3.8.5}$$

It is easy to see that order-by-order in $1/c$ this gives the same result as working with spin j representations and then setting $j = -h$ at the end.

3.9 $SL(2, \mathbb{R})$ matrix elements

In this appendix, we derive a recursion relation for the $SL(2, \mathbb{R})$ group theory factors which enter into the calculation of the large c expansion of matrix elements of the gravitational Wilson line operator. The factors of interest are the functions F_n defined by,

$$z^{2j} F_n(z; y_n, \cdots, y_1) = \langle j, -j|e^{zL_1} X(y_n) \cdots X(y_1)|j, j\rangle\tag{3.9.1}$$

where $X(y) = L_{-1} - 2yL_0 + y^2 L_1$. Furthermore, $|j, j\rangle$ denotes the highest weight state of a representation of $SL(2, \mathbb{R})$ with finite dimension $2j + 1 \in \mathbb{N}$ and thus satisfies $L_{-1}|j, j\rangle = 0$. Choosing unit norm for $|j, j\rangle$ sets $F_0(z) = 1$. To obtain a recursion relation for the matrix elements F_n we recursively define the states $\mathfrak{S}_n$ by,

$$\mathfrak{S}_n(y_n, \cdots, y_1) = X(y_n)\,\mathfrak{S}_{n-1}(y_{n-1}, \cdots, y_1), \qquad\qquad \mathfrak{S}_0 = |j, j\rangle, \qquad (3.9.2)$$

or equivalently $\mathfrak{S}_n(y_n, \cdots, y_1) = X(y_n) \cdots X(y_1)|j, j\rangle$. Commuting the operators L_{-1} and L_0 in each X-factor to the right and evaluating the result on $|j, j\rangle$ shows that $\mathfrak{S}_n$ is a linear combination of states $L_1^k|j, j\rangle$ with coefficients $S_n^{(k)}$,

$$\mathfrak{S}_n(y_n, \cdots, y_1) = \sum_{k=0}^{n} S_n^{(k)}(y_n, \cdots, y_1)L_1^k\,|j, j\rangle. \qquad (3.9.3)$$

Implementing the recursion relations on the states $\mathfrak{S}_n$ given by (3.9.2) produces the following recursion relations on the coefficients $S_n^{(k)}$,

$$\sum_{k=0}^{n+1} S_{n+1}^{(k)}L_1^k\,|j, j\rangle = \sum_{\ell=0}^{n} S_n^{(\ell)}\Big(\ell(\ell - 2j - 1)L_1^{\ell-1} - 2y_{n+1}(j - \ell)L_1^{\ell} + y_{n+1}^2 L_1^{\ell+1}\Big)|j, j\rangle. \quad (3.9.4)$$

Assuming that j is large enough, namely for $n + 1 < 2j$, the states $L_1^k|j, j\rangle$ for $0 \leq k \leq n+1$ will all be linearly independent. Identifying their coefficients on both sides gives the following recursion relations for $0 \leq k \leq n + 1$,

$$S_{n+1}^{(k)} = y_{n+1}^2\,S_n^{(k-1)} - 2(j - k)y_{n+1}S_n^{(k)} + (k + 1)(k - 2j)S_n^{(k+1)} \qquad (3.9.5)$$

where $S_0^{(0)} = 1$ and we set $S_n^{(k)} = 0$ whenever $k < 0$ or $k > n$. The truncations $S_n^{(k)} = 0$ which arise for $n \geq k > 2j$, follow automatically from the recursion relations for j. Finally, we derive the formula for F_n in terms of $S_n^{(k)}$ by using the matrix elements $\langle j, -j|e^{zL_1}|j, j\rangle = z^{2j}$ and their z-derivatives,

$$F_n(z; y_n, \cdots, y_1) = \sum_{k=0}^{n} \frac{\Gamma(2j + 1)z^{-k}}{\Gamma(2j + 1 - k)}S_n^{(k)}(y_n, \cdots, y_1). \qquad (3.9.6)$$

By construction, the combination $z^n F_n(z; y_n, \ldots, y_1)$ is a homogeneous polynomial in the variables $z, y_1, \ldots, y_n$ of combined degree $2n$.

3.10 Gravitational Wilson line computations

In this appendix we discuss the calculations of the coefficients $W_{1\cdots n}$ and W_P required to evaluate $W_\varepsilon(z)$ in (3.5.22). The numerator functions $\mathcal{N}_n$ are given by (3.5.24) in terms of the functions F_n computed in Appendix 3.9. They are polynomials in $\alpha_1, \cdots \alpha_{n-1}$ with coefficients which have simple poles in ε. Their expressions rapidly become lengthy as n increases, and were handled by MAPLE. The denominator functions $\mathcal{D}_P$ will be listed below.

3.10.1 Computation of W_{12} and W_{123}

The denominator functions for $n = 2, 3$ are given as follows,

$$\mathcal{D}_{12} = 1, \qquad\qquad \mathcal{D}_{123} = (1 - \alpha_2)^{2-\varepsilon}\alpha_2^{2-\varepsilon}. \qquad\qquad (3.10.1)$$

The integration over α_1 may be carried out by using the δ-function, and we have,

$$W_{12} = \frac{6^2}{2c}\mathcal{N}_2(1), \qquad\qquad W_{123} = \frac{6^3}{c^2}\int_0^1 d\alpha_2\, \frac{\mathcal{N}_3(1-\alpha_2, \alpha_2)}{\alpha_2^{2-\varepsilon}(1-\alpha_2)^{2-\varepsilon}} \qquad (3.10.2)$$

which leads to the results on the first two lines of (3.5.25). Since $\mathcal{N}_3(1-\alpha_2, \alpha_2)$ is polynomial in α_2, the only integrals required to evaluate W_{123} are of the Euler type given in (3.12.1).

3.10.2 Calculation of W_{1234}

For $n = 4$ the different partitions give the following denominator functions,

$$\mathcal{D}_{[12][34]} = \alpha_1^{4-2\varepsilon}\alpha_3^{4-2\varepsilon} \qquad\qquad\qquad \alpha_2$$

$$\mathcal{D}_{[13][24]} = (\alpha_1 + \alpha_2)^{4-2\varepsilon}(\alpha_2 + \alpha_3)^{4-2\varepsilon} \qquad\qquad \alpha_2$$

$$\mathcal{D}_{[14][23]} = \alpha_2^{4-2\varepsilon} \qquad\qquad\qquad \alpha_3$$

$$\mathcal{D}_{[1234]} = \alpha_1^{2-\varepsilon}\alpha_2^{2-\varepsilon}\alpha_3^{2-\varepsilon} \qquad\qquad\qquad \alpha_3$$

$$\mathcal{D}_{[1324]} = (\alpha_1 + \alpha_2)^{2-\varepsilon}\alpha_2^{2-\varepsilon}(\alpha_2 + \alpha_3)^{2-\varepsilon} \qquad\qquad \alpha_2$$

$$\mathcal{D}_{[1342]} = \alpha_1^{2-\varepsilon}(\alpha_1 + \alpha_2)^{2-\varepsilon}(\alpha_2 + \alpha_3)^{2-\varepsilon}\alpha_3^{2-\varepsilon} \qquad \alpha_2 \qquad (3.10.3)$$

where the right column lists a convenient choice of variable to be eliminated with the help of the δ-function. Since $\mathcal{N}_4$ is polynomial in $\alpha_1, \alpha_2, \alpha_3$, the integrals required to evaluate

$W_{[12][34]}, W_{[13][24]}$ and $W_{[14][23]}$ are of the Euler beta function type of (3.12.1). They may be readily evaluated and produce the results on the third line of (3.5.25).

The evaluation of $W_{[1234]}$ proceeds analogously. For $W_{[1324]}$, however, a set of non-standard integrals is required. They are denoted by $\mathcal{K}_\varepsilon(a, b, c)$ and are calculated in Appendix 3.12.2. Similarly, for $W_{[1342]}$ another set of non-standard integrals is required which are denoted $\mathcal{J}_\varepsilon(a, b)$ and evaluated in Appendix 3.12.1. Here and below, the nature of these non-standard integrals is dictated by the structure of the denominator functions.

3.10.3   Calculation of W_{12345}

For $n = 5$ the denominator functions are given by,

$$\mathcal{D}_{[12][345]} = \alpha_1^{4-2\varepsilon}\alpha_3^{2-\varepsilon}(\alpha_3 + \alpha_4)^{2-\varepsilon}\alpha_4^{2-\varepsilon} \qquad\qquad \alpha_2$$

$$\mathcal{D}_{[13][245]} = (\alpha_1 + \alpha_2)^{4-2\varepsilon}(\alpha_2 + \alpha_3)^{2-\varepsilon}(\alpha_2 + \alpha_3 + \alpha_4)^{2-\varepsilon}\alpha_4^{2-\varepsilon} \qquad\qquad \alpha_3$$

$$\mathcal{D}_{[14][235]} = (\alpha_1 + \alpha_2 + \alpha_3)^{4-2\varepsilon}\alpha_2^{2-\varepsilon}(\alpha_2 + \alpha_3 + \alpha_4)^{2-\varepsilon}(\alpha_3 + \alpha_4)^{2-\varepsilon} \qquad\qquad \alpha_3$$

$$\mathcal{D}_{[15][234]} = \alpha_2^{2-\varepsilon}\alpha_3^{2-\varepsilon}(\alpha_2 + \alpha_3)^{2-\varepsilon} \qquad\qquad \alpha_4$$

$$\mathcal{D}_{[23][145]} = (\alpha_1 + \alpha_2 + \alpha_3)^{2-\varepsilon}\alpha_2^{4-2\varepsilon}\alpha_4^{2-\varepsilon} \qquad\qquad \alpha_3$$

$$\mathcal{D}_{[24][135]} = (\alpha_1 + \alpha_2)^{2-\varepsilon}(\alpha_2 + \alpha_3)^{4-2\varepsilon}(\alpha_3 + \alpha_4)^{2-\varepsilon} \qquad\qquad \alpha_4$$

$$\mathcal{D}_{[25][134]} = (\alpha_1 + \alpha_2)^{2-\varepsilon}(\alpha_1 + \alpha_2 + \alpha_3)^{2-\varepsilon}(\alpha_2 + \alpha_3 + \alpha_4)^{4-2\varepsilon}\alpha_3^{2-\varepsilon} \qquad\qquad \alpha_2$$

$$\mathcal{D}_{[34][125]} = \alpha_1^{2-\varepsilon}(\alpha_2 + \alpha_3 + \alpha_4)^{2-\varepsilon}\alpha_3^{4-2\varepsilon} \qquad\qquad \alpha_4$$

$$\mathcal{D}_{[35][124]} = \alpha_1^{2-\varepsilon}(\alpha_1 + \alpha_2 + \alpha_3)^{2-\varepsilon}(\alpha_2 + \alpha_3)^{2-\varepsilon}(\alpha_3 + \alpha_4)^{4-2\varepsilon} \qquad\qquad \alpha_2$$

$$\mathcal{D}_{[45][123]} = \alpha_1^{2-\varepsilon}(\alpha_1 + \alpha_2)^{2-\varepsilon}\alpha_2^{2-\varepsilon}\alpha_4^{4-2\varepsilon} \qquad\qquad \alpha_3 \qquad (3.10.4)$$

The integrals required for the coefficients $W_{[12][345]}, W_{[15][234]}, W_{[23][145]}, W_{[34][125]}, W_{[34][125]}$, and $W_{[45][123]}$ may be reduced to integrals of the Euler type in (3.12.1) using judicious choices of variables. For example, in $W_{[45][123]}$ we integrate over α_3, keep the variable α_4, and change variable from α_1, α_2 to t, β with $\alpha_1 = (1-\alpha_4)t\beta$ and $\alpha_2 = (1-\alpha_4)t(1-\beta)$, so that $0 \le t, \beta \le 1$. In terms of these variables, and letting $\alpha_4 \to 1 - \alpha_4$, the integral becomes,

$$c^3 W_{[45][123]} = \frac{6^5}{2}\int_0^1 d\alpha_4 \int_0^1 dt \int_0^1 d\beta \frac{\mathcal{N}_5(\alpha_4 t\beta, \alpha_4 t(1 - \beta), \alpha_4(1 - t), 1 - \alpha_4)}{\alpha_4^{4-3\varepsilon}(1 - \alpha_4)^{4-2\varepsilon} t^{5-3\varepsilon} \beta^{2-\varepsilon}(1 - \beta)^{2-\varepsilon}} \ . \quad (3.10.5)$$

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To evaluate the decoupled integrals we expand the numerator $\mathcal{N}_5$ into powers of α_4, t, β,

$$\mathcal{N}_5(\alpha_4 t\beta, \alpha_4 t(1-\beta), \alpha_4(1-t), 1-\alpha_4) = \sum_{A=0}^{4}\sum_{B=0}^{6}\sum_{C=0}^{2}\alpha_4^A\, t^B\, \beta^C \mathcal{M}_{A,B,C} \qquad (3.10.6)$$

and use,

$$\int_0^1 d\alpha_4 \int_0^1 dt \int_0^1 d\beta \frac{\alpha_4^A\, t^B\, \beta^C}{\alpha_4^{4-3\varepsilon}(1-\alpha_4)^{4-2\varepsilon}\, t^{5-3\varepsilon}\, \beta^{2-\varepsilon}(1-\beta)^{2-\varepsilon}}$$
$$= \frac{\Gamma(A-3+3\varepsilon)\Gamma(-3+2\varepsilon)\Gamma(C-1+\varepsilon)\Gamma(-1+\varepsilon)}{(B-4+3\varepsilon)\Gamma(A-6+5\varepsilon)\Gamma(C-2+2\varepsilon)} \quad . \qquad (3.10.7)$$

The integrals for the remaining partitions $W_{[13][245]}$, $W_{[35][124]}$, $W_{[14][235]}$, $W_{[25][134]}$ are closely related to one another. They may be evaluated in terms of nested integrals $\mathcal{L}_\varepsilon(a,b,c,f)$ computed in the Appendix 3.12.3. For example, in $W_{[13][245]}$ we integrate over α_3 with the help of the δ-function, and change variables from α_2 to $\beta = \alpha_1 + \alpha_2$,

$$W_{[13][245]} = \frac{6^5}{2}\int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_4 \int_{\alpha_1}^{1-\alpha_4} d\beta \frac{\mathcal{N}_5(\alpha_1, \beta-\alpha_1, 1-\beta-\alpha_4, \alpha_4)}{\beta^{4-2\varepsilon}(1-\alpha_1)^{2-\varepsilon}(1-\alpha_1-\alpha_4)^{2-\varepsilon}\alpha_4^{2-\varepsilon}} \quad (3.10.8)$$

The polynomial $\mathcal{N}_5$ is a quadratic in each variable $\alpha_1, \alpha_4, \beta$. Expanding in powers of β, for fixed α_1, α_4, we obtain,

$$\frac{6^5}{2}\mathcal{N}_5(\alpha_1, \beta-\alpha_1, 1-\beta-\alpha_4, \alpha_4) = \sum_{B=0}^{2}\beta^B \mathcal{M}_B(\alpha_1, \alpha_4) \qquad (3.10.9)$$

where the functions $\mathcal{M}_B(\alpha_1, \alpha_4)$ are quadratic polynomials in α_1 and α_4. The integral over β may now be performed term by term in powers of β,

$$W_{[13][245]} = \sum_{B=0}^{2}\frac{W_B^{(4)} - W_B^{(1)}}{B-3+2\varepsilon} \qquad (3.10.10)$$

where,

$$W_B^{(1)} = \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_4 \frac{\mathcal{M}_B(\alpha_1, \alpha_4)}{\alpha_1^{3-B-2\varepsilon}(1-\alpha_1)^{2-\varepsilon}(1-\alpha_1-\alpha_4)^{2-\varepsilon}\alpha_4^{2-\varepsilon}} \quad ,$$
$$W_B^{(4)} = \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_4 \frac{\mathcal{M}_B(\alpha_1, \alpha_4)}{(1-\alpha_1)^{2-\varepsilon}(1-\alpha_1-\alpha_4)^{2-\varepsilon}\alpha_4^{2-\varepsilon}(1-\alpha_4)^{3-B-2\varepsilon}} \quad (3.10.11)$$

In the integral for $W_B^{(1)}$, we decouple the integrations by changing variables from α_4 to $\alpha_4 = (1-\alpha_1)t$, and then perform the integrations using (3.12.1). The evaluation of $W_B^{(4)}$ is

considerably more complicated. We expand $\mathcal{M}_B$ is powers of $(1 - \alpha_1)$ and $(1 - \alpha_4)$,

$$\mathcal{M}_B(\alpha_1, \alpha_4) = \sum_{A=0}^{2} \sum_{C=0}^{2} (1 - \alpha_1)^A (1 - \alpha_4)^C \mathcal{M}_{A,B,C}^{(4)}$$

$$W_B^{(4)} = \sum_{A,C=0}^{2} \mathcal{M}_{A,B,C}^{(4)} \mathcal{L}_\varepsilon(A - 1, -1, B + C - 2, -1) \qquad (3.10.12)$$

where the family of integrals $\mathcal{L}_\varepsilon(a, b, c, f)$ is defined and evaluated in Appendix 3.12.3.

3.10.4 Calculation of W_{123456}

Finally, the denominator functions for $n = 6$ are given by,

$$\mathcal{D}_{[12][34[56]} = \alpha_1^{4-2\varepsilon} \alpha_3^{4-2\varepsilon} \alpha_5^{4-2\varepsilon} \qquad\qquad \alpha_4$$

$$\mathcal{D}_{[12][35[46]} = \alpha_1^{4-2\varepsilon} (\alpha_3 + \alpha_4)^{4-2\varepsilon} (\alpha_4 + \alpha_5)^{4-2\varepsilon} \qquad\qquad \alpha_2$$

$$\mathcal{D}_{[12][36[45]} = \alpha_1^{4-2\varepsilon} (1 - \alpha_1 - \alpha_2)^{4-2\varepsilon} \alpha_4^{4-2\varepsilon} \qquad\qquad \alpha_5$$

$$\mathcal{D}_{[13][24[36]} = (\alpha_1 + \alpha_2)^{4-2\varepsilon} (\alpha_2 + \alpha_3)^{4-2\varepsilon} \alpha_5^{4-2\varepsilon} \qquad\qquad \alpha_4$$

$$\mathcal{D}_{[13][25[46]} = (\alpha_1 + \alpha_2)^{4-2\varepsilon} (1 - \alpha_1 - \alpha_5)^{4-2\varepsilon} (\alpha_4 + \alpha_5)^{4-2\varepsilon} \qquad\qquad \alpha_3$$

$$\mathcal{D}_{[13][26[45]} = (\alpha_1 + \alpha_2)^{4-2\varepsilon} (1 - \alpha_1)^{4-2\varepsilon} \alpha_4^{4-2\varepsilon} \qquad\qquad \alpha_5$$

$$\mathcal{D}_{[14][23[56]} = (\alpha_1 + \alpha_2 + \alpha_3)^{4-2\varepsilon} \alpha_2^{4-2\varepsilon} \alpha_5^{4-2\varepsilon} \qquad\qquad \alpha_4$$

$$\mathcal{D}_{[14][25[36]} = (\alpha_1 + \alpha_2 + \alpha_3)^{4-2\varepsilon} (1 - \alpha_1 - \alpha_5)^{4-2\varepsilon} (\alpha_3 + \alpha_4 + \alpha_5)^{4-2\varepsilon} \qquad \alpha_3$$

$$\mathcal{D}_{[14][26[35]} = (\alpha_1 + \alpha_2 + \alpha_3)^{4-2\varepsilon} (\alpha_3 + \alpha_4)^{4-2\varepsilon} (1 - \alpha_1)^{4-2\varepsilon} \qquad\qquad \alpha_5$$

$$\mathcal{D}_{[15][23[46]} = (1 - \alpha_5)^{4-2\varepsilon} \alpha_2^{4-2\varepsilon} (\alpha_4 + \alpha_5)^{4-2\varepsilon} \qquad\qquad \alpha_3$$

$$\mathcal{D}_{[15][24[36]} = (1 - \alpha_5)^{4-2\varepsilon} (\alpha_2 + \alpha_3)^{4-2\varepsilon} (1 - \alpha_1 - \alpha_2)^{4-2\varepsilon} \qquad\qquad \alpha_4$$

$$\mathcal{D}_{[15][26[34]} = (1 - \alpha_5)^{4-2\varepsilon} \alpha_3^{4-2\varepsilon} (1 - \alpha_1)^{4-2\varepsilon} \qquad\qquad \alpha_4$$

$$\mathcal{D}_{[16][23[45]} = \alpha_2^{4-2\varepsilon} \alpha_4^{4-2\varepsilon} \qquad\qquad \alpha_5$$

$$\mathcal{D}_{[16][24[35]} = (\alpha_2 + \alpha_3)^{4-2\varepsilon} (\alpha_3 + \alpha_4)^{4-2\varepsilon} \qquad\qquad \alpha_5$$

$$\mathcal{D}_{[16][25[34]} = (\alpha_2 + \alpha_3 + \alpha_4)^{4-2\varepsilon} \alpha_3^{4-2\varepsilon} \qquad\qquad \alpha_5 \qquad (3.10.13)$$

The integrals required for the coefficients $W_{[12][34][56]}$, $W_{[12][35][46]} = W_{[13][24][56]}$, $W_{[12][36][25]}$, $W_{[13][25][46]}$, $W_{[13][26][45]}$, $W_{[14][23][56]}$, $W_{[15][23][46]}$, $W_{[16][23][45]}$, $W_{[16][24][35]}$, $W_{[16][25][34]}$ may be evaluated using judicious variables and the Euler formula of (3.12.1). The integrals required

for the coefficients $W_{[14][25][36]}$, $W_{[14][26][35]}$, $W_{[15][24][36]}$, $W_{[15][26][34]}$ may be evaluated using the family of integrals with $\mathcal{K}_{2\varepsilon}$ evaluated in Appendix 3.12. Putting everything together we get the result reported in the first line of (3.5.26).

3.11 Calculation of $\langle T_\infty W_\varepsilon(z) \rangle$

The calculation of $\langle T_\infty W_\varepsilon(z) \rangle$ is parallel to the calculation of $\langle W_\varepsilon(z) \rangle$ already given. The expansion of the path ordered exponential (3.5.14) may be organized as follows,

$$\langle T_\infty W(z) \rangle = z^{2j+2} \sum_{n=0}^{\infty} \alpha^n \, z^{(n-1)\varepsilon} \, TW_{x1\cdots n}(z) \tag{3.11.1}$$

where the coefficients $TW_{x1\cdots n}$ are independent of z and given by,

$$TW_{x1\cdots n}(z) = \frac{6^n}{c^n} \int_0^1 dy_n \cdots \int_0^{y_2} dy_1 F_n(1; y_n, \cdots, y_1) \langle T_\infty T^n \rangle_{x1\cdots n} \tag{3.11.2}$$

where we use the following notation,

$$\langle T_\infty T^n \rangle_{x1\cdots n} = \lim_{x \to \infty} \left(x^{4-2\varepsilon} \langle T(x) T(y_1) \cdots T(y_n) \rangle \right). \tag{3.11.3}$$

The symbol x used in the subscript to $TW_{x1\cdots n}$ and $\langle T_\infty T^n \rangle_{x1\cdots n}$ stands for a place-holder indicating the position of the operator $T(x)$ in the correlator.

The stress tensor correlators are evaluated using the same decomposition into partitions of one-loop cycles that we have used for the calculation of $W_{1\cdots n}$, and the relevant correlators are given as follows. Evidently we have $\langle T_\infty T^0 \rangle_x = 0$ and $\langle T_\infty T^1 \rangle_{x1} = \frac{c}{2}$, as well as the following formula for cycles of arbitrary length $n+1$,

$$\langle T_\infty T^n \rangle_{[x1\cdots n]} = \frac{c}{|y_1 - y_2|^{2-\varepsilon} |y_2 - y_3|^{2-\varepsilon} \cdots |y_{n-1} - y_n|^{2-\varepsilon}} \ . \tag{3.11.4}$$

The correlators we need are as follows,

$$\begin{aligned}
\langle T_\infty T^3 \rangle_{x123} &= \langle T_\infty T^3 \rangle_{[x1][23]} + \langle T_\infty T^3 \rangle_{[x2][31]} + \langle T_\infty T^3 \rangle_{[x3][12]} \\
&\quad + \langle T_\infty T^3 \rangle_{[x123]} + \langle T_\infty T^3 \rangle_{[x132]} + \langle T_\infty T^3 \rangle_{[x213]}, \\
\langle T_\infty T^4 \rangle_{x1235} &= \langle T_\infty T^4 \rangle_{[x1][234]} + \ 3 \text{ more partitions} \\
&\quad + \langle T_\infty T^4 \rangle_{[x12][34]} + \ 5 \text{ more partitions}, \\
\langle T_\infty T^5 \rangle_{x12345} &= \langle T_\infty T^5 \rangle_{[x1][23][45]} + \ 14 \text{ more partitions}. \tag{3.11.5}
\end{aligned}$$

The contribution from a partition P is given by the product of the contributions of all the cycles in the partition, just as in (3.5.10), but including now also the point x.

The integrals in (3.11.2) may again be simplified with the help of the change of variables used for W in (3.5.21), and we obtain the following final formula,

$$TW_P = \frac{6^n}{2^{p_2} c^{n-p}} \int_0^1 d\alpha_{n-1} \cdots \int_0^1 d\alpha_1 \, \delta\left(1 - \sum_{k=1}^{n-1} \alpha_k\right) \frac{\mathcal{TN}_n(\alpha_1, \cdots, \alpha_{n-1})}{\mathcal{TD}_P(\alpha_1, \cdots, \alpha_{n-1})} \qquad (3.11.6)$$

where p is the total number of cycles in the partition P and p_2 is the number of 2-cycles. The function $\mathcal{TN}_n$ is given by,

$$\mathcal{TN}_n(\alpha_1, \cdots, \alpha_{n-1}) = \int_0^1 du \, u^{-n+(n-1)\varepsilon} \int_u^1 dx_n \, F_n(1; y_n, y_{n-1}, \cdots, y_1). \qquad (3.11.7)$$

Note that the integrand of $\mathcal{TN}_n$ differs in the variable u from the one for $\mathcal{N}_n$ used in the calculation of W. The function $\mathcal{TD}_P$ is given in terms of the stress tensor correlators by,

$$\langle T_\infty T^n \rangle_P = \frac{c^p}{2^{p_2}} \frac{u^{-(2-\varepsilon)(n-1)}}{\mathcal{TD}_P(\alpha_1, \cdots, \alpha_{n-1})} . \qquad (3.11.8)$$

One of the α_k-integrals may be carried out by satisfying the δ-function, so that the number of non-trivial integrals left over is $n - 2$.

3.11.1   Calculation of $TW_{x1 \cdots n}$

Since TW_x involves the expectation value of a single stress tensor, it vanishes. One also readily shows that $TW_{x1} = -j$. For higher values of n, the expressions for $\mathcal{TN}_n$ rapidly become lengthy and the corresponding calculations have been carried out using MAPLE. The α-integrals involved are less exotic than the ones that were needed for the calculation of $W_\varepsilon(z)$, and may easily be worked out. To orders $1/c$ and ε the coefficients are given by,

$$\begin{aligned} c\,TW_{x12} &= \frac{12j}{\varepsilon} - j(18j + 13) + \frac{j}{12}(162j + 259)\varepsilon, \\ c\,TW_{x123}^{(1)} &= -\frac{6j(j^2 + j + 1)}{\varepsilon} - 2j(5j^2 - 4j - 5) - \frac{j}{3}(74j^2 + 128j + 107)\varepsilon. \end{aligned} \qquad (3.11.9)$$

To order $1/c^2$ we have the following contributions,

$$c^2 \, TW^{(2)}_{x123} = -\frac{144j}{\varepsilon^2} + \frac{48j}{\varepsilon}(9j+8) - \frac{48\pi^2 j}{5}(j^2+j-2) - 8j(36j^2+144j+131),$$

$$c^2 \, TW_{x1234} = \frac{24j}{\varepsilon^2}(7j^2+7j+8) - \frac{2j}{\varepsilon}(78j^3-75j^2+279j+274)$$
$$-\frac{j}{6}(1242j^3-10863j^2-21717j-15880) - 16\pi^2 j(j^2+j+4),$$

$$c^2 \, TW_{x12345} = -\frac{18j}{\varepsilon^2}(j^2+j+1)(j^2+j+3) - \frac{3j}{\varepsilon}(20j^4-8j^3+25j^2-55j-59)$$
$$-2j(99j^4+102j^3+753j^2+1065j+622) + 24\pi^2 j(j^2+j+2) \quad (3.11.10)$$

where $TW_{x123} = TW^{(1)}_{x123} + TW^{(2)}_{x123}$.

3.12 Non-standard integrals

The most basic integral we use throughout is Euler's beta function formula,

$$\int_0^1 d\alpha \, \alpha^{s-1}(1-\alpha)^{t-1} = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \, . \quad (3.12.1)$$

Next, we evaluate various non-standard integrals, needed in an expansion in powers of ε.

3.12.1 The $\mathcal{J}_\varepsilon(A, B)$ integrals

The integrals are defined by,

$$\mathcal{J}_\varepsilon(A, B) = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \, \frac{\alpha^A \beta^B}{\alpha^{2-\varepsilon}(1-\alpha)^{2-\varepsilon}\beta^{2-\varepsilon}(1-\beta)^{2-\varepsilon}} \quad (3.12.2)$$

for integers A, B in the range $0 \le A, B \le 2$. In view of the symmetry of the integration under the interchange of α and β, we have $\mathcal{J}_\varepsilon(A, B) = \mathcal{J}_\varepsilon(B, A)$, reducing the number of integrals needed from 9 to 6. We begin by evaluating the following auxiliary integrals,

$$I_{a,b}(s, t) = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta (1-2\alpha)^a (1-2\beta)^b \alpha^{s-1}(1-\alpha)^{s-1}\beta^{t-1}(1-\beta)^{t-1} \quad (3.12.3)$$

for positive integers a, b. Clearly, we have $I_{a,b}(s, t) = I_{b,a}(t, s)$ and the integrals $J_\varepsilon(A, B)$ are linear combinations of the integrals $I_{a,b}(\varepsilon-1, \varepsilon-1)$ for various values of a, b. In view of the identity $(1-2\alpha)^2 = 1 - 4\alpha(1-\alpha)$ and its analogue for β, we have the following relations,

$$I_{a+2,b}(s, t) = I_{a,b}(s, t) - 4I_{a,b}(s+1, t),$$
$$I_{a,b+2}(s, t) = I_{a,b}(s, t) - 4I_{a,b}(s, t+1) \quad (3.12.4)$$

allowing us to restrict the range to $0 \leq a, b \leq 1$. In view of these symmetries and relations, the remaining integrals may be evaluated using (3.12.1), and we have $I_{1,1}(s,t) = 0$ as well as,

$$I_{0,0}(s,t) = \frac{\Gamma(s)^2 \, \Gamma(t)^2}{2 \, \Gamma(2s) \, \Gamma(2t)} \,, \qquad I_{0,1}(s,t) = \frac{\Gamma(s+t)^2}{t \, \Gamma(2s+2t)} \,. \qquad (3.12.5)$$

Explicit expressions for the required $J_\varepsilon(A, B)$ in terms of $I_{a,b}(s,t)$ are given as follows,

$$
\begin{aligned}
\mathcal{J}_\varepsilon(0,0) &= I_{0,0}(\varepsilon-1, \varepsilon-1), \\
\mathcal{J}_\varepsilon(1,0) &= -\frac{1}{2}I_{1,0}(\varepsilon-1, \varepsilon-1) + \frac{1}{2}\mathcal{J}_\varepsilon(0,0), \\
\mathcal{J}_\varepsilon(2,0) &= -I_{0,0}(\varepsilon, \varepsilon-1) + \mathcal{J}_\varepsilon(1,0), \\
\mathcal{J}_\varepsilon(1,1) &= \mathcal{J}_\varepsilon(1,0) - \frac{1}{4}\mathcal{J}_\varepsilon(0,0), \\
\mathcal{J}_\varepsilon(2,1) &= \frac{1}{2}I_{0,1}(\varepsilon, \varepsilon-1) + \mathcal{J}_\varepsilon(1,1) + \frac{1}{2}\mathcal{J}_\varepsilon(2,0) - \frac{1}{2}\mathcal{J}_\varepsilon(1,0), \\
\mathcal{J}_\varepsilon(2,2) &= \mathcal{J}_{\varepsilon+1}(0,0) + 2\mathcal{J}_\varepsilon(2,1) - \mathcal{J}_\varepsilon(1,1). \qquad (3.12.6)
\end{aligned}
$$

3.12.2 The $\mathcal{K}_\varepsilon(a,b,c)$ integrals

We shall also need integrals of the following form,

$$\mathcal{K}_\varepsilon(a,b,c) = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \, (1-\alpha)^{a-1+\varepsilon}(1-\beta)^{b-1+\varepsilon}(1-\alpha-\beta)^{c-1+\varepsilon}. \qquad (3.12.7)$$

for several sets of integers a, b, c. Clearly we have $\mathcal{K}_\varepsilon(a,b,c) = \mathcal{K}_\varepsilon(b,a,c)$. We use the identity $(1-\alpha) + (1-\beta) - (1-\alpha-\beta) = 1$, and integration by parts in α and in β to find the following formulas,

$$
\begin{aligned}
(a+b+c+3\varepsilon)\mathcal{K}_\varepsilon(a+1,b,c) &= (a+\varepsilon)\mathcal{K}_\varepsilon(a,b,c) + \frac{1}{a+c+2\varepsilon} \,, & (3.12.8) \\
(a+b+c+3\varepsilon)\mathcal{K}_\varepsilon(a,b+1,c) &= (b+\varepsilon)\mathcal{K}_\varepsilon(a,b,c) + \frac{1}{b+c+2\varepsilon} \,, \\
(a+b+c+3\varepsilon)\mathcal{K}_\varepsilon(a,b,c+1) &= -(c+\varepsilon)\mathcal{K}_\varepsilon(a,b,c) + \frac{1}{a+c+2\varepsilon} + \frac{1}{b+c+2\varepsilon} \,.
\end{aligned}
$$

To initialize the recursion relations in all three integers a, b, c it suffices to compute $\mathcal{K}_\varepsilon(a,b,c)$ at a point in the domain of the variables a, b, c, ε where it is given by a convergent integral. For example, $\mathcal{K}_\varepsilon(1,1,1)$ is given by an absolutely convergent integral for $-1 < Re(\varepsilon)$, and

admits a convergent Taylor expansion in ε around $\varepsilon = 0$. To order ε^2, it is given as follows,

$$\mathcal{K}_\varepsilon(1, 1, 1) = \frac{1}{2} - \frac{5}{4}\varepsilon + \frac{11}{8}\varepsilon^2 + \frac{\pi^2}{12}\varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{3.12.9}$$

The expressions for $\mathcal{K}_\varepsilon(a, b, c)$ for the values $-1 \leq a, b \leq 1$ and $c = -1$ needed for the evaluation of $W_{[1324]}$ are obtained using the recursion relations through MAPLE.

3.12.3 The $\mathcal{L}_\varepsilon(a, b, c, f)$ integrals

The integrals are defined by,

$$\mathcal{L}_\varepsilon(a, b, c, f) = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \, (1-\alpha)^{a-1+\varepsilon} \beta^{b-1+\varepsilon} (1-\beta)^{c-1+2\varepsilon} (1-\alpha-\beta)^{f-1+\varepsilon} \tag{3.12.10}$$

for integer values of a, b, c, f. A first pair of recursion relations on the indices a, b, c, f is obtained by inserting the identities $(1-\alpha) + (1-\beta) - (1-\alpha-\beta) = 1$ and $\beta + (1-\beta) = 1$ into the integrand, while a second set is obtained by evaluating the α and β-derivatives of the integrand, in each case expressing the result in terms of $\mathcal{L}_\varepsilon$-functions. We may solve this linear system to obtain four one-step recursion relations, given by,

$$
\begin{aligned}
Z\mathcal{L}_\varepsilon(a, b+1, c, f) &= (Z - c - 2\varepsilon)\mathcal{L}_\varepsilon(a, b, c, f) - R, \tag{3.12.11} \\
Z\mathcal{L}_\varepsilon(a, b, c+1, f) &= (c + 2\varepsilon)\mathcal{L}_\varepsilon(a, b, c, f) + R, \\
(a + f + 2\varepsilon)Z\mathcal{L}_\varepsilon(a+1, b, c, f) &= (a + \varepsilon)(Z - c - 2\varepsilon)\mathcal{L}_\varepsilon(a, b, c, f) + (Z - a - \varepsilon)R, \\
(a + f + 2\varepsilon)Z\mathcal{L}_\varepsilon(a, b, c, f+1) &= -(f + \varepsilon)(Z - c - 2\varepsilon)\mathcal{L}_\varepsilon(a, b, c, f) + (Z + f + \varepsilon)R.
\end{aligned}
$$

where we have used the following abbreviations,

$$Z = a + b + c + f - 1 + 5\varepsilon, \qquad\qquad R = \frac{\Gamma(b + \varepsilon)\Gamma(c + f + 3\varepsilon)}{\Gamma(b + c + f + 4\varepsilon)}. \tag{3.12.12}$$

The recursion relations may be initialized by the absolutely convergent integral $\mathcal{L}_\varepsilon(1, 1, 1, 1)$ for ε near 0, in an expansion in powers of ε,

$$\mathcal{L}_\varepsilon(1, 1, 1, 1) = \frac{1}{2} - \frac{9}{4}\varepsilon - \frac{\pi^2 \varepsilon^2}{12} + \frac{53}{8}\varepsilon^2 + \mathcal{O}(\varepsilon^3). \tag{3.12.13}$$

The expressions for $\mathcal{L}_\varepsilon(a, b, c, f)$ for the other required values of a, b, c, f are obtained using the recursion relations through MAPLE.

3.12.4 Evaluating the integrals $\mathcal{Q}^{(i)}(a,b)$ for $i = 1, 2, 3$

The integrals are defined as follows,

$$\mathcal{Q}^{(1)}_\varepsilon(a,b) = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\, \alpha^{a-1+\varepsilon}\beta^{b-1+\varepsilon}(1-\alpha)^\varepsilon(1-\beta)^\varepsilon(1-\alpha-\beta)^\varepsilon,$$

$$\mathcal{Q}^{(2)}_\varepsilon(a,b) = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\, \alpha^\varepsilon\beta^\varepsilon(1-\alpha)^{a-1+\varepsilon}(1-\beta)^{b-1+\varepsilon}(1-\alpha-\beta)^\varepsilon,$$

$$\mathcal{Q}^{(3)}_\varepsilon(a,b) = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\, \alpha^\varepsilon(1-\alpha)^{a-1+\varepsilon}\beta^\varepsilon(1-\beta)^\varepsilon(1-\alpha-\beta)^{b-1+\varepsilon}. \qquad (3.12.14)$$

The integrals are absolutely convergent for $\varepsilon > -1$ and $Re\,(a), Re\,(b) > -\varepsilon$. We shall be interested in evaluating these integrals in a small neighborhood of $\varepsilon = 0$, where they are absolutely convergent for $Re\,(a), Re\,(b) > 0$. Beyond their ranges of convergence, the integrals need to be analytically continued.

3.12.4.1 Recursion relations for $\mathcal{Q}^{(1)}_\varepsilon(a,b)$ and $\mathcal{Q}^{(2)}_\varepsilon(a,b)$

The integrals $\mathcal{Q}^{(1)}(a,b)$ and $\mathcal{Q}^{(2)}(a,b)$ satisfy the symmetry relation,

$$\mathcal{Q}^{(i)}_\varepsilon(b,a) = \mathcal{Q}^{(i)}_\varepsilon(a,b) \qquad\qquad i = 1, 2. \qquad (3.12.15)$$

To obtain recursion relations for $\mathcal{Q}^{(1)}(a,b)$ we consider the following identity,

$$\int_0^1 d\alpha \int_0^{1-\alpha} d\beta\, \frac{\partial}{\partial\alpha}\left(\alpha^{a+\varepsilon}\beta^{b-1+\varepsilon}(1-\alpha)^{1+\varepsilon}(1-\beta)^\varepsilon(1-\alpha-\beta)^{1+\varepsilon}\right) = 0, \qquad (3.12.16)$$

and its β-derivative counterpart, and express the individual contributions in terms of $\mathcal{Q}^{(1)}(a,b)$. As its turns out, $\mathcal{Q}^{(2)}(a,b)$ satisfies the same recursion relations, and we have for $i = 1, 2$,

$$\begin{aligned}
(a+\varepsilon)\mathcal{Q}^{(i)}_\varepsilon(a,b) &= (2a+2+4\varepsilon)\mathcal{Q}^{(i)}_\varepsilon(a+1,b) + (a+\varepsilon)\mathcal{Q}^{(i)}_\varepsilon(a,b+1) \qquad (3.12.17)\\
&\quad -(a+2+3\varepsilon)\mathcal{Q}^{(i)}_\varepsilon(a+2,b) - (a+1+2\varepsilon)\mathcal{Q}^{(i)}_\varepsilon(a+1,b+1),\\
(b+\varepsilon)\mathcal{Q}^{(i)}_\varepsilon(a,b) &= (2b+2+4\varepsilon)\mathcal{Q}^{(i)}_\varepsilon(a,b+1) + (b+\varepsilon)\mathcal{Q}^{(i)}_\varepsilon(a+1,b)\\
&\quad -(b+2+3\varepsilon)\mathcal{Q}^{(i)}_\varepsilon(a,b+2) - (b+1+2\varepsilon)\mathcal{Q}^{(i)}_\varepsilon(a+1,b+1).
\end{aligned}$$

The integrals we need (in a short series expansion in ε) are for $a, b \geq -3$. The above recursion relations do not proceed by single-steps, and are considerably more complicated than those

for the earlier integrals. In particular, they cannot be initialized at a single pair (a, b). Instead, the above recursion relations allow us to express $\mathcal{Q}_\varepsilon^{(i)}(a, b)$ for integer $a, b \geq -3$ as a linear combination of $\mathcal{Q}_\varepsilon^{(i)}(a, b)$ with $a \geq -3$ and $b \geq 1$. These relations are relatively involved and were handled with MAPLE.

3.12.4.2 Recursion relation for $\mathcal{Q}_\varepsilon^{(3)}(a, b)$

Contrarily to $\mathcal{Q}_\varepsilon^{(1)}(a, b)$ and $\mathcal{Q}_\varepsilon^{(2)}(a, b)$, the function $\mathcal{Q}_\varepsilon^{(3)}(a, b)$ is not symmetric in its arguments a, b. By expressing the vanishing of the integral over partial derivatives with respect to α and β in terms of $\mathcal{Q}_\varepsilon^{(3)}(a, b)$, we obtain two recursion relations,

$$
\begin{aligned}
0 = {} & (a + 1 + 2\varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a + 1, b + 1) - (a + \varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a, b + 1) \\
& + (b + \varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a + 2, b) - (b + \varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a + 1, b), \\
0 = {} & (2b + 2 + 4\varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a + 1, b + 1) - (b + 1 + 2\varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a, b + 1) \\
& - (b + 2 + 3\varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a, b + 2) \\
& - (b + \varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a + 2, b) + (b + \varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a + 1, b).
\end{aligned} \tag{3.12.18}
$$

The last lines of both equations are the only terms whose second argument is b. Adding the equations eliminates those terms. Shifting the resulting equation by $b + 1 \to b$, shifting the first equation by $a + 1 \to a$, and eliminating $\mathcal{Q}_\varepsilon^{(3)}(a + 1, b)$ we obtain a formula for $\mathcal{Q}_\varepsilon^{(3)}(a, b)$ in terms of functions with second argument $b + 1$, and thus a recursion relation in b,

$$
\mathcal{Q}_\varepsilon^{(3)}(a, b) = \mathcal{Q}_\varepsilon^{(3)}(a, b + 1) + \frac{a + 2b + 1 + 6\varepsilon}{(b + \varepsilon)(b + 1 + 3\varepsilon)}\Big((a + 2\varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a, b + 1) \\
- (a - 1 + \varepsilon)\mathcal{Q}_\varepsilon^{(3)}(a - 1, b + 1)\Big). \tag{3.12.19}
$$

Applying this recursion relation, the required quantities $\mathcal{Q}_\varepsilon^{(3)}(a, b)$, for $b = -3, -2, -1, 0$ may be obtained from $\mathcal{Q}_\varepsilon^{(3)}(a, 1)$, which we evaluate by convergent series.

3.12.4.3 Initializing $\mathcal{Q}_\varepsilon^{(1)}(a, b)$

To evaluate the integrals $\mathcal{Q}_\varepsilon^{(1)}(a, b)$ for $a \geq -3$ and $b \geq 1$ near $\varepsilon = 0$, we change variables from α to t by setting $\alpha = (1 - \beta)t$ for $0 \leq t \leq 1$, expand the factor $(1 - (1 - \beta)t)^\varepsilon$ in powers

of $(1 - \beta)t$, and use the Euler relation (3.12.1) to evaluate the decoupled integrals over β, t. It will be convenient to recast the result as follows,

$$\mathcal{Q}_{\varepsilon}^{(1)}(a, b) = \sum_{k=0}^{\mathfrak{a}} \frac{\Gamma(k - \varepsilon)}{\Gamma(-\varepsilon)\, k!} \frac{\Gamma(b + \varepsilon)\Gamma(k + a + 1 + 3\varepsilon)}{\Gamma(k + a + b + 1 + 4\varepsilon)} \frac{\Gamma(1 + \varepsilon)\Gamma(k + a + \varepsilon)}{\Gamma(k + a + 1 + 2\varepsilon)} \quad (3.12.20)$$

$$-\varepsilon \sum_{k=\mathfrak{a}+1}^{\infty} \frac{\Gamma(k - \varepsilon)\Gamma(b + \varepsilon)\Gamma(1 + \varepsilon)\Gamma(k + a + 1 + 3\varepsilon)\Gamma(k + a + \varepsilon)}{k!\, \Gamma(1 - \varepsilon)\Gamma(k + a + b + 1 + 4\varepsilon)\Gamma(k + a + 1 + 2\varepsilon)}$$

where $\mathfrak{a} = \max(0, -a)$. The finite sum is readily expanded in powers of ε. The summand of the infinite series grows as $k^{-2-b-3\varepsilon}$ for large k. Therefore the series converges absolutely and uniformly in ε for $b + 3\, Re\,(\varepsilon) > -1$ which allows for $Re\,(\varepsilon) > -2/3$ in view of the assumption $b \geq 1$. The region of convergence includes the neighborhood of $\varepsilon = 0$ needed here, so that the expansion of $\mathcal{Q}_{\varepsilon}^{(1)}(a, b)$ is obtained by expanding the series term by term.

3.12.4.4 Initializing $\mathcal{Q}_{\varepsilon}^{(2)}(a, b)$

The expansion for $\mathcal{Q}_{\varepsilon}^{(2)}(a, b)$ for $b \geq 1$ may be obtained by the same methods and is similar, but not identical, to the one for $\mathcal{Q}_{\varepsilon}^{(1)}(a, b)$. Starting with its definition in (3.12.14), we change variables from β to t with $\beta = (1 - \alpha)t$ for $0 \leq t \leq 1$, expand the factor $(1 - t(1 - \alpha))^{b-1+\varepsilon}$ in powers of $t(1 - \alpha)$, and perform the decoupled integrals using Euler's formula. It will be convenient to recast the result as follows,

$$\mathcal{Q}_{\varepsilon}^{(2)}(a, b) = \sum_{k=0}^{\mathfrak{b}} \frac{\Gamma(k - b + 1 - \varepsilon)}{\Gamma(-b + 1 - \varepsilon)\, k!} \frac{\Gamma(1 + \varepsilon)^2 \Gamma(k + a + 1 + 3\varepsilon)\Gamma(k + 1 + \varepsilon)}{\Gamma(k + a + 2 + 4\varepsilon)\Gamma(k + 2 + 2\varepsilon)} \quad (3.12.21)$$

$$+ \sum_{k=\mathfrak{b}+1}^{\infty} \frac{\Gamma(k - b + 1 - \varepsilon)}{\Gamma(-b + 1 - \varepsilon)\, k!} \frac{\Gamma(1 + \varepsilon)^2 \Gamma(k + a + 1 + 3\varepsilon)\Gamma(k + 1 + \varepsilon)}{\Gamma(k + a + 2 + 4\varepsilon)\Gamma(k + 2 + 2\varepsilon)}$$

where $\mathfrak{b} = \max(b - 1, -a - 1)$. The summand of the infinite series grows as $k^{-2-b-3\varepsilon}$ for large k and therefore the series converges absolutely and uniformly in ε for $b \geq 1$ and ε near zero. The expansion of $\mathcal{Q}_{\varepsilon}^{(2)}(a, b)$ in powers of ε is obtained as it was for $\mathcal{Q}_{\varepsilon}^{(1)}(a, b)$.

3.12.4.5 Initializing $\mathcal{Q}_{\varepsilon}^{(3)}(a, b)$

The recursion relation for $\mathcal{Q}_{\varepsilon}^{(3)}(a, b)$ is initialized by the value of the integrals $\mathcal{Q}_{\varepsilon}^{(3)}(a, 1)$. To evaluate it, we change variables from β to t with $\beta = (1 - \alpha)t$ for $0 \leq t \leq 1$, expand the

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factor $(1 - t(1 - \alpha))^\varepsilon$ in powers of $t(1 - \alpha)$, and perform the integrals using Euler's formula. The result is conveniently presented as follows,

$$
\begin{aligned}
\mathcal{Q}_\varepsilon^{(3)}(a, 1) \;=\; & \sum_{k=0}^{\mathfrak{a}'} \frac{\Gamma(k - \varepsilon)}{\Gamma(-\varepsilon)\, k!} \frac{\Gamma(1 + \varepsilon)^2\, \Gamma(k + a + 1 + 3\varepsilon)\Gamma(k + 1 + \varepsilon)}{\Gamma(k + a + 2 + 4\varepsilon)\Gamma(k + 2 + 2\varepsilon)} \\
& - \varepsilon \sum_{k=\mathfrak{a}'+1}^{\infty} \frac{\Gamma(k - \varepsilon)}{\Gamma(1 - \varepsilon)\, k!} \frac{\Gamma(1 + \varepsilon)^2\, \Gamma(k + a + 1 + 3\varepsilon)\Gamma(k + 1 + \varepsilon)}{\Gamma(k + a + 2 + 4\varepsilon)\Gamma(k + 2 + 2\varepsilon)}
\end{aligned}
\tag{3.12.22}
$$

where $\mathfrak{a}' = \max(0, -a - 1)$. The summand behaves as $k^{-3-3\varepsilon}$ for large k and the infinite series is absolutely and uniformly convergent in the neighborhood of $\varepsilon = 0$, and may be expanded in ε.

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