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A Bimodal Perspective on Possibility Semantics

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Abstract

In this paper we develop a bimodal perspective on *possibility semantics*, a framework allowing partiality of states that provides an alternative modeling for classical propositional and modal logics [Humberstone, 1981, Holliday, 2015]. In particular, we define a full and faithful *translation* of the basic modal logic **K** over possibility models into a bimodal logic of partial functions over partial orders, and we show how to modulate this analysis by varying across logics and model classes that have independent topological motivations. This relates the two realms under comparison both semantically and syntactically at the level of derivations. Moreover, our analysis clarifies the interplay between the complexity of translations and axiomatizations of the corresponding logics: adding axioms to the target bimodal logic simplifies translations, or vice versa, complex translations can simplify frame conditions. We also investigate a transfer of first-order correspondence theory between possibility semantics and its bimodal counterpart. Finally, we discuss the conceptual trade-off between giving translations and giving new semantics for logical systems, and we identify a number of further research directions to which our analysis gives rise.

Keywords: classical modal logic, intuitionistic modal logic, possibility semantics, embeddings into bimodal logic, topological logics

Contents

1 Introduction

A standard view of the semantics of classical and intuitionistic logic associates classical logic with models based on *complete* states and intuitionistic logic with models based on partially ordered sets of *partial* states. Yet natural semantics for classical logic can also be given using models based on posets of partial states [Fine, 1975, van Benthem, 1981, Humberstone, 1981]. In place of intuitionistic Beth or Kripke-style semantics [Beth, 1956, Kripke, 1965], classical partial-state semantics uses a modified definition of satisfaction, according to which a partial state leaves the truth value of a formula undetermined just in case the formula is true at some refinement of the state and false at some other refinement of the state. Technically, this definition of satisfaction is like the notion of "weak forcing" in set theory, which in effect builds into the semantics the double negation translation of classical into intuitionistic logic. As a result, the truth of a formula φ at a classical partial state x is equivalent to its *cofinal truth*: for every refinement x' of x there is a further refinement x'' of x' such that φ is true at x''. Topologically, this means that propositions in the classical picture are *regular open* sets in the upset topology on the poset, i.e., sets that are equal to the interior of their closure, in contrast to the intuitionistic picture where propositions can be arbitrary open sets. Motivations for this remodeling of classical logic range from philosophical views about the partiality of situations to a mathematical desire for simple completeness proofs that do not involve maximality or choice principles.

The application of these ideas to the semantics of classical *modal* logic involves a generalization of possible world semantics to a "possibility semantics", introduced in Humberstone 1981 and further developed in Holliday 2014, 2015. In Holliday 2015, a notion of *possibility frame* is shown to provide a more general semantics than standard possible world frames for characterizing classical normal modal logics, while still retaining appealing features of modal semantics, such as correspondences between modal axioms and first-order properties of frames. Like frames for intuitionistic modal logic, possibility frames for classical modal logic are based on a partially ordered set of states with an accessibility relation. Just as the requirement that propositions be open sets in intuitionistic modal frames imposes conditions on the interplay of the partial order and accessibility relation in such frames, the requirement that propositions be regular open sets in possibility frames imposes conditions on the interplay of the partial order and accessibility relation in these frames. In §2.3, we will review the notions of frames and of satisfaction in possibility semantics.

The two distinct relations found in possibility frames suggest an alternative medium of description: a natural *bimodal* language with one modality for the ordering relation and one for the accessibility relation, and with the propositional connectives treated as in standard possible world semantics. The resulting bimodal perspective on possibility semantics will be our main focus in this paper.

This perspective extends earlier work. Intuitionistic modal logic can be fruitfully analyzed in terms of classical bimodal logic [Fischer Servi, 1977, Wolter and Zakharyaschev, 1999]. Bimodal interaction axioms capture conditions on the interplay of the ordering and accessibility relations in intuitionistic modal frames; and the translation of the intuitionistic unimodal language into the classical bimodal language parallels the definition of intuitionistic satisfaction. Possibility frames involve a different interplay of ordering and accessibility, and a different notion of satisfaction. Even so, it makes sense to continue in the bimodal vein, and indeed, to observe parallels between the results for possibility models and intuitionistic models.

Our bimodal analysis has several dimensions. Semantically, we can relate models directly between systems, and syntactically, we can relate axiomatic derivations; but underlying both connections is a notion of relative interpretation. The main result in this paper is a full and faithful translation of the basic modal logic **K** over possibility models into a bimodal logic with two components: an **S4**-type modal logic of inclusion and a logic for a partially functional modality over the inclusion structure. To this base we can add various axioms to regulate the interaction between the components, for instance, axioms expressing that the function is topologically continuous, or open, or an R-map, i.e., such that the inverse image of a regular open set is regular open. Adding interaction axioms to the target bimodal logic allows us to simplify our translation.

The paper is organized as follows. In §2, we fix our unimodal language and logics and present the basic background on intuitionistic modal semantics and classical possibility semantics. We also introduce a key "possibilization" construction taking standard relational frames to possibility frames that will be used repeatedly later on. In §3, we introduce and develop a bimodal perspective on possibility frames and their logics. We analyze a range of special axioms for functions on inclusion orders, using Sahlqvist correspondence techniques, and identify their content in the context of dynamic topological spaces [Artemov et al., 1997, Kremer and Mints, 2005]. In our core §4, we present our syntactic translation showing, via an argument inspired by possibility semantics, how the minimal modal logic for arbitrary relations can be decomposed into a bimodal logic of preorders plus partial functions taking states to unique alternatives. We thereby arrive in §4.1 at the following informal slogan:

$$\text{RELATION} \quad \hookrightarrow \quad \text{PREORDER} + \text{PARTIAL FUNCTION},$$

where $\hookrightarrow$ means that the modal logic of what appears on the left can be embedded in the bimodal logic of what appears on the right. In the process of establishing our results in §4, we give detailed combinatorial information about the relevant bimodal logics. In §4.2, we relate our translation to a composition of two famous embeddings from the classical literature on modal and intuitionistic logic: the Gödel-McKinsey-Tarski translation and the Gödel-Gentzen translation. In §4.3, we show how we can simplify our translation by strengthening the target bimodal logic. Using either our original translation or a simplified one, we show how the modal logic of arbitrary serial relations can be decomposed into a bimodal logic of continuous functions over topological spaces. We thereby arrive at another informal slogan:

$$\text{SERIAL RELATION} \quad \hookrightarrow \quad \text{TOPOLOGICAL SPACE} + \text{CONTINUOUS FUNCTION},$$

with the same interpretation as above. Finally, the simplest of our translations, based directly on possibility semantics, embeds unimodal logic into a stronger bimodal logic of topological R-maps in §4.4.

In §§5-6, we discuss the broader conceptual significance of our results, with an emphasis on the duality between system translation and designing alternative semantics for given logics. We also list a few of the many further directions that are suggested by our style of analysis, including a transfer of frame correspondence results between classical bimodal semantics and possibility semantics, possible language extensions, and connections with logics of topologies endowed with operators as found in dynamical systems.

2 Intuitionistic Semantics and Possibility Semantics

In this section, we give a brief introduction to possibility semantics for classical modal logic (§§2.3-2.4), facilitated by a brief review of the standard semantics for intuitionistic modal logic (§2.2).

2.1 Language and Logics

We begin by fixing the first of our two languages.

Definition 2.1 (Unimodal Language and Logics)**.** Fixing a nonempty set Prop of propositional variables, let $\mathcal{L}_1$ be the language defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Box\varphi,$$

where $p \in \mathsf{Prop}$.

A *classical normal modal logic* for $\mathcal{L}_1$ is a set $\mathbf{L} \subseteq \mathcal{L}_1$ of formulas that is closed under uniform substitution, contains all classical propositional tautologies, contains the K axiom $\Box(p \to q) \to (\Box p \to \Box q)$, and is closed under necessitation: if $\varphi \in \mathbf{L}$, then $\Box\varphi \in \mathbf{L}$. An *intuitionistic normal modal logic* for $\mathcal{L}_1$ is defined in the same way, except with "theorems of Heyting propositional calculus" in place of "classical propositional tautologies". As usual, let $\mathbf{K}$ be the smallest classical normal modal logic for $\mathcal{L}_1$. Let $\mathbf{HK}$ be the smallest intuitionistic normal modal logic for $\mathcal{L}_1$ [Božic and Došen, 1984].

For the semantics of classical normal modal logics, we assume familiarity with the standard relational frames $\mathfrak{F} = \langle W, R\rangle$, models $\mathfrak{M} = \langle W, R, V\rangle$, and the satisfaction relation $\vDash$ relating pointed relational models $\mathfrak{M}, w$ for $w \in W$ to formulas of $\mathcal{L}_1$. When we want to contrast standard relational models with possibility models, we may call the former "possible world models".

2.2 Intuitionistic Frames and Models

For the semantics of intuitionistic normal modal logics, we adopt a standard starting point in Definition 2.2. As usual, for any binary relation Q on a set S and $X \subseteq S$, we define $\Box_Q X = \{x \in S \mid Q(x) \subseteq X\}$ and $\Diamond_Q X = \{x \in S \mid Q(x) \cap X \neq \emptyset\}$, where $Q(x) = \{y \in S \mid xQy\}$.

Definition 2.2 (Intuitionistic Modal Frame)**.** An *intuitionistic modal frame* is a tuple $\mathcal{F} = \langle S, \leq, R, P\rangle$ where $\langle S, \leq\rangle$ is a poset, R is a binary relation on S, P is the set of all *upsets* in $\langle S, \leq\rangle$, and P is closed under $\Box_R$. An *intuitionistic model based on* $\mathcal{F}$ is a tuple $\mathcal{M} = \langle S, \leq, R, \pi\rangle$ such that $\pi\colon \mathsf{Prop} \to P$.

In the literature on intuitionistic modal logic, authors have considered a variety of first-order conditions on the interplay of $\leq$ and R, which ensure that P is closed under $\Box_R$. Rather than building such a condition into the definition of frames, we will deduce a condition in Proposition 2.5 below.

Definition 2.3 (Intuitionistic Satisfaction)**.** The intuitionistic satisfaction relation $\Vdash_{\mathrm{i}}$ between pointed intuitionistic models and formulas of $\mathcal{L}_1$ is defined recursively as follows:

1. $\mathcal{M}, x \Vdash_{\mathrm{i}} p$ iff $x \in \pi(p)$;

2. $\mathcal{M}, x \Vdash_{\mathrm{i}} \neg\varphi$ iff $\forall x' \geq x$: $\mathcal{M}, x' \nVdash_{\mathrm{i}} \varphi$;

3. $\mathcal{M}, x \Vdash_{\mathrm{i}} \varphi \to \psi$ iff $\forall x' \geq x$: if $\mathcal{M}, x' \Vdash_{\mathrm{i}} \varphi$, then $\mathcal{M}, x' \Vdash_{\mathrm{i}} \psi$;

4. $\mathcal{M}, x \Vdash_{\mathrm{i}} \varphi \wedge \psi$ iff $\mathcal{M}, x \Vdash_{\mathrm{i}} \varphi$ and $\mathcal{M}, x \Vdash_{\mathrm{i}} \psi$;

5. $\mathcal{M}, x \Vdash_{\mathrm{i}} \varphi \vee \psi$ iff $\mathcal{M}, x \Vdash_{\mathrm{i}} \varphi$ or $\mathcal{M}, x \Vdash_{\mathrm{i}} \psi$;

6. $\mathcal{M}, x \Vdash_{\mathrm{i}} \Box\varphi$ iff $\forall y \in R(x)$: $\mathcal{M}, y \Vdash_{\mathrm{i}} \varphi$.

Let $[\![\varphi]\!]_{\mathrm{i}}^{\mathcal{M}} = \{x \in S \mid \mathcal{M}, x \Vdash_{\mathrm{i}} \varphi\}$.

If $\mathcal{M}$ is based on an intuitionistic modal frame $\mathcal{F} = \langle S, \leq, R, P\rangle$, then an easy induction shows that for all $\varphi \in \mathcal{L}_1$, $[\![\varphi]\!]_{\mathrm{i}}^{\mathcal{M}} \in P$, using the fact that the set of all upsets is closed under the $\Vdash_{\mathrm{i}}$-semantic operations for the connectives, plus the assumption that P is closed under $\Box_R$.

Proposition 2.4 (Božic and Došen 1984)**. HK** is sound and complete with respect to the class of intuitionistic modal frames.

Let us now return to the issue raised above about the interplay of $\leq$ and R. We first identify the bimodal frame condition C that underlies the above closure condition on upsets (cf. Fischer Servi 1980, §2). It is a familiar commutativity condition that can also be viewed as expressing a sort of bisimulation behavior.

$$\begin{tikzcd} x' \arrow[r, dashed] & y' & & x' \arrow[r, dashed] & y' \\ x \arrow[u] & & \Rightarrow & x \arrow[u] \arrow[r, dashed, "\exists"'] & y \arrow[u] \end{tikzcd}$$

Figure 1: condition C on the interplay of $\leq$ and R from Proposition 2.5. A solid arrow from s to t indicates that $s \leq t$. A dashed arrow from s to t indicates that sRt.

Proposition 2.5. For any poset $\langle S, \leq\rangle$ and binary relation R on S, the following are equivalent:

1. The set of all upsets in $\langle S, \leq\rangle$ is closed under $\Box_R$;

2. $\leq$ and R satisfy the following condition (see Figure 1)

   C – if $x \leq x'Ry'$, then $\exists y$: $xRy \leq y'$.

Proposition 2.5 can be viewed as a correspondence observation, in the sense of modal correspondence theory, although this will only become explicit with the bimodal language to be introduced later. But the argument that is needed here involves a twist. In ordinary correspondence theory, the admissible valuations for propositional variables range over all sets. Here they only range over upward-closed sets, and this gives intuitionistic correspondence theory a special flavor.[1] For instance, minimal valuations in Sahlqvist axioms (cf. Blackburn et al. 2001, §3.6) now need adjustment, and the resulting frame conditions become slightly more complex (cf. Rodenburg 1986). We give a proof of Proposition 2.5 to increase the reader's familiarity with the concepts involved, and as a warm-up for our later analysis of possibility models.

[1]Another difference is the modal character of intuitionistic implication and negation, at least as seen from a classical viewpoint, making simple-looking propositional formulas complex with stacked modalities in their explicit modal form.

Proof of Proposition 2.5. From part 2 to part 1, consider an upset X and points x, x' with $x \in \Box_R X$ and $x \leq x'$. We show that $x' \in \Box_R X$, i.e., $R(x') \subseteq X$. Let $x'Ry'$: then by C, there is a y such that $xRy \leq y'$. Since $x \in \Box_R X$, we have $y \in X$, and then since X is an upset, we also have $y' \in X$.

From part 1 to part 2, we derive C. Suppose that $x \leq x'Ry'$. Now let $\downarrow y'$ be the principal downset generated by y': clearly then, $V = S \setminus \downarrow y'$ is an upset. Now $x' \notin \Box_R V$, since $x'Ry'$. By assumption 1, $\Box_R V$ is an upset too, so we also have $x \notin \Box_R V$. Unpacking this, we get the desired point y for condition C. $\Box$

Additional conditions on the interaction of $\leq$ and R make sense for intuitionistic modal logic (see, e.g., Wolter and Zakharyaschev 1997). For our story here, we will only mention one such condition, which fills in the other corner of the commutative diagram suggested by the C condition:

O – if $xRy \leq y'$, then $\exists x'$: $x \leq x'Ry'$ (see Figure 2).

It is straightforward to show that any intuitionistic modal frame can be transformed into a modally equivalent one satisfying the O condition, e.g., by defining a new relation R' by: $xR'y$ iff for every upset X, $x \in \Box_R X$ implies $y \in X$. In this sense, O may be assumed without loss of generality. Yet we will not build O into the definition of frames, since we would like to analyze its effect separately.

$$\begin{tikzcd}
 & y' & & x' \arrow[r, dashed] & y' \\
x \arrow[r, dashed] & y \arrow[u] & \Rightarrow & x \arrow[u, "\exists"] \arrow[r, dashed] & y \arrow[u]
\end{tikzcd}$$

Figure 2: condition O on the interplay of $\leq$ and R.

Remark 2.6 (Non-standard semantics for $\Box$)**.** We can give semantics for intuitionistic modal logic using frames $\mathfrak{F} = \langle S, \leq, R, P\rangle$ with no required interaction conditions relating $\leq$ and R, but with P still the set of all upsets, provided we modify the satisfaction relation $\Vdash_{\mathrm{i}}$ with a non-standard clause for $\Box$, namely: $[\![\Box\varphi]\!]^{\mathcal{M}} = \Box_{\leq}\Box_R[\![\varphi]\!]^{\mathcal{M}}$ (cf. Wijesekera 1990). Assuming such a semantics, any frame can be turned into a modally equivalent one satisfying C and O by defining a new relation R' by: $xR'y$ iff for every upset X, $x \in \Box_{\leq}\Box_R X$ implies $y \in X$. We shall see a syntactic analogue of this non-standard semantics in §4.2.

2.3 Possibility Frames and Models

Possibility semantics for classical modal logic, though motivated independently, is formally similar to the semantics for intuitionistic modal logic from §2.2, and we will exploit this analogy. (For a more general intuitionistic semantics that is more deeply related to possibility semantics, see Bezhanishvili and Holliday Forthcoming.) Our treatment and results from this section are due to Holliday 2015, to which we refer for further details.[2] Although here we consider only possibility semantics for propositional modal logic, one can also give natural possibility semantics for quantified modal logic (see Harrison-Trainor 2016a).

One good way of understanding how classical logic arises in possibility semantics is through the following mathematical notion. In the topology of all upsets in $\langle S, \leq\rangle$, the *regular open* sets form a subfamily of special importance, being those sets $X \subseteq S$ such that $X = \mathrm{int}(\mathrm{cl}(X))$, where int and cl are the interior and

[2]Note that Holliday 2015 works with downsets rather than upsets of a poset $\langle S, \sqsubseteq\rangle$, with $y \sqsubseteq x$ meaning that y is a refinement of x, following the common convention for set-theoretic forcing.

closure operations, respectively, on the upset topology. For $X \subseteq S$, $\mathrm{int}(X) = \{y \in S \mid \forall x \geq y \colon x \in X\}$ and $\mathrm{cl}(X) = \{y \in S \mid \exists x \geq y \colon x \in X\}$; so $\mathrm{cl}(X)$ is simply $\downarrow X$, the downset generated by X. Using the notation of the previous section, $\mathrm{int}(X) = \Box_{\leq} X$ and $\mathrm{cl}(X) = \Diamond_{\leq} X$. It is straightforward to check that the regular open sets are exactly the sets X satisfying two conditions, one that we had with intuitionistic propositions and one that is new:

- *Persistence* – if $x \in X$ and $x \leq x'$, then $x' \in X$;
- *Refinability* – if $x \notin X$, then $\exists x' \geq x \ \forall x'' \geq x' \colon x'' \notin X$.

Note that the converse of *Refinability* follows from *Persistence*. Also note that the two conditions together are equivalent to the condition that $x \in X$ iff $\forall x' \geq x \ \exists x'' \geq x' \colon x'' \in X$.

Let $\mathrm{RO}(S, \leq)$ be the set of all regular open sets in the upset topology on $\langle S, \leq \rangle$.

Definition 2.7 (Modal Possibility Frame)**.** A *modal possibility frame* is a tuple $\mathcal{F} = \langle S, \leq, R, P \rangle$ where $\langle S, \leq \rangle$ is a poset, R is a binary relation on S, $P = \mathrm{RO}(S, \leq)$, and P is closed under $\Box_R$.[3] A *possibility model based on* $\mathcal{F}$ is a tuple $\mathcal{M} = \langle S, \leq, R, \pi \rangle$ such that $\pi \colon \mathsf{Prop} \to P$.

The possibility satisfaction relation is essentially as for intuitionistic modal logic, with one twist: the interpretation of disjunction is like that found in "weak" forcing in set theory (see, e.g., Jech 2008, §5.1.3).

Definition 2.8 (Possibility Satisfaction)**.** The possibility satisfaction relation $\Vdash_{\mathrm{p}}$ between pointed possibility models and formulas of $\mathcal{L}_1$ is as in Definition 2.3, with $\Vdash_{\mathrm{p}}$ in place of $\Vdash_{\mathrm{i}}$, except for a different clause for $\vee$:

- $\mathcal{M}, x \Vdash_{\mathrm{p}} \varphi \vee \psi$ iff $\forall x' \geq x \ \exists x'' \geq x' \colon \mathcal{M}, x'' \Vdash_{\mathrm{p}} \varphi$ or $\mathcal{M}, x'' \Vdash_{\mathrm{p}} \psi$.

In other words, where $[\![\varphi]\!]^{\mathcal{M}}_{\mathrm{p}} = \{x \in S \mid \mathcal{M}, x \Vdash_{\mathrm{p}} \varphi\}$, we have $[\![\varphi \vee \psi]\!]^{\mathcal{M}}_{\mathrm{p}} = \mathrm{int}(\mathrm{cl}(([\![\varphi]\!]^{\mathcal{M}}_{\mathrm{p}} \cup [\![\psi]\!]^{\mathcal{M}}_{\mathrm{p}}))$.

If $\mathcal{M}$ is based on a possibility frame $\mathcal{F} = \langle S, \leq, R, P \rangle$, then an easy induction shows that for all $\varphi \in \mathcal{L}_1$, $[\![\varphi]\!]^{\mathcal{M}}_{\mathrm{p}} \in P$, using the fact that the set of all regular open sets is closed under the $\Vdash_{\mathrm{p}}$-semantic operations for the connectives, plus the assumption that P is closed under $\Box_R$.

The basic completeness result for modal possibility frames is as follows.

Proposition 2.9. **K** is sound and complete with respect to the class of modal possibility frames.

The soundness of classical propositional logic follows from the observation that $\mathrm{RO}(S, \leq)$ forms a (complete) Boolean algebra with the join given by the interior of the closure of the union, as in Definition 2.8, the complement given by the interior of the set-theoretic complement, as in Definition 2.3, and the meet given by intersection (see Tarski 1938, 1956, Givant and Halmos 2009, Ch. 10). Then the soundness of **K** follows from the observation that $\Box_R$ is an operator on $\mathrm{RO}(S, \leq)$ that preserves intersections and maps S to S. For a quick proof of completeness, we can appeal to the completeness of **K** with respect to possible world frames, because these are a special case of possibility frames, namely those in which $\leq$ is the identity relation, and over possible world frames the possibility satisfaction relation $\Vdash_{\mathrm{p}}$ agrees with the standard satisfaction relation $\vDash$ for possible world semantics. Alternatively, one can prove completeness directly, building a canonical possibility model out of finite consistent sets of formulas (see Holliday 2014, 2015), rather than infinite maximally consistent sets of formulas as in the standard canonical possible world model.

[3]In Holliday 2015, these frames are called *full* possibility frames, to distinguish them from (general) possibility frames that relax the first requirement on P from $P = \mathrm{RO}(S, \leq)$ to $P \subseteq \mathrm{RO}(S, \leq)$ with appropriate closure conditions.

We can now do for modal possibility frames what we did for intuitionistic modal frames in §2.2, analyzing the conditions on the interplay of $\leq$ and R that hold for all possibility frames, as well as stronger conditions that we may assume without loss of generality.

Again we start with a correspondence result appropriate to this new setting, where admissible valuations are now restricted to regular open sets. Compare the following result to Proposition 2.5, and especially, note the additional complexity in the correspondence proof given below. For notation, let $x \between y$ (x and y are *compatible*) iff $\exists z$: $x \leq z$ and $y \leq z$. Then $x \perp y$ (x and y are *incompatible*) iff it is not the case that $x \between y$.

Proposition 2.10. For any poset $\langle S, \leq \rangle$ and binary relation R on S, the following are equivalent:

1. $\mathrm{RO}(S, \leq)$ is closed under $\Box_R$;

2. $\leq$ and R satisfy the following conditions:

   R.1 – if $x \leq x'Ry' \leq z$, then $\exists y$: $xRy \between z$ (see Figure 3);

   R.2 – if xRy, then $\forall y' \geq y \; \exists x' \geq x \; \forall x'' \geq x' \; \exists y'' \between y'$: $x''Ry''$ (see Figure 4).

In particular, it follows that any modal possibility frame satisfies conditions R.1 and R.2. Also note that the condition R.1 follows from the condition C of intuitionistic modal frames.

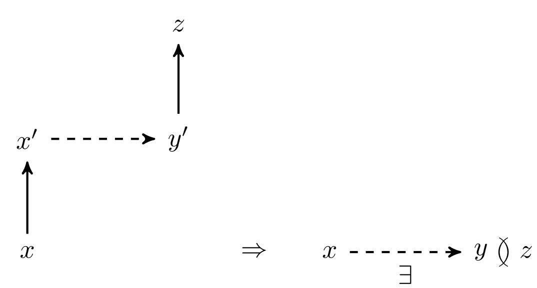

Figure 3: condition R.1 on the interplay of $\leq$ and R.

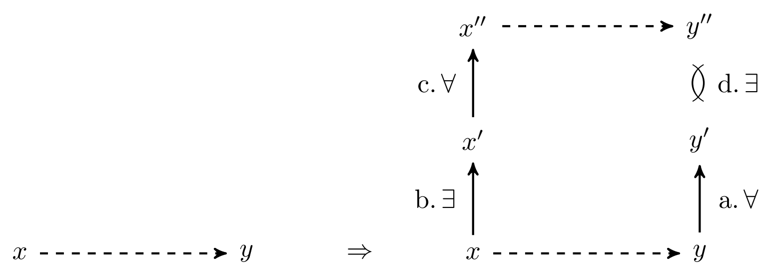

Figure 4: condition R.2 on the interplay of $\leq$ and R.

We include a proof of Proposition 2.10 as in Holliday 2015 to convey the flavor of possibility semantics and for comparison with the proof of Proposition 2.5. The reader will find it instructive to see how the restriction to regular open sets again modifies the frame constraints to come out of the analysis.

Proof of Proposition 2.10. From 2 to 1, consider an X satisfying *Persistence* and *Refinability* and $x \leq x'$. Suppose $x' \notin \Box_R X$, so there is a y' with $x'Ry'$ and $y' \notin X$. Then by *Refinability* for X, there is a $z \geq y'$

such that (i) for all $z' \geq z$, $z' \notin X$. Since $x \leq x'Ry' \leq z$, by R.1 we have a y with $xRy \between z$. Given $y \between z$, (i), and *Persistence* for X, we have $y \notin X$, which with xRy implies $x \notin \Box_R X$. Hence $\Box_R X$ satisfies *Persistence*.

Next suppose that $x \notin \Box_R X$, so there is a y with xRy and $y \notin X$. Then by *Refinability* for X, there is a $y' \geq y$ such that (ii) for all $z \geq y'$, $z \notin X$. Since $xRy \leq y'$, we have an x' as in R.2. From $y'' \between y'$, (ii), and *Persistence* for X, we have $y'' \notin X$, which with $x''Ry''$ implies $x'' \notin \Box_R X$. Hence we have shown that if $x \notin \Box_R X$, then there is an $x' \geq x$ such that for all $x'' \geq x$, $x'' \notin \Box_R X$, so $\Box_R X$ satisfies *Refinability*.

From 1 to 2, suppose R.1 does not hold, so we have $x \leq x'Ry' \leq z$ and for all y, xRy implies $y \perp z$. Let $V = \{v \in S \mid v \perp z\}$, so $x \in \Box_R V$. One can check that V satisfies *Persistence* and *Refinability*, so $V \in \mathrm{RO}(S, \leq)$. Since $y' \leq z$, we have $y' \notin V$, which with $x'Ry'$ implies $x' \notin \Box_R V$. It follows that $\Box_R V$ does not satisfy *Persistence*, so $\Box_R V \notin \mathrm{RO}(S, \leq)$. Hence $V \in \mathrm{RO}(S, \leq)$ is not closed under $\Box_R$.

Next suppose that R.2 does not hold, so we have xRy and a $y' \geq y$ such that (iii) $\forall x' \geq x\ \exists x'' \geq x'\ \forall y''$: $x''Ry''$ implies $y'' \perp y'$. Let $V = \{v \in S \mid v \perp y'\}$, so V satisfies *Persistence* and *Refinability* as above. Since $y' \geq y$, $y \notin V$, which with xRy implies $x \notin \Box_R V$. But by (iii), $\forall x' \geq x\ \exists x'' \geq x'$: $x'' \in \Box_R V$. Thus, $\Box_R V$ does not satisfy *Refinability*, so $\Box_R V \notin \mathrm{RO}(S, \leq)$. Hence $V \in \mathrm{RO}(S, \leq)$ is not closed under $\Box_R$.[4] $\Box$

In stark contrast to the case of intuitionistic frames, with possibility frames we can assume without loss of generality that something much stronger holds, namely that the accessibility relation R is partially *functional*. This creates a connection with logics of functions on topological spaces, which we discuss in §3.4. The full proof of the next proposition can be found in Holliday 2015 (§4.4, §5.3).

Proposition 2.11. For every modal possibility frame $\mathcal{F}$, there is a modal possibility frame $\mathcal{F}'$ with an accessibility relation R' such that:

1. R' is partially functional and satisfies C and R.2;
2. for all formulas $\varphi \in \mathcal{L}_1$, $\mathcal{F} \Vdash_{\mathrm{p}} \varphi$ iff $\mathcal{F}' \Vdash_{\mathrm{p}} \varphi$.

Proof. (Sketch) Given a modal possibility frame $\mathcal{F} = \langle S, \leq, R, P\rangle$, define the new frame $\mathcal{F}' = \langle S', \leq', R', P'\rangle$ as follows. Let $S' = P \setminus \{\emptyset\}$, recalling that $P = \mathrm{RO}(S, \leq)$. For $X, Y \in S'$, let $X \leq' Y$ iff $Y \subseteq X$, and let $XR'Y$ iff $Y = \mathrm{int}(\mathrm{cl}(\uparrow R[X]))$, where $R[X] = \{y \in S \mid \exists x \in X \colon xRy\}$ and $\uparrow R[X]$ is the upset generated by $R[X]$. Finally, let $P' = \mathrm{RO}(S', \leq')$. For part 1 of the proposition, clearly R' is partially functional, and it can be shown that R' satisfies C and R.2. For part 2, the map $h \colon S \to S'$ defined by $h(x) = \mathrm{int}(\mathrm{cl}(\uparrow x))$ is the kind of morphism between possibility frames that implies that $\mathcal{F}$ and $\mathcal{F}'$ validate the same modal formulas (see Holliday 2015). $\Box$

By contrast, we cannot assume without loss of generality that our *intuitionistic* modal frames are such that R is partially functional, because over such frames the principle $\Box(\varphi \vee \psi) \to (\Box\varphi \vee \Box\psi)$ is valid. While this principle might make sense for some interpretations of $\Box$ (e.g., intuitionistic provability), it is not a theorem of the minimal intuitionistic normal modal logic **HK**. Note how the departure from intuitionistic disjunction in Definition 2.8 is crucial for opening up the functional option in possibility semantics.

Also note that in the case of functional possibility frames, the C condition says that the function is *monotonic* with respect to the ordering $\leq$.

Observation 2.12 (Monotonicity)**.** If R is partially functional, and f is the associated partial function, then the C condition is equivalent to:

[4] This proof can also be given in a direct format without contraposition, but the essential feature remains the same. We use an appropriate choice of sets V that correspond to minimal valuations in the usual modal correspondence algorithm, but now subject to our two semantic constraints of *Persistence* and *Refinability* on admissible subsets.

- if $x \leq x'$ and $f(x')$ is defined, then $f(x)$ is defined and $f(x) \leq f(x')$.

If moreover R is functional, then the C condition is equivalent to:

- if $x \leq x'$, then $f(x) \leq f(x')$.

Thus, Proposition 2.11 shows that we can assume without loss of generality that the accessibility relations in our modal possibility frames are (partial) monotonic functions.

The interplay conditions between $\leq$ and R that we have reviewed for intuitionistic frames in §2.2 and for possibility frames in this section clearly cry out for a bimodal analysis, which we will give in §3.

Remark 2.13 (Non-standard semantics for $\Box$)**.** We can give semantics for classical modal logic using frames $\mathfrak{F} = \langle S, \leq, R, P \rangle$ with no required interaction conditions relating $\leq$ and R, but with P still the set of all regular open sets, which we will call *quasi possibility frames.* We can use these frames provided we modify the satisfaction relation $\Vdash_{\mathrm{p}}$ to a satisfaction relation $\Vdash_{\mathrm{q}}$ with a non-standard clause for $\Box$, namely: $[\![\Box\varphi]\!]^{\mathcal{M}}_{\mathrm{q}} = \mathrm{int}(\mathrm{cl}(\mathrm{int}(\Box_R[\![\varphi]\!]^{\mathcal{M}}_{\mathrm{q}}))) = \Box_{\leq}\Diamond_{\leq}\Box_{\leq}\Box_R[\![\varphi]\!]^{\mathcal{M}}_{\mathrm{q}}$ (cf. Remark 2.6). Since $\leq$ is transitive, the operator $\Box_{\leq}\Diamond_{\leq}\Box_{\leq}\Box_R$ on $\mathrm{RO}(S, \leq)$ still preserves finite intersections and sends S to S, so $\mathbf{K}$ is sound with respect to quasi possibility frames with the $\Vdash_{\mathrm{q}}$ semantics; and it is also complete, because possible world frames are a special case of quasi possibility frames, namely those in which $\leq$ is identity, and over possible world frames $\Vdash_{\mathrm{q}}$ agrees with the standard satisfaction relation for possible world semantics.

Note that if we consider quasi possibility frames that at least satisfy the condition C, then the $\Vdash_{\mathrm{q}}$ clause for $\Box$ can be simplified to: $[\![\Box\varphi]\!]^{\mathcal{M}} = \mathrm{int}(\mathrm{cl}(\Box_R[\![\varphi]\!]^{\mathcal{M}})) = \Box_{\leq}\Diamond_{\leq}\Box_R[\![\varphi]\!]^{\mathcal{M}}$.

We shall see syntactic analogues of these non-standard semantics in §4.1 and §4.3.

2.4 From World Models to Possibility Models

Possibility models and possible world models for modal logic are systematically related. A key construction for our purposes will be the following from Holliday 2015.

Definition 2.14 (Functional Powerset Possibilization)**.** Given a possible world model $\mathfrak{M} = \langle W, R, V \rangle$, define its *functional powerset possibilization* $\mathfrak{M}^{\wp}_f = \langle W^{\wp}, \leq, R^{\wp}, V^{\wp} \rangle$ as follows:

1. $W^{\wp} = \wp(W) \setminus \{\emptyset\}$;
2. $X \leq X'$ iff $X \supseteq X'$;
3. $XR^{\wp}Y$ iff $R[X] = Y$, where $R[X] = \{y \in W \mid \exists x \in X : xRy\}$;
4. $V^{\wp}(p) = \{X \in W^{\wp} \mid X \subseteq V(p)\}$.

It is straightforward to check that $V^{\wp}(p)$ satisfies *Persistence* and *Refinability*, and if $\mathcal{X} \subseteq W^{\wp}$ satisfies *Persistence* and *Refinability*, then so does $\Box_{R^{\wp}}\mathcal{X}$. Thus, this construction produces a possibility model. In addition, it preserves satisfaction of formulas as in part 2 of the following.

Lemma 2.15 (Possibilization Lemma)**.** For any possible world model $\mathfrak{M} = \langle W, R, V \rangle$:

1. $\mathfrak{M}^{\wp}_f$ is a possibility model such that $R^{\wp}$ is partially functional and satisfies C and $\mathsf{R.2}$;
2. for all $X \in W^{\wp}$ and $\varphi \in \mathcal{L}_1$, $\mathfrak{M}^{\wp}_f, X \Vdash_{\mathrm{p}} \varphi$ iff $\forall x \in X$: $\mathfrak{M}, x \vDash \varphi$.

Such special possibility frames based on posets of the form $\langle \wp(W) \setminus \{\emptyset\}, \supseteq \rangle$ have an independent interest. The underlying poset frames have a long history in the literature on intermediate propositional logics. The intuitionistic propositional logic of such frames for finite W is Medvedev's [1966] "logic of finite problems", and the intuitionistic propositional logic of such frames for arbitrary W is Skvortsov's [1979] "logic of infinite problems". We will return to this connection below (see Remark 3.11).

The powerset possibilization can also be carried out at the level of frames, taking a possible world frame $\langle W, R \rangle$ to a possibility frame $\langle W^{\wp}, \leq, R^{\wp}, P^{\wp} \rangle$, where $P^{\wp} = \mathrm{RO}(W^{\wp}, \leq)$ is the set of all principal upsets in the poset $\langle W^{\wp}, \leq \rangle$ plus $\emptyset$. It is easy to see from Lemma 2.15 that these frames validate exactly the same modal formulas. However, there can be no such general construction in the other direction. For there are possibility frames whose logics are not only Kripke-incomplete but even Kripke-unsound, i.e., not sound with respect to any possible world frame (see Holliday 2015).

If we go to "general frame" versions of possible world frames and possibility frames, then there is a general duality going back and forth (see Holliday 2015), but we will not need this further analysis here. For further discussion of transformations between possible world models and possibility models, as well as general frame versions thereof, see Harrison-Trainor 2016b.

3 Bimodal Perspective on Possibility Frames and Their Logics

In this section, we develop a bimodal approach to possibility frames that is analogous to bimodal approaches to intuitionistic modal frames (see, e.g., Wolter and Zakharyaschev 1999). For a detailed study of modal logics with families of operators and their properties, we refer to Gabbay et al. 2003.

3.1 Language and Semantics

We now move from the unimodal language of Definition 2.1 to the following bimodal language.

Definition 3.1 (Bimodal Language)**.** Given a nonempty set Prop of propositional variables, let $\mathcal{L}_2$ be the language defined by the grammar

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \to \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid [\leq]\varphi \mid [R]\varphi,$$

where $p \in \mathsf{Prop}$. We define the classical existential dual modalities by $\langle \leq \rangle \varphi := \neg[\leq]\neg\varphi$ and $\langle R \rangle \varphi := \neg[R]\neg\varphi$. We will also find many uses for the "cofinality modality" defined by $[co]\varphi := [\leq]\langle \leq \rangle \varphi$.

The semantics of this bimodal language is totally standard over models with two accessibility relations. We interpret formulas of $\mathcal{L}_2$ in models $\mathfrak{M} = \langle S, \leq, R, V \rangle$ based on birelational frames $\mathfrak{F} = \langle S, \leq, R \rangle$, with $[\leq]$ as the box modality for the $\leq$ relation and $[R]$ as the box modality for the R relation.

Given an intuitionistic frame or possibility frame $\mathcal{F} = \langle S, \leq, R, P \rangle$, we obtain a standard birelational frame $\mathfrak{F} = \langle S, \leq, R \rangle$ by dropping the set P of admissible propositions. As usual, one may think of the set of admissible propositions in a standard birelational frame as $\wp(S)$.

The following fact about the semantics of the cofinality modality $[co]$ and its connection to regular open sets as in §2.3 will be important in what follows.

Fact 3.2. For any birelational model $\mathfrak{M} = \langle S, \leq, R \rangle$ with $\leq$ a preorder and $\varphi \in \mathcal{L}_2$:

1. $[\![[co]\varphi]\!]^{\mathfrak{M}} = \mathrm{int}(\mathrm{cl}([\![\varphi]\!]^{\mathfrak{M}}))$, where as before, for $X \subseteq S$, $\mathrm{int}(X) = \{y \in S \mid \forall x \geq y \colon x \in X\}$ and $\mathrm{cl}(X) = \{y \in S \mid \exists x \geq y \colon x \in X\}$;

2. $[\![[co]\varphi]\!]^{\mathfrak{M}}$ is a regular open set in the upset topology on $\langle S, \leq \rangle$;

3. $[\![\varphi]\!]^{\mathfrak{M}}$ is a regular open set in the upset topology on $\langle S, \leq \rangle$ iff $[\![\varphi]\!]^{\mathfrak{M}} = [\![[co]\varphi]\!]^{\mathfrak{M}}$, or equivalently, iff $\varphi \leftrightarrow [co]\varphi$ is globally true in $\mathfrak{M}$.

3.2 Logics, Axioms, and Proofs

Thinking of $\leq$ as an ordering relation and R as an accessibility relation as in intuitionistic models and possibility models, we can consider a wide variety of bimodal logics, starting from the plain fusion $\mathbf{S4} \otimes \mathbf{K}$ of $\mathbf{S4}$ for $[\leq]$ and $\mathbf{K}$ for $[R]$, and then adding various axioms that specialize the ordering component or the accessibility component, or the bridge between the two. The axioms that follow all reflect semantic constraints that we have encountered before, but as we shall see, they also have an independent interest.

Definition 3.3 (Bimodal Logics)**.** We will consider normal extensions of $\mathbf{S4} \otimes \mathbf{K}$ obtained by adding some of the following axioms:

P(artial function) $\langle R \rangle p \to [R]p$;

F(unction) $\langle R \rangle p \leftrightarrow [R]p$;

C(ontinuous) $[R][\leq]p \to [\leq][R]p$; R.1 $[R][\leq]p \to [\leq][R][co]p$

O(pen) $[\leq][R]p \to [R][\leq]p$;

R(egular) $[co][R][co]p \leftrightarrow [R][co]p$; R.2 $[co][R][\leq]p \to [R][co]p$.

We adopt the following naming convention for extensions of the basic fusion system $\mathbf{S4} \otimes \mathbf{K}$: the logic

$$\mathbf{X}\text{-}\mathbf{Y}\text{-}\mathbf{Z}_1 \ldots \mathbf{Z}_n$$

is the smallest normal bimodal logic that extends the fusion $\mathbf{X} \otimes \mathbf{Y}$ of the logic $\mathbf{X}$ for $[\leq]$ and the logic $\mathbf{Y}$ for $[R]$ with the bimodal interaction axioms $\mathbf{Z}_1 \ldots \mathbf{Z}_n$. Standard notation for this would be

$$(\mathbf{X} \otimes \mathbf{Y}) \oplus \mathbf{Z}_1 \oplus \cdots \oplus \mathbf{Z}_n$$

but for cleanliness we will use the $\mathbf{X}\text{-}\mathbf{Y}\text{-}\mathbf{Z}_1 \ldots \mathbf{Z}_n$ format. Let $\mathbf{P}$ be the extension of $\mathbf{K}$ for $[R]$ with the P axiom for $[R]$. Let $\mathbf{F}$ be the extension of $\mathbf{K}$ for $[R]$ with the F axiom for $[R]$. Then **S4-F** is the extension of **S4-K** ($\mathbf{S4} \otimes \mathbf{K}$) with the F axiom, **S4-F-C** is the extension of **S4-F** with the interaction axiom C, etc.

Intuitively, the axioms listed above express properties linking the two relations so that intuitionistic modal semantics or possibility semantics gets the right preservation properties, as studied earlier. Alternatively, the axioms may be viewed as describing properties of functions on topological spaces that preserve less or more natural structure. We will explain the topological perspective in §3.4. For now, let us observe how one of the key semantic results of §2.3 has a precise syntactic analogue in this bimodal setup.

With the context of Fact 3.2, the R axiom has a clear meaning: *the result of applying the* $[R]$ *operator to a regular open set* $[\![[co]\varphi]\!]^{\mathfrak{M}}$*, i.e.,* $[\![[R][co]\varphi]\!]^{\mathfrak{M}}$*, is also regular open.* Since $[\![[R][co]\varphi]\!]^{\mathfrak{M}} = \Box_R [\![[co]\varphi]\!]^{\mathfrak{M}}$, where $\Box_R$ is the operation from Definitions 2.2 and 2.7, the meaning of R can be equivalently stated as: the set of regular open sets is closed under $\Box_R$. Recall that the closure of the set of regular open sets under $\Box_R$ was exactly the topic of Proposition 2.10, which showed that such closure is equivalent to the frame satisfying the conditions $\mathsf{R.1}$ and $\mathsf{R.2}$. It is no accident that we also have axioms labeled as 'R.1' and 'R.2' in Definition 3.3. In §3.3, we will show that the axioms R.1 and R.2 *correspond* to the frame conditions $\mathsf{R.1}$ and $\mathsf{R.2}$, respectively. Thus, the *syntactic analogue* of the semantic Proposition 2.10 is the following.

Fact 3.4. The least normal extension of **S4-K** with the R axiom and the least normal extension of **S4-K** with the R.1 and R.2 axioms are the same, as sets of theorems.

Proof. We prove the fact in three parts. First, we show that given **S4** for $[\leq]$, adding R.2 is equivalent to adding the left-to-right direction of the R axiom:

$\mathrm{R}_{\rightarrow}$ $[co][R][co]p \rightarrow [R][co]p$.

From R.2 to $\mathrm{R}_{\rightarrow}$, as an instance of R.2, substituting $\langle\leq\rangle p$ for p, we have $[co][R][\leq]\langle\leq\rangle p \rightarrow [R][co]\langle\leq\rangle p$, i.e., $[co][R][co]p \rightarrow [R][\leq]\langle\leq\rangle\langle\leq\rangle p$, which with the 4 axiom for $[\leq]$ implies $[co][R][co]p \rightarrow [R][co]p$.

From $\mathrm{R}_{\rightarrow}$ to R.2, the antecedent of R.2 implies the antecedent of $\mathrm{R}_{\rightarrow}$ given the T axiom for $[\leq]$.

Second, we show that given **S4** for $[\leq]$, adding R.1 gives us the right-to-left direction of the R axiom:

$\mathrm{R}_{\leftarrow}$ $[R][co]p \rightarrow [co][R][co]p$.

As an instance of R.1, substituting $\langle\leq\rangle p$ for p, we have $[R][\leq]\langle\leq\rangle p \rightarrow [\leq][R][co]\langle\leq\rangle p$, i.e., $[R][co]p \rightarrow [\leq][R][\leq]\langle\leq\rangle\langle\leq\rangle p$, which with the T axiom for $[\leq]$ implies $[R][co]p \rightarrow [\leq]\langle\leq\rangle[R][\leq]\langle\leq\rangle\langle\leq\rangle p$, which with the 4 axiom for $[\leq]$ implies $[R][co]p \rightarrow [\leq]\langle\leq\rangle[R][\leq]\langle\leq\rangle p$, i.e., $[R][co]p \rightarrow [co][R][co]p$.

Third, we show that given **S4** for $[\leq]$, adding the R axiom gives us the R.1 axiom. From $\mathrm{R}_{\rightarrow}$ and the normality of $[\leq]$, we have $[\leq][co][R][co]p \rightarrow [\leq][R][co]p$. Given the T axiom and the normality of $[R]$ and $[\leq]$, $[R][\leq]p$ implies $[R][\leq]\langle\leq\rangle p$, i.e., $[R][co]p$, which with $\mathrm{R}_{\leftarrow}$ gives us $[co][R][co]p$, which with the 4 axiom and the normality of $[\leq]$ gives us $[\leq][co][R][co]p$, which with the previous sentence gives us $[\leq][R][co]p$. Then since $[R][\leq]p$ is the antecedent of the R.1 axiom and $[\leq][R][co]p$ is the consequent, we are done. □

Fact 3.4 is a striking example of how we can drive semantic facts from the metatheory of possibility models down into syntactic facts with our bimodal object language. We will see more examples in §4.3. For now we note just one more, simpler example. Recall the observation after Proposition 2.10 that the condition $\mathsf{R.1}$ of possibility frames follows from the condition C of intuitionistic modal frames. The syntactic analogue of that observation, in light of the correspondences we will establish in §3.3, is the following.

Fact 3.5. The R.1 axiom is a theorem of **S4-K-C**.

Proof. As an instance of the C axiom, substituting $[co]p$ for p, we have $[R][\leq][co]p \rightarrow [\leq][R][co]p$. Since $[\leq]p \rightarrow [\leq][co]p$, i.e., $[\leq]p \rightarrow [\leq][\leq]\langle\leq\rangle p$, is a theorem of **S4**, we have $[R][\leq]p \rightarrow [R][\leq][co]p$, which with the theorem of the previous sentence gives us $[R][\leq]p \rightarrow [\leq][R][co]p$, which is R.1. □

Remark 3.6. One further general perspective on what we are doing here, and also elsewhere in this paper, is the remarkable fact that inside simple-looking logics such as modal **S4**, defined modalities of the special forms $[\leq]\langle\leq\rangle\varphi$ and $\langle\leq\rangle[\leq]\varphi$ generate rich sublogics whose validities can be surprising. We supply typical deductive information of this kind when proving the correctness of our main translation in §4.1. For one example already, observe that $[co][co]\varphi \leftrightarrow [co]\varphi$ is a theorem of **S4**.

3.3 Correspondence over Birelational Frames

We can analyze all of the axioms listed in Definition 3.3 in terms of standard modal *frame correspondence* (see Blackburn et al. 2001, Ch. 3). In fact, this analysis is straightforward, since all of the axioms have a syntactic *Sahlqvist form*, either explicitly, or via some simple manipulation.

Proposition 3.7 (Frame Correspondence)**.** For any birelational frame $\mathfrak{F} = \langle S, \leq, R \rangle$:

1. the C axiom $[R][\leq]p \to [\leq][R]p$ is valid over $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies

   C: if $x \leq x'Ry'$, then $\exists y$: $xRy \leq y'$;

2. if R is functional, then the C axiom is valid over $\mathfrak{F}$ iff R is *monotonic* as a function with respect to $\leq$ (recall Observation 2.12);

3. the O axiom $[\leq][R]p \to [R][\leq]p$ is valid over $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies

   O: if $xRy \leq y'$, then $\exists x'$: $x \leq x'Ry'$;

4. the R.1 axiom $[R][\leq]p \to [\leq][R][co]p$ is valid over $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies

   $\mathsf{R.1}$: if $x \leq x'Ry' \leq z$, then $\exists y$: $xRy \between z$;

5. the R.2 axiom $[co][R][\leq]p \to [R][co]p$ is valid over $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies

   $\mathsf{R.2}$: if xRy, then $\forall y' \geq y\ \exists x' \geq x\ \forall x'' \geq x'\ \exists y'' \between y'$: $x''Ry''$;

6. if $\leq$ is a preorder, then the R axiom $[co][R][co]p \leftrightarrow [R][co]p$ is valid over $\mathfrak{F}$ iff $\mathfrak{F}$ satisfies $\mathsf{R.1}$ and $\mathsf{R.2}$, which by Proposition 2.10 is equivalent to $\mathrm{RO}(S, \leq)$ being closed under $\Box_R$.

Proof. We do one case explicitly just to familiarize the reader with the setting. But we emphasize once more that there is a general algorithm transforming Sahlqvist-type axioms into first-order frame equivalents.

For part 4, suppose $\mathfrak{F}$ satisfies $\mathsf{R.1}$ and $\mathfrak{M}$ is a model based on $\mathfrak{F}$ with a state x such that $\mathfrak{M}, x \vDash [R][\leq]p$. To show that $\mathfrak{M}, x \vDash [\leq][R][co]p$, we must show that $x \leq x'Ry' \leq z$ implies $\mathfrak{M}, z \vDash \langle \leq \rangle p$. By $\mathsf{R.1}$, $x \leq x'Ry' \leq z$ implies that there is a y such that $xRy \between z$, so there is a $z' \geq y$ such that $z' \geq z$. Since $\mathfrak{M}, x \vDash [R][\leq]p$, $xRy \leq z'$ implies $\mathcal{M}, z' \vDash p$, which with $z' \geq z$ implies $\mathfrak{M}, z \vDash \langle \leq \rangle p$, as desired. Now suppose $\mathfrak{F}$ does not satisfy $\mathsf{R.1}$, so we have $x \leq x'Ry' \leq z$ but for all y, xRy implies $y \perp z$. Then following the proof of Proposition 2.10, define a model $\mathfrak{M}$ on $\mathfrak{F}$ with a valuation V such that $V(p) = \{v \in S \mid v \perp z\}$, recalling that $v \perp z$ means there is no u such that $u \geq v$ and $u \geq z$. Then observe that $\mathfrak{M}, x \vDash [R][\leq]p$ but $\mathfrak{M}, x \nvDash [\leq][R][co]p$.

Similarly, one can prove part 5 by following the relevant parts of the proof of Proposition 2.10. Alternatively, it suffices to note that the equivalent contrapositive form of R.2 is a Sahlqvist formula.

Part 6 follows from parts 4-5 together with the semantic analogue of the observation in Fact 3.4 that relative to **S4** for $[\leq]$, taking R as an axiom is equivalent to taking R.1 and R.2 as axioms. $\Box$

These correspondence results establish soundness of the earlier bimodal logics for their intended models. In addition, the Sahlqvist Completeness Theorem (see again Blackburn et al. 2001, p. 210) applied to the above axioms yields something more.

Theorem 3.8 (Completeness)**.** For any normal extension **L** of **S4-K** obtained by adding axioms among those in Proposition 3.7, **L** is sound and complete with respect to the class of birelational frames with the corresponding properties.

From Propositions 3.7.6 and 2.10 and Theorem 3.8, it follows that **S4-K-R** is *The Bimodal Logic of Possibility Frames* in the following sense.

Theorem 3.9 (The Bimodal Logic of Possibility Frames)**.** **S4-K-R** is sound and complete with respect to the class of all birelational frames $\langle S, \leq, R\rangle$ obtained from possibility frames $\langle S, \leq, R, P\rangle$.

Similarly, **S4-P-R** is *The Bimodal Logic of Partially Functional Possibility Frames* and **S4-F-R** is *The Bimodal Logic of Functional Possibility Frames*.

Now that we know the frame conditions corresponding to our bimodal axioms, we can see that for any possible world model, its *functional powerset possibilization* as in Definition 2.14 is a model of a strong bimodal logic. The following lemma is a key tool that we will use repeatedly.

Lemma 3.10 (Powerset Possibilizations Bimodally)**.** For any model $\mathfrak{M} = \langle W, R, V\rangle$:

1. $\mathfrak{M}_f^{\wp}$ (Definition 2.14) is an **S4-P-CR** model;
2. if R is serial, then $\mathfrak{M}_f^{\wp}$ is an **S4-F-CR** model.

Proof. It is straightforward to check that $\mathfrak{M}_f^{\wp}$ satisfies the frame conditions that correspond to the specified axioms according to Proposition 3.7. □

Remark 3.11. If $\mathfrak{M}$ is finite, then $\mathfrak{M}_f^{\wp}$ is a model of still stronger bimodal logics. Most obviously, over finite powerset possibilizations, the logic of $[\leq]$ can be strengthened from **S4** to **Grz**, which extends **K** with the Grz axiom $[\leq]([\leq](p \to [\leq]p) \to p) \to p$. Grz is valid on a frame $\langle S, \leq, R\rangle$ iff $\leq$ is a Noetherian partial order, i.e., a partial order that contains no infinite chain of distinct elements (see, e.g., Chagrov and Zakharyaschev 1997, p. 83). In fact, the logic of $[\leq]$ over finite powerset possibilizations is exactly the modal logic **Medv** of Medvedev frames, i.e., frames $\langle S, \leq\rangle$ where $S = \wp(W) \setminus \{\emptyset\}$ for some nonempty finite W and $x \leq y$ iff $x \supseteq y$. **Medv** is a proper extension of **Grz**, though it is not finitely axiomatizable [Prucnal, 1979] and even its recursive axiomatizability is an open question (cf. Shehtman 1990, Holliday Forthcoming). Medvedev frames arise naturally in the analysis of constructive mathematics as a logic of "finite problems" [Medvedev, 1966], and they have re-emerged recently in the semantics of questions in natural language as setting directions of inquiry [Ciardelli, 2009]. However, since our main focus in this paper is not on the pure logic of $[\leq]$, we will formulate most of our results to follow in terms of **S4** for the sake of familiarity.

3.4 Dynamic Topological Spaces

Our bimodal frames and the above axioms may have seemed to be merely generated by the technical needs of possibility semantics. However, there is an independent interest to the structures we have found, especially when the accessibility relation is a partial or total function. One interesting connection is with an earlier framework extending the usual topological semantics for modal logic to spaces where the topology also has a "dynamics" in the form of a transformation taking the space to itself, usually a continuous map as in dynamical systems (see Artemov et al. 1997, Kremer and Mints 2005).

Definition 3.12 (Dynamic Topological Spaces and Models)**.** A *dynamic topological space* is a tuple $\mathcal{F} = \langle S, \mathcal{T}, f\rangle$ where $\langle S, \mathcal{T}\rangle$ is a topological space and $f\colon S \to S$. A *dynamic topological model* based on $\mathcal{F}$ is a tuple $\mathcal{M} = \langle S, \mathcal{T}, f, V\rangle$ where $V\colon \mathsf{Prop} \to \wp(S)$.

Definition 3.13 (Dynamic Topological Semantics)**.** Given a dynamic topological model $\mathcal{M} = \langle S, \mathcal{T}, f, V\rangle$, we define an interpretation function $[\![\cdot]\!]^{\mathcal{M}}\colon \mathcal{L}_2 \to \wp(S)$ as follows:

1. $[\![p]\!]^{\mathcal{M}} = V(p)$;

2. $[\![\neg\varphi]\!]^{\mathcal{M}} = S \setminus [\![\varphi]\!]^{\mathcal{M}}$; $[\![\varphi \to \psi]\!]^{\mathcal{M}} = (S \setminus [\![\varphi]\!]^{\mathcal{M}}) \cup [\![\psi]\!]^{\mathcal{M}}$;

3. $[\![\varphi \wedge \psi]\!]^{\mathcal{M}} = [\![\varphi]\!]^{\mathcal{M}} \cap [\![\psi]\!]^{\mathcal{M}}$; $[\![\varphi \vee \psi]\!]^{\mathcal{M}} = [\![\varphi]\!]^{\mathcal{M}} \cup [\![\psi]\!]^{\mathcal{M}}$;

4. $[\![[\leq]\varphi]\!]^{\mathcal{M}} = \mathrm{int}([\![\varphi]\!]^{\mathcal{M}})$;

5. $[\![[R]\varphi]\!]^{\mathcal{M}} = f^{-1}[[\![\varphi]\!]^{\mathcal{M}}]$.

Again we can analyze what relevant structure means in terms of bimodal axioms.

Proposition 3.14 (Frame Correspondence)**.** For any dynamic topological space $\mathcal{F} = \langle S, \mathcal{T}, f \rangle$:

1. the C axiom $[R][\leq]p \to [\leq][R]p$ is valid over $\mathcal{F}$ iff f is a continuous map;

2. the O axiom $[\leq][R]p \to [R][\leq]p$ is valid over $\mathcal{F}$ iff f is an open map;

3. the R axiom $[co][R][co]p \leftrightarrow [R][co]p$ is valid over $\mathcal{F}$ iff f is an R-map [Carnahan, 1973], i.e., if O is regular open, then $f^{-1}[O]$ is regular open.

These correspondences now work over topological spaces, which provide a generalized setting compared to the relational frames for bimodal logic that we considered earlier on. The preorders $\langle S, \leq \rangle$ in bimodal frames give rise to special topological spaces: the upset topology arising from $\langle S, \leq \rangle$ gives us an *Alexandrov* space, i.e., a topological space in which the intersection of any family of open sets is open. By contrast, Definition 3.12 allows any topological space. Nonetheless, similar correspondence reasoning applies. A proof for part 1 of Proposition 3.14 is available in the literature (see again Kremer and Mints 2005). To familiarize the reader with this reasoning style, we give the proofs for parts 2 and 3.

Proof. For part 2, from left to right, given an open set O in $\mathcal{T}$, we must show that $f[O]$ is also open. Take a model $\mathcal{M} = \langle S, \mathcal{T}, f, V \rangle$ where $V(p) = f[O]$, so $[\![[R]p]\!]^{\mathcal{M}} = f^{-1}[f[O]]$. Then since $O \subseteq f^{-1}[f[O]]$ and O is open, $O \subseteq \mathrm{int}(f^{-1}[f[O]]) = [\![[\leq][R]p]\!]^{\mathcal{M}}$, so by the assumption that the O axiom is valid, $O \subseteq [\![[R][\leq]p]\!]^{\mathcal{M}}$, which implies $f[O] \subseteq [\![[\leq]p]\!]^{\mathcal{M}} = \mathrm{int}(f[O])$, which implies that $f[O]$ is open in $\mathcal{T}$. In the other direction, suppose f is an open map. From $[\![[\leq][R]p]\!]^{\mathcal{M}} = \mathrm{int}([\![[R]p]\!]^{\mathcal{M}}) \subseteq [\![[R]p]\!]^{\mathcal{M}}$, we have $f[[\![[\leq][R]p]\!]^{\mathcal{M}}] \subseteq [\![p]\!]^{\mathcal{M}}$. Since f is an open map, $f[\mathrm{int}([\![[R]p]\!]^{\mathcal{M}})]$ is open, so $f[\mathrm{int}([\![[R]p]\!]^{\mathcal{M}})] = f[[\![[\leq][R]p]\!]^{\mathcal{M}}] \subseteq [\![p]\!]^{\mathcal{M}}$ implies $f[[\![[\leq][R]p]\!]^{\mathcal{M}}] \subseteq \mathrm{int}([\![p]\!]^{\mathcal{M}}) = [\![[\leq]p]\!]^{\mathcal{M}}$, which means $[\![[\leq][R]p]\!]^{\mathcal{M}} \subseteq [\![[R][\leq]p]\!]^{\mathcal{M}}$.

For part 3, from left to right, assuming that O is regular open, we must show that $f^{-1}[O]$ is regular open. Take a model $\mathcal{M} = \langle S, \mathcal{T}, f, V \rangle$ where $V(p) = O$. Since O is regular open, $O = \mathrm{int}(\mathrm{cl}(O)) = \mathrm{int}(\mathrm{cl}([\![p]\!]^{\mathcal{M}}) = [\![[co]p]\!]^{\mathcal{M}}$, so $f^{-1}[O] = [\![[R][co]p]\!]^{\mathcal{M}}$. Then since the R axiom is valid, $f^{-1}[O] = [\![[R][co]p]\!]^{\mathcal{M}} = [\![[co][R][co]p]\!]^{\mathcal{M}} = \mathrm{int}(\mathrm{cl}([\![[R][co]p]\!]^{\mathcal{M}})) = \mathrm{int}(\mathrm{cl}(f^{-1}[O]))$, so $f^{-1}[O]$ is regular open. In the other direction, suppose f is an R-map. Then since $[\![[co]p]\!]^{\mathcal{M}} = \mathrm{int}(\mathrm{cl}([\![p]\!]^{\mathcal{M}}))$ is regular open and f is an R-map, $f^{-1}[[\![[co]p]\!]^{\mathcal{M}}] = [\![[R][co]p]\!]^{\mathcal{M}}$ is regular open, so $[\![[R][co]p]\!]^{\mathcal{M}} = \mathrm{int}(\mathrm{cl}([\![[R][co]p]\!]^{\mathcal{M}})) = [\![[co][R][co]p]\!]^{\mathcal{M}}$. $\square$

Here is one basic completeness theorem from the literature.

Theorem 3.15 (Artemov et al. 1997)**.** The logic **S4-F-C** is sound and complete with respect to the class of dynamic topological spaces with continuous maps.

Our analysis adds new results of this sort, of which we formulate one.

Theorem 3.16 (Logic of R-maps)**.** The logic **S4-F-R** is sound and complete with respect to the class of dynamic topological spaces with R-maps.

Proof. By Fact 3.4, taking R as an axiom is equivalent to taking R.1 and R.2, which are Sahlqvist formulas. Therefore, **S4-F-R** is relationally sound and complete. This implies that **S4-F-R** is sound and complete with respect to dynamic topological spaces based on Alexandrov spaces with R-maps. Thus, by Proposition 3.14.3, **S4-F-R** is also sound and complete with respect to all dynamic topological spaces with R-maps. □

4 Bimodal Perspective on Possibility Semantics via Translations

Many of our earlier observations can be summed up in a particularly simple format, that of a *relative interpretation* of possibility logics into bimodal logics via suitable *translations*. In this section we state the main results of the paper, providing a translation of basic modal logic into bimodal logics based on possibility semantics. We will prove that this translation is full and faithful. We will also show how our translation is related to two classic ones from the literature, being essentially a composition of a Modal Gödel-Gentzen translation of the basic modal logic into intuitionistic modal logic and an Extended Gödel-McKinsey-Tarski translation of intuitionistic modal logic into bimodal logic. Finally, we will show how the translation can be simplified if we strengthen the bimodal logic to a logic of intuitionistic modal frames or possibility frames.

4.1 Translating Possibility Logic into Bimodal Logic

Below we give our first translation of unimodal logic over possibility models into bimodal logic. The translation of propositional variables is based on the requirement in possibility semantics that propositions be regular open sets; the translations of the Boolean connectives are based on the possibility satisfaction relation $\Vdash_{\mathrm{p}}$ (Definition 2.8); and the translation of $\Box$ is based on the first of the non-standard semantics introduced in Remark 2.13. In §4.4, we will give a much simpler translation based on the standard semantics for $\Box$, but which requires a stronger target bimodal logic with axioms corresponding to the interaction conditions between $\leq$ and R in possibility frames.

Definition 4.1 (g translation)**.** Define a function $\mathrm{g}\colon \mathcal{L}_1 \to \mathcal{L}_2$ recursively as follows:

1. $\mathrm{g}(p) = [co]p$;
2. $\mathrm{g}(\neg\varphi) = [\leq]\neg\mathrm{g}(\varphi)$;
3. $\mathrm{g}(\varphi \to \psi) = [\leq](\mathrm{g}(\varphi) \to \mathrm{g}(\psi))$;
4. $\mathrm{g}(\varphi \wedge \psi) = \mathrm{g}(\varphi) \wedge \mathrm{g}(\psi)$;
5. $\mathrm{g}(\varphi \vee \psi) = [co](\mathrm{g}(\varphi) \vee \mathrm{g}(\psi))$;
6. $\mathrm{g}(\Box\varphi) = [co][\leq][R]\mathrm{g}(\varphi)$.

A key observation about this translation is that given **S4** for $[\leq]$ and **K** for $[R]$, the modality $[co][\leq][R]$ is a normal modality, i.e., it distributes over implication and admits necessitation (cf. Remark 2.13).

Using the g translation, we will show that the unimodal logic of an arbitrary relation can be "decomposed" as a bimodal logic of two very special relations: a *preorder* for the inclusion modality $[\leq]$ and a (partial) *function* for the accessibility modality $[R]$. (On the point of partial vs. total functions, we will show that g embeds **K** into a logic where the $[R]$ modality is partially functional and embeds **KD** into a logic where the $[R]$ modality is totally functional.) This should remind the reader of the semantic fact from §2.3 that in possibility semantics, we may assume without loss of generality that the accessibility relation is (partially) functional. We thus arrive at the first of our informal slogans:

$$\text{RELATION} \quad \hookrightarrow \quad \text{PREORDER} + \text{PARTIAL FUNCTION},$$

where $\hookrightarrow$ means that the modal logic of what appears on the left can be embedded in the bimodal logic of what appears on the right. This slogan is supported by the following theorem, the first of our main results.

Theorem 4.2 (Embedding I)**.** For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{K}} \varphi$ iff $\vdash_{\mathbf{S4\text{-}P}} \mathrm{g}(\varphi)$.

We will initially prove Theorem 4.2 without the use of possibility semantics, i.e., without facts involving possibility frames and the possibility satisfaction relation $\Vdash_{\mathrm{p}}$. Afterward we will give a possibility-semantic proof of the left-to-right direction of Theorem 4.2 (see Remark 4.5).

First we give a more syntactic proof of the left-to-right direction. For this we need a lemma whose proof tell us quite a bit about the deductive power of even the weak fusion logic **S4-K**.

Lemma 4.3 (Persistence and Refinability)**.** For all formulas $\varphi \in \mathcal{L}_1$:

1. $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi) \to [\leq]\mathrm{g}(\varphi)$;
2. $\vdash_{\mathbf{S4\text{-}K}} [co]\mathrm{g}(\varphi) \to \mathrm{g}(\varphi)$.

Proof. The proof of part 1 is by induction on φ.

For the atomic case, $\mathrm{g}(p) \to [\leq]\mathrm{g}(p)$ is $[\leq]\langle\leq\rangle p \to [\leq][\leq]\langle\leq\rangle p$, which is an instance of the 4 axiom.

For the $\neg$ case, $\mathrm{g}(\neg\varphi) \to [\leq]\mathrm{g}(\neg\varphi)$ is $[\leq]\neg\mathrm{g}(\varphi) \to [\leq][\leq]\neg\mathrm{g}(\varphi)$, which is an instance of the 4 axiom.

For the $\Box$ case, $\mathrm{g}(\Box\varphi) \to [\leq]\mathrm{g}(\Box)\varphi$ is $[co][\leq][R]\mathrm{g}(\varphi) \to [\leq][co][\leq][R]\mathrm{g}(\varphi)$, which unpacks to the formula $[\leq]\langle\leq\rangle[\leq][R]\mathrm{g}(\varphi) \to [\leq][\leq]\langle\leq\rangle[\leq][R]\mathrm{g}(\varphi)$, which is an instance of the 4 axiom.

For the $\wedge$ case, the inductive hypothesis is that $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi) \to [\leq]\mathrm{g}(\varphi)$ and $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\psi) \to [\leq]\mathrm{g}(\psi)$. Then since $\mathrm{g}(\varphi\wedge\psi)$ is $\mathrm{g}(\varphi)\wedge\mathrm{g}(\psi)$, we have $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi\wedge\psi) \to ([\leq]\mathrm{g}(\varphi) \wedge [\leq]\mathrm{g}(\psi))$ and hence $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi\wedge\psi) \to [\leq](\mathrm{g}(\varphi)\wedge\mathrm{g}(\psi))$ by the normality of $[\leq]$, so $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi\wedge\psi) \to [\leq]\mathrm{g}(\varphi\wedge\psi)$.

The proof of part 2 is also by induction on φ.

For the atomic case, $[co]\mathrm{g}(p) \to \mathrm{g}(p)$ is $[co][co]p \to [co]p$, which is a theorem of **S4** for $[\leq]$.

For the $\neg$ case, $[co]\mathrm{g}(\neg\varphi) \to \mathrm{g}(\neg\varphi)$ is $[co][\leq]\neg\mathrm{g}(\varphi) \to [\leq]\neg\mathrm{g}(\varphi)$. We will show that the contrapositive of this formula, $\langle\leq\rangle\mathrm{g}(\varphi) \to \langle\leq\rangle[co]\mathrm{g}(\varphi)$, is a theorem of **S4-K**. First, $[\leq]\mathrm{g}(\varphi) \to [co]\mathrm{g}(\varphi)$ is a theorem of **S4** for $[\leq]$, so

$$\vdash_{\mathbf{S4\text{-}K}} \langle\leq\rangle[\leq]\mathrm{g}(\varphi) \to \langle\leq\rangle[co]\mathrm{g}(\varphi) \tag{1}$$

by the normality of $[\leq]$. Next, part 1 gives us $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi) \to [\leq]\mathrm{g}(\varphi)$ and hence

$$\vdash_{\mathbf{S4\text{-}K}} \langle\leq\rangle\mathrm{g}(\varphi) \to \langle\leq\rangle[\leq]\mathrm{g}(\varphi) \tag{2}$$

by the normality of $[\leq]$. Then (1)-(2) imply $\vdash_{\mathbf{S4\text{-}K}} \langle\leq\rangle\mathrm{g}(\varphi) \to \langle\leq\rangle[co]\mathrm{g}(\varphi)$, as desired.

For the $\wedge$ case, the inductive hypothesis is that $\vdash_{\mathbf{S4\text{-}K}} [co]\mathrm{g}(\varphi) \to \mathrm{g}(\varphi)$ and $\vdash_{\mathbf{S4\text{-}K}} [co]\mathrm{g}(\psi) \to \mathrm{g}(\psi)$, which implies that $\vdash_{\mathbf{S4\text{-}K}} [co](\mathrm{g}(\varphi)\wedge\mathrm{g}(\psi)) \to (\mathrm{g}(\varphi)\wedge\mathrm{g}(\psi))$ and hence $\vdash_{\mathbf{S4\text{-}K}} [co]\mathrm{g}(\varphi\wedge\psi) \to \mathrm{g}(\varphi\wedge\psi)$.

For the $\Box$ case, $[co]\mathrm{g}(\Box\varphi) \to \mathrm{g}(\Box\varphi)$ is $[co][co][\leq][R]\mathrm{g}(\varphi) \to [co][\leq][R]\mathrm{g}(\varphi)$, and for any ψ, $[co][co][\leq]\psi \to [co][\leq]\psi$ is derivable from the **S4** axioms. $\Box$

While Lemma 4.3 will be key to proving the left-to-right direction of Theorem 4.2, showing that if **K** proves φ then even the weak fusion **S4-K** proves $\mathrm{g}(\varphi)$, the following lemma is the key to proving the right-to-left direction, showing that if the stronger logic **S4-P** proves $\mathrm{g}(\varphi)$, then **K** proves φ. Recall that $\vDash$ is the standard satisfaction relation for bimodal or unimodal possible world semantics.

Lemma 4.4 (Powerset Possibilization and the g Translation)**.** For any possible world model $\mathfrak{M} = \langle W, R, V\rangle$, its functional powerset possibilization $\mathfrak{M}_f^{\wp}$ (Definition 2.14) is such that for all $\varphi \in \mathcal{L}_1$ and $X \in W^{\wp}$:

$$\mathfrak{M}_f^{\wp}, X \vDash \mathrm{g}(\varphi) \text{ iff } \forall x \in X : \mathfrak{M}, x \vDash \varphi.$$

Proof. The proof is an easy induction on φ using the definition of g and the fact that $\mathfrak{M}_f^{\wp}, X \vDash [co]\varphi$ holds iff $\forall X' \subseteq X\ \exists X'' \subseteq X'$ such that $\mathfrak{M}_f^{\wp}, X'' \vDash \varphi$, which holds iff $\forall x \in X$, $\mathfrak{M}_f^{\wp}, \{x\} \vDash \varphi$. In the $\Box$ case, where $\mathrm{g}(\Box\varphi) = [co][\leq][R]\mathrm{g}(\varphi)$, by the previous observation we have that $\mathfrak{M}_f^{\wp}, X \vDash [co][\leq][R]\mathrm{g}(\varphi)$ iff for all $x \in X$, $\mathfrak{M}_f^{\wp}, \{x\} \vDash [\leq][R]\mathrm{g}(\varphi)$, which is equivalent to $\mathfrak{M}_f^{\wp}, \{x\} \vDash [R]\mathrm{g}(\varphi)$. By definition of $\mathfrak{M}_f^{\wp}$, we have $\mathfrak{M}_f^{\wp}, \{x\} \vDash [R]\mathrm{g}(\varphi)$ iff either $R(x) = \emptyset$, in which case $\mathfrak{M}, x \vDash \Box\varphi$, or $\mathfrak{M}_f^{\wp}, R(x) \vDash \mathrm{g}(\varphi)$, which by the inductive hypothesis is equivalent to the condition that for all $y \in R(x)$, $\mathfrak{M}, y \vDash \varphi$, so $\mathfrak{M}, x \vDash \Box\varphi$. $\Box$

We are now prepared to prove Theorem 4.2.

Proof of Theorem 4.2. For the right-to-left direction of Theorem 4.2, if $\nvdash_{\mathbf{K}} \varphi$, then by the completeness of **K** with respect to the class of possible world models, there is such a model that falsifies φ. Thus, by Lemmas 3.10.1 and 4.4, there is an **S4-P** model $\mathfrak{M}_f^{\wp}$ (indeed, this is a model of a much stronger logic, a point to which we will return below) that falsifies $\mathrm{g}(\varphi)$, so $\nvdash_{\mathbf{S4\text{-}P}} \mathrm{g}(\varphi)$. In all, this is a simple semantic argument using standard completeness plus the powerset possibilization construction.

The argument for the converse direction is a direct combinatorial analysis of axioms and proofs, but its details are less obvious than the reader might expect from other translations, since we need to investigate how even simple classical propositional reasoning transforms under our translation with various added modalities. First, we show that the translation of any propositional tautology is a theorem of **S4-K**.

We start with some auxiliary observations. Given any propositional formula φ, consider an equivalent disjunctive normal form

$$\varphi^D := \bigvee_{i \in I} \Big(\bigwedge_{p \in P_i} p \wedge \bigwedge_{q \in Q_i} \neg q \Big),$$

where $I \subseteq \mathbb{N}$ and $P_i, Q_i \subseteq \mathsf{Prop}$. Then $\mathrm{g}(\varphi^D)$ is equivalent to a formula of the form

$$[co]\Big(\bigvee_{i \in I} \Big(\bigwedge_{p \in P_i} [co]p \wedge \bigwedge_{q \in Q_i} [\leq]\neg[co]q \Big) \Big).$$

Now suppose there is an **S4-K** model $\mathfrak{M}$ such that $\mathfrak{M}, x \vDash \mathrm{g}(\varphi^D)$, so there is some $x' \geq x$ such that

$$\mathfrak{M}, x' \vDash \bigvee_{i \in I} \Big(\bigwedge_{p \in P_i} [co]p \wedge \bigwedge_{q \in Q_i} [\leq]\neg[co]q \Big)$$

and hence

$$\mathfrak{M}, x' \vDash \bigwedge_{p \in P_i} [co]p \wedge \bigwedge_{q \in Q_i} [\leq]\neg[co]q$$

for some $i \in I$, which with the reflexivity of $\leq$ implies

$$\mathfrak{M}, x' \vDash \bigwedge_{p \in P_i} [co]p \wedge \bigwedge_{q \in Q_i} \neg[co]q. \tag{3}$$

Define a propositional valuation $v\colon \mathsf{Prop} \to \{0,1\}$ by $v(p) = 1$ iff $\forall y \geq x'\ \exists y' \geq y$: $\mathfrak{M}, y' \vDash p$. Then it follows from (3) that $v(p) = 1$ for all $p \in P_i$, and $v(q) = 0$ for all $q \in Q_i$, and this in turn implies $\hat{v}(\varphi^D) = \hat{v}(\varphi) = 1$, where $\hat{v}$ is the usual classical extension of v.

So we have shown that if the translation $\mathrm{g}(\varphi)$ of a propositional formula φ is satisfiable in an **S4-K** model, then φ is satisfiable by an ordinary propositional valuation.

Now if there is a propositional formula ψ such that $\nvdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\psi)$, then by the completeness of **S4-K**, there is an **S4-K** model with $\mathfrak{M}, x \nvDash \mathrm{g}(\psi)$ and hence $\mathfrak{M}, x \vDash \neg\mathrm{g}(\psi)$. By Lemma 4.3.2, $\mathfrak{M}, x \vDash [\leq]\langle\leq\rangle\mathrm{g}(\psi) \to \mathrm{g}(\psi)$, so $\mathfrak{M}, x \vDash \neg\mathrm{g}(\psi)$ implies $\mathfrak{M}, x \vDash \langle\leq\rangle[\leq]\neg\mathrm{g}(\psi)$, i.e., $\mathfrak{M}, x \vDash \langle\leq\rangle\mathrm{g}(\neg\psi)$, so there is an $x' \geq x$ such that $\mathfrak{M}, x' \vDash \mathrm{g}(\neg\psi)$. Then what we showed above implies that $\neg\psi$ is satisfiable by a propositional valuation. Thus, ψ is not a propositional tautology. Hence if ψ is a propositional tautology, then $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\psi)$.

Second, we observe that by using the forms produced by the g translation and the laws of **S4-K**, we can match applications of modus ponens that were made in **K**. Suppose $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi)$ and $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi \to \psi)$, i.e., $\vdash_{\mathbf{S4\text{-}K}} [\leq](\mathrm{g}(\varphi) \to \mathrm{g}(\psi))$. Then by the T axiom for $[\leq]$, $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi) \to \mathrm{g}(\psi)$, so by modus ponens in **S4-K**, we have $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\psi)$.

Third, we show that the g translation of the K axiom for $\Box$ is a theorem of **S4-K**. The translation is

$$[\leq]\big(\big([co][\leq][R]\mathrm{g}(\varphi) \wedge [co][\leq][R][\leq](\mathrm{g}(\varphi) \to \mathrm{g}(\psi))\big) \to [co][\leq][R]\mathrm{g}(\psi)\big). \tag{4}$$

We claim that

$$\vdash_{\mathbf{S4\text{-}K}} \big([co][\leq][R]\mathrm{g}(\varphi) \wedge [co][\leq][R][\leq](\mathrm{g}(\varphi) \to \mathrm{g}(\psi))\big) \to [co][\leq][R]\mathrm{g}(\psi)\big), \tag{5}$$

from which it follows by necessitation for $[\leq]$ that (4) is a theorem of **S4-K**.

By Lemma 4.3.1, $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi) \to [\leq]\mathrm{g}(\varphi)$, so by the normality of $[co][\leq][R]$ in **S4-K**, $\vdash_{\mathbf{S4\text{-}K}} [co][\leq][R]\mathrm{g}(\varphi) \to [co][\leq][R][\leq]\mathrm{g}(\varphi)$. Thus, from the antecedent of the main conditional in (5) we can derive in **S4-K** that

$$
\begin{array}{ll}
 & [co][\leq][R][\leq]\mathrm{g}(\varphi) \wedge [co][\leq][R][\leq]\big(\mathrm{g}(\varphi) \to \mathrm{g}(\psi)\big) \\
\Rightarrow & [co][\leq][R][\leq]\mathrm{g}(\varphi) \wedge [co][\leq][R]\big([\leq]\mathrm{g}(\varphi) \to [\leq]\mathrm{g}(\psi)\big) \text{ by normality of } [\leq]\text{, monotonicity of } [co][\leq][R] \\
\Rightarrow & [co][\leq][R][\leq]\mathrm{g}(\psi) \text{ by normality of } [co][\leq][R] \\
\Rightarrow & [co][\leq][R]\mathrm{g}(\psi) \text{ by the T axiom for } [\leq] \text{ and normality of } [co][\leq][R],
\end{array}
$$

so we have established (5).

Finally, we observe that with g and **S4-K** we can match applications of necessitation in **K**: if $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi)$, then $\vdash_{\mathbf{S4\text{-}K}} [co][\leq][R]\mathrm{g}(\varphi)$ by necessitation for $[R]$ and $[\leq]$, the T axiom $\psi \to \langle\leq\rangle\psi$, and then the normality of $[\leq]$. Therefore, $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\Box\varphi)$ by definition of g. $\Box$

Remark 4.5 (A Possibility-Semantic Proof)**.** As noted above, we can also give a possibility-semantic proof of the left-to-right direction of Theorem 4.2, which avoids the direct combinatorial analysis of axioms and proofs given above. If $\nvdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi)$, then there is a bimodal model $\mathfrak{M} = \langle S, \leq, R, V\rangle$ that falsifies $\mathrm{g}(\varphi)$. Let $\mathcal{F} = \langle S, \leq, R, P\rangle$ with $P = \mathrm{RO}(S, \leq)$, so $\mathcal{F}$ is what we called a *quasi* possibility frame in Remark 2.13. We define a valuation π on $\mathcal{F}$ such that $\pi(p)$ is the interior of the closure of $V(p)$:

$$\pi(p) = \{x \in S \mid \forall x' \geq x\, \exists x'' \geq x' : x'' \in V(p)\},$$

which implies $\pi\colon \mathsf{Prop} \to P$. Let $\mathcal{M} = \langle S, \leq, R, \pi\rangle$. Then an easy induction shows that for any $s \in S$,

$$\mathfrak{M}, s \vDash \mathrm{g}(\varphi) \text{ iff } \mathcal{M}, s \Vdash_{\mathrm{q}} \varphi,$$

where $\vDash$ is the standard satisfaction relation for possible world semantics and $\Vdash_{\mathrm{q}}$ is the quasi possibility satisfaction relation from Remark 2.13 with the non-standard clause for $\Box$. Then since $\mathfrak{M}$ falsifies $\mathrm{g}(\varphi)$ according to $\vDash$, $\mathcal{M}$ falsifies φ according to $\Vdash_{\mathrm{q}}$. Hence $\nvdash_{\mathbf{K}} \varphi$, since as we noted in Remark 2.13, **K** is sound with respect to quasi possibility frames with the $\Vdash_{\mathrm{q}}$ semantics. Thus, $\nvdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi)$ implies $\nvdash_{\mathbf{K}} \varphi$.

The proof of Theorem 4.2 shows that g also embeds **K** into **S4-K**. Recall that in possibility semantics, we may assume without loss of generality that the accessibility relation is partially functional—but this is not *required*. Non-functional relations serve as well. Similarly, the g translation works fine into a bimodal logic without the partial functionality axiom P for $[R]$. Of course, there are completely trivial translations of **K** into **S4-K**, such as the translation that simply switches $\Box$ to $[R]$, so the fact that g embeds **K** into **S4-K** is nothing special. By contrast, the fact that g embeds **K** into the logic **S4-P** with a partially functional modality is something special—not just any translation can pull this off.

Let us now move from partial to *total* functionality. The g translation cannot embed **K** into the bimodal logic **S4-F** with the functionality axiom F for $[R]$, because while $\Box\bot$ is consistent in **K**, $\mathrm{g}(\Box\bot) = [co][\leq][R]\bot$ is inconsistent in any bimodal logic with the F axiom for $[R]$ and the D axiom for $[\leq]$. However, the g translation is able to embed the logic **KD** with the D axiom for $\Box$ into **S4-F**. We thus arrive at the second of our informal slogans:

$$\text{SERIAL RELATION} \quad \hookrightarrow \quad \text{PREORDER} + \text{FUNCTION},$$

which is supported by the following theorem.

Theorem 4.6 (Embedding II)**.** For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{KD}} \varphi$ iff $\vdash_{\mathbf{S4\text{-}F}} \mathrm{g}(\varphi)$.

Proof. In the right-to-left direction, the proof is the same as for Theorem 4.2, except we use the completeness of **KD** with respect to the class of *serial* relational models and then Lemmas 3.10.2 and 4.4.

For the left-to-right direction, we do not need the full deductive power of **S4-F**. The deductive power of its sublogic **S4-KD** is enough. Again we could give a possibility-semantic proof that $\nvdash_{\mathbf{S4\text{-}KD}} \mathrm{g}(\varphi)$ implies $\nvdash_{\mathbf{KD}} \varphi$ as in Remark 4.5. For the more syntactic route, all we need to add to the proof of Theorem 4.2 is that the g-translation of the D axiom $\Box\varphi \to \Diamond\varphi$ is derivable in **S4-KD**. To work out the translation, first observe that since we defined $\Diamond\varphi := \neg\Box\neg\varphi$, we have:

$$\begin{array}{rcl}
\mathrm{g}(\Diamond\varphi) & = & \mathrm{g}(\neg\Box\neg\varphi) \\
& = & [\leq]\neg\mathrm{g}(\Box\neg\varphi) \\
& = & [\leq]\neg[co][\leq][R]\mathrm{g}(\neg\varphi) \\
& = & [\leq]\neg[co][\leq][R][\leq]\neg\mathrm{g}(\varphi) \\
& \Leftrightarrow & [\leq]\langle\leq\rangle[\leq]\langle\leq\rangle\langle R\rangle\langle\leq\rangle\mathrm{g}(\varphi) \\
& = & [co][co]\langle R\rangle\langle\leq\rangle\mathrm{g}(\varphi),
\end{array}$$

which is equivalent to $[co]\langle R\rangle\langle\leq\rangle\mathrm{g}(\varphi)$ given **S4** for $[\leq]$. Thus, the g-translation of D is equivalent to

$$[\leq]([co][R]\mathrm{g}(\varphi) \to [co]\langle R\rangle\langle\leq\rangle\mathrm{g}(\varphi)). \tag{6}$$

As an instance of the T axiom for $[\leq]$, we have $\mathrm{g}(\varphi) \to \langle\leq\rangle\mathrm{g}(\varphi)$, which gives us $[R]\mathrm{g}(\varphi) \to [R]\langle\leq\rangle\mathrm{g}(\varphi)$. As an instance of the D axiom for $[R]$, we have $[R]\langle\leq\rangle\mathrm{g}(\varphi) \to \langle R\rangle\langle\leq\rangle\mathrm{g}(\varphi)$. Then from the previous two steps, we obtain $[R]\mathrm{g}(\varphi) \to \langle R\rangle\langle\leq\rangle\mathrm{g}(\varphi)$, which gives us $\langle\leq\rangle[R]\mathrm{g}(\varphi) \to \langle\leq\rangle\langle R\rangle\langle\leq\rangle\mathrm{g}(\varphi)$ and then $[\leq]\langle\leq\rangle[R]\mathrm{g}(\varphi) \to [\leq]\langle\leq\rangle\langle R\rangle\langle\leq\rangle\mathrm{g}(\varphi)$, which gives us (6) by necessitation for $[\leq]$. $\square$

We need not stop at **S4-F**. We can make the target bimodal logic even stronger, e.g., the logic **S4-F-CR** of topological spaces with *continuous R-maps* as in §3.4. The general reason is the following.

Theorem 4.7 (More Embeddings)**.** For any bimodal logic **L** extending **S4-K** and any bimodal logic $\mathbf{L}'$ extending **S4-KD**:

1. if **L** is sound over the class of powerset possibilizations of possible world models, then for all $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{K}} \varphi$ iff $\vdash_{\mathbf{L}} \mathrm{g}(\varphi)$;
2. if $\mathbf{L}'$ is sound over the class of powerset possibilization of serial possible world models, then for all $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{KD}} \varphi$ iff $\vdash_{\mathbf{L}'} \mathrm{g}(\varphi)$.

Proof. We have already seen that if $\vdash_{\mathbf{K}} \varphi$, then $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi)$, and that if $\vdash_{\mathbf{KD}} \varphi$, then $\vdash_{\mathbf{S4\text{-}KD}} \mathrm{g}(\varphi)$.

In the other direction, if $\nvdash_{\mathbf{K}} \varphi$ (resp. $\nvdash_{\mathbf{KD}} \varphi$) then as in the proof of Theorem 4.2 (resp. 4.6), there is a powerset possibilization of a possible world model (resp. serial possible world model) that falsifies $\mathrm{g}(\varphi)$, so if **L** (resp. $\mathbf{L}'$) is sound over such powerset possibilizations, then $\nvdash_{\mathbf{L}} \mathrm{g}(\varphi)$ (resp. $\nvdash_{\mathbf{L}'} \mathrm{g}(\varphi)$). $\square$

Once we sufficiently strengthen the target bimodal logic, we can also simplify the g translation to a translation with a much cleaner clause for $\Box$. We will demonstrate this in §§4.3-4.4.

Using Theorem 4.7, not only can we strengthen the logic of the $[R]$ modality, but in light of Remark 3.11, we can also strengthen the logic of the $[\leq]$ modality. Since **K** and **KD** have the finite model property in the standard sense of possible world semantics, it follows from Remark 3.11 that we can strengthen the logic of the $[\leq]$ modality all the way to the strong logic **Medv**.

But of course there is a limit to strengthening the target bimodal logic. As an example, let us show that the g translation does not faithfully embed **K** into **S4-P-O**, where O is the axioms for *open* maps in the topological context of §3.4.

Fact 4.8. $\nvdash_{\mathbf{K}} \Diamond p \to \Box p$ but $\vdash_{\mathbf{S4\text{-}P\text{-}O}} \mathrm{g}(\Diamond p \to \Box p)$.

Proof. The contrapositive of the O axiom is $\langle R\rangle\langle\leq\rangle p \to \langle\leq\rangle\langle R\rangle p$, which with the partial functionality axiom $\langle R\rangle p \to [R]p$ gives us $\vdash_{\mathbf{S4\text{-}P\text{-}O}} \langle R\rangle\langle\leq\rangle\varphi \to \langle\leq\rangle[R]\varphi$ for any φ. Then using the translation of $\Diamond$ from the proof of Theorem 4.6, the monotonicity of $[co]$, and the fact that $\vdash_{\mathbf{S4\text{-}K}} [co]\langle\leq\rangle\psi \to [co]\psi$, we have:

$$\begin{aligned} \mathrm{g}(\Diamond p) &\Leftrightarrow [co]\langle R\rangle\langle\leq\rangle[co]p \\ &\Rightarrow [co]\langle\leq\rangle[R][co]p \\ &\Rightarrow [co][R][co]p \\ &= \mathrm{g}(\Box p). \end{aligned}$$

Hence $\vdash_{\mathbf{S4\text{-}P\text{-}O}} \mathrm{g}(\Diamond p \to \Box p)$. $\square$

Remark 4.9 (A Related Embedding)**.** If we restrict the domain of the g translation to the propositional language with $\neg$, $\wedge$, and $\vee$, then it becomes a translation to a modal language with just one modal operator, namely $[\leq]$, which we will write for the moment as $\Box_1$:

- $\mathrm{g}(p) = \Box_1\Diamond_1 p$; $\mathrm{g}(\neg\varphi) = \Box_1\neg\mathrm{g}(\varphi)$; $\mathrm{g}(\varphi \wedge \psi) = \mathrm{g}(\varphi) \wedge \mathrm{g}(\psi)$; $\mathrm{g}(\varphi \vee \psi) = \Box_1\Diamond_1(\mathrm{g}(\varphi) \vee \mathrm{g}(\psi))$.

Goldblatt [1974] shows that this translation embeds *orthologic*, the sublogic of classical propositional logic corresponding to the equational theory of ortholattices, into the modal logic **KTB** for $\Box_1$. The reason that the $\Box_1\Diamond_1$ pattern appears for Goldblatt, just as it does for us, becomes clear when we consider Goldblatt's semantics for orthologic.[5] On one version of the semantics, models are tuples $\mathcal{M} = \langle S, C, \pi\rangle$ where S is a nonempty set, C is a reflexive and symmetric relation on S, and $\pi\colon \mathsf{Prop} \to \wp(S)$ is such that if for all x' with xCx', there is an $x'' \in \pi(p)$ with $x'Cx''$, then $x \in \pi(p)$. Note that any possibility model gives rise to one of Goldblatt's models with C as the compatibility relation $\between$ from the possibility model, defined as before by $x \between y$ iff $\exists z$: $x \leq z$ and $y \leq z$. Clearly $\between$ is reflexive and symmetric, and Goldblatt's valuation condition follows from the possibility-semantic conditions of *Persistence* and *Refinability*; if for all x' with $x \between x'$, there is an $x'' \in \pi(p)$ with $x' \between x''$, then $x \in \pi(p)$. Goldblatt's semantic clause for negation is that $\mathcal{M}, x \Vdash \neg\varphi$ iff $\forall y \in S$: if xCy, then $\mathcal{M}, y \nVdash \varphi$. If $\mathcal{M}$ is a possibility model and C is $\between$, then Goldblatt's clause for $\neg$ is equivalent to the possibility-semantic clause for $\neg$ that uses $\leq$, given *Persistence*. But unlike possibility models, Goldblatt's models with C as a primitive need not validate all classical inferences, such as distribution from $p \wedge (q \vee r)$ to $(p \wedge q) \vee (p \wedge r)$, with the classical definition of $\vee$ as $(\alpha \vee \beta) := \neg(\neg\alpha \wedge \neg\beta)$. Of course, Goldblatt's models can also be viewed as models for a classical unimodal logic, with $\Box_1$ as the box modality for C. The reflexivity and symmetry of C explains the appearance of **KTB**, and Goldblatt's valuation condition explains the appearance of $\Box_1\Diamond_1$ in the translation. Given that the g translation embeds classical logic into **KT4** (**S4**) for $\Box_1$ and orthologic into **KTB** for $\Box_1$, the semantic fact above about distribution is reflected in the syntactic fact that $\mathrm{g}(p \wedge (q \vee r)) \to \mathrm{g}((p \wedge q) \vee (p \wedge r))$ is a theorem of **KT4** but not of **KTB**. As all of this shows, there are interesting aspects of the translation even at the propositional level.

4.2 Decomposing the Translation

The preceding translation can be decomposed into two steps that both have a long history in the analysis of constructive and classical logic. The first step goes from intuitionistic modal logic to classical bimodal logic.

Definition 4.10 (Extended Gödel-McKinsey-Tarski Translation)**.** Define a function $(\cdot)^G\colon \mathcal{L}_1 \to \mathcal{L}_2$ recursively as follows:

1. $p^G = [\leq]p$;
2. $(\neg\varphi)^G = [\leq]\neg\varphi^G$;
3. $(\varphi \to \psi)^G = [\leq](\varphi^G \to \psi^G)$;
4. $(\varphi \wedge \psi)^G = \varphi^G \wedge \psi^G$;
5. $(\varphi \vee \psi)^G = \varphi^G \vee \psi^G$;
6. $(\Box\varphi)^G = [\leq][R]\varphi^G$.

[5]We thank Lloyd Humberstone for suggesting that we draw a connection here with Goldblatt 1974.

The original translation of Gödel [1933b] and McKinsey and Tarski [1948] was from the language of propositional logic to the language of unimodal propositional logic, for which they obtained the following famous result, where **IPC** is the intuitionistic propositional calculus.

Theorem 4.11 (GMT)**.** For all propositional formulas φ, $\vdash_{\mathbf{IPC}} \varphi$ iff $\vdash_{\mathbf{S4}} \varphi^G$.

In the extended $(\cdot)^G$ translation (cf. Fischer Servi 1977), the $\Box$ clause mirrors the non-standard semantics for $\Box$ mentioned in Remark 2.6. It is not difficult to show that the extended $(\cdot)^G$ translation embeds the intuitionistic modal logic **HK** into the classical bimodal logic **S4-K** as in Theorem 4.12 (cf. Theorem 4.23 below). For more general results on embedding intuitionistic normal modal logics into normal extensions of **S4-K** via a translation like $(\cdot)^G$, see Wolter and Zakharyaschev 1999.

Theorem 4.12. For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{HK}} \varphi$ iff $\vdash_{\mathbf{S4\text{-}K}} \varphi^G$.

Proof. From left to right, the propositional part is given by Theorem 4.11. For the modal part, one can easily check that the $(\cdot)^G$ translation of the K axiom is a theorem of **S4-K**, and applications of $\Box$-necessitation in **HK** can be matched by applications of $[\leq][R]$-necessitation in **S4-K**. Of course, there is also a semantic proof from $\nvdash_{\mathbf{S4\text{-}K}} \varphi^G$ to $\nvdash_{\mathbf{HK}} \varphi$ using Remark 2.6, analogous to the semantic proof in Remark 4.5.

From right to left, if $\nvdash_{\mathbf{HK}} \varphi$, then by Theorem 2.4 there is an intuitionistic modal model $\mathcal{M}$ that falsifies φ according to intuitionistic semantics, and it is easy to see that $\mathcal{M}$ is also a bimodal model for **S4-K** that falsifies φ^G according to classical semantics. $\Box$

The second translation that we need takes us from classical modal logic to intuitionistic modal logic.

Definition 4.13 (Modal Gödel-Gentzen Translation)**.** Define a function $(\cdot)_G \colon \mathcal{L}_1 \to \mathcal{L}_1$ recursively as follows:

1. $p_G = \neg\neg p$;
2. $(\neg\varphi)_G = \neg\varphi_G$;
3. $(\varphi \to \psi)_G = \varphi_G \to \psi_G$;
4. $(\varphi \wedge \psi)_G = \varphi_G \wedge \psi_G$;
5. $(\varphi \vee \psi)_G = \neg\neg(\varphi_G \vee \psi_G)$;
6. $(\Box\varphi)_G = \neg\neg\Box\varphi_G$.

The translations of Gödel [1933a] and Gentzen [1933, 1936, 1974] were for the language of first-order logic, with the clauses $(\forall x\varphi)_G = \forall x\varphi_G$ and $(\exists x\varphi)_G = \neg\neg\exists x\varphi_G$. For the language of propositional logic, their results establish the following, where **CPC** is the classical propositional calculus.

Theorem 4.14. For all propositional formulas φ, $\vdash_{\mathbf{CPC}} \varphi$ iff $\vdash_{\mathbf{IPC}} \varphi_G$.

As Glivenko [1929] showed, one can embed **CPC** into **IPC** by simply translating φ as $\neg\neg\varphi$. However, this does not extend to predicate logic, as $\forall x(P(x) \vee \neg P(x))$ is a classical theorem, while $\neg\neg\forall x(P(x) \vee \neg P(x))$ is not an intuitionistic theorem. It also does not extend to the modal logics **K** and **HK**, as $\Box(p \vee \neg p)$ is a theorem of **K**, while $\neg\neg\Box(p \vee \neg p)$ is not a theorem of **HK** (cf. Bezhanishvili 2001). We need to apply the translation recursively past the modality, as in the Gödel-Gentzen translation. We also need to add $\neg\neg$ in front of $\Box$, for otherwise the theorem $\neg\neg\Box p \to \Box p$ of **K** would translate to the non-theorem $\neg\neg\Box\neg\neg p \to \Box\neg\neg p$ of **HK**. As Božic and Došen [1984, pp. 231-2] observe, this suffices to obtain an embedding of **K** into **HK**.

Theorem 4.15 (Božic and Došen 1984)**.** For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{K}} \varphi$ iff $\vdash_{\mathbf{HK}} \varphi_G$.

Now it is easy to see that our earlier translation g from Definition 4.1 is essentially a composition of the translations of Definitions 4.13 and 4.10, though slightly simpler. For $\psi \in \mathcal{L}_2$, let ψ_B be the result of replacing each propositional variable p in ψ by $[\leq]p$. Then for $\varphi \in \mathcal{L}_1$, applying the $(\cdot)_G$ translation followed by the $(\cdot)^G$ translation is equivalent to applying the g translation followed by the $(\cdot)_B$ translation.

Proposition 4.16 (Composition)**.** For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{K\text{-}K}} (\varphi_G)^G \leftrightarrow \mathrm{g}(\varphi)_B$.

Proof. By the definitions of the translations, we have:

- $(p_G)^G = (\neg\neg p)^G = [\leq]\neg[\leq]\neg p^G = [\leq]\neg[\leq]\neg[\leq]p$ and $\mathrm{g}(p)_B = ([co]p)_B = [co][\leq]p$.
- $((\neg\varphi)_G)^G = (\neg\varphi_G)^G = [\leq]\neg(\varphi_G)^G$ and $\mathrm{g}(\neg\varphi)_B = ([\leq]\neg\mathrm{g}(\varphi))_B = [\leq]\neg\mathrm{g}(\varphi)_B$. The $\to$ case is similar.
- $((\varphi \wedge \psi)_G)^G = (\varphi_G \wedge \psi_G)^G = (\varphi_G)^G \wedge (\psi_G)^G$ and $\mathrm{g}(\varphi \wedge \psi)_B = (\mathrm{g}(\varphi) \wedge \mathrm{g}(\psi))_B = \mathrm{g}(\varphi)_B \wedge \mathrm{g}(\psi)_B$.
- $((\varphi \vee \psi)_G)^G = (\neg\neg(\varphi_G \vee \psi_G))^G = [\leq]\neg[\leq]\neg((\varphi_G)^G \vee (\psi_G)^G)$ and $\mathrm{g}(\varphi \vee \psi)_B = ([co](\mathrm{g}(\varphi) \vee \mathrm{g}(\psi)))_B = [co](\mathrm{g}(\varphi)_B \vee \mathrm{g}(\psi)_B)$.
- $((\Box\varphi)_G)^G = (\neg\neg\Box\varphi_G)^G = [\leq]\neg[\leq]\neg(\Box\varphi_G)^G = [\leq]\neg[\leq]\neg[\leq][R](\varphi_G)^G$ and $\mathrm{g}(\Box\varphi)_B = ([co][\leq][R]\mathrm{g}(\varphi))_B = [co][\leq][R]\mathrm{g}(\varphi)_B$.

Given these equations, the proposition is easily proved by induction on φ. ☐

4.3 Simplifying the Translation I

We can give simpler translations of unimodal logic into bimodal logics of preorders plus partial functions—in particular, simpler translations for $\Box$—provided we strengthen the target bimodal logic. In this section, we will consider a natural strengthening of the target bimodal logic with the C axiom $[R][\leq]p \to [\leq][R]p$ of intuitionistic modal frames (recall Fact 2.5 and Proposition 3.7.1), and we will show how this allows a modest simplification of our translation. In §4.4, we will consider a substantial simplification of the translation and then identify the necessary associated strengthening of the target bimodal logic.

If we add the C axiom to **S4-F**, we obtain the well-known dynamic topological logic **S4-F-C**, the logic of dynamic topological spaces with continuous functions (recall §3.4). By Theorem 4.7, we know that the g translation embeds **KD** into **S4-F-C** and **K** into **S4-P-C**. We will show that a simpler translation suffices for these embeddings, so the simpler translation is enough to justify the informal slogans:

$$\text{SERIAL RELATION} \quad \hookrightarrow \quad \text{PREORDER} + \text{MONOTONIC FUNCTION}$$

$$\text{SERIAL RELATION} \quad \hookrightarrow \quad \text{TOPOLOGICAL SPACE} + \text{CONTINUOUS FUNCTION.}$$

The $\Box$ clause of the following translation is based on the non-standard semantics mentioned in the second paragraph of Remark 2.13.

Definition 4.17 (h Translation)**.** Define a function $\mathrm{h}\colon \mathcal{L}_1 \to \mathcal{L}_2$ with the same recursive clauses as for g in Definition 4.1 except with h in place of g and with $\mathrm{h}(\Box\varphi) = [co][R]\mathrm{h}(\varphi)$ instead of $\mathrm{g}(\Box\varphi) = [co][\leq][R]\mathrm{g}(\varphi)$.

The main result of this section is the following analogue of Theorems 4.2 and 4.6 combined.

Theorem 4.18 (Embedding III)**.** For all $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{KD}} \varphi$ iff $\vdash_{\mathbf{S4\text{-}F\text{-}C}} \mathrm{h}(\varphi)$; and $\vdash_{\mathbf{K}} \varphi$ iff $\vdash_{\mathbf{S4\text{-}P\text{-}C}} \mathrm{h}(\varphi)$.

As before, there is a possibility-semantic proof of the left-to-right direction of these embeddings, as well as a more syntactic proof. The possibility-semantic proof is as in Remark 4.5. For the syntactic proof, we first need the analogue of Lemma 4.3.1 for the h translation.

Lemma 4.19 (Persistence)**.** For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{S4\text{-}K}} \mathrm{h}(\varphi) \to [\leq]\mathrm{h}(\varphi)$.

Proof. The proof is the same as that of Lemma 4.3.1 except in the $\Box$ case. Observe that $\mathrm{h}(\Box\varphi) \to [\leq]\mathrm{h}(\Box)\varphi$ is $[co][R]\mathrm{h}(\varphi) \to [\leq][co][R]\mathrm{h}(\varphi)$, which unpacks to the formula $[\leq]\langle\leq\rangle[R]\mathrm{h}(\varphi) \to [\leq][\leq]\langle\leq\rangle[R]\mathrm{h}(\varphi)$, which is an instance of the 4 axiom. ☐

Now we can show that in **S4-K-C**, our earlier g translation is equivalent to the simplified h translation.

Lemma 4.20 (Simplifying I)**.** For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{S4\text{-}K\text{-}C}} \mathrm{g}(\varphi) \leftrightarrow \mathrm{h}(\varphi)$.

Proof. The proof is by induction on φ. The only case to check is that of $\Box$. We must show that $\vdash_{\mathbf{S4\text{-}K\text{-}C}} [co][\leq][R]\mathrm{g}(\varphi) \leftrightarrow [co][R]\mathrm{h}(\varphi)$. By the inductive hypothesis, $\vdash_{\mathbf{S4\text{-}K\text{-}C}} \mathrm{g}(\varphi) \leftrightarrow \mathrm{h}(\varphi)$, so it suffices to show that $\vdash_{\mathbf{S4\text{-}K\text{-}C}} [co][\leq][R]\mathrm{h}(\varphi) \leftrightarrow [co][R]\mathrm{h}(\varphi)$. The left-to-right implication is obvious given the T axiom for $[\leq]$. From right to left, by Lemma 4.19, $\vdash_{\mathbf{S4\text{-}K\text{-}C}} \mathrm{h}(\varphi) \to [\leq]\mathrm{h}(\varphi)$, so $\vdash_{\mathbf{S4\text{-}K\text{-}C}} [R]\mathrm{h}(\varphi) \to [R][\leq]\mathrm{h}(\varphi)$. Then since $[R][\leq]\mathrm{h}(\varphi) \to [\leq][R]\mathrm{h}(\varphi)$ is an instance of the C axiom, we have $\vdash_{\mathbf{S4\text{-}K\text{-}C}} [R]\mathrm{h}(\varphi) \to [\leq][R]\mathrm{h}(\varphi)$ and hence $\vdash_{\mathbf{S4\text{-}K\text{-}C}} [co][R]\mathrm{h}(\varphi) \to [co][\leq][R]\mathrm{h}(\varphi)$, which completes the proof. ☐

Finally, we need an analogue of Lemma 4.4 for the h translation.

Lemma 4.21 (Powerset Possibilization and the h Translation)**.** For any possible world model $\mathfrak{M} = \langle W, R, V\rangle$, its functional powerset possibilization $\mathfrak{M}_f^{\wp}$ (Definition 2.14) is such that for all $\varphi \in \mathcal{L}_1$ and $X \in W^{\wp}$:

$$\mathfrak{M}_f^{\wp}, X \vDash \mathrm{h}(\varphi) \text{ iff } \forall x \in X\colon \mathfrak{M}, x \vDash \varphi.$$

Proof. The proof is an easy induction on φ using the definition of h and fact that $\mathfrak{M}_f^{\wp}, X \vDash [co]\varphi$ holds iff $\forall X' \subseteq X\ \exists X'' \subseteq X'$ such that $\mathfrak{M}_f^{\wp}, X'' \vDash \varphi$, which holds iff $\forall x \in X$, $\mathfrak{M}_f^{\wp}, \{x\} \vDash \varphi$. The proof for the $\Box$ case is just a slight simplification of the corresponding part of the proof of Lemma 4.4. ☐

We can now put everything together to prove Theorem 4.18.

Proof of Theorem 4.18. From left to right, if $\vdash_{\mathbf{KD}} \varphi$, then $\vdash_{\mathbf{S4\text{-}F}} \mathrm{g}(\varphi)$ by Theorem 4.6, so $\vdash_{\mathbf{S4\text{-}F\text{-}C}} \mathrm{h}(\varphi)$ by Lemma 4.20. From right to left, if $\nvdash_{\mathbf{KD}} \varphi$, then by the completeness of **KD** with respect to the class of serial possible worlds models, there is such a model that falsifies φ. Thus, by Lemmas 3.10.2 and 4.21, there is an **S4-F-C** model $\mathfrak{M}_f^{\wp}$ that falsifies $\mathrm{h}(\varphi)$, so $\nvdash_{\mathbf{S4\text{-}F\text{-}C}} \mathrm{h}(\varphi)$.

The argument for **K** and **S4-P-C** is of the same form. ☐

From here we can also prove an exact analogue of Theorem 4.7, showing the range of bimodal logics into which **K** and **KD** embed, respectively, via the h translation. In the statement of Theorem 4.7, simply replace **S4-K** with **S4-K-C**, **S4-KD** with **S4-KD-C**, and g with h.

Finally, it is worth noting that like g, h can also be viewed in terms of a composition of translations.

Definition 4.22 (Extended Gödel-McKinsey-Tarski Translation II)**.** Define a function $(\cdot)^H\colon \mathcal{L}_1 \to \mathcal{L}_2$ with the same recursive clauses as for $(\cdot)^G$ in Definition 4.10 but with H in place of G and $(\Box\varphi)^H = [R]\varphi^H$ instead of $(\Box\varphi)^G = [\leq][R]\varphi^G$.

That this H translation embeds the intuitionistic modal logic **HK** into the classical bimodal **S4-K-C** is Theorem 3 of Božic and Došen 1984 (p. 231).

Theorem 4.23 (Božic and Došen 1984)**.** For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{HK}} \varphi$ iff $\vdash_{\mathbf{S4\text{-}K\text{-}C}} \varphi^H$.

Now we obtain the following analogue of Proposition 4.16.

Proposition 4.24 (Composition II)**.** For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{K\text{-}K}} (\varphi_G)^H \leftrightarrow \mathrm{h}(\varphi)_B$.

Proof. The proof is the same as that of Proposition 4.16 except in the $\Box$ case, for which we observe that $((\Box\varphi)_G)^H = (\neg\neg\Box\varphi_G)^H = [\leq]\neg[\leq]\neg(\Box\varphi_G)^H = [\leq]\neg[\leq]\neg[R](\varphi_G)^H$ and $\mathrm{h}(\Box\varphi)_B = ([co][R]\mathrm{h}(\varphi))_B = [co][R]\mathrm{h}(\varphi)_B$. $\Box$

4.4 Simplifying the Translation II

Our final translation is the simplest of all. As we shall see, it works provided we strengthen the target bimodal logic with the key axiom R of possibility frames.

Definition 4.25 (p Translation)**.** Define a function $\mathrm{p}\colon \mathcal{L}_1 \to \mathcal{L}_2$ recursively as follows:

1. $\mathrm{p}(p) = [co]p$;
2. $\mathrm{p}(\neg\varphi) = [\leq]\neg\mathrm{p}(\varphi)$;
3. $\mathrm{p}(\varphi \to \psi) = [\leq](\mathrm{p}(\varphi) \to \mathrm{p}(\psi))$;
4. $\mathrm{p}(\varphi \wedge \psi) = \mathrm{p}(\varphi) \wedge \mathrm{p}(\psi)$;
5. $\mathrm{p}(\varphi \vee \psi) = [co](\mathrm{p}(\varphi) \vee \mathrm{p}(\psi))$;
6. $\mathrm{p}(\Box\varphi) = [R]\mathrm{p}(\varphi)$.

The simplification of our earlier translations occurs in the $\mathrm{p}(\Box\varphi)$ clause, which replaces $\Box$ by $[R]$ and then pushes the translation through. Otherwise the clauses for p are the same as for g in §4.1. Note the exact parallel between the p translation and the definition of truth in possibility semantics (Definitions 2.3 and 2.8). Also note that

$$\mathrm{p}(\Diamond\varphi) = \mathrm{p}(\neg\Box\neg\varphi) = [\leq]\neg\mathrm{p}(\Box\neg\varphi) = [\leq]\neg[R]\mathrm{p}(\neg\varphi) = [\leq]\neg[R][\leq]\neg\mathrm{p}(\varphi) \Leftrightarrow [\leq]\langle R\rangle\langle\leq\rangle\mathrm{p}(\varphi).$$

The price we pay for the simplified p translation is that we must strengthen the bimodal logic into which we embed **K**. One can see this by noting that while $\neg\neg\Box p \to \Box p$ is a theorem of **K**, its translation,

$$\mathrm{p}(\neg\neg\Box p \to \Box p) = [\leq]([co][R][co]p \to [R][co]p),$$

is not a theorem of **S4-P**, since one can obviously construct a model of **S4-P** that falsifies the formula. Does this formula look familiar? Indeed,

$$\mathrm{p}(\neg\neg\Box p \leftrightarrow \Box p) = [\leq]([co][R][co]p \leftrightarrow [R][co]p)$$

is exactly the $[\leq]$-necessitation of the R axiom from Definition 3.3!

This observation shows that any bimodal extension of **S4** into which we embed **K** via the p translation must include the axiom R. Not only is R necessary, but also adding R to the base logic **S4-K** is sufficient: we will show that the p translation embeds **K** into **S4-K-R**, *The Logic of Possibility Frames*. This is a syntactic analogue of the semantic fact that **K** is sound and complete with respect to the class of all possibility models.

Theorem 4.26 (Embedding III)**.** For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{K}} \varphi$ iff $\vdash_{\mathbf{S4\text{-}K\text{-}R}} \mathrm{p}(\varphi)$.

Once again, there is a possibility-semantic proof of the left-to-right direction, as well as a more syntactic proof. In this case, the possibility-semantic proof uses standard possibility frames and standard possibility satisfaction, in contrast to the possibility-semantic proof of Theorem 4.2 in Remark 4.5, which used quasi possibility frames and the non-standard satisfaction relation $\Vdash_{\mathrm{q}}$. If $\nvdash_{\mathbf{S4\text{-}K\text{-}R}} \mathrm{p}(\varphi)$, then there is a bimodal **S4-K-R** frame $\mathfrak{F} = \langle S, \leq, R\rangle$ that refutes $\mathrm{p}(\varphi)$. Since $\mathfrak{F} = \langle S, \leq, R\rangle$ is an **S4-K-R** frame, it follows by the correspondence-theoretic fact in Proposition 3.7.6 that $\mathrm{RO}(S, \leq)$ is closed under $\Box_R$. Thus, the frame $\mathcal{F} = \langle S, \leq, R, P\rangle$ with $P = \mathrm{RO}(S, \leq)$ is a standard possibility frame as in Definition 2.7. Now exactly as in Remark 4.5, we can turn the valuation V that refutes $\mathrm{p}(\varphi)$ over $\mathfrak{F}$, according to $\vDash$, into a valuation π that refutes φ over $\mathcal{F}$, now according to the standard possibility satisfaction relation $\Vdash_{\mathrm{p}}$.

For the syntactic proof of the left-to-right direction of Theorem 4.26, we first need an analogue of Lemma 4.3 for the p translation.

Lemma 4.27 (Persistence and Refinability)**.** For any $\varphi \in \mathcal{L}_1$:

1. $\vdash_{\mathbf{S4\text{-}K\text{-}R}} \mathrm{p}(\varphi) \to [\leq]\mathrm{p}(\varphi)$;[6]

2. $\vdash_{\mathbf{S4\text{-}K\text{-}R}} [co]\mathrm{p}(\varphi) \to \mathrm{p}(\varphi)$.

Proof. The proof is by induction on φ. The atomic, $\neg$, and $\wedge$ cases are the same as in the proof of Fact 4.3.1. For the $\Box$ case, $\mathrm{p}(\Box\varphi) \to [\leq]\mathrm{p}(\Box\varphi)$ is $[R]\mathrm{p}(\varphi) \to [\leq][R]\mathrm{p}(\varphi)$. First, the inductive hypothesis give us that $\vdash_{\mathbf{S4\text{-}K\text{-}R}} \mathrm{p}(\varphi) \to [\leq]\mathrm{p}(\varphi)$, which with the normality of $[R]$ gives us that $\vdash_{\mathbf{S4\text{-}K\text{-}R}} [R]\mathrm{p}(\varphi) \to [R][\leq]\mathrm{p}(\varphi)$. Second, $[R][\leq]\mathrm{p}(\varphi) \to [\leq][R][co]\mathrm{p}(\varphi)$ is an instance of the R.1 axiom, which follows from the R axiom given **S4** for $[\leq]$ (Fact 3.4). Third, the inductive hypothesis gives us that $\vdash_{\mathbf{S4\text{-}K\text{-}R}} [co]\mathrm{p}(\varphi) \to \mathrm{p}(\varphi)$. Given the previous three implications and the normality of $[\leq]$ and $[R]$, we have $\vdash_{\mathbf{S4\text{-}K\text{-}R}} [R]\mathrm{p}(\varphi) \to [\leq][R]\mathrm{p}(\varphi)$.

Next, observe that $[co]\mathrm{p}(\Box\varphi) \to \mathrm{p}(\Box\varphi)$ is $[co][R]\mathrm{p}(\varphi) \to [R]\mathrm{p}(\varphi)$. The inductive hypothesis is that $\vdash_{\mathbf{S4\text{-}K\text{-}R}} [co]\mathrm{p}(\varphi) \to \mathrm{p}(\varphi)$, which by the normality of $[R]$ implies

$$\vdash_{\mathbf{S4\text{-}K\text{-}R}} [R][co]\mathrm{p}(\varphi) \to [R]\mathrm{p}(\varphi). \tag{7}$$

Part 1 gives us $\vdash_{\mathbf{S4\text{-}K\text{-}R}} \mathrm{p}(\varphi) \to [\leq]\mathrm{p}(\varphi)$, so by the normality of $[R]$ and $[co]$,

$$\vdash_{\mathbf{S4\text{-}K\text{-}R}} [co][R]\mathrm{p}(\varphi) \to [co][R][\leq]\mathrm{p}(\varphi). \tag{8}$$

Since $[co][R][\leq]\mathrm{p}(\varphi) \to [R][co]\mathrm{p}(\varphi)$ is an instance of the R.2 axiom, which follows from the R axiom given **S4** for $[\leq]$ (Fact 3.4), (7)-(8) together imply that $\vdash_{\mathbf{S4\text{-}K\text{-}R}} [co][R]\mathrm{p}(\varphi) \to [R]\mathrm{p}(\varphi)$. $\Box$

Now we can show that in **S4-K-R**, our earlier g translation is equivalent to the simplified p translation.

Lemma 4.28 (Simplifying II)**.** For all formulas $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{S4\text{-}K\text{-}R}} \mathrm{g}(\varphi) \leftrightarrow \mathrm{p}(\varphi)$.

[6]Note that if we drop R but strengthen R.1 to C, then we have $\vdash_{\mathbf{S4\text{-}K\text{-}C}} \mathrm{p}(\varphi) \to [\leq]\mathrm{p}(\varphi)$.

Proof. The proof is by induction on φ. The only case to check is that of $\Box$. We must show that $\vdash_{\mathbf{S4\text{-}K\text{-}R}}$ $[co][\leq][R]\mathrm{g}(\varphi) \leftrightarrow [R]\mathrm{p}(\varphi)$. By the inductive hypothesis, $\vdash_{\mathbf{S4\text{-}K\text{-}R}} \mathrm{g}(\varphi) \leftrightarrow \mathrm{p}(\varphi)$, so it suffices to show that $\vdash_{\mathbf{S4\text{-}K\text{-}R}} [co][\leq][R]\mathrm{p}(\varphi) \leftrightarrow [R]\mathrm{p}(\varphi)$, i.e., $\vdash_{\mathbf{S4\text{-}K\text{-}R}} [co][\leq]\mathrm{p}(\Box\varphi) \leftrightarrow \mathrm{p}(\Box\varphi)$. For the left-to-right implication, by the T axiom for $[\leq]$, $\vdash_{\mathbf{S4\text{-}K}} [co][\leq]\mathrm{p}(\Box\varphi) \to [co]\mathrm{p}(\Box\varphi)$, and by Lemma 4.27.2, $\vdash_{\mathbf{S4\text{-}K\text{-}R}} [co]\mathrm{p}(\Box\varphi) \to \mathrm{p}(\Box\varphi)$. From right to left, by Lemma 4.27.1, $\vdash_{\mathbf{S4\text{-}K\text{-}R}} \mathrm{p}(\Box\varphi) \to [\leq]\mathrm{p}(\Box\varphi)$, and $\vdash_{\mathbf{S4\text{-}K}} [\leq]\mathrm{p}(\Box\varphi) \to [co][\leq]\mathrm{p}(\Box\varphi)$. $\Box$

Finally, we need an analogue of Lemma 4.4 for the p translation.

Lemma 4.29 (Powerset Possibilization and the p Translation)**.** For any possible world model $\mathfrak{M} = \langle W, R, V\rangle$, its functional powerset possibilization $\mathfrak{M}_f^{\wp}$ (Definition 2.14) is such that for all $\varphi \in \mathcal{L}_1$ and $X \in W^{\wp}$:

$$\mathfrak{M}_f^{\wp}, X \vDash \mathrm{p}(\varphi) \text{ iff } \forall x \in X\colon \mathfrak{M}, x \vDash \varphi.$$

Proof. The proof is an easy induction on φ using the definition of p and fact that $\mathfrak{M}_f^{\wp}, X \vDash [co]\varphi$ holds iff $\forall X' \subseteq X\ \exists X'' \subseteq X'$ such that $\mathfrak{M}_f^{\wp}, X'' \vDash \varphi$, which holds iff $\forall x \in X$, $\mathfrak{M}_f^{\wp}, \{x\} \vDash \varphi$. In the $\Box$ case, by the definition of $\mathfrak{M}_f^{\wp}$, we have $\mathfrak{M}_f^{\wp}, X \Vdash [R]\mathrm{p}(\varphi)$ iff either $R[X] = \emptyset$, in which case $\mathfrak{M}, x \vDash \Box\varphi$ for all $x \in X$, or $\mathfrak{M}_f^{\wp}, R[X] \vDash \mathrm{p}(\varphi)$, which by the inductive hypothesis is equivalent to the condition that for all $y \in R[X]$, $\mathfrak{M}, y \vDash \varphi$, which is in turn equivalent to the condition that for all $x \in X$, $\mathfrak{M}, x \vDash \Box\varphi$. $\Box$

We can now put everything together to prove Theorem 4.26.

Proof of Theorem 4.26. From left to right, if $\vdash_{\mathbf{K}} \varphi$, then $\vdash_{\mathbf{S4\text{-}K}} \mathrm{g}(\varphi)$ by Theorem 4.7, so $\vdash_{\mathbf{S4\text{-}K\text{-}R}} \mathrm{p}(\varphi)$ by Lemma 4.28. From right to left, if $\nvdash_{\mathbf{K}} \varphi$, then by the completeness of **K** with respect to the class of possible world models, there is such a model that falsifies φ. Thus, by Lemmas 3.10.1 and 4.29, there is an **S4-K-R** model $\mathfrak{M}_f^{\wp}$ that falsifies $\mathrm{p}(\varphi)$, so $\nvdash_{\mathbf{S4\text{-}K\text{-}R}} \mathrm{p}(\varphi)$. $\Box$

As before, we can significantly strengthen the bimodal logic into which we embed **K** or **KD**. For example, we can embed **K** into **S4-P-R**, *The Logic of Partially Functional Possibility Frames*, and we can embed **KD** into **S4-F-R**, *The Logic of Functional Possibility Frames*, which we showed in §3.4 is also the logic of dynamic topological spaces with R-maps. These claims are consequences of the following general result.

Theorem 4.30 (More Embeddings II)**.** For any bimodal logic **L** extending **S4-K-R** and bimodal logic **L′** extending **S4-KD-R**:

1. if **L** is sound over the class of powerset possibilizations of possible world models, then for all $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{K}} \varphi$ iff $\vdash_{\mathbf{L}} \mathrm{p}(\varphi)$;
2. if **L′** is sound over the class of powerset possibilization of serial possible world models, then for all $\varphi \in \mathcal{L}_1$, $\vdash_{\mathbf{KD}} \varphi$ iff $\vdash_{\mathbf{L'}} \mathrm{p}(\varphi)$.

Proof. The argument is the same as in the proof of Theorem 4.7, only using Theorem 4.26 instead of 4.2. In part 2, if $\vdash_{\mathbf{KD}} \varphi$, then as in the proof of Theorem 4.6, $\vdash_{\mathbf{S4\text{-}KD}} \mathrm{g}(\varphi)$, so by Lemma 4.28, $\vdash_{\mathbf{S4\text{-}KD\text{-}R}} \mathrm{p}(\varphi)$. $\Box$

Also as before, there is a limit to strengthening the target bimodal logic. For example, the p translation does not faithfully embed **K** into **S4-P-OR**. In the topological setting of §3.4, this is the logic of topological spaces with *open* R-maps.

Fact 4.31. $\nvdash_{\mathbf{K}} \Diamond p \to \Box p$ but $\vdash_{\mathbf{S4\text{-}P\text{-}OR}} \mathrm{p}(\Diamond p \to \Box p)$.

Proof. The contrapositive of the O axiom is $\langle R\rangle\langle\leq\rangle p \to \langle\leq\rangle\langle R\rangle p$, which with the partial functionality axiom $\langle R\rangle p \to [R]p$ gives us $\vdash_{\mathbf{S4\text{-}P\text{-}O}} \langle R\rangle\langle\leq\rangle\varphi \to \langle\leq\rangle[R]\varphi$ for any φ. Thus,

$$\begin{aligned}
\mathrm{p}(\Diamond p) &\Leftrightarrow [\leq]\langle R\rangle\langle\leq\rangle[co]p \\
&\Rightarrow [\leq]\langle\leq\rangle[R][co]p \\
&= [co][R][co]p \\
&= [co]\mathrm{p}(\Box p) \\
&\Rightarrow \mathrm{p}(\Box p),
\end{aligned}$$

where the last line uses Lemma 4.27.2 for **S4-K-R**. Hence $\vdash_{\mathbf{S4\text{-}P\text{-}OR}} \mathrm{p}(\Diamond p \to \Box p)$. □

5 Further Directions

The notions and results presented in this paper invite further investigation along a number of lines.

5.1 Exploiting the Translation: Correspondence

We have established our translations and embeddings for theoretical reasons, without an eye to practical purposes. Still, there are several further uses that could be made of the translations. In particular, it would be possible to make concrete comparisons of proofs in possibility style with proofs in bimodal style. Our arguments in §4 already provide many relevant examples, where issues arise such as the role of intermediate bimodal formulas that do not themselves occur as translations of possibility formulas. Somewhat more technically, we can also use our translation to define bimodal companions for possibility logics and then to compare the two landscapes for automatic transfer of properties of logics. This is in analogy with *modal companions* of intermediate logics for the Gödel-McKinsey-Tarski translation (see, e.g., Chagrov and Zakharyaschev 1997, Sec. 9.6). We can call a bimodal logic $\mathbf{L}'$ a *bimodal companion* of a modal logic $\mathbf{L}$ if for each modal formula φ, we have $\vdash_{\mathbf{L}} \varphi$ iff $\vdash_{\mathbf{L}'} \mathrm{g}(\varphi)$. Similarly, we can define bimodal companions for the p translation. The theory of these companions deserves special attention and will be discussed elsewhere.

Given the analysis of semantic conditions and axioms in this paper, perhaps the most obvious topic for further analysis through our translation is *frame correspondence*. We provide a bit of detail on this, though a full analysis is beyond the scope of this paper; cf. Holliday 2015, Yamamoto 2016 for further results.

Let us say that a unimodal formula φ is *first-order definable* over possibility frames iff there is a sentence in the language of first-order logic with binary relation symbols for $\leq$ and R such that for any possibility frame $\mathcal{F} = \langle S, \leq, R, P\rangle$, φ is valid over $\mathcal{F}$ according to possibility semantics iff ψ is true in $\langle S, \leq, R\rangle$ as a first-order structure. By contrast, we say that a *bimodal* formula φ is first-order definable over a class F of bimodal frames iff there is a sentence ψ of the same first-order language such that for any $\mathfrak{F} \in \mathsf{F}$, φ is valid over $\mathfrak{F}$ according to standard possible world semantics iff ψ is true in $\mathfrak{F}$ as a first-order structure.

Can we translate possibility formulas φ and then find out their frame properties, such as first-orderness, using standard correspondence results for the formula $\mathrm{p}(\varphi)$ in our bimodal language? There is a difficulty in doing so, since our translation adds extra levels of modality for negations and for atoms, so even simple syntactic Sahlqvist forms may end up looking quite complex. Even so, one observation can be made.

Proposition 5.1 (Transferring Correspondence)**.** A unimodal formula φ is first-order definable over possibility frames iff its translation $\mathrm{p}(\varphi)$ is first-order definable over possibility frames viewed as bimodal frames.

Proof. The proof of this result uses the following observation, relying on the special syntactic form of our translation that substituted cofinality formulas $[\leq]\langle\leq\rangle p$ for atoms p. Any possibility frame $\mathcal{F} = \langle S, \leq, R, P\rangle$ is at the same time a bimodal frame $\langle S, \leq, R\rangle$, and as in Remark 4.5, it is easy to show by induction on formulas that for any valuation V on the bimodal frame, the resulting bimodal model makes a formula $\mathrm{p}(\varphi)$ true at state s iff φ is true (now in the sense of possibility semantics) at s in the matching possibility model whose valuation π is such that $\pi(p)$ is the interior of the closure of $V(p)$. Conversely, any admissible valuation on the possibility frame is already a valuation on the bimodal frame such that $\mathrm{p}(\varphi)$ is true at s in the bimodal model iff φ is true at s in the possibility model. $\square$

Of course, if $\mathrm{p}(\varphi)$ is first-order definable over all bimodal frames, then in particular it is first-order definable over those bimodal frames coming from possibility frames, so we can apply the right-to-left direction of Proposition 5.1 to show that φ is first-order definable over possibility frames. In this way, general bimodal correspondence theory may be brought to bear on correspondence theory for possibility semantics.

Unfortunately, as noted above, bimodal translations of possibility axioms tend to be complex, and they only yield to general Sahlqvist-style results in very simple cases. As a positive example, to see that $\Box p \to p$ is first-order definable over possibility frames, we simply note that its p translation $[\leq]([R][\leq]\langle\leq\rangle p \to [\leq]\langle\leq\rangle p)$ is equivalent to the Sahlqvist formula $[\leq](\langle\leq\rangle[\leq]p \to \langle R\rangle\langle\leq\rangle[\leq]p)$ and is therefore first-order definable over all bimodal frames. Thus, $\Box p \to p$ is first-order definable over possibility frames by Proposition 5.1. Even if for a given φ, $\mathrm{p}(\varphi)$ is not provably equivalent to a Sahlqvist formula in the basic bimodal logic **K-K**, this is not the end of the story. For if $\mathrm{p}(\varphi)$ is provably equivalent to a bimodal formula ψ in the bimodal logic **S4-K-R** of possibility frames, then $\mathrm{p}(\varphi)$ is semantically equivalent to ψ over possibility frames regarded as bimodal frames, so we may substitute ψ for $\mathrm{p}(\varphi)$ in Proposition 5.1. In short, we may be able to use our bimodal logic to simplify $\mathrm{p}(\varphi)$ into a ψ that is Sahlqvist and then apply Proposition 5.1. Moreover, if we are interested in correspondence relative to, e.g., *functional* possibility frames, then we may use the stronger bimodal logic **S4-F-R** with the axiom $\langle R\rangle p \leftrightarrow [R]p$ to try to reduce $\mathrm{p}(\varphi)$ to a Sahlqvist formula.

To understand the general situation, an analogy may be helpful with correspondence theory for *intuitionistic logic*. Here valuations only assign persistent sets (upsets) to propositional variables, and the analysis takes place on special frame classes: pre-orders, partial orders, or even trees. Both of these differences matter. Allowing only special semantic values for propositions may make certain valuations used in classical correspondence arguments unavailable, such as the "minimal valuations" that are crucial to Sahlqvist-style analysis.[7] And working on special frame classes may change correspondence behavior drastically—witness the result in van Benthem 1976 that the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$, and in fact all modal reduction principles in the unimodal language, become first-order definable over transitive frames.

Instead of investigating this issue further here, we cite some relevant results from Rodenburg 1986. First of all, many complex intuitionistic axioms turn out to be first-order when restrictions on valuations plus special frame conditions are combined. There are indeed second-order axioms too, but these live higher up in syntactic complexity, an example being "Scott's Axiom" $((\neg\neg p \to p) \to (p \vee \neg p)) \to (\neg p \vee \neg\neg p)$. In this setting, Rodenburg develops a tableau method for describing refutation patterns of formulas which allows him to prove, among many other things, that every intuitionistic formula in the sublanguage with $\neg, \wedge$ only defines a first-order frame condition. Now our possibility semantics is different in two ways: it puts more restrictions on admissible valuations, and it adds an ordinary modality with its own accessibility relation, often a partial function. Nevertheless, some analogies may continue to hold.

[7]However, there are often fixes for this, for which we refer again to Holliday 2015, Yamamoto 2016.

The following point should be stressed: every possible world frame is at the same time a modally equivalent possibility frame, in which $\leq$ is the identity relation, so if a modal formula is not first-order definable over possible world frames, then it is not first-order definable over possibility frames either—at least not over *arbitrary* possibility frames. This argument does not show that, e.g., non-first-orderness over possible world frames implies non-first-orderness over possibility frames in which the modal accessibility relation is a (partial) *function*. Indeed, it is an open question whether we may get more first-order correspondence over functional possibility frames. Given Proposition 5.1, one way to pursue this question would be to use results about bimodal correspondence relative to frames in which one relation is a preorder and the other is function. As noted above, there are strong results for unimodal correspondence relative to preordered frames, so a natural follow-up would be to look for results in the bimodal case of a preorder plus function.

5.2 Translations, Semantics, and Logical Systems

We have presented a dense array of formal results on system translations, but what is the main thrust? We briefly list a few perhaps unusual ways of looking at our findings, without going into sustained discussion. The way we see our analysis, several things seem noteworthy.

First, as we have shown, our approach extends well-known existing translations from classical logic into intuitionistic logic, and from intuitionistic logic into classical modal **S4**. We believe that this adds motivation for a more general study of modal system embeddings, where our results highlighted the linkage between syntactic translations and axiomatic strength of the logics involved. The same translation may work while the target logic gets strengthened progressively, but we also found transition points. Moreover, we showed how sometimes, stronger logics may support syntactically simpler embeddings. All of this seems suggestive material for further general theory of translations between logical systems.

Our translation also changes the usual view of bimodal logics as simple extensions of unimodal ones. It shows that one can decompose unimodal logics for very general classes of structures into bimodal logics for much more constrained model classes. How far does this phenomenon go? Other examples exist, such as the reanalysis of **S4** into modal **S5** for epistemically accessible worlds plus a temporal modality over strict partial orders [van Benthem, 2009]. This shows that there is more to modal logics as usually given with their *prima facie* semantics than meets the eye, and one would want to understand this phenomenon in general.

Perhaps of greatest interest to us, however, is another trade-off. Our bimodal embedding of possibility logic highlights the connection between two options for conceptual analysis. One can give meaning to a standard logical language in terms of a new "nonstandard" semantics, or one can translate the language and its logic into some other standard system. This is of course well-known in the case of Kripke and Beth semantics for intuitionistic logic, but our results extend the range of examples. Again, this seems a phenomenon that needs to be understood more generally: when and how can non-standard semantics be mimicked faithfully by standard translations? There may be a trivial sense in which this can always be done, as soon as precise non-standard truth conditions are given, since we can translate into the meta-language of the models, often a first-order language.[8] But our results use only a small part of that meta-language, staying close to the original object language, and the issue is when such small steps suffice.

Finally, here are some concrete technical open problems beyond the positive results we have presented. We embedded unimodal logics (**K**, **KD**) into bimodal ones (**S4-P**, **S4-K-R**, etc.). What about the converse direction, from these bimodal logics back to these unimodal ones? We suspect that no embeddings exist of this kind. Also, we worked with a whole landscape of bimodal logics, but we only compared these logics in a

[8]The well-known "standard translation" for modal logic shows how fruitful this method can be.

weak way, namely, qua power for proving translated formulas from our unimodal logic. What about relative interpretations, or lack thereof, between the various bimodal logics themselves that we have introduced? A well-known difficulty in the area of relative interpretation is finding *impossibility results* showing non-embeddability of logics. It would be good to complement our analysis with one that also offers such tools, to see what equalities and non-equalities of logics we have really established modulo embeddability.

5.3 Further Mathematical Perspectives

While our presentation has concentrated on logics, and in particular, classical normal modal logics, other perspectives seem worth exploring. In particular, we have seen that our bimodal embeddings use systems that make sense in *topology* and even the theory of dynamical systems. We see our results, therefore, as also adding to the tradition of "dynamic topological logic" [Artemov et al., 1997, Kremer and Mints, 2005], and we believe that topological models may add valuable intuitions to possibility semantics.

But one can also generalize from topology and ask how our results will fare on weaker base logics. From a modal perspective, the obvious candidate here is *neighborhood models* that support weaker modal logics where $\Box$ does not distribute over conjunction and only upward monotonicity remains valid [Chellas, 1980, Hansen, 2003]. Can our results be extended to this weaker modal base?

One can also look at weakenings of the propositional base of our logics, going from Boolean algebra to distributive lattices or even lower. In this case, *algebraic methods* may become most suited, and it is relevant that an algebraic representation theory for possibility frames has been given in Holliday 2015.[9] Also relevant here may be the general algebraic correspondence techniques of Conradie et al. 2014.

5.4 Language Redesign

We conclude with a standard question about non-standard semantics. Whenever we extend a class of models for a given language, the question arises whether other languages would be more appropriate for these models, perhaps making finer distinctions than the original one.[10]

One obvious extension of our modal base language would add an *iteration modality* $\Box^*$ that allows us to talk about iterated beliefs, or iterated actions on a topological space under our functional translation. The semantic clause for $\Box^*$ in possibility semantics is the standard one, saying that φ is true at any state reachable from the current one by taking R-steps. Our main results all hold for such a **PDL**-type extension of our logics, starting with the key Possibilization Lemma (Lemma 2.15), which now also tells us that $\mathfrak{M}_f^{\wp}, X \Vdash_{\mathrm{p}} \Box^*\varphi$ iff $\forall x \in X$: $\mathfrak{M}, x \vDash \Box^*\varphi$.

But perhaps more interesting are *new propositional connectives* arising in possibility models. The original paper of Humberstone [1981] appeared at a time when "interval semantics" for temporal expressions was attracting interest [Humberstone, 1979, Kamp, 1979, van Benthem, 1980, Röper, 1980]. The conditions of Persistence and Refinability essentially restrict attention to what are called "distributive properties" of intervals, which reduce to truth in all points in the interval, assuming there are such points.[11] This is precisely what is expressed by the Possibilization Lemma. While this is an important case, one might argue that the properties that make intervals come into their own are non-distributive "collective" ones such as

[9]An order-topological duality for this algebraic semantics via *descriptive possibility frames* will appear in a future paper.

[10]Examples of this are ubiquitous: intuitionistic logic over stage models, linear logic over state spaces, dependence logic over sets of assignments, and more.

[11]It is a crucial aspect of possibility semantics that it does not require that every state be refined by an endpoint (instant, world). Indeed, this is what makes possibility frames more general than possible world frames for characterizing modal logics (see Holliday 2015).

“lasting for an hour” or “building a house together”. Many examples of such further connectives can be found in the literature, such as the modalities for suprema and infima introduced in van Benthem 1995. We do not know what becomes of our results in a setting extended with collective properties and matching logical operators, and what translations can still be made. It would in fact be interesting to delimit the scope of our translation method more precisely by determining which logical operations, with what sort of truth conditions, support the kind of analysis we have given.

6 Conclusion

We started from the recently reviving semantic paradigm of possibility models for modal logic. We then analyzed this paradigm in terms of a new embedding of modal logic into a classical bimodal logic of an inclusion order with a partial function acting on it. Using this first bridge between new and old systems, we broadened our analysis to a greater variety of bimodal logics, as well as simplified translations. Our analysis brought to light several new systems and technical questions of axiomatizability, frame correspondence, and relative interpretability. Moreover, our results connect existing systems in new ways, for instance, linking possibility semantics to dynamic topological logic.

This can be seen as a contribution to technical modal logic, but we also see a more general philosophical thrust. In much of the philosophical literature, conceptual innovation is equated with providing new semantics, or put in other terms, moving toward “non-standard logics”. While this is indeed one fruitful methodology, there is always the alternative of deconstructing alternative semantics and logics in terms of translations into more classical systems. This paper is one more instance of this illuminating duality.

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