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UNIVERSITY OF CALIFORNIA, SAN DIEGO

**Generalized Shuffle Conjectures for the Garsia-Haiman Delta Operator**

A dissertation submitted in partial satisfaction of the  
requirements for the degree  
Doctor of Philosophy

in

Mathematics

by

Andrew Timothy Wilson

Committee in charge:

Professor Jeffrey B. Remmel, Chair  
Professor Adriano M. Garsia  
Professor Ronald Graham  
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Professor Brendon Rhoades

2015

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The dissertation of Andrew Timothy Wilson is approved, and it is acceptable in quality and form for publication on micro-film and electronically:

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Chair

University of California, San Diego

2015

DEDICATION

For Helen.

## EPIGRAPH

"Do you mean ter tell me," he growled at the Dursleys, "that this boy - this boy! - knows nothin' abou' - about ANYTHING?"

Harry thought this was going a bit far. He had been to school, after all, and his marks weren't bad.

"I know some things," he said. "I can, you know, do math and stuff."

-J. K. Rowling, *Harry Potter and the Sorcerer's Stone*

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A. T. Wilson, "An extension of MacMahon's Equidistribution Theorem to ordered multiset partitions," *Proceedings of FPSAC 2014*, pp. 345-356. Extended abstract.

J. B. Remmel and A. T. Wilson, "Block patterns in Stirling permutations," *J. of Combinatorics*, **6** (2015), pp. 179-204.

J. B. Remmel and A. T. Wilson, "An extension of MacMahon's Equidistribution Theorem to ordered set partitions," accepted to *J. of Combinatorial Theory, Series A*, 2015.

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ABSTRACT OF THE DISSERTATION

**Generalized Shuffle Conjectures for the Garsia-Haiman Delta Operator**

by

Andrew Timothy Wilson

Doctor of Philosophy in Mathematics

University of California, San Diego, 2015

Professor Jeffrey B. Remmel, Chair

We conjecture two combinatorial interpretations for the symmetric function  $\Delta_{e_k} e_n$ , where  $\Delta_f$  is an eigenoperator for the modified Macdonald polynomials defined by Garsia and Haiman. The first interpretation is due to Haglund and the second is due to the author. Both interpretations can be seen as generalizations of the Shuffle Conjecture of Haglund, Haiman, Remmel, Loehr, and Ulyanov. The primary goal of this dissertation is to prove various special cases of these conjectures. We accomplish this goal by connecting the interpretations to objects such as ordered set partitions, rook placements, Tesler matrices, and LLT polynomials. These connections lead to many new results about these objects, such as an extension of MacMahon's classical equidistribution theorem from permutations to ordered set partitions.

# Chapter 1

## Introduction

Many of the central results in algebraic combinatorics deal with correspondences between symmetric functions, representations of the symmetric group, and combinatorial generating functions. The study of Macdonald polynomials [Mac95] and the Shuffle Conjecture [HHL<sup>+</sup>05b] are two such examples. In this chapter, we explain how to view Macdonald polynomials and the Shuffle Conjecture from this perspective and how these examples have inspired the work contained in the remainder of this dissertation.

We begin by reviewing the theory of symmetric functions in Section 1.1. In Section 1.2 we review the history of the Shuffle Conjecture and we generalize the Shuffle Conjecture to obtain our main focus, which we call the Delta Conjectures. Finally, in Section 1.3 we give several combinatorial reformulations of the Delta Conjectures which will be useful in subsequent chapters.

### 1.1 Symmetric Functions

In this section, we give a brief review of the combinatorics of symmetric functions. This material can also be found in Chapter 7 of [Sta99] or in [Sag02, Hag08]; readers looking for a more leisurely introduction may wish to consult those sources. We begin by defining the *symmetric group of order  $n$* , which we will denote  $\mathfrak{S}_n$ , as the set of bijections from  $\{1, 2, \dots, n\}$  to itself with composition as the group operation. Elements of the symmetric group are called *permutations*. Most often, we will write a permutation  $\sigma \in \mathfrak{S}_n$  as the word whose  $i$ th entry  $\sigma_i$ , reading from left to right, equals

$\sigma(i)$ . This is called *one-line notation*. Occasionally we will need to think of  $\sigma$  in *cycle notation*, where we use a series of parentheses to denote the cycles in the map  $\sigma$ . For example,  $\sigma = 52143$  is written as  $(1, 5, 3)(2)(4)$  in cycle notation. The *cycle type* of  $\sigma$  is the weakly decreasing list of sizes of cycles in  $\sigma$ . In the previous example,  $\sigma$  has cycle type  $(3, 1, 1)$ .

Given a formal power series  $f$  in the infinite set of variables  $x_1, x_2, x_3, \dots$ , any permutation  $\sigma \in \mathfrak{S}_n$  acts on  $f$  by sending  $x_i$  to  $x_{\sigma_i}$  for each  $i = 1, 2, \dots, n$ . The *ring of symmetric functions*  $\Lambda$  is the ring of such formal power series that are invariant under this action for every permutation of every order. For reasons that will become clear later, we will set  $\mathbb{C}(q, t)$  as the field of coefficients of  $\Lambda$ .  $\Lambda$  can be graded as  $\Lambda = \bigoplus_{n \geq 0} \Lambda^{(n)}$ , where  $\Lambda^{(n)}$  consists of the formal power series in  $\Lambda$  that are homogeneous of degree  $n$ .

### 1.1.1 The classical bases

There are several classical bases for  $\Lambda^{(n)}$  when it is viewed as a vector space. Each basis is indexed by the set of *partitions* of  $n$ , which are the weakly decreasing sequences of positive integers that sum to  $n$ . We write  $\lambda \vdash n$  to specify that  $\lambda$  is a partition of  $n$  and  $\ell(\lambda)$  to denote the length of  $\lambda$ . For example,  $(3, 1, 1) \vdash 5$  and this partition has length 3. We define a partial order on the set of partitions of  $n$  by  $\lambda \leq \mu$  if and only if  $\ell(\lambda) \leq \ell(\mu)$  and each part of  $\lambda$  is less than or equal to the corresponding part in  $\mu$ . For example,  $(3, 1, 1) \leq (3, 2)$ , while  $(3, 1, 1)$  and  $(2, 2, 1)$  are incomparable. Later, we will also need to consider *compositions*  $\alpha \vDash n$ , which are essentially partitions but with ordered parts. For example,  $(1, 3, 1)$  and  $(1, 1, 3)$  are two different compositions of 5.

For any partition  $\lambda \vdash n$  and any positive integer  $k$ , we set

$$\begin{aligned} e_k(x) &= \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k} & e_\lambda(x) &= \prod_{i=1}^{\ell(\lambda)} e_{\lambda_i}(x) \\ h_k(x) &= \sum_{i_1 \leq \dots \leq i_k} x_{i_1} \dots x_{i_k} & h_\lambda(x) &= \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i}(x) \\ p_k(x) &= \sum_{i \geq 1} x_i^k & p_\lambda(x) &= \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(x). \end{aligned}$$

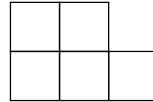


and

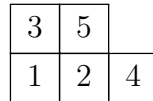
$$m_\lambda(x) = \sum_{i_1, \dots, i_{\ell(\lambda)} \geq 1} x_{i_1}^{\lambda_1} \dots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}}$$

These are the *elementary*, *homogeneous*, and *power sum*, and *monomial symmetric functions*, respectively. When the set of variables is clear, we will often omit  $x$  from the notation for these symmetric functions. It is a classical fact that taking the union over any of these sets of symmetric functions for all  $\lambda \vdash n$  gives a basis for  $\Lambda^{(n)}$ . For example,  $\{p_3, p_{2,1}, p_{1,1,1}\}$  is a basis for  $\Lambda^{(3)}$ .

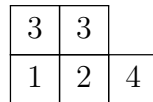
There are two more bases for  $\Lambda^{(n)}$  which will be instrumental in our work. The first is the *Schur functions*  $s_\lambda(x)$ , which are deeply connected to topics such as the representation theory of  $\mathfrak{S}_n$  and the geometry of the Grassmannian variety. Schur functions have several nontrivially equivalent definitions. For our purposes, it will be most convenient to define them combinatorially. Given a partition  $\lambda \vdash n$ , the *Ferrers diagram* of  $\lambda$  is the diagram consisting of  $\lambda_1$  left-justified squares in the bottom row,  $\lambda_2$  left-justified squares in the second row from the bottom, and so on. Below we have drawn the Ferrers diagram for the partition  $(3, 2) \vdash 5$ .



A *standard Young tableau* of shape  $\lambda$  is a bijective filling of the cells of the Ferrers diagram of  $\lambda$  with the integers  $1, 2, \dots, n$  such that the entries increase from left to right in each row and from bottom to top in each column. For example,



is a standard Young tableau of shape  $(3, 2)$ . A *semistandard Young tableau* is a (not necessarily bijective) filling with positive integers such that the entries increase weakly from left to right in each row and strictly from bottom to top in each column. Given a tableau  $T$ , the monomial  $x^T$  is the product  $\prod_{c \in \lambda} x_{T(c)}$  over all cells  $c$  in  $\lambda$ . Below, we have drawn a semistandard Young tableau of shape  $(3, 2)$  with monomial  $x_1 x_2 x_3^2 x_4$ .



The set of all standard and semistandard Young tableaux with respect to  $\lambda$  are denoted  $\text{SYT}(\lambda)$  and  $\text{SSYT}(\lambda)$ , respectively. Now we can define the Schur function:

$$s_\lambda(x) = \sum_{T \in \text{SSYT}(\lambda)} x^T.$$

We will define one more basis for  $\Lambda$  in Subsection 1.1.5.

## 1.1.2 Properties of symmetric functions

There are number of properties of the ring of symmetric functions that will be valuable in the sequel. The first is that they have an inner product, usually called the *Hall inner product*, which can be defined by any of the following:

$$\begin{aligned} \langle s_\lambda, s_\mu \rangle &= \chi(\lambda = \mu) \\ \langle m_\lambda, h_\mu \rangle &= \chi(\lambda = \mu) \\ \langle p_\lambda, p_\mu \rangle &= z_\lambda \chi(\lambda = \mu) \end{aligned}$$

where  $\chi$  evaluates to 1 if the statement inside it is true and 0 if the statement is false and  $z_\lambda = \prod_{i \geq 1} 1^{m_i} m_i!$ , where  $m_i$  is the multiplicity of  $i$  in  $\lambda$ .

Next, we define an algebra endomorphism  $\omega$  from  $\Lambda$  to itself by  $\omega(e_\lambda) = h_\lambda$  for all partitions  $\lambda$ . It follows that  $\omega$  is actually an involution, i.e.  $\omega(h_\lambda) = e_\lambda$ . One can also show that  $\omega(s_\lambda) = s_{\lambda'}$ , where  $\lambda'$  is the partition obtained by reflecting the Ferrers diagram of  $\lambda$  around the diagonal line  $y = x$ .

We will find that the concept of plethysm is quite valuable, as it has been throughout the study of Macdonald polynomials. Given a power series  $E$  in the variables  $q, t$  and  $x_1, x_2, x_3, \dots$ , we consider  $E$  as a sum of signed monomials. We define the plethysm  $p_k[E]$  to be the sum of all the monomials in  $E$  raised to the  $k$ th power. Extending by multiplication, this defines  $p_\lambda[E]$  for any partition  $\lambda$ . Finally, for any symmetric function  $f$  we compute  $f[E]$  by expanding  $f$  into the power sum basis and then replacing each  $p_\lambda$  with  $p_\lambda[E]$ . Sometimes we will use  $X$  to denote the sum  $x_1 + x_2 + x_3 + \dots$ . With this notation, we can state a useful identity that is sometimes called Cauchy's Formula: for any bases  $\{a_\lambda : \lambda \vdash n\}$  and  $\{b_\lambda : \lambda \vdash n\}$  that are dual with respect to the Hall inner

product and two sums  $X$  and  $Y$ ,

$$e_n[XY] = \sum_{\lambda \vdash n} \omega(a_\lambda[X]) b_\lambda[Y]. \quad (1.1)$$

More information about plethysm can be found in [LR11].

### 1.1.3 Connections to representation theory

One of the most celebrated applications of symmetric functions is to representation theory, specifically the representation theory of the symmetric and general linear groups. We will focus on the symmetric group. Any group representation  $\mathbf{M}$  is uniquely associated with a function called the *character*, denoted  $\chi^{\mathbf{M}}$ , which is computed by taking traces of the matrices associated with the representation. Every character is a *class function*, meaning that it is constant on conjugacy classes. There is a natural scalar product on the class functions of a group  $G$ , defined by

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{\sigma \in G} f(\sigma) \overline{g(\sigma)}$$

where the bar denotes complex conjugation. The *Frobenius map* sends class functions of  $\mathfrak{S}_n$  to  $\Lambda$  and is defined by

$$\text{Frob}(f) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(\sigma) p_{\text{cycle}(\sigma)}$$

where  $\text{cycle}(\sigma)$  is the cycle type of  $\sigma$ . One can show that the Frobenius map is a bijective ring homomorphism as well as an isometry with respect to the inner product defined above and the Hall inner product on  $\Lambda$  [Sta99]. It is the key tool to understanding the irreducible characters of  $\mathfrak{S}_n$ . In particular, the irreducible characters of  $\mathfrak{S}_n$  are in bijection with partitions  $\lambda \vdash n$ . If  $\chi^\lambda$  is the irreducible character associated with  $\lambda$ , then  $\text{Frob}(\chi^\lambda) = s_\lambda$ .

We will find that many of the natural  $\mathfrak{S}_n$ -modules that arise in our work have a natural grading or bigrading, i.e. we can write  $\mathbf{M} = \bigoplus_{i \geq 0} \mathbf{M}_i$  or  $\mathbf{M} = \bigoplus_{i, j \geq 0} \mathbf{M}_{i, j}$  where each summand is also an  $\mathfrak{S}_n$ -module. In these cases, we will often want to know

about the *Hilbert series* of  $\mathbf{M}$ , defined by

$$\begin{aligned}\text{Hilb}(\mathbf{M}; q) &= \sum_{i \geq 0} q^i \dim(\mathbf{M}_i) \\ \text{Hilb}(\mathbf{M}; q, t) &= \sum_{i, j \geq 0} q^i t^j \dim(\mathbf{M}_{i, j})\end{aligned}$$

as well as the *Frobenius series* of  $\mathbf{M}$ , defined by

$$\begin{aligned}\text{Frob}(\mathbf{M}; q) &= \sum_{i \geq 0} q^i \text{Frob}(\mathbf{M}_i) \\ \text{Frob}(\mathbf{M}; q, t) &= \sum_{i, j \geq 0} q^i t^j \text{Frob}(\mathbf{M}_{i, j}).\end{aligned}$$

The presence of the variables  $q$  and  $t$  should clarify any ambiguity between Frobenius series and the Frobenius map. Note that any Frobenius series is necessarily *Schur positive*, meaning that it can be expressed in the Schur basis for  $\Lambda$  with coefficients in  $\mathbb{N}[q, t]$ . This is because every  $\mathfrak{S}_n$ -module decomposes as a sum of irreducible modules, and irreducible characters correspond to Schur functions via the Frobenius map. We should also mention that one can recover the Hilbert series of a module from its Frobenius series by taking the scalar product with  $p_1^n$ .

### 1.1.4 Quasisymmetric functions

The ring of quasisymmetric functions consists of the formal power series in variables  $x_1, x_2, x_3, \dots$  that are invariant under any permutation of the indices that preserves the order of the indices. We will primarily use the monomial basis  $\{M_\alpha : \alpha \vDash n\}$  for the quasisymmetric functions that are homogeneous of degree  $n$ . These functions are defined

$$M_\alpha(x) = \sum_{1 \leq i_1 < \dots < i_{\ell(\alpha)}} x_{i_1}^{\alpha_1} \dots x_{i_{\ell(\alpha)}}^{\alpha_{\ell(\alpha)}}.$$

In the Shuffle Conjecture as well as the Delta Conjectures, one can reformulate the combinatorial side in terms of Gessel fundamental quasisymmetric function; however, since we opt to use the monomial approach for the remainder of the paper, we will not need the fundamental quasisymmetric functions. [Sta99] contains more information on quasisymmetric functions.

### 1.1.5 Macdonald polynomials

In [Mac95], Macdonald introduced a basis for  $\Lambda^{(n)}$  that generalized several existing classes of symmetric functions, such as Schur functions, Jack polynomials, Hall-Littlewood polynomials. In his honor, these new polynomials came to be known as Macdonald polynomials. One way to define Macdonald polynomial associated with a partition  $\mu$  is as the unique symmetric function  $\tilde{H}_\mu[X; q, t]$  satisfying the conditions

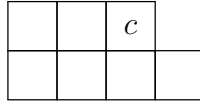
$$\begin{aligned}\tilde{H}_\mu[X(q-1); q, t] &\in \mathbb{C}[q, t] \{m_\lambda : \lambda \leq \mu'\} \\ \tilde{H}_\mu[X(t-1); q, t] &\in \mathbb{C}[q, t] \{m_\lambda : \lambda \leq \mu\} \\ \langle \tilde{H}_\mu[X; q, t], s_n \rangle &= 1.\end{aligned}$$

We have used brackets because the definition involves plethysm; when there are no plethystic computations in sight, we will often write the Macdonald polynomial as  $\tilde{H}_\mu(x; q, t)$  or even just  $\tilde{H}_\mu$ .

There are two properties of Macdonald polynomials which motivate our work. The first is that Macdonald polynomials are Schur positive, meaning they can be expanded into the Schur function basis as

$$\tilde{H}_\mu(x; q, t) = \sum_{\lambda \vdash n} \tilde{K}_{\lambda, \mu}(q, t) s_\lambda(x)$$

for polynomials  $\tilde{K}_{\lambda, \mu}(q, t) \in \mathbb{N}[q, t]$ . These polynomials are known as the  $q, t$ -Kostka coefficients. Although Macdonald himself noticed that his polynomials seemed to be Schur positive, he was unable to prove that this was the case. In [GH93], Garsia and Haiman defined a module that they believed had Frobenius series equal to  $\tilde{H}_\mu$ . Given a partition  $\mu \vdash n$  and a cell  $c$  in the Young diagram of  $\mu$  (drawn in French notation), we set  $a'(c)$  and  $\ell'(c)$  to be the number of cells in  $\mu$  that are strictly to the left and strictly below  $c$  in  $\mu$ , respectively. For example, the cell  $c$  in the following partition has  $a'(c) = 2$  and  $\ell'(c) = 1$ .



Garsia and Haiman defined  $\tilde{\mathbf{H}}_\mu$  to be the linear span of all partial derivatives of the determinant

$$\left| x_i^{\ell'(c)} y_i^{a'(c)} \right|_{1 \leq i \leq n, c \in \mu}.$$

$\tilde{\mathbf{H}}_\mu$  is an  $\mathfrak{S}_n$ -module under the *diagonal action*, where  $\sigma \in S_n$  simultaneously sends  $x_i$  to  $x_{\sigma_i}$  and  $y_i$  to  $y_{\sigma_i}$ . After a long program of work, Haiman proved that  $\text{Frob}(\tilde{\mathbf{H}}_\mu; q, t) = \tilde{H}_\mu(x; q, t)$  in [Hai01], implying the Schur positivity of Macdonald polynomials.

Haiman's proof relies heavily on modern techniques in algebraic geometry; as a result the proof does not provide much insight into the  $q, t$ -Kostka coefficients  $\tilde{K}_{\lambda, \mu}(q, t)$ . The main result in this direction is a theorem of Haglund, Haiman, and Loehr [HHL05a]. In this paper, the authors prove that

$$\tilde{H}_\mu(x; q, t) = \sum_T q^{\text{inv}(T)} t^{\text{maj}(T)} x^T$$

where the sum is over all (not necessarily semistandard) fillings of the Ferrers diagram of  $\mu$  with (not necessarily unique) positive integers and  $\text{inv}$  and  $\text{maj}$  are two statistics on such fillings. Haglund, Haiman, and Loehr relate their formula to Lascoux, Leclerc, and Thibon's polynomials [LLT97] in order to show that their formula is symmetric and that the coefficient of  $m_\lambda$  in their formula corresponds to restricting the sum to fillings with exactly  $\lambda_i$   $i$ 's for all  $i \geq 1$ . Thus, the formula gives the monomial expansion of  $\tilde{H}_\mu(x; q, t)$ .

The important takeaway of this work for our purposes is that Macdonald polynomials can be considered from three points of view: as symmetric functions (like in Macdonald's original work), as Frobenius series (of Garsia-Haiman modules), or as combinatorial generating functions (of fillings of Ferrers diagrams). In the next section, we will explore other polynomials from each of these three perspectives.

### 1.1.6 Notation

We will often use the standard combinatorial notation for  $q$ - and  $q, t$ -analogs. To begin, for any nonzero integer  $n$  we set

$$[n]_q = \frac{q^n - 1}{q - 1} \qquad [n]_{q,t} = \frac{q^n - t^n}{q - t}.$$

Note that when  $n > 0$  we can rewrite these expressions as

$$[n]_q = q^{n-1} + q^{n-2} + \dots + q + 1 \qquad [n]_{q,t} = q^{n-1} + q^{n-2}t + \dots + qt^{n-2} + t^{n-1}.$$

Analogues of factorials and binomial coefficients are defined as follows.

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q \quad [n]_{q,t}! = [n]_{q,t} [n-1]_{q,t} \cdots [1]_{q,t}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

## 1.2 Shuffle Conjectures

In the course of their research, Garsia and Haiman noticed that their modules were intimately connected with another  $\mathfrak{S}_n$ -module, the ring of diagonal coinvariants. Later, Bergeron and Garsia noticed that the Frobenius series of this module could be expressed simply in terms of a Macdonald polynomial eigenoperator. Haglund, Haiman, Loehr, Remmel, and Ulyanov conjectured a combinatorial formula for this Frobenius series similar to Haglund, Haiman, and Loehr's formula for Macdonald polynomials [HHL<sup>+</sup>05b]. Their formula uses objects known as parking functions. In this section, we describe this conjectured formula as well as its extensions and refinements.

### 1.2.1 Diagonal coinvariants

The *ring of invariants*  $\mathbf{I}_n$  of  $\mathfrak{S}_n$  is the set of polynomials in the variables  $x_1, \dots, x_n$  which are invariant under the action of  $\mathfrak{S}_n$ . These are simply the symmetric functions in  $n$  variables. The *ring of coinvariants*  $\mathbf{R}_n$  is the quotient ring obtained by modding out by the invariants, i.e.

$$\mathbf{R}_n = \mathbb{C}[x_1, \dots, x_n] / \mathbf{I}_n.$$

The ring of coinvariants is an  $\mathfrak{S}_n$ -module and it has a natural grading by degree. With this grading, it has Hilbert series

$$\text{Hilb}(\mathbf{R}_n; q) = [n]_q!$$

and Frobenius series

$$\text{Frob}(\mathbf{R}_n; q) = \sum_{\lambda \vdash n} s_\lambda \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)}$$

where  $\text{maj}$  is a certain statistic on standard Young tableaux [Hag08].

Garsia and Haiman considered the *ring of diagonal invariants*  $\mathbf{DI}_n$ , which are the polynomials in variables  $x_1, \dots, x_n$  and in  $y_1, \dots, y_n$  which are invariant under the diagonal action. The *ring of diagonal coinvariants* is the quotient ring (and  $\mathfrak{S}_n$ -module)

$$\mathbf{DR}_n = \mathbb{C}[x_1, \dots, x_n; y_1, \dots, y_n] / \mathbf{DI}_n.$$

Garsia and Haiman noticed that the dimension of  $\mathbf{DR}_n$  was  $(n+1)^{n-1}$ , which also gives the number of parking functions of order  $n$  (which we define in Subsection 1.2.3). Their hope was to discover information about the bigraded Hilbert and Frobenius series of  $\mathbf{DR}_n$ .

$\mathbf{DR}_n$  is isomorphic to another module known as the module of *diagonal harmonics*, denoted  $\mathbf{DH}_n$ . We define  $\mathbf{DH}_n$  to be the set of all polynomials  $f$  in the ring  $\mathbb{C}[x_1, \dots, x_n; y_1, \dots, y_n]$  with the property that

$$\sum_{k=1}^n \frac{\partial^i}{x_k^i} \frac{\partial^j}{y_k^j} f = 0$$

for any nonnegative integers  $i, j$  with  $i + j > 0$ . In some ways, the definition of  $\mathbf{DH}_n$  is reminiscent of the Garsia-Haiman modules. In fact, Garsia-Haiman modules are submodules of the space of diagonal harmonics.

## 1.2.2 Macdonald eigenoperators

While exploring the module of diagonal coinvariants, Garsia and Bergeron noticed that its Frobenius series could be written in terms of certain operators defined on Macdonald polynomials. Given a partition  $\mu \vdash n$  and a cell  $c$  in the Young diagram of  $\mu$  (drawn in French notation), recall that  $a'(c)$  and  $\ell'(c)$  are the number of cells in  $\mu$  that are strictly to the left and strictly below  $c$  in  $\mu$ , respectively. We define

$$B_\mu = \sum_{c \in \mu} q^{a'(c)} t^{\ell'(c)} \quad T_\mu = \prod_{c \in \mu} q^{a'(c)} t^{\ell'(c)}.$$

Given any symmetric function  $f \in \Lambda$ , we define operators  $\nabla$ ,  $\Delta_f$ , and  $\Delta'_f$  on  $\Lambda$  by their action on the Macdonald polynomial basis:

$$\nabla \tilde{H}_\mu = T_\mu \tilde{H}_\mu \quad \Delta_f \tilde{H}_\mu = f[B_\mu] \tilde{H}_\mu \quad \Delta'_f \tilde{H}_\mu = f[B_\mu - 1] \tilde{H}_\mu.$$



Here, we have used the notation that, for a symmetric function  $f$  and a sum  $A = a_1 + \dots + a_N$  of monic monomials,  $f[A]$  is equal to the specialization of  $f$  at  $x_1 = a_1, \dots, x_N = a_N$ , where the remaining variables are set equal to zero. We note that  $\nabla = \Delta_{e_n} = \Delta'_{e_{n-1}}$  as operators on  $\Lambda^{(n)}$ .

Since these operators act by constant multiplication on Macdonald polynomials, we call them Macdonald eigenoperators. The fundamental relationship between these operators and diagonal coinvariants was noticed by Bergeron and Garsia and proved by Haiman [Hai02]:

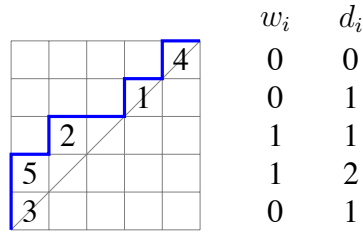
$$\text{Frob}(\mathbf{DR}_n; q, t) = \nabla e_n.$$

This result provides a symmetric function corresponding to the bigraded Frobenius series of  $\mathbf{DR}_n$ .

### 1.2.3 Parking functions

To state the combinatorial generating function connected to diagonal coinvariants, we will need to make several definitions. A *Dyck path* of order  $n$  is a lattice path from  $(0, 0)$  to  $(n, n)$  consisting of north and east steps that remains weakly above the line  $y = x$ , which is sometimes called the diagonal, main diagonal, or 0-diagonal. A *parking function* of order  $n$  consists of a Dyck path of order  $n$  whose north steps have been labeled uniquely with the integers  $1, 2, \dots, n$  such that the labels increase going up each column. A *word parking function* of order  $n$  has the same condition about increasing columns, but the labeling set can be any multiset of positive integers. We write  $\mathcal{D}_n$ ,  $\mathcal{PF}_n$ , and  $\mathcal{WPF}_n$  for the Dyck paths, parking functions, and word parking functions of order  $n$ , respectively.

To see why these objects are called parking functions, consider  $n$  cars and parking spots, each labeled  $1, 2, \dots, n$ . The labeled north steps correspond to cars, and the column that contains car  $i$  is car  $i$ 's preferred parking spot. For example, Figure 1.1 corresponds to a parking function where cars  $1, 2, \dots, 5$  have preferred spots  $4, 2, 1, 5$ , and  $1$ , respectively. In increasing order of label, each car drives into a parking lot with spots labeled  $1, 2, \dots, n$ . The car drives by spots  $1, 2, \dots, n$  in increasing order. If the car's preferred spot is available, it parks there; otherwise, it parks in the next available



**Figure 1.1:** A parking function  $P \in \mathcal{PF}_5$  with  $\text{area}(P) = 2$ ,  $\text{dinv}(P) = 5$ , and  $\text{Val}(P) = \{4, 5\}$ .

spot. A list of preferences is a parking function if every car parks successfully, i.e. no car "drives off the lot." One can check that the condition that the underlying Dyck path remains above the diagonal is equivalent to the condition that every car parks successfully.

Given a Dyck path  $D \in \mathcal{D}_n$ , we number the rows of  $D$  with  $1, 2, \dots, n$  from bottom to top. Then, for each row  $i$ , we set the *area* of the row  $i$ , written  $w_i(D)$ , to be the number of full squares between  $D$  and the diagonal. A (word) parking function  $P$  inherits the values  $w_i(P)$  from its underlying Dyck path  $D(P)$ . We also set

$$d_i(P) = |\{i < j \leq n : w_i(P) = w_j(P), \ell_i(P) < \ell_j(P)\}| \\ + |\{i < j \leq n : w_i(P) = w_j(P) + 1, \ell_i(P) > \ell_j(P)\}|.$$

where  $\ell_i(P)$  is the label in the  $i$ th row of  $P$ . These are called the *primary* and *secondary diagonal inversions* beginning in row  $i$ , respectively. The area and  $\text{dinv}$  statistics are defined by  $\text{area}(P) = \sum_{i=1}^n w_i(P)$  and  $\text{dinv}(P) = \sum_{i=1}^n d_i(P)$ . The *contractible valleys* of  $P$  are

$$\text{Val}(P) = \{2 \leq i \leq n : w_i(P) < w_{i-1}(P)\} \\ \cup \{2 \leq i \leq n : w_i(P) = w_{i-1}(P), \ell_i(P) > \ell_{i-1}(P)\}.$$

Visually, these are the rows  $i$  that are immediately preceded by an east step and, if we were to remove this east step and shift everything beyond it one step to the west, the resulting labeled path would still have increasing labels in its columns. Finally, by  $x^P$  we mean the monomial  $\prod_{i=1}^n x_{\ell_i(P)}$ .

### 1.2.4 The Shuffle Conjecture, extensions, and refinements

In [HHL<sup>+</sup>05b], Haglund, Haiman, Loehr, Remmel, and Ulyanov proposed that a certain combinatorial generating function over word parking functions was equal to  $\nabla e_n$ ; in particular, their conjecture was that

$$\nabla e_n = \sum_{P \in \mathcal{WPF}_n} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P.$$

This has come to be known as the Shuffle Conjecture, since taking scalar products with  $h_\mu$  involves "shuffles" of sequences of labels. We will discuss this perspective more in Chapter 5.

Since its appearance, the Shuffle Conjecture has been refined in a number of ways. In [Hag04], Haglund defined a new symmetric function  $E_{n,k}$  and conjectured that  $\nabla E_{n,k}$  was equal to the same generating function restricted to parking functions with  $k$  returns to the diagonal. Haglund, Morse, and Zabrocki refined this conjecture further in [HMZ12] by defining polynomials  $C_\alpha$  for any composition  $\alpha \vDash n$  and conjecturing that  $\nabla C_\alpha$  was equal to the generating function restricted to word parking functions whose returns to the diagonal occurred after exactly  $\alpha_1 + \dots + \alpha_i$  north steps for each  $i$ . We call these the Return and Compositional Shuffle Conjectures, respectively. At this point, we do not know of any modules similar to the module of diagonal coinvariants whose Frobenius series give these symmetric functions.

The Shuffle Conjecture has also been extended to more general settings. Several authors noticed what happens when  $\nabla e_n$  is replaced by  $\nabla^m e_n$  for any positive integer  $m$ . In this case, the generating function involves labeled paths from  $(0, 0)$  to  $(mn, n)$  that remain weakly above the line  $y = x/m$ . There is also a module whose Frobenius series equals  $\nabla^m e_n$ , as proved by Haiman in [Hai02]. Recently, Bergeron, Garsia, Leven, and Xin have begun to study parking functions whose Dyck paths travel from  $(0, 0)$  to  $(a, b)$  for any positive integers  $a, b$  and stay above the associated "diagonal" [BGLX14].

Various special cases of the Shuffle Conjecture have been proved, but in general it is still quite open. Perhaps the first major case that was proved was the Catalan case by Garsia and Haglund in [GH03]. In this work, Garsia and Haglund prove that the two sides of the Shuffle Conjecture are equal after taking scalars product with  $e_n$ . On

the combinatorial side, this has the effect of restricting the generating function to Dyck paths, which are counted by the Catalan numbers  $\frac{1}{n+1} \binom{2n}{n}$ . Haglund extended this work in [Hag04] by proving the Return Shuffle Conjecture after taking scalar products with either  $e_{n-d}h_d$  or  $h_{n-d}h_d$  for any integer  $0 \leq d \leq n$ . Garsia, Xin, and Zabrocki improved upon these results by proving the Compositional Shuffle Conjecture after taking scalar products with any symmetric function of the form  $e_a h_b h_{n-a-b}$  [GXZ14].

### 1.2.5 The Delta Conjectures

The primary goal of this dissertation is to study two new extensions of the Shuffle Conjecture. In these extensions, the symmetric function  $\nabla e_n$  is replaced by the symmetric function  $\Delta'_{e_k} e_n$  or  $\Delta_{e_k} e_n$ . Note that, by definition, for any  $1 \leq k \leq n$

$$\Delta_{e_k} e_n = \Delta'_{e_k+e_{k-1}} e_n = \Delta'_{e_k} e_n + \Delta'_{e_{k-1}} e_n. \quad (1.2)$$

Furthermore, for any  $k > n$ ,  $\Delta_{e_k} e_n = \Delta'_{e_{k-1}} e_n = 0$ . Therefore  $\Delta_{e_n} e_n = \Delta'_{e_{n-1}} e_n$ . We will often refer to the following conjectures as the Delta Conjectures.

**Conjecture 1.2.5.1** (Delta Conjectures). *For any integers  $n > k \geq 0$ ,*

$$\Delta'_{e_k} e_n = \sum_{P \in \mathcal{WPF}_n} q^{\text{dinv}(P)} t^{\text{area}(P)} \prod_{i: w_i(P) > w_{i-1}(P)} (1 + z/t^{w_i(P)}) x^P \Big|_{z^{n-k-1}} \quad (1.3)$$

$$= \sum_{P \in \mathcal{WPF}_n} q^{\text{dinv}(P)} t^{\text{area}(P)} \prod_{i \in \text{Val}(P)} (1 + z/q^{d_i(P)+1}) x^P \Big|_{z^{n-k-1}}. \quad (1.4)$$

*Equivalently, we can replace the left-hand side with  $\Delta_{e_k} e_n$  for integers  $n \geq k \geq 0$ , multiply both right-hand sides by  $(1 + z)$ , and then take the coefficient of  $z^{n-k}$ .*

We will often refer to (1.3) as the Rise Version and (1.4) as the Valley Version of the Delta Conjectures. It will also be useful to set  $\text{Rise}_{n,k}(x; q, t)$  equal to the right-hand side of (1.3) and  $\text{Val}_{n,k}(x; q, t)$  equal to the right-hand side of (1.4).

The remainder of this dissertation is devoted to the Delta Conjectures. We summarize the current status of our progress on these conjectures in Figure 1.2.  $M_\alpha$  is the monomial quasisymmetric function associated with the composition  $\alpha$ , which is defined in Section 1.1.4.

Conditions	LHS of (1.3)	RHS of (1.3)	RHS of (1.4)
Coefficient of $M_{1^n}$ at $q = 0$	Chapter 3	Chapter 2	Chapter 2
Coefficient of $M_{1^n}$ at $t = 0$	Chapter 3	Chapter 2	?
$\langle \cdot, e_{n-d}h_d \rangle$ at $t = 1/q$	Chapter 5	Chapter 5	?
$\langle \cdot, h_{n-d}h_d \rangle$ at $t = 1/q$	Chapter 5	Chapter 5	?
$k = 1$	Chapter 4	Chapter 4	?

**Figure 1.2:** This table summarizes the progress of our work on the Delta Conjectures. For example, the first line indicates that we can prove all three statements in the Delta Conjectures are equal after setting  $q = 0$  and taking the coefficient of  $M_{1^n}$ .

## 1.3 Combinatorial Objects for the Delta Conjectures

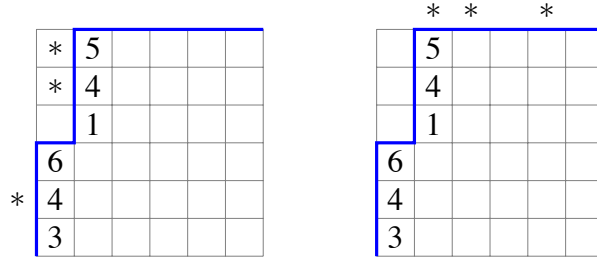
In this section, we provide classes of objects which we use to give several completely combinatorial interpretations of  $\text{Rise}_{n,k}(x; q, t)$  and  $\text{Val}_{n,k}(x; q, t)$ . These objects will be useful in working through the combinatorics of these polynomials in later chapters.

### 1.3.1 Decorated parking functions

We begin by decorating parking functions. Specifically, given  $P \in \mathcal{WPF}_n$ , let the *double rises* of  $P$  be the set

$$\text{Rise}(P) = \{2 \leq i \leq n : w_i(P) > w_{i-1}(P)\}.$$

We make the same definition for  $P \in \mathcal{PF}_n$ . In general, in this section we will often refer elements of  $\mathcal{PF}_n$  as well as  $\mathcal{WPF}_n$  as parking functions; we will only refer to the latter elements as word parking functions when disambiguation is necessary. These are the rows whose north step is immediately preceded by another north step. Similarly, we define the *double falls* of  $P$ , written  $\text{Fall}(P)$ , to be the columns of  $P$  whose east step is immediately followed by another east step. Then we can define the *double rise-decorated*, *double fall-decorated*, and *contractible valley-decorated parking functions*,



**Figure 1.3:** An example of the bijection between  $\mathcal{WPF}_{n,k}^{\text{Rise}}$  and  $\mathcal{WPF}_{n,k}^{\text{Fall}}$  for  $n = 6$  and  $k = 3$ .

respectively, as follows:

$$\begin{aligned} \mathcal{WPF}_{n,k}^{\text{Rise}} &= \{(P, R) : P \in \mathcal{WPF}_n, R \subseteq \text{Rise}(P), |R| = k\} \\ \mathcal{WPF}_{n,k}^{\text{Fall}} &= \{(P, F) : P \in \mathcal{WPF}_n, F \subseteq \text{Fall}(P), |F| = k\} \\ \mathcal{WPF}_{n,k}^{\text{Val}} &= \{(P, V) : P \in \mathcal{WPF}_n, V \subseteq \text{Val}(P), |V| = k\}. \end{aligned}$$

There is a trivial bijection between  $\mathcal{WPF}_{n,k}^{\text{Rise}}$  and  $\mathcal{WPF}_{n,k}^{\text{Fall}}$ ; namely, given a row  $i \in R$  with  $w_i(P) = a$ , send  $i$  to the column which contains the first east step north of  $i$  that is  $a$  lattice steps away from the diagonal. Figure 1.3 contains an example of this map, which is equivalent to matching open and closed parentheses in Dyck words. We will give a bijection connecting each of these sets to  $\mathcal{WPF}_{n,k}^{\text{Val}}$  later in this section. For now, we define statistics on these objects as follows. For  $P \in \mathcal{WPF}_n$ ,  $R \subseteq \text{Rise}(P)$ ,  $F \subseteq \text{Fall}(P)$ , and  $V \subseteq \text{Val}(P)$ , we set

$$\begin{aligned} \text{area}^-((P, R)) &= \sum_{i \in \{1, 2, \dots, n\} \setminus R} w_i(P) \\ \text{area}^-((P, F)) &= \sum_{i \in \{1, 2, \dots, n\} \setminus F} c_i(P) \\ \text{dinv}^-((P, V)) &= \sum_{i \in \{1, 2, \dots, n\} \setminus V} d_i(P) - |V|. \end{aligned}$$

where  $c_i(P)$  is the number of full squares between  $P$  and the diagonal in the  $i$ th column.

It is not immediately clear from its definition that  $\text{dinv}^-((P, V))$  is always non-negative. To see this, consider a (contractible) valley  $v$  of a parking function  $P \in \mathcal{WPF}_n$ . We will show that there is always at least one diagonal inversion of the form

$(i, v)$  for  $i < v$  with  $i \notin \text{Val}(P)$ . By definition, we must have  $v > 1$ . If  $w_{v-1} = w_v$ , then by the definition of contractible valleys  $(v-1, v)$  is a diagonal inversion. Now assume that  $w_{v-1} > w_v$ . Then there must be a row  $j < v$  with  $w_j = w_v$  such that  $j+1 \in \text{Rise}(P)$ . Choose the smallest such  $j$ . If  $j \in \text{Val}(P)$ , choose  $i$  to be as large as possible so that each of  $i+1, i+2, \dots, j \in \text{Val} P$ . By the definition of  $i$  and by the choice of  $j$ ,  $i$  cannot be a valley. Since  $j+1 \in \text{Rise}(P)$ ,  $j+1 \notin \text{Val}(P)$ . We claim that at least one of  $(i, v)$  and  $(j+1, v)$  is a diagonal inversion.  $(i, v)$  is a primary diagonal inversion unless  $\ell_i(P) \geq \ell_v(P)$ ; in that case,  $\ell_{j+1}(P) > \ell_j(P) > \ell_i(P)$ , so  $(j+1, v)$  is a secondary diagonal inversion.

The following identities follow directly from the definitions given above. They give alternate expressions for the right-hand sides of Conjecture 1.2.5.1 and, thanks to the argument above, show that the powers of  $q$  and  $t$  in (1.4) are always nonnegative.

**Proposition 1.3.1.1.** *For integers  $n > k \geq 0$ ,*

$$\begin{aligned} \text{Rise}_{n,k}(x; q, t) &= \sum_{(P,R) \in \mathcal{WPF}_{n,n-k-1}^{\text{Rise}}} q^{\text{dinv}(P)} t^{\text{area}^-((P,R))} x^P \\ &= \sum_{(P,F) \in \mathcal{WPF}_{n,n-k-1}^{\text{Fall}}} q^{\text{dinv}(P)} t^{\text{area}^-((P,F))} x^P. \\ \text{Val}_{n,k}(x; q, t) &= \sum_{(P,V) \in \mathcal{WPF}_{n,n-k-1}^{\text{Rise}}} q^{\text{dinv}^-((P,V))} t^{\text{area}(P)} x^P. \end{aligned}$$

### 1.3.2 Leaning stacks

In this section, we define a class of objects which will allow us to state the two forms of the Delta Conjecture on a single set of objects. We consider what we call *leaning stacks*. A leaning stack is a sequence of  $n$  unit lattice square boxes, each of which is either just northeast of the box below it or directly north of the box below it. We denote the set of leaning stacks with  $n$  boxes,  $k$  of which are diagonally above the square blow them, by  $\text{Stack}_{n,k}$ .

For a fixed leaning stack  $S \in \text{Stack}_{n,k}$ , the *parking functions* with respect to  $S$ , denoted  $\mathcal{WPF}(S)$ , are the lattice paths consisting of north and east steps from  $(0, 0)$  to  $(k+1, n)$  that remain weakly to the left of the left border of  $S$  and which are labeled

	5		$w_i$	$h_i$
	4		1	4
	1		0	3
	6		0	2
	4		1	2
	3		1	1
			0	0

**Figure 1.4:** An example  $P \in \mathcal{WPF}_{6,2}^{\text{Stack}}$  with stack  $S$  given by  $\text{Diag}(S) = \{1, 2, 6\}$ . The boxes in the stack are shaded yellow. We have  $\text{area}(P) = 3$ ,  $\text{wdinv}(P) = 1$ , and  $\text{hdinv}(P) = 0$ .

according to the same rules as ordinary parking functions. We denote the unlabeled versions of these objects by  $\mathcal{D}(S)$ . We set  $\mathcal{WPF}_{n,k}^{\text{Stack}} = \cup_{S \in \text{Stack}_{n,k}} \mathcal{WPF}(S)$ .

We claim that  $\mathcal{WPF}_{n,k}^{\text{Stack}}$  is in bijection with each of  $\mathcal{WPF}_{n,n-k-1}^{\text{Rise}}$ ,  $\mathcal{WPF}_{n,n-k-1}^{\text{Fall}}$ , and  $\mathcal{WPF}_{n,n-k-1}^{\text{Val}}$ . Furthermore, we can translate the statistics from these sets of objects to  $\mathcal{WPF}_{n,k}^{\text{Stack}}$ . Given  $P \in \mathcal{WPF}(S)$  with leaning stack  $S \in \text{Stack}_{n,k}$ , for each row of  $P$  set  $w_i(P)$  to be the number of squares between  $P$  and  $S$  and  $h_i(P)$  to be the number of squares strictly below the square just to the right of the north step in row  $i$  and weakly above the bottom square of  $S$  in the same column. Then  $\text{area}(P) = \sum_{i=1}^n w_i(P)$  is simply the number of squares between  $P$  and  $S$ . (Note that this is not equal to  $\sum_{i=1}^n h_i(P)$ .) Set  $\text{Diag}(S)$  to be the rows of  $S$  which are diagonally above the square below them along with row 1. Then we can define

$$\begin{aligned} \text{wdinv}(P) &= |\{1 \leq i < j \leq n : i \in \text{Diag}(S), w_i(P) = w_j(P), \ell_i(P) < \ell_j(P)\}| \\ &\quad + |\{1 \leq i < j \leq n : i \in \text{Diag}(S), w_i(P) = w_j(P) + 1, \ell_i(P) > \ell_j(P)\}| \\ &\quad - (n - k - 1) \\ \text{hdinv}(P) &= |\{1 \leq i < j \leq n : h_i(P) = h_j(P), \ell_i(P) < \ell_j(P)\}| \\ &\quad + |\{1 \leq i < j \leq n : h_i(P) = h_j(P) + 1, \ell_i(P) > \ell_j(P)\}|. \end{aligned}$$

**Proposition 1.3.2.1.** *We can construct bijections*

$$\begin{aligned} \phi_{n,k} &: \mathcal{WPF}_{n,n-k-1}^{\text{Fall}} \rightarrow \mathcal{WPF}_{n,k}^{\text{Stack}} \\ \psi_{n,k} &: \mathcal{WPF}_{n,n-k-1}^{\text{Val}} \rightarrow \mathcal{WPF}_{n,k}^{\text{Stack}} \end{aligned}$$



such that

$$\text{area}(\phi_{n,k}((P, F))) = \text{area}^-((P, F)) \quad (1.5)$$

$$\text{hdinv}(\phi_{n,k}((P, F))) = \text{dinv}(P) \quad (1.6)$$

$$\text{area}(\psi_{n,k}((P, F))) = \text{area}(P) \quad (1.7)$$

$$\text{wdinv}(\psi_{n,k}((P, V))) = \text{dinv}^-((P, V)). \quad (1.8)$$

and  $x^P$  is preserved. As a result,

$$\begin{aligned} \text{Rise}_{n,k}(x; q, t) &= \sum_{P \in \mathcal{WPF}_{n,k}^{\text{Stack}}} q^{\text{hdinv}(P)} t^{\text{area}(P)} x^P \\ \text{Val}_{n,k}(x; q, t) &= \sum_{P \in \mathcal{WPF}_{n,k}^{\text{Stack}}} q^{\text{wdinv}(P)} t^{\text{area}(P)} x^P. \end{aligned}$$

*Proof.* To define  $\phi_{n,k}$ , we take some  $P \in \mathcal{WPF}_n$ ,  $F \subseteq \text{Fall}(P)$  with  $|F| = n - k - 1$ . We begin with the leaning stack that consists entirely of diagonal steps between squares. Then, for each column  $j \in F$ , we remove the east step in column  $j + 1$  and move the square of  $S$  in column  $j + 1$  one space to the left. The result is  $\phi_{n,k}((P, F))$ . To invert  $\phi_{n,k}$ , we simply "push" over all squares of the stack that appear directly above the square below them and insert east steps in the columns that were occupied by these squares. To see that  $\phi_{n,k}$  cooperates with the statistics as proposed, we note that, for each  $j \in F$ , the process above removes  $j$  squares from between  $P$  and the diagonal. This proves (1.5). (1.6) follows from the fact that  $h_i(\phi_{n,k}((P, F))) = w_i(P)$  and the definitions given above.

Now we define  $\psi_{n,k}$  for  $P \in \mathcal{WPF}_n$ ,  $V \subseteq \text{Val}(P)$ . We begin with the completely diagonal leaning stack again. For each  $i \in V$ , we remove the east step preceding the north step in row  $i$  and move the square of  $S$  in row  $i$  one space to the left. To invert  $\psi_{n,k}$ , we push over all vertical squares in the stack and insert east steps preceding the rows that were occupied by these squares. We notice that, for each row  $i$ ,  $w_i(P) = w_i(\psi_{n,k}((P, V)))$ , so  $\psi_{n,k}$  preserves area. (1.8) follows from the definitions of  $\text{wdinv}$  and  $\text{dinv}^-$ .

□

Figure 1.5 contains examples of the maps  $\phi_{n,k}$  and  $\psi_{n,k}$ . We note that the compo-

sition  $\psi_{n,k} \circ \phi_{n,k}$  is a bijection  $\mathcal{WPF}_{n,n-k-1}^{\text{Fall}} \rightarrow \mathcal{WPF}_{n,n-k-1}^{\text{Val}}$  that preserves the monomial  $x^P$ . Furthermore, Proposition 1.3.2.1 implies  $\text{Rise}_{n,k}(x; 1, t) = \text{Val}_{n,k}(x; 1, t)$ .

### 1.3.3 Densely labeled parking functions

For our final combinatorial formulation, we again begin with integers  $n > k \geq 0$ . We use a shorter Dyck path  $D \in \mathcal{D}_{k+1}$ . Now we label each lattice square that occurs weakly above the line  $y = x$  whose northwest corner intersects  $D$ . A square whose west edge is a north step of  $D$  is called a north square; the other labeled squares are called east squares. Furthermore, we label these squares with sets of positive integers such that

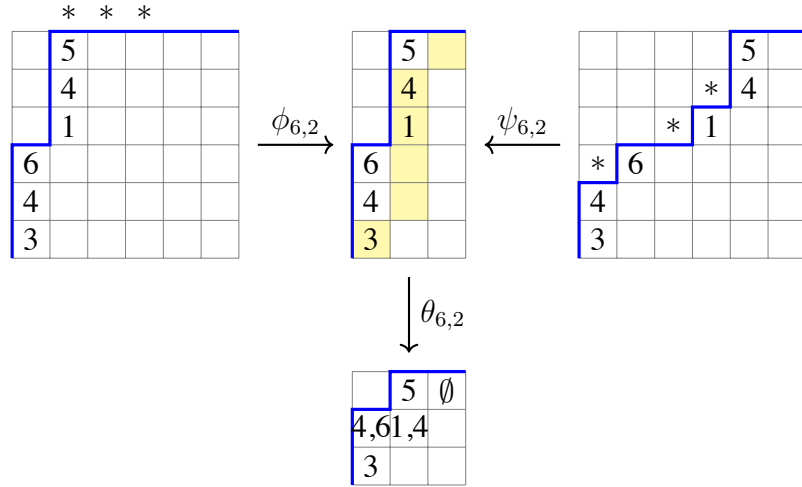
1. no north square receives the label  $\emptyset$ ,
2. for two north squares in the same column, every entry in the label of the lower square is less than every entry in the label of the upper square, and
3. there are  $n$  total elements used in the labels.

We call the resulting collection of objects *densely labeled parking functions*, written  $\mathcal{WPF}_{n,k}^{\text{Dense}}$ . Figure 1.5 contains an example of a densely labeled parking function.

In order to move the statistics from our previous objects, for each element  $r$  of any label in some  $P \in \mathcal{WPF}_{n,k}^{\text{Dense}}$  we set  $w(r, P)$  to be the number of full squares between  $r$ 's square and the diagonal. It is quite difficult to define the height of an entry in this setting, so we focus only on the area and  $\text{wdinv}$  statistics. We say

- $\text{area}(P) = \sum_{\text{label entries } r} w(r, P)$ ,
- $\text{wdinv}(P)$  is equal to the number of pairs of label entries  $(r, s)$  with  $r$  minimal in its square,  $r$ 's square appearing strictly west of  $s$ 's square, and either
  - $r < s$  and  $w(r, P) = w(s, P)$ , or
  - $r > s$  and  $w(r, P) = w(s, P) + 1$

minus the number of entries in labels in east squares in  $P$ .



**Figure 1.5:** Examples of the maps  $\phi_{6,2}$ ,  $\psi_{6,2}$ , and  $\theta_{6,2}$ . We have marked the selected double falls and contractible valleys with stars.

**Proposition 1.3.3.1.** *We can construct a bijection  $\theta_{n,k} : \mathcal{WPF}_{n,k}^{\text{Stack}} \rightarrow \mathcal{WPF}_{n,k}^{\text{Dense}}$  such that*

$$\begin{aligned} \text{area}(\theta_{n,k}(P)) &= \text{area}(P) \\ \text{wdiv}(\theta_{n,k}(P)) &= \text{wdiv}(P). \end{aligned}$$

As a result,

$$\text{Val}_{n,k}(x; q, t) = \sum_{P \in \mathcal{WPF}_{n,k}^{\text{Dense}}} q^{\text{wdiv}(P)} t^{\text{area}(P)} x^P.$$

*Proof.* We define  $\theta_{n,k}$  by contracting every north step of  $P$  that shares a row with a vertical square of the leaning stack. The labels whose north steps are removed are simply combined with the remaining labels to form the set labels. The inverse is direct and the assertions about the statistic follow from the definitions above.  $\square$

We summarize all of our bijections in Figure 1.5.

### 1.3.4 The $q = t = 1$ case

As an immediate application of these interpretations, we obtain a formula for  $\text{Rise}_{n,k}(x; 1, 1) + \text{Rise}_{n,k-1}(x; 1, 1)$ . Thanks to the leaning stacks interpretation, we already know that  $\text{Rise}_{n,k}(x; 1, t) = \text{Val}_{n,k}(x; 1, t)$ . In Chapter 5, we will see that  $\Delta_{e_k} e_n$

also obeys the formula we prove here. This proves the  $q = t = 1$  case of the Delta Conjecture.

**Proposition 1.3.4.1.** *For any integers  $1 \leq k \leq n$ ,*

$$\begin{aligned} \text{Rise}_{n,k}(x; 1, 1) + \text{Rise}_{n,k}(x; 1, 1) &= \frac{1}{k+1} \binom{n}{k} \left( \sum_{i \geq 0} e_i u^i \right)^{k+1} \Big|_{u^n} \\ &= \frac{1}{k+1} \binom{n}{k} e_n [(k+1)X]. \end{aligned}$$

*Proof.* By the definition of  $\text{Rise}_{n,k}(x; q, t)$  appearing in the Delta Conjectures,

$$\text{Rise}_{n,k}(x; 1, 1) + \text{Rise}_{n,k-1}(x; 1, 1) = \sum_{P \in \text{WPF}_n} (1+z)^{|\text{Rise}(P)|+1} x^P \Big|_{z^{n-k}}. \quad (1.9)$$

Given a partition  $\lambda \vdash n$ , set  $m_i = m_i(\lambda)$  to be the multiplicity of  $i$  in  $\lambda$ . As mentioned in Equation 4 of [ALW14], the number of Dyck paths with exactly  $m_i$  vertical runs of length  $i$  for each  $i$  is

$$\frac{1}{n+1} \binom{n+1}{m_1, m_2, \dots, m_n, n - \ell(\lambda) + 1}. \quad (1.10)$$

Furthermore, such a Dyck path has  $n - \ell(\lambda)$  double rises. We label each of the vertical runs of such a Dyck path with increasing sequences of integers, contributing an  $e_\lambda$  term. Hence (1.9)

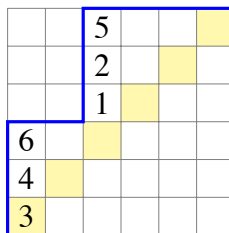
$$= \sum_{\lambda \vdash n} \frac{1}{n+1} \binom{n+1}{m_1, m_2, \dots, m_n, n - \ell(\lambda) + 1} (1+z)^{n-\ell(\lambda)+1} e_\lambda \Big|_{z^{n-k}} \quad (1.11)$$

$$= \frac{1}{n+1} \binom{n+1}{n-k} \left( \sum_{i \geq 0} e_i u^i \right)^{k+1} \Big|_{u^n} \quad (1.12)$$

which proves the first identity in the proposition. The second identity follows from Cauchy's Formula.  $\square$

### 1.3.5 Preference functions and undesirable spaces

Finally, we would like to address the relationship between these objects and the classical notion of a parking function. Parking functions were first introduced in computer science by Konheim and Weiss in the following setting [KW66]. Consider  $n$  cars



**Figure 1.6:** An example labeled Dyck path corresponding to the parking function 331131.

and  $n$  parking spots, each labeled bijectively with the set  $\{1, 2, \dots, n\}$ . We are given a function  $F : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  that maps each car to its preferred spot.  $F$  is called a *preference function*. Then, from car 1 through car  $n$ , each car drives into the parking lot towards its preferred spot. Say the preferred spot is labeled  $p$ . If this spot is unoccupied, the car parks in spot  $p$ . If it is occupied, the car examines spots  $p + 1, p + 2, \dots$  until it finds an unoccupied spot and it parks in that spot. If none of the spots  $p + 1, \dots, n$  are available, the car does not park. The preference function  $F$  is called a *parking function* if this process results in every car parking successfully. Often, a parking function is written as the word  $F(1)F(2) \dots F(n)$ . For example, 121 is a parking function but 322 is not, since the latter preference function does not allow car 3 to park. To see how these objects biject to what we have been calling parking functions, we form a labeled Dyck path by writing  $F^{-1}(1)$  in column 1, then  $F^{-1}(2)$  in column 2, and so on, writing the cars by increasing label from bottom to top. For example, the parking function 331131 corresponds to the labeled Dyck path in Figure 1.6.

With this in mind, we say that a *parking preference function* is a function  $F$  from the integers  $\{1, 2, \dots, n\}$  to  $\{1, 2, \dots, n\}$  such that the  $i$ th entry in the increasing rearrangement of  $F(1), F(2), \dots, F(n)$  is at most  $i$  for every  $1 \leq i \leq n$ . One can check that this is equivalent to each car parking successfully.

Our goal is to define an analogous set of objects that corresponds to our Delta Conjectures. We note that decorating double falls is equivalent to decorating empty columns. These correspond to elements in the set  $\{1, 2, \dots, n\}$  that are not in the image of the preference function. Therefore, we will allow each spot that is not in the image of  $F$ , i.e. not "preferred," to be either marked or unmarked. We can accomplish this through

the notion of "undesirable spaces," which are spaces that are not allowed to be preferred. We say that the *parking preference functions of order  $n$  with  $k$  undesirable spaces* are the set of pairs  $(U, F)$  with  $U \subseteq \{1, 2, \dots, n\}$ ,  $|U| = k$ , and  $F : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  such that  $\text{image}(F) \cap U = \emptyset$  and  $F$  satisfies the same weakly increasing property as before. Then parking preference functions of order  $n$  with  $k$  undesirable spaces correspond to the any of the combinatorial objects in this section that are related to the symmetric function  $\Delta'_{e_{n-k-1}} e_n$ .

## 1.4 Outline

In Chapter 2, we focus on the specializations  $\text{Rise}_{n,k}(x; q, 0)$ ,  $\text{Rise}_{n,k}(x; 0, q)$ ,  $\text{Val}_{n,k}(x; q, 0)$ , and  $\text{Val}_{n,k}(x; 0, q)$ . We show that each of these polynomials is equal to the distribution of a statistic on the set of *ordered multiset partitions*. That is, given a composition  $\alpha$ , we define  $\mathcal{OP}_{\alpha,k}$  to be the ordered partitions of the multiset  $\{i^{\alpha_i} : i = 1, 2, \dots, \ell(\alpha)\}$  into  $k$  sets. We define four statistics  $\text{dinv}$ ,  $\text{maj}$ ,  $\text{inv}$ , and  $\text{minimaj}$  that map  $\mathcal{OP}_{\alpha,k}$  to  $\mathbb{Z}_{\geq 0}$  and prove that

$$\begin{aligned} \text{Rise}_{n,k}(x; q, 0)|_{M_\alpha} &= \sum_{\pi \in \mathcal{OP}_{\alpha,k+1}} q^{\text{dinv}(\pi)} \\ \text{Rise}_{n,k}(x; 0, q)|_{M_\alpha} &= \sum_{\pi \in \mathcal{OP}_{\alpha,k+1}} q^{\text{maj}(\pi)} \\ \text{Val}_{n,k}(x; q, 0)|_{M_\alpha} &= \sum_{\pi \in \mathcal{OP}_{\alpha,k+1}} q^{\text{inv}(\pi)} \\ \text{Val}_{n,k}(x; 0, q)|_{M_\alpha} &= \sum_{\pi \in \mathcal{OP}_{\alpha,k+1}} q^{\text{minimaj}(\pi)}. \end{aligned}$$

Then we prove that the first three distributions are equal. We also explain why we are currently unable to show that the fourth polynomial is equal to the other tree. Thus

$$\text{Rise}_{n,k}(x; q, 0) = \text{Rise}_{n,k}(x; 0, q) = \text{Val}_{n,k}(x; q, 0).$$

When  $k = |\alpha|$ , our result reduces to the famous result of MacMahon that inversion number and major index are equidistributed over permutations. From this perspective, our theorem is a generalization of MacMahon's theorem to ordered set partitions. Our

method is a bijective generalization of Carlitz's insertion method for proving MacMahon's theorem. We also explain some applications of our result to a problem of Steingrímsson and to generalizations of Macdonald polynomials.

In Chapter 3, we show how the symmetric function  $\Delta'_{e_k} e_n$  can be connected to certain upper triangular matrices we call Tesler matrices. Special cases of what we call Tesler matrices were introduced in [Hag11] and studied in [AGR<sup>+</sup>12, GH14, GHX14, MMR14]. We use the connection between Tesler matrices and  $\Delta'_{e_k} e_n$  to give simple formulas for  $\langle \Delta_{e_1} e_n, p_{1^n} \rangle$  and  $\langle \Delta'_{e_k} e_n, p_{1^n} \rangle|_{t=0}$ . We use the second formula to show that

$$\langle \Delta'_{e_k} e_n, p_{1^n} \rangle|_{t=0} = \langle \text{Rise}_{n,k}(x; q, 0), p_{1^n} \rangle = \langle \text{Val}_{n,k}(q, 0), p_{1^n} \rangle,$$

verifying two cases of the Delta Conjectures. Symmetry gives us another case of the Delta Conjectures.

Chapter 4 is a short chapter in which we prove the  $k = 1$  case of the Rise Version of the Delta Conjecture. We accomplish this by using a reciprocity identity of Haglund [Hag04] to show

$$\Delta_{e_1} e_n = \sum_{m=0}^{\lfloor n/2 \rfloor} s_{2^m, 1^{n-2m}} \sum_{p=m}^{n-m} [p]_{q,t}.$$

We manipulate LLT polynomials [LLT97] to show that  $\text{Rise}_{n,0}(x; q, t) + \text{Rise}_{n,1}(x; q, t)$  also satisfies this identity.

In Chapter 5, we prove recursions for the polynomials that result from taking inner products between  $\text{Rise}_{n,k}(x; q, t)$  or  $\text{Val}_{n,k}(x; q, t)$  and  $e_{n-d} h_d$  or  $h_{n-d}, h_d$  for an integer  $0 \leq d \leq n$ . These recursions involve parking functions with only 2 types of cars. If we set  $t = 1/q$ , we can use these recursions to obtain  $q$ -binomial formulas for the inner products involving  $\text{Rise}_{n,k}(x; q, t)$ . We also compute the same inner products on the symmetric function side when  $t = 1/q$ , proving cases of the Delta Conjecture.

Finally, we close by mentioning some potential directions for future research in Chapter 6. Some of these problems are straightforward conjectures that arose while pursuing the research in this dissertation, while others are more general long-term goals for this subject area.

Chapter 1 is currently being prepared for submission for publication. Haglund, J.; Remmel, J.; Wilson, A.T. The dissertation author was the primary investigator and author of this work.

# Chapter 2

## Combinatorics at $q = 0$ or $t = 0$

In this chapter, we focus on the specializations  $\text{Rise}_{n,k}(x; q, 0)$ ,  $\text{Rise}_{n,k}(x; 0, q)$ ,  $\text{Val}_{n,k}(x; q, 0)$ , and  $\text{Val}_{n,k}(x; 0, q)$ . In Section 2.2, we show that these polynomials can be expressed as distributions of certain statistics on ordered multiset partitions, which we also define in that section. Section 2.3 contains a bijective proof that

$$\langle \text{Rise}_{n,k}(x; q, 0), p_{1^n} \rangle = \langle \text{Rise}_{n,k}(x; 0, q), p_{1^n} \rangle = \langle \text{Val}_{n,k}(x; q, 0), p_{1^n} \rangle.$$

We prove that these three polynomials are equal in general (i.e. without taking inner products with  $p_{1^n}$ ) in Section 2.4. Our method of proof is a generalization of a bijection on permutations due to Carlitz, which is sometimes known as the insertion method. We begin by reviewing Carlitz's insertion method in Section 2.1.

### 2.1 MacMahon's equidistribution theorem and Carlitz's insertion method

#### 2.1.1 Permutation statistics

Given a composition  $\alpha$  of length  $n$ , we let  $\mathfrak{S}_\alpha$  be the set of all permutations of the multiset  $\{i^{\alpha_i} : 1 \leq i \leq n\}$ . Given a permutation  $\sigma \in \mathfrak{S}_\alpha$  in one-line notation, the *descent* and *ascent sets* of  $\sigma$  are

$$\text{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\} \quad \text{Asc}(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$$



The *inversions* of  $\sigma$  are the pairs

$$\text{Inv}(\sigma) = \{(i, j) : 1 \leq i < j \leq n, \sigma_i > \sigma_j\}.$$

It will be convenient to refine the set of inversions in the following manner:

$$\text{Inv}^{i,\square} = \{(i, j) : i < j \leq n, \sigma_i > \sigma_j\} \quad \text{Inv}^{\square,j} = \{(i, j) : 1 \leq i < j, \sigma_i > \sigma_j\}.$$

These are the elements of  $\text{Inv}(\sigma)$  whose first (resp. second) coordinate is  $i$  (resp.  $j$ ).

These sets allow us to define several statistics on  $\mathfrak{S}_\alpha$ :

$$\text{des}(\sigma) = |\text{Des}(\sigma)| \quad \text{asc}(\sigma) = |\text{Asc}(\sigma)| \quad \text{inv}(\sigma) = |\text{Inv}(\sigma)| \quad \text{maj}(\sigma) = \sum_{i \in \text{Des}(\sigma)} i.$$

These statistics are known as the *descent number*, *ascent number*, *inversion number*, and *major index* of  $\sigma$ , respectively. We will also make use of two refinements of inversion number:

$$\text{inv}^{i,\square}(\sigma) = |\text{Inv}^{i,\square}(\sigma)| \quad \text{inv}^{\square,j}(\sigma) = |\text{Inv}^{\square,j}(\sigma)|.$$

Given a statistic  $\text{stat}$  on  $\mathfrak{S}_\alpha$ , the *distribution* of  $\text{stat}$  over  $\mathfrak{S}_\alpha$  is the polynomial

$$D_\alpha^{\text{stat}}(q) = \sum_{\sigma \in \mathfrak{S}_\alpha} q^{\text{stat}(\sigma)}.$$

When  $\alpha = 1^n$ , we will simply write  $D_n^{\text{stat}}(q)$ . Two statistics, say  $\text{stat}$  on  $\text{Obj}$  and  $\text{stat}'$  on  $\text{Obj}'$ , are said to be *equidistributed* if their distributions are equal. One particularly nice way to prove equidistribution is to give a bijection  $f : \text{Obj} \rightarrow \text{Obj}'$  such that  $\text{stat}'(f(\sigma)) = \text{stat}(\sigma)$  for every  $\sigma \in \text{Obj}$ . Our main result will be a bijection of this form. It will also be convenient to use interval notation for the integers, i.e.  $[a, b]$  is the set of integers at least  $a$  and at most  $b$ .

In [Mac15], MacMahon showed that inversion number and major index are equidistributed over  $\mathfrak{S}_\alpha$ , and that

$$D_\alpha^{\text{inv}}(q) = D_\alpha^{\text{maj}}(q) = \left[ \begin{matrix} |\alpha| \\ \alpha_1, \alpha_2, \dots, \alpha_n \end{matrix} \right]_q = \frac{[|\alpha|]_q!}{[\alpha_1]_q! [\alpha_2]_q! \dots [\alpha_n]_q!}.$$

MacMahon's proof was not bijective; the first bijective proof of this fact was given in [Foa68]. A second proof, essentially due to Carlitz [Car75], is sometimes known as the *insertion method*. It will be the template for our bijections in Sections 2.3 and 2.4.

### 2.1.2 The insertion method for $\mathfrak{S}_n$

One consequence of MacMahon's equidistribution theorem is a pair of recursions for the distributions of the inversion number and the major index over  $\mathfrak{S}_n$ :

$$D_n^{\text{inv}}(q) = [n]_q D_{n-1}^{\text{inv}}(q) \quad D_n^{\text{maj}}(q) = [n]_q D_{n-1}^{\text{maj}}(q). \quad (2.1)$$

On the other hand, these two statements imply MacMahon's result. Carlitz's insertion method gives bijective proofs of these statements which can be combined to build a recursive bijection  $\psi_n : \mathfrak{S}_n \rightarrow \mathfrak{S}_n$  such that  $\text{maj}(\psi(\sigma)) = \text{inv}(\sigma)$ . We say that  $\psi_n$  maps the inversion number to the major index. We outline Carlitz's insertion method below.

To prove the left statement in (2.1), one simply considers all the possible ways to insert  $n$  into a permutation in  $\mathfrak{S}_{n-1}$  to create a permutation in  $\mathfrak{S}_n$ . It is clear that, for  $\sigma \in \mathfrak{S}_{n-1}$ , inserting  $n$  after the first  $i$  elements of  $\sigma$  creates  $n - i - 1$  new inversions and does not affect the previously existing inversions. For example, for  $\sigma = 5167324 \in \mathfrak{S}_7$ , we can "label" these positions with subscripts that give the number of new inversions created by inserting an 8 at that position:

$${}_7 5_6 1_5 6_4 7_3 3_2 2_1 4_0.$$

This proves the inversion side of (2.1). The key to the insertion method is that something similar is true for the major index. In particular, we can label the spaces between elements of  $\sigma \in \mathfrak{S}_{n-1}$ , along with the left and right ends, according to the following scheme:

1. Label the position after  $\sigma_{n-1}$  with a zero.
2. Label the descents of  $\sigma$  right to left with  $1, 2, \dots, \text{des}(\sigma)$ .
3. Label the position before  $\sigma_1$  with  $\text{des}(\sigma) + 1$ .
4. Label the ascents of  $\sigma$  from left to right with  $\text{des}(\sigma) + 2, \dots, n - 1$ .

For example,  $\sigma = 5167324$  receives the following labels in this setting:

$${}_4 5_3 1_5 6_6 7_2 3_1 2_7 4_0$$

$n$	$\sigma$	Change in inv	$\psi_n(\sigma)$
5	52143		24153
4	2143	4	<i>223411430</i>
3	213	1	<i>2211330</i>
2	21	0	<i>22110</i>
1	1	1	<i>110</i>

**Figure 2.1:** We compute  $\psi_5(52143)$ .

These labels give the change in major index that comes from inserting  $n$  at that position; one proof of this fact can be found in [HLR05]. This completes the proof of (2.1) and also gives a bijection that takes the inversion number to the major index. We include an example of this bijection in Figure 2.1. To compute  $\psi_5(52143)$ , we remove the 5 and count the number of inversions lost by removing 5. In this case, we have lost 4 inversions. We record this number in the third column and the resulting permutation in the  $\sigma$  column. We repeat this process until we have reached  $n = 1$  and filled the first three columns of the table. To build our new permutation, we recursively place  $n$  at the position that receives label  $i$  in the major index labeling. These labels have been italicized in the example.

### 2.1.3 The insertion method on $\mathfrak{S}_\alpha$

It is natural to hope that this proof can be extended to permutations that may contain multiple copies of the same number. That is, we would like to give insertion proofs that

$$D_\alpha^{\text{inv}}(q) = \begin{bmatrix} |\alpha| \\ \alpha_n \end{bmatrix}_q D_{\alpha^-}^{\text{inv}}(q) \quad D_\alpha^{\text{maj}}(q) = \begin{bmatrix} |\alpha| \\ \alpha_n \end{bmatrix}_q D_{\alpha^-}^{\text{maj}}(q). \quad (2.2)$$

where  $\alpha^- = (\alpha_1, \alpha_2, \dots, \alpha_{n-1})$ . Such proofs would imply MacMahon's equidistribution theorem and provide a bijection between inversion number and major index.

The inversion side cooperates nicely. As before, inserting an  $n$  to the right of  $i$  elements of  $\sigma \in \mathfrak{S}_\alpha$  increases the inversion number by  $|\alpha^-| - i$ . Hence this insertion can create between 0 and  $|\alpha^-|$  inversions. Furthermore, the position of a new  $n$  has no affect on the number of inversions added by other  $n$ 's; in other words, each insertion is

independent of the other insertions. This allows us to compute  $D_\alpha^{\text{inv}}(q)$  from  $D_{\alpha^-}^{\text{inv}}(q)$ :

$$D_\alpha^{\text{inv}}(q) = D_{\alpha^-}^{\text{inv}}(q) \left( \prod_{i=0}^{|\alpha^-|} \frac{1}{1 - q^k x} \right) \Big|_{x^{\alpha_n}} = \begin{bmatrix} |\alpha| \\ \alpha_n \end{bmatrix}_q D_{\alpha^-}^{\text{inv}}(q).$$

To prove the major index side of (2.2), we essentially recreate the bijection constructed in [FH08, CPYY10]. Let  $\binom{S}{k}$  denote the family of  $k$ -element multisets containing elements drawn from the set  $S$ . We would like to establish a bijection

$$\phi_\alpha^{\text{maj}} : \mathfrak{S}_{\alpha^-} \times \left( \binom{[0, |\alpha^-|]}{\alpha_n} \right) \rightarrow \mathfrak{S}_\alpha$$

such that

$$\text{maj}(\phi_\alpha^{\text{maj}}(\sigma, B)) = \text{maj}(\sigma) + \sum_{b \in B} b.$$

Such a map would provide a combinatorial proof of the major index side of (2.2). Before inserting any  $n$ 's, we label  $\sigma \in \mathfrak{S}_{\alpha^-}$  in a manner reminiscent of Section 2.1.2:

1. Label the position after  $\sigma_{|\alpha^-|}$  with a zero.
2. Label the descents of  $\sigma$  right to left with  $1, 2, \dots, \text{des}(\sigma)$ .
3. Label the position before  $\sigma_1$  with  $\text{des}(\sigma) + 1$ .
4. Label the non-descents of  $\sigma$  from left to right with  $\text{des}(\sigma) + 2, \dots, |\alpha^-|$ .

Write  $B = \{b_1 \geq b_2 \geq \dots \geq b_{\alpha_n}\}$ . We insert an  $n$  into the position labeled  $b_1$ . Then we go through the labeling process again, stopping once we have used the label  $b_1$ . We insert an  $n$  into the position labeled  $b_2$ . We repeat this process until we have processed each element of  $B$ . We omit the proof that this map satisfies the desired properties, which can be found in [FH08, CPYY10]. Instead, we will work through an example.

Let  $\alpha = \{2, 1, 3, 2\}$ ,  $\sigma = 323113 \in \mathfrak{S}_{\alpha^-}$ , and  $B = \{5^2\}$ . We note that  $\text{maj}(\sigma) = 4$ . We begin by labeling  $\sigma$  according to the labeling associated with the major index.

$${}_3 3_2 2_4 3_1 1_5 1_6 3_0.$$

$\alpha$	$\sigma$	$B$	$\psi_A(\sigma)$
$\{1, 2, 1, 3\}$	2443214		4432124 4 4 <sub>3</sub> 3 <sub>2</sub> 2 <sub>1</sub> 2 <sub>0</sub> 4 <sub>3</sub> 3 <sub>2</sub> 2 <sub>1</sub> 2 <sub>0</sub>
$\{1, 2, 1\}$	2321	$\{3, 3, 0\}$	<sub>3</sub> 3 <sub>2</sub> 2 <sub>1</sub> 1 <sub>4</sub> 2 <sub>0</sub>
$\{1, 2\}$	212	$\{2\}$	<sub>2</sub> 2 <sub>1</sub> 1 <sub>3</sub> 2 <sub>0</sub> 2 <sub>1</sub> 1 <sub>0</sub>
$\{1\}$	1	$\{1, 0\}$	<sub>1</sub> 1 <sub>0</sub>

**Figure 2.2:** An example of the map  $\psi_\alpha$  for  $\alpha = \{1, 2, 1, 3\}$ .

We place a 4 at the label 5 to get 3231413. We relabel this permutation, stopping when we use the label 5.

$${}_4 3_3 2_5 3_2 1 \ 4_1 1 \ 3_0$$

Then we insert a 4 at the position labeled 5 to get 32431413. As desired,

$$\text{maj}(32431413) = 14 = \text{maj}(\sigma) + \sum_{b \in B} b = 4 + 5 + 5.$$

Just as before, these insertion maps can be combined to yield a bijection  $\psi_\alpha : \mathfrak{S}_\alpha \rightarrow \mathfrak{S}_\alpha$  that takes inversion number to major index. We illustrate  $\psi_\alpha$  with the example in Figure 2.2. As in Section 2.1.2, we fill the first three columns of the table from top to bottom by removing all copies of the largest element and recording the multiset of inversions lost during each removal, which we call  $B$ . Then we fill the fourth column by using the labeling associated with the major index to repeatedly insert a new element at the position that received the largest remaining label in  $B$ .

## 2.2 Ordered Set and Multiset Partitions

### 2.2.1 Definitions

The *ordered set partitions* of order  $n$  with  $k$  blocks are partitions of the set  $\{1, 2, \dots, n\}$  into  $k$  subsets (called *blocks*) with some order on the blocks. We write

this set as  $\mathcal{OP}_{n,k}$ . For example,  $13|45|2 \in \mathcal{OP}_{5,3}$ , where we have listed each block as an increasing sequence and we have used bars to separate blocks. It is not difficult to see that  $\mathcal{OP}_{n,n} = \mathfrak{S}_n$ , so ordered set partitions are a natural extension of permutations.

More generally, given a composition  $\alpha$  of length  $n$ , the ordered multiset partitions  $\mathcal{OP}_{\alpha,k}$  are the partitions of the multiset  $A(\alpha) = \{i^{\alpha_i} : 1 \leq i \leq n\}$  into  $k$  ordered sets, which we still call blocks. For example,  $24|134|2 \in \mathcal{OP}_{(1,2,1,2),3}$ . Note that, although we are dealing with the elements of a multiset, each block is still a set. The analogous objects where blocks are also multisets will not arise in our work.

So far, we have written each block of an ordered set or multiset partition in increasing order from left to right. We will often wish to use the opposite notation, i.e. we will write each block in decreasing order from left to right. Furthermore, we will use stars as subscripts to "connect" elements in the same block instead of bars to separate blocks. For example, the ordered multiset partition  $24|134|2$  is written as  $4_*2 \ 4_*3_*1 \ 2$  in this new notation. We will refer to an ordered multiset partition written this way as a *descent-starred permutation*, since every permutation of the given multiset with some (but maybe not all) of its descents "starred" corresponds to an ordered multiset permutation in this fashion. More formally, we define the *descent-starred permutations* of  $A(\alpha)$  with  $k$  stars as follows:

$$\mathfrak{S}_{\alpha,k}^> = \{(\sigma, S) : \sigma \in \mathfrak{S}_A, S \subseteq \text{Des}(\sigma), |S| = k\}.$$

The set  $S$  corresponds to the entries of  $\sigma$  which are followed by stars. Then there is a straightforward bijection  $\mathcal{OP}_{\alpha,k} \leftrightarrow \mathfrak{S}_{\alpha,|\alpha|-k}^>$ ; given an ordered multiset partition, we write its blocks in decreasing order from left to right, add stars between adjacent elements that share a block, and remove the bars.

## 2.2.2 Statistics

Setting  $q = 0$  or  $t = 0$  in  $\text{Rise}_{n,k}(x; q, t)$  or  $\text{Val}_{n,k}(x; q, t)$  will lead to four different statistics on ordered multiset partitions. We define these statistics below.

First, given  $\pi \in \mathcal{OP}_{\alpha,k}$ ,  $\text{inv}(\pi)$  counts the number of pairs  $a > b$  such that  $a$ 's block is strictly to the left of  $b$ 's block in  $\pi$  and  $b$  is minimal in its block in  $\pi$ . We call these pairs *inversions*. For example,  $15|23|4$  has two inversions, between the 5 and the

2 and the 5 and the 4.

For any  $\pi \in \mathcal{OP}_{\alpha,k}$ , we number  $\pi$ 's blocks  $\pi_1, \pi_2, \dots, \pi_k$  from left to right. Let  $\pi_i^h$  by the  $h$ th smallest element in  $\pi_i$ , beginning at  $h = 1$ . Then the *diagonal inversions* of  $\pi$ , written  $\text{Dinv}(\pi)$ , are the triples

$$\{(h, i, j) : 1 \leq i < j \leq k, \pi_i^h > \pi_j^h\} \cup \{(h, i, j) : 1 \leq i < j \leq k, \pi_i^h < \pi_j^{h+1}\}.$$

The triples of the first type are *primary diagonal inversions*, and the triples of the second type are *secondary diagonal inversions*. We set  $\text{dinv}(\pi)$  to be the cardinality of  $\text{Dinv}(\pi)$ . For example, consider the ordered multiset permutation  $24|134|2$ . It is helpful to "stack" the elements in each block vertically, obtaining the diagram

	4	
4	3	
2	1	2

The primary diagonal inversions are  $(1, 1, 2)$  (between the leftmost 2 and the 1 in the first row) and  $(2, 1, 2)$  (between the 4 and the 3 in the second row) and the only secondary diagonal inversion is  $(1, 1, 2)$  (between the leftmost 2 in the first row and the 3 in the second row), for a total of three diagonal inversions.

To define the *major index* of  $\pi$ , we consider the permutation  $\sigma = \sigma(\pi)$  obtained by writing each block of  $\pi$  in decreasing order. Then we recursively form a word  $w$  by setting  $w_0 = 0$  and  $w_i = w_{i-1} + \chi(\sigma_i \text{ is minimal in its block in } \pi)$  for each  $i > 0$ . Then we set

$$\text{maj}(\pi) = \sum_{i: \sigma_i > \sigma_{i+1}} w_i.$$

Using the ordered multiset permutation  $\pi = 24|134|2$  again, we obtain  $\sigma = 424312$  and  $w = 0011123$ , beginning with  $w_0 = 0$ . The descents of  $\sigma$  occur at positions 1, 3, and 4, so  $\text{maj}(\pi) = w_1 + w_3 + w_4 = 0 + 1 + 1 = 2$ .

There is an alternate definition of the major index which we will use in some of our proofs. It is clear from the definition above that for any  $\pi \in \mathcal{OP}_{n,n}$ , the major index defined here is equivalent to the major index defined on permutations in Section 2.1. Now we consider what happens to  $\text{maj}(\pi)$  if we decide to insert a star after a descent at position  $d$ . We note that  $w_i$  decreases by 1 for each  $i \geq d$ . Therefore the major index

of  $\pi$  has decreased by 1 for each descent weakly to the right of position  $d$ . Therefore, if  $(\sigma, S)$  is the descent-starred permutation representation of  $\pi$ , we can write

$$\text{maj}(\pi) = \text{maj}(\sigma) - \sum_{i \in S} |\text{Des}(\sigma) \cap \{i, i+1, \dots\}|. \quad (2.3)$$

Finally, we define the *minimum major index* of  $\pi$  as follows. We begin by writing the elements of  $\pi_k$  in increasing order from left to right. Then, recursively for  $i = k - 1$  to 1, we choose  $r$  to be the largest element in  $\pi_i$  that is less than or equal to the leftmost element in  $\pi_{i+1}$ , as previously recorded. If there is no such  $r$ , we write  $\pi_i$  in increasing order. If there is such an  $r$ , beginning with  $\pi_i$  in increasing order, we cycle its elements until  $r$  is the rightmost element in  $\pi_i$ . We write down  $\pi_i$  in this order. We continue this process until we have processed each block of  $\pi$ . For example, consider the ordered multiset permutation  $\pi = 13|23|14|234$ . Processing the blocks of  $\pi$  from right to left, we obtain 312341234. We consider the result as a permutation, which we denote  $\tau = \tau(\pi)$ , and define

$$\text{minimaj}(\pi) = \sum_{i: \tau_i > \tau_{i+1}} i$$

i.e. the major index of the permutation  $\tau$ . The name *minimaj* comes from the fact that  $\text{minimaj}(\pi)$  is equal to the minimum major index achieved by any permutation that can be obtained by permuting elements within the blocks of  $\pi$ .

### 2.2.3 From parking functions to ordered multiset partitions

Now we will show how setting  $q = 0$  or  $t = 0$  in the Delta Conjectures leads naturally to ordered multiset partitions. We will use the following notation for the distribution of a statistic  $\text{stat}$  on the class of ordered set and multiset partitions:

$$D_{n,k}^{\text{stat}}(q) = \sum_{\pi \in \mathcal{OP}_{n,k}} q^{\text{stat}(\pi)} \quad D_{\alpha,k}^{\text{stat}}(q) = \sum_{\pi \in \mathcal{OP}_{\alpha,k}} q^{\text{stat}(\pi)}.$$





**Figure 2.3:** A parking function with area equal to 0 that corresponds to the ordered set partition  $14|2|35$ .

**Proposition 2.2.3.1.**

$$\text{Rise}_{n,k}(x; q, 0)|_{M_\alpha} = D_{\alpha,k}^{\text{dinv}}(q) \quad (2.4)$$

$$\text{Rise}_{n,k}(x; 0, q)|_{M_\alpha} = D_{\alpha,k}^{\text{maj}}(q) \quad (2.5)$$

$$\text{Val}_{n,k}(x; q, 0)|_{M_\alpha} = D_{\alpha,k}^{\text{inv}}(q) \quad (2.6)$$

$$\text{Val}_{n,k}(x; 0, q)|_{M_\alpha} = D_{\alpha,k}^{\text{minimaj}}(q). \quad (2.7)$$

*Proof.* To prove (2.4), it is easiest to use the interpretation of  $\text{Rise}_{n,k}(x; q, 0)$  involving leaning stacks given in Subsection 1.3.2, which gives

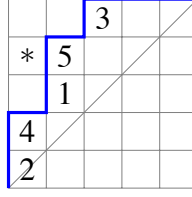
$$\text{Rise}_{n,k}(x; q, 0)|_{M_\alpha} = \sum_P q^{\text{hdinv}(P)}$$

where the sum is over  $P \in \mathcal{PF}_{n,k}^{\text{Stack}}$  with  $\text{area}(P) = 0$  and  $x^P = \prod_{i=1}^{\ell(\alpha)} x_i^{\alpha_i}$ . We consider the map from such paths  $P$  to ordered multiset partitions  $\pi \in \mathcal{OP}_{\alpha,k+1}$  where  $\pi_i$  consists of the elements in the  $i$ th column of  $P$ , counting from right to left. This is clearly a bijection, and it follows from the definitions that  $\text{hdinv}(P) = \text{dinv}(\pi)$ , proving (2.4). For example, the parking function in Figure 2.3 corresponds to the ordered set partition  $14|2|35$ .

To prove (2.5), we consider the interpretation of  $\text{Rise}_{n,k}(x; q, t)$  from Subsection 1.3.1 in which we decorated double rises. This allows us to write

$$\text{Rise}_{n,k}(x; 0, q)|_{M_\alpha} = \sum_P q^{\text{area}(P)}$$

where the sum is over  $(P, R) \in \mathcal{PF}_{n,n-k-1}^{\text{Rise}}$  with  $\text{dinv}(P) = 0$  and  $x^P = \prod_{i=1}^{\ell(\alpha)} x_i^{\alpha_i}$ . We note that  $P$  can only have  $\text{dinv}(P) = 0$  if  $w_i(P)$  is weakly increasing from bottom to



**Figure 2.4:** This rise-decorated parking function with  $\text{dinv}$  equal to 0 is sent to the descent-starred permutation  $35_*142$ , which corresponds to the ordered set partition  $3|15|4|2$ .



**Figure 2.5:** This densely labeled Dyck path is sent to the ordered set partition  $5|23|14$ .

top; furthermore,  $w_{i+1}(P) > w_i(P)$  if and only if  $\ell_{i+1}(P) > \ell_i(P)$ . To form an ordered multiset partition from such a path  $P$ , we record the labels of  $P$  from top to bottom as a multiset permutation  $\sigma$ . Then, for each  $i \in R$ , we join the corresponding entry of  $\sigma$  with the entry to its right to form a block. This map gives a bijection to  $\mathcal{OP}_{\alpha, k+1}$  and sends  $\text{area}$  to  $\text{maj}$ . For example, the rise-decorated parking function in Figure 2.4 corresponds to  $3|15|4|2$ .

For (2.6), we consider the interpretation of  $\text{Val}_{n,k}(x; q, t)$  from Subsection 1.3.3 involving densely labeled Dyck paths, which implies

$$\text{Val}_{n,k}(x; q, 0)|_{M_\alpha} = \sum_P q^{\text{wdinv}(P)}$$

where the sum is over  $P \in \mathcal{PF}_{n,k}^{\text{Dense}}$  with  $\text{area}(P) = 0$  and  $x^P = \prod_{i=1}^{\ell(\alpha)} x_i^{\alpha_i}$ .  $\text{area}(P) = 0$  implies that the underlying Dyck path of  $P$  is the path  $(NE)^{k+1}$  that never leaves the diagonal. To form  $\pi \in \mathcal{OP}_{\alpha,k}$ , we simply make each label set of  $P$  from right to left into a block. This is a bijection and it is clear that  $\text{wdinv}(P) = \text{inv}(\pi)$ . For example, the densely labeled Dyck path in Figure 2.5 is sent to the ordered set partition  $5|23|14$ .

The proof of (2.7) is quite technical, so we have placed it in Appendix A.

□

We will prove results about the polynomials in (2.4), (2.5), and (2.6) in the remainder of this chapter. The polynomial in (2.7) has been more difficult to study; we discuss these issues in Subsection 2.4.4.

## 2.3 Equidistribution on Ordered Set Partitions

In this section, we prove that  $\text{inv}$ ,  $\text{dinv}$ , and  $\text{maj}$  are equidistributed on  $\mathcal{OP}_{n,k}$ . We accomplish this by generalizing the insertion maps defined for permutations in Section 2.1. We will recursively build ordered set partitions in such a way that allows us to keep track of the relevant statistic. We address the statistics  $\text{inv}$ ,  $\text{dinv}$ , and  $\text{maj}$  in Subsections 2.3.1, 2.3.2, and 2.3.3, respectively. We will also see that the distribution shared by these statistics has a simple form. If we define a  $q$ -analog of the Stirling numbers of the second kind via the recursion

$$S_{n,k}(q) = S_{n-1,k-1}(q) + [k]_q S_{n-1,k}(q)$$

with initial conditions  $S_{0,0}(q) = 1$  and  $S_{n,k}(q) = 0$  if  $k < 0$  or  $n < k$ , then we will prove

$$D_{n,k}^{\text{inv}}(q) = D_{n,k}^{\text{dinv}}(q) = D_{n,k}^{\text{maj}}(q) = [k]_q! S_{n,k}(q).$$

We will refer to this distribution as the *Mahonian distribution* on  $\mathcal{OP}_{n,k}$ . Finally, we will show how these results are related to a new type of rook placement in Subsection 2.3.4 and to the Euler-Mahonian distribution in Subsection 2.3.5.

### 2.3.1 Insertion for $\text{inv}$

We consider how to build an ordered set partition of order  $n$  from an ordered set partition of order  $n - 1$ . Perhaps the most straightforward way to do this is to "insert" an  $n$  somewhere. We can either insert  $n$  as its own block or we can place it into an existing block. Say that we are trying to build an ordered set partition in  $\mathcal{OP}_{n,k}$ . If we want to insert  $n$  as its own block, we must start with an element of  $\mathcal{OP}_{n-1,k-1}$ ; given such an ordered set partition, there are  $k$  places to insert the new block. If we want to insert  $n$  into an existing block, we start with an element of  $\mathcal{OP}_{n-1,k}$ , and there are  $k$  blocks into

which we could place  $n$ . With this in mind, our insertion maps for ordered set partitions will be bijections

$$(\mathcal{OP}_{n-1,k-1} \cup \mathcal{OP}_{n-1,k}) \times \{0, 1, \dots, k-1\} \rightarrow \mathcal{OP}_{n,k}.$$

We begin by defining

$$\phi_{n,k}^{\text{inv}} : (\mathcal{OP}_{n-1,k-1} \cup \mathcal{OP}_{n-1,k}) \times \{0, 1, \dots, k-1\} \rightarrow \mathcal{OP}_{n,k}.$$

Say we are given an ordered set partition  $\pi \in \mathcal{OP}_{n-1}$ . If  $\pi \in \mathcal{OP}_{n-1,k-1}$ , we number the spaces between the blocks of  $\pi$  from right to left with the integers  $0, 1, \dots, k-1$ . We call these *gap labels*. Then, to obtain  $\phi_{n,k}^{\text{inv}}((\pi, i))$  we simply place  $n$  as a new block in the position labeled  $i$ . If  $\pi \in \mathcal{OP}_{n-1,k}$ , we number the blocks of  $\pi$  from right to left with  $0, 1, \dots, k-1$ ; these are *block labels*. We set  $\phi_{n,k}^{\text{inv}}((\pi, i))$  equal to  $\pi$  with  $n$  inserted into the block labeled  $i$ . Inverting this map is straightforward and  $\phi_{n,k}^{\text{inv}}$  is clearly a bijection.

For example, say  $n = 5$ ,  $k = 3$ , and  $\pi = 14|23 \in \mathcal{OP}_{4,2}$ . Writing the gap labels, we obtain  ${}_214_1|23_0$ . Then  $\phi_{5,3}^{\text{inv}}(\pi, 1) = 14|5|23$ .

**Lemma 2.3.1.1** (Insertion for inv). *For any  $\pi \in \mathcal{OP}_{n-1,k-1} \cup \mathcal{OP}_{n-1,k}$  and  $0 \leq i < k$ , we have*

$$\text{inv}(\phi_{n,k}^{\text{inv}}(\pi, i)) = \text{inv}(\pi) + i.$$

*Proof.* It is clear that inserting  $n$  does not affect any of the inversions that existed in  $\pi$ . Therefore we only need to show that inserting  $n$  at the label  $i$  creates  $i$  new inversions. If  $i$  is a gap label, there are exactly  $i$  blocks to the left of the new block containing  $n$ . Each of these creates one new inversion between its minimal element and  $n$ . If  $i$  is a block label, there are still  $i$  blocks to the right of  $n$ 's block, and we still get  $i$  new inversions.  $\square$

Lemma 2.3.1.1 implies that the distribution of  $\text{inv}$  on  $\mathcal{OP}_{n,k}$  obeys the recursion

$$D_{n,k}^{\text{inv}}(q) = [k]_q (D_{n-1,k-1}^{\text{inv}}(q) + D_{n-1,k}^{\text{inv}}(q)).$$

One can also show that the polynomial  $[k]_q! S_{n,k}(q)$  obeys this recursion by using the definition of  $S_{n,k}(q)$ . This implies  $D_{n,k}^{\text{inv}}(q) = [k]_q! S_{n,k}(q)$ .

### 2.3.2 Insertion for $\text{dinv}$

In this subsection, we define a bijection

$$\phi_{n,k}^{\text{dinv}} : (\mathcal{OP}_{n-1,k-1} \cup \mathcal{OP}_{n-1,k}) \times \{0, 1, \dots, k-1\} \rightarrow \mathcal{OP}_{n,k}$$

with the property

$$\text{dinv}(\phi_{n,k}^{\text{dinv}}(\pi, i)) = \text{dinv}(\pi) + i.$$

We will also show how these maps provide a bijection from  $\mathcal{OP}_{n,k}$  to itself which shows that  $\text{inv}$  and  $\text{dinv}$  are equidistributed.

To define  $\phi_{n,k}^{\text{dinv}}$ , we begin with an ordered set partition  $\pi \in \mathcal{OP}_{n-1,k-1}$ . We label the positions between the blocks, as well as the positions at either end of  $\pi$ , with the labels  $0, 1, \dots, k$  from right to left. In other words, we use the same gap labels as in the inversion case. Our block labels will be quite different from before. If  $\pi \in \mathcal{OP}_{n-1,k}$ , set  $h$  to be the maximum size of any block in  $\pi$ . We obtain the block labels of  $\pi$  by, beginning with the label 0, repeatedly

1. labeling blocks of  $\pi$  of size  $h$  from left to right with increasing labels, and
2. decrementing  $h$ .

We repeat until  $h = 0$ . For example, consider the ordered set partition pictured below, where we have stacked each block vertically.

	4			
5	3			
2	1	7	6	

The gap labels are 4,3,2,1, and 0 and the block labels are 1, 0, 2, and 3, reading from left to right.

**Lemma 2.3.2.1** (Insertion for  $\text{dinv}$ ). *For any  $\pi \in \mathcal{OP}_{n-1,k-1} \cup \mathcal{OP}_{n-1,k}$  and  $0 \leq i < k$ , we have*

$$\text{dinv}(\phi_{n,k}^{\text{dinv}}(\pi, i)) = \text{dinv}(\pi) + i.$$

*Proof.* It is clear that inserting an  $n$  at a gap labeled  $b$  creates  $b$  new diagonal inversions, one with each block to the right of the gap, and does not affect any other diagonal inversions.

Now say we insert an  $n$  into a block  $\pi_i$  labeled  $u$ . Say that  $\pi_i$  had size  $s$  before we added an  $n$ . After inserting an  $n$ , it has size  $s + 1$  with an  $n$  at height  $s + 1$ . We claim that we have created one new diagonal inversion for each block of size greater than  $s$  and for each block of size  $s$  that is to the left of  $\pi_i$ . First, consider a block  $\pi_j$  with  $|\pi_j| > s$ . If  $j > i$ , then  $(s + 1, i, j)$  is a new primary diagonal inversion; if  $j < i$ , then  $(s, j, i)$  is a new secondary diagonal inversion. There are no other new diagonal inversions between  $\pi_i$  and  $\pi_j$ . Now we consider  $\pi_j$  with  $j < i$  and  $|\pi_j| = s$ . There is a new secondary diagonal inversion  $(s, j, i)$ . By the definition of our insertion map, there are exactly  $u$  such blocks, so we have created  $u$  new diagonal inversions. □

Now we will recursively define a bijection  $\theta_{n,k} : \mathcal{OP}_{n,k} \rightarrow \mathcal{OP}_{n,k}$  with the property that  $\text{dinv}(\theta_{n,k}(\rho)) = \text{inv}(\rho)$  for all  $\rho \in \mathcal{OP}_{n,k}$ . If  $n > 1$ ,  $\theta_{n,k}$  is defined by

$$\theta_{n,k}(\rho) = \begin{cases} \left( \phi_{n,k}^{\text{dinv}} \circ (\theta_{n-1,k-1}, \text{id}) \circ (\phi_{n,k}^{\text{inv}})^{-1} \right) (\rho) & n \text{ is in its own block in } \rho \\ \left( \phi_{n,k}^{\text{dinv}} \circ (\theta_{n-1,k}, \text{id}) \circ (\phi_{n,k}^{\text{inv}})^{-1} \right) (\rho) & n \text{ shares a block in } \rho. \end{cases}$$

For  $n = 1$ ,  $\theta_{n,k}$  is simply the identity. Figure 2.6 contains an example of how to compute  $\theta_{n,k}$  for  $n = 5, k = 2$ . We begin by taking an ordered set partition  $\rho = 134|25 \in \mathcal{OP}_{5,2}$ . We repeatedly take out the largest element in  $\rho$  and write down how many  $\text{inv}$  this costs and whether or not this decreases the number of blocks in the ordered set partition. Then we use this information to build up the image of  $\rho$  under  $\theta_{5,3}$ . In particular, we use the gap labels for  $\text{dinv}$  if we need to create a new block and block labels otherwise. We have italicized the label that we chose at each step.

**Proposition 2.3.2.1.**  $\theta_{n,k}$  is a bijection with

$$\text{dinv}(\theta_{n,k}(\rho)) = \text{inv}(\rho)$$

for all  $\rho \in \mathcal{OP}_{n,k}$ .

*Proof.* Let  $(\pi, i)$  be the inverse of  $\rho$  under  $\phi_{n,k}^{\text{inv}}$  and  $\pi'$  be the image of  $\pi$  under either  $\theta_{n-1,k-1}$  or  $\theta_{n-1,k}$ , depending on whether  $n$  is in its own block in  $\rho$  or not. Then, using

$\rho$	inv Lost	Label Type	$\theta_{n,k}(\rho)$
134 25			145 23
134 2	0	block	14 <sub>0</sub>  23 <sub>1</sub>
13 2	1	block	1 <sub>l</sub>  23 <sub>0</sub>
1 2	1	block	1 <sub>0</sub>  2 <sub>l</sub>
1	0	gap	<sub>1</sub> 1 <sub>0</sub>

**Figure 2.6:** An example of the recursive bijection  $\theta_{5,2}$ .

Lemmas 2.3.1.1 and 2.3.2.1, we compute

$$\text{dinv}(\theta_{n,k}(\rho)) = \text{dinv}(\pi') + i \quad (2.8)$$

$$= \text{inv}(\pi) + i \quad (2.9)$$

$$= \text{inv}(\rho). \quad \square$$

### 2.3.3 Insertion for maj

In this subsection, we define a bijection

$$\phi_{n,k}^{\text{maj}} : (\mathcal{OP}_{n-1,k-1} \cup \mathcal{OP}_{n-1,k}) \times \{0, 1, \dots, k-1\} \rightarrow \mathcal{OP}_{n,k}$$

with the property

$$\text{maj}(\phi_{n,k}^{\text{maj}}(\pi, i)) = \text{maj}(\pi) + i.$$

This map will give us a bijection from  $\mathcal{OP}_{n,k}$  to  $\mathcal{OP}_{n,k}$  that carries inv to maj. The insertion map for maj will be quite different from the maps for inv and dinv. In fact, in order to deal with maj we will need to rely on the insertion lemma for the major index over the symmetric group.

First, we will prefer to deal with descent-starred permutations instead of ordered set partitions. Recall that every  $\pi \in \mathcal{OP}_{n,k}$  has a unique representation as a descent-starred permutation  $(\sigma, S) \in \mathfrak{S}_{n,n-k}^>$  where we write each block in decreasing order from left to right and we use stars to indicate elements that share a block instead of using bars to separate blocks. For example, the ordered set partition 5|12|4|367 is represented  $5 \ 2_* \ 1 \ 4 \ 7_* \ 6_* \ 3$ .

To define  $\phi_{n,k}^{\text{maj}}$ , we begin with a  $\pi \in \mathcal{OP}_{n-1}$ . If  $\pi \in \mathcal{OP}_{n-1,k-1}$ , then its corresponding descent-starred permutation  $(\sigma, S)$  is in  $\mathfrak{S}_{n-1,n-k}^>$ . We label the rightmost position with a zero, and then we label the unstarred descents from right to left with  $1, 2, \dots$ . We label the leftmost position with the next number. Then we label the unstarred ascents from left to right with increasing labels. If  $\pi \in \mathcal{OP}_{n-1,k}$ , the only difference in our labeling is that we skip the rightmost position.

For example, say  $(\sigma, S) = 5\ 2_*1\ 4\ 7_*6_*3$ , which corresponds to an element of  $\mathcal{OP}_{7,4}$ . If  $k = 5$ , we label  $(\sigma, S)$  as

$${}_25_12_*1_34_47_*6_*3_0. \quad (2.10)$$

If  $k = 4$ , we label  $(\sigma, S)$  as

$${}_15_02_*1_24_37_*6_*3. \quad (2.11)$$

Now we need to define how to build a new descent-starred permutation after choosing a certain label. This process will be quite different from the process we established for the inversion number. We consider  $\pi \in \mathcal{OP}_{n-1,k-1} \cup \mathcal{OP}_{n-1,k}$  as a descent-starred permutation  $(\sigma, S)$ . If  $\pi \in \mathcal{OP}_{n-1,k-1}$ , to obtain  $\phi_{n,k}^{\text{maj}}(\pi, i)$  we

1. insert  $n$  at the label  $i$ , and then
2. move each star to the right of  $n$  one descent to its left.

This second step is well-defined because, after step 1,  $n$  will always be an unstarred descent. If  $\pi \in \mathcal{OP}_{n-1,k}$ , we add a third step, which is to

3. add a star to the rightmost descent.

Since every star weakly to the right of  $n$  was just moved one descent to the left, this step is always possible. For example, with  $(\sigma, S) = 5\ 2_*1\ 4\ 7_*6_*3$  as above, (2.10) gives

$$\phi_{8,5}^{\text{maj}}((\sigma, S), 3) = 5\ 2_*1\ 8_*4\ 7_*6\ 3$$

and (2.11) gives

$$\phi_{8,4}^{\text{maj}}((\sigma, S), 3) = 5\ 2_*1\ 4\ 8_*7_*6_*3.$$

The key is that we can track the change in major index through this insertion process.



**Lemma 2.3.3.1** (Insertion for maj). *For any  $\pi \in \mathcal{OP}_{n-1,k-1} \cup \mathcal{OP}_{n-1,k}$  and  $0 \leq i < k$ , we have*

$$\text{maj}\left(\phi_{n,k}^{\text{maj}}(\pi, i)\right) = \text{maj}(\pi) + i.$$

*Proof.* We begin with  $\pi \in \mathcal{OP}_{n-1,k-1}$ . We let  $(\sigma, S)$  be the descent-starred permutation representation of  $\pi$  and  $(\tau, T)$  be the descent-starred permutation representation of  $\phi_{n,k}^{\text{maj}}(\pi, i)$ . Recall that

$$\text{maj}((\sigma, S)) = \text{maj}(\sigma) - \sum_{j \in S} |\text{Des}(\sigma) \cap \{j, j+1, \dots\}|.$$

If  $i = 0$ , we insert  $n$  at the far right end. This does not change either term in the above expression, so we have  $\text{maj}((\tau, T)) = \text{maj}((\sigma, S)) + 0$ , as desired.

Now suppose that the space labeled  $i$  is the space immediately following  $\sigma_p$  where  $\sigma_p > \sigma_{p+1}$  and  $p \notin S$ . Suppose that there are  $a$  starred descents and  $b$  unstarred descents to the left of  $\sigma_p$  and  $c$  starred descents and  $d$  unstarred descents weakly to the right of  $\sigma_p$  in  $(\sigma, S)$ . Then we must have  $i = d$ . We know from Section 2.1 that inserting  $n$  at this position increases the major index of the permutation  $\sigma$  by  $c + d$ , i.e.  $\text{maj}(\tau) = \text{maj}(\sigma) + c + d$ . Now we need to consider how moving stars changes the term

$$\sum_{j \in S} |\text{Des}(\sigma) \cap \{j, j+1, \dots\}|.$$

Since we have inserted  $n$  at a descent, we have not increased the number of descents of  $\sigma$ , i.e.  $\text{des}(\tau) = \text{des}(\sigma)$ . Therefore each star to the left of position  $p$  has the same number of descents weakly to its right, so their contribution to this sum does not change. However, each of the  $c$  starred descents to the right of position  $p$  gains an additional descent to its right (since we have moved the stars themselves to the left). Therefore this sum increases by  $c$ , and we have

$$\text{maj}((\tau, T)) = \text{maj}(\tau) - \sum_{j \in T} |\text{Des}(\tau) \cap \{j, j+1, \dots\}| \tag{2.12}$$

$$= \text{maj}(\sigma) + c + d - \left( c + \sum_{j \in T} |\text{Des}(\sigma) \cap \{j, j+1, \dots\}| \right) \tag{2.13}$$

$$= \text{maj}((\sigma, S)) + d \tag{2.14}$$

$$= \text{maj}((\sigma, S)) + i. \tag{2.15}$$

Next, suppose that the space labeled  $i$  is the space at the start of  $(\sigma, S)$ . Assume that there are  $c$  starred descents and  $d$  unstarred descents in  $(\sigma, S)$ . Then  $i = d + 1$ . As discussed in Section 2.1, inserting  $n$  at this position increases the major index of the underlying permutation by  $c + d + 1$ , i.e.  $\text{maj}(\tau) = \text{maj}(\sigma) + c + d + 1$ . Now every starred descent in  $\tau$  is weakly to the right of  $n$ , so moving stars decreases the major index of the ordered set partition by  $c$ . Then, as above, we argue that

$$\text{maj}((\tau, T)) = \text{maj}(\tau) + \sum_{j \in T} |\text{Des}(\tau) \cap \{j, j + 1, \dots\}| \quad (2.16)$$

$$= \text{maj}(\sigma) + c + d + 1 - \left( c + \sum_{j \in T} |\text{Des}(\sigma) \cap \{j, j + 1, \dots\}| \right) \quad (2.17)$$

$$= \text{maj}((\sigma, S)) + d + 1 \quad (2.18)$$

$$= \text{maj}((\sigma, S)) + i. \quad (2.19)$$

Now suppose that the space labeled  $i$  is the space following  $\sigma_p$  where  $\sigma_p < \sigma_{p+1}$ . Suppose that there are  $a$  starred descents and  $b$  unstarred descents in  $(\sigma, S)$  strictly to the left of  $\sigma_p$  and  $c$  starred descents and  $d$  unstarred descents in  $(\sigma, S)$  weakly to the right of  $\sigma_p$ . Then

$$i = b + d + 1 + p - (a + b) = d + p - a + 1. \quad (2.20)$$

Furthermore, we have

$$\text{maj}(\tau) = \text{maj}(\sigma) + a + b + c + d + p - (a + b) + 1 = c + d + p + 1. \quad (2.21)$$

Now, we claim that moving stars decreases the major index by  $a + c$ , i.e. one for each star. This is because each star to the left of position  $p$  picks up an additional descent to its right with the insertion of  $n$  at an ascent. We also note that each star to the right of

position  $p$  picks up another descent to its right after being moved. Hence

$$\text{maj}((\tau, T)) = \text{maj}(\tau) - \sum_{j \in T} |\text{Des}(\tau) \cap \{j, j+1, \dots\}| \quad (2.22)$$

$$= \text{maj}(\sigma) + c + d + p + 1 - \left( a + c + \sum_{j \in S} |\text{Des}(\sigma) \cap \{j, j+1, \dots\}| \right) \quad (2.23)$$

$$= \text{maj}((\sigma, S)) + d + p - a + 1 \quad (2.24)$$

$$= \text{maj}((\sigma, S)) + i. \quad (2.25)$$

Finally, we need to consider what happens when  $\pi \in \mathcal{OP}_{n-1, k}$ . Note that we obtain the labels for this case by subtracting 1 from each label (except 0) from the  $\pi \in \mathcal{OP}_{n-1, k-1}$  case. We also notice that starring the rightmost descent subtracts exactly 1 from the major index, since it adds a new star with 1 descent weakly to its right. This completes the proof.  $\square$

Now we can recursively define a bijection  $\psi_{n, k} : \mathcal{OP}_{n, k} \rightarrow \mathcal{OP}_{n, k}$  by setting it equal to the identity if  $n = k = 1$  and declaring

$$\psi_{n, k}(\rho) = \begin{cases} \left( \phi_{n, k}^{\text{maj}} \circ (\psi_{n-1, k-1}, \text{id}) \circ (\phi_{n, k}^{\text{inv}})^{-1} \right) (\rho) & n \text{ is in its own block in } \rho \\ \left( \phi_{n, k}^{\text{maj}} \circ (\psi_{n-1, k}, \text{id}) \circ (\phi_{n, k}^{\text{inv}})^{-1} \right) (\rho) & n \text{ shares a block in } \rho. \end{cases}$$

An analogous argument to the one used to prove Lemma 2.3.2.1 proves that  $\text{maj}(\psi_{n, k}(\rho)) = \text{inv}(\rho)$  for every  $\rho \in \mathcal{OP}_{n, k}$ . Thus  $D_{n, k}^{\text{inv}}(q) = D_{n, k}^{\text{maj}}(q)$ . We have worked through an example of  $\psi_{n, k}$  in Figure 2.7 for  $n = 5$  and  $k = 2$ . We begin by taking an ordered set partition  $\rho = 134|25 \in \mathcal{OP}_{5, 2}$ . We repeatedly take out the largest element in  $\rho$  and write down how many inv this costs and whether or not this decreases the number of blocks in the ordered set partition. Then we use this information to build up the image of  $\rho$  under  $\psi_{5, 3}$ . In particular, we use the labels for maj, adding a new star if and only if the corresponding step did *not* reduce the block count. We have italicized the label that we chose at each step.

Furthermore, the equidistribution of inv and maj on  $\mathcal{OP}_{n, k}$  can be reformulated as an identity about permutations. By the alternate definition of maj in (2.3),

$$\sum_{k=1}^n z^{n-k} D_{n, k}^{\text{maj}}(q) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} \prod_{j=1}^{\text{des}(\sigma)} (1 + z/q^j). \quad (2.26)$$

$\rho$	inv Lost	Add Star?	$\psi_{n,k}(\rho)$
134 25			$5_*14_*3_*2$
134 2	0	yes	$01_14_*3_*2$
13 2	1	yes	$01_13_*2$
1 2	1	yes	$01_I2$
1	0	no	$11_0$

**Figure 2.7:** Here we have an example of the recursive bijection  $\psi_{5,2}$ .

We also notice that, for a descent-starred permutation  $(\sigma, S)$ ,

$$\text{inv}((\sigma, S)) = \text{inv}(\sigma) - \sum_{i \in S} (\text{inv}^{\square, i}(\sigma) + 1) \quad (2.27)$$

where  $\text{inv}^{\square, i}(\sigma)$  is the number of inversions in the permutation  $\sigma$  whose right entry is at position  $i$ . Therefore we have shown

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} \prod_{i \in \text{Des}(\sigma)} \left(1 + z/q^{\text{inv}^{\square, i}(\sigma)+1}\right) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} \prod_{j=1}^{\text{des}(\sigma)} \left(1 + z/q^j\right). \quad (2.28)$$

This statement was originally conjectured by Haglund via personal communication. By considering inversions on ordered set partitions as ascent-starred permutations, we also see that these expressions are equal to

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} \prod_{i \in \text{Asc}(\sigma)} \left(1 + z/q^{\text{inv}^{\square, i+1}(\sigma)}\right). \quad (2.29)$$

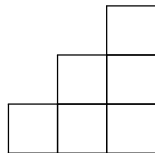
### 2.3.4 Mixed rook placements

Since we have proved that the following three distributions are equal, we can define

$$D_{n,k}(q) = D_{n,k}^{\text{inv}}(q) = D_{n,k}^{\text{dinv}}(q) = D_{n,k}^{\text{maj}}(q).$$

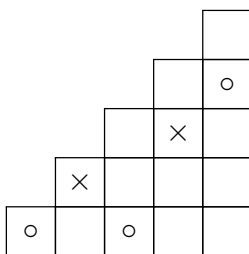
We refer to this polynomial as the *Mahonian distribution* on  $\mathcal{OP}_{n,k}$ . We will see that the Mahonian distribution on  $\mathcal{OP}_{n,k}$  also appears as the distribution of a certain statistic on mixed rook placements, which are a new type of rook placement in the staircase board. These rook placements will give a direct combinatorial proof that  $D_{n,k}(q) = [k]_q! S_{n,k}(q)$ .

We give by defining the *staircase board*  $\text{Stair}_n$ , which is the bottom-justified collection of cells with 1 cell in the first column, 2 cells in the second column, and so on until there are  $n$  cells in the  $n$ th column. Here is the staircase board  $\text{Stair}_3$ :



We will place two types of objects into these boards, both of which we will call rooks because no two of these objects can be placed in the same column. A *right-canceling rook* does not allow other rooks to be placed in its column or in its row to the right. A *column rook* does not have the condition about rows; it simply does not allow other rooks to be placed in its column. The cases where one places only right-canceling rooks or only column rooks in the staircase board are well-studied [BCHR]. In particular, there are exactly  $S_{n,n-k}$  ways to place  $k$  right-canceling rooks in the board  $\text{Stair}_{n-1}$ .

We will place both of these types of rooks into the same board, which is a case that has not been studied previously. We define  $\mathcal{M}_{n,k}$  to be the set of all placements of  $k$  column rooks and  $n - k$  right-canceling rooks into the staircase board  $\text{Stair}_n$  such that no right-canceling rooks appear in the bottom row of  $\text{Stair}_n$ . Since no rooks can share a column, this means that there are no empty columns in such a placement. We call these *mixed rook placements*. Below, we have drawn a mixed placement in  $\mathcal{M}_{5,3}$ . We have used circles to represent column rooks and crosses for right-canceling rooks.



First, we claim that  $|\mathcal{M}_{n,k}| = k!S_{n,k}$ . To see this, we first place  $n - k$  right-canceling rooks in  $\text{Stair}_n$ ; there are  $S_{n,k}$  ways to accomplish this [BCHR]. Then we consider the  $k$  remaining empty columns. We say a cell is *canceled* if it is to the right of a right-canceling rook. We claim that, from left to right, the  $k$  empty rows have 1, 2, ...,  $k$  uncanceled cells. If this is true, then there are clearly  $k!$  ways to place the  $k$  column rooks into these columns. Say that the empty columns are columns  $c_1, \dots, c_k$

in  $\text{Stair}_n$ . We must have  $c_1 = 1$ . Now consider column  $c_i$  for  $i > 1$ . It has  $c_i$  cells and  $c_i - 1 - (i - 1) = c_i - i$  right-canceling rooks to its left. Each of these rooks cancels one cell in column  $c_i$ , so column  $c_i$  has  $i$  uncanceled cells. This proves the claim.

Next, we will define a statistic on mixed rook placements such that the distribution of this statistic over  $\mathcal{M}_{n,k}$  equals  $[k]_q! S_{n,k}(q)$ . We will also give recursive bijections between mixed rook placements and ordered set partitions that sends our new statistic to any of the statistics  $\text{inv}$ ,  $\text{dinv}$ , and  $\text{minimaj}$  on ordered set partitions. The statistic  $\text{unc}$  counts the number of cells that are not canceled and that are below some rook but not in the bottom row of a column containing a right-canceling rook. For example, the mixed rook placement in  $\mathcal{M}_{5,3}$  pictured above has  $\text{unc} = 1$ , since only the bottom right cell meets all these criteria.

Now we define an insertion map for mixed rook placement and the statistic  $\text{unc}$ . We define

$$\phi_{n,k}^{\text{unc}} : (\mathcal{M}_{n-1,k-1} \cup \mathcal{M}_{n-1,k}) \times \{0, 1, \dots, k-1\} \rightarrow \mathcal{M}_{n,k}$$

such that

$$\text{unc}(\phi_{n,k}^{\text{unc}}(P, i)) = \text{unc}(P) + i.$$

Say we begin with some placement  $P \in \mathcal{M}_{n-1,\ell}$  for  $\ell = k$  or  $\ell = k - 1$ . We add a column with  $n$  cells to the right of  $P$ . If  $\ell = k$ , then we must add a right-canceling rook to this new column. Since  $P$  had  $n - k - 1$  right-canceling rooks, there are  $k + 1$  uncanceled cells in the new column. We cannot place a right-canceling rook in the bottom row, so there are  $k$  places we can put the new right-canceling rook. Furthermore, if we number these places  $0, 1, \dots, k - 1$  from bottom to top, this numbering gives the increase in  $\text{unc}$  in the new mixed placement. If, instead,  $\ell = k - 1$ , we need to add a column rook to the new column. In this case,  $P$  has  $n - k$  right-canceling rooks, so there are  $k$  places we can put this new column rook. Again, numbering these places from bottom to top gives the potential increase in  $\text{unc}$ .

These insertion maps can be used to recursively construct bijections between

$\mathcal{M}_{n,k}$  and  $\mathcal{OP}_{n,k}$  that send  $\text{unc}$  to any one of  $\text{inv}$ ,  $\text{maj}$ , or  $\text{dinv}$ . As a result, we have

$$\sum_{P \in \mathcal{M}_{n,k}} q^{\text{unc}(P)} = D_{n,k}(q).$$

Furthermore, we also have  $\sum_{P \in \mathcal{M}_{n,k}} q^{\text{unc}(P)} = [k]_q! S_{n,k}(q)$ . To see this, we use the fact that the distribution of  $\text{unc}$  over placements of  $n - k$  right-attacking rooks into  $\text{Stair}_{n-1}$  is  $S_{n,k}(q)$  [GR86]. Since right-attacking rooks are not allowed in the bottom row of a mixed placement, we can think of this step as placing our  $n - k$  right-attacking rooks. We then wish to "complete" such a placement to a placement in  $\mathcal{M}_{n,k}$ . As we proved earlier, the empty columns have exactly  $1, 2, \dots, k$  uncanceled cells, from left to right. The distribution of  $\text{unc}$  over all ways to place  $k$  column rooks into these columns is clearly  $[k]_q!$ . Therefore  $\sum_{P \in \mathcal{M}_{n,k}} q^{\text{unc}(P)} = [k]_q! S_{n,k}(q)$ , and we have shown that  $D_{n,k}(q) = [k]_q! S_{n,k}(q)$ . We also could have proved this fact by induction, using the recursive definition of  $S_{n,k}(q)$ , but this proof is more direct.

### 2.3.5 An application to the Euler-Mahonian distribution

In this subsection, we show that our equidistribution theorem, paired with the alternate definition of the major index in (2.3), solves a problem posed by Steingrímsson about the Euler-Mahonian distribution. Let  $A_{n,d}(q)$  be the sum of  $q^{\text{maj}(\sigma)}$  over the permutations  $\sigma \in \mathfrak{S}_n$  with exactly  $d$  descents. Given any permutation  $\sigma \in \mathfrak{S}_n$  with at least  $n - k$  descents, we can obtain an ordered set partition in  $\mathcal{OP}_{n,k}$  by placing stars after  $n - k$  of the descents of  $\sigma$ . By (2.3), placing a star after the  $i$ th descent of  $\sigma$ , counting from 1 and from right to left, removes  $i$  from the major index of the resulting ordered set partition. Therefore

$$\begin{aligned} D_{n,k}^{\text{maj}}(q) &= \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{des}(\sigma) \geq n-k}} q^{\text{maj}(\sigma)} \prod_{i=1}^{\text{des}(\sigma)} (1 + uq^{-i}) \Big|_{u^{n-k}} \\ &= \sum_{d=n-k}^{n-1} \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{des}(\sigma)=d}} q^{\text{maj}(\sigma)} \prod_{i=1}^d (1 + uq^{-i}) \Big|_{u^{n-k}} \\ &= \sum_{d=n-k}^{n-1} A_{n,d}(q) \begin{bmatrix} d \\ n-k \end{bmatrix}_q q^{\binom{n-k}{2} - d(n-k)}. \end{aligned}$$

Therefore our equidistribution results on  $\mathcal{OP}_{n,k}$  provide a combinatorial proof that

$$[k]_q! S_{n,k}(q) = \sum_{d=n-k}^{n-1} A_{n,d}(q) \begin{bmatrix} d \\ n-k \end{bmatrix}_q q^{\binom{n-k}{2} - d(n-k)}.$$

Steingrímsson asked for such a proof in [Ste07].

## 2.4 Equidistribution on Ordered Multiset Partitions

In this section, we prove the following equidistribution theorem for ordered multiset partitions.

**Theorem 2.4.0.1.** *For any composition  $\alpha$ ,*

$$D_{\alpha,k}^{\text{inv}}(q) = D_{\alpha,k}^{\text{maj}}(q) = D_{\alpha,k}^{\text{dinv}}(q).$$

As a result of this theorem and Proposition 2.2.3.1, we have

$$\text{Rise}_{n,k}(x; q, 0) = \text{Rise}_{n,k}(x; 0, q) = \text{Val}_{n,k}(x; q, 0).$$

The reader may have noticed that  $D_{n,k}^{\text{minimaj}}(q)$  and  $\text{Val}_{n,k}(x; 0, q)$  are not included in the list of equidistributed polynomials; that is because we have not proved this case at this point. We describe the unique difficulties of this case in Subsection 2.4.4.

Our proof of Theorem 2.4.0.1 is bijective and employs a generalization of Carlitz's insertion method from permutations to ordered multiset partitions. We describe "insertion maps" for inv, dinv, and maj in Subsections 2.4.1, 2.4.3, 2.4.2, respectively. These maps will be of the form

$$\begin{aligned} \phi_{\alpha,k,\ell}^{\text{inv}} &: \mathcal{OP}_{\alpha^-, \ell} \times \binom{[0, \ell-1]}{\alpha_n - k + \ell} \times \left( \binom{[0, \ell]}{k - \ell} \right) \rightarrow \mathcal{OP}_{\alpha,k} \\ \phi_{\alpha,k,\ell}^{\text{maj}} &: \mathcal{OP}_{\alpha^-, \ell} \times \binom{[0, \ell-1]}{\alpha_n - k + \ell} \times \left( \binom{[0, \ell]}{k - \ell} \right) \rightarrow \mathcal{OP}_{\alpha,k} \\ \phi_{\alpha,k,\ell}^{\text{dinv}} &: \mathcal{OP}_{\alpha^-, \ell} \times \binom{[0, \ell-1]}{\alpha_n - k + \ell} \times \left( \binom{[0, \ell]}{k - \ell} \right) \rightarrow \mathcal{OP}_{\alpha,k} \end{aligned}$$

for any composition  $\alpha$  of length  $n$  where  $\alpha^-$  is the composition obtained by removing



the rightmost entry in  $\alpha$ . By "insertion maps," we mean that they satisfy the properties

$$\begin{aligned} \text{inv}(\phi_{\alpha,k,\ell}^{\text{inv}}(\pi, U, B)) &= \text{inv}(\pi) + \sum_{u \in U} u + \sum_{b \in B} b \\ \text{maj}(\phi_{\alpha,k,\ell}^{\text{maj}}(\pi, U, B)) &= \text{maj}(\pi) + \sum_{u \in U} u + \sum_{b \in B} b \\ \text{dinv}(\phi_{\alpha,k,\ell}^{\text{dinv}}(\pi, U, B)) &= \text{dinv}(\pi) + \sum_{u \in U} u + \sum_{b \in B} b. \end{aligned}$$

When  $k = |\alpha|$  these maps will reduce to the insertion processes defined in Subsection 2.1.3. We will use these maps to construct a bijective proof of Theorem 2.4.0.1. Subsection 2.4.5 contains more information about the shared distribution of the polynomials in Theorem 2.4.0.1.

### 2.4.1 Insertion for inv

Recall that we need to define a map of the form

$$\phi_{\alpha,k,\ell}^{\text{inv}} : \mathcal{OP}_{\alpha^-, \ell} \times \binom{[0, \ell-1]}{\alpha_n - k + \ell} \times \left( \binom{[0, \ell]}{k - \ell} \right) \rightarrow \mathcal{OP}_{\alpha, k}.$$

We can think of the set  $U \in \binom{[0, \ell-1]}{\alpha_n - k + \ell}$  as providing the increases in  $\text{inv}$  that come from adding a new  $n$  without creating a new block, and the multiset  $B \in \left( \binom{[0, \ell]}{k - \ell} \right)$  as providing the increases in the statistic that come from adding a new  $n$  while creating a new block. These maps will cooperate with our inversion statistic in the following manner:

$$\text{inv}(\phi_{\alpha,k,\ell}^{\text{inv}}(\pi, U, B)) = \text{inv}(\pi) + \sum_{u \in U} u + \sum_{b \in B} b \quad (2.30)$$

Given  $\pi \in \mathcal{OP}_{\alpha^-, \ell}$ , we label each block of  $\pi$  from right to left with the numbers  $0, 1, \dots, \ell - 1$ . We repeatedly remove the largest element from the multiset  $U \cup B$ , taking the element from  $U$  if the largest elements are equal. Call this element  $i$ . If  $i$  came from  $U$ , we place an  $n$  in the block that received the label  $i$ . If  $i$  came from  $B$  and is equal to  $\ell$ , we place an  $n$  as a new block to the left of the block that received the label  $i$ . If  $i$  came from  $B$  and is less than  $\ell$ , we place an  $n$  as a new block just to the right of the block labeled  $i$ . The resulting ordered multiset partition is  $\phi^{\text{inv}}(\pi, U, B)$ .

For example, say  $\alpha = \{1, 2, 2, 3\}$ ,  $k = 6$ , and  $\ell = 5$ . We consider  $\pi = 3|1|2|2|13 \in \mathcal{OP}_{\alpha^-, 5}$ ,  $U = \{2, 0\}$ , and  $B = \{3\}$ . Then  $U \cup B = \{3, 2, 0\}$ . We label  $\pi$  as follows, with the labels written as subscripts at the end of each block:

$$3_4|1_3|2_2|2_1|13_0$$

The largest element in  $U \cup B$  is 3 and it comes from  $B$ , so we insert a 4 at the position labeled 3 as a new block to the right of the block labeled with the 3.

$$3_4|1_3|4|2_2|2_1|13_0$$

Now the largest remaining element of  $U \cup B$  is 2 and it comes from  $U$ , so we put a 4 into the block labeled 2.

$$3_4|1_3|4|2_4|2_1|13_0$$

Finally, we insert a 4 into the block labeled 0 to obtain  $\phi^{\text{inv}}(\pi, U, B)$ .

$$3_4|1_3|4|2_4|2_1|134_0$$

We can check that (2.30) holds here.

$$\begin{aligned} 10 &= \text{inv}(3|1|4|4|2|4|3|1) \\ &= \text{inv}(3|1|2|2|3|1) + \sum_{u \in U} u + \sum_{b \in B} b \\ &= 5 + (2 + 0) + 3. \end{aligned}$$

**Lemma 2.4.1.1** (Insertion for inv). *For any composition  $\alpha$  of length  $n$  and positive integers  $\ell \leq k$ ,  $\phi_{\alpha, k, \ell}^{\text{inv}}$  is well-defined and injective. The image of  $\phi_{\alpha, k, \ell}^{\text{inv}}$  is the ordered multiset partitions  $\pi \in \mathcal{OP}_{\alpha, k}$  with exactly  $k - \ell$  singleton blocks containing  $n$ . Furthermore, for any  $\pi \in \mathcal{OP}_{\alpha, \ell}$ ,  $U \in \binom{[0, \ell-1]}{\alpha_n - k + \ell}$ , and  $B \in \binom{[0, \ell]}{k - \ell}$ ,*

$$\text{inv}(\phi_{\alpha, k, \ell}^{\text{inv}}(\pi, U, B)) = \text{inv}(\pi) + \sum_{u \in U} u + \sum_{b \in B} b.$$

*Proof.* The only way  $\phi_{\alpha, k, \ell}^{\text{inv}}$  could not be well-defined is if we tried to insert two  $n$ 's into the same block. Since  $U$  is a set, this does not occur. The statement about the image

of  $\phi_{\alpha,k,\ell}^{\text{inv}}$  follows from the definition. Furthermore, the insertion map is clearly bijective, and any function that bijects onto its image is an injection.

We will prove that whenever we remove the largest element  $i$  from  $U \cup B$  and introduce a new  $n$  to our ordered multiset partition as described in the insertion map, we introduce  $i$  new inversions to the ordered multiset partition. Specifically, we create inversions between this new  $n$  and the minimal elements of the  $i$  labeled blocks of  $\pi$  that are to the right of the block that received label  $i$ . Since  $n$  is the largest entry, we do not create any new inversions that end at  $n$ . Finally, we note that we do not destroy any inversions that existed before we inserted this new  $n$ .  $\square$

## 2.4.2 Insertion for maj

To define  $\phi_{\alpha,k,\ell}^{\text{maj}}$ , we will view  $\pi \in \mathcal{OP}_{\alpha^-, \ell}$  as a descent-starred multiset permutation  $(\sigma, S)$ . We will label the unstarred positions of  $\sigma$  as in Section 2.1. Specifically, we label the unstarred descents from right to left, then the non-descents from left to right, using increasing labels  $0, 1, \dots, \ell$ . Let  $U^+ = \{u + 1 : u \in U\}$ . We repeatedly remove the largest element  $i$  from  $U^+ \cup B$ , taking  $i$  from  $B$  if the largest elements are equal. Then we proceed through the following algorithm:

1. Insert an  $n$  at the position labeled  $i$ .
2. Move each star to the right of the new  $n$  one descent to the left.
3. If  $i$  came from  $U^+$ , star the rightmost descent.
4. Relabel as before, stopping at the label  $i$  if  $i$  came from  $B$  and  $i - 1$  if  $i$  came from  $U^+$ .

When we have used each element of  $U^+ \cup B$ , the result is  $\phi_{\alpha,k,\ell}^{\text{maj}}((\sigma, S), U, B)$ .

For example, let us again consider  $\alpha = \{1, 2, 2, 3\}$ ,  $k = 6$ , and  $\ell = 5$  with  $(\sigma, S) = 3\ 1\ 2\ 2\ 3_*1 \in \mathcal{OP}_{\alpha^-, 5}$ ,  $U = \{2, 0\}$ , and  $B = \{3\}$ . Then  $U^+ \cup B = \{3, 3, 1\}$ . We label  $(\sigma, S)$  as follows.

$${}_2 3_1 1_3 2_4 2_5 3_* 1_0.$$

We insert a 4 at the position labeled 3 and then move all stars to the right of that position one spot to their left. Since we took 3 from  $B$ , we do not star the rightmost descent, resulting in  $3\ 1\ 4_*\ 2\ 2\ 3\ 1$ . We re-label to obtain the following.

$${}_3 3_2 1\ 4_* 2\ 2\ 3_1 1_0$$

Again we choose the position labeled 3. This time we star the rightmost descent after shifting stars because 3 came from  $U^+$ , yielding  $4\ 3_*\ 1\ 4\ 2\ 2\ 3_*\ 1$ . Finally we label this element

$$4_2 3_* 1\ 4_1 2\ 2\ 3_* 1_0.$$

We insert a 4 at the position labeled 1, shift stars, and star the rightmost descent to get  $4\ 3_*\ 1\ 4\ 4_*\ 2\ 2\ 3_*\ 1$ . We check that this new permutation has the desired major index.

$$\begin{aligned} 10 &= \text{maj}(4\ 3_*\ 1\ 4\ 4_*\ 2\ 2\ 3_*\ 1) = 1 + 1 + 3 + 5 \\ &= \text{maj}(3\ 1\ 2\ 2\ 3_*\ 1) + \sum_{u \in U} u + \sum_{b \in B} b \\ &= (1 + 4) + (2 + 0) + 3. \end{aligned}$$

Now we prove that this process satisfies the necessary properties.

**Lemma 2.4.2.1** (Insertion Lemma for maj). *For any composition  $\alpha$  of length  $n$  and positive integers  $\ell \leq k$ ,  $\phi_{\alpha, k, \ell}^{\text{maj}}$  is well-defined and injective. Furthermore, for any  $\pi \in \mathcal{OP}_{\alpha^-, \ell}$ ,  $U \in \binom{[0, \ell-1]}{\alpha_n - k + \ell}$ , and  $B \in \binom{[0, \ell]}{k - \ell}$ ,*

$$\text{maj}\left(\phi_{\alpha, k, \ell}^{\text{maj}}(\pi, U, B)\right) = \text{maj}(\pi) + \sum_{u \in U} u + \sum_{b \in B} b.$$

*Proof.* To show that  $\phi_{\alpha, k, \ell}^{\text{maj}}$  is injective, we describe its inverse. Given some descent-starred multiset permutation  $(\tau, T)$ , we first check if the rightmost descent of  $(\tau, T)$  is starred. If it is, we remove the star and prepare to add an element to  $U$ . Otherwise, we prepare to add an element to  $B$ . We scan  $\tau$  for the rightmost  $n$  which is either at the right end of  $\tau$  or between two entries such that the entry to the left of  $n$  is greater than the entry to the right of  $n$ . If there is no such  $n$ , we choose the leftmost  $n$  in  $\tau$ . We move all stars that are weakly to the right of this  $n$ 's position one descent to their right and

then remove  $n$ . Say that, at this point, we have decreased the original major index of  $(\tau, T)$  by  $i$ . We add  $i$  to either  $U$  or  $B$ , as decided above. Then we repeat this process until we have removed all  $n$ 's. It is important to note that the inverse does not depend on knowledge of  $\ell$ ; therefore, each  $\rho \in \mathcal{OP}_{\alpha, k}$  is in the image of  $\phi_{\alpha, k, \ell}^{\text{maj}}$  for a unique value of  $\ell$ .

In the remainder of the proof, we show that the map cooperates with the statistic  $\text{maj}$  as proposed in the lemma. We consider the labeling of the descent-starred multiset permutation  $(\sigma, S)$  equivalent to  $\pi$  at any step during the insertion process. Say  $i$  is currently the largest element of  $U^+ \cup B$ .

Assume first that the position labeled  $i$  is a descent. We use  $d$  to denote the number of starred descents to the right of this position. By the insertion method discussed in Subsection 2.1.3, inserting  $n$  into the position labeled  $i$  increases  $\text{maj}(\sigma)$  by  $i + d$ . Furthermore, we have not created a new descent, so the number of descents weakly to the right of any starred position has remained the same. Therefore, by the alternate definition of  $\text{maj}(\pi)$  in (2.3), we have increased  $\text{maj}(\pi)$  by  $i + d$  after Step 1.

For Step 2, we move all stars to the right of the position labeled  $i$  one descent to their left. Since position  $i$  contains an unstarred descent, this is always possible. Furthermore, each of these  $d$  stars have picked up an additional descent that is weakly to their right. Using (2.3) again, we see that the change in  $\text{maj}(\pi)$  after Step 2 is  $i + d - d = i$ .

Finally, we need to consider if  $i$  came from  $U^+$  or  $B$ . If  $i$  came from  $U^+$ , we star the rightmost descent. This subtracts 1 from  $\text{maj}(\pi)$ . In either case, we have increased the major index of  $\pi$  by the amount equal to the element from  $U$  or  $B$  corresponding to  $i$ .

By the insertion process from Subsection 2.1.3, we can relabel the resulting descent-starred permutation and repeat the process as long as we bound the labels as described in Step 4. Then, by the same argument as we used above, the insertion process will modify the major index as described in the statement of the lemma.

Now we consider where the argument must change when the position labeled  $i$  is not a descent. We still use  $d$  to denote the number of starred descents to the right of position  $i$ , and we set  $c$  to be the number of starred descent to this position's left. Since every starred descent occurs before the position labeled  $i$  in the labeling order for

$\text{maj}(\sigma)$ , Step 1 increases  $\text{maj}(\pi)$  by  $i + c + d$ . For Step 2, the position labeled  $i$  still contains an unstarred descent, so we can still move the stars as described. As before, this means that each of the  $d$  stars to the right of the position labeled  $i$  adds a descent to its right, contributing  $-d$  to  $\text{maj}(\pi)$ . Furthermore, inserting  $n$  at a non-descent creates a new descent, so each of the  $c$  stars to the left of the position labeled  $i$  has added a descent to its right, contributing  $-c$  to  $\text{maj}(\pi)$ . Therefore the total increase of  $\text{maj}(\pi)$  is  $i + c + d - c - d = i$ . Steps 3 and 4 do not depend on whether we are inserting at a descent or a non-descent.  $\square$

These two insertion maps work together to provide a bijection  $\psi_{\alpha,k} : \mathcal{OP}_{\alpha,k} \rightarrow \mathcal{OP}_{\alpha,k}$  that takes inversion number to major index. The bijection is described recursively as follows.

1. Given an ordered multiset partition  $\rho \in \mathcal{OP}_{\alpha,k}$ , choose  $\ell$  such that  $\rho$  has  $k - \ell$  singleton blocks containing  $n$ .
2. Set  $(\pi, U, B)$  to be the inverse of  $\rho$  under  $\phi_{\alpha,k,\ell}^{\text{inv}}$ .
3. Recursively send  $\pi$  to  $\pi' = \psi_{\alpha^-, \ell}(\pi)$ .
4. Set  $\psi_{\alpha,k}(\rho) = \phi_{\alpha,k,\ell}^{\text{maj}}(\pi', U, B)$ .

Finally, in order to begin the recursion, we declare that  $\psi_{1^m, m}$  is the identity map. We work through an example of this bijection in Figure 2.8.

**Proposition 2.4.2.1.** *For any composition  $\alpha$  and positive integer  $k$ ,  $\psi_{\alpha,k}$  is a bijection with the property*

$$\text{maj}(\psi_{\alpha,k}(\rho)) = \text{inv}(\rho)$$

for any  $\rho \in \mathcal{OP}_{\alpha,k}$ .

*Proof.* We will work by induction on  $n$ , the length of  $\alpha$ . If  $n = 1$ , Then the multiset is  $\{1^{\alpha_1}\}$  and  $k$  must be equal to  $\alpha_1$ . In this case  $\mathcal{OP}_{\alpha,k}$  only has 1 element, which has  $\alpha_1$  parts all equal to 1. This element clearly has  $\text{inv} = \text{maj} = 0$ . We defined  $\psi_{\alpha,k}$  so that it is the identity in this case, which clearly is a bijection and satisfies  $\text{maj}(\psi_{\alpha,k}(\pi)) = \text{inv}(\pi)$  for the unique  $\pi \in \mathcal{OP}_{\alpha,k}$ .

$\alpha$	$k$	$\pi$	$U$	$B$	$\psi_{A,k}(\pi)$
$\{3, 1, 2, 1\}$	3	134 1 3 12			$3_*2_*1\ 1\ 4\ 3\ 1$
$\{3, 1, 2\}$	2	13 1 3 12	$\{4\}$	$\emptyset$	$2_3_*2_*1_3\ 1_4\ 3_1\ 1_0$ $1_2_*1_2\ 1\ 3_*1_0$
$\{3, 1\}$	1	1 1 12	$\{2\}$	$\{1\}$	$1_2_*1_2\ 1_3\ 1_0$
$\{3\}$	0	1 1 1	$\{0\}$	$\emptyset$	$1_2\ 1_3\ 1_0$

**Figure 2.8:** An example of the map  $\psi_{A,k}(\pi)$ .

If  $n > 1$ , take any element  $\rho \in \mathcal{OP}_{\alpha,k}$ . We choose  $\ell, \pi, \pi', U$ , and  $B$  as instructed in the definition of  $\psi_{\alpha,k}$ . The images of  $\phi_{\alpha,k,\ell}^{\text{inv}}$  for  $\ell = 1$  to  $k$  partition  $\mathcal{OP}_{\alpha,k}$  into the subsets consisting of elements which have  $k - \ell$  singleton  $n$  blocks. (Since we assume each  $\alpha_i > 0$ , an element cannot consist entirely of singleton  $n$  blocks.) We also noted while proving Lemma 2.4.2.1 that each  $\rho \in \mathcal{OP}_{\alpha,k}$  is in the image of  $\phi_{\alpha,k,\ell}^{\text{maj}}$  for a unique value of  $\ell$ . Since each of these insertion maps is invertible,  $\psi_{\alpha,k}$  is a bijection.

Finally, we use Lemmas 2.4.1.1 and 2.4.2.1 along with the inductive hypothesis to compute

$$\begin{aligned}
\text{maj}(\psi_{\alpha,k}(\rho)) &= \text{maj}(\pi') + \sum_{u \in U} u + \sum_{b \in B} b \\
&= \text{inv}(\pi) + \sum_{u \in U} u + \sum_{b \in B} b \\
&= \text{inv}(\rho). \quad \square
\end{aligned}$$

We work through an example of the map  $\psi_{A,k}(\pi)$  in Figure 2.8. We repeatedly remove all of the largest elements (and their stars) from the starred permutation and recording the number of inversions lost in the  $U$  and  $B$  columns. Starred elements contribute to the  $U$  column and unstarred elements contribute to the  $B$  column. Once we have reached the final row, we use this information to build the  $\psi_{\alpha,k}((\sigma, S))$  column from bottom to top. We use the elements of  $U^+ \cup M$  to select the positions at which to insert new largest elements. This insertion follows the procedure laid out in the definition of  $\phi^{\text{maj}}$ .

There are a number of other consequences of our proof. For example, the *right-to-left minima* of a permutation  $\sigma$  are the entries  $\sigma_i$  such that, for all  $j > i$ ,  $\sigma_i < \sigma_j$ . In [RW15], Remmel and the author proved that the  $\alpha = 1^n$  case of  $\psi_{\alpha,k}$  preserves the right-to-left minima of  $\sigma$ , where  $(\sigma, S)$  is the descent-starred permutation representation of an ordered partition of  $\alpha$ . The same is true for general  $\psi_{\alpha,k}$ .

**Corollary 2.4.2.1.** *For any element of  $\pi \in \mathcal{OP}_{\alpha,k}$  considered as a descent-starred permutation  $(\sigma, S)$ , consider its image  $\psi_{\alpha,k}(\pi)$  as the descent-starred permutation  $(\tau, T)$ . Then  $\sigma$  and  $\tau$  have the same right-to-left minima. In particular,  $\sigma_n = \tau_n$ .*

The crux of the proof is that the insertion algorithms only change the last element of  $\sigma$  when 0 is an element of the multiset  $B$ .

### 2.4.3 Insertion for $\text{dinv}$

In order to prove that  $\text{dinv}$  is equidistributed with  $\text{inv}$  and  $\text{maj}$ , we define an insertion map for  $\text{dinv}$

$$\phi_{\alpha,k,\ell}^{\text{dinv}} : \mathcal{OP}_{\alpha^-, \ell} \times \binom{[0, \ell - 1]}{\alpha_n - k + \ell} \times \binom{[0, \ell]}{k - \ell} \rightarrow \mathcal{OP}_{\alpha,k}.$$

Given  $\pi \in \mathcal{OP}_{\alpha^-, \ell}$ , we will use two different labelings to insert the  $n$ 's into  $\pi$ . We label the positions between the blocks, as well as the positions at either end of  $\pi$ , with the labels  $0, 1, \dots, \ell$  from right to left. We will call these the *gap labels*.

We will label the  $\ell$  blocks of  $\pi$  with the labels  $0, 1, \dots, \ell - 1$ . Set  $h$  to be the maximum height of any element in  $\pi$ . We obtain the *block labels* of  $\pi$  by, beginning with the label 0, repeatedly

1. labeling the elements of  $\pi$  at height  $h$  from left to right with increasing labels, and
2. decrementing  $h$ .

We repeat until  $h < 0$ . For example, if  $\pi = 124|2|13|134|1$ , the block labels of  $\pi$  are  $0|3|2|1|4$ .

With these labels in hand, we define  $\phi_{\alpha,k,\ell}^{\text{dinv}}(\pi, U, B)$  by inserting an  $n$  into each block that receives a block label  $u \in U$  and into each gap that receives a gap label  $b \in B$ . As usual, the key is to prove that this map cooperate with the statistic  $\text{dinv}$ .



**Lemma 2.4.3.1** (Insertion for  $\text{dinv}$ ). *For any composition  $\alpha$  of length  $n$  and positive integers  $\ell \leq k$ ,  $\phi_{\alpha,k,\ell}^{\text{dinv}}$  is well-defined and injective. The image of  $\phi_{\alpha,k,\ell}^{\text{dinv}}$  is the ordered multiset partitions  $\pi \in \mathcal{OP}_{\alpha,k}$  with exactly  $k - \ell$  singleton blocks containing  $n$ . Furthermore, for any  $\pi \in \mathcal{OP}_{\alpha,\ell}$ ,  $U \in \binom{[0,\ell-1]}{\alpha_n - k + \ell}$ , and  $B \in \binom{[0,\ell]}{k - \ell}$ ,*

$$\text{dinv}(\phi_{\alpha,k,\ell}^{\text{dinv}}(\pi, U, B)) = \text{dinv}(\pi) + \sum_{u \in U} u + \sum_{b \in B} b.$$

*Proof.* The proof is essentially the same as that of Lemma 2.3.2.1. We just have to note that, since two  $n$ 's cannot form a diagonal inversion, we can insert  $n$ 's according to the  $u \in U$  and  $b \in B$  in any order and create the correct number of diagonal inversions.  $\square$

It follows from Lemma 2.4.3.1 that  $D_{\alpha,k}^{\text{dinv}}(q) = D_{\alpha,k}^{\text{inv}}(q) = D_{\alpha,k}^{\text{maj}}(q)$ . We can form a bijection between any pair of these statistics by using the definition of  $\psi_{\alpha,k}$  as a template. Furthermore, any such bijection preserves right-to-left minima.

## 2.4.4 The statistic $\text{minimaj}$

Data computed in Sage suggests that  $\text{minimaj}$  shares the distribution of  $\text{inv}$ ,  $\text{maj}$ , and  $\text{dinv}$ ; unfortunately, we cannot prove this with the techniques currently available to us. In fact, we conjecture that  $\text{minimaj}$  is equidistributed with  $\text{inv}$  in a particularly strong way. If we set the *shape* of an ordered multiset partition  $\pi \in \mathcal{OP}_{\alpha,k}$  to be

$$\text{shape}(\pi) = (|\pi^1|, |\pi^2|, \dots, |\pi^k|),$$

then we have the following conjecture.

**Conjecture 2.4.4.1.** *For any compositions  $\alpha, \beta$  of length  $n$  with  $|\alpha| = |\beta|$ ,*

$$\sum_{\substack{\pi \in \mathcal{OP}_{\alpha} \\ \text{shape}(\pi) = \beta}} q^{\text{inv}(\pi)} = \sum_{\substack{\pi \in \mathcal{OP}_{\alpha} \\ \text{shape}(\pi) = \beta}} q^{\text{minimaj}(\pi)}.$$

*This implies  $D_{\alpha,k}^{\text{minimaj}}(q) = D_{\alpha,k}^{\text{inv}}(q) = D_{\alpha,k}^{\text{maj}}(q) = D_{\alpha,k}^{\text{dinv}}(q)$ .*

Of the four statistics we have defined on ordered multiset partitions, data implies that  $(\text{inv}, \text{minimaj})$  is the only pair for which Conjecture 2.4.4.1 may hold.

### 2.4.5 The Mahonian distribution on $\mathcal{OP}_{\alpha,k}$

In this subsection, we describe the distribution shared by the statistics  $\text{inv}$ ,  $\text{maj}$ , and  $\text{dinv}$  on  $\mathcal{OP}_{\alpha,k}$ . Define the *Mahonian distribution* on  $\mathcal{OP}_{\alpha,k}$  to be the polynomial

$$D_{\alpha,k}(q) = D_{\alpha,k}^{\text{inv}}(q) = D_{\alpha,k}^{\text{maj}}(q) = D_{\alpha,k}^{\text{dinv}}(q).$$

We know from MacMahon's theorem that  $D_{\alpha,|\alpha|}(q) = \left[ \begin{smallmatrix} |\alpha| \\ \alpha_1, \dots, \alpha_n \end{smallmatrix} \right]_q$ . In general, we can only give a recursive description of  $D_{\alpha,k}(q)$ . Applying standard  $q$ -binomial identities to the insertion maps given above, we see

$$D_{\alpha,k}(q) = \sum_{\ell=1}^k q^{\binom{\alpha_n - k + \ell}{2}} \left[ \begin{smallmatrix} \ell \\ \alpha_n - k + \ell \end{smallmatrix} \right]_q \left[ \begin{smallmatrix} k \\ \ell \end{smallmatrix} \right]_q D_{\alpha^-, \ell}(q) \quad (2.31)$$

with initial condition

$$D_{(\alpha_1),k}(q) = \chi(k = \alpha_1). \quad (2.32)$$

We can cancel terms of (2.31) to obtain the identity

$$D_{\alpha,k}(q) = \sum_{\ell=1}^k q^{\binom{\alpha_n - k + \ell}{2}} \left[ \begin{smallmatrix} k \\ \alpha_n - k + \ell, k - \alpha_n, k - \ell \end{smallmatrix} \right]_q D_{\alpha^-, \ell}(q). \quad (2.33)$$

We can obtain another expression for this polynomial in the special case  $\alpha_1 = \dots = \alpha_n = a$ . Before we can state this expression, we must define a  $q$ -analog of the (generalized) Stirling numbers of the second kind. The  $q = 1$  case of these polynomials appear in [BPS03], Equations (20) and (21). We define these polynomials recursively by

$$S_{n,k}^{(a)}(q) = \sum_{i=1}^k q^{\binom{a-k+i}{2}} \left[ \begin{smallmatrix} i \\ a - k + i \end{smallmatrix} \right]_q \frac{[a]_q!}{[k-i]_q!} S_{n-1,i}^{(a)}(q) \quad (2.34)$$

$$S_{1,k}^{(a)}(q) = \chi(k = a). \quad (2.35)$$

Note that, at  $a = 1$ , the recursion simplifies to the  $q$ -Stirling numbers  $S_{n,k}(q)$ .

**Proposition 2.4.5.1.** *When  $\alpha = a^n$ ,*

$$D_{\alpha,k}(q) = \frac{[k]_q!}{([a]_q!)^n} S_{n,k}^{(a)}(q).$$

*When  $a = 1$ , this formula reduces to the formula*

$$D_{n,k}(q) = [k]_q! S_{n,k}(q).$$

*Proof.* We work by induction on  $n$ . When  $n = 1$ , the right-hand side of Proposition 2.4.5.1 equals

$$\frac{[k]_q!}{([a]_q!)^n} S_{1,k}^{(a)}(q) = \chi(k = a)$$

which is equal to the left-hand side by (2.32).

If  $n > 1$ , we use the induction hypothesis with the recursion (2.33) to compute

$$\begin{aligned} D_{\alpha,k}(q) &= \sum_{\ell=1}^k q^{\binom{a-k+\ell}{2}} \left[ a-k+\ell, k-a, k-\ell \right]_q \left( \frac{[\ell]_q!}{([a]_q!)^{n-1}} S_{n-1,\ell}^{(a)}(q) \right) \\ &= \frac{[k]_q!}{([a]_q!)^{n-1}} \sum_{\ell=1}^k q^{\binom{a-k+\ell}{2}} \frac{[\ell]_q!}{[a-k+\ell]_q! [k-a]_q! [k-\ell]_q!} S_{n-1,\ell}^{(a)}(q) \\ &= \frac{[k]_q!}{([a]_q!)^n} \sum_{\ell=1}^k q^{\binom{a-k+\ell}{2}} \left[ a-k+\ell \right]_q \frac{[a]_q!}{[k-\ell]_q!} S_{n-1,\ell}^{(a)}(q) \\ &= \frac{[k]_q!}{([a]_q!)^n} S_{n,k}^{(a)}(q) \end{aligned}$$

by (2.34) with  $i = \ell$ . □

It would be interesting to give a more combinatorial proof of Proposition 2.4.5.1, especially one that would shed light on why each of the three terms  $[k]_q!$ ,  $([a]_q!)^n$ , and  $S_{n,k}^{(a)}(q)$  appears.

## 2.4.6 Extending Macdonald polynomials

In this subsection, we apply our  $\text{inv}$  and  $\text{maj}$  statistics to the combinatorial definition of Macdonald polynomials for hook shapes, as given (for any shape) in [HHL05a]. This yields functions whose coefficients are four-variable polynomials instead of the usual two-variable polynomial coefficients. Our previous work allows us to prove that these polynomials are symmetric and to expand them into Schur functions.

For convenience, for any statistic  $\text{stat}$ , let

$$\text{stat}_{[a,b]}(\sigma) = \text{stat}(\sigma_a \sigma_{a+1} \dots \sigma_b).$$

We will also need two new statistics, one on permutations and one on descent-starred permutations:

$$\begin{aligned} \text{rlmaj}(\sigma) &= \sum_{i \in \text{Des}(\sigma)} (n - i) \\ \text{rlmaj}((\sigma, S)) &= \text{rlmaj}(\sigma) - \sum_{i \in S} |\text{Des}(\sigma) \cap [1, i]|. \end{aligned}$$

We set  $\tilde{H}_{n,m}(x; q, t, u, v)$  equal to

$$\begin{aligned} &\sum_{\sigma \in \{1, 2, \dots\}^n} q^{\text{inv}_{[m+1, n]}(\sigma)} t^{\text{maj}_{[1, m]}(\sigma)} x^\sigma \\ &\times \prod_{i \in \text{Des}(\sigma) \cap [m+1, n]} \left(1 + u/q^{\text{inv}_{[m+1, n]}^{\square, i}(\sigma)+1}\right) \prod_{j=1}^{|\text{Des}(\sigma) \cap [1, m]|} (1 + v/t^j). \end{aligned}$$

We will refer to these polynomials as *starred Macdonald polynomials of hook shape*. These polynomials can be thought of as a sum over all descent-starred multiset permutations  $(\sigma, S)$  in the Young diagram for the shape  $(n - m, 1^m)$  where we calculate our maj and inv statistics down the column and across the row, respectively. When  $u = v = 0$ , we obtain the (modified) Macdonald polynomial for the shape  $(n - m, 1^m)$ , as proven in [HHL05a]. Here is an example filling for  $n = 8$  and  $m = 3$ .

2				
5*				
3				
6*	1	7	4	8

The numbers followed by stars are the starred descents. The weight of this filling would be  $q^2 t$ , since the descent-starred permutation  $6_* 1 7 4 8$  has 2 inversions, both ending at the 4, and  $2 5_* 3 6$  has major index equal to 1.

Our main result in this section allows us to transfer many important properties of the Macdonald polynomials to the starred Macdonald polynomials of hook shape. The  $u = v = 0$  case of this result was originally proved in [Ste94]. The result requires some definitions on standard Young tableaux. The *descent set* of a standard Young tableau  $T$  (in French notation) is the set of all  $i$  such that  $i + 1$  is strictly north (and weakly west) of  $i$  in  $T$ . Then, for any standard Young tableau  $T$  with  $n$  entries,

$$\text{maj}(T) = \sum_{i \in \text{Des}(T)} i \quad \text{rlmaj}(T) = \sum_{i \in \text{Des}(T)} (n - i).$$

For example, here is a standard Young tableau with descent set  $\{2, 5\}$ .

3	4	6	
1	2	5	7

**Theorem 2.4.6.1.** *The starred Macdonald polynomials of hook shape are symmetric. Furthermore, for  $\lambda \vdash n$  the coefficient of the Schur function  $s_\lambda(x)$  in  $\tilde{H}_{n,m}(x; q, t, u, v)$  is equal to*

$$\sum_{T \in \text{SYT}(\lambda)} q^{\text{rlmaj}_{[m+1,n]}(T)} t^{\text{maj}_{[1,m]}(T)} \prod_{i=1}^{|\text{Des}(T) \cap [m+1,n]|} (1 + uq^{-i}) \prod_{j=1}^{|\text{Des}(T) \cap [1,m]|} (1 + vt^{-j}).$$

*Proof.* We begin with a descent-starred multiset permutation  $(\sigma, S)$ . Then we apply the bijection

$$\gamma = \text{complement} \circ \text{reverse} \circ \psi_{\beta, \ell} \circ \text{reverse} \circ \text{complement}$$

to the descent-starred multiset permutation  $(\sigma_{m+1} \dots \sigma_n, S \cap [m+1, n])$  for suitable  $\beta, \ell$ . Since preserves the rightmost letter,  $\gamma$  preserves  $\sigma_{k+1}$  and sends the inv of  $(\sigma_{m+1} \dots \sigma_n, S \cap [m+1, n])$  to the rlma $j$  of the resulting descent-starred multiset permutation. Hence,  $\tilde{H}_{n,m}$  equals

$$\sum_{\sigma \in \{1, 2, \dots, n\}^n} x^\sigma q^{\text{rlmaj}_{[k+1,n]}(\sigma)} t^{\text{maj}_{[1,k]}(\sigma)} \prod_{i=1}^{|\text{Des}(\sigma) \cap [k+1,n]|} (1 + u/q^i) \prod_{j=1}^{|\text{Des}(\sigma) \cap [1,k]|} (1 + v/t^j). \quad (2.36)$$

For any composition  $\alpha$ , the coefficient of  $x^\alpha$  in this expression is just the sum over  $\sigma$  that are permutations of the multiset  $\{1^{\alpha_1}, 2^{\alpha_2}, \dots\}$ . We would like to show that the coefficients of  $x^\alpha$  and  $x^{\alpha^{(r)}}$  are equal, where  $\alpha^{(r)}$  is obtained from  $\alpha$  by switching  $\alpha_r$  and  $\alpha_{r+1}$ . To do this, we apply a procedure known as *r-pairing* to the permutation  $\tau$ . We illustrate *r-pairing* via the example in Figure 2.9. To perform *r-pairing* on a sequence, we begin by temporarily ignoring all entries not equal to  $r$  or  $r+1$ . Then we pair off adjacent occurrences of  $r+1$  and  $r$ , ignoring previously paired entries and iterating this pairing procedure. When there are no more such pairs, we replace each un-paired occurrence of  $r$  with an  $r+1$  and vice versa. Finally, we re-insert the entries we had temporarily removed in their initial positions. We provide an example above with  $r = 2$ .

24231243331324123321  
 2 23 2 333 32 2332  
 2 2     333         23  
 2 2     33           3  
 3 3     22           2  
 3 3     223         22  
 3 33 2 223 32 2232  
 34331242231324122321

**Figure 2.9:** An example of  $r$ -pairing with  $r = 2$ .

For our purposes, it is enough to know that  $r$ -pairing replaces  $\sigma$  with a permutation of the multiset  $\{1^{\alpha_1}, \dots, r^{\alpha_{r+1}}, (r+1)^{\alpha_r}, \dots\}$ , and that this permutation has the same descent set as  $\sigma$ . Therefore  $r$ -pairing does not alter any of the expressions in (2.36), so  $\tilde{H}_{n,m}$  is symmetric.

Furthermore, if we apply the Robinson-Schensted-Knuth correspondence ([Sta99]) to each permutation  $\sigma$  involved in the coefficient of  $x^\lambda$  for a partition  $\lambda$ , we see that the coefficient of  $x^\lambda$  in  $\tilde{H}_{n,m}$  is

$$\begin{aligned}
 & K_{\lambda, (n-m, 1^m)} \sum_{T \in \text{SYT}(\lambda)} q^{\text{rlmaj}_{[m+1, n]}(T)} t^{\text{maj}_{[1, m]}(T)} \\
 & \times \prod_{i=1}^{|\text{Des}(T) \cap [m+1, n]|} (1 + u/q^i) \prod_{j=1}^{|\text{Des}(T) \cap [1, m]|} (1 + v/t^j).
 \end{aligned}$$

Here  $K_{\lambda, (n-m, 1^m)}$  is the Kostka number. Translating from the coefficient of  $x^\lambda$  to the coefficient of  $s_\lambda(x)$  exactly consists of removing this Kostka number, so the theorem follows.  $\square$

One consequence of Theorem 2.4.6.1 is the identity

$$\tilde{H}_{n,m}(x; q, t, u, v) = \tilde{H}_{n, n-m}(x; t, q, v, u). \tag{2.37}$$

This parallels the well-known identity

$$\tilde{H}_\mu(x; q, t) = \tilde{H}_{\mu'}(x; t, q) \quad (2.38)$$

for Macdonald polynomials. For Macdonald polynomials of hook shape, there is a direct bijective proof of (2.38). We would like to find a similar bijective proof of (2.37).

In future work, we hope to define and explore starred Macdonald polynomials for non-hook shapes. The full combinatorial formulation of Macdonald polynomials in [HHL05a] essentially allows one to either star horizontally or vertically, giving analogs of Macdonald polynomials with three-variable polynomials for coefficients. At this point, we have been unable to define a four-variable generalization in the non-hook case that retains desirable properties such as symmetry. It would also be interesting to develop a connection between these polynomials and some sort of generalization of Garsia-Haiman modules.

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# Chapter 3

## Generalized Tesler matrices and virtual Hilbert series

In this chapter, we generalize previous definitions of Tesler matrices to allow negative matrix entries and hook sums. Our main result is an algebraic interpretation of a certain weighted sum over these matrices. Our interpretation uses *virtual Hilbert series*, a new class of symmetric function specializations which are defined by their values on Macdonald polynomials. As a result of this interpretation, we obtain a Tesler matrix expression for the Hall inner product  $\langle \Delta_f e_n, p_{1^n} \rangle$ . We use our Tesler matrix expression, along with various facts about Tesler matrices, to provide simple formulas for  $\langle \Delta_{e_1} e_n, p_{1^n} \rangle$  and  $\langle \Delta_{e_k} e_n, p_{1^n} \rangle|_{t=0}$  involving  $q, t$ -binomial coefficients and ordered set partitions, respectively. This allows us to conclude the proofs of the  $q = 0$  and  $t = 0$  cases of the Rise Version of the Delta Conjectures and the  $t = 0$  case of the Valley Version in the Hilbert setting, i.e. after taking scalar products with  $p_{1^n}$ .

### 3.1 Introduction

Given a vector  $\alpha \in \mathbb{Z}^n$ , we define the *Tesler matrices with hook sums*  $\alpha$  to be the set of all  $n \times n$  upper triangular matrices  $U$  with entries in  $\mathbb{Z}$  such that

1.  $U$  has no zero rows,
2. each row of  $U$  is either entirely non-negative or entirely non-positive, and



3. the  $k$ th hook sum of  $U$ , defined by

$$(u_{k,k} + u_{k,k+1} + \dots + u_{k,n}) - (u_{k,1} + u_{k,2} + \dots + u_{k,k-1}),$$

equals  $\alpha_k$  for every  $1 \leq k \leq n$ .

We will sometimes refer to a matrix that satisfies condition 1 as *essential* and a matrix that satisfies condition 2 as *signed*. Since previous work on Tesler matrices primarily addresses matrices with positive hook sums, and conditions 1 and 2 are trivial in that setting, our definition generalizes previous definitions of Tesler matrices. We denote the set of Tesler matrices with hook sums  $\alpha$  by  $\mathcal{T}(\alpha)$ .

The cases  $\alpha = (1, 1, \dots, 1)$  and  $\alpha = (1, m, \dots, m)$  for any positive integer  $m$  are studied in [Hag11], where they are used to give an expression for the Hilbert series of the (generalized) module of diagonal harmonics. More values of  $\alpha$  have appeared in the study of Hall-Littlewood polynomials [AGR<sup>+</sup>12], Macdonald polynomial operators [GHX14], and flow polytopes [MMR14]. It would be particularly interesting to see if the polytope approach in [MMR14] can be extended to our (essential, signed) Tesler matrices.

We set the *weight* of an  $n \times n$  Tesler matrix  $U$  to be

$$\text{wt}(U; q, t) = (-1)^{\text{entries}^+(U) - \text{rows}^+(U)} M^{\text{nonzero}(U) - n} \prod_{u_{i,j} \neq 0} [u_{i,j}]_{q,t}$$

where  $M = (1 - q)(1 - t)$ ,  $\text{entries}^+(U)$  is the number of positive entries in  $U$ ,  $\text{rows}^+(U)$  is the number of rows of  $U$  whose nonzero entries are all positive,  $\text{nonzero}(U)$  is the number of nonzero entries of  $U$ , and  $[k]_{q,t} = \frac{q^k - t^k}{q - t}$ , the usual  $q, t$ -analogue of an integer  $k$ . Since  $U$  is essential, the exponent of  $M$  is nonnegative and  $\text{wt}(U; q, t) \in \mathbb{Z}[q, t, 1/q, 1/t]$ . When  $U$  has no negative entries, this weight function is equal to the weight function defined in [Hag11]. It is also worth noticing that the weight of a Tesler matrix is independent of  $\alpha$ . We define the *Tesler polynomial with hook sums*  $\alpha$  to be

$$\text{Tes}(\alpha; q, t) = \sum_{U \in \mathcal{T}(\alpha)} \text{wt}(U; q, t).$$

In [Hag11], Haglund showed that  $\text{Tes}(1^n; q, t)$  is equal to the Hilbert series of the module of diagonal harmonics, which can also be written in terms of Macdonald polynomial

operators as  $\langle \nabla e_n, p_{1^n} \rangle$  or  $\langle \Delta_{e_n} e_n, p_{1^n} \rangle$ . [GHX14] contains an algebraic interpretation for  $\text{Tes}(\alpha; q, t)$  for any  $\alpha$  with positive integer entries. We summarize these results, along with the necessary notation, in Section 3.2.

In Section 3.3, we develop an algebraic interpretation for  $\text{Tes}(\alpha; q, t)$  for any  $\alpha \in \mathbb{Z}^n$  in terms of new symmetric function specializations which we call *virtual Hilbert series*. Our interpretation is equivalent to the interpretation in [GHX14] for positive hook sums; in this sense, the Tesler matrix definition we have used here is the natural extension of previous definitions. These specializations generalize the map that sends a symmetric function  $f$  that is homogeneous of degree  $n$  to its inner product with  $p_{1^n}$ . In the case that  $f$  is the Frobenius image of an  $\mathfrak{S}_n$ -module, this inner product extracts the module's Hilbert series. With this in mind, for *any* symmetric function  $f$  that is homogenous of degree  $n$ , we will often use the notation

$$\text{Hilb } f = \langle f, p_{1^n} \rangle.$$

In Section 3.4, we show that certain sums of virtual Hilbert series appear in the study of diagonal harmonics, especially in connection with the Macdonald polynomial operators  $\Delta_f$  and  $\Delta'_f$ . We use the algebraic interpretation of Tesler polynomials from Section 3.3 to produce a number of new results about these operators.

## 3.2 Background

First, we fix some notation. We will use  $\underline{\Lambda}$  to refer to the algebra of symmetric Laurent polynomials. Occasionally, we will use  $\mathbb{Z}[q, t]$  as a subscript to refer to the subalgebra of symmetric functions or symmetric Laurent polynomials consisting of the functions with coefficients in  $\mathbb{Z}[q, t]$ .

The only basis we will use for the symmetric Laurent polynomials is the monomial basis  $\{m_\rho\}$ , defined as the sum of all monomials whose exponents, when arranged in weakly decreasing order, equal the finite, weakly decreasing vector of nonzero integers  $\rho$ . We will refer to a finite vector of weakly decreasing nonzero integers as a *Laurent partition*. By definition, if all of  $\rho$ 's entries are positive then  $m_\rho$  is equal to the symmetric function  $m_\rho$ , so our notation is consistent.

The main result in [Hag11] is that

$$\text{Hilb } \nabla e_n = \text{Tes}(1^n; q, t).$$

For  $f \in \underline{\Lambda}$ , the operator  $\Delta'_f$  can be connected to  $\Delta_f$  via the identity

$$\Delta_{m_\rho} = \Delta'_{m_\rho} + \sum_{\xi} \Delta'_{m_\xi} \quad (3.1)$$

where the sum is over all  $\xi$  which can be obtained by removing one part from  $\rho$ . In particular,

$$\Delta_{e_k} = \Delta'_{e_k} + \Delta'_{e_{k-1}} \quad (3.2)$$

which, combined with the fact that  $\Delta'_f g = 0$  if the degree of  $f$  is greater than or equal to the degree of  $g$ , implies  $\nabla e_n = \Delta'_{e_{n-1}} e_n$ . We allow for  $f$  to be a symmetric Laurent polynomial because it provides a way to obtain negative powers of  $\nabla$  in terms of our operators via the identity

$$\nabla^{-1} = \Delta_{m_{(-1)^n}} \quad (3.3)$$

on  $\Lambda^{(n)}$ .

We will make use of the Pieri and skew Pieri coefficients of  $\tilde{H}_\mu$ . We define the skewing operator on  $\Lambda$  by insisting that

$$\langle f^\perp g, h \rangle = \langle g, fh \rangle$$

for any symmetric functions  $f, g$ , and  $h$ . Then the skew Pieri coefficients  $c_{\mu, \nu}$  are defined by

$$e_1^\perp \tilde{H}_\mu = \sum_{\nu \rightarrow \mu} c_{\mu, \nu} \tilde{H}_\nu$$

where the sum is over all partitions  $\nu$  that can be obtained by removing a single cell from  $\mu$ . In [GHX14], the authors use a constant term algorithm to provide a formula for  $\text{Tes}(\alpha; q, t)$  for any vector  $\alpha$  of positive integers in terms of the skewing operator. We will also use the Pieri coefficients<sup>1</sup>  $d_{\mu, \nu}$ , defined by

$$\frac{e_1}{M} \tilde{H}_\nu = \sum_{\mu \leftarrow \nu} d_{\mu, \nu} \tilde{H}_\mu$$

---

<sup>1</sup>Some authors do not divide by  $M$  in the definition of the Pieri coefficients.

where the sum is over all  $\mu$  that can be created by adding a cell to  $\nu$ .

Finally, we will employ the following standard notation for  $q$ ,  $t$ - and  $q$ -analogs of integers:

$$[n]_{q,t} = \frac{q^n - t^n}{q - t}$$

$$[n]_q = [n]_{q,1} = \frac{q^n - 1}{q - 1}.$$

Note that  $[n]_{q,t} \in \mathbb{N}[q, t]$  if  $n \geq 0$  and  $[n]_{q,t} \in \mathbb{Z}[1/q, 1/t]$  if  $n \leq 0$ . This implies  $\text{Tes}(\alpha; q, t) \in \mathbb{Z}[q, t, 1/q, 1/t]$  for any  $\alpha \in \mathbb{Z}^n$ . We will also use overlines to indicate the operation of replacing  $q$  by  $1/q$  and  $t$  by  $1/t$ ; for example,  $\overline{q + t + qt} = 1/q + 1/t + 1/(qt)$ .

### 3.3 Virtual Hilbert Series

In this section, we use new symmetric function specializations to derive an algebraic interpretation for  $\text{Tes}(\alpha; q, t)$  for any vector of integers  $\alpha$ . Our interpretation generalizes the formulas in [Hag11, GHX14].

#### 3.3.1 Definitions and connections to Tesler polynomials

Given any  $\alpha \in \mathbb{Z}^{n-1}$  and  $\mu \vdash n$ , we make the following recursive definition.

$$F_\mu^\alpha = \sum_{\nu \rightarrow \mu} c_{\mu,\nu} (T_\mu/T_\nu)^{\alpha_1} F_\nu^{(\alpha_2, \dots, \alpha_{n-1})}$$

$$F_{(1)}^0 = 1$$

It is worth noting that  $F_\mu^{0^{n-1}} = \text{Hilb } \tilde{H}_\mu$ , the Hilbert series of the Garsia-Haiman module associated with  $\tilde{H}_\mu$ , which is sometimes denoted  $F_\mu$ . As a result,  $F_\mu^\alpha$  can be thought of as a modification of this Hilbert series. The famous  $n!$  conjecture of Garsia and Haiman, proved in [Hai01], is simply the statement that setting  $q = t = 1$  in  $F_\mu$  yields  $n!$ . We list some open questions about the  $F_\mu^\alpha$  below.

- Computations in Sage suggest that  $F_\mu^\alpha \in \mathbb{Q}[q, t]$ . Is this true?
- For which  $\alpha, \mu$  is  $F_\mu^\alpha \in \mathbb{N}[q, t]$ ?

- [HHL05a] provides a combinatorial formula for  $F_\mu$ . Is there a similar combinatorial formula for  $F_\mu^\alpha$ ?

Now we define a map

$$\begin{aligned}\widetilde{\text{Hilb}}_\alpha : \Lambda^{(n)} &\rightarrow \mathbb{C}(q, t) \\ \tilde{H}_\mu &\mapsto F_\mu^\alpha\end{aligned}$$

We will sometimes refer to  $\widetilde{\text{Hilb}}_\alpha f$  as the *virtual Hilbert series* of  $f$  with respect to  $\alpha$ . We can justify this terminology by noting that  $F_\mu^{0^{n-1}} = F_\mu$  implies

$$\widetilde{\text{Hilb}}_{0^{n-1}} = \text{Hilb} \tag{3.4}$$

on  $\Lambda^{(n)}$ . Furthermore, we have

$$\widetilde{\text{Hilb}}_{k^{n-1}} = \text{Hilb} \nabla^k. \tag{3.5}$$

for any  $k \in \mathbb{Z}$  on  $\Lambda^{(n)}$ . The following result gives an algebraic interpretation for  $\text{Tes}(\alpha; q, t)$  for any  $\alpha \in \mathbb{Z}^{n-1}$ . We note that the right-hand side is equivalent to the right-hand side of I.9 in [GHX14] if each entry of  $\alpha$  is positive.

**Theorem 3.3.1.1.** *For any  $\alpha \in \mathbb{Z}^{n-1}$ , we have*

$$\text{Tes}(\alpha; q, t) = \frac{(-1)^{n-1}}{[n]_q [n]_t} \widetilde{\text{Hilb}}_\alpha p_n.$$

If we are willing to restrict our attention to vectors that begin with a 1, we can simplify the right-hand side of Theorem 3.3.1.1 slightly. We also obtain a direct generalization of the results in [Hag11].

**Corollary 3.3.1.1.** *For any  $\alpha \in \mathbb{Z}^{n-1}$ , we have*

$$\text{Tes}((1, \alpha); q, t) = \widetilde{\text{Hilb}}_\alpha e_n.$$

### 3.3.2 Proof of Theorem 3.3.1.1

We will need the following lemmas in order to prove Theorem 3.3.1.1. For  $\mu \leftarrow \nu$ , we abbreviate  $T_\mu/T_\nu$  by  $T$ .

**Lemma 3.3.2.1.** For any partition  $\nu$  and  $k \in \mathbb{Z}$ , we have

$$\sum_{\mu \leftarrow \nu} d_{\mu, \nu} T^k = \begin{cases} (-1)^{k-1} e_{k-1}[MB_\nu - 1]/M & k > 0 \\ 1/M & k = 0 \\ \frac{(-1)^{-k}}{qt} e_{-k}[MB_\nu - 1]/M & k < 0 \end{cases} \quad (3.6)$$

and, as a result,

$$\sum_{\mu \leftarrow \nu} d_{\mu, \nu} (1 - T) T^k = \begin{cases} (-1)^{k-1} e_k[MB_\nu]/M & k > 0 \\ 0 & k = 0 \\ \frac{(-1)^{-k}}{qt} e_{-k}[MB_\nu]/M & k < 0. \end{cases} \quad (3.7)$$

*Proof.* The  $k \geq 0$  case of (3.6) was first noticed by Zabrocki and proved in [GHXZ14]. The  $k \geq 0$  of (3.7) was shown to follow from (3.6) in [Hag11]. We begin by proving the  $k < 0$  case of (3.6), which follows from the  $k \geq 0$  case of (3.6) due to the following argument of Garsia (personal communication, 2015).

First, we need to relate  $d_{\mu, \nu}$  to  $\overline{d_{\mu, \nu}}$ . We will use the identity  $\tilde{H}_\mu = T_\mu \omega \overline{\tilde{H}_\mu}$  [Mac95]. By definition, we have

$$\sum_{\mu \leftarrow \nu} d_{\mu, \nu} \tilde{H}_\mu = \frac{e_1}{M} \tilde{H}_\nu \quad (3.8)$$

$$= \frac{e_1}{M} T_\nu \omega \overline{\tilde{H}_\nu} \quad (3.9)$$

$$= \frac{T_\nu}{qt} \omega \left( \frac{e_1}{M} \overline{\tilde{H}_\nu} \right) \quad (3.10)$$

$$= \frac{T_\nu}{qt} \omega \left( \sum_{\mu \leftarrow \nu} \overline{d_{\mu, \nu} \tilde{H}_\mu} \right) \quad (3.11)$$

$$= \frac{T_\nu}{qt} \sum_{\mu \leftarrow \nu} \overline{d_{\mu, \nu} \omega \tilde{H}_\mu} \quad (3.12)$$

$$= \sum_{\mu \leftarrow \nu} \overline{d_{\mu, \nu}} \frac{1}{qtT} \tilde{H}_\mu \quad (3.13)$$

which implies  $d_{\mu, \nu} = \frac{1}{qtT} \overline{d_{\mu, \nu}}$ . Now, for  $k < 0$ , we have

$$\sum_{\mu, \nu} d_{\mu, \nu} T^k = \frac{1}{qt} \sum_{\mu, \nu} \overline{d_{\mu, \nu} T^{k-1}} \quad (3.14)$$

$$= \frac{1}{qt} \sum_{\mu \leftarrow \nu} \overline{d_{\mu, \nu} T^{-k+1}} \quad (3.15)$$

$$= \frac{(-1)^{-k}}{qt} e_{-k}[MB_\nu - 1]/M. \quad (3.16)$$

This proves (3.6). The same plethystic computation used to derive Lemma 1 from (13) in [Hag11] can be used to prove (3.7).  $\square$

**Lemma 3.3.2.2** (Lemma 2 in [Hag11]). *For any positive integer  $k$ ,*

$$(-1)^{k-1} e_k[M]/M = [k]_{q,t}.$$

Our proof of Theorem 3.3.1.1 will closely follow the main proof in [Hag11]. First, we note that [Mac95]

$$\frac{(-1)^{n-1}}{[n]_q[n]_t} p_n = \sum_{\mu \vdash n} \frac{M\Pi_\mu}{w_\mu} \tilde{H}_\mu \quad (3.17)$$

where

$$\begin{aligned} \Pi_\mu &= \prod_{\substack{c \in \mu \\ c \neq (0,0)}} (1 - q^{a'(c)} t^{\ell'(c)}) \\ w_\mu &= \prod_{c \in \mu} (q^{a(c)} - t^{\ell(c)+1})(t^{\ell(c)} - q^{a(c)+1}). \end{aligned}$$

Thus, the right-hand side of Theorem 3.3.1.1 equals

$$\widetilde{\text{Hilb}}_\alpha \left( \sum_{\mu \vdash n} \frac{M\Pi_\mu}{w_\mu} \tilde{H}_\mu \right) = \sum_{\mu \vdash n} \frac{M\Pi_\mu}{w_\mu} \sum_{\nu \rightarrow \mu} c_{\mu,\nu} T^{\alpha_1} F_\nu^{(\alpha_2, \dots, \alpha_{n-1})} \quad (3.18)$$

$$= \sum_{\nu \vdash n-1} M F_\nu^{(\alpha_2, \dots, \alpha_{n-1})} \sum_{\mu \leftarrow \nu} \frac{\Pi_\mu}{w_\mu} c_{\mu,\nu} T^{\alpha_1} \quad (3.19)$$

where we have used the definition of  $\widetilde{\text{Hilb}}_\alpha$  and switched the order of the sums. Using

$$\Pi_\mu = (1 - T)\Pi_\nu \quad (3.20)$$

$$\frac{c_{\mu,\nu}}{w_\mu} = \frac{d_{\mu,\nu}}{w_\nu} \quad (3.21)$$

from the definition of  $\Pi_\mu$  and from [GH03], respectively, (3.19) equals

$$\sum_{\nu \vdash n-1} \frac{M\Pi_\nu}{w_\nu} F_\nu^{(\alpha_2, \dots, \alpha_{n-1})} \sum_{\mu \leftarrow \nu} d_{\mu,\nu} (1 - T) T^{\alpha_1}. \quad (3.22)$$

Now we use Lemma 3.3.2.1 to simplify the inner sum. For the sake of compactness, let  $b_k = b_k(\nu)$  equal the right-hand side of Lemma (3.7) that corresponds to  $k \in \mathbb{Z}$  and  $a_k = (-1)^{k-1} e_k[M]/M$ . Then (3.22) equals

$$\sum_{\nu \vdash n-1} \frac{M \Pi_\nu}{w_\nu} F_\nu^{(\alpha_2, \dots, \alpha_{n-1})} b_{\alpha_1}. \quad (3.23)$$

We would like to iterate this argument. The only real difficulty comes from  $b_{\alpha_1}$ . In particular, we need to know how to simplify expressions of the form  $b_k|_{B_\nu \mapsto B_\nu + T}$ . From [Hag11], for  $k > 0$  we get

$$b_k|_{B_\nu \mapsto B_\nu + T} = b_k + T^k a_k - \sum_{j=1}^{k-1} M T^{k-j} a_{k-j} b_j. \quad (3.24)$$

From Lemmas 3.3.2.1 and 3.3.2.2, we have

$$\overline{b_k} = -q t b_{-k} \quad (3.25)$$

$$\overline{a_k} = -q t a_k. \quad (3.26)$$

We also have  $\overline{M} = \frac{M}{q t}$  by definition. Using these identities, for  $k > 0$  we compute

$$b_{-k}|_{B_\nu \mapsto B_\nu + T} = -\frac{\overline{b_k|_{B_\nu \mapsto B_\nu + T}}}{q t} \quad (3.27)$$

$$= -\frac{\overline{b_k} + T^{-k} \overline{a_k} - \sum_{j=1}^{k-1} \overline{M} T^{j-k} \overline{a_{k-j}} \overline{b_j}}{q t} \quad (3.28)$$

$$= b_{-k} + T^{-k} a_k + \sum_{j=1}^{k-j} M T^{j-k} a_{k-j} b_{-j}. \quad (3.29)$$

Iterating this procedure  $r$  times, we obtain an expression for the right-hand side of Theorem 3.3.1.1 of the form

$$\sum_{\nu \vdash n-r+1} \frac{M \Pi_\nu}{w_\nu} F_\nu^{(\alpha_{r+1}, \dots, \alpha_{n-1})} A_r^\alpha \quad (3.30)$$

where  $A_r^\alpha$  is some expression in the  $a_k$ 's and  $b_k$ 's. Moreover, we can compute  $A_{r+1}^\alpha$  from  $A_r^\alpha$  by the following recursive procedure.

1. Replace the  $b_k$ 's in  $A_r^\alpha$  with

$$\begin{aligned} & b_k + T^k a_k - \sum_{j=1}^{k-1} M T^{k-j} a_{k-j} b_j && \text{if } k > 0 \\ & b_k + T^k a_{-k} + \sum_{j=1}^{-k-j} M T^{-k+j} a_{-k-j} b_{-j} && \text{if } k < 0. \end{aligned}$$



2. Expand to form a Laurent polynomial in  $T$ , say  $\sum_j \gamma_j T^j$ .
3. Replace each  $T^j$  with  $b_{j+\alpha_{r+1}}$ .

At  $r = n$ , we obtain

$$\frac{(-1)^{n-1}}{[n]_q[n]_t} \widetilde{\text{Hilb}}_{\alpha p_n} = \sum_{\nu \vdash 1} \frac{M \Pi_\nu}{w_\nu} A_n^\alpha = A_n^\alpha \quad (3.31)$$

since  $\Pi_{(1)} = 1$  and  $w_{(1)} = M$ . Our next goal is to prove the following expression for  $A_n^\alpha$  in terms of Tesler matrices

**Lemma 3.3.2.3.**

$$A_n^\alpha = \sum_{U \in \mathcal{T}(\alpha)} (-1)^{\text{entries}^+(U) - \text{rows}^+(U)} M^{\text{nonzero}(U) - n} \prod_{u_{i,i} \neq 0} b_{u_{i,i}} \prod_{u_{i,j} \neq 0: i \neq j} a_{u_{i,j}}$$

where we have set  $\nu = (1)$ .

*Proof.* We will prove this claim by induction on  $n = \ell(\alpha) + 1$ . The crux of the induction step is noticing that there is a recursion on Tesler matrices. Given a Tesler matrix  $U \in \mathcal{T}((\alpha_1, \dots, \alpha_{p-1}))$ , we create a Tesler matrix  $V \in \mathcal{T}((\alpha_1, \dots, \alpha_p))$  as follows:

1. For each row  $i$  in  $U$ , we "move" some of the diagonal entry  $u_{i,i}$  to the far right to create  $v_{i,p}$ .
2. To create row  $p$ , we choose  $v_{p,p}$  such that the  $p$ th hook sum of  $V$  is  $\alpha_p$ .

Now we check that this recursion matches the recursive procedure for generating  $A_n^\alpha$ . We should think of each  $b_k$  as representing some integer  $k$  on the diagonal in  $U$  and each  $a_k$  as representing an off-diagonal entry  $k$ . Then Step 1 of the procedure for generating  $A_n^\alpha$  corresponds to moving some part of each diagonal entry into the new rightmost column. The power of  $T$  tracks the entries in this new rightmost column. Note that a new  $M$  appears each time we increase the number of nonzero entries in the matrix and a new  $-1$  appears each time we create a new positive entry. Step 3 of the procedure corresponds to choosing the new bottom right entry such that the final hook sum is correct.  $\square$

Finally, we note that, for  $\nu = (1)$ ,

$$a_{|k|} = b_k = [k]_{q,t} \quad (3.32)$$

for any integer  $k$  by Lemma 3.3.2.2. This fact, along with Lemma 3.3.2.3, concludes the proof of Theorem 3.3.1.1. Corollary 3.3.1.1 follows by essentially the same argument except we use the expansion

$$e_n = \sum_{\mu \vdash n} \frac{MB_\mu \Pi_\mu}{w_\mu} \tilde{H}_\mu. \quad (3.33)$$

instead of the expansion for  $p_n$ .

## 3.4 Applications to Delta Operators

The right-hand sides of Theorem 3.3.1.1 and Corollary 3.3.1.1 bear some similarity to symmetric function expressions popular in the study of diagonal harmonics. In this section, we explore these connections and use the connections to prove new results about the Macdonald polynomial operators  $\Delta_f$  and  $\Delta'_f$ .

### 3.4.1 From virtual Hilbert series to delta operators

Recall that we have defined an operator  $\Delta'_f$  on  $\Lambda^{(n)}$  by stating that it acts on the Macdonald polynomials by

$$\Delta'_f \tilde{H}_\mu = f[B_\mu - 1] \tilde{H}_\mu.$$

Although we will not be able to describe every virtual Hilbert series in terms of this operator, we do have the following result involving symmetric sums of virtual Hilbert series. We let  $\text{sort}$  be the map that takes  $\alpha \in \mathbb{Z}^n$ , removes the zeros from  $\alpha$ , and then sorts the remaining entries in weakly decreasing order.

**Theorem 3.4.1.1.** *For any Laurent partition  $\rho$  and positive integer  $n$ ,*

$$\text{Hilb } \Delta'_{m_\rho} = \sum_{\substack{\alpha \in \mathbb{Z}^{n-1} \\ \text{sort}(\alpha) = \rho}} \widetilde{\text{Hilb}}_\alpha \quad (3.34)$$

*as operators on  $\Lambda^{(n)}$ .*

*Proof.* We will show that these operators are equal by showing that their actions are equal on the Macdonald polynomial  $\tilde{H}_\mu$  for any  $\mu \vdash n$ . The right-hand side of the statement in the theorem equals

$$\sum_{\substack{\alpha \in \mathbb{Z}^{n-1} \\ \text{sort}(\alpha) = \rho}} F_\mu^\alpha. \quad (3.35)$$

Now we iterate through the recursive definition of  $F_\mu^\alpha$  in order to obtain an alternative definition. Rather than just considering  $\nu \rightarrow \mu$ , we can consider all the saturated chains in Young's lattice from  $\emptyset$  to  $\mu$ . These are in bijection with the standard Young tableaux of shape  $\mu$ , denoted  $\text{SYT}(\mu)$ . Let  $c_S$  equal the product of all  $c_{\mu, \nu}$ 's that we encounter on the saturated chain from  $\emptyset$  to  $\mu$  associated with a given  $S \in \text{SYT}(\mu)$ . Given a cell  $d$  in the Young diagram of  $\mu$ , let  $S(d)$  denote the entry in cell  $d$  in  $S$ . Then (3.35) equals

$$\sum_{\substack{\alpha \in \mathbb{Z}^{n-1} \\ \text{sort}(\alpha) = \rho}} \sum_{S \in \text{SYT}(\mu)} c_S \prod_{\substack{d \in \mu \\ d \neq (0,0)}} \left( q^{a'(d)} t^{\ell'(d)} \right)^{\alpha_{n+1-S(d)}} \quad (3.36)$$

$$= \sum_{S \in \text{SYT}(\mu)} c_S \sum_{\substack{\alpha \in \mathbb{Z}^{n-1} \\ \text{sort}(\alpha) = \rho}} \prod_{\substack{d \in \mu \\ d \neq (0,0)}} \left( q^{a'(d)} t^{\ell'(d)} \right)^{\alpha_{n+1-S(d)}} \quad (3.37)$$

$$= \sum_{S \in \text{SYT}(\mu)} c_S m_\rho[B_\mu - 1] \quad (3.38)$$

$$= F_\mu m_\rho[B_\mu - 1] \quad (3.39)$$

$$= \text{Hilb } \Delta'_{m_\rho} \tilde{H}_\mu. \quad \square$$

Applying Theorem 3.4.1.1 to the Tesler polynomial expressions obtained in Section 3.3, we obtain the following identities.

**Corollary 3.4.1.1.**

$$\frac{(-1)^{n-1}}{[n]_q [n]_t} \text{Hilb } \Delta'_{m_\rho} p_n = \sum_{\alpha: \text{sort}(\alpha) = \rho} \text{Tes}(\alpha; q, t) \quad (3.40)$$

$$\text{Hilb } \Delta'_{m_\rho} e_n = \sum_{\alpha: \text{sort}(\alpha) = \rho} \text{Tes}((1, \alpha); q, t) \quad (3.41)$$

As a result, both left-hand sides are in  $\mathbb{Z}[q, t, 1/q, 1/t]$ . Furthermore, by the linearity in

the subscript of  $\Delta'_f$  we have

$$\frac{(-1)^{n-1}}{[n]_q[n]_t} \text{Hilb } \Delta'_f p_n, \text{ Hilb } \Delta'_f e_n \in \mathbb{Z}[q, t, 1/q, 1/t] \quad (3.42)$$

$$\frac{(-1)^{n-1}}{[n]_q[n]_t} \text{Hilb } \Delta'_g p_n, \text{ Hilb } \Delta'_g e_n \in \mathbb{Z}[q, t] \quad (3.43)$$

for any  $f \in \underline{\Lambda}_{\mathbb{Z}[q,t]}$ ,  $g \in \Lambda_{\mathbb{Z}[q,t]}$ . Finally, by (3.1) we could replace  $\Delta'$  by  $\Delta$  in (3.42), (3.43).

Corollary 3.4.1.1 can be thought of as a more concrete version of a special case of Theorem 1.3 in [BGHT99], in which the authors showed that  $\Delta_f \Lambda_{\mathbb{Z}[q,t]} \subseteq \Lambda_{\mathbb{Z}[q,t]}$  for any  $f \in \Lambda_{\mathbb{Z}[q,t]}$ . Corollary 3.4.1.1 provides the first direct formulas for Hilbert series of expressions of this type.

It may be of interest to the reader to use Corollary 3.4.1.1 in order to explicitly compute some  $\text{Hilb } \Delta_f e_n$ . Rather than state the exact analog of (3.41) for this case, we mention that the following process accomplishes this task.

1. Expand  $f$  into variables  $x_1, x_2, \dots, x_{n-1}, 1$ .
2. Replace each monomial  $x^\alpha$  in this expansion with  $\text{Tes}((1, \alpha); q, t)$ .

As an example, we compute  $\text{Hilb } \Delta_{s_{3,2,1}} e_3$  using

$$s_{3,2,1}(x_1, x_2, 1) = x_1^3 x_2^2 + x_1^2 x_2^3 + x_1^3 x_2 + 2x_1^2 x_2^2 + x_1 x_2^3 + x_1^2 x_2 + x_1 x_2^2. \quad (3.44)$$

Replacing each monomial with its associated Tesler polynomial, we get

$$\begin{aligned} \text{Hilb } \Delta_{s_{3,2,1}} e_3 &= \text{Tes}((1, 3, 2); q, t) + \text{Tes}((1, 2, 3); q, t) + \text{Tes}((1, 3, 1); q, t) \\ &\quad + 2 \text{Tes}((1, 2, 2); q, t) + \text{Tes}((1, 1, 3); q, t) \\ &\quad + \text{Tes}((1, 2, 1); q, t) + \text{Tes}((1, 1, 2); q, t). \end{aligned} \quad (3.45)$$

We have not explored how this method compares to current methods for computing  $\text{Hilb } \Delta_f e_n$  from a computational perspective.

### 3.4.2 Positive formulas

In this subsection, we use Corollary 3.4.1.1 to obtain formulas for  $\text{Hilb } \Delta_{e_1} e_n$ ,  $\text{Hilb } \frac{(-1)^{n-1}}{[n]_q[n]_t} \text{Hilb } \Delta_{e_2} p_n$ , and  $\text{Hilb } \Delta_{m-1} e_n$ . Each formula shows that the Hilbert series

of the given symmetric function is positive with respect to some set of variables. The operator  $\Delta_{e_1}$  can be thought of as a translation of Macdonald's original  $E$  operator in [Mac95] to the context of the modified Macdonald polynomials  $\tilde{H}_\mu$ .

**Corollary 3.4.2.1.**

$$\text{Hilb } \Delta_{e_1} e_n = \sum_{k=1}^n \binom{n}{k} [k]_{q,t} \quad (3.46)$$

$$\frac{(-1)^{n-1}}{[n]_q [n]_t} \text{Hilb } \Delta_{e_2} p_n = \sum_{k=1}^{n-1} \binom{n-1}{k} [k]_{q,t}. \quad (3.47)$$

In particular, the left-hand sides of both statements are in  $\mathbb{N}[q, t]$ .

**Corollary 3.4.2.2.**

$$\text{Hilb } \Delta_{m_{-1}} e_n = \left(1 - \frac{1}{qt}\right)^{n-1}.$$

As a result, the left-hand side is in  $\mathbb{N}\left[-\frac{1}{qt}\right]$ .

In [HRW], Haglund, Remmel, and the author use a reciprocity identity to obtain the full Schur expansion of  $\Delta_{e_1} e_n$ , implying the  $e_1$  statement in Corollary 3.4.2.1. In general, none of these symmetric functions are currently associated with modules, which means that direct formulas such as these are the only way to give positivity results at this point. In order to prove Corollaries 3.4.2.1 and 3.4.2.2, we need the following lemma.

**Lemma 3.4.2.1.** For any  $\alpha \in \mathbb{Z}^n$ ,

$$\text{Tes}((1, \alpha); q, t) = \text{Tes}(\alpha; q, t) + \sum_{i=1}^n \text{Tes}((\alpha_1, \dots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \dots, \alpha_n); q, t).$$

*Proof.* Consider a Tesler matrix  $U$  with hook sums  $(1, \alpha)$ . Its first row must consist of a single nonzero entry, which must be equal to 1. Say this entry occurs in column  $j$ , i.e.  $u_{1,j} = 1$ . If  $j = 1$ , removing the first row of  $U$  produces a Tesler matrix with hook sums  $\alpha$ , and this process produces a new matrix with hook sums  $(\alpha_1, \dots, \alpha_n)$ . If  $j > 1$ , we produce a Tesler matrix with hook sums  $(\alpha_1, \dots, \alpha_{j-2}, \alpha_{j-1} + 1, \alpha_j, \dots, \alpha_n)$ . Finally, we note that removing the first row does not change the weight of such a Tesler matrix.  $\square$

*Proof of Corollary 3.4.2.1.* By (3.2), the left-hand side (3.46) is equal to

$$\text{Tes}((1, 0^{n-1}); q, t) + \sum_{i=0}^{n-2} \text{Tes}((1, 0^i, 1, 0^{n-i-2}); q, t). \quad (3.48)$$

In order to simplify this expression, we use Lemma 3.4.2.1 along with the fact that  $\text{Tes}(\alpha; q, t) = 0$  if  $\alpha_1 = 0$ . As a result, (3.48) equals

$$= \text{Tes}((1); q, t) + \sum_{i=0}^{n-2} \text{Tes}((1, 1, 0^i); q, t) \quad (3.49)$$

$$= \text{Hilb } \Delta_{e_1} e_{n-1} + \text{Tes}((1, 1, 0^{n-2}); q, t). \quad (3.50)$$

Applying Lemma 3.4.2.1 again, we get

$$\begin{aligned} \text{Tes}((1, 1, 0^{n-2}); q, t) &= \text{Tes}((2, 0^{n-2}); q, t) + \text{Tes}((1, 0^{n-2}); q, t) \\ &\quad + \sum_{i=0}^{n-3} \text{Tes}((1, 0^i, 1, 0^{n-i-3}); q, t) \end{aligned} \quad (3.51)$$

$$= \text{Tes}((2, 0^{n-2}); q, t) + \text{Hilb } \Delta_{e_1} e_{n-1}. \quad (3.52)$$

Therefore

$$\text{Hilb } \Delta_{e_1} e_n = 2 \text{Hilb } \Delta_{e_1} e_{n-1} + \text{Tes}((2, 0^{n-2}); q, t). \quad (3.53)$$

We claim that

$$\text{Tes}((2, 0^k); q, t) = \sum_{i=1}^{k+2} \left( \binom{k+1}{i-1} - \binom{k+1}{i} \right) [i]_{q,t}. \quad (3.54)$$

If we can prove this, induction on (3.53) implies

$$\text{Hilb } \Delta_{e_1} e_n = 2 \sum_{k=1}^{n-1} \binom{n-1}{k} [k]_{q,t} + \sum_{k=1}^n \left( \binom{n-1}{k-1} - \binom{n-1}{k} \right) [k]_{q,t} \quad (3.55)$$

$$= \sum_{k=1}^n \left( 2 \binom{n-1}{k} + \binom{n-1}{k-1} - \binom{n-1}{k} \right) [k]_{q,t} \quad (3.56)$$

$$= \sum_{k=1}^n \binom{n}{k} [k]_{q,t} \quad (3.57)$$

concluding the proof. We consider what happens when we remove the first row of a Tesler matrix with hook sums  $(2, 0^{n-2})$ . The only way such a matrix can avoid having

a zero row is if its first row contains a 2 in column 2 or a 1 in column 2 and a 1 in some other column. By removing the first row and inducting, we get

$$\text{Tes}((2, 0^k); q, t) = [2]_{q,t} \text{Tes}((2, 0^{k-1}); q, t) - M \text{Hilb } \Delta_{e_1} e_k. \quad (3.58)$$

Since  $k < n$ , we can use induction to write this as

$$\text{Tes}((2, 0^k); q, t) = [2]_{q,t} \sum_{i=1}^{k+1} \left( \binom{k}{i-1} - \binom{k}{i} \right) [i]_{q,t} - M \sum_{i=1}^k \binom{k}{i} [i]_{q,t} \quad (3.59)$$

$$= \sum_{i=1}^{k+1} \left( (q+t) \left( \binom{k}{i-1} - \binom{k}{i} \right) - (1-q)(1-t) \binom{k}{i} \right) [i]_{q,t} \quad (3.60)$$

$$= \sum_{i=1}^{k+1} \left( (q+t) \binom{k}{i-1} - (1+qt) \binom{k}{i} \right) [i]_{q,t}. \quad (3.61)$$

Now all that remains to show is that, for any  $a, b \geq 0$ , the coefficient of  $q^a t^b$  in the previous statement is  $\binom{k+1}{a+b} - \binom{k+1}{a+b+1}$ . This coefficient equals

$$-\binom{k}{a+b+1} + 2\binom{k}{a+b-1} - \binom{k}{a+b-1} \quad (3.62)$$

$$= \binom{k}{a+b-1} - \binom{k}{a+b+1} \quad (3.63)$$

$$= \left( \binom{k}{a+b-1} + \binom{k}{a+b} \right) - \left( \binom{k}{a+b} + \binom{k}{a+b+1} \right) \quad (3.64)$$

$$= \binom{k+1}{a+b} - \binom{k+1}{a+b+1}. \quad (3.65)$$

We omit the proof of (3.47), as it follows directly from the argument above and Theorem 3.3.1.1.  $\square$

To prove Corollary 3.4.2.2, we will need another lemma about Tesler polynomials.

**Lemma 3.4.2.2.** *Given  $\alpha \in \mathbb{Z}^n$ , let  $-\alpha = (-\alpha_1, \dots, -\alpha_n)$ . Then*

$$\text{Tes}(-\alpha; q, t) = \left( -\frac{1}{qt} \right)^n \text{Tes}(\alpha; 1/q, 1/t).$$

*Proof.* Given  $U \in \mathcal{T}(\alpha)$ , consider the matrix  $-U$ . Clearly  $-U \in \mathcal{T}(-\alpha)$ . Furthermore,  $\text{wt}(-U; q, t)$

$$= (-1)^{\text{entries}^-(U) - \text{rows}^-(U)} M^{\text{nonzero}(U) - n} \prod_{u_{i,j} \neq 0} [-u_{i,j}]_{q,t} \quad (3.66)$$

$$= (-1)^{\text{entries}^-(U) - \text{rows}^-(U)} (qt\overline{M})^{\text{nonzero}(U) - n} \quad (3.67)$$

$$\begin{aligned} & \times \prod_{u_{i,j} \neq 0} \left( -\frac{1}{qt} \right) [u_{i,j}]_{1/q, 1/t} \\ & = \left( -\frac{1}{qt} \right)^n (-1)^{\text{entries}^+(U) - \text{rows}^+(U)} \overline{M}^{\text{nonzero}(U) - n} \prod_{u_{i,j} \neq 0} \overline{[u_{i,j}]_{q,t}} \end{aligned} \quad (3.68)$$

$$= \left( -\frac{1}{qt} \right)^n \text{wt}(U; 1/q, 1/t) \quad (3.69)$$

where in (3.68) we have used the fact that

$$(-1)^{\text{entries}^-(U) - \text{rows}^-(U) + \text{nonzero}(U)} = (-1)^{\text{entries}^-(U) - (n - \text{rows}^+(U)) + (\text{entries}^+(U) + \text{entries}^-(U))} \quad (3.70)$$

$$= (-1)^{-n + \text{entries}^+(U) + \text{rows}^+(U)} \quad (3.71)$$

$$= (-1)^{n + \text{entries}^+(U) - \text{rows}^+(U)}. \quad \square$$

*Proof of Corollary 3.4.2.2.* We write

$$\Delta_{m-1} e_n = \text{Tes}(1, 0^{n-1}; q, t) + \sum_{i=0}^{n-2} \text{Tes}((1, 0^i, -1, 0^{n-i-2}); q, t) \quad (3.72)$$

$$= \text{Tes}((1); q, t) + \sum_{i=0}^{n-2} \text{Tes}((1, -1, 0^i); q, t) \quad (3.73)$$

$$= \Delta_{m-1} e_{n-1} + \text{Tes}((1, -1, 0^{n-2}); q, t) \quad (3.74)$$

by induction. It is enough to show that

$$\text{Tes}((1, -1, 0^k); q, t) = -\frac{1}{qt} \left( 1 - \frac{1}{qt} \right)^k \quad (3.75)$$

for  $k \leq n - 2$ . To accomplish this, we use Lemmas 3.4.2.1 and 3.4.2.2 to write



$\text{Tes}((1, -1, 0^k); q, t)$  as

$$\text{Tes}((-1, 0^k); q, t) + \sum_{i=0}^{k-1} \text{Tes}((-1, 0^i, 1, 0^{k-i-1}); q, t) \quad (3.76)$$

$$= \left(-\frac{1}{qt}\right)^{k+1} \times \left( \text{Tes}((1, 0^k); 1/q, 1/t) + \sum_{i=0}^{k-1} \text{Tes}((1, 0^i, -1, 0^{k-i-1}); 1/q, 1/t) \right) \quad (3.77)$$

$$= \left(-\frac{1}{qt}\right)^{k+1} \overline{\text{Hilb } \Delta_{m-1} e_{k+1}} \quad (3.78)$$

$$= \left(-\frac{1}{qt}\right)^{k+1} (1 - qt)^k \quad (3.79)$$

$$= \left(-\frac{1}{qt}\right) (1 - 1/qt)^k \quad (3.80)$$

by induction on  $k$ .  $\square$

### 3.4.3 The $t = 0$ case

In this subsection, we show how to relate  $\text{Hilb } \Delta'_{e_k} e_n|_{t=0}$  to the distribution the inversion statistic on ordered set partitions studied in Chapter 2. Together with Section 2.3, this verifies the case of the Delta Conjectures where we take scalar products with  $p_{1^n}$  and set  $q = 0$  or  $t = 0$  in the Rise Version of  $t = 0$  in the Valley Version.

Recall that we set  $\mathcal{OP}_{n,k}$  to be the ordered partitions of the set  $\{1, 2, \dots, n\}$  into exactly  $k$  blocks. Given a subset  $S$  of  $\{1, 2, \dots, n\}$ , we set  $\mathcal{OP}_{n,S}$  to be the ordered set partitions in which the minimal elements of the blocks are exactly the elements of  $S$ . For example,  $7|236|45|1$  is an element of  $\mathcal{OP}_{7,\{1,2,4,7\}}$  and  $\mathcal{OP}_{7,4}$ . For any  $\beta \in \{0, 1\}^{n-1}$ , we write

$$\text{set}(\beta) = \{1\} \cup \{i + 1 : \beta_i = 1\}. \quad (3.81)$$

#### Corollary 3.4.3.1.

$$\begin{aligned} \widetilde{\text{Hilb}}_{\beta} e_n|_{t=0} &= \sum_{\pi \in \mathcal{OP}_{n, \text{set}(\beta)}} q^{\text{inv}(\pi)} \\ \text{Hilb } \Delta'_{e_k} e_n|_{t=0} &= \sum_{\pi \in \mathcal{OP}_{n,k+1}} q^{\text{inv}(\pi)}. \end{aligned}$$

Furthermore, [RW15] implies that we can restate these results as

$$\begin{aligned}\widetilde{\text{Hilb}}_{\beta} e_n \Big|_{t=0} &= \sum_{j=1}^{n-1} [1 + \beta_1 + \dots + \beta_j]_q \\ \text{Hilb } \Delta'_{e_k} e_n \Big|_{t=0} &= [k + 1]_q! S_{n,k+1}(q).\end{aligned}$$

In order to prove Corollary 3.4.3.1, we first note that Levande defined a map from Tesler matrices with hook sums  $1^n$  to  $\mathfrak{S}_n$  [Lev12]. We will denote this map by  $L_n$ . Furthermore, Levande used a weight-preserving, sign-reversing involution to prove that

$$\sum_{\substack{U \in \mathcal{T}(1^n) \\ L_n(U) = \sigma}} \text{wt}(U; q, 0) = q^{\text{inv}(\sigma)}. \quad (3.82)$$

for any  $\sigma \in \mathfrak{S}_n$ . Summing (3.82) over all permutations  $\sigma \in \mathfrak{S}_n$  yields

$$\text{Tes}(1^n; q, 0) = [n]_q!. \quad (3.83)$$

We extend Levande's results to our setting as follows. For any  $\beta \in \{0, 1\}^{n-1}$ , we define a map  $L_\beta$  from Tesler matrices with hook sums  $(1, \beta)$  onto  $\mathcal{OP}_{n, \text{set}(\beta)}$ . To define  $L_\beta$ , we first map a Tesler matrix  $U$  with hook sums  $(1, \beta)$  to an intermediary array. This array is created as follows:

1. First, read the entries of the diagonal  $u_{j,j}$  for  $j = n$  to 1. If  $u_{j,j} > 0$ , record a  $j$  in the rightmost column of the array  $u_{j,j}$  times, recording from top to bottom. After this step, the array will have a single column of length  $|\beta| + 1$  which weakly decreases from top to bottom.
2. For every  $j = n$  to 1, read up the  $j$ th column from  $u_{j-1,j}$  to  $u_{1,j}$ . For every  $u_{i,j} > 0$ , find the highest  $j$  in the array that currently has no entries to its left. Place an  $i$  to its left. Place an  $i$  in this manner  $u_{i,j}$  times.

For example, we send

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mapsto \begin{pmatrix} 1 & 4 \\ & 4 \\ 2 & 3 \end{pmatrix}.$$

Given such an array, we produce an ordered set partition by the following process.

1. Read the leftmost entries in each row of the array from bottom to top. Make these the minimal elements in  $k$  different blocks, from left to right.
2. For each  $i = 1$  to  $n$  which is not yet placed into the ordered set partition, find the lowest row in the array in which it appears. Place it in the block which contains the leftmost entry of that row.

Continuing our example, we obtain the ordered set partition  $23|4|1$ .

**Lemma 3.4.3.1.** *For any  $\beta \in \{0, 1\}^{n-1}$ ,  $L_\beta$  is well-defined. Furthermore, for any  $\pi \in \mathcal{OP}_{n, \text{set}(\beta)}$ ,*

$$\sum_{\substack{U \in \mathcal{T}((1, \beta)) \\ L_\beta(U) = \pi}} \text{wt}(U; q, 0) = q^{\text{inv}(\pi)}. \quad (3.84)$$

*Proof.* First, we argue that an array can always be created from  $U$  in the manner described above. Consider the  $j$ th column of  $U$ . At this point in the process of creating the array associated with  $U$ , we have processed all entries of the form  $u_{j,k}$  for any  $k \geq j$ . Since the  $j$ th hook sum of  $U$  is nonnegative, there are enough  $j$ 's available for us to place  $u_{i,j}$   $i$ 's to the left of a  $j$  for each  $i$ .

Next, we show that we can always create an ordered set partition  $\pi \in \mathcal{OP}_{n, \text{set}(\beta)}$  from such an array. The number of leftmost elements  $j$  in the rows of the array is equal to the  $j$ th hook sum of  $U$ . Since  $\beta \in \{0, 1\}^{n-1}$ , the minimal elements of  $\pi$  are unique and they are indeed equal to  $\text{set}(\beta)$ . This proves  $\pi \in \mathcal{OP}_{n, \text{set}(\beta)}$ .

Now that  $L_\beta$  is well-defined, we wish to create an involution that concludes the proof of the lemma. Our proof is quite similar to the proof for the  $\beta = 1^{n-1}$  case in [Lev12]. We begin by noting that, at  $t = 0$ , the weight of  $U$  is

$$(q - 1)^{\text{entries}(U) - \text{rows}(U)} \prod_{u_{i,j} \neq 0} q^{u_{i,j} - 1} \quad (3.85)$$

since  $U$  may not have any negative entries. We will assign a weight to the associated array in a way that corresponds to this weight. This weight will take the form of an array of the same shape as the associated array, except it will be filled with entries  $q$ ,  $1$ , or  $-1$ . For every entry  $a$  in the array, let  $b$  be the entry directly to its right. (If  $a$  is in the rightmost column of the array, set  $b = n + 1$ .) If this is not the lowest appearance of the

adjacent pair  $ab$  in the array, we assign a weight of  $q$ . If this is the lowest appearance of  $ab$  but it is not the lowest appearance of  $a$ , we assign a weight of  $q$  or  $-1$ . Otherwise, we assign the weight 1. Then we define the weight of the array to be the product of these individual weights. For example, the only way to assign weights to the above array is

$$\begin{pmatrix} 1 & 4 \\ & 4 \\ 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & q \\ & 1 \\ 1 & 1 \end{pmatrix}$$

where the weights are in parentheses. The total weight of this assignment is  $q$ .

Now, we wish to define an involution  $\Phi_\pi$  on these weighted arrays. Let  $c$  be the highest leading (i.e. leftmost in its row) entry such that there exists a  $d > c$  with the property that either

- $d$  appears below  $c$ 's row but does not appear in  $c$ 's row, or
- $d$  appears in  $c$ 's row and  $d$  has a weight of  $-1$ .

Choose  $d$  to be the smallest (and therefore leftmost) entry in  $c$ 's row that satisfies one of these conditions. In the first case,  $\Phi_\pi$  inserts a  $d$  into  $c$ 's row along with a  $-1$  weight. In the second case,  $\Phi_\pi$  removes the  $d$  from  $c$ 's row along with its  $-1$  weight. If no such  $c$  and  $d$  exist,  $\Phi_\pi$  leaves the array as a fixed point.

To prove that  $\Phi_\pi$  is an involution, first consider the case where  $d$  appears below  $c$ 's row but does not appear in  $c$ 's row. Set  $c'$  (respectively  $d'$ ) to be the largest (resp. smallest) element in  $c$ 's row that is less than (resp. greater than)  $d$ ; in other words, if  $d$  were in  $c$ 's row  $c'$  and  $d'$  would be its left and right neighbors, respectively. ( $d'$  may be empty, in which case we consider it to be  $n+1$ .) We need to argue that  $d$  can be inserted between  $c'$  and  $d'$  with a weight of  $-1$ , which can only happen if the lowest appearance of the successive pair  $dd'$  is in  $c$ 's row. Say that there is some lower occurrence of  $dd'$ . This must occur below  $c$ 's row, which contains the adjacent pair  $c'd'$ . This cannot happen, by the way in which we create these arrays. By a similar argument, the resulting array is valid, i.e. the new adjacent pairs  $c'd$  and  $dd'$  obey the defining property of our matrices: if  $ai$  and  $bi$  are pairs with  $ai$  occurring above  $bi$ , then  $a > b$ .

Now assume that  $d$  appears in  $c$ 's row and  $d$  has a weight of  $-1$ . Removing  $d$  creates the adjacent pair  $c'd'$ . We need to show that any weight that had been assigned

to  $c'$  is still valid now that its right neighbor is  $d'$  instead of  $d$ . If  $c'$  had been assigned a weight of  $q$ , then there must have been a lower occurrence of  $c'$ , which must still exist. If  $c'$  had been assigned a 1, then we must be considering the lowest appearance of  $c'$ , and removing  $d$  does not alter this. Finally,  $c'$  cannot be weighted with a  $-1$  by the minimality of  $d$ .

It is clear that  $\Phi_\pi$  is an involution that reverses signs of its non-fixed points. It only remains to investigate the fixed points of  $\Phi_\pi$ . In such a fixed point, for every  $a$  that is the leftmost entry in a row of the array and  $b > a$ ,  $b$  must appear in  $a$ 's row. Furthermore, the weight associated with  $b$  must either be  $q$  or 1 (which can only occur if the entry immediately to the left of  $b$  contains the lowest appearance of that element). Fixed points also contain no  $-1$  weights. From this, we can see that each  $\pi$  has a unique fixed point. It is the array created by the following process:

1. Write the blocks of  $\pi$  as rows in the array, from left to right in  $\pi$  and bottom to top in the array.
2. For each entry in the array  $b$  and each leftmost element  $a$  that appears above  $b$ , add  $b$  to  $a$ 's row if  $a < b$ .
3. Add a weight of  $q$  at all possible positions.

For example, the fixed point associated with  $\pi = 23|4|1$  is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & 4 & \\ & & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ & & 1 & \\ & & q & q \end{pmatrix}.$$

The weight of this fixed point is equal to the number of minimal elements to the right of  $b$  in  $\pi$  that are less than  $b$  for each  $b$ . This is exactly the  $\text{inv}$  statistic on ordered set partitions.  $\square$

Combined with our results in Chapter 2, we have proved the following special cases of the Delta Conjectures.

$$\text{Hilb } \Delta'_{e_k} e_n \Big|_{t=0} = \text{Hilb } \Delta'_{e_k} e_n \Big|_{q=0, t=q} \quad (3.86)$$

$$= \text{Hilb Rise}_{n,k}(x; q, 0) = \text{Hilb Rise}_{n,k}(x; 0, q) \quad (3.87)$$

$$= \text{Hilb Val}_{n,k}(x; q, 0). \quad (3.88)$$

### 3.5 Future Work

In our results so far, we have relied heavily on the fact that we are taking Hilbert series of the various symmetric functions at hand. It is reasonable to ask how Tesler matrices can be used to give formulas for the symmetric functions themselves. For example, in [GH14], Garsia and Haglund use Tesler matrices to give a formula for the symmetric function  $\nabla e_n$ . A similar (but not equivalent) formula for (rational extensions of)  $\nabla e_n$  is given in [GN13].

In a similar vein, it would be interesting to obtain symmetric functions whose Hilbert series are equal to  $F_\mu^\alpha$ . Such a result would allow us to replace virtual Hilbert series with the actual Hilbert series of these symmetric functions.

Finally, it seems possible that the methods used in Subsection 3.4.2 and 3.4.3 could be applied when  $e_1$  is replaced by a slightly more complicated function ( $e_2$  or  $m_2$ , for example). Similarly, we may be able to extend the results in Subsection 3.4.3 to  $\beta$  with entries not equal to 0 or 1. The computations will be more difficult in these cases, but they may still be tractable.

Chapter 3 is currently being prepared for submission for publication. An extended abstract of this work will be published in the Proceedings of Formal Power Series and Algebraic Combinatorics 2015. The dissertation author was the primary investigator and author of this work.

# Chapter 4

## The Rise Version at $k = 1$

In this chapter, we prove the following special case of the Rise Version of the Delta Conjectures.

**Theorem 4.0.0.1.** *For any positive integer  $n$ ,*

$$\begin{aligned}\Delta_{e_1} e_n &= \text{Rise}_{n,0}(x; q, t) + \text{Rise}_{n,1}(x; q, t) \\ &= \sum_{m=0}^{\lfloor n/2 \rfloor} s_{2^m, 1^{n-2m}} \sum_{p=m}^{n-m} [p]_{q,t}.\end{aligned}$$

*This verifies the Rise Version of the Delta Conjectures for  $k = 1$ .*

We deal with the symmetric function component of Theorem 4.0.0.1 in Section 4.1 and the combinatorial component in Section 4.2.

### 4.1 The Symmetric Side

In this subsection, we prove the "symmetric side" of Theorem 4.0.0.1, restated below.

**Proposition 4.1.0.1.** *For any positive integer  $n$ ,*

$$\Delta_{e_1} e_n = \sum_{m=0}^{\lfloor n/2 \rfloor} s_{2^m, 1^{n-2m}} \sum_{p=m}^{n-m} [p]_{q,t}.$$

Our main tool will be the following reciprocity rule for the operator  $\Delta$ , which was proven by Haglund as Corollary 2 in [Hag04].

**Lemma 4.1.0.2** (Corollary 2 in [Hag04]). *For positive integers  $d, n$  and any symmetric function  $f \in \Lambda^{(n)}$ ,*

$$\langle \Delta_{e_{d-1}} e_n, f \rangle = \langle \Delta_{\omega f} e_d, s_d \rangle.$$

We set  $d = 2$  and  $f = s_\lambda$  for  $\lambda \vdash n$ , since taking the scalar product of a symmetric function with  $s_\lambda$  yields the coefficient of  $s_\lambda$  in the Schur expansion of that symmetric function. Lemma 4.1.0.2 implies that

$$\langle \Delta_{e_1} e_n, s_\lambda \rangle = \langle \Delta_{s_{\lambda'}} e_2, s_2 \rangle. \quad (4.1)$$

We can compute the right-hand side by hand. First, we expand  $e_2$  into the modified Macdonald polynomial basis:

$$e_2 = \frac{1}{t-q} \tilde{H}_{1,1} - \frac{1}{t-q} \tilde{H}_2. \quad (4.2)$$

Then we apply the operator  $\Delta_{s_{\lambda'}}$ .

$$\Delta_{s_{\lambda'}} e_2 = \frac{s_{\lambda'}[1+t]}{t-q} \tilde{H}_{1,1} - \frac{s_{\lambda'}[1+q]}{t-q} \tilde{H}_2. \quad (4.3)$$

Now we expand this expression into the Schur basis and take the coefficient of  $s_2$ , yielding

$$\langle \Delta_{s_{\lambda'}} e_2, s_2 \rangle = \frac{s_{\lambda'}[1+t] - s_{\lambda'}[1+q]}{t-q}. \quad (4.4)$$

It is already clear that the above expression is a polynomial in  $q$  and  $t$ . Moreover, for any monomial  $u$  the principal specialization  $s_{\lambda'}[1+u]$  is equal to the sum  $\sum_T u^{\# 2's \text{ in } T}$  over all semi-standard tableaux  $T$  of shape  $\lambda'$  filled with 1's and 2's. This sum is zero if  $\lambda'$  has more than two rows, so we can restrict our attention to  $\lambda' = (n-m, m)$  for some integer  $0 \leq m \leq \lfloor n/2 \rfloor$ . For such a tableau  $T$  of shape  $(n-m, m)$ , it is clear that the first  $m$  entries in the first row of  $T$  must be 1's and all entries in the second row of  $T$  must be 2's. Of the remaining  $n-2m$  entries, we are free to choose an integer  $0 \leq i \leq n-2m$  such that the left  $i$  entries are 1's and the right  $n-2m-i$  entries are 2's. Hence

$$s_{n-m, m}[1+u] = \sum_{p=m}^{n-m} u^p \quad (4.5)$$



Since  $(n - m, m)' = (2^m, 1^{n-2m})$ , we have

$$\langle \Delta_{e_1} e_n, s_{2^m, 1^{n-2m}} \rangle = \frac{\sum_{p=m}^{n-m} (t^p - q^p)}{t - q} = \sum_{p=m}^{n-m} [p]_{q,t} \quad (4.6)$$

which proves Proposition 4.1.0.1.

In theory, this method can be used to compute  $\Delta_{e_k} e_n$  for any fixed value of  $k$ . For example,  $\langle \Delta_{e_2} e_n, s_\lambda \rangle$  equals

$$\frac{(t - q^2) s_\lambda [1 + t + t^2] - (q + t + 1)(t - q) s_\lambda [1 + q + t] + (t^2 - q) s_\lambda [1 + q + q^2]}{(t - q)(t^2 - q)(t - q^2)} \quad (4.7)$$

which is a polynomial in  $q$  and  $t$ . Unfortunately, it is not clear why the resulting expression should be a *positive* polynomial in  $q$  and  $t$ , and this problem only gets more difficult as  $k$  grows.

## 4.2 The Combinatorial Side

In this subsection, we prove the following proposition, completing the proof of Theorem 4.0.0.1.

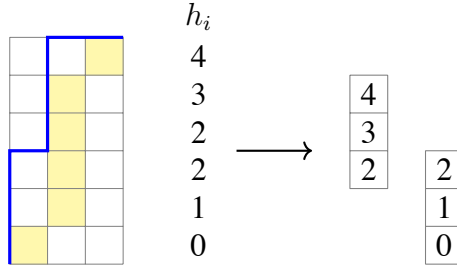
**Proposition 4.2.0.2.** *For any positive integer  $n$ ,*

$$\text{Rise}_{n,0}(x; q, t) + \text{Rise}_{n,1}(x; q, t) = \sum_{m=0}^{\lfloor n/2 \rfloor} s_{2^m, 1^{n-2m}} \sum_{p=m}^{n-m} [p]_{q,t}.$$

First, we note that  $\text{Rise}_{n,0}(x; q, t) = s_{1^n}$ , which accounts for the  $m = p = 0$  term above. We will need to work harder to expand  $\text{Rise}_{n,1}(x; q, t)$ . Recall the interpretation for  $\text{Rise}_{n,k}(x; q, t)$  given in terms of labeled Dyck paths and leaning stacks in Subsection 1.3.2:

$$\text{Rise}_{n,k}(x; q, t) = \sum_{P \in \mathcal{P}\mathcal{F}_{n,k}^{\text{Stack}}} q^{\text{hdinv}(P)} t^{\text{area}(P)} x^P. \quad (4.8)$$

We note that this interpretation is closely related to the LLT polynomials of [LLT97]. Namely, we can refine the sum on the right-hand side by fixing a leaning stack  $S$  and



**Figure 4.1:** A leaning stack  $S$  and a Dyck path  $D$  are mapped to a tuple of skew diagrams  $\nu = (\nu^{(1)}, \nu^{(2)})$ . We have filled the cells of the skew diagrams with their contents.

then a Dyck path  $D \in \mathcal{D}(S)$  and considering all ways of labeling the Dyck path  $D$ .

$$\text{Rise}_{n,k}(x; q, t) = \sum_{S \in \text{Stack}_{n,k}} \sum_{D \in \mathcal{D}(S)} t^{\text{area}(D)} \sum_{P \in \mathcal{PF}(S): D(P)=D} q^{\text{hdinv}(P)} x^P \quad (4.9)$$

$$= \sum_{S \in \text{Stack}_{n,k}} \sum_{D \in \mathcal{D}(S)} t^{\text{area}(D)} LLT_{S,D}(x; q) \quad (4.10)$$

where we have defined

$$LLT_{S,D}(x; q) = \sum_{P \in \mathcal{PF}(S): D(P)=D} q^{\text{hdinv}(P)} x^P. \quad (4.11)$$

We call this the *LLT polynomial* with respect to  $S$  and  $D$ , since these are special cases of the polynomials introduced in [LLT97]. We can relate our versions of LLT polynomials more precisely to the notation for LLT polynomials appearing in [HHL05a] as follows. Say that the north steps of  $D$  appear in  $d$  different columns. Furthermore, for  $j = 1, 2, \dots$  from right to left, if the  $j$ th column's bottom row is row  $i$ , then we define  $c_j = h_i(D)$ . Consider the tuple of skew diagrams  $\nu = (\nu^{(1)}, \dots, \nu^{(d)})$  where the number of squares in  $\nu^{(j)}$  is equal to the number of north steps in the  $j$ th column of  $D$  with the content of the bottom square equal to  $c_j$ . Then  $LLT_{S,D}(x; q) = G_\nu(x; q)$ , where the latter appears as Definition 3.2 of [HHL05a]. We show an example in Figure 4.1.

There are many benefits of this connection between  $\text{Rise}_{n,k}(x; q, t)$  and LLT polynomials. The first is that LLT polynomials are known to be symmetric; this fact, along with (4.10), implies that  $\text{Rise}_{n,k}(x; q, t)$  is symmetric. On the other hand, we are still unable to prove that  $\text{Val}_{n,k}(x; q, t)$  is symmetric. More pertinent to our current case, when  $D$  has two columns, much is known about the LLT polynomial  $LLT_{S,D}(x; q)$ . In

the remainder of this subsection, we leverage this information to complete the proof of Theorem 4.0.0.1.

We use the notation that the *reading word* of a labeled Dyck path  $P \in \mathcal{WPF}_{n,k}^{\text{Stack}}$ , written  $w(P)$ , is obtained by reading its labels from maximum  $h_i$  value down to  $h_i = 0$  from right to left. We say that a word whose entries are positive integers is *Yamanouchi* if each of its suffixes has more  $i + 1$ 's than  $i$ 's for every positive integer  $i$ .

**Lemma 4.2.0.3** (Carré and Leclerc [CL95], van Leeuwen [vL00]). *For any  $S \in \text{Stack}_{n,1}$  and  $D \in \mathcal{PF}(S)$ , the coefficient of  $s_\lambda$  in the Schur expansion of  $LLT_{S,D}(x; q)$  is equal to the sum*

$$\sum_P q^{\text{hdinv}(P)}$$

*over all  $P \in \mathcal{WPF}(S)$  with  $D(P) = D$  such that  $x^P = \prod_{i=1}^{\ell(\lambda)} x_i^{\lambda_i}$  and  $w(P)$  is Yamanouchi.*

For any such  $P$ , each integer can be used as a label at most twice. Thus the only Schur functions appearing in the expansion of  $LLT_{S,D}(x; q)$  are of the form  $s_{2^m, 1^{n-2m}}$  for some integer  $0 \leq m \leq \lfloor n/2 \rfloor$ . Furthermore, we can uniquely represent a labeled Dyck path  $P$  that satisfies the conditions in Lemma 4.2.0.3 by filling a certain two-column array with  $X$ 's and  $Y$ 's according to the following procedure. For each height that occurs in  $P$  from 0 up to the maximum height, consider the two columns of  $P$ . If the left column of  $P$  contains a label at that height, place a square into the left column of the array. If we have already come across the value of the label while creating our array, we place a  $Y$  in the new square; otherwise, we place an  $X$ . Then we do the same for the right column. We continue until all heights have been processed. We call this the *XY diagram* of  $P$ .

Since each label appears at most twice in  $P$ , this process is well-defined. Furthermore, it is invertible; to obtain the original labeled Dyck path  $P$ , we scan the *XY diagram* from bottom to top and left to right. For each  $X$  or  $Y$ , we place a label in the corresponding column at the corresponding height that counts the number of times (including the current letter) that we have observed the current letter so far.

It is clear by definition that all *XY diagrams* have two columns, that the left column may extend below the right column (but not vice versa), and that the lower left

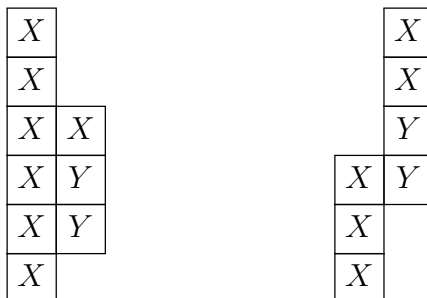


**Figure 4.2:** To the left, we have drawn a two-column labeled Dyck path whose word is Yamanouchi with its leaning stack shaded yellow. To the right, we have drawn the corresponding  $XY$  diagram.

square of a diagram always contains an  $X$ . The crux of the proof of Proposition 4.2.0.2 is that we can use the Yamanouchi restriction on  $w(P)$  to completely classify the possible  $XY$  diagrams. We note that  $w(P)$  is Yamanouchi if and only if, reading the diagram from bottom to top and left to right, we have always seen at least as many  $X$ 's as  $Y$ 's. Furthermore, the labels of  $P$  are increasing up columns if and only if there are no  $Y$ 's on top of  $X$ 's. Since the entry in the lowest entry in the left column must be an  $X$ , this implies that  $Y$ 's always occur in the right column.

These conditions are enough to allow us to classify the possible  $XY$  diagrams. From bottom to top, every diagram begins with  $a \geq 0$  rows consisting of only a left square which contains an  $X$ . Then it has  $b \geq 0$  rows which have two squares where the left square contains an  $X$  and the right square contains a  $Y$ . From this point on, the diagram can have one of two types. We say that *Type I  $XY$  diagrams* have a sequence of  $c \geq 0$  rows with two squares, both of which contain  $X$ 's, followed by a sequence of  $d \geq 0$  rows with a single square containing an  $X$ . The final  $d$  rows must either consist entirely of left squares or of right squares. In *Type II  $XY$  diagrams*, the  $b$   $XY$  rows are followed by  $c'$  rows with only a right square containing a  $Y$ . (For Type II diagrams, we must have  $b \geq 1$ .) Here,  $c'$  is an integer satisfying  $1 \leq c' \leq a$ . Finally, a Type II diagram has  $d' \geq 0$  rows with only an  $X$  in the right square.

We would like to recover the area and  $\text{hdinv}$  of the original labeled Dyck path  $P$  from its  $XY$  diagram. It is not hard to see that  $\text{area}(P)$  is equal to  $a$ , the number of rows at the bottom of the diagram containing only an  $X$ . The  $\text{hdinv}$  of a diagram is equal to



**Figure 4.3:** A Type I diagram on the left and a Type II diagram on the right.

the number of pairs of left and right squares such that either

- the left square appears immediately northwest of the right square, or
- the two squares are in the same row and both contain  $X$ 's.

Now we can use this characterization to find the coefficient of  $s_{2m,1^{n-2m}}$  in  $\text{Rise}_{n,1}(x; q, t)$  for any  $0 \leq m \leq \lfloor n/2 \rfloor$ . Since there are always at least as many  $X$ 's as  $Y$ 's in a diagram, we restrict our attention to diagrams with  $m$   $Y$ 's and  $n - m$   $X$ 's. Clearly the area of such a diagram may range between 0 and  $n - m - 1$ , which corroborates the formula in Proposition 4.2.0.2. More precisely, we fix the area to be some value  $0 \leq j \leq n - m - 1$ . If we can show that there is exactly one diagram with area  $j$ ,  $n - m$   $X$ 's, and  $m$   $Y$ 's with  $\text{hdinv} = i$  for each  $\max(0, m - j - 1) \leq i \leq n - m - j - 1$ , then we have completed the proof of Proposition 4.2.0.2.

First, we consider the possible Type I diagrams. We know that such a diagram must begin with  $j$  rows consisting only of  $X$ 's in the left square followed by  $m$  rows consisting of an  $X$  in the left square and a  $Y$  in the right square. Let us assume  $m \geq 1$  for now. Then we have already accumulated  $m - 1$   $\text{hdinv}$ . We must place  $n - 2m - j$  more  $X$ 's. There are exactly  $n - 2m - j + 1$  ways to accomplish this task. Namely, we choose any integer  $0 \leq r \leq n - 2m - j$ . We repeatedly place an  $X$  in the left square, then the right square, then the next left square up, and so on, placing  $r$   $X$ 's this way. After this, we stack the remaining  $X$ 's above the last of the  $r$   $X$ 's we had just placed. (If  $r = 0$ , we place every  $X$  in a stack above the highest  $Y$ .) We have created every Type I diagram with area  $j$ ,  $n - m$   $X$ 's and  $m$   $Y$ 's. Furthermore, the resulting diagram has

hdinv =  $m - 1 + r$ , so have contributed

$$\sum_{i=m-1}^{n-m-j-1} q^i \quad (4.12)$$

to the coefficient of  $t^j s_{2m, 1^{n-2m}}$ . If  $m = 0$ , the same logic shows that we have contributed

$$\sum_{i=0}^{n-j-1} q^i \quad (4.13)$$

to the coefficient of  $t^j s_{1^n}$ .

Now we consider the Type II diagrams with area  $j$ ,  $n - m$   $X$ 's, and  $m$   $Y$ 's. A Type II diagram only exists if  $m \geq 2$ . Such a diagram must begin with  $j$  rows of just an  $X$  in the left square, followed by  $1 \leq b \leq m - 1$  rows of an  $X$  and a  $Y$ , contributing  $b - 1$  hdinv. Then the rest of the diagram is determined, as it must have  $m - b$  rows that just have a  $Y$  on the right followed by  $n - m - j - b$  rows consisting of an  $X$  on the right. Recall from the characterization of Type II diagrams that we must have  $1 \leq m - b \leq j$ , so actually  $\max(1, m - j) \leq b \leq m - 1$ . This yields a contribution of

$$\sum_{i=\max(0, m-j-1)}^{m-2} q^i \quad (4.14)$$

to the coefficient of  $t^j s_{2m, 1^{n-2m}}$ . Gathering (4.12), (4.13), and (4.14), the coefficient of  $t^j s_{2m, 1^{n-2m}}$  in  $\text{Rise}_{n,1}(x; q, t)$  is

$$\sum_{i=\max(0, m-j-1)}^{n-m-j-1} q^i. \quad (4.15)$$

This concludes the proof of Proposition 4.2.0.2.

Chapter 4 is currently being prepared for submission for publication. Haglund, J.; Remmel, J.; Wilson, A.T. The dissertation author was the primary investigator and author of this work.

# Chapter 5

## Decorated Schröder Paths and Two-Car Parking Functions at $t = 1/q$

The goal of this chapter is to investigate cases of the Delta Conjecture of the form  $\langle -, e_{n-d}h_d \rangle|_{t=1/q}$  and  $\langle -, h_{n-d}h_d \rangle|_{t=1/q}$ . First, we use plethystic techniques to obtain a formula for the symmetric function  $\Delta_f e_n$  at  $t = 1/q$  for any  $f \in \Lambda$ . We spend the remainder of the chapter addressing the combinatorial sides of the Delta Conjectures. We develop recursions for refinements of the polynomials  $\langle \text{Rise}_{n,k}(q, t), g \rangle, \langle \text{Val}_{n,k}(q, t), g \rangle$  for any  $g$  of the form  $e_{n-d}h_d$  or  $h_{n-d}h_d$ . These generalize the recursions used in [Hag04, HL13]. At  $t = 1/q$ , these recursions allow us to manipulate  $q$ -binomial coefficients to resolve the Rise Version of the Delta Conjecture at  $t = 1/q$  after taking scalar products with  $e_{n-d}h_d$  or  $h_{n-d}h_d$ . We hope that these recursions may be used to prove the same results without the assumption  $t = 1/q$  in the future, fully generalizing Haglund's results in [Hag04].

### 5.1 $\Delta_f e_n$ at $t = 1/q$

In this section, we obtain a plethystic formula for  $\Delta_{e_k} e_n$  at  $t = 1/q$ . We prove the plethystic formula below and then use it to show that  $\Delta_{e_k} e_n$  is Schur positive at  $t = 1/q$  up to a power of  $q$ . This formula also completes the proof of the  $q = t = 1$  case of the Delta Conjectures; we proved the corresponding combinatorial statements in Subsection 1.3.4.

**Theorem 5.1.0.2.** For any symmetric function  $f \in \Lambda^{(k)}$ ,

$$\Delta_f e_n|_{t=1/q} = \frac{f[[n]_q] e_n[X[k+1]_q]}{q^{k(n-1)} [k+1]_q}.$$

*Proof.* First we note that

$$\tilde{H}_\mu[X; q, 1/q] = C s_\mu \left[ \frac{X}{1-q} \right] \quad (5.1)$$

for some constant  $C$ . This fact can be derived from [Mac95]. We use Cauchy's Formula to write

$$e_n[X] = e_n \left[ (1-q) \frac{X}{1-q} \right] = \sum_{\mu \vdash n} s_{\mu'} \left[ \frac{X}{1-q} \right] s_\mu[1-q]. \quad (5.2)$$

For any monomial  $u$ ,  $s_\mu[1-u]$  is zero if  $\mu$  is not a hook shape and

$$s_\mu[1-u] = (-u)^r (1-u) \quad (5.3)$$

if  $\mu = (n-r, 1^r)$  [Mac95]. Therefore, summing over hook shapes  $\mu$ , we have

$$e_n[X] = \sum_{\mu=(n-r, 1^r)} s_{\mu'} \left[ \frac{X}{1-q} \right] (-q)^r (1-q). \quad (5.4)$$

Next, we note that, for  $\mu = (n-r, 1^r)$ ,  $\mu' = (r+1, 1^{n-r-1})$  and

$$B_{\mu'}(q, 1/q) = q^{-(n-r-1)} [n]_q. \quad (5.5)$$

Therefore

$$f[B_{\mu'}(q, 1/q)] = q^{-k(n-r-1)} f[[n]_q]. \quad (5.6)$$

Combining (5.4) with (5.6), we see that  $\Delta_f e_n[X]|_{t=1/q}$  is equal to

$$\sum_{\mu=(n-r, 1^r)} q^{-k(n-r-1)} (-q)^r (1-q) f[[n]_q] s_{\mu'} \left[ \frac{X}{1-q} \right] \quad (5.7)$$

$$= \frac{f[[n]_q]}{q^{k(n-1)} [k+1]_q} \sum_{\mu=(n-r, 1^r)} (-q^{k+1})^r (1-q^{k+1}) s_{\mu'} \left[ \frac{X}{1-q} \right]. \quad (5.8)$$

Applying Cauchy's Formula again, we get

$$\sum_{\mu=(n-r, 1^r)} (-q^{k+1})^r (1-q^{k+1}) s_{\mu'} \left[ \frac{X}{1-q} \right] = e_n[X[k+1]_q]. \quad (5.9)$$

□



From Theorem 5.1.0.2, it is easy to compute

$$\Delta_{e_k} e_n \Big|_{t=1/q} = \frac{q^{\binom{k}{2} - k(n-1)}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q e_n[X[k+1]_q]. \quad (5.10)$$

We can also use Theorem 5.1.0.2 along with a recent result of Garsia, Leven, Wallach, and Xin to prove a Schur positivity result for our symmetric function at  $t = 1/q$ .

**Corollary 5.1.0.2.**  $q^{k(n-1) - \binom{k}{2}} \Delta_{e_k} e_n \Big|_{t=1/q}$  is a Schur positive symmetric polynomial.

*Proof.* Let  $d = \gcd(k+1, n)$ . Then Theorem 2.1 in [GLWX15a] implies that

$$\frac{[d]_q}{[k+1]_q} e_n[X[k+1]_q]$$

is a Schur positive symmetric polynomial. By Theorem 5.1.0.2, it is enough to show that  $\frac{1}{[d]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \in \mathbb{N}[q]$ . Furthermore, Proposition 2.4 in [GLWX15a] implies that if  $\frac{1}{[d]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$  is a polynomial then it must have nonnegative coefficients (since  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  is known to be a unimodal positive polynomial). Therefore we only need to show that  $\frac{1}{[d]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$  is a polynomial.

To accomplish this, we will use the  $q$ -Lucas Theorem, first proved in [Oli65] and given a nice combinatorial proof in [Sag92]. To state the  $q$ -Lucas Theorem, given integers  $n, k$ , and  $p$ , we divide  $n$  and  $k$  by  $p$  to obtain  $n = n_1 p + n_0$  and  $k = k_1 p + k_0$  for  $n_0, k_0 < p$ . Then

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv \binom{n_1}{k_1} \begin{bmatrix} n_0 \\ k_0 \end{bmatrix}_q \pmod{\Phi_p(q)}. \quad (5.11)$$

where  $\Phi_p(q)$  is  $p$ th cyclotomic polynomial. Consider any  $p$  such that  $\Phi_p(q)$  divides  $[d]_q$ . If we can show that all such  $\Phi_p(q)$  divide  $\begin{bmatrix} n \\ k \end{bmatrix}_q$ , we are done. Since  $p$  divides  $d$  and  $d$  divides both  $n$  and  $k+1$ ,  $p$  divides  $n$  but it does not divide  $k$ . This means that  $n_0 = 0$  and  $k_0 > 0$ . By the  $q$ -Lucas Theorem,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv \binom{n_1}{k_1} \begin{bmatrix} n_0 \\ k_0 \end{bmatrix}_q \equiv 0 \pmod{\Phi_p(q)} \quad (5.12)$$

so  $\Phi_p(q)$  divides  $\begin{bmatrix} n \\ k \end{bmatrix}_q$ . □

## 5.2 Combinatorial Objects

In this section, we describe the combinatorial effect of taking scalar products with functions of the form  $e_\lambda h_\mu$ . Recall that, for a parking function  $P \in \mathcal{WPF}_n$ , the *reading word* of  $P$  is the word obtained by reading the labels of  $P$  from maximum to minimum area and from right to left. To understand the necessary scalar products, we need to consider parking functions with a larger alphabet of labels

$$\mathcal{A} = \{1, 2, 3, \dots\} \cup \{\bar{1}, \bar{2}, \dots\}$$

along with the relation  $<_{\mathcal{A}}$  defined as follows:

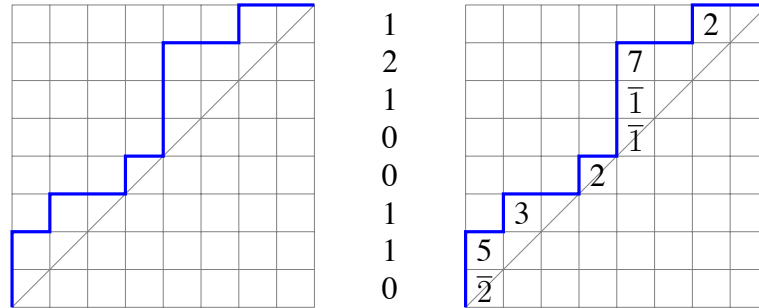
1.  $a <_{\mathcal{A}} b$  if and only if  $a < b$  as integers.
2.  $\bar{a} <_{\mathcal{A}} \bar{b}$  if and only if  $a \leq b$  as integers.
3.  $\bar{a} <_{\mathcal{A}} b$  for any integers  $a$  and  $b$ .

Then we can define  $\mathcal{WPF}_{\lambda, \mu}$  to be the set of parking functions consisting of labeled Dyck paths from  $(0, 0)$  to  $(|\lambda| + |\mu|, |\lambda| + |\mu|)$  that are labeled with words consisting of exactly  $\lambda_i \bar{i}$ 's and  $\mu_j j$ 's for every  $i$  and  $j$  such that, in each column, the labels are strictly increasing from bottom to top according to the relation  $<_{\mathcal{A}}$ . We extend the definitions of  $\text{dinv}$  and  $\text{area}$  to these sets of objects with the relation  $<_{\mathcal{A}}$ . See Figure 5.1 for an example. On the left we have drawn a Dyck path of order 8. We have written the corresponding entries of its area vector to its right. One can see that its total area is 6. On the right, we have labeled the Dyck path to obtain a word parking function of order 8 over the alphabet  $\mathcal{A}$ . One can check that the parking function has 7 diagonal inversions, 2 of which are primary and 5 secondary.

The Shuffle Conjecture got its name from the following fact: if we choose partitions  $\lambda$  and  $\mu$  and set  $n = |\lambda| + |\mu|$ , then we have

$$\left\langle \sum_{P \in \mathcal{WPF}_n} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P, e_\lambda h_\mu \right\rangle = \sum_{P \in \mathcal{WPF}_{\lambda, \mu}} q^{\text{dinv}(P)} t^{\text{area}(P)}. \quad (5.13)$$

Analogously, if we take scalar products with  $e_\lambda h_\mu$  in the Delta Conjectures, we get sums over rise- or valley-decorated elements of  $\mathcal{WPF}_{\lambda, \mu}$ . We will be particularly interested in the cases  $\lambda = (a)$ ,  $\mu = (b)$  and  $\lambda = 0$ ,  $\mu = (a, b)$  where  $a + b = n$ . We discuss these cases in the next two subsections.



**Figure 5.1:** A Dyck path of order 8 with area 6 and 7 diagonal inversions.

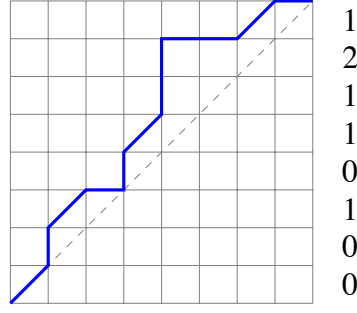
### 5.2.1 Decorated Schröder Paths

We note that taking a scalar product with  $e_a h_b$  means that we only need to consider word parking functions labeled by exactly  $a$   $\bar{1}$ 's and  $b$  1's. By the definition of  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ ,  $\bar{1}$ 's may be placed on top of one another and 1's may be placed on top of  $\bar{1}$ 's, but no labels may be placed above 1's. This allows us to replace north steps labeled with 1's (and their following east steps) with diagonal steps and north steps labeled with  $\bar{1}$ 's with unlabeled north steps. As a result, we obtain a bijection between such parking functions and *Schröder paths*  $\mathcal{S}_{a,b}$ , the set of lattice paths from  $(0, 0)$  to  $(a + b, a + b)$  that consist of  $a$  north steps,  $a$  east steps, and  $b$  diagonal steps while remaining weakly above the main diagonal. We can define  $\text{div}_i$  and  $\text{area}_i$  on Schröder paths using this bijection. That is, given a Schröder path  $P \in \mathcal{S}_{a,b}$ , we set  $\text{area}_i(P)$  to be the number of squares whose lower right corner is between  $P$  and the line  $y = x$  in the  $i$ th row from the bottom and  $\text{div}_i(P)$  counts the number of pairs  $i < j$  such that

- $\text{area}_i(P) = \text{area}_j(P)$  and row  $i$  contain a north step, or
- $\text{area}_i(P) = \text{area}_j(P) + 1$  and row  $j$  contains a north step.

For an example of a Schröder path, see Figure 5.2.

In order to relate Schröder paths to the Delta Conjectures, we need to define decorated versions. Double rises and valleys are defined as before; we set  $\text{Rise}(P)$  to be the set of  $2 \leq i \leq n$  such that  $\text{area}_i(P) > \text{area}_{i-1}(P)$  and  $\text{Val}(P)$  to be the set of



**Figure 5.2:** A Schröder path of order 8 with 4 diagonal steps, area 6, and 3 diagonal inversions, 2 of which are primary and 1 secondary.

removable valleys of  $P$ . Then

$$\begin{aligned}\mathcal{RS}_{a,b,k} &= \{(P, R) : P \in \mathcal{S}_{a,b}, R \subseteq \text{Rise}(P), |R| = k\} \\ \mathcal{VS}_{a,b,k} &= \{(P, V) : P \in \mathcal{S}_{a,b}, V \subseteq \text{Val}(P), |V| = k\}.\end{aligned}$$

These are the double rise- and removable valley-decorated Schröder paths, respectively.

We define

$$\begin{aligned}\text{area}^-((P, R)) &= \text{area}(P) - \sum_{i \in R} \text{area}_i(P) \\ \text{dinv}^-((P, V)) &= \text{dinv}(P) - |V| - \sum_{i \in V} \text{dinv}_i(P).\end{aligned}$$

Now we can state the restriction of the Delta Conjectures to these objects. We will prove the  $t = 1/q$  case of the first statement in this conjecture later in this chapter.

**Conjecture 5.2.1.1.** *For any nonnegative integers  $a$ ,  $b$ , and  $k$ ,*

$$\begin{aligned}\langle \Delta'_{e_{a+b-k-1} e_{a+b}, e_a h_b} \rangle &= \sum_{(P,R) \in \mathcal{RS}_{a,b,k}} q^{\text{dinv}(P)} t^{\text{area}^-((P,R))} \\ &= \sum_{(P,V) \in \mathcal{VS}_{a,b,k}} q^{\text{dinv}^-((P,V))} t^{\text{area}(P)}.\end{aligned}$$

We will need several refinements of the set  $\mathcal{RS}_{a,b,k}$  and  $\mathcal{VS}_{a,b,k}$ . We define

$$\begin{aligned} \mathcal{RS}_{a,b,k}^{(h)} &= \{(P, R) \in \mathcal{RS}_{a,b,k} : P \text{ returns to the main diagonal } h \text{ times}\} \\ \mathcal{RS}_{a,b,k}^{(r,s,i)} &= \{(P, R) \in \mathcal{RS}_{a,b,k} : (P, R) \text{ has } r \text{ north steps, } s \text{ diagonal steps,} \\ &\quad \text{and } i \text{ decorated rises on the main diagonal}\} \\ \overline{\mathcal{RS}}_{a,b,k}^{(r,s,i)} &= \{(P, R) \in \mathcal{RS}_{a,b,k}^{(r,s,i)} : \text{the lowest rise in } P \text{ is not in } R\} \\ \mathcal{VS}_{a,b,k}^{(g)} &= \{(P, V) \in \mathcal{VS}_{a,b,k} : P \text{ has } g \text{ returns to the diagonal that are not in } V\} \\ \mathcal{VS}_{a,b,k}^{(r,s,i)} &= \{(P, V) \in \mathcal{VS}_{a,b,k} : (P, V) \text{ has } r \text{ north steps, } s \text{ diagonal steps,} \\ &\quad \text{and } i \text{ decorated valleys on the diagonal}\}. \end{aligned}$$

For each of these sets, we will remove the italics and append variables  $q$  and  $t$  to mean the polynomial obtained by summing  $q^{\text{dinv}(P)}t^{\text{area}^-((P,R))}$  (if the set is rise-decorated) or  $q^{\text{dinv}^-((P,V))}t^{\text{area}(P)}$  (if the set is valley-decorated) over all the objects in the set. For example,

$$RS_{a,b,k}^{(r,s,i)}(q, t) = \sum_{(P,R) \in \mathcal{RS}_{a,b,k}^{(r,s,i)}} q^{\text{dinv}(P)}t^{\text{area}^-((P,R))}.$$

## 5.2.2 Two-Car Parking Functions

In this subsection, we deal with the case of taking scalar products with  $h_a h_b$ . This corresponds to word parking functions of order  $a + b$  that use  $a$  1's and  $b$  2's as labels and no other labels. We write the class of all of these parking functions as  $\mathcal{T}_{a,b}$ . The usual definitions of area, dinv, double rises, and blank valleys apply to these objects with no modifications necessary. We set

$$\begin{aligned} \mathcal{RT}_{a,b,k} &= \{(P, R) : P \in \mathcal{T}_{a,b}, R \subseteq \text{Rise}(P), |R| = k\} \\ \mathcal{VT}_{a,b,k} &= \{(P, V) : P \in \mathcal{T}_{a,b}, V \subseteq \text{Val}(P), |V| = k\}. \end{aligned}$$

The relevant restriction of the Delta Conjectures is as follows. As with the Schröder case, we prove the  $t = 1/q$  case of the first statement in this conjecture later in this chapter.

**Conjecture 5.2.2.1.** *For any nonnegative integers  $a$ ,  $b$ , and  $k$ ,*

$$\begin{aligned} \langle \Delta'_{e_{a+b-k-1}} e_{a+b}, h_a h_b \rangle &= \sum_{(P,R) \in \mathcal{RT}_{a,b,k}} q^{\text{dinv}(P)} t^{\text{area}^-((P,R))} \\ &= \sum_{(P,V) \in \mathcal{VT}_{a,b,k}} q^{\text{dinv}^-((P,V))} t^{\text{area}(P)}. \end{aligned}$$

As in the Schröder case, it will be convenient later to have several refinements of  $\mathcal{RT}_{a,b,k}$  and  $\mathcal{VT}_{a,b,k}$ .

$$\mathcal{RT}_{a,b,k}^{(h)} = \{(P, R) \in \mathcal{RT}_{a,b,k} : P \text{ has } h \text{ 2's on the main diagonal}\}$$

$$\mathcal{RT}_{a,b,k}^{(r,s,i)} = \{(P, R) \in \mathcal{RT}_{a,b,k} : (P, R) \text{ has } r \text{ 1's, } s \text{ 2's, and } i \text{ decorated rises on the main diagonal}\}$$

$$\overline{\mathcal{RT}}_{a,b,k}^{(r,s,i)} = \{(P, R) \in \mathcal{RT}_{a,b,k}^{(r,s,i)} : \text{the first rise in } P \text{ is not in } R\}$$

$$\mathcal{VT}_{a,b,k}^{(g)} = \{(P, V) \in \mathcal{VT}_{a,b,k} : P \text{ has } g \text{ 2's on the diagonal that are not in } V\}$$

$$\mathcal{VT}_{a,b,k}^{(r,s,i,j)} = \{(P, V) \in \mathcal{VT}_{a,b,k} : (P, V) \text{ has } r \text{ 1's, } s \text{ 2's, and } i \text{ decorated 1's, and } j \text{ decorated 2's on the diagonal}\}.$$

As above, we remove italics to obtain polynomials equal to the relevant  $q, t$ -sum over the class of objects.

## 5.3 Recursions

In this section, we state and prove recursions for the polynomials defined in the previous section. We will use these recursions in Section 5.4 to prove  $q$ -binomial formulas for the rise versions of these polynomials at  $t = 1/q$ . However, we note that these recursions do not use the  $t = 1/q$  assumption, so they may be useful in future work that removes this assumption.

We will use two well-known results which can be thought of as  $q$ -analogs of the binomial theorem:

$$\prod_{i=0}^{n-1} (1 + q^i x) = \sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \quad (5.14)$$

$$\prod_{i=0}^{n-1} \frac{1}{1 - q^i x} = \sum_{k=0}^n \begin{bmatrix} n + k - 1 \\ k \end{bmatrix}_q x^k. \quad (5.15)$$

These identities have the following combinatorial significance. In (5.14), we are choosing  $k$  objects without replacement out of  $n$  objects which have "weights"  $0, 1, \dots, n-1$ . Then the sum of  $q$  to the sum of the chosen objects' weights over all possible choices is equal to  $q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$ . For (5.15), we have the same setup but we are now choosing with replacement. The identity asserts that the same generating function in  $q$  is now equal to  $\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$ . We will use these identities repeatedly to develop our recursions.

### 5.3.1 Rise-decorated Schröder paths

#### Proposition 5.3.1.1.

$$\begin{aligned} \overline{RS}_{a,b,k}^{(r,s,i)}(q,t) &= q^{\binom{r}{2} + \binom{i+1}{2}} t^{a-r+b-s-k} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \\ &\quad \times \sum_{h=1}^{a-r+b-s} \begin{bmatrix} h-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ h \end{bmatrix}_q RS_{a-r,b-s,k-i}^{(h)}(q,t). \end{aligned}$$

*Proof.* We claim that the following process uniquely creates each path in  $\overline{RS}_{a,b,k}^{(r,s,i)}$ .

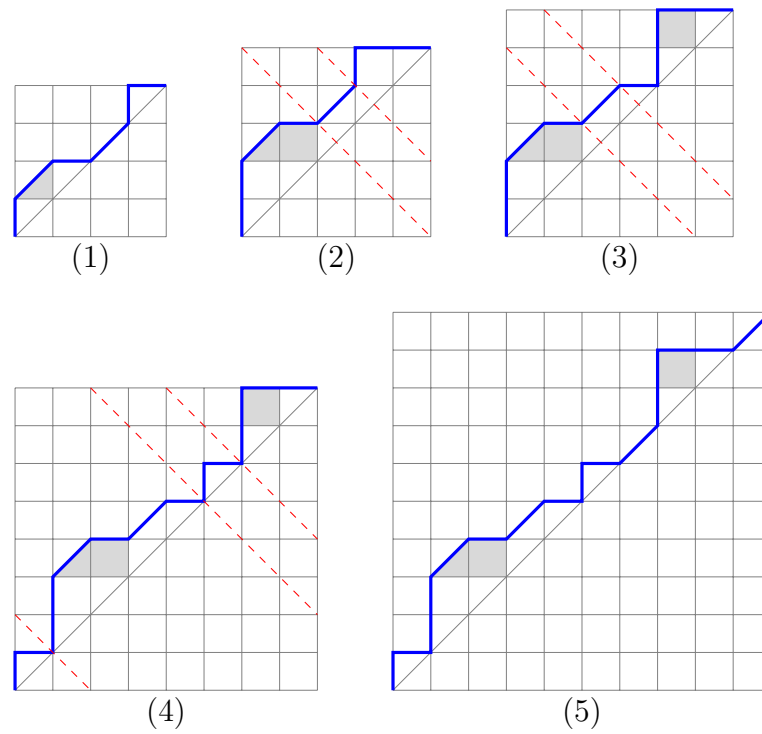
1. First, we choose  $h \in \{1, \dots, a-r+b-s\}$ . We begin with any path  $(P_1, R_1) \in \mathcal{RS}_{a-r,b-s,k-i}^{(h)}$ .
2. We prepend a north step and append an east step to  $P_1$  to obtain  $(P_2, R_2) \in \overline{\mathcal{RS}}_{a-r+1,b-s,k-i}^{(1,0,0)}$ . Since this increases the area of each row in  $P_1$  that is not in  $R_1$  by 1, we have

$$\text{area}^-((P_2, R_2)) - \text{area}^-((P_1, R_1)) = a-r+b-s - (k-i).$$

3. Next, we break  $P_2$  into  $h$  segments, where each segment begins with a north or diagonal step from the superdiagonal and each segment ends with the next return to the superdiagonal. (The first segment includes the initial north step.) We ignore the first segment and choose  $i$  out of the remaining segments to which we prepend an east step followed by a north step. We decorate the first row of each of these chosen segments. This results in a path  $(P_3, R_3) \in \overline{\mathcal{RS}}_{a-r+i+1,b-s,i}^{(i+1,0,i)}$ .

We need to check how we have altered our area and dinv statistics. We note that

$$\text{area}^-((P_3, R_3)) - \text{area}^-((P_2, R_2)) = -i$$



**Figure 5.3:** An example of the process in this proof with  $a = 6$ ,  $b = 4$ ,  $r = 4$ ,  $s = 2$ ,  $h = 3$ ,  $k = 2$ ,  $i = 1$ ,  $u = 01101$ , and  $v = 111010$ .



so we have

$$\text{area}^-(P_3, R_3) - \text{area}^-(P_1, R_1) = a - r + b - s - k.$$

None of the remaining steps will affect the area, so we have explained the  $t^{a-r+b-s-k}$  term in our recursion. Skipping the first segment (which we were not allowed to choose), we note that choosing the  $j$ th segment from the bottom contributes  $j$  secondary diagonal inversions. No matter which segments we chose, we obtain  $\binom{i}{2}$  primary diagonal inversions on the main diagonal. As a result, summing over all possible choices yields a contribution of  $q^{\binom{i}{2} + \binom{i+1}{2}} \left[ \begin{matrix} h-1 \\ i \end{matrix} \right]_q$ .

4. Now we will add the  $r - i - 1$  remaining north steps to the diagonal of  $P_3$ . We accomplish this by dividing  $P_3$  into segments once again. We proceed from the bottom to the top of  $P_3$ , ending each segment (and then beginning the next segment) whenever we encounter an east or a diagonal step which ends at the superdiagonal. (We include the final east step in the last segment.) We have  $h$  segments in all. Given any word  $u$  consisting of  $r - i - 1$  0's and  $h$  1's, we build a new path  $P_4$  by beginning at  $(0,0)$ , reading the word  $u$  from left to right, and acting as follows:

- Assume we are considering  $u_j$ , the  $j$ th letter of  $u$ . Say  $u_j = 0$ . If we are on the superdiagonal, we add an east step. Then we must be on the main diagonal. We add a north step. If the next unplaced segment (from bottom to top) of  $P_3$  was on the main diagonal in  $P_3$ , place an east step immediately after this north step.
- Otherwise,  $u_j = 1$ . If we are currently on the superdiagonal and the next unplaced segment was on the main diagonal in  $P_3$ , we add an east step. We append the next unplaced segment from  $P_3$ . One may note that, if this segment was on the diagonal in  $P_3$ , then it gets placed on the diagonal of this new path, and if this segment was on the superdiagonal in  $P_3$ , then it gets placed on the superdiagonal of this new path.

We denote this new path  $(P_4, R_4)$ , which must be in  $\overline{\mathcal{RS}}_{a,b-s,i}^{(r,0,i)}$ . We note that  $\text{area}^-(P_4, R_4) = \text{area}^-(P_3, R_3)$ . To count diagonal inversions, we must notice

that the new secondary diagonal inversions in  $P_4$  are in bijection with the inversions of  $u$ . Since we automatically gain  $\binom{r}{2} - \binom{i}{2}$  new primary diagonal inversions, summing over all words  $u$  contributes  $q^{\binom{r}{2} - \binom{i}{2}} \left[ \begin{smallmatrix} h+r-i-1 \\ h \end{smallmatrix} \right]_q$  to the recursion.

5. Finally, we need to place  $s$  diagonal steps on the main diagonal of  $P_4$ . To do this, we break  $P_4$  into segments upon each return to the main diagonal and we choose any word  $v$  consisting of  $s$  0's and  $r$  1's. We build a new path  $(P_5, R_5)$  by reading  $v$  from left to right and adding a diagonal step for each 0 and a segment of  $P_4$  for each 1. Summing over all words and counting the new (primary) diagonal inversions, we get a new term of  $\left[ \begin{smallmatrix} r+s \\ r \end{smallmatrix} \right]_q$ .

By reversing these steps and recording the words  $u$  and  $v$ , one can see that this process is bijective. Figure 5.3 depicts a particular example of this process. We have drawn the path created at the end of each step of a particular example with  $a = 6, b = 4, r = 4, s = 2, h = 3, k = 2, i = 1, u = 01101$ , and  $v = 111010$ . We have shaded in the canceled area cells and used dashed lines to separate the path into segments when the next step requires it. To go from (2) to (3) we chose segment 3 out of the possible choices of segments 2 and 3.

□

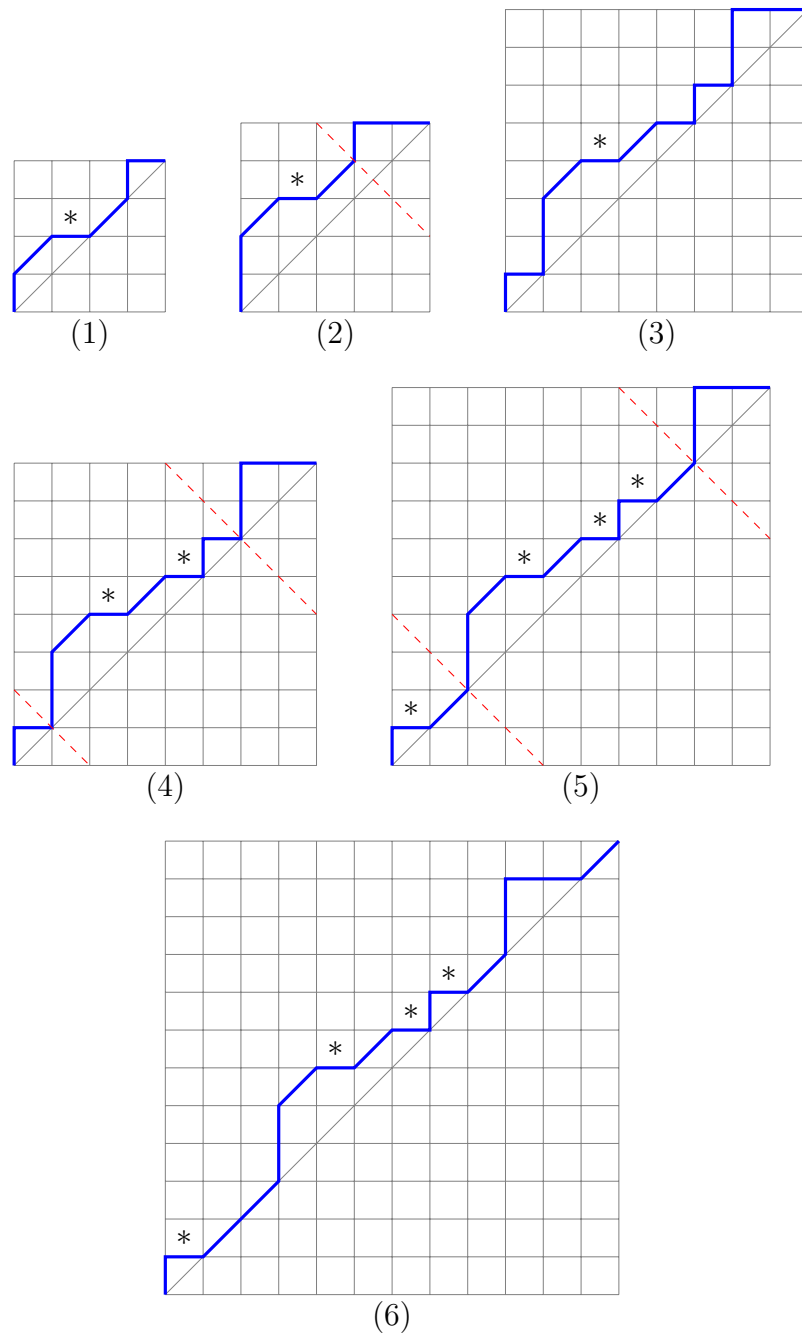
### 5.3.2 Valley-decorated Schröder paths

#### Proposition 5.3.2.1.

$$VS_{a,b,k}^{(r,s,i,j)}(q, t) = q^{\binom{r-i}{2} + \binom{j}{2}} t^{a-r+b-s} \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} r-i \\ j \end{bmatrix}_q \begin{bmatrix} r-i+s-j \\ r-i \end{bmatrix}_q \\ \times \sum_{g=1}^{a-r+b-s} \begin{bmatrix} g+r-1 \\ g \end{bmatrix}_q VS_{a-r,b-s,k-i-j}^{(g)}(q, t).$$

*Proof.* We build up valley-decorated Schröder paths in a bijective fashion, as in the example in Figure 5.4.

1. We choose  $h \in \{1, 2, \dots, a-r+b-s\}$  and a valley-decorated Schröder path  $(P_1, V_1) \in \mathcal{VS}_{a-r,b-s,k-i-j}^{(h)}$ .



**Figure 5.4:** An example of this process with  $a = b = 6$ ,  $k = 4$ ,  $r = 3$ ,  $s = 4$ ,  $i = 1$ ,  $j = 2$ , and  $h = 1$ .

2. We prepend a north step and append an east step to  $P_1$  to obtain  $(P_2, V_2)$  in  $\mathcal{VS}_{a-r+1, b-s, k-i-j}^{(1,0,0,0)}$ . This increases the area of each of the  $a - r + b - s$  rows in  $P_1$  by 1, for a total increase of  $a - r + b - s$ .
3. We break  $(P_2, V_2)$  into  $h$  segments by breaking wherever  $P_2$  returns to the super-diagonal unless this return is in  $V_2$ . We insert the remaining  $r - 1$  north steps into the  $h + 1$  spaces below, between, or above these  $g$  segments. Call the resulting decorated Schröder path  $(P_3, V_3) \in \mathcal{VS}_{a, b-s, k-i-j}^{(r,0,0,0)}$ . Since we can insert multiple north steps at a single space, this step contributes  $\begin{bmatrix} g+r-1 \\ g \end{bmatrix}_q$  secondary diagonal inversions. We postpone counting the new primary diagonal inversions until the next step.
4. Now we will decide which of these new north steps on the diagonal are decorated. It is clear that each of them except the very first north step occurs at a removable valley. We choose  $i$  of the remaining  $r - 1$  north steps to decorate. From a complementary viewpoint, we choose to  $r - j - 1$  of the north steps and we count the primary diagonal inversions ending at these steps. These north steps have  $0, 1, \dots, r - 2$  primary diagonal inversions above them, from top to bottom. We also must count the diagonal inversions ending at the bottom north step. Hence, this choice contributes  $\begin{bmatrix} r-1 \\ i \end{bmatrix}_q q^{\binom{r-i-1}{2} + r - i - 1} = \begin{bmatrix} r-1 \\ i \end{bmatrix}_q q^{\binom{r-i}{2}}$  primary diagonal inversions. The result is a decorated path  $(P_4, V_4) \in \mathcal{VS}_{a, b-s, k-j}^{(r,0,i,0)}$ .
5. Now we place the  $j$  diagonal steps on the diagonal that will be decorated valleys. We can do this after any east step that returns to the diagonal that is not already part of a decorated valley. There are  $r - i$  such places, we have  $j$  decorated diagonal steps to insert, and we can put at most one decorated diagonal step at each place. Furthermore, the places contribute  $0, 1, 2, \dots, r - i - 1$  new primary dinv, from bottom to top. Therefore this step contributes  $\begin{bmatrix} r-i \\ j \end{bmatrix}_q q^{\binom{j}{2}}$  new primary diagonal inversions. We call the resulting decorated path  $(P_5, V_5) \in \mathcal{VS}_{a, b, k}^{(r,j,i,j)}$ .
6. Finally, we wish to place the  $s - j$  remaining diagonal steps on the main diagonal. We can do this at the very bottom of the path or after any return to the diagonal that is not part of a decorated valley, and we can place more than 1 step in each place. There are  $r - i + 1$  such places, and we have  $s - j$  diagonal steps to insert.

We note that each such place, from bottom to top, contributes  $0, 1, \dots, r - i$  new primary diagonal inversions. Thus, this step contributes a term of  $\begin{bmatrix} r-i+s-j \\ r-i \end{bmatrix}_q$  new primary diagonal inversions, and the result is our final path  $(P_6, V_6) \in \mathcal{VS}_{a,b,k}^{(r,s,i,j)}$ .

As before, this process is directly invertible.  $\square$

### 5.3.3 Rise-decorated two-car parking functions

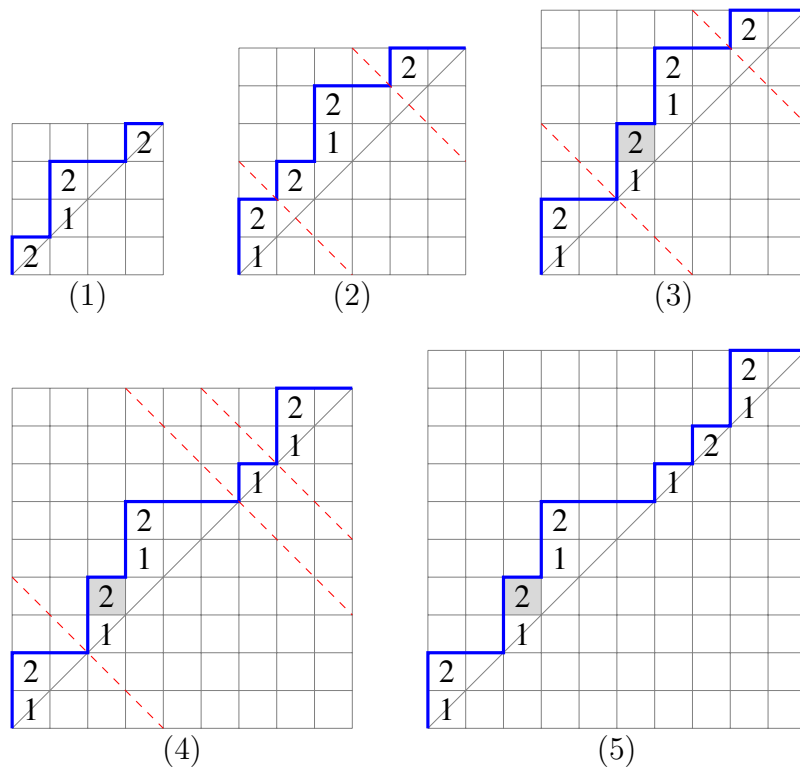
#### Proposition 5.3.3.1.

$$\begin{aligned} \overline{RT}_{a,b,k}^{(r,s,i)}(q, t) &= q^{\binom{i+1}{2}} t^{a-r+b-s-k} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \\ &\quad \times \sum_{h=1}^{b-s} \begin{bmatrix} h-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ h \end{bmatrix}_q RT_{a-r,b-s-1,k-i}^{(h-1)}(q, t). \end{aligned}$$

*Proof.* As is evident by the formula, the proof will be quite similar to the proofs of previous recursions. The main difference is that we can only place 1's underneath 2's, whereas before we could place north steps under both north steps and diagonal steps.

Figure 5.5 shows an example of this process.

1. Choose  $h \in \{1, \dots, b - s\}$ . We begin with any parking function  $(P_1, R_1) \in \mathcal{RT}_{a-r,b-s-1,k-i}^{(h-1)}$ .
2. To form  $(P_2, R_2) \in \overline{\mathcal{RT}}_{a+r+1,b-s,k-i}^{(1,0,0)}$ , we prepend a north-north-east sequence and append an east step to  $P_1$ , labeling the two new north steps 1 and 2 from bottom to top. This introduces  $a + b - r - s - (k - i)$  new area.
3. We split  $P_2$  into  $h$  segments, beginning a new segment each time we see a 2 on the superdiagonal. (We include the initial 1 as part of the first segment.) We ignore the first segment and then choose  $i$  of the remaining segments to prepend with an east step and a north step labeled 1. We cancel the rows following these new north steps. This process yields a parking function  $(P_3, R_3) \in \overline{\mathcal{RT}}_{a+b-r-s+i+1,b-s,k}^{(i+1,0,i)}$ . This step contributes a factor of  $t^{-i} q^{\binom{i+1}{2}} \begin{bmatrix} h-1 \\ i \end{bmatrix}_q$ .
4. Now we place the remaining  $r - i - 1$  1's on the main diagonal. We do this by separating  $(P_3, R_3)$  into  $h$  segments which begin each time we encounter either a



**Figure 5.5:** We have drawn the path created at the end of each step of a particular example with  $a = 5$ ,  $b = 5$ ,  $r = 4$ ,  $s = 1$ ,  $h = 3$ ,  $k = 1$ ,  $i = 1$ ,  $u = 11001$ , and  $v = 11101$ .

1 on the main diagonal (which is either the first 1 or is followed by a canceled row) or a 2 on the superdiagonal in a row that is not canceled (discounting the very first 2). We place 1's on the main diagonal by the same process used in Step 4 in the proof of Proposition 5.3.1.1, labeling each new north step with a 1. The result is a parking function  $(P_4, R_4) \in \overline{\mathcal{RT}}_{a,b-s,k}^{(r,0,i)}$  and a factor of  $\left[ \begin{smallmatrix} h+r-i-1 \\ h \end{smallmatrix} \right]_q$ .

5. Finally, we break  $(P_4, R_4)$  into segments each time it hits the main diagonal. We interlace the  $s$  new 2's the same way we introduced the new diagonal steps in Step 4 in the proof of Proposition 5.3.1.1, resulting in a factor  $\left[ \begin{smallmatrix} r+s \\ s \end{smallmatrix} \right]_q$  and the final parking function in  $\overline{\mathcal{RT}}_{a,b,k}^{(r,s,i)}$ .

As before, it is straightforward to verify that this process is bijective.  $\square$

### 5.3.4 Valley-decorated two-car parking functions

#### Proposition 5.3.4.1.

$$\begin{aligned} VT_{a,b,k}^{(r,s,i,j)}(q, t) &= q^{\binom{i}{2} + \binom{j}{2}} t^{a-r+b-s} \left[ \begin{smallmatrix} r-i \\ j \end{smallmatrix} \right]_q \left[ \begin{smallmatrix} r-i+s-j \\ r-i \end{smallmatrix} \right]_q \\ &\quad \times \sum_{g=1}^{b-s} \left[ \begin{smallmatrix} g+r-i-1 \\ g \end{smallmatrix} \right]_q \left[ \begin{smallmatrix} g \\ i \end{smallmatrix} \right]_q VS_{a-r,b-s-1,k-i-j}^{(g-1)}(q, t). \end{aligned}$$

*Proof.* As usual, we build up the relevant objects recursively. We show an example of this process in Figure 5.6.

1. We begin with  $(P_1, V_1) \in \mathcal{VT}_{a-r,b-s-1,k-i-j}^{(g-1)}$  for some  $g \in \{1, \dots, b-s\}$ .
2. As in the proof of Proposition 5.3.3.1, we prepend two north steps and an east step to  $P_1$  and append an east step. We label the first two north steps 1 and 2 and the resulting path  $(P_2, V_2) \in \mathcal{VT}_{a-r+1,b-s,k-i-j}^{(1,0,0,0)}$ . This step introduces a  $t^{a-r+b-s}$  term to the recursion.
3. Next, we break our path into segments at each undecorated 2 on the superdiagonal (except for the lowest 2 that we just introduced). We will place the  $r-i-1$  undecorated 1's on the diagonal. We can insert these before any segment or at the end of the sequence, for a total of  $g+1$  total slots. Since we are inserting





"with multiplicity," i.e. more than one 1 can go in each slot, and each slot yields  $0, 1, \dots, g$  new secondary divv, from bottom to top, this yields a term of  $\left[ \begin{matrix} g+r-i-1 \\ g \end{matrix} \right]_q$ .

We call the resulting path  $(P_3, V_3) \in \mathcal{VT}_{a-i, b-s, k-i-j}^{(r-i, 0, 0, 0)}$ .

4. Now we place the  $j$  decorated 2's on the diagonal. These can precede any 1 on the diagonal except the first or they can go at the end of the sequence, for a total of  $r-i$  possibilities. These possibilities introduce  $0, 1, \dots, r-i-1$  new primary divv, from bottom to top, and we can choose each possibility at most once, so we get a new term of  $\left[ \begin{matrix} r-i \\ j \end{matrix} \right]_q q^{\binom{j}{2}}$ . The new parking function is  $(P_4, V_4) \in \mathcal{VT}_{a-i, b-s+j, k-i}^{(r-i, j, 0, j)}$ .
5. The next step is to place the  $i$  decorated 1's on the diagonal. These 1's can go at any of the  $g$  returns to the diagonal from the superdiagonal. At most one 1 can go in each of these spots, and such an insertion introduces  $0, 1, \dots, g-1$  secondary divv, from bottom to top, yielding a term of  $\left[ \begin{matrix} g \\ i \end{matrix} \right]_q q^{\binom{i}{2}}$ .
6. Finally, we place the  $s-j$  undecorated 2's on the diagonal. They can be placed (with multiplicity) before any of the  $r-i$  undecorated 1's on the diagonal or at the very top, for a total of  $r-i+1$  spots. Inserting at each spot contributes  $0, 1, \dots, r-i$  new secondary divv, from bottom to top, so we get a contribution of  $\left[ \begin{matrix} r-i+s-j \\ r-i \end{matrix} \right]_q$ .

As in all other cases, inverting this bijection is straightforward.  $\square$

## 5.4 Inductions at $t = 1/q$

In this section, we prove  $q$ -binomial formulas for some of the polynomials introduced in the previous section with the additional assumption that  $t = 1/q$ . Together with Theorem 5.1.0.2, this completes the proof of two more cases of the Delta Conjectures. We will need the following three lemmas to simplify the various  $q$ -binomial and  $q$ -integer expressions which will appear in our calculations. All three lemmas are elementary and previously known, but we provide a citation or a short proof for each lemma for the sake of completeness.

**Lemma 5.4.0.1.** *If  $1 \leq m \leq n$  and  $\ell \geq 0$ , then*

$$\begin{bmatrix} n + \ell \\ n \end{bmatrix}_q = \sum_{j=0}^{\ell} q^{m(\ell-j)} \begin{bmatrix} m + j - 1 \\ m - 1 \end{bmatrix}_q \begin{bmatrix} n - m + \ell - j \\ n - m \end{bmatrix}_q$$

Lemma 5.4.0.1 is proven (as Lemma 3.1) in [HL13].

**Lemma 5.4.0.2.** *If  $0 \leq m \leq n + \ell$  and  $\ell \geq 0$ , then*

$$\begin{bmatrix} n + \ell \\ \ell \end{bmatrix}_q = \sum_{j=0}^{\ell} q^{j(n-m+j)} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n + \ell - m \\ \ell - j \end{bmatrix}_q$$

*Proof.* We can think of  $\begin{bmatrix} n + \ell \\ \ell \end{bmatrix}_q$  as  $\sum_{\sigma \in \mathfrak{S}_{1^n, 2^\ell}} q^{\text{inv}(\sigma)}$ . We fix some index  $m$  and assume there are  $j$  2's in the first  $m$  entries of the permutation. Then we choose a permutation for the first  $m$  entries and a permutation for the remaining  $n + \ell - m$  entries.  $\square$

**Lemma 5.4.0.3.** *For  $m, n \geq 0$  and  $0 \leq \ell \leq m, n$ ,*

$$q^\ell [m - \ell]_q [n - \ell]_q + [\ell]_q [m + n - \ell]_q = [m]_q [n]_q$$

*Proof.* Using the definition of  $q$ -integers, we see that the left-hand side is

$$\begin{aligned} &= [m - \ell]_q ([n]_q - [\ell]_q) + [\ell]_q [m + n - \ell]_q \\ &= [m - \ell]_q [n]_q + [\ell]_q ([m + n - \ell]_q - [m - \ell]_q) \\ &= [m - \ell]_q [n]_q + q^{m-\ell} [\ell]_q [n]_q \\ &= ([m - \ell]_q + q^{m-\ell} [\ell]_q) [n]_q \\ &= [m]_q [n]_q. \end{aligned}$$

$\square$

It will also be convenient to abbreviate the set of variables  $(q, 1/q)$  with just  $(q)$ .

For example, by  $RS_{a,b,k}(q)$  we mean  $RS_{a,b,k}(q, 1/q)$ .

## 5.4.1 Rise-decorated Schröder paths

**Proposition 5.4.1.1.** *For each formula, we assume  $a \geq 0, b \geq 0, 0 \leq r \leq a, 0 \leq s \leq b, 0 \leq p \leq a + b$ , and  $0 \leq k \leq r, a - r + b - s, a + b - p$ . If any of these conditions fails, then the polynomial at hand is equal to zero.*

(i) If  $r = 0$ , then

$$\overline{RS}_{a,b,k}^{(r,s,i)}(q) = \chi(a = k = i = 0)\chi(b = s).$$

Otherwise,

$$\begin{aligned} \overline{RS}_{a,b,k}^{(r,s,i)}(q) &= q^{\binom{a}{2} + \binom{k+2}{2} - 1 - (a-r)(a+b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\ &\times \begin{bmatrix} 2a-r+b-s-k-1 \\ a \end{bmatrix}_q q^{i(a-r-k+i)} \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} a-r \\ k-1 \end{bmatrix}_q. \end{aligned}$$

(ii) If  $a = 0$ , then

$$RS_{a,b,k}^{(h)}(q) = \chi(k = 0)\chi(b = h).$$

Otherwise

$$\begin{aligned} RS_{a,b,k}^{(h)}(q) &= q^{\binom{a+1}{2} + \binom{k+1}{2} - a(a+b-h+1)} \frac{[h]_q}{[a+b]_q} \begin{bmatrix} a \\ k \end{bmatrix}_q \\ &\times \begin{bmatrix} a+b \\ a \end{bmatrix}_q \begin{bmatrix} 2a+b-h-k-1 \\ a-1 \end{bmatrix}_q. \end{aligned}$$

**Corollary 5.4.1.1.** (i) We have

$$RS_{a,b,k}(q) = \frac{q^{\binom{a+1}{2} + \binom{k+1}{2} - a(a+b)}}{[a+b]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} 2a+b-k \\ a+1 \end{bmatrix}_q$$

(ii) We have

$$\begin{aligned} RS_{a,b,k}(q) + RS_{a,b,k-1}(q) &= \frac{q^{\binom{b}{2} + \binom{k}{2} - \binom{a+b}{2}}}{[a+b-k+1]_q} \begin{bmatrix} a+b \\ k \end{bmatrix}_q \\ &\times \begin{bmatrix} a+b-k+1 \\ b \end{bmatrix}_q \begin{bmatrix} 2a+b-k \\ a \end{bmatrix}_q \end{aligned}$$

*Proof of Proposition 5.4.1.1.* We proceed by induction on  $N = a + b$ . In particular, we will assume Proposition 5.4.1.1(ii) and use that to prove Proposition 5.4.1.1(i). Then we will prove Proposition 5.4.1.1(ii).

By Proposition 5.3.1.1,  $\overline{RS}_{a,b,k}^{(r,s,i)}(q, t)$  equals

$$q^{\binom{r}{2} + \binom{i+1}{2}} t^{a-r+b-s-k} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \sum_{h=1}^{a-r+b-s} \begin{bmatrix} h-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ h \end{bmatrix}_q RS_{a-r,b-s,k-i}^{(h)}(q, t).$$

If  $a = r$  then, by induction and Proposition 5.4.1.1.(ii), the only nonzero term occurs when  $i = k$  and  $h = b - s$ . Thus

$$\begin{aligned}\overline{RS}_{a,b,k}^{(r,s)}(q) &= q^{\binom{a}{2} + \binom{k+1}{2} - (b-s-k)} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} b-s-1 \\ k \end{bmatrix}_q \begin{bmatrix} a+b-s-k-1 \\ b-s \end{bmatrix}_q \\ &= q^{\binom{a+1}{2} + \binom{k+2}{2} - b+s-1} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \frac{[a]_q}{[b-s]_q} \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \begin{bmatrix} a+b-s-k-1 \\ a \end{bmatrix}_q\end{aligned}$$

after canceling out  $[b-s]_q!$  terms and introducing  $[a]_q!$  terms to the  $q$ -binomial coefficients. This expression is equal to that in Proposition 5.4.1.1.(i) when  $a = r$ .

If  $a > r$ , by induction,

$$\begin{aligned}\overline{RS}_{a,b,k}^{(r,s,i)}(q) &= q^{\binom{r}{2} + \binom{i+1}{2}} t^{a-r+b-s-k} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \sum_{h=1}^{a-r+b-s} \begin{bmatrix} h-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ h \end{bmatrix}_q \\ &\quad \times q^{\binom{a+1}{2} + \binom{k+1}{2} - a(a+b-h+1)} \frac{[h]_q}{[a+b]_q} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} a+b \\ a \end{bmatrix}_q \begin{bmatrix} 2a+b-h-k-1 \\ a-1 \end{bmatrix}_q.\end{aligned}$$

We will rearrange some of the terms in this last expression. We can write

$$[h]_q \begin{bmatrix} h-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ h \end{bmatrix}_q = [r]_q \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ r \end{bmatrix}_q.$$

At this point, only two of the  $q$ -binomials depend on  $h$ . Next, we carefully rewrite the power of  $q$  as follows.

$$\begin{aligned}& \binom{r}{2} - (a-r+b-s-k) + \binom{i+1}{2} + \binom{k-i+1}{2} + \binom{a-r}{2} \\ & \quad - (a-r)(a-r+b-s-h) \\ &= \left( \binom{r}{2} + \binom{a-r}{2} \right) + \left( \binom{i+1}{2} + \binom{k-i+1}{2} \right) \\ & \quad - (a-r)(a-r+b-s-h) - a+r-b+s+k \\ &= \binom{a}{2} + r^2 - ar + \binom{k+1}{2} + k + i^2 - ik \\ & \quad - (a-r)(a-r+b-s-h) - a+r-b+s \\ &= \binom{a}{2} + \binom{k+2}{2} - 1 - (a-r)(a+b-s) \\ & \quad + i(a-r-k+i) + (a-r)(h-i-1).\end{aligned}$$

This allows us to write  $\overline{RS}_{a,b,k}^{(r,s,i)}(q)$  as

$$\begin{aligned} & q^{\binom{a}{2} + \binom{k+2}{2} - 1 - (a-r)(a+b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\ & \times q^{i(a-r-k+i)} \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} a-r \\ k-1 \end{bmatrix}_q \\ & \times \sum_{h=i+1}^{a-r+b-s-k+i} q^{(a-r)(h-i-1)} \begin{bmatrix} h+r-i-1 \\ r \end{bmatrix}_q \begin{bmatrix} 2(a-r)+b-s-h-k+i-1 \\ a-r+b-s-h-k+i \end{bmatrix}_q. \end{aligned}$$

Let us examine the sum over  $h$ . By putting  $j = a - r + b - s - h - k + i$ ,  $\ell = a - r + b - s - k - 1$ ,  $m = a - r$ , and  $n = a$ , we can apply Lemma 5.4.0.1 replace the sum over  $h$  with

$$\begin{bmatrix} 2a - r + b - s - k - 1 \\ a \end{bmatrix}_q.$$

Hence

$$\begin{aligned} \overline{RS}_{a,b,k}^{(r,s,i)}(q) &= q^{\binom{a}{2} + \binom{k+2}{2} - 1 - (a-r)(a+b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\ &\times \begin{bmatrix} 2a - r + b - s - k - 1 \\ a \end{bmatrix}_q q^{i(a-r-k+i)} \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} a-r \\ k-1 \end{bmatrix}_q. \end{aligned}$$

This verifies Proposition 5.4.1.1.(i).

Next, we sum over  $i$  to obtain

$$\begin{aligned} \overline{RS}_{a,b,k}^{(r,s)}(q) &= \sum_{i=0}^k \overline{RS}_{a,b,k}^{(r,s,i)}(q) \\ &= q^{\binom{a}{2} + \binom{k+2}{2} - 1 - (a-r)(a+b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\ &\times \begin{bmatrix} 2a - r + b - s - k - 1 \\ a \end{bmatrix}_q \sum_{i=0}^k q^{i(a-r-k+i)} \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} a-r \\ k-1 \end{bmatrix}_q \\ &= q^{\binom{a}{2} + \binom{k+2}{2} - 1 - (a-r)(a+b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\ &\times \begin{bmatrix} 2a - r + b - s - k - 1 \\ a \end{bmatrix}_q \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \end{aligned}$$

by Lemma 5.4.0.2 with  $j = i$ ,  $\ell = k$ ,  $m = r - 1$ , and  $n = a - k - 1$ .

Our next goal is to remove the bar from over our polynomials. If  $k = 0$ , we notice that, if  $a > r$  and  $b > s$ , then  $RS_{a,b,0}^{(r,s)}(q) = \overline{RS}_{a,b,0}^{(r,s)}(q)$ . Otherwise, we must have

$a = r$  and  $b = s$ . In this case, we only consider paths that remain on the diagonal. It follows that  $RS_{a,b,0}^{(a,b)}(q) = q^{\binom{a}{2}} \begin{bmatrix} a+b \\ a \end{bmatrix}_q$ . Now we deal with  $k > 0$ . By definition,

$$RS_{a,b,k}^{(r,s)}(q) = \overline{RS}_{a,b,k}^{(r,s)}(q) + q\overline{RS}_{a,b,k-1}^{(r,s)}(q).$$

We now know that the right-hand side is equal to

$$\begin{aligned} & q^{\binom{a+1}{2} + \binom{k+1}{2} - (a-r+1)(a+b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\ & \times \left( q^k \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \begin{bmatrix} 2a-r+b-s-k-1 \\ a \end{bmatrix}_q + \begin{bmatrix} a-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 2a-r+b-s-k \\ a \end{bmatrix}_q \right) \\ & = q^{\binom{a+1}{2} + \binom{k+1}{2} - (a-r+1)(a+b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\ & \times \frac{[a-1]_q! [2a-r+b-s-k-1]_q!}{[a]_q! [a-r+b-s-k]_q! [k]_q! [a-k]_q!} \\ & \times (q^k [a-r+b-s-k]_q a-k + [k]_q [2a-r+b-s-k]_q). \end{aligned}$$

After setting  $\ell = k$ ,  $m = a$ , and  $n = a - r + b - s$  in Lemma 5.4.0.3, we see that

$$q^k [a-r+b-s-k]_q a-k + [k]_q [2a-r+b-s-k]_q = [a]_q [a-r+b-s]_q.$$

Making this substitution and gathering terms into  $q$ -binomial coefficients, we obtain

$$\begin{aligned} RS_{a,b,k}^{(r,s)}(q) & = q^{\binom{a+1}{2} + \binom{k+1}{2} - (a-r+1)(a+b-s)} \frac{[r]_q}{[2a-r+b-s-k]_q} \begin{bmatrix} a \\ k \end{bmatrix}_q \\ & \times \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \begin{bmatrix} 2a-r+b-s-k \\ a \end{bmatrix}_q. \end{aligned}$$

Now we sum over  $r$  and  $s$  to get the desired formula for  $RS_{a,b,k}^{(h)}(q)$ . The  $a = 0$  case of Proposition 5.4.1.1.(ii) follows from inspection. If  $a > 0$ , by definition we have

$$RS_{a,b,k}^{(h)}(q) = \sum_{r=1}^a RS_{a,b,k}^{(r,h-r)}(q).$$

The following identity will be helpful in rewriting our previous formula.

$$\begin{aligned} & \frac{[r]_q}{[2a-r+b-s-k]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} 2a-r+b-s-k \\ a \end{bmatrix}_q \\ & = \frac{[r+s]_q}{[a]_q} \begin{bmatrix} r+s-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} 2a-r+b-s-k-1 \\ a-1 \end{bmatrix}_q \end{aligned}$$

Now we have

$$\begin{aligned}
RS_{a,b,k}^{(h)}(q) &= \sum_{r=1}^a RS_{a,b,k}^{(r,h-r)}(q) \\
&= q^{\binom{a+1}{2} + \binom{k+1}{2} - a(a+b-h+1)} \frac{[h]_q}{[a]_q} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} 2a+b-h-k-1 \\ a-1 \end{bmatrix}_q \\
&\quad \times \sum_{r=1}^a q^{(r-1)(b-h+r)} \begin{bmatrix} h-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} a+b-h \\ a-r \end{bmatrix}_q.
\end{aligned}$$

Applying Lemma 5.4.0.2 with  $j = r - 1$ ,  $\ell = a - 1$ ,  $m = p - 1$ , and  $n = b$  completes the proof of Proposition 5.4.1.1.(ii).  $\square$

*Proof of Corollary 5.4.1.1.* We sum over  $h$  to get a formula for  $RS_{a,b,k}(q)$ . If  $a = 0$ , every parking function must remain on the main diagonal, so we cannot cancel any rows. Therefore

$$RS_{0,b,k}(q) = \chi(k = 0)$$

which is equal to the given formula when  $a = 0$ . Now assume  $a > 0$ . Using Proposition 5.4.1.1.(ii), we have

$$\begin{aligned}
RS_{a,b,k}(q) &= \sum_{h=1}^{a+b-k} S_{a,b,k}^{(h)}(q) \\
&= \frac{q^{\binom{a+1}{2} + \binom{k+1}{2} - a(a+b)}}{[a+b]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q \begin{bmatrix} a \\ k \end{bmatrix}_q \\
&\quad \times \sum_{h=1}^{a+b-k} q^{a(h-1)} \begin{bmatrix} h \\ 1 \end{bmatrix}_q \begin{bmatrix} 2a+b-h-k-1 \\ a-1 \end{bmatrix}_q \\
&= \frac{q^{\binom{a+1}{2} + \binom{k+1}{2} - a(a+b)}}{[a+b]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} 2a+b-k \\ a+1 \end{bmatrix}_q.
\end{aligned}$$

using Lemma 5.4.0.1 with  $j = a + b - p - k$ ,  $\ell = a + b - k - 1$ ,  $m = a$ , and  $n = a + 1$ .

Finally, we wish to prove Corollary 5.4.1.1.(ii). First, let us examine the case where  $k = 0$ . Then our desired coefficient is equal to  $RS_{a,b,0}(q)$ . By Corollary 5.4.1.1.(i),  $RS_{a,b,0}(q)$  is equal to the desired formula. Now we assume  $k > 0$ . By the definition of  $RS_{a,b,k}(q)$ , we want to show that  $RS_{a,b,k}(q) + RS_{a,b,k-1}(q)$  is equal to the formula given in the statement of Corollary 5.4.1.1.(ii). By Corollary 5.4.1.1.(i),  $RS_{a,b,k}(q) + RS_{a,b,k-1}(q)$

equals

$$\begin{aligned}
& \frac{q^{\binom{a+1}{2} + \binom{k}{2} - a(a+b)}}{[a+b]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q \left( q^k \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} 2a+b-k \\ a+1 \end{bmatrix}_q + \begin{bmatrix} a \\ k-1 \end{bmatrix}_q \begin{bmatrix} 2a+b-k+1 \\ a+1 \end{bmatrix}_q \right) \\
&= \frac{q^{\binom{a+1}{2} + \binom{k}{2} - a(a+b)} [a]_q! [2a+b-k]_q!}{[a+b]_q [k]_q! [a-k+1]_q! [a+1]_q! [a+b-k]_q!} \\
&\quad \times \left( q^k [a-k+1]_q [a+b-k]_q + [2a+b-k+1]_q [k]_q \right).
\end{aligned}$$

If we set  $\ell = k$ ,  $m = a + 1$ , and  $n = a + b$  in Lemma 5.4.0.3, we see that the inner sum is equal to  $[a+1]_q [a+b]_q$ . Therefore the entire expression is

$$\begin{aligned}
&= \begin{bmatrix} a+b \\ a \end{bmatrix}_q \frac{q^{\binom{a+1}{2} + \binom{k}{2} - a(a+b)} [a+1]_q! [2a+b-k]_q!}{[k]_q! [a-k+1]_q! [a+1]_q! [a+b-k]_q!} \\
&= \frac{q^{\binom{a+1}{2} + \binom{k}{2} - a(a+b)} [a+b]_q! [a+1]_q! [2a+b-k]_q!}{[a]_q! [b]_q! [k]_q! [a-k+1]_q! [a+1]_q! [a+b-k]_q!}.
\end{aligned}$$

Replacing both occurrences of  $[a+1]_q!$  with  $[a+b-k+1]_q!$  and partitioning  $q$ -factorials into  $q$ -binomial coefficients, we can get

$$= \frac{q^{\binom{a+1}{2} + \binom{k}{2} - a(a+b)}}{[a+b-k+1]_q} \begin{bmatrix} a+b \\ k \end{bmatrix}_q \begin{bmatrix} a+b-k+1 \\ b \end{bmatrix}_q \begin{bmatrix} 2a+b-k \\ a \end{bmatrix}_q.$$

Finally, we notice that  $\binom{a+1}{2} - a(a+b) = \binom{b}{2} - \binom{a+b}{2}$ . This gives the result stated in Corollary 5.4.1.1.(ii).  $\square$

## 5.4.2 Rise-decorated two-car parking functions

**Proposition 5.4.2.1.** *For each formula, we assume  $a \geq 0$ ,  $b \geq 0$ ,  $0 \leq r \leq a$ ,  $0 \leq s \leq b$ , and  $0 \leq k \leq r$ ,  $a - r + b - s$ . If any of these conditions fails, then the polynomial at hand is equal to zero.*

(i) *If  $r = 0$ , then*

$$\overline{RT}_{a,b,k}^{(r,s,i)}(q) = \chi(a = k = i = 0) \chi(b = s).$$

*Otherwise,*

$$\begin{aligned}
\overline{RT}_{a,b,k}^{(r,s,i)}(q) &= q^{\binom{k+2}{2} - (a-r+1)(b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\
&\quad \times \begin{bmatrix} a+b-s-k-1 \\ a \end{bmatrix}_q q^{i(a-r-k+i)} \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} a-r \\ k-i \end{bmatrix}_q.
\end{aligned}$$



(ii) If  $a = 0$ , then

$$RT_{a,b,k}^{(s)}(q) = \chi(b = s)\chi(k = 0).$$

Otherwise,

$$RT_{a,b,k}^{(s)}(q) = q^{\binom{k+1}{2} - a(b-s)} \frac{[s+1]_q}{[b+1]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q \begin{bmatrix} a+b-s-k-1 \\ a-1 \end{bmatrix}_q \begin{bmatrix} a \\ k \end{bmatrix}_q.$$

**Corollary 5.4.2.1.** (i)

$$RT_{a,b,k}(q) = \frac{q^{\binom{k+1}{2} - ab}}{[a+b-k+1]_q} \begin{bmatrix} a+b \\ k \end{bmatrix}_q \begin{bmatrix} a+b-k+1 \\ a+1 \end{bmatrix}_q \begin{bmatrix} a+b-k+1 \\ b+1 \end{bmatrix}_q.$$

(ii)

$$\begin{aligned} RT_{a,b,k}(q) + RT_{a,b,k-1}(q) &= \frac{q^{\binom{k}{2} - ab}}{[a+b-k+1]_q} \begin{bmatrix} a+b \\ k \end{bmatrix}_q \\ &\quad \times \begin{bmatrix} a+b-k+1 \\ a \end{bmatrix}_q \begin{bmatrix} a+b-k+1 \\ b \end{bmatrix}_q. \end{aligned}$$

*Proof of Proposition 5.4.2.1.* We will follow the proof of Proposition 5.4.1.1 quite closely.

Again, we work by induction on  $a + b = N$ . By Proposition 5.3.3.1,

$$\begin{aligned} \overline{RT}_{a,b,k}^{(r,s,i)}(q) &= q^{-(a-r+b-s-k-1)} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \\ &\quad \times \sum_{h=i+1}^{b-s-k+i} q^{\binom{i+1}{2}} \begin{bmatrix} h-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ h \end{bmatrix}_q RT_{a-r,b-s-1,k-i}^{(h-1)}(q). \end{aligned}$$

If  $r = a$  then, by induction and Proposition 5.4.2.1.(ii), the only nonzero terms occurs at

$i = k$  and  $h = b - s$ . Thus  $\overline{RT}_{a,b,k}^{(r,s)}(q)$  equals

$$\begin{aligned} & q^{-(b-s-k-1) + \binom{k+1}{2}} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} b-s-1 \\ k \end{bmatrix}_q \begin{bmatrix} b-s+a-k-1 \\ b-s-k \end{bmatrix}_q RT_{0,b-s-1,0}^{(b-s-1)}(q) \\ &= q^{\binom{k+1}{2} - (b-s-k-1)} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} b-s-1 \\ k \end{bmatrix}_q \begin{bmatrix} a+b-s-k-1 \\ b-s-k \end{bmatrix}_q \\ &= q^{\binom{k+2}{2} - (b-s)} \frac{[a]_q}{[b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a+b-s-k-1 \\ a \end{bmatrix}_q \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \end{aligned}$$

which is the desired formula.

Now we assume that  $a > r$ . The induction hypothesis gives

$$\begin{aligned}
\overline{RT}_{a,b,k}^{(r,s,i)}(q) &= q^{-(a-r+b-s-k-1)} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \\
&\quad \times \sum_{h=i+1}^{b-s-k+i} q^{\binom{i+1}{2}} \begin{bmatrix} h-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ h \end{bmatrix}_q RT_{a-r,b-s-1,k-i}^{(h-1)}(q) \\
&= q^{-(a-r+b-s-k-1)} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \sum_{h=i+1}^{b-s-k+i} q^{\binom{i+1}{2}} \begin{bmatrix} h-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ h \end{bmatrix}_q \\
&\quad \times q^{\binom{k-i+1}{2} - (a-r)(b-s-h)} \frac{[h]_q}{[b-s]_q} \begin{bmatrix} a-r+b-s-1 \\ a-r \end{bmatrix}_q \\
&\quad \times \begin{bmatrix} a-r+b-s-h-k+i-1 \\ a-r-1 \end{bmatrix}_q \begin{bmatrix} a-r \\ k-i \end{bmatrix}_q.
\end{aligned}$$

We will rearrange some of the terms in this last expression. We can write

$$[h]_q \begin{bmatrix} h-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ h \end{bmatrix}_q = [r]_q \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} h+r-i-1 \\ r \end{bmatrix}_q$$

At this point, only two of the  $q$ -binomials depend on  $h$ . Therefore we have

$$\begin{aligned}
\overline{RT}_{a,b,k}^{(r,s)}(q) &= q^{k-(a-r-1)(b-s+1)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\
&\quad \times q^{\binom{i+1}{2} + \binom{k-i+1}{2} + (a-r)(i+1)} \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} a-r \\ k-i \end{bmatrix}_q \\
&\quad \times \sum_{h=i+1}^{b-s-k+i} q^{(a-r)(h-i-1)} \\
&\quad \times \begin{bmatrix} r+h-i-1 \\ r \end{bmatrix}_q \begin{bmatrix} a-r-1+b-s-h-k+i \\ a-r-1 \end{bmatrix}_q.
\end{aligned}$$

Let us examine the sum over  $h$ . By putting  $\ell = b-s-k-1$ ,  $j = b-s-h-k+i$ ,  $m = a-r$ , and  $n = a$ , we can apply Lemma 5.4.0.1 to this sum, resulting in the identity

$$\begin{aligned}
&\sum_{h=i+1}^{b-s-k+i} q^{(a-r)(h-i-1)} \begin{bmatrix} r+h-i-1 \\ r \end{bmatrix}_q \begin{bmatrix} a-r-1+b-s-h-k+i \\ a-r-1 \end{bmatrix}_q \\
&= \begin{bmatrix} a+b-s-k-1 \\ a \end{bmatrix}_q.
\end{aligned}$$

Now we have

$$\begin{aligned} \overline{RT}_{a,b,k}^{(r,s,i)}(q) &= q^{k-(a-r-1)(b-s+1)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\ &\quad \times \begin{bmatrix} a+b-s-k-1 \\ a \end{bmatrix}_q q^{\binom{i+1}{2} + \binom{k-i+1}{2} + (a-r)(i+1)} \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} a-r \\ k-i \end{bmatrix}_q \end{aligned}$$

We compute that

$$\binom{i+1}{2} + \binom{k-i+1}{2} = \binom{k+1}{2} - ik + i^2.$$

Therefore

$$\begin{aligned} \overline{T}_{a,b,k}^{(r,s,i)}(q) &= q^{\binom{k+2}{2} - (a-r+1)(b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \\ &\quad \times \begin{bmatrix} a+b-s-k-1 \\ a \end{bmatrix}_q q^{i(a-r-k+i)} \begin{bmatrix} r-1 \\ i \end{bmatrix}_q \begin{bmatrix} a-r \\ k-i \end{bmatrix}_q. \end{aligned}$$

Now we want to show that the sum over  $i$  is equal to  $\begin{bmatrix} a-1 \\ k \end{bmatrix}_q$ . This follows from an application of Lemma 5.4.0.2 with  $j = i$ ,  $\ell = k$ ,  $m = r - 1$ , and  $n = a - k - 1$ .

If  $k = 0$ ,  $\overline{RT}_{a,b,k}^{(r,s)}(q) = RT_{a,b,k}^{(r,s)}(q)$ . Otherwise, we know that

$$T_{a,b,k}^{(r,s)}(q) = q \overline{T}_{a,b,k-1}^{(r,s)}(q) + \overline{T}_{a,b,k}^{(r,s)}(q).$$

We now know that the right-hand side is equal to

$$\begin{aligned} & q^{\binom{k+1}{2} - (a-r+1)(b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \times \\ & \left( \begin{bmatrix} a+b-s-k \\ a \end{bmatrix}_q \begin{bmatrix} a-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} a+b-s-k-1 \\ a \end{bmatrix}_q \begin{bmatrix} a-1 \\ k \end{bmatrix}_q \right). \end{aligned}$$

Canceling common terms from both sides and using the identity

$$[a]_q [b-s]_q = [k]_q [a+b-s-k]_q + q^k [a-k]_q [b-s-k]_q.$$

which follows from Lemma 5.4.0.3 with  $m = a$ ,  $n = b - s$ , and  $\ell = k$ , we get

$$\begin{aligned} RT_{a,b,k}^{(r,s)}(q) &= q^{\binom{k+1}{2} - (a-r+1)(b-s)} \frac{[r]_q}{[a-r+b-s]_q} \begin{bmatrix} r+s \\ r \end{bmatrix}_q \\ &\quad \times \begin{bmatrix} a-r+b-s \\ a-r \end{bmatrix}_q \begin{bmatrix} a-k+b-s-1 \\ a-k \end{bmatrix}_q \begin{bmatrix} b-s \\ k \end{bmatrix}_q. \end{aligned}$$

Finally, we sum over  $r$  to get the desired formula for  $T_{a,b}^{(s)}(q)$ .

$$\begin{aligned} T_{a,b,k}^{(s)}(q) &= \sum_{r=1}^a T_{a,b,k}^{(r,s)}(q) \\ &= q^{\binom{k+1}{2}-ab} \frac{[s+1]_q}{[b-s]_q} \begin{bmatrix} a-k+b-s-1 \\ a-k \end{bmatrix}_q \begin{bmatrix} b-s \\ k \end{bmatrix}_q \\ &\quad \times \sum_{r=1}^a q^{(r-1)(b-s)} \begin{bmatrix} r+s \\ r-1 \end{bmatrix}_q \begin{bmatrix} a-r+b-s-1 \\ a-r \end{bmatrix}_q. \end{aligned}$$

Applying Lemma 5.4.0.1 with  $m = b - s$ ,  $j = a - r$ ,  $\ell = a - 1$ , and  $n = b + 1$ , we obtain

$$\begin{aligned} &= q^{\binom{k+1}{2}-ab} \frac{[s+1]_q}{[b-s]_q} \begin{bmatrix} a-k+b-s-1 \\ a-k \end{bmatrix}_q \begin{bmatrix} b-s \\ k \end{bmatrix}_q \begin{bmatrix} a+b \\ a-1 \end{bmatrix}_q \\ &= q^{\binom{k+1}{2}-ab} \frac{[s+1]_q}{[b+1]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q \begin{bmatrix} a+b-s-k-1 \\ a \end{bmatrix}_q \begin{bmatrix} a \\ k \end{bmatrix}_q. \end{aligned}$$

□

*Proof of Corollary 5.4.2.1.* If  $a = 0$ , every parking function must remain on the main diagonal, so we cannot cancel any rows. Therefore

$$RT_{0,b,k}(q) = \chi(k=0)$$

which is equal to the given formula when  $a = 0$ .

Now assume  $a > 0$ . Using Proposition 5.4.2.1(ii), we have

$$\begin{aligned} RT_{a,b,k}(q) &= \sum_{s=0}^{b-k} RT_{a,b,k}^{(s)}(q) \\ &= \frac{q^{\binom{k+1}{2}-ab}}{[b+1]_q} \begin{bmatrix} a+b \\ a \end{bmatrix}_q \begin{bmatrix} a \\ k \end{bmatrix}_q \sum_{s=0}^{b-k} q^{as} \begin{bmatrix} s+1 \\ 1 \end{bmatrix}_q \begin{bmatrix} a+b-s-k-1 \\ a-1 \end{bmatrix}_q \\ &= \frac{q^{\binom{k+1}{2}-ab}}{[a+b-k+1]_q} \begin{bmatrix} a+b \\ k \end{bmatrix}_q \begin{bmatrix} a+b-k+1 \\ b+1 \end{bmatrix}_q \\ &\quad \times \sum_{s=0}^{b-k} q^{as} \begin{bmatrix} s+1 \\ 1 \end{bmatrix}_q \begin{bmatrix} a+b-s-k-1 \\ a-1 \end{bmatrix}_q. \end{aligned}$$

To simplify the sum, we use Lemma 5.4.0.1 with  $\ell = b - k$ ,  $j = b - s - k$ ,  $m = a$ , and  $n = a + 1$ . Thus

$$RT_{a,b,k}(q) = \frac{q^{\binom{k+1}{2}-ab}}{[a+b-k+1]_q} \begin{bmatrix} a+b \\ k \end{bmatrix}_q \begin{bmatrix} a+b-k+1 \\ a+1 \end{bmatrix}_q \begin{bmatrix} a+b-k+1 \\ b+1 \end{bmatrix}_q.$$

To prove Corollary 5.4.2.1(ii), we examine the case where  $k = 0$ . Then our desired coefficient is equal to  $RT_{a,b,0}(q)$ . By Proposition 5.4.2.1(i),  $RT_{a,b,0}(q)$  is equal to the desired formula.

Now we assume  $k > 0$ . By Proposition 5.4.2.1(i),  $RT_{a,b,k-1}(q) + RT_{a,b,k}(q)$  equals

$$\begin{aligned} & q^{\binom{k}{2}-ab} \left( \frac{q^k}{[a+b-k+1]_q} \begin{bmatrix} a+b \\ k \end{bmatrix}_q \begin{bmatrix} a+b-k+1 \\ a+1 \end{bmatrix}_q \begin{bmatrix} a+b-k+1 \\ b+1 \end{bmatrix}_q \right. \\ & \left. + \frac{1}{[a+b-k+2]_q} \begin{bmatrix} a+b \\ k-1 \end{bmatrix}_q \begin{bmatrix} a+b-k+2 \\ a+1 \end{bmatrix}_q \begin{bmatrix} a+b-k+2 \\ b+1 \end{bmatrix}_q \right) \\ & = \frac{q^{\binom{k}{2}-ab} [a+b]_q! [a+b-k+1]_q! [a+b-k+1]_q}{[a+b-k+1]_q! [k]_q! [a+1]_q! [b-k+1]_q! [b+1]_q! [a-k+1]_q!} \\ & \quad \times (q^k [b-k+1]_q a - k + 1 + [k]_q [a+b-k+2]_q). \end{aligned}$$

We can write the inner sum simply as  $[a+1]_q [b+1]_q$  after setting  $\ell = k$ ,  $m = a+1$ , and  $n = b+1$  in Lemma 5.4.0.3. Gathering terms into binomial coefficients concludes the proof.  $\square$

### 5.4.3 Valley-decorated cases

One may notice that we have only proved  $q$ -binomial formulas for the rise-decorated polynomials in this section. We should explain why we do not have analogous results for the valley-decorated polynomials at this point. The main reason is the historical development of the Delta Conjectures. The Rise Version of the Delta Conjecture has been circulated since early 2014, while the Valley Version is only a few months old and has not received the same amount of study.

However, there are some real barriers in the Valley Version case. First, computations performed in Sage suggest that  $VS_{a,b,k}^{(h)}(q)$  does not always factor as a product of  $q$ -binomial coefficients<sup>1</sup>. For example,

$$VS_{3,2,2}^{(2)}(q) = q^{-3} [4]_q (q^4 + 2q^3 + 2q^2 + q + 1).$$

<sup>1</sup>Interestingly enough, the polynomials  $VS_{a,b,k}^{(r,s,i,j)}(q)$  do seem to factor into  $q$ -binomial coefficients, so the summation is the problem.

This would mean that the methods we used above would be unlikely to work in this case. These computations are still preliminary, but they suggest that this case may be difficult.

In the valley-decorated two-car case, we do not have the same issue. In fact, the same computations suggest the surprising identity

$$VT_{a,b,k}^{(h)}(q) = RT_{a,b,k}^{(h)}(q)$$

which we cannot prove at this moment. The difficulty in this case comes from the recursion in Proposition 5.3.4.1. In particular, we have not been able to find a way to rearrange the  $q$ -binomial coefficients so that fewer than three  $q$ -binomial coefficients depend on  $g$ . All of our current  $q$ -binomial coefficient lemmas involve sums over at most two  $q$ -binomial coefficients. Perhaps this issue can be addressed by finding another recursion or by employing more sophisticated techniques, such as those coming from the study of hypergeometric series or the automated techniques developed by Petkovsek, Wilf, and Zeilberger [PWZ96].

# Chapter 6

## Future Directions

In this final chapter, we discuss several ways in which our research could be extended in the future. While we intend on investigating these areas ourselves, we welcome other mathematicians to think about these problems on their own or to ask their own questions. Although we have done our best to extend the knowledge of the Shuffle Conjecture to our Delta Conjectures, the huge amount of existing work on the Shuffle Conjecture makes this an impossible task for a single mathematician.

### 6.1 A Refinement of the Delta Conjectures

The original Shuffle Conjecture has seen a number of refinements based on how the underlying Dyck paths return to the main diagonal. In this section, we review these refinements and explain the issues involved in their transition to the setting of the Delta Conjectures.

In [GH01], Garsia and Haglund define polynomials  $E_{n,r}$  via the plethystic equation

$$e_n \left[ X \frac{1-z}{1-q} \right] = \sum_{r=1}^n \frac{(z; q)_r}{(q; q)_r} E_{n,r}$$

where  $(a; q)_n = (1-a)(1-aq) \dots (1-aq^{n-1})$ . One can show that  $\sum_{r=1}^n E_{n,r} = e_n$ . In [Hag04], Haglund conjectured that

$$\nabla E_{n,r} = \sum_P q^{\text{dinv}(P)} t^{\text{area}(P)} x^P$$

where the sum is over word parking functions  $P \in \mathcal{WPF}_n$  with exactly  $r$  returns to the main diagonal.

In [HMZ12], Haglund, Morse, and Zabrocki refine this conjecture further. Given a composition  $\alpha \vDash n$  of length  $r$ , they define

$$C_\alpha[X; q] = \mathbb{C}_{\alpha_1} \dots \mathbb{C}_{\alpha_r} 1$$

via the operator

$$\mathbb{C}_m f[X] = (-1/q)^{m-1} \sum_{i \geq 0} q^{-i} h_{m+i}[X] h_i[X(1-q)]^\perp f[X]$$

for any symmetric function  $f$ . We have used the skewing operator, which is defined by

$$\langle f^\perp g, h \rangle = \langle g, fh \rangle.$$

The authors then conjecture that

$$\nabla C_\alpha[X; q] = \sum_P q^{\text{dinv}(P)} t^{\text{area}(P)} x^P$$

where the sum is over word parking functions  $P \in \mathcal{WPF}_n$  whose  $r$  returns to the main diagonal come after exactly  $\alpha_1, \alpha_1 + \alpha_2, \dots$  north steps.

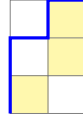
Unfortunately, the full generality of these conjectures do not hold in the delta setting. In particular, the symmetric functions  $\Delta_{e_k} E_{n,r}$  are not in  $\Lambda_{\mathbb{N}[q,t]}$  or even  $\Lambda_{\mathbb{Z}[q,t]}$ , i.e. they are not positive or even polynomial in the monomial basis. For example,

$$\Delta_{e_1} E_{2,1} = \left( \frac{qt-1}{q} \right) m_{1,1} - \frac{1}{q} m_2.$$

This makes it essentially impossible to provide a combinatorial formula for these polynomials.

However, we can salvage these conjectures in the Catalan case. In order to formulate these conjectures, we consider unlabeled Dyck paths with respect to leaning stacks, as defined in Subsection 1.3.2. We compute statistics on these objects by taking their maximum possible value among all labelings; this is analogous to how  $\text{dinv}$  is computed on Dyck paths. Then we have the following conjectures.





**Figure 6.1:** This Dyck path and stack pair is assigned to the composition  $\alpha = (2, 1)$ .

**Conjecture 6.1.0.1** (Compositional Catalan Delta Conjectures). *For any integers  $1 \leq k < n$  and any composition  $\alpha \vDash n$ ,*

$$\begin{aligned} \langle \Delta'_{e_k} C_\alpha, e_n \rangle &= \sum_{S \in \text{Stack}_{n,k}} \sum_D q^{\text{hdinv}(D)} t^{\text{area}(D)} \\ &= \sum_{S \in \text{Stack}_{n,k}} \sum_D q^{\text{wdinv}(D)} t^{\text{area}(D)}. \end{aligned}$$

where the second sums are over  $D \in \mathcal{D}(S)$  whose intersections with southeast edges of boxes in  $S$  occur exactly after  $\alpha_1, \alpha_1 + \alpha_2, \dots$  north steps.

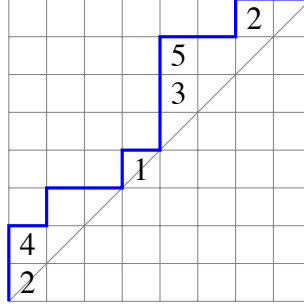
For example, the Dyck path and stack pair in Figure 6.1 would be assigned the composition  $\alpha = (2, 1)$ .

We obtain the  $\langle -, e_n \rangle$  case of the Compositional Shuffle Conjecture when  $k = n - 1$ . It would be interesting to see what other refinements can be made and if any of these refinements are amenable to proof. For example, it was obtaining a recursion for  $\langle \nabla E_{n,r}, e_n \rangle$  that finally allowed Garsia and Haglund to prove the Catalan case of the Shuffle Conjecture [GH03].

## 6.2 An Extension of the Delta Conjectures

In this section, we propose a generalization of the Delta Conjectures that we hope will lead to a better understanding of the delta operators in general. In particular, we will conjecture a formula for the symmetric function  $\Delta'_{e_k} \Delta_{h_\ell} e_n$ . In order to state this conjecture, we need to define a new generalization of parking functions. Given a Dyck path  $D \in \mathcal{D}_N$ , we obtain a *parking function with  $\ell$  blank valleys* by labeling  $N - \ell$  of the north steps of  $D$  with positive integers such that

- the labels strictly increase up each column, and



**Figure 6.2:** An element  $P \in \mathcal{WPF}_{8,2}^{\text{Blank}}$  with  $\text{area}(P) = 6$  and  $\text{dinv}(P) = 4$ .

- each of the  $\ell$  north steps that does not receive a label is a valley, i.e. it occurs immediately after an east step.

We have drawn an example  $P \in \mathcal{WPF}_{8,2}^{\text{Blank}}$  with  $\text{area}(P) = 6$  and  $\text{dinv}(P) = 4$  in Figure 6.2. More precisely, to compute  $\text{dinv}$  we count the pairs  $(1, 5)$ ,  $(2, 3)$ ,  $(2, 4)$ ,  $(2, 5)$ ,  $(4, 5)$ , and  $(7, 8)$  and then subtract  $\ell = 2$ .

We will write the set of all such labelings as  $\mathcal{WPF}_{N,\ell}^{\text{Blank}}$ . Each  $P \in \mathcal{WPF}_{N,\ell}^{\text{Blank}}$  inherits the area statistic from its underlying Dyck path, i.e.  $\text{area}(P) = \text{area}(D(P))$  and  $\text{area}_i(P) = \text{area}_i(D(P))$ . The diagonal inversion statistic must be stated more carefully. We say that  $\text{dinv}(P)$  counts the number of pairs  $(i, j)$  with  $1 \leq i < j \leq N$  such that

- $\text{area}_i(P) = \text{area}_j(P)$  and either row  $j$  is blank or neither row is blank and  $\ell_i(P) > \ell_j(P)$ , or
- $\text{area}_i(P) = \text{area}_j(P) + 1$  and either row  $j$  is blank or neither row is blank and  $\ell_i(P) < \ell_j(P)$

minus  $\ell$ . Recall that  $\ell_i(P)$  is the label in the  $i$ th row of  $P$ . One can check that when  $\ell = 0$  this recovers the usual notion of diagonal inversions on parking functions. Now we can state our conjectured combinatorial interpretation.

**Conjecture 6.2.0.2.** For any positive integers  $n, k$ , and  $\ell$  with  $k < n$ ,

$$\Delta'_{e_k} \Delta_{h_\ell} e_n = \sum_{P \in \mathcal{WPF}_{n+\ell,\ell}^{\text{Blank}}} q^{\text{dinv}(P)} t^{\text{area}(P)} x^P \prod_{i: w_i(P) > w_{i-1}(P)} (1 + z/t^{w_i(P)}) \Big|_{z^{n-k-1}}$$

We note that Conjecture 6.2.0.2 directly generalizes the Rise Version of the Delta Conjecture but not the Valley Version; it would be interesting to develop a connection between the Valley Version and this conjecture.

Currently, all evidence we have for Conjecture 6.2.0.2 is data computed in Sage. It would be interesting to prove some special cases of the conjecture, like we have done for the Delta Conjectures. In particular, it would be interesting to investigate the  $q = 0$  or  $t = 0$  cases of this new conjecture. The relevant objects are an extension of ordered set partitions, but it is unclear what statistics appear and if we can show these statistics are equidistributed. It also seems as if the Catalan case of this conjecture, i.e. taking scalar products of both sides with  $e_n$ , has some approachable recursions. In fact, any of the work in this dissertation could feasibly be extended from the Delta Conjectures to Conjecture 6.2.0.2.

Finally, it is straightforward to adjust Conjecture 6.2.0.2 to obtain a conjectured formula for the symmetric function  $\Delta_{e_k h_\ell} e_n$ . It would be nice to use the identity  $e_k h_\ell = s_{\ell, 1^k} + s_{\ell+1, 1^{k-1}}$  to obtain a conjectured formula for  $\Delta_{s_\lambda} e_n$  for any hook shape  $\lambda$ . Further in this direction, we may hope to eventually have some combinatorial understanding of any symmetric function of the form  $\Delta_f e_n$  for any  $f \in \Lambda$ , perhaps by understanding  $\Delta_{s_\lambda} e_n$  or possibly  $\Delta_{e_\lambda} e_n$  for all partitions  $\lambda$ .

### 6.3 Cyclic Sieving

We would also like to investigate whether our objects have any relationship to the cyclic sieving phenomenon of [RSW04]. Given a set of objects  $X$ , an action of the cyclic group  $C = \langle c \rangle$  of order  $n$  on  $X$ , and a polynomial  $X(q)$ , the triple  $(X, C, X(q))$  is said to exhibit the *cyclic sieving phenomenon* if

$$\#\{x \in X : c^d(x) = x\} = X(e^{2\pi id/n})$$

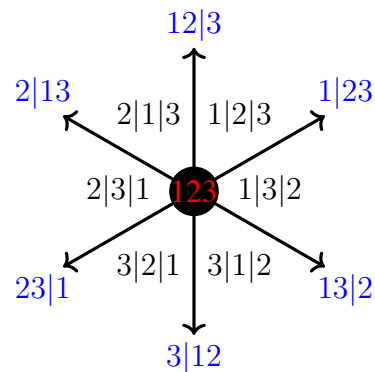
for every integer  $1 \leq d \leq n$ . In words, plugging roots of unity into the polynomial  $X(q)$  recovers the sizes of fixed point sets for the action of  $C$  on  $X$ . This phenomenon suggests a particularly strong relationship between the generating function  $X(q)$  and the set  $X$ .

Since its introduction, many authors have shown that various sets of combinatorial objects exhibit the cyclic sieving phenomenon; their work is nicely surveyed in [Sag11]. Perhaps the most relevant instance for us is the Catalan case. For any positive integer  $n$ , we set  $X$  to be the set of *non-crossing partitions* of the integers  $\{1, 2, \dots, n\}$ . We can draw such a partition by writing the integers 1 through  $n$  around a circle and connecting adjacent integers that share a block in the partition. Non-crossing partitions are the ones whose connecting arcs do not cross in such a depiction. The group  $\mathbb{Z}_n = \langle c \rangle$  acts on  $X$  by rotation by  $2\pi/n$  radians. If we set  $X(q) = \frac{1}{[n+1]_q} [2n]_q$ , then  $(X, C, X(q))$  exhibit the cyclic sieving phenomenon [RSW04]. Much work has been done to refine this case (by number of blocks, for example). Conjecture 6.2.0.2 suggests a natural generalization of Dyck paths: Dyck paths with some number of decorated double rises and some number of decorated valleys. One can translate these objects to decorated non-crossing partitions via standard bijections between the non-crossing partitions and Dyck paths. It would be interesting to see if these decorated objects, or some similar set of objects, exhibit a cyclic sieving phenomenon.

## 6.4 Other Problems

We have already seen at least two straightforward ways in which our work could be extended. The first is Conjecture 2.4.4.1, in which we proposed a distribution for the statistic  $\text{minimaj}$  on ordered multiset partitions. The second is in Chapter 5, where we were able to prove formulas for the Rise Version of the Delta Conjecture at  $t = 1/q$  but not the Valley Version; it would be nice to develop the same results for the Valley Version.

On the other hand, there are several potential extensions of our work that are currently less concrete. For example, there have been several efforts to extend (at least part of) the Shuffle Conjecture to other reflection groups [Stu10]. It would be interesting to see if the Delta Conjectures can be generalized in the same way. In particular, there is a notion of an ordered set partition for any reflection group; given the Coxeter hyperplane arrangement for a reflection group, an ordered set partition is defined to be an intersection of some number of regions which have been closed along their adjacent hyperplanes.



**Figure 6.3:** The set  $\mathcal{OP}_3$ , depicted as intersections of closed Coxeter regions.

In Figure 6.3, we have drawn the Coxeter arrangement for the symmetric group  $\mathfrak{S}_3$ . The regions are labeled by permutations, i.e. ordered set partitions with 3 blocks. The intersections of their closures are the 6 rays, which are each labeled (in blue) by an ordered set partition with 2 blocks. The intersection of all the rays is the origin, which receives the label 123 (in red), the only ordered set partition with 1 block.

It would be interesting to see any of our ordered set partition statistics could be generalized to this setting, particularly if the generalized statistics were equidistributed.

Finally, there has been a huge amount of research recently on rational extensions of the Shuffle Conjecture [BGLX14, ALW14, GLWX15b]. In Chapter 5, we even used some of this work to prove a Schur positivity result for our delta operator at  $t = 1/q$ . It seems possible that there is a stronger connection between the "rational world" and our "delta world," and this possibility should be investigated. At the very least, one can explore if decorating double rises or falls of rational parking functions leads to any interesting conjectures.

# Appendix A

## Completing the Proof of Proposition

### 2.2.3.1

In this appendix, we prove the following statement, which appears as (2.7) in Proposition 2.2.3.1:

$$\text{Val}_{n,k}(x; 0, q)|_{M_\alpha} = \sum_{\pi \in \mathcal{OP}_{\alpha, k+1}} q^{\text{minimaj}(\pi)}.$$

We will define a map

$$\gamma_{\alpha, k} : \mathcal{OP}_{\alpha, k+1} \rightarrow \{P \in \mathcal{PF}_{n, k}^{\text{Dense}} : \text{wdiv}(P) = 0, x^P = \prod_{i=1}^{\ell(\alpha)} x_i^{\alpha_i}\}.$$

Then we will prove that this map is a bijection which satisfies  $\text{area}(\gamma_{\alpha, k}(\pi)) = \text{minimaj}(\pi)$ . Given  $\pi \in \mathcal{OP}_{\alpha, k+1}$ , we consider the permutation  $\tau = \tau(\pi)$  as in the definition of  $\text{minimaj}$ . Let  $T$  be the positions of  $\tau$  of the entries which are minimal in their blocks in  $\pi$ . We define the *runs* of  $\tau$  to be its maximal, contiguous, weakly increasing sequences. For convenience, we label the runs from right to left, saying that the rightmost run is the 0th run. Say that  $\tau$  has  $s$  runs, and define positive integers  $n = r_0 > r_1 > \dots > r_s = 0$  such that the  $i$ th run of  $\tau$  is equal to  $\tau_{r_{i+1}+1} \dots \tau_{r_i}$ . Define  $b_1^i < \dots < b_{p_i}^i$  to be the positions of entries in the  $i$ th run of  $\tau$  which are the leftmost entries in blocks which are entirely contained in the  $i$ th run of  $\tau$ . Finally, for each  $i < s - 1$ , set  $b_0^i$  to be the position of the leftmost entry in  $\tau$  which shares a block with  $\tau_{r_{s-i}}$ .

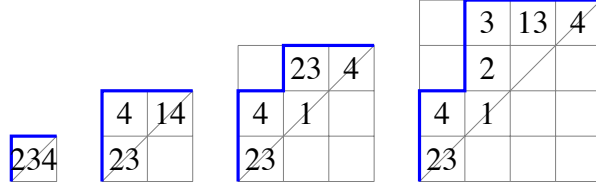
For example, set  $\pi = 13|23|14|234$  with  $\tau = 312341234$ . We decorate  $\tau$  with bars after its minimal elements to obtain  $31|23|41|234$ .  $\tau$  has 3 runs with  $r_3 = 0, r_2 = 1, r_1 = 5$ , and  $r_4 = 9$ . Using dashes to separate the runs, we get  $3 - 1|23|4 - 1|234$ . We compute  $b_1^0 = 7, b_0^0 = 5, b_1^1 = 3$  and  $b_0^1 = 1$ . Since the leftmost run does not contain any blocks, there are no  $b_j^2$ 's.

We define  $\gamma_{\alpha,k}(\pi)$  as follows. For  $i = 0$  to  $s-1$ , we will insert the elements of the  $i$ th run of  $\tau$  such that their rows in  $P$  each have area equal to  $i$ . After each  $i$ , we will obtain a *partial densely labeled Dyck path*  $P^{(i+1)}$ , which is densely labeled Dyck path whose set of labels does not necessarily form a composition. We begin with the empty densely labeled Dyck path  $P^{(0)}$ . To create  $P^{(1)}$ , we begin with the Dyck path  $(NE)^{p_0}$ . We label the squares from top to bottom with the sets  $\tau_{b_1^0} \dots \tau_{b_2^0-1}, \tau_{b_2^0} \dots \tau_{b_3^0-1}, \dots, \tau_{b_{p_0}^0} \dots \tau_n$ . Now we insert the entries  $\tau_{b_0^0} \dots \tau_{b_1^0-1}$  in a slightly more complicated fashion. We find the maximum entry in the northernmost square which is less than  $\tau_{b_0^0}$ ; call this element  $c$ . By the definition of  $\tau$ , such a  $c$  must exist. We insert a north step and then an east step immediately after the north step adjacent to this northernmost square. The new north square receives the label  $\tau_{b_0^0} \dots \tau_{r_1}$ . The new east square's label contains  $\tau_{r_1+1} \dots \tau_{b_1^0-1}$  along with the entries in  $c$ 's square which are greater than  $c$ . In other words, we move these entries from  $c$ 's square to the new east square. The result is  $P^{(1)}$ . We can check  $\text{wdiv}(P^{(1)}) = 0$ .

For greater values of  $i$ , we "repeat" this process as follows. We repeatedly insert  $\tau_{b_j^i} \dots \tau_{b_{j+1}^i-1}$  for  $j = p_i$  down to 1 just above the last east step added above. We leave the labels  $\tau_{r_i+1} \dots \tau_{b_1^{i-1}-1}$  in their east square and push the labels that were originally in  $c$ 's square that are greater than  $c$  so that they are always in the highest east square with area equal to  $i-1$ . Then we find the maximum entry  $c$  in the northernmost square with area  $i$  such that  $c < \tau_{b_0^i}$  and add new north and east squares as described above. The only remaining case to consider is if there is no  $b_0^i$ ; then the new east squares label is just the entries in  $c$ 's square which are greater than  $c$ . We produce an example in Figure A.1.

We note that, at each step, we have introduced zero  $\text{wdiv}$ , so  $\gamma_{\alpha,k}$  indeed maps to the paths

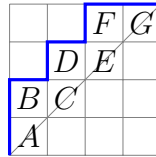
$$\{P \in \mathcal{PF}_{n,k}^{\text{Dense}} : \text{wdiv}(P) = 0, x^P = \prod_{i=1}^{\ell(\alpha)} x_i^{\alpha_i}\}.$$



**Figure A.1:** We compute  $\phi_{(2,2,3,2),3}(13|23|14|234)$ . From left to right, we depict  $P^{(1)}$ ,  $P^{(2)}$ ,  $P^{(3)}$ , and finally  $P^{(4)}$ .

To see that  $\gamma_{\alpha,k}$  is injective, we construct its inverse. We begin with the squares at maximum area in  $P$ . We remove them from top to bottom, using their labels to construct the blocks in the leftmost run in  $\tau$ . When we only have one square remaining at that area, we remove that square and form a block that consists of the labels in that square along with the smaller labels in the east square just to the right of that square (if there are any such labels). Then we move the larger labels into the north square below the square we just removed. We continue at the next largest area until all squares have been removed.

Next, we claim that  $\gamma_{\alpha,k}$  is surjective. Carefully inspecting the image of  $\gamma_{\alpha,k}$ , we note that it contains any  $P \in \mathcal{PF}_{n,k}^{\text{Dense}}$  with  $\text{w} \text{dinv}(P) = 0$  with the additional condition that every nonempty east square occurs either adjacent to the lowest two north squares with a given area or to the right of the uppermost north square at a given area. Essentially, if we see something of the form



then we must have  $E = \emptyset$ . It only remains to show that this condition is necessary in order to have  $\text{w} \text{dinv}(P) = 0$ . If  $E \neq \emptyset$ , it contains some element  $e$ . For  $f = \min(F)$  and  $d = \min(D)$ , we must have  $e < f \leq d$ , since we have zero total  $\text{w} \text{dinv}$ , so  $e < d$ . In order to have zero  $\text{w} \text{dinv}$ ,  $e$  cannot be involved in any more diagonal inversions. However, either  $e > a = \min(A)$  or  $e \leq a < b = \min(B)$ , so  $e$  is involved in at least one more diagonal inversion, meaning that the total  $\text{w} \text{dinv}$  cannot be zero. Thus we must have  $E = \emptyset$ .

Finally, we need to show that  $\text{area}(\gamma_{\alpha,k}(\pi)) = \text{minimaj}(\pi)$ . By definition, the



minimum major index of  $\pi$  is equal to  $\text{maj}(\tau)$ , which is equivalent to the sum

$$\sum_{i=0}^{s-1} i(\# \text{ of elements in run } i \text{ in } \tau).$$

Since each element of the  $i$ th run in  $\tau$  is placed in a square with area  $i$  in  $\gamma_{\alpha,k}(\pi)$ , we have  $\text{area}(\gamma_{\alpha,k}(\pi)) = \text{minimaj}(\pi)$ .

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