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Assorted Control Algorithms using Hybrid System Tools

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by

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To Naveen, Steven, Corrado, Sven, Kristen, Fabio, Xue Fang, Fulvio, Michelanglo, Iman, Ananth and Jorge.

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Abstract

Assorted Control Algorithms using Hybrid System Tools

by

Matthew J. Hartman

Hybrid systems are dynamical systems where the state is allowed to both flow continuously and jump discretely. This dissertation addresses three somewhat disparate problems that are united by using hybrid system tools for their resolution. The first topic is a parameter estimation algorithm where an existing algorithm is embedded within a hybrid system, thereby permitting the use of several relevant tools. It is shown that a set corresponding to correct parameter estimation has a robust stability property and that a persistency of excitation condition leads to convergence to the correct parameter estimate in finite time. The second topic involves implementing hybrid control through high-gain observers. Here results on robustness of stability are repeatedly exploited on the way to providing a semi-global practical stability result. Finally, the third topic is a distributed algorithm for synchronizing agents on a circle. Here a hybrid system approach is necessary to overcome the topological obstruction of being confined to a circle, while randomness is needed to overcome symmetry issues.

Contents

Notation

- $\mathbb{R}$ denotes the real numbers.
- $\mathbb{Z}$ denotes the integers.
- $\mathbb{R}^n$ denotes n-dimensional Euclidean space.
- $\mathbb{Z}_{\geq i}$ denotes the integers greater than or equal to i.
- $\mathbb{B}$ (resp. $\mathbb{B}^o$), denotes the closed (resp. open) unit ball in the Euclidian norm.
- Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_S := \inf_{y \in S} |x - y|$.
- A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing.
- A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and is unbounded.
- A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if it is continuous, non-decreasing in its first argument, non-increasing in its second argument, and $\lim_{s \searrow 0} \beta(s,t) = \lim_{t \to \infty} \beta(s,t) = 0$.

Chapter 1

Introduction

Hybrid systems are useful for dealing with a variety of control problems. From overcoming topological obstacles [29] to modeling systems that experience impacts, there are many situations where using a purely continuous or discrete-time system is inadequate to the task. The framework for hybrid systems developed by the authors of [18], emphasizing robustness of stability, has led to an extensive literature, expanding both theory and applications for the framework. This thesis adds to that literature by presenting three topics, each of which uses or expands the toolkit for hybrid systems.

1.1 Organization and Contributions

The remainder of this chapter introduces hybrid systems more broadly and presents many of the tools used in later chapters.

Chapter Two presents a parameter estimation algorithm that owes much to the work of Adetola and Guay. In [1], the authors present a parameter estimation algorithm that consists of several differential equations such that if a particular matrix in the state space becomes invertible, one can immediately solve for the parameter in question. The contribution this chapter makes is first to embed this algorithm into a hybrid system so that it runs perpetually rather than being 'one-shot'. We go on to show the existence of a globally asymptotically stable set that corresponds to estimating the parameter correctly, and that this set has robustness properties. We further give a Persistency of Excitation condition, under which we can ensure that the parameter estimation happens in finite time.

Chapter Three deals with the problem of implementing hybrid controllers for a continuous time plant when you have less than full state feedback available. Here we show that if you have an observability condition related to uniform complete observability [14], then you can implement such controllers using a high-gain observer, and obtain a semi-global practical result with respect to several parameters.

Chapter Four presents a synchronization algorithm for agents on a circle under limited communication. Here each agent has a state space that includes two circles: one on which only jumps are possible, and one on which only flowing is possible. The agents first converge in the 'jump-only' space by jumping to the position of one of their neighbors according to a particular criterion, leading to synchronization. Concurrently, for each agent the position in the 'flow-only' space tracks the position of the 'jump-only' space using a hybrid algorithm, thus leading to synchronization in both subspaces. We go on to show through heuristic argument that the algorithm performs well under a variety of conditions.

1.2 Hybrid Systems

There are many ways to model systems having states capable of both continuous and discrete change. Options include hybrid automata [22] and impulsive differential equations [26]. The motivation for considering an alternative to these models is to have a general framework that extends useful results from nonlinear systems; in particular, results on invariance principles [34], converse theorems [7], robustness of stability [16], singular perturbation theory [35] and averaging theory [46] are easily accessible through the framework we use. Moreover, systems modeled as hybrid automata can be reformulated into this model [15].

We consider hybrid systems modeled as follows:

$$\mathcal{H}: \quad x \in \mathbb{R}^n \quad \begin{cases} \dot{x} &= f(x), \quad x \in C \\ x^+ &= g(x), \quad x \in D. \end{cases} \tag{1.1}$$

When the state is in the set C (the "flow set") it is allowed to change continuously according to $\dot{x} = f(x)$; when the state is in the set D (the "jump set") it is allowed to jump according to $x^+ = g(x)$ where x^+ is the post-jump value of x. More generally, $\dot{x} = f(x)$ can be replaced by $\dot{x} \in F(x)$ where F is a set-valued mapping. Similarly, $x^+ = g(x)$ can be replaced by $x^+ \in G(x)$. Generalizing to set-valued mappings is useful for capturing the behavior of some physical phenomena, as well as modeling perturbations of (1.1).

To describe solutions to $\mathcal{H}$, we need the concept of a hybrid time domain. A subset E of $\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a hybrid time domain if it is the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times j$, where $0 = t_0 \leq t_1 \leq t_2 \leq ...$, or finitely many of such intervals, with the last one possibly of the form $[t_j, t_{j+1}] \times \{j\}$, $[t_j, t_{j+1}) \times \{j\}$, or $[t_j, \infty) \times \{j\}$. Hybrid time domains can be parameterized by (t, j), which means that $x(t, j)$ represents the state of the hybrid system after t amount of ordinary time and j jumps.

Solutions to $\mathcal{H}$ take the form of hybrid arcs. A hybrid arc is a function $\phi : E \to \mathbb{R}^n$ such that E is a hybrid time domain and $t \to \phi(t, j)$ is locally absolutely continuous for fixed j. We define *dom* ϕ as the hybrid time domain associated with hybrid arc ϕ.

A hybrid arc is a solution to $\mathcal{H}$ if $\phi(0,0) \in C \cup D$ and:

1) for all $j \in \mathbb{Z}_{\geq 0}$ and almost all t such that $(t,j) \in dom\ \phi$: $\phi(t,j) \in C$, $\dot{\phi}(t,j) \in F(\phi(t,j))$

2) for all $(t,j) \in dom\ \phi$ such that $(t,j+1) \in dom\ \phi$: $\phi(t,j) \in D$, $\phi(t,j+1) \in G(\phi(t,j))$.

A solution ϕ to a hybrid system is *complete* if *dom* ϕ is unbounded (in either the t or j direction). A solution ϕ is *maximal* if it cannot be extended, i.e., it is not a truncation of another solution ϕ' to some proper subset of *dom* ϕ'.

There is a set of regularity conditions on $\mathcal{H}$ useful for establishing various results including invariance principles, converse Lyapunov theorems, and robustness of stability (see [15]). Labeling these conditions the *Basic Assumptions*, they are given as follows: The sets C and D are closed; the mappings F and G are outer semicontinuous[1]and locally bounded[2]; $F(x)$ is non-empty and convex for all $x \in C$; $G(x)$ is non-empty for all $x \in D$. These assumptions ensure that the set of solutions is sequentially compact and semicontinuous with respect to initial conditions.

The following definitions pertain to stability in hybrid systems.

Definition 1 *A compact set $\mathcal{A}$ of a hybrid system is*

- stable *if for each $\epsilon > 0$ there exists $\delta > 0$ such that every solution x of $\mathcal{H}$ satisfies*

$$|x(0,0)|_{\mathcal{A}} \leq \delta \ \Rightarrow \ |x(t,j)|_{\mathcal{A}} \leq \epsilon \quad (t,j) \in dom\ x.$$

- attractive *if there exists a neighborhood of $\mathcal{A}$ from which each solution is bounded, and each complete solution converges to $\mathcal{A}$.*

[2]A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous if its graph $\{(x,y) : x \in \mathbb{R}^n, y \in F(x)\} \subset \mathbb{R}^{2n}$ is closed.

[3]The mapping F is locally bounded on a set C if for each compact set $K \subset C$, the set $F(K) := \underset{x \in K}{\cup} F(x)$ is bounded.

- asymptotically stable *if it is both stable and attractive.*

For a compact asymptotically stable set $\mathcal{A} \subset \mathbb{R}^n$, its *basin of attraction* is the set of points in $\mathbb{R}^n$ from which each solution is bounded, and each complete solution converges to $\mathcal{A}$. By definition, each point not in $C \cup D$ is in the basin of attraction. When the basin of attraction is $\mathbb{R}^n$, $\mathcal{A}$ is said to be *globally asymptotically stable* (GAS).

Let $\mathcal{A}$ be compact and $\mathcal{O}$ be an open set containing $\mathcal{A}$. A continuous function $\omega : \mathcal{O} \to \mathbb{R}_{\geq 0}$ is called a proper indicator for $\mathcal{A}$ on $\mathcal{O}$ when $\omega(x) = 0$ if and only if $x \in \mathcal{A}$, and also $\omega(x_i)$ tends to infinity when x_i tends to infinity or tends to the boundary of $\mathcal{O}$. Every open set $\mathcal{O}$ and compact set $\mathcal{A} \subset \mathcal{O}$ admits a proper indicator [25].

A characterization of asymptotic stability in terms of $\mathcal{KL}$ functions is as follows:

Proposition 1 *[15, Theorem 14] For a hybrid system $\mathcal{H}$ satisfying the Basic Assumptions, if a compact set $\mathcal{A}$ is globally asymptotically stable, then there exists $\beta \in \mathcal{KL}$ such that all solutions satisfy*

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j) \qquad \forall (t,j) \in dom\, x.$$

Thus, for compact sets, asymptotic stability is equivalent to uniform asymptotic stability.

For $\mathcal{P}$ and a continuous function $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, we define the σ-perturbation of $\mathcal{P}$, denoted $\mathcal{P}_\sigma$, through the data in (1.2) where $\overline{con}\,\Sigma$ is the closure of the convex hull of a set Σ.

Theorem 1 *For system $\mathcal{P}$ under Assumptions 3 and 4, suppose that the compact set $\mathcal{A}$ is asymptotically stable. Then there exists a continuous function $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying $\sigma(x) > 0$ for all $x \in \mathbb{R}^n \setminus \mathcal{A}$, such that the compact set $\mathcal{A}$ is asymptotically stable for the system $\mathcal{P}_\sigma$.*

$$
\mathcal{P}_\sigma \quad \left\{ \begin{array}{rcl} C_\sigma & := & \{x : (x+\sigma(x)\mathbb{B}) \cap C \neq \emptyset\}, \\ F_\sigma(x) & := & \overline{con} f((x+\sigma(x)\mathbb{B}) \cap C) + \sigma(x)\mathbb{B} \qquad \forall x \in C_\sigma, \\ D_\sigma & := & \{x : (x+\sigma(x)\mathbb{B}) \cap C \neq \emptyset\}, \\ G_\sigma(x) & := & \{v : v \in h + \sigma(h)\mathbb{B},\ h \in g((x+\sigma(x)\mathbb{B}) \cap D)\} \qquad \forall x \in D_\sigma. \end{array} \right\} \tag{1.2}
$$

The following represents a perturbation of a nominal hybrid system in a manner that covers both temporal regularization-type perturbations and "inner perturbations". Consider

$$
\begin{gathered}
\begin{bmatrix} \dot{x} \\ \dot{\tau} \end{bmatrix} \in \begin{bmatrix} f(x+\delta\mathbb{B}) \\ \sigma(\tau) \end{bmatrix}, \\
(x,\tau) \in \{(x,\tau) : (\{x\}+\delta\mathbb{B}) \cap C \neq \emptyset, \tau \in [0,2]\} \cup (\mathbb{R}^n \times [0,\delta]) \\
\begin{bmatrix} x^+ \\ \tau^+ \end{bmatrix} \in \begin{bmatrix} G(x+\delta\mathbb{B}) \\ 0 \end{bmatrix}, \\
(x,\tau) \in \{(x,\tau) : (\{x\}+\delta\mathbb{B}) \cap D \neq \emptyset, \tau \in [0,2]\},
\end{gathered} \tag{1.3}
$$

where $\delta \geq 0$ and σ is continuous. For this system we state a constant perturbations robustness result.

Proposition 2 *[17, Theorem 6.6] Suppose, for system* (1.3) *with $\delta = 0$, satisfying the Basic Assumptions, that a compact set $\mathcal{A} \times [0,2]$ is globally asymptotically stable. In particular, suppose that there exist $\beta \in \mathcal{KL}$ such that, for all solutions,*

$$
|x(t,j)|_{\mathcal{A}\times[0,2]} \leq \beta(|x(0,0)|_{\mathcal{A}\times[0,2]}, t+j) \qquad \forall (t,j) \in dom\,(x,\tau).
$$

Then, for each $\epsilon > 0$ and compact set K, there exists $\delta > 0$ such that each solution to

(1.2) *starting in K satisfies*

$$|x(t,j)|_{\mathcal{A}\times[0,2]} \leq \beta(|x(0,0)|_{\mathcal{A}\times[0,2]}, t+j) + \epsilon \qquad \forall (t,j) \in dom\,(x,\tau).$$

1.3 Stochastic Hybrid Systems

Stochastic hybrid systems have been extensively studied in the literature and several frameworks have been proposed. One of the important distinguishing factors of these frameworks relates to how the randomness affects the dynamics of the system. A summary of the various ways by which stochastic elements can be introduced in hybrid systems is listed in [9, Ch. 1].

In [11], [10] piecewise deterministic Markov process are modeled, in which the continuous-time dynamics are deterministic, discrete-time dynamics are random and jumps of the state occur either at random times or when the state exits an open domain. In [48],[20] hybrid switching diffusions are analyzed where the continuous-valued states are driven by a stochastic differential equation associated with a certain discrete-valued state and jumps occurring at random time lead to changes in this discrete-valued state. A general framework for stochastic hybrid systems is proposed in [23],[4] that can model hybrid systems with randomness affecting both the continuous and discrete dynamics while allowing for random jump times.

Most of the existing frameworks in the literature for modeling stochastic hybrid systems do not encompass systems that permit non-unique solutions. Non-unique solutions arise in the case of stochastic systems when analyzing robustness of stability properties [45],[39] and while defining notions of generalized random solutions [19]. In [42] the framework is extended to hybrid systems with non-unique solutions while allowing for stochastic elements in the jump map. This class of systems can be used to model systems with spontaneous transitions as illustrated in [42]. We note that this model does not allow for stochasticity in the flow map; such a generalization is considered in [43] where the author adds diffusive flows through constrained stochastic differential inclusions.

1.3.1 Modeling Framework

For the rest of this paper we will adopt the mathematical framework in [42]. We consider stochastic hybrid systems with a state $x \in \mathbb{R}^n$ and random variable $v \in \mathbb{R}^m$ written formally as

$$x \in C \qquad \dot{x} \in F(x) \tag{1.4a}$$

$$x \in D \qquad x^+ \in G(x, v^+) \tag{1.4b}$$

$$v \sim \mu(\cdot). \tag{1.4c}$$

As before, C and D denote the flow and jump sets while F and G denote the flow and jump maps. The distribution function μ is derived from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of independent, identically distributed (i.i.d.) input random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbf{v}_i : \Omega \to \mathbb{R}^m$, $i \in \mathbb{Z}_{\geq 1}$ denote a sequence of i.i.d random variables and $\mathbf{B}(\mathbb{R}^m)$ denote the Borel-σ field of $\mathbb{R}^m$. Then μ is defined as $\mu(A) := \mathbb{P}(\omega \in \Omega : \mathbf{v}_i(\omega) \in A)$ for every $A \in \mathbf{B}(\mathbb{R}^m)$ and is independent of i because the sequence of random variables $\{\mathbf{v}\}_{i=1}^{\infty}$ are i.i.d. We denote by $\mathcal{F}_i$ the collection of sets $\{\omega : (\mathbf{v}_1(\omega), ..., \mathbf{v}_i(\omega)) \in A\}$, $A \in \mathbf{B}((\mathbb{R}^m)^i)$ which are the sub-σ fields of $\mathcal{F}$ that form the natural filtration of $\mathbf{v} = \{\mathbf{v}_i\}_{i=1}^{\infty}$. The data of the system (1.4) will be represented as (C, F, D, G, μ) for simplicity. The definition of random solution to (1.4) requires concepts of measurability for set-valued mappings. For a measurable space (T, Γ), a mapping $M : T \rightrightarrows \mathbb{R}^n$ is *measurable* [32, Def. 14.1], if for each open set $\mathcal{O} \subset \mathbb{R}^n$, the set

$M^{-1}(\mathcal{O}) := \{t \in T : M(t) \cap \mathcal{O} \neq \varnothing\} \in \Gamma$. When the values of M are closed, measurability is equivalent to $M^{-1}(\mathcal{C})$ being measurable for each closed set $C \subseteq \mathbb{R}^n$ [32, Thm. 14.3].

We now define the notion of random solution to the stochastic hybrid systems. A mapping $\mathbf{x}$ from Ω to the set of hybrid arcs is a *random solution* of (1.4) starting at x, denoted by $\mathbf{x} \in \mathcal{S}_{\mathbf{r}}(x)$, if it satisfies the following properties.

1. (Feasibility) For every $\omega \in \Omega$, the pair $(\mathbf{x}_\omega, \mathbf{u}_\omega)$ with $\mathbf{x}_\omega := \mathbf{x}(\omega)$ and $\mathbf{u}_\omega$ a hybrid arc with $\operatorname{dom} \mathbf{u}_\omega = \operatorname{dom} \mathbf{x}_\omega$ and $\mathbf{u}_\omega(t,j) := \mathbf{v}_j(\omega)$ for all $(t,j) \in \operatorname{dom} \mathbf{x}(\omega) \cap (\mathbb{R} \times \mathbb{Z}_{\geq 1})$ is a standard solution staring at x.

2. (Causal measurability) For each $i \in \mathbb{Z}_{\geq 0}$, the mapping $\omega \mapsto \operatorname{graph}_{\leq i}(\mathbf{x}(\omega)) := \operatorname{graph}(\mathbf{x}(\omega)) \cap (\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\leq i} \times \mathbb{R}^n)$ has closed values and is $\mathcal{F}_i$- measurable with $\mathcal{F}_0 = \{\varnothing, \Omega\}$, and $(\mathcal{F}_1, \mathcal{F}_2, ...)$ the natural filtration of $\mathbf{v}$.

To guarantee the existence of random solutions defined above, we impose the following regularity property on the data (C, F, D, G, μ) as in [42].

Assumption 1 *(Hybrid Basic conditions)*

1. The sets $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^n$ are closed.

2. F is outer semicontinuous, locally bounded and for each $x \in C$, $F(x)$ is nonempty and convex.

3. G is locally bounded and for each $v \in \mathbb{R}^m$, the mapping $x \mapsto G(x,v)$ is outer semicontinuous.

Assumption 2 *(Stochastic Hybrid Basic condition) The set-valued mapping*

$$v \mapsto graph(G(\cdot, v)) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in G(x,v)\}$$

is measurable.

The existence of random solutions to the stochastic hybrid system (1.4) under Assumptions 2-3 is established in [42, Thm 3.1].

1.3.2 Stochastic stability notions

In this section we review some of the stochastic stability concepts explored in [42]. The definition of a random solution to (1.4) states that the graphs of the solutions are measurable and so we express the stability notions in terms of probabilities on the solution graphs. The dependence of random solution on ω is suppressed and we write "$\mathbf{x}(t,j) \in S$ for $(t,j) \in \operatorname{dom} \mathbf{x}$" in place of "$\mathbf{x}_\omega(t,j) \in S$ for $(t,j) \in \operatorname{dom} \mathbf{x}_\omega$" where $\mathbf{x}_\omega := \mathbf{x}(\omega)$ to save on notation. The stability notions defined in this section are "strong" stability notions, meaning they hold for every random process $\mathbf{x}$ generated by the stochastic hybrid system (1.4) from a particular initial condition.

Referring to the statement

$$\mathbb{P}(\operatorname{graph}_{\geq \tau}(\mathbf{x}) \subset (\mathbb{R}^2 \times (\mathcal{A} + \epsilon \mathbb{B}^o))) \geq 1 - \rho, \quad \forall \zeta \in \mathcal{A} + \delta \mathbb{B}^o, \quad \mathbf{x} \in \mathcal{S}(\zeta), \tag{1.5}$$

a compact set $\mathcal{A} \subset \mathbb{R}^n$ for a stochastic hybrid system is

- *uniformly Lyapunov stable in probability* if for $\tau = 0$ and each $\epsilon > 0$ and $\rho > 0$ there exists $\delta > 0$ such that (1.5) holds,
- *uniformly Lagrange stable in probability* if for $\tau = 0$ and each $\delta > 0$ and $\rho > 0$ there exists $\epsilon > 0$ such that (1.5) holds,
- *uniformly attractive in probability* if for each $\epsilon > 0$, $\delta > 0$, and $\rho > 0$, there exists $\tau > 0$ such that (1.5) holds,
- *uniformly globally asymptotically stable (UGAS) in probability* if it is uniformly

Lyapunov stable in probability, uniformly Lagrange stable in probability, and uniformly attractive in probability.

Next we present sufficient Lyapunov conditions as established in [42] to certify uniform global asymptotic stability in probability for stochastic hybrid systems.

Let $\mathcal{V} := \bigcup_{\omega\in\Omega, i\in\mathbb{Z}_{\geq 0}} \mathbf{v}_{i+1}(\omega)$. A function $V : \mathrm{dom}\, V \to \mathbb{R}$ is a certification candidate for (C, D, G, μ) if

C1. $C \cup D \cup G(D \times \mathcal{V}) \subset \mathrm{dom}\, V$,

C2. $0 \leq V(x)$ for all $x \in C \cup D \cup G(D \times \mathcal{V})$, and

C3. $\int_{\mathbb{R}^m} \sup_{g\in G(x,v)} V(g)\mu(dv)$ is well defined for each $x \in D$, with the convention that $\sup_{g\in G(x,v)} V(g) = 0$ when $G(x, v) = \varnothing$.

Under sufficient regularity assumptions (Assumptions 2-3) on the data of the stochastic hybrid systems it is established in [42, Lemma 4.1] if V : dom $V \to \mathbb{R}$ is upper semicontinuous [3] and satisfies conditions C1-C2 then it satisfies the condition C3.

Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. A continuously differentiable certification candidate for (C, D, G, μ) is a *Lyapunov function* for $\mathcal{A}$ if there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned} &\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A}) \quad \forall x \in C \cup D \cup G(D \times \mathcal{V}) \\ &\langle \nabla V(x), f(x)\rangle \leq -\rho(|x|_\mathcal{A}) \qquad\qquad \forall x \in C, f \in F(x) \\ &\int_{\mathbb{R}^m} \sup_{g\in G(x,v)} V(g)\mu(dv) \leq V(x) - \rho(|x|_\mathcal{A}) \quad \forall x \in D. \end{aligned} \tag{1.6}$$

The nest result then follows from [42, Thm 4.4].

[3]A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is *upper semicontinuous* if, for each sequence x_i converging to x, $\limsup_{i\to\infty} \phi(x_i) \leq \phi(x)$.

Theorem 2 *Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. If Assumptions 2-3 hold for the system* (1.4)*, then the existence of a Lyapunov function for $\mathcal{A}$ implies uniform global asymptotic stability in probability of $\mathcal{A}$ for* (1.4)*.*

The results developed in [42] also pertain to establishing certification candidates for Lyapunov stability and Lagrange stability in probability. Weakened sufficient conditions for uniform global asymptotic stability in probability and uniform global recurrence using nested Matrosov functions are also developed in [42].

Chapter 2

Finite-time Parameter Estimation

2.1 Introduction

In a parameter estimation algorithm, one desires speed, accuracy and robustness. There have been many contributions to such algorithms for nonlinear systems that focus on rate of convergence and robustness [13], [3], [33]. However, the advantage of the algorithm presented in [1] is that it transcends the idea of rate of convergence by identifying a parameter vector exactly and in finite time. This finite-time characteristic suggests a discontinuity in the parameter estimation state, lending itself to a hybrid system interpretation.

By analyzing a modified version of [1] in a hybrid setting, we take advantage of recent theoretical advances to show that the parameter estimation has robust properties. For instance, small uncertainties in aspects of system data assumed to be known lead to only small errors in the parameter estimate. In fact, our main result shows that the hybrid algorithm induces a robustly asymptotically stable set where the projection of this set in the direction of the parameter estimate is equal to the unknown parameter.

2.2 The Algorithm

We begin with the nonlinear system containing the parameter we want to estimate. Consider

$$\dot{x} = \tilde{f}(x,u) + \tilde{g}(x,u)\theta. \tag{2.1}$$

Functions $\tilde{f}$ and $\tilde{g}$ are continuous, x belongs to a compact set $X \subset \mathbb{R}^n$, and $\theta \in \mathbb{R}^p$. We know x, X, and u, but not θ. Our goal is to estimate θ such that the parameter estimate $\hat{\theta}$ equals θ in finite time, and given small uncertainties in $\tilde{f}$, $\tilde{g}$, x, X, and u, we will have only small errors in $\hat{\theta}$.

The algorithm we use to meet this goal is inspired by [1] and consists of embedding (2.1) into a hybrid system. To get an idea of how this works, consider a system with state $\bar{x} = \{x, \hat{x}, \hat{\theta}, w, Q, \eta, \Gamma, \tau\}$ and continuous dynamics

$$\begin{aligned}
\dot{x} &= \tilde{f}(x,u) + \tilde{g}(x,u)\theta & \dot{\hat{\theta}} &= h(\bar{x}) \\
\dot{\hat{x}} &= \tilde{f}(x,u) + \tilde{g}(x,u)\hat{\theta} + k(x-\hat{x}) & \dot{w} &= \tilde{g}(x,u) - kw \\
\dot{Q} &= w^\top w & \dot{\eta} &= -k\eta + wh(\bar{x}) \\
\dot{\Gamma} &= w^\top(w\hat{\theta} + x - \hat{x} - \eta) & \dot{\tau} &= 1
\end{aligned} \tag{2.2}$$

where $k > 0$, and $h(\bar{x})$ is an update law for the parameter estimate $\hat{\theta}$. These continuous dynamics apply when $\bar{x}$ is in the set[1] $C := X \times \mathbb{R}^n \times \mathbb{R}^p \times M\mathbb{B} \times \mathcal{SP}^{p\times p} \times \mathbb{R}^n \times \mathbb{R}^p \times [0,T]$, $T > 0$. The system also has discrete dynamics

$$\begin{aligned}
x^+ &= x & \hat{x}^+ &= x & \hat{\theta}^+ &= Q^{-1}\Gamma & w^+ &= 0 \\
Q^+ &= 0 & \eta^+ &= 0 & \Gamma^+ &= 0 & \tau^+ &= 0
\end{aligned} \tag{2.3}$$

[1] $\mathcal{SP}^{p\times p}$ indicates the set of positive semi-definite matrices of size $p \times p$; $M\mathbb{B}$ is the closed ball in the Frobenius norm of appropriate dimension and radius M.

that apply when $\bar{x}$ is in $D := \{\bar{x} \in C : det(Q) \geq \epsilon\}$, $\epsilon > 0$.

It turns out that if a solution following the dynamics (2.2), (2.3) with constraints C, D jumps twice, then at the second jump we have $\hat{\theta}^{+} = Q^{-1}\Gamma = \theta$. Indeed, after a jump given by (2.3) at a time t_0 followed by continuous change according to the differential equation (2.2) on the interval $[t_0, t_1]$, we have $\eta(t) = x(t) - \hat{x}(t) - w(t)[\theta - \hat{\theta}(t)] \quad \forall t \in [t_0, t_1]$. This can be verified by noting that the equation holds at t_0, and that the time derivatives of each side are equal on the interval. With another jump at time t_1, $Q(t_1)$ is invertible due to the definition of the jump set D. And so, using (2.2) and (2.3),

$$
\begin{aligned}
\theta &= Q(t_1)^{-1}Q(t_1)\theta = Q(t_1)^{-1}\int_{t_0}^{t_1} w^{\top}(\tau)w(\tau)\theta d\tau \\
&= Q(t_1)^{-1}\int_{t_0}^{t_1} w^{\top}(\tau)[w(\tau)\hat{\theta}(\tau) + x(\tau) - \hat{x}(\tau) - \eta(\tau)]d\tau \\
&= Q(t_1)^{-1}\int_{t_0}^{t_1} \dot{\Gamma}(\tau)d\tau = Q(t_1)^{-1}(\Gamma(t_1) - \Gamma(t_0)) \\
&= Q(t_1)^{-1}\Gamma(t_1)
\end{aligned}
$$

which gives the result.

In the algorithm presented in [1], the parameter estimate is obtained by starting a system similar to (2.2) from a particular initial condition, waiting until Q becomes invertible, and then calculating $\theta = Q^{-1}\Gamma$. Our algorithm differs: rather than a one-time estimation, we have an autonomous system that implements the particular initial condition as a jump map and can therefore start from any initial condition. Another difference is that in the continuous dynamics (2.2) we move $wh(\bar{x})$ from the $\dot{\hat{x}}$ equation to the $\dot{\eta}$ equation in order to relax the assumption on the parameter update law ("$\dot{\hat{\theta}}$ is bounded" relaxes to the inequality in Assumption 3).

$$
\mathcal{P}\quad\left\{\begin{array}{rcl}
C & := & X \times \mathbb{R}^n \times \mathbb{R}^p \times M\mathbb{B} \times \mathcal{SP}^{p\times p} \times \mathbb{R}^n \times \mathbb{R}^p \times [0,T] \\
D & := & \{\bar{x} \in C : det(Q) \geq \epsilon\} \\
f(\bar{x},u) & := & \begin{bmatrix} \tilde{f}(x,u) + \tilde{g}(x,u)\theta \\ \tilde{f}(x,u) + \tilde{g}(x,u)\hat{\theta} + k(x-\hat{x}) \\ h(\bar{x}) \\ \tilde{g}(x,u) - kw \\ w^\top w \\ -k\eta + wh(\bar{x}) \\ w^\top (w\hat{\theta} + x - \hat{x} - \eta) \\ 1 \end{bmatrix} \quad g(\bar{x}) := \begin{bmatrix} x \\ x \\ Q^{-1}\Gamma \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{array}\right. \tag{2.4}
$$

2.3 Main Results

We can express the algorithm of Section 2.2 through the framework of Chapter One. In particular, the algorithm agrees with the hybrid system $\mathcal{P} := \{f, C, g, D\}$ with state $\bar{x} = \{x, \hat{x}, \hat{\theta}, w, Q, \eta, \Gamma, \tau\}$ and data given in (2.4). Note that although the previous chapter deals with autonomous systems, and $\mathcal{P}$ is a system with an input, analogous definitions apply to $\mathcal{P}$ (see [8]). We add the following assumptions[2]:

Assumption 3 *X is compact, $k, \epsilon, T, M > 0$, and there exists $L > 0$ such that*

$$|h(\bar{x})| \leq L|x - \hat{x}| \quad \forall \bar{x} \in C.$$

Assumption 4 *The functions $\tilde{f}$, $\tilde{g}$, and h are continuous, u is measurable and belongs to a compact set U, and*

$$M \geq \frac{1}{k} \underset{x\in X\ u\in U}{sup} |\tilde{g}(x,u)|.$$

On the choice of parameters: the purpose of the function h is to give the user the option of having a continuous-time parameter update law, since this might be useful during the interval before the parameter estimate converges. However, a choice of $h(\bar{x}) \equiv$

[4] $|\cdot|$ indicates the Frobenius norm, which reduces to the Euclidean norm for vectors.

0 will work perfectly well, as it satisfies Assumption 3. The choice of constant ϵ is a balance between small enough so that convergence time is fast, but large enough so that the inversion of Q is numerically stable. The choice of T should be sufficiently large so that in the presence of a Persistency of Excitation condition (see Assumption 5), the condition $det(Q) \geq \epsilon$ is met. Finally, choosing k small tends to decrease the convergence time.

We proceed with stating the main stability result:

Theorem 3 *For the system $\mathcal{P}$ under Assumptions 3 and 4, there exists a compact, possibly empty, globally asymptotically stable set contained in $S \times \{\theta\} \times M\mathbb{B} \times \mathcal{SP}^{p \times p} \times \{0\} \times \mathbb{R}^p \times [0, T]$ where $S = \{x, \hat{x} : x \in X, \hat{x} = x\}$. Furthermore, for all solutions $\bar{x}$ to $\mathcal{P}$, $t + j \geq 2 + 2T$ implies $\hat{\theta}(t, j) = \theta \quad \forall (t, j) \in dom\ \bar{x}$.*

Proof: Given in Appendix. ■

This result constrains how solutions to (2.4) behave, and particularly the part of the solution corresponding to the parameter estimate. We can say more though — that this stability property is robust, both to constant perturbations and state-dependent perturbations.

Similarly, when a bounded disturbance is added to the $\dot{x}$ equation in system $\mathcal{P}$, the overall state of $\mathcal{P}$ remains bounded (Cf. [1, eq. 27]). This can be shown as follows: during flows, which last for at most T seconds, a sector condition $|f(\bar{x}, u)| \leq L_1|\bar{x}| + L_2 \quad \forall \bar{x} \in C, u \in U$ holds (see Appendix A), ensuring a bounded reachable set for T seconds from each bounded set of initial conditions. Then, during jumps, the state enters a compact set that is independent of the size of initial conditions (for $\hat{\theta}$, note $|Q^{-1}\Gamma| \leq |adjugate(Q)||\Gamma|/det(Q)$ and $det(Q) \geq \epsilon$ at jumps). This gives the result.

Finally, adding a Persistency of Excitation (PE) condition on $\tilde{g}$ for system $\mathcal{P}$ ensures that solutions to $\mathcal{P}$ are complete.

Assumption 5 *The function $\tilde{g}$ is differentiable, $t \mapsto u(t)$ is globally Lipschitz, maximal solutions to $\dot{x} = \tilde{f}(x,u) + \tilde{g}(x,u)\theta$ starting from X are complete, and there exists $\sigma, \beta > 0$ such that*

$$\int_{t_0}^{t_0+\sigma} \tilde{g}^\top(x(s),u(s))\tilde{g}(x(s),u(s))ds \geq \beta I \qquad \forall\ t_0 \geq 0.$$

This leads to the following result:

Theorem 4 *For the system $\mathcal{P}$ under Assumptions 3, 4 and 5, maximal solutions starting from $\{\bar{x} \in C \cup D : \tau \in [0,1]\}$ are complete for sufficiently large T.*

Proof: Given in Appendix. ■

The significance of this theorem is that knowing solutions to $\mathcal{P}$ are complete ensures that the parameter estimate converges in finite time, and that the globally asymptotically stable set in Theorem 3 is not empty.

2.4 Simulation

We simulate the algorithm using an example similar to that in [1]. For the system

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta_1 x_1 & \dot{x}_2 &= x_3 + \theta_2 x_1 \\ \dot{x}_3 &= \theta_3 x_1^3 + \theta_4 x_2 + \theta_5 x_3 + (1 + x_1^2)u & y &= x_1 \end{aligned}$$

we would like to identify the parameter $\theta^\top = [\theta_1, \ldots, \theta_5]$, while having the output y track a reference signal y_r. Moreover, our measurements of x_1, x_2, x_3 each contain uniformly distributed noise on the interval $[-0.15, 0.15]$. Using control laws given in [27] along with our algorithm, we simulate this system in Matlab for both the case where we have noisy measurements of x, and the case where we know x exactly. Choosing parameters $\theta = [-1\ -2\ 1\ 2\ 3]$, $k = 0.1$, $\epsilon = 10^{-4}$, $T = 12$, $y_r = 1 + 0.1 sin(\tau)$, and non-zero initial conditions, we show results in Fig. 2.1.

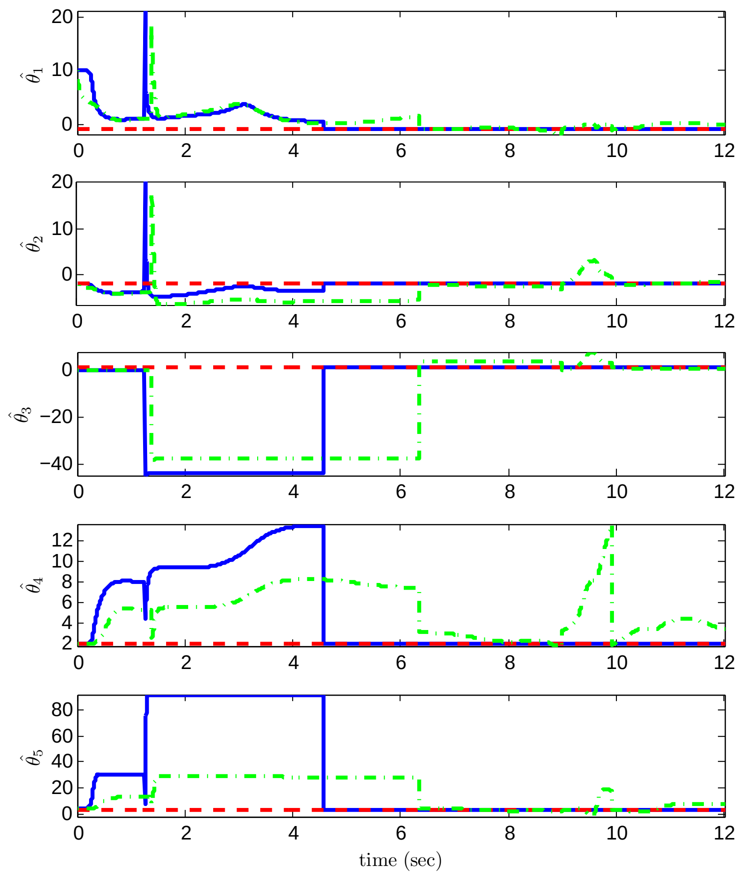

Figure 2.1: Parameter Estimate with noise uniformly distributed on [-0.15, 0.15] in the measurement of x (dash-dot), without noise (solid line), and the actual parameter (dashed).

In the case where x is known exactly, we see that after one jump at $t = 1.3$, and a second jump at $t = 4.6$, the parameter estimate arrives at to the parameter, and stays there for future time. In the case with noisy measurements of x, we can expect from robustness that small perturbations in the system lead to small errors in $\hat{\theta}$. This is reflected in Fig. 2.1 since after the second jump, $\hat{\theta}$ lands near θ, and stays near for future time. In fact, for noise levels on the order of 10^{-2} or less, $\hat{\theta}$ is virtually indistinguishable from the zero noise case $\hat{\theta}$. This illustrates the convergence of the algorithm from non-zero initial conditions, as well as its robustness to measurement noise.

Chapter 3

Implementing hybrid state feedback through high-gain observers

3.1 Introduction

Hybrid-system-based controllers are useful for solving many control problems (see [29], [24], and [15]). This invites the question: under what observability conditions can one implement such controllers using only plant measurements? In [44], this question is answered in terms of a local separation principle, where the plant, controller, and observer are all allowed to be hybrid. Drawing on [41], this chapter presents a semiglobal practical result, where we have a continuous-time plant with a hybrid controller, and an observability assumption somewhat stronger than Uniform Complete Observability (UCO).

We note that the authors of [28] also implement hybrid state feedback through high gain observers, and so we emphasize some differences between that work and the current chapter. In [28] they consider a plant that is SISO and affine in control input and assume a hybrid state feedback with a single-valued jump map. Access to a suitable

Lyapunov function is needed in order to provide an explicit output feedback algorithm. By contrast, this chapter considers a plant that is MIMO and general in control input, assumes a hybrid state feedback where the jump map may be set-valued, and does not require access to a Lyapunov function to provide an explicit output feedback algorithm.

3.2 Setting and Assumptions

We begin with the following system:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x) \end{aligned} \tag{3.1}$$

$x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, where X and U are closed, and $f : X \times U \to \mathbb{R}^n$, $h : X \to \mathbb{R}^p$ are sufficiently smooth functions.

We assume an observability property.

Assumption 6 *There exist an integer n_y, smooth functions $\Phi : \mathbb{R}^{p(n_y+1)} \to \mathbb{R}^n$ and $\varphi_i : \mathbb{R}^n \to \mathbb{R}^p$, $i = 0 \ldots n_y$, and for each function $u : \mathbb{R} \to U$ and each solution to* (3.1)*, we have, for all t where the solution to* (3.1) *makes sense, $x(t) = \Phi(y(t), \ldots, y^{(n_y)}(t))$, $y^{(i)}(t) = \varphi_i(x(t))$, and $y^{(n_y+1)}(t) = \varphi_{n_y+1}(x(t), u(t))$, where $y^{(i)}(t)$ indicates the ith time derivative of y at time t.*

We also assume the existence of a hybrid state feedback controller:

Assumption 7 *There exist closed sets C_c, D_c, X, and compact set X_c satisfying C_c, $D_c \subset X \times X_c \subset \mathbb{R}^n \times \mathbb{R}^c$, continuous functions $f_c : C_c \to \mathbb{R}^c$, $\alpha : C_c \to U$, and set-valued mapping $G_c : \mathbb{R}^{n+c} \rightrightarrows \mathbb{R}^c$ that is outer semicontinuous, locally bounded and nonempty on*

D_c, such that the system

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} f(x, \alpha(x, x_c)) \\ f_c(x, x_c) \end{bmatrix}, (x, x_c) \in C_c, \\
\begin{bmatrix} x^+ \\ x_c^+ \end{bmatrix} &\in \begin{bmatrix} x \\ G_c(x, x_c) \end{bmatrix}, \qquad (x, x_c) \in D_c,
\end{aligned} \tag{3.2}$$

contains a globally asymptotically stable compact set $\mathcal{A}$.

We proceed to build an output feedback based on these two assumptions. As a first step, we add temporal regularization to the state feedback (3.2); this is needed to give the (continuous-time) observer enough ordinary time to develop a good state estimate. This can be written as:

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{x}_c \\ \dot{\tau} \end{bmatrix} &= \begin{bmatrix} f(x, \alpha(x, x_c)) \\ f_c(x, x_c) \\ \sigma(\tau) \end{bmatrix}, (x, x_c, \tau) \in (C_c \times [0, 2]) \cup (X \times \mathbb{R}^c \times [0, T]) \\
\begin{bmatrix} x^+ \\ x_c^+ \\ \tau^+ \end{bmatrix} &\in \begin{bmatrix} x \\ G_c(x, x_c) \\ 0 \end{bmatrix}, \qquad (x, x_c, \tau) \in D_c \times [T, 2]
\end{aligned} \tag{3.3}$$

where $T \in (0, 1]$ is a parameter to be chosen, f_c and α are continuously extended so as to be defined on $X \times \mathbb{R}^c$, and

$$\sigma(\tau) := \begin{cases} 1 & \tau \in [0,1] \\ 2-\tau & \tau \in (1,2]. \end{cases}$$

We proceed to build a high gain observer for (3.1). Denoting $y_i = y^{(i)}$ and using Assumption 6, we have

$$\begin{aligned} \dot{y}_0 &= y_1 \\ &\vdots \\ \dot{y}_{n_y-1} &= y_{n_y} \\ \dot{y}_{n_y} &= \varphi_{n_y+1}(x,u). \end{aligned} \tag{3.4}$$

Using ideas from [12] and [41], we build the state observer:

$$\begin{aligned} \dot{\hat{y}}_0 &= \hat{y}_1 + Ll_0(y-\hat{y}_0) \\ &\vdots \\ \dot{\hat{y}}_{n_y-1} &= \hat{y}_{n_y} + L^{n_y} l_{n_y-1}(y-\hat{y}_0) \\ \dot{\hat{y}}_{n_y} &= L^{n_y+1} l_{n_y}(y-\hat{y}_0) + \varphi_{n_y+1}(\hat{x}(\hat{y}),u). \\ \hat{x}(\hat{y}) &= \mathrm{sat}_{x_{\max}}(\Phi(\hat{y}_0,\ldots,\hat{y}_{n_y})) \end{aligned} \tag{3.5}$$

where

$$\mathrm{sat}_{x_{\max}}(\cdot) := \min\{1, \frac{x_{\max}}{|\cdot|}\}(\cdot), \tag{3.6}$$

$x_{\max} > 0$, $L > 0$ are parameters to be chosen, and the l_i's are coefficients of a Hurwitz polynomial.

Implementing the state feedback (3.2) with the estimate $\hat{x}(\hat{y})$ and temporal regularization, we obtain the following output feedback system:

$$
\begin{bmatrix} \dot{x} \\ \dot{x}_c \\ \dot{\tau} \\ \dot{\hat{y}} \end{bmatrix} = \begin{bmatrix} f(x, \alpha(\hat{x}(\hat{y}), x_c)) \\ f_c(\hat{x}(\hat{y}), x_c) \\ \sigma(\tau) \\ s_L(y, x_c, \hat{y}) \end{bmatrix},
$$
$$
(x, x_c, \tau, \hat{y}) \in \{(x, x_c, \tau, \hat{y}) \in X \times X_c \times [0, 2] \times \mathbb{R}^{p(n_y+1)} : (\mathrm{proj}_X(\hat{x}(\hat{y})) \times \{x_c\}) \cap C_c \neq \emptyset\}
$$
$$
\cup (X \times \mathbb{R}^c \times [0, T] \times \mathbb{R}^{p(n_y+1)})
$$
$$
\begin{bmatrix} x^+ \\ x_c^+ \\ \tau^+ \\ \hat{y}^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_c(\hat{x}(\hat{y}), x_c) \\ 0 \\ \hat{y} \end{bmatrix},
$$
$$
(x, x_c, \tau, \hat{y}) \in \{(x, x_c, \tau, \hat{y}) \in X \times X_c \times [T, 2] \times \mathbb{R}^{p(n_y+1)} : (\mathrm{proj}_X(\hat{x}(\hat{y})) \times \{x_c\}) \cap D_c \neq \emptyset\} \tag{3.7}
$$

where $\mathrm{proj}_X(x) := \arg\min_{\tilde{x} \in X} |x - \tilde{x}|$ is a closed set, $y = h(x)$ and

$$
s_L(y, x_c, \hat{y}) := \begin{bmatrix} \hat{y}_1 + L l_0 (y - \hat{y}_0) \\ \vdots \\ \hat{y}_{n_y} + L^{n_y} l_{n_y - 1}(y - \hat{y}_0) \\ L^{n_y+1} l_{n_y}(y - \hat{y}_0) + \varphi_{n_y+1}(\hat{x}(\hat{y}), \alpha(\hat{x}(\hat{y}), x_c)) \end{bmatrix}.
$$

With our candidate output feedback defined, we state the main result [1].

Theorem 5 *For system (3.7) under Assumptions 6 and 7, for each $\epsilon > 0$ and each compact set $K_x \subset X$, $K_{\hat{y}} \subset \mathbb{R}^{p(n_y+1)}$, there exists $x^*_{max} > 0$ and for each $x_{max} \geq x^*_{max}$ there exists $T^* \in (0, 1]$ and for each $T \in (0, T^*]$ there exists $L^* > 0$, and for each $L \geq L^*$*

there exists $\mathcal{A}_1 \subset (\mathcal{A} + \epsilon\mathbb{B}) \times [0, 2]$ *such that the set* $\{(x, x_c, \tau, \hat{y}) : (x, x_c, \tau) \in \mathcal{A}_1, x = \Phi(\hat{y}, \ldots, \hat{y}_{n_y})\}$ *is asymptotically stable with basin of attraction including* $K_x \times X_c \times [0, 2] \times K_{\hat{y}}$.

Consequently, global asymptotic stability of $\mathcal{A}$ under state feedback translates into semi-global practical stability of $\mathcal{A}$ under the output feedback (3.7). We note that compared to the results in [41] where they show semi-global asymptotic stability, we have the weaker result of semi-global practical stability; this is due to the addition of temporal regularization. However, for the case of $\mathcal{A} \cap D_c = \emptyset$, i.e. when there are no jumps near the attractor, we will obtain semi-global asymptotic stability.

[1] $M\mathbb{B}$ indicates the closed ball in Euclidean space of appropriate dimension and radius M.

3.3 Analysis

To analyze (3.7), we change coordinates. Defining $e_i := L^{n_y-i}(y_i - \hat{y}_i)$, $i = 0 \dots n_y$, (3.7) can be written as:

$$
\begin{aligned}
&\begin{bmatrix} \dot{x} \\ \dot{x}_c \\ \dot{\tau} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} f(x, \alpha(\hat{x}(\hat{h}(x,e)), x_c)) \\ f_c(\hat{x}(\hat{h}(x,e)), x_c) \\ \sigma(\tau) \\ LAe + \phi(x, x_c, \hat{x}(\hat{h}(x,e))) \end{bmatrix}, (x, x_c, \tau, e) \in C_{(3.8)} \\
&C_{(3.8)} := \{(x, x_c, \tau, e) \in X \times X_c \times [0,2] \times \mathbb{R}^{p(n_y+1)} : \\
&(\mathrm{proj}_X\{\hat{x}(\hat{h}(x,e))\}) \times \{x_c\} \cap C_c \neq \emptyset\} \cup (X \times \mathbb{R}^c \times [0,T] \times \mathbb{R}^{p(n_y+1)}) \\
&\begin{bmatrix} x^+ \\ x_c^+ \\ \tau^+ \\ e^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_c(\hat{x}(\hat{h}(x,e)), x_c) \\ 0 \\ e \end{bmatrix}, \ (x, x_c, \tau, e) \in D_{(3.8)} \\
&D_{(3.8)} := \{(x, x_c, \tau, e) \in X \times X_c \times [T,2] \times \mathbb{R}^{p(n_y+1)} : \\
&(\mathrm{proj}_X\{\hat{x}(\hat{h}(x,e))\} \times \{x_c\}) \cap D_c \neq \emptyset\}
\end{aligned}
\tag{3.8}
$$

where A is the Hurwitz companion matrix of the l_i's, $T, L > 0$, and

$$
\hat{h}(x,e) \ := \begin{bmatrix} \varphi_0(x) - L^{-n_y} e_0 \\ \varphi_1(x) - L^{1-n_y} e_1 \\ \vdots \\ \varphi_{n_y}(x) - e_{n_y} \end{bmatrix},
$$

$$\phi(x, x_c, \hat{x}(\hat{h}(x,e))) := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Delta(x, x_c, \hat{x}(\hat{h}(x,e))) \end{bmatrix},$$

$$\Delta(x, x_c, \hat{x}(\hat{h}(x,e))) := \varphi_{n_y+1}(x, \alpha(\hat{x}(\hat{h}(x,e)), x_c)) - \varphi_{n_y+1}(\hat{x}(\hat{h}(x,e)), \alpha(\hat{x}(\hat{h}(x,e)), x_c)).$$

We note from the definition of e and $\hat{h}$ that $\hat{y} = \hat{h}(x,e)$.

Next, for each $M, \delta > 0$ and $\mathcal{A} \subset X \times X_c$ we define the following sets:

$$\begin{aligned}
K_{K_x,K_{\hat{y}},L} &:= \{(x, x_c, \tau, e) \in K_x \times X_c \times [0,2] \times \mathbb{R}^{p(n_y+1)} : \\
&\qquad e_i = L^{n_y-i}(\varphi_i(x) - \hat{y}_i), i = 0 \ldots n_y, \hat{y} \in K_{\hat{y}}\}, \\
E_\delta &:= \{(x, x_c, \tau, e) \in X \times X_c \times [0,2] \times \mathbb{R}^{p(n_y+1)} : |x - \hat{x}(\hat{h}(x,e))| < \delta\}, \\
\overline{E}_\delta &:= \{(x, x_c, \tau, e) \in X \times X_c \times [0,2] \times \mathbb{R}^{p(n_y+1)} : |x - \hat{x}(\hat{h}(x,e))| \leq \delta\}, \\
K_M &:= \mathrm{int}(\mathcal{A} + M\mathbb{B}) \times [0,2] \times \mathbb{R}^{p(n_y+1)}, \\
\overline{K}_M &:= (\mathcal{A} + M\mathbb{B}) \times [0,2] \times \mathbb{R}^{p(n_y+1)},
\end{aligned} \tag{3.9}$$

where $K_{K_x,K_{\hat{y}},L}$ corresponds to the set of initial conditions, E_δ and $\overline{E}_\delta$ correspond to small state estimate errors, and $K_M, \overline{K}_M$ correspond to sets that we expect solutions starting from $K_{K_x,K_{\hat{y}},L}$ to remain in.

We proceed with a series of lemmas that will aid us in the analysis of (3.8). The first lemma gives conditions under which the (x, x_c) component of solutions to (3.8) stays bounded.

Lemma 1 *For system* (3.8) *under Assumption 7, for each $\epsilon > 0$ and each compact set $K_x \subset X$ there exists $M > 0$, and for each $x_{max} > 0$, there exists $\delta \in (0,1]$ such that for* *each* $T \in (0, \delta]$, $L > 0$, $\bar{x} \in \mathcal{S}_{(3.8)}(K_{K_x, K_{\hat{y}}, L})$ *and* $(\tilde{t}, \tilde{j}) \in dom\, \bar{x}$,

$$\bar{x}(t,j) \in K_M \qquad \forall (t,j) \in dom\, \bar{x} \cap ([0,T] \times \mathbb{N}), \tag{3.10}$$

and if

$$\bar{x}(t,j) \in \overline{E}_\delta \qquad \forall (t,j) \in dom\, \bar{x} \cap ([T,\tilde{t}] \times [0,\tilde{j}]), \tag{3.11}$$

then

$$\bar{x}(t,j) \in K_M \qquad \forall (t,j) \in dom\, \bar{x} \cap ([0,\tilde{t}] \times [0,\tilde{j}]). \tag{3.12}$$

Proof: See Appendix. ■

The next lemma states that as long as the (x, x_c) component of solutions to the output feedback system stays bounded, we can choose L so that $|x - \hat{x}|$ becomes arbitrarily small, arbitrarily quickly.

Lemma 2 *For system* (3.8) *under Assumptions 6 and 7, for each* $\epsilon, M > 0$ *and each compact set* $K_x \subset X$, $K_{\hat{y}} \subset \mathbb{R}^{p(n_y+1)}$, *there exists* $x^*_{max} > 0$, *and for each* $x_{max} \geq x^*_{max}$, $\delta > 0$, $T \in (0,1]$, *there exists* $L^* > 0$, *such that, for each* $L \geq L^*$ *and* $\bar{x} \in \mathcal{S}_{(3.8)}(K_{K_x, K_{\hat{y}}, L})$ *and* $(\tilde{t}, \tilde{j}) \in dom\, \bar{x}$, *if*

$$\bar{x}(t,j) \in \overline{K}_M \qquad \forall (t,j) \in dom\, \bar{x} \cap ([0,\tilde{t}] \times [0,\tilde{j}]), \tag{3.13}$$

then

$$\bar{x}(t,j) \in E_\delta \qquad \forall (t,j) \in dom\, \bar{x} \cap ([T,\tilde{t}] \times [0,\tilde{j}]). \tag{3.14}$$

Proof: See Appendix. ■

The following proposition gives a stability result for the system (3.15), which differs from the output feedback system (3.8) in that the space of (x, x_c) is constrained to a compact set. The notation $C_{(3.8)}$ and $D_{(3.8)}$ indicates the flow and jump sets respectively, for system (3.8).

$$
\begin{bmatrix} \dot{x} \\ \dot{x}_c \\ \dot{\tau} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} f(x, \alpha(\hat{x}(\hat{h}(x,e)), x_c)) \\ f_c(\hat{x}(\hat{h}(x,e)), x_c) \\ \sigma(\tau) \\ LAe + \phi(x, x_c, \hat{x}(\hat{h}(x,e))) \end{bmatrix}, \ (x, x_c, \tau, e) \in C_{(3.8)} \cap \overline{K}_M
$$

$$
\begin{bmatrix} x^+ \\ x_c^+ \\ \tau^+ \\ e^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_c(\hat{x}(\hat{h}(x,e)), x_c) \\ 0 \\ e \end{bmatrix}, \ (x, x_c, \tau, e) \in D_{(3.8)} \cap \overline{K}_M. \tag{3.15}
$$

Proposition 3 *For system* (3.15) *under Assumptions 6 and 7, for each* $\epsilon, M > 0$*, there exists* x^*_{max} *and for each* $x_{max} \geq x^*_{max}$ *there exists* $T^* \in (0, 1]$ *and for each* $T \in (0, T^*]$ *there exists* $L^* > 0$*, such that for each* $L \geq L^*$*, there exists* $\mathcal{A}_1 \subset (\mathcal{A} + \epsilon\mathbb{B}) \times [0, 2]$ *such that the set* $\mathcal{A}_1 \times \{0\}$ *is globally asymptotically stable.*

Proof: See Appendix. ■

We are now ready to prove Theorem 5.

Proof: Let K_x and $K_{\hat{y}}$ be compact sets satisfying $K_x \subset X$ and $K_{\hat{y}} \subset \mathbb{R}^{p(n_y+1)}$, let $\epsilon > 0$, and let $M > 0$ be chosen according to Lemma 1. Let $x^*_{\max1} > 0$ and $x^*_{\max2} > 0$ be chosen according to Lemma 2 and Proposition 3 respectively, set $x^*_{\max} := \max\{x^*_{\max1}, x^*_{\max2}\}$ and let $x_{\max} \geq x^*_{\max}$. Let $T_1^* > 0$ and $\delta > 0$ be chosen according to Proposition 3 and Lemma 1 repectively, set $T^* := \min\{T_1^*, \delta\}$ and let $T \in (0, T^*]$. Let $L_1^* > 0$ and $L_2^* > 0$ be chosen according to Lemma 2 and Proposition 3 respectively, set $L^* := \max\{L_1^*, L_2^*\}$ and let $L \geq L^*$.

We proceed to show that solutions to (3.8) starting from $K_{K_x,K_{\hat{y}},L}$ stay in K_M.

Let $\bar{x} \in \mathcal{S}_{(3.8)}(K_{K_x,K_{\hat{y}},L})$. By Lemma 1, $\bar{x}$ satisfies (3.10), and then by Lemma 2, $\bar{x}(T,j) \in E_\delta \cap K_M$ for any j that satisfies $(T,j) \in \operatorname{dom} \bar{x}$.

Now suppose there exists $(\tilde{t},\tilde{j}) \in \operatorname{dom} \bar{x}$ such that $\bar{x}(t,j) \in \overline{E}_\delta \cap \overline{K}_M$ for $(t,j) \in \operatorname{dom} \bar{x} \cap ([T,\tilde{t}] \times [0,\tilde{j}])$ and either

1. $\bar{x}(\tilde{t},\tilde{j}+1) \notin \overline{E}_\delta \cap \overline{K}_M$ and $(\tilde{t},\tilde{j}+1) \in \operatorname{dom} \bar{x}$, or

2. $\bar{x}(r_i,\tilde{j}) \notin \overline{E}_\delta \cap \overline{K}_M$ for each i, for some monotonically decreasing sequence r_i satisfying $\lim_{i\to\infty} r_i = \tilde{t}$.

Due to the dynamics of (3.8), $\bar{x}(\tilde{t},\tilde{j}+1) \in \overline{E}_\delta$. Then by Lemma 1 we have $\bar{x}(\tilde{t},\tilde{j}+1) \in K_M$, ruling out the first case. Due to Lemmas 1 and 2, $\bar{x}(\tilde{t},\tilde{j}) \in E_\delta \cap K_M$. Then by continuity of $\bar{x}(\cdot,\tilde{j})$ the second case cannot occur.

Therefore, $\bar{x}$ is also a solution to (3.15). This, along with Proposition 3, implies that for system (3.8), there exists $\mathcal{A}_1 \subset (\mathcal{A} + \epsilon\mathbb{B}) \times [0,2]$ such that $\mathcal{A}_1 \times \{0\}$ is asymptotically stable with basin of attraction including $K_{K_x,K_{\hat{y}},L}$. Switching back to the coordinates of (3.7), we have the result. ■

3.4 Simulation Study

This example uses a hybrid planar rotations controller from [31] with the control input backstepped through an integrator using the ideas in [30], and furthermore implemented through a high-gain observer. Consider a system with state $(x,w,q,\tau,\hat{y}) \in \mathcal{S}^1 \times \mathbb{R} \times$

$\{1,2\} \times [0,2] \times \mathbb{R}^4$ and dynamics

$$\begin{aligned}
&\begin{bmatrix} \dot{x} \\ \dot{w} \\ \dot{q} \\ \dot{\tau} \\ \dot{\hat{y}} \end{bmatrix} = \begin{bmatrix} wSx \\ \kappa(\hat{x},\hat{w},q) \\ 0 \\ \sigma(\tau) \\ s_L(y,q,\hat{y}) \end{bmatrix}, \\
&(x,w,q,\tau,\hat{y}) \in \{(x,w,q,\tau,\hat{y}) \in \mathcal{S}^1 \times \mathbb{R} \times \{1,2\} \times [0,2] \times \mathbb{R}^4 : \\
&(\mathrm{proj}_{\mathcal{S}^1}(\hat{x}(\hat{y})) \times \{q\}) \cap C_c \neq \emptyset\} \cup (\mathcal{S}^1 \times \mathbb{R} \times \{1,2\} \times [0,T] \times \mathbb{R}^4) \\
&\begin{bmatrix} x^+ \\ w^+ \\ q^+ \\ \tau^+ \\ \hat{y}^+ \end{bmatrix} \in \begin{bmatrix} x \\ w \\ 3-q \\ 0 \\ \hat{y} \end{bmatrix}, \\
&(x,w,q,\tau,\hat{y}) \in \{(x,w,q,\tau,\hat{y}) \in \mathcal{S}^1 \times \mathbb{R} \times \{1,2\} \times [T,2] \times \mathbb{R}^4 : \\
&(\mathrm{proj}_{\mathcal{S}^1}(\hat{x}(\hat{y})) \times \{q\}) \cap D_c \neq \emptyset\}
\end{aligned} \tag{3.16}$$

where the output $y = x$, parameters $x_{max} = 1$, $w_{max} = 10$, $T = 0.1$, $L = 10$, and definitions

$$S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$C_c := \{(x,q) \in \mathcal{S}^1 \times \{0,1\} : V(x,q) - \rho_V(x) \leq 0.5\},$$

$$D_c := \{(x,q) \in \mathcal{S}^1 \times \{0,1\} : V(x,q) - \rho_V(x) \geq 0.5\},$$

$$s_L(y,q,\hat{y}) := \begin{bmatrix} \hat{y}_1 + 2L(y-\hat{y}_0) \\ L^2(y-\hat{y}_0) + \varphi_2(\hat{x}(\hat{y}),\hat{w}(\hat{y}),\kappa(\hat{x}(\hat{y}),\hat{w},q)) \end{bmatrix},$$

$$\varphi_2(\hat{x}, \hat{w}, u) := \begin{bmatrix} -u\hat{x}_2 - \hat{w}^2\hat{x}_1 \\ u\hat{x}_1 - \hat{w}^2\hat{x}_2 \end{bmatrix},$$

$$\kappa(x, w, q) := -w - \nabla_x V^\top(x, q) S x,$$

$$V(x, q) := P(\exp((2q-3)P(x)S)x),$$

$$\rho_V(x, q) := \min_{q \in \{1,2\}} V(x, q),$$

$$P(x) := \frac{1}{2}(1 - x_1),$$

$$\hat{x}(\hat{y}) := \mathrm{sat}_{x_{\max}}(\hat{y}_0),$$

$$\hat{w}(\hat{y}) := \mathrm{sat}_{w_{\max}}(\lambda(\hat{y})),$$

$$\lambda(\hat{y}) := \begin{cases} \frac{-\hat{y}_{11}}{\hat{y}_{02}} & \hat{y}_{02} \geq 0.7 \\ \frac{\hat{y}_{12}}{\hat{y}_{01}} & \text{otherwise.} \end{cases}$$

The control objective is to globally asymptotically stabilize a point on a circle, which in this case amounts to making the set $\mathcal{A} := \{x^*\} \times \{0\} \times \{1, 2\} \times [0, 2] \times \{0\}$ asymptotically stable, where $x^* = (1, 0)$. Figure 3.1 shows the trajectory of a solution starting from initial condition $(0, 1, 0, 1, 0, 1, 0, 2, 2)$. One can see peaking in the observer states and a jump at $t = 0.1$ as the controller state switches before the solution converges to $\mathcal{A}$.

A note on parameter choice: the saturation level should be chosen outside of where one expects the plant state to be, given the desired basin of attraction. The dwell-time parameter should be chosen sufficiently small such that the effect of temporal regularization on the full state feedback is small. Finally, the high gain parameter should be sufficiently high such that the error in the state estimate is small within the dwell-time period. Generally some trial and error is needed before finding suitable parameters.

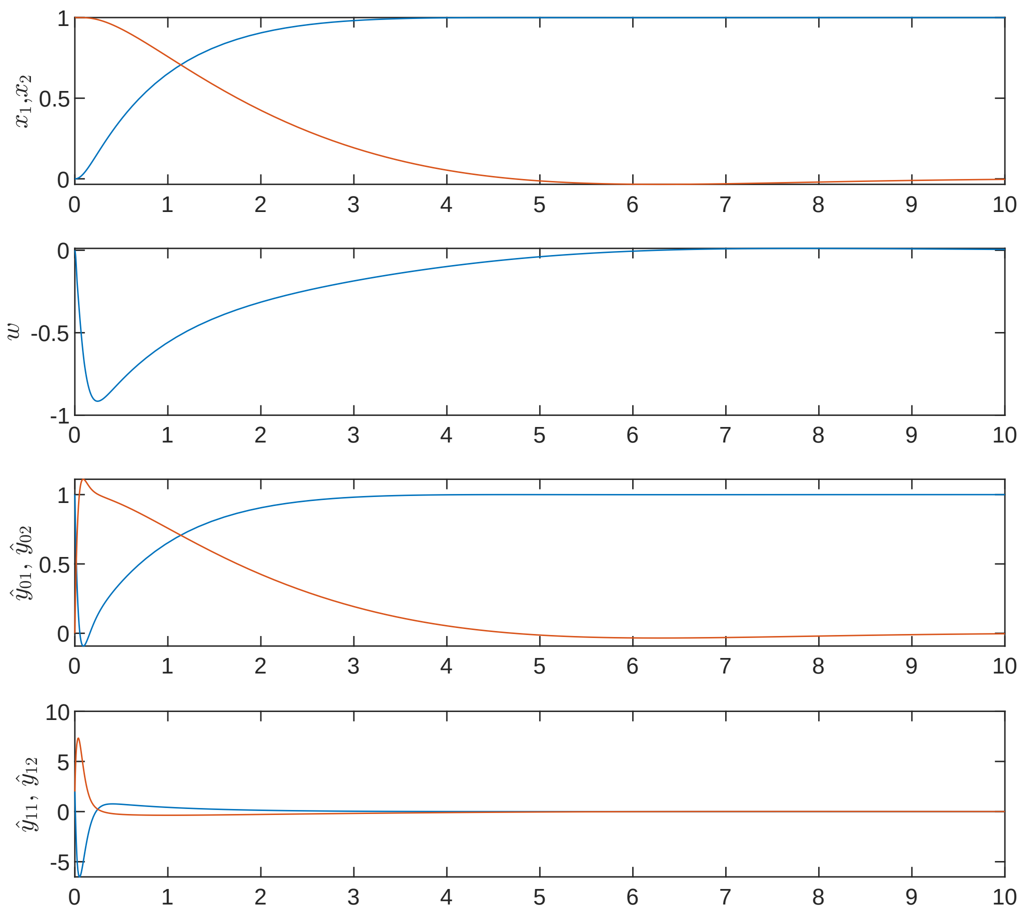

Figure 3.1: Example trajectory for planar rotations controller.

Chapter 4

Global Synchronization on the Circle under Limited Communication

4.1 Introduction

In this chapter we turn to the topic of consensus of agents on nonlinear manifolds. The topological constraints of nonlinear manifolds provide interesting design challenges (an excellent overview of this topic is given in [37]). If one desires robust, global synchronization for agents that evolve continuously on a circle, for instance, a stochastic, hybrid controller is required (cf. [40] [21]). In this chapter, we present a class of controllers that lead to global synchronization on the circle, where in contrast to the all-to-all communication assumption in [40] and [21], we assume only a connected communication graph.

4.2 Preliminaries

For $\xi^* \in \mathbb{S}^1$, we define $R_{\xi^*}(\theta) := R(\theta)\xi^*$, where $R(\cdot)$ is the rotation matrix

$$R(\phi) := \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix},$$

and in particular, $R_{\xi^*}([-\epsilon, \epsilon])$ is the arc of the unit circle centered at ξ^* of length 2ϵ.

A graph $\mathbb{G}$ is composed of a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$. In this chapter, each agent is identified with a vertex of the graph and designated with a positive integer so that $\mathcal{V} = \{1, ..., N\}$. For each $i \in \mathcal{V}$ we define the neighbor set $\mathcal{N}_i := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. When $\mathcal{N}_i$ is used as a subscript, it indicates a vector comprising data of the neighbors of agent i, and in particular, $\xi_{\mathcal{N}_i} := [\xi_{n_{1,i}}, \xi_{n_{2,i}}, ..., \xi_{n_{|\mathcal{N}_i|,i}}]$ where $n_{1,i} < n_{2,i} < ... < n_{|\mathcal{N}_i|,i}$ and $\bigcup_{j \in \{1,...,|\mathcal{N}_i|\}} n_{j,i} \subset \mathcal{N}_i$.

4.3 Timer and Circle Systems

Consider the following timer system:

$$\begin{aligned} \dot{\tau} &= -1 \quad && \tau \in [0, T], \\ \tau^+ &= v && \tau = 0 \end{aligned} \tag{4.1}$$

where $T > 0$ and the random input v is uniformly distributed on $[0, T]$. The timer decreases steadily until it hits zero, at which point it jumps randomly to a point in the interval $[0, T]$.

For each $i \in \mathbb{Z}_{\geq 0}$ we define

$$\begin{aligned}
\mathbf{t}_i(\omega) &:= \inf_{(t,i)\in \operatorname{dom} \boldsymbol{\tau}(\omega)} t \\
\Omega_{\dashv,i} &:= \{\omega \in \Omega : \operatorname{graph}(\boldsymbol{\tau}(\omega)) \subset (\mathbb{R} \times \mathbb{Z}_{\leq i+2} \times \mathbb{R}^n)\} \\
\Omega_{(a,b),i} &:= \{\omega \in \Omega : (\operatorname{graph}(\boldsymbol{\tau}(\omega)) \cap (\{\mathbf{t}_i(\omega) + T\} \times \{i+1, i+2\} \times (a,b)) \neq \varnothing\} \cup \Omega_{\dashv,i}.
\end{aligned} \tag{4.2}$$

The event $\Omega_{(a,b),i}$ corresponds to the timer belonging to (a,b) at regular time T after the i-th jump. The following claim characterizes a useful subevent of $\Omega_{(a,b),i}$.

Claim 1 *For* (4.1) *with* $T > 0$, v *uniformly distributed on* $[0,T]$, *and constants* a, b *satisfying* $T \geq b > a \geq 0$, *there exists* $\epsilon > 0$ *and for each* $i \in \mathbb{Z}_{\geq 0}$ *and random solution* $\boldsymbol{\tau} \in \mathcal{S}_r([0,T])$ *there exists* $\Omega'_{(a,b),i} \subset \Omega$ *such that*

$$\Omega'_{(a,b),i} \subset \Omega_{(a,b),i} \tag{4.3a}$$

$$\Omega'_{(a,b),i} \in \mathcal{F}_{i+2} \tag{4.3b}$$

$$\textit{the pair } (\Omega'_{(a,b),i}, \mathcal{F}_i) \textit{ is independent} \tag{4.3c}$$

$$\mathbb{P}(\Omega'_{(a,b),i}) = \epsilon. \tag{4.3d}$$

Proof: See Appendix. ■

This claim is of interest as a preliminary step towards showing recurrence of a set corresponding to the synchronization of agents.

We now consider the case of N agents, each of which has a timer state τ_i and a state which belongs to the unit circle $\tilde{\xi}_i \in \mathbb{S}^1$. These agents are connected via a communication graph $\mathbb{G}$ so that each agent i knows the position $\tilde{\xi}_j$ of each of its neighbors $j \in \mathcal{N}_i$. For each agent $i \in \{1, ..., N\}$ we have

$$\begin{aligned}
&\tilde{x}_i := (\tilde{\xi}_i^\top, \tau_i)^\top, \quad v_i := (v_{1,i}, v_{2,i}, v_{3,i}), \quad \tilde{x} := (x_1^\top, ..., x_N^\top)^\top \in \mathbb{R}^{3N} \\
&\left.\begin{bmatrix} \dot{\tilde{\xi}}_i \\ \dot{\tau}_i \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} =: \tilde{f}(\tilde{x}_i) \right\} \quad (\tilde{\xi}_i, \tau_i) \in \mathbb{S}^1 \times [0, T] =: \tilde{C}, \\
&\left.\begin{bmatrix} \tilde{\xi}_i^+ \\ \tau_i^+ \end{bmatrix} \in \begin{bmatrix} \breve{G}_{\mathcal{N}_i}(\tilde{\xi}, v_i) \\ v_{3,i} \end{bmatrix} =: \tilde{G}_{\mathcal{N}_i}(\tilde{x}, v_i) \right\} \quad (\tilde{\xi}_i, \tau_i) \in \mathbb{S}^1 \times \{0\} =: \tilde{D},
\end{aligned} \tag{4.4}$$

where $\mathcal{N}_i$ is derived from a connected graph $\mathbb{G}$ containing N vertices,

$$\breve{G}_{\mathcal{N}_i}(\tilde{\xi}, v_i) := \begin{cases} \tilde{\xi}_{v_{2,i}}, & v_{1,i} = 0 \\ \breve{G}_{\mathcal{N}_i}(\tilde{\xi}), & v_{1,i} = 1, \end{cases} \tag{4.5}$$

and $\breve{G}_{\mathcal{N}_i} : (\mathbb{S}^1)^{|\mathcal{N}_i|} \rightrightarrows \mathbb{S}^1$ is an outer semi-continuous set-valued mapping that satisfies

$$\breve{G}_{\mathcal{N}_i}(\tilde{\xi}) \subset \bigcup_{j \in \mathcal{N}_i} \tilde{\xi}_j, \qquad \forall \tilde{\xi} \in \operatorname{dom} \breve{G}_{\mathcal{N}_i}. \tag{4.6}$$

We make the following assumptions on the random variables:

Assumption 8 *The inputs $v_{1,i}$, $v_{2,i}$, $v_{3,i}$ are generated from i.i.d random variables. The random variable generating $v_{1,i}$ has a distribution μ_1 defined in (4.7), where $p_{rc} \in (0, 1]$. The random variable generating $v_{2,i}$ has a distribution μ_2 which is uniform on $\mathcal{N}_i$. The random variable generating $v_{3,i}$ has a distribution μ_3 which is uniform on $[0, T]$.*

$$\mu_1(a) := \begin{cases} p_{rc}, & a = 0 \\ 1 - p_{rc}, & a = 1. \end{cases} \tag{4.7}$$

The overall system is $\{C_c, D_c, f_c, G_c, (\mu_1, \mu_2, \mu_3)\}$ defined as follows:

$$\begin{aligned}
C_c :=& \tilde{C} \times ... \times \tilde{C} \\
D_c :=& \{\tilde{x} \in C_c : \tilde{x}_i \in \tilde{D} \text{ for some } i \in \{1, ..., N\}\} \\
f_c(\tilde{x}) :=& \tilde{f}(\tilde{x}_1) \times \tilde{f}(\tilde{x}_2) ... \times \tilde{f}(\tilde{x}_N) \\
\gamma_i(\tilde{x}, v) :=& [\tilde{x}_1, ..., \tilde{x}_{i-1}, \tilde{G}_{\mathcal{N}_i}(\tilde{x}, v_i), \tilde{x}_{i+1}, ..., \tilde{x}_N] \\
G_c(\tilde{x}, v) :=& \bigcup_{i \in \{1,...,N\} : x_i \in \tilde{D}} \gamma_i(\tilde{x}, v).
\end{aligned} \tag{4.8}$$

4.4 Tracking Algorithm

With each agent $i \in \{1, ..., N\}$ we associate a non-stochastic hybrid control system with state $x_i = (\xi_i, \beta_i) \in \mathbb{S}^1 \times \{a, b\}$, input $w_i \in \mathbb{S}^1$, and dynamics

$$\begin{aligned}
(w_i, x_i) \in C \qquad & \dot{x}_i = f(w_i, x_i) \\
(w_i, x_i) \in D \qquad & x_i^+ = G(w_i, x_i)
\end{aligned} \tag{4.9}$$

with definitions

$$C_a := \{(w_i, \xi_i) \in \mathbb{S}^1 \times \mathbb{S}^1 : \xi_i \in R(\theta) w_i,\ \theta \in [\tfrac{\pi}{3}, \tfrac{5\pi}{3}]\} \tag{4.10a}$$

$$C_b := \{(w_i, \xi_i) \in \mathbb{S}^1 \times \mathbb{S}^1 : \xi_i \in R(\theta) w_i,\ \theta \in [-\tfrac{2\pi}{3}, \tfrac{2\pi}{3}]\} \tag{4.10b}$$

$$C := \bigcup_{\beta_i \in \{a,b\}} C_{\beta_i} \times \{\beta_i\} \tag{4.10c}$$

$$f(w_i, x_i) := \begin{bmatrix} (\mathbb{I}_a(\beta_i) + \mathbb{I}_b(\beta_i) w_i^\top J \xi_i) J \xi_i \\ 0 \end{bmatrix} \tag{4.10d}$$

$$D_a := \{(w_i, \xi_i) \in \mathbb{S}^1 \times \mathbb{S}^1 : \xi_i \in R(\theta) w_i,\ \theta \in [-\tfrac{\pi}{3}, \tfrac{\pi}{3}]\} \tag{4.10e}$$

$$D_b := \{(w_i, \xi_i) \in \mathbb{S}^1 \times \mathbb{S}^1 : \xi_i \in R(\theta) w_i,\ \theta \in [\tfrac{2\pi}{3}, \tfrac{4\pi}{3}]\} \tag{4.10f}$$

$$D := \bigcup_{\beta_i \in \{a,b\}} D_{\beta_i} \times \{\beta_i\} \tag{4.10g}$$

$$G(w_i, x_i) := \begin{bmatrix} \xi_i \\ \alpha \in \{a,b\} \backslash \beta_i : (w_i, \xi_i) \in C_\alpha \end{bmatrix}. \tag{4.10h}$$

This system has x_i track w_i via two modes: mode a is engaged when w_i is far from x_i and moves x_i at a constant rate clockwise; mode b is engaged when w_i is close to x_i and uses a local tracking rule. These two modes are combined with hysteresis.

We proceed by defining an autonomous system based on (4.9) where the input w_i belongs to an arc of the unit circle:

$$C_{\xi^*,\epsilon} := \{x_i : (w_i, x_i) \in C, w_i \in R_{\xi^*}([-\epsilon, \epsilon])\} \tag{4.11a}$$

$$F_{\xi^*,\epsilon}(x_i) := \{y_i : (w_i, x_i, y_i) \in \operatorname{gph} f, w_i \in R_{\xi^*}([-\epsilon, \epsilon])\} \tag{4.11b}$$

$$D_{\xi^*,\epsilon} := \{x_i : (w_i, x_i) \in D, w_i \in R_{\xi^*}([-\epsilon, \epsilon])\} \tag{4.11c}$$

$$G_{\xi^*,\epsilon}(x_i) := \{y_i : (w_i, x_i, y_i) \in \operatorname{gph} G, w_i \in R_{\xi^*}([-\epsilon, \epsilon]).\} \tag{4.11d}$$

For this system we have the following result:

Proposition 4 *For system* (4.11) *under the basic assumptions, for all* $\xi^* \in \mathbb{S}^1$ *and* $\epsilon \in [0, \frac{\pi}{12}]$*, the set* $R_{\xi^*}([-\epsilon, \epsilon]) \times \{b\}$ *is UGAS.*

Proof: See Appendix. ∎

4.5 Combined Algorithm

For each agent $i \in \{1, ..., N\}$, the combined algorithm has a state $\bar{x}_i := (\tilde{\xi}_i, \tau_i, \xi_i, \beta_i) = (\tilde{x}_i, x_i) \in \mathbb{S}^1 \times [0, T] \times \mathbb{S}^1 \times \{a, b\}$, flow map and flow set

$$\begin{gathered}\begin{bmatrix} \dot{\tilde{x}}_i \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} \tilde{f}(\tilde{x}_i) \\ f(\tilde{\xi}_i, x_i) \end{bmatrix} := \bar{f}(\bar{x}_i), \\ \bar{C} := \{(\tilde{\xi}_i, \tau_i, \xi_i, \beta_i) \in \mathbb{R}^6 : (\tilde{\xi}_i, \tau_i) \in \tilde{C},\ (\tilde{\xi}_i, \xi_i, \beta_i) \in C\},\end{gathered} \tag{4.12}$$

and jump map and jump set

$$\begin{gathered}\begin{bmatrix} \tilde{x}_i^+ \\ x_i^+ \end{bmatrix} \in \begin{bmatrix} \mathbb{I}_{\tilde{D}}(\tilde{x}_i)\tilde{G}(\tilde{\xi}, v_{1,i}, v_{2,i}) + (1 - \mathbb{I}_{\tilde{D}}(\tilde{x}_i))\tilde{x}_i \\ \mathbb{I}_D(x_i)G(\tilde{\xi}_i, x_i) + (1 - \mathbb{I}_D(x_i))x_i \end{bmatrix} := \bar{G}(\bar{x}, v_{1,i}, v_{2,i}), \\ \bar{D} := \{(\tilde{\xi}_i, \tau_i, \xi_i, \beta_i) \in \bar{C} : (\tilde{\xi}_i, \tau_i) \in \tilde{D} \text{ or } (\tilde{\xi}_i, \xi_i, \beta_i) \in D\}.\end{gathered} \tag{4.13}$$

Define the overall state $\bar{x}$ and permutation matrix $\mathcal{T}$ via

$$\bar{x} := (\tilde{x}, x) \tag{4.14a}$$

$$\zeta_i := (\tilde{\xi}_i, \tau_i, \xi_i, \beta_i) \tag{4.14b}$$

$$\zeta := (\zeta_1^\top, ..., \zeta_N^\top)^\top \tag{4.14c}$$

$$\bar{x} = \mathcal{T}\zeta. \tag{4.14d}$$

Using (4.12)-(4.13), we define the overall system:

$$C_{ov} := \mathcal{T}(\bar{C} \times ... \times \bar{C}) \tag{4.15a}$$

$$D_{ov} := \mathcal{T}(\{x \in \bar{C} : x_i \in \bar{D} \text{ for some } i \in \{1, ..., N\}\}) \tag{4.15b}$$

$$f_{ov}(\bar{x}) := \mathcal{T}(\bar{f}(\bar{x}_1) \times \bar{f}(\bar{x}_2) \times ... \times \bar{f}(\bar{x}_N)) \tag{4.15c}$$

$$\Gamma_i(\bar{x}, v_{1,i}, v_{2,i}) := [\bar{x}_1, ..., \bar{x}_{i-1}, \bar{G}(\bar{x}, v_{1,i}, v_{2,i}), \bar{x}_{i+1}, ..., \bar{x}_N] \tag{4.15d}$$

$$G_{ov}(\bar{x}, v) := \mathcal{T}\Big(\bigcup_{i \in \{1,...,N\}: x_i \in \bar{D}} \Gamma_i(\bar{x}, v_{1,i}, v_{2,i})\Big). \tag{4.15e}$$

4.6 Choices of jump map and simulation results

In this section, we discuss how different choices of $\breve{G}_{|\mathcal{N}_i|}$ and probability distributions affect the performance of the core algorithm (4.8).

We refer to the mapping $\breve{G}_{|\mathcal{N}_i|}$ (cf. (4.6)) together with distribution μ as a *jump rule*. We refer to the jump rule as *random neighbor* when $p_{rc} = 1$, that is, ξ_i jumps to the position of a neighbor based on a uniform distribution over neighbors. We refer to the jump rule as *minimax neighbor* when $p_{rc} \in (0, 1)$ and

$$\breve{G}_{|\mathcal{N}_i|}(\xi_{\mathcal{N}_i}) = \breve{G}_{mm|\mathcal{N}_i|}(\xi_{\mathcal{N}_i}) := \arg\min_{\zeta \in \xi_{\mathcal{N}_i}} \max_{j \in \mathcal{N}_i} |\zeta - \xi_j|. \tag{4.16}$$

With this rule, the agent jumps to the position of a neighbor such that the maximum distance to a neighbor is minimized.

We refer to the jump rule as *minimum sum of squares neighbor* when $p_{rc} \in (0, 1)$ and

$$\breve{G}_{|\mathcal{N}_i|}(\xi_{\mathcal{N}_i}) = \breve{G}_{sos|\mathcal{N}_i|}(\xi_{\mathcal{N}_i}) := \arg\min_{\zeta \in \xi_{\mathcal{N}_i}} \sum_{j \in \mathcal{N}_i} |\zeta - \xi_j|^2. \tag{4.17}$$

With this rule, the agent jumps to the position of a neighbor such that the sum of squares of distances to neighbors is minimized.

We are also interested in jump rules that depart slightly from the definition (4.5) but remain within a suitable perturbation. We refer to the jump rule as *near-minimax neighbor* when $p_{rc} \in (0,1)$ and

$$\check{G}_{|\mathcal{N}_i|}(\xi_{\mathcal{N}_i}) = \check{G}_{nmm|\mathcal{N}_i|}(\xi_{\mathcal{N}_i}, \xi_i) := \underset{\zeta \in \check{G}_{mm|\mathcal{N}_i|}(\xi_{\mathcal{N}_i}) + \gamma\mathbb{B}}{\arg\min} |\zeta - \xi_i|. \tag{4.18}$$

With this rule, the agent jumps to the closest position on the circle that lies within a γ-perturbation of minimax neighbor.

We compare these jump rules over a number of different communication graphs. Following the ideas in [47], we consider a family of communication graphs inspired by the 'small-world' phenomena. In particular, we consider a ring lattice graph where each agent has an in-degree of four, and then subsequent graphs which are randomly 'rewired' versions of the ring lattice graph, where the amount of rewiring is parameterized by a variable $p \in [0,1]$, with $p = 0$ indicating no rewiring, and $p = 1$ indicating that every edge gets randomly reassigned to another node.

Figures 4.1-4.3 show histograms of each jump rule under three different communication graphs. We see that as the average graph distance between agents becomes lower, the relative effectiveness of the near-minimax jump rule becomes larger. This is because the near-minimax rule combines the fast convergence to a semi-circle characteristic of conformist-type algorithms with fast convergence once on a convex manifold characteristic of averaging-type algorithms.

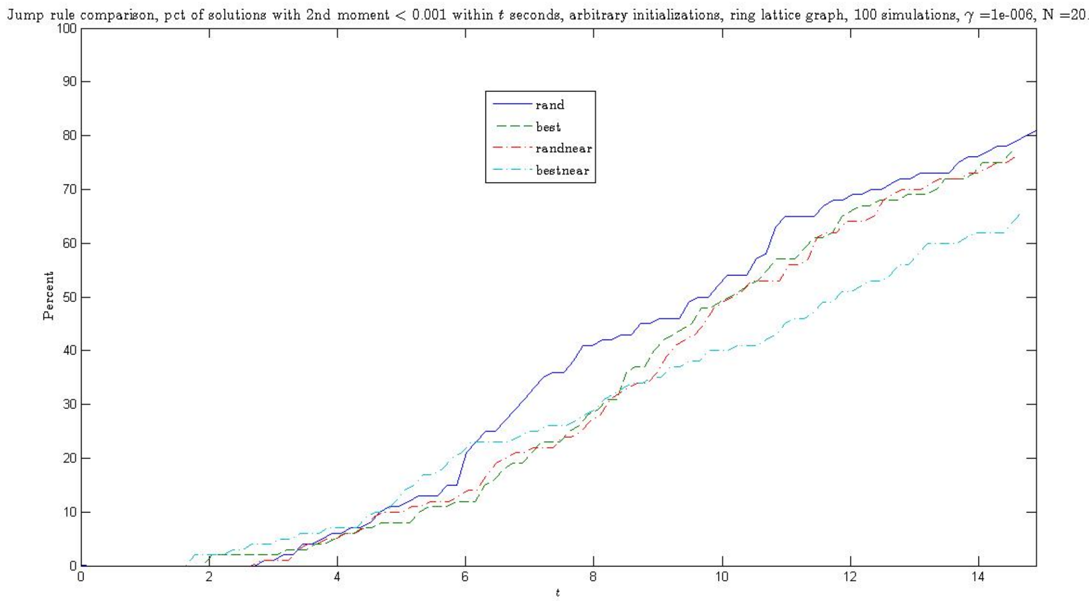

Figure 4.1: Comparison of jump rules under a ring lattice communication graph

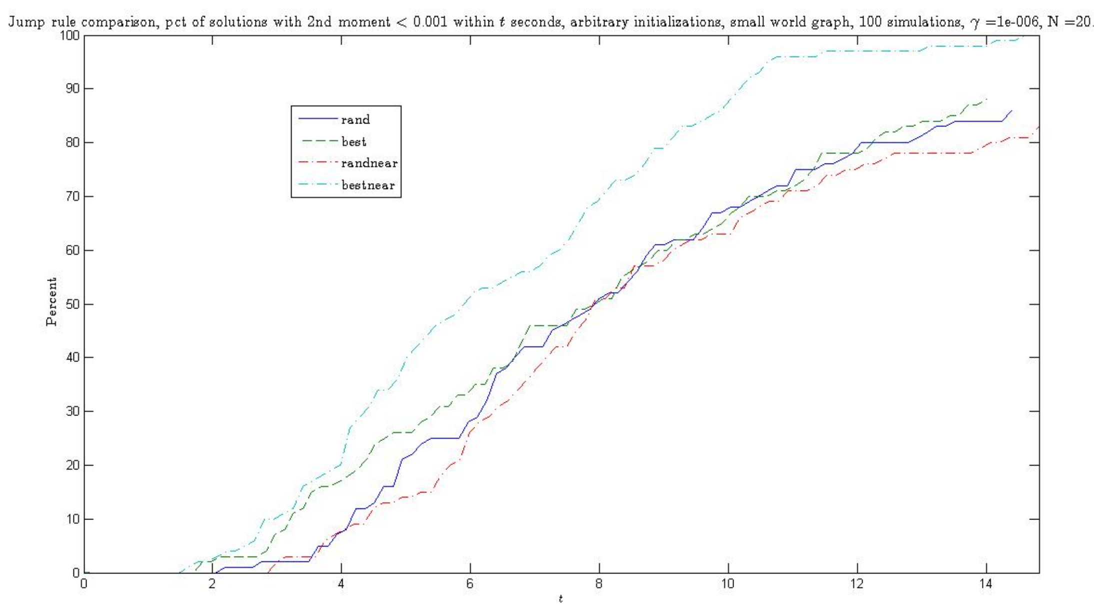

Figure 4.2: Comparison of jump rules under a small world communication graph

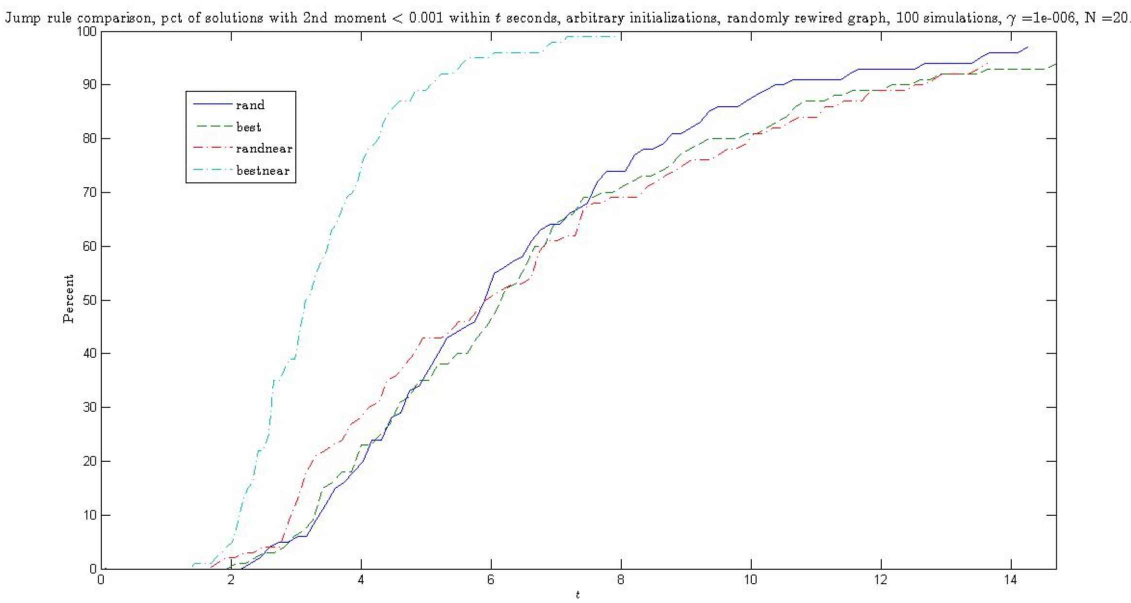

Figure 4.3: Comparison of jump rules under a randomized communication graph

Chapter 5

Conclusions and Future Directions

In chapter two we presented a finite-time parameter estimation algorithm and showed that a set corresponding to the correct parameter estimate had the property of robust asymptotic stability. We further gave a persistency of excitation condition under which solutions are guaranteed to be complete.

In chapter three we showed that, under a reasonable observability condition, one can implement hybrid controllers of continuous-time systems through a high-gain observer to obtain a semi-global practical result. It would be interesting to see if a similar result could be extended to the case where the plant is also hybrid.

In chapter four we presented an algorithm for synchronization on the circle under limited communication. We compared the performance of different versions of the algorithm under a variety of communication graphs. Further work needs to be done showing that the set corresponding to synchronization state has the global asymptotic stability in probability property. It would also be interesting to extend the algorithm idea into other topologies beyond the circle, e.g. $\mathbb{S}^2$ or $SO(3)$.

Appendix A

Proofs

A.1 Proof of Theorem 3

We begin by recalling two definitions.

Definition 2 *For a hybrid system* $\mathcal{H}$ (1.1) *and set* $\mathcal{X} \subset \mathbb{R}^n$*, we define the* reachable set *from* $\mathcal{X}$ *as*[1]

$$\mathcal{R}^0_{\mathcal{H}}(\mathcal{X}) := \{y \in \mathbb{R}^n : y = \phi(t, j), \phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}), (t, j) \in dom\ \phi\},$$

and the omega limit set *from* $\mathcal{X}$ *as*

$$\Omega_{\mathcal{H}}(\mathcal{X}) := \{y \in \mathbb{R}^n : y = \lim_{i\to\infty} \phi_i(t_i, j_i),\ \phi_i \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}),$$
$$(t_i, j_i) \in dom\ \phi_i,\ t_i + j_i \to \infty\}.$$

We now consider a proposition, introduced in [5][2]:

[1] $\mathcal{S}_{\mathcal{H}}(\mathcal{X})$ indicates the set of maximal solutions to $\mathcal{H}$ from the set of initial conditions $\mathcal{X}$.

Proposition 5 *Let $\mathcal{X}$ be compact. Suppose that for a hybrid system $\mathcal{H}$ satisfying the basic assumptions, $\mathcal{R}^0_{\mathcal{H}}(\mathcal{X})$ is bounded and $\Omega_{\mathcal{H}}(\mathcal{X}) \subset int(\mathcal{X})$. Then $\Omega_{\mathcal{H}}(\mathcal{X})$ is compact and asymptotically stable with basin of attraction containing $\mathcal{X}$. Moreover, it is the smallest such set contained in $int(\mathcal{X})$.*

This result will be the main tool in the proof of Theorem 3. We add the following definition:

Definition 3 *Given sets A and S where $A \subset S = S_1 \times S_2 \times \cdots \times S_k$ and variable $x = (x_1, x_2, \ldots, x_k) \in S$, we define the* projection *of A in the x_j direction as*

$$proj_{x_j}(A) := \{a_j \in S_j : (a_1, a_2, \ldots, a_j, \ldots, a_k) \in A\}.$$

This leads to the following lemma:

Lemma 3 *Suppose $A, B \subset S = S_1 \times S_2 \times \cdots \times S_k$ and $B = B_1 \times B_2 \times \cdots \times B_k$. Then*

$$\begin{pmatrix} proj_{x_1}(A) \subset proj_{x_1}(B), \\ \vdots \\ proj_{x_k}(A) \subset proj_{x_k}(B) \end{pmatrix} \Rightarrow (A \subset B).$$

Now we are ready to prove the stability result.

Proof: (Theorem 3) Throughout this proof, we analyze a system that is a generalization of $\mathcal{P}$. Consider the hybrid system $\hat{\mathcal{P}} := \{\hat{F}, C, \hat{G}, D\}$ with data given in (A.1), satisfying Assumption 3. In this system, we replace the functions $\tilde{f}$ and $\tilde{g}$ with free variables that live in a closed ball, giving us a differential inclusion rather than equation. This allows us to ignore the control signal u and focus on the parameter identification problem. After analyzing $\hat{\mathcal{P}}$, we will apply the results to $\mathcal{P}$, noting that solutions to $\mathcal{P}$ are contained in the solution set to $\hat{\mathcal{P}}$ when M_1 is sufficiently large.

[2] $int(\mathcal{X})$ indicates the set containing all the interior points of $\mathcal{X}$.

$$
\hat{\mathcal{P}} \quad \left\{ \begin{array}{l}
\begin{array}{rcl} C & := & X \times \mathbb{R}^n \times \mathbb{R}^p \times M\mathbb{B} \times \mathcal{SP}^{p\times p} \times \mathbb{R}^n \times \mathbb{R}^p \times [0,T] \\ D & := & \{\bar{x} \in C : det(Q) \geq \epsilon\} \end{array} \\
\quad \hat{F}(\bar{x}) := \left\{ \hat{f} : \hat{f} = \begin{bmatrix} v_0 + \sum_{i=1}^{p} v_i\theta_i \\ v_0 + \sum_{i=1}^{p} v_i\hat{\theta}_i + k(x-\hat{x}) \\ h(\bar{x}) \\ [v_1\ v_2\ ...\ v_p] - kw \\ w^\top w \\ -k\eta + wh(\bar{x}) \\ w^\top(w\hat{\theta} + x - \hat{x} - \eta) \\ 1 \end{bmatrix} \right\}, \\
\qquad \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_p \end{bmatrix} \in M_1\mathbb{B} \quad \hat{G}(\bar{x}) := \left\{ \begin{bmatrix} x \\ x \\ Q^{-1}\Gamma \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}
\end{array} \right. \tag{A.1}
$$

We apply Proposition 5 to system $\hat{\mathcal{P}}$ and compact set[3]

$$
\mathcal{X} = M_2\mathbb{B} \times M_2\mathbb{B} \times M_3\mathbb{B} \times (M+1)\mathbb{B} \times \mathcal{SP}^{p\times p}(M_4)
$$

$$
\times M_5\mathbb{B} \times M_6\mathbb{B} \times [-1, T+1]
$$

where constants $M_2 - M_6$ are determined by T, M_1, and X and will be described in greater detail later on.

First of all, since C and D are closed, $\hat{F}$ and $\hat{G}$ are outer-semicontinuous and locally bounded, $\hat{F}$ is non-empty and convex for all $\bar{x} \in C$, and $G\hat{(x)}$ is non-empty for all $\bar{x} \in D$, the Basic Assumptions are satisfied.

Next, we show that $\mathcal{R}^0_{\hat{\mathcal{P}}}(\mathcal{X})$ is bounded by demonstrating that $\hat{F}(\bar{x})$ obeys a sector condition, ruling out finite escape times. Then, since solutions flow for a maximum time

[3] $\mathcal{SP}^{p\times p}(M)$ indicates the set of positive semi-definite matrices with maximum eigenvalue upper bounded by M.

T, and jump to a bounded set, we have that $\mathcal{R}^0_{\hat{\mathcal{P}}}(\mathcal{X})$ is bounded.

We proceed to show that $\hat{F}(\bar{x})$ obeys the sector condition $|\hat{f}| \leq L_1|\bar{x}| + L_2 \quad \forall \hat{f} \in \hat{F}(\bar{x}), \quad \forall \bar{x} \in C$ for some positive constants L_1 and L_2, where we choose $|\bar{x}| := |x| + |\hat{x}| + |\hat{\theta}| + |w| + |Q| + |\eta| + |\Gamma| + |\tau|$ as the norm in the space to which $\bar{x}$ belongs. In the following series of inequalities, note that x, w, and the v's are bounded and $|h(\bar{x})| \leq L|x - \hat{x}|$ from Assumption 3,

$$\begin{aligned} |\hat{f}| &\leq |\dot{x}| + |\dot{\hat{x}}| + |\dot{\hat{\theta}}| + |\dot{w}| + |\dot{Q}| + |\dot{\eta}| + |\dot{\Gamma}| + |\dot{\tau}| \\ &\leq L_1(|\hat{x}| + |\eta|) + L_2 \leq L_1|\bar{x}| + L_2. \end{aligned}$$

This implies that $\hat{\mathcal{P}}$ will not exhibit finite escape time.

To see what happens to $\bar{x}$ at jump times, note that after one jump followed by continuous change according to $\hat{F}$ on the interval $(t_1, 1)$ to $(t_2, 1)$, we have $\eta(t,1) = x(t,1) - \hat{x}(t,1) - w(t,1)[\theta - \hat{\theta}(t,1)] \quad \forall t \in [t_1, t_2]$. With another jump at time $(t_2, 1)$, $Q(t_2, 1)$ is invertible due to the definition of the jump set D. And so,

$$\begin{aligned} \theta &= Q(t_2,1)^{-1}Q(t_2,1)\theta = Q(t_2,1)^{-1}\int_{t_1}^{t_2} w^\top(\tau,1)w(\tau,1)\theta d\tau \\ &= Q(t_2,1)^{-1}\int_{t_1}^{t_2} w^\top(\tau,1)[w(\tau,1)\hat{\theta}(\tau,1) + x(\tau,1) \\ &\qquad - \hat{x}(\tau,1) - \eta(\tau,1)]d\tau = Q(t_2,1)^{-1}\int_{t_1}^{t_2} \dot{\Gamma}(\tau,1)d\tau \\ &= Q(t_2,1)^{-1}(\Gamma(t_2,1) - \Gamma(t_1,1)) = Q(t_2,1)^{-1}\Gamma(t_2,1) \end{aligned}$$

which is the value that $\hat{G}$ assigns $\hat{\theta}$ at time $(t_2, 2)$. Then for all $j \geq 2$, $|\dot{\hat{\theta}}| = |h(\bar{x})| \leq L|x - \hat{x}| = 0$ since $\dot{\hat{x}} = \dot{x}$ during flows and $\hat{x}^+ = x$ during jumps. And so, noting that flow time is bounded by T, we have for all solutions $\bar{x}$ to $\mathcal{P}$, $t + j \geq 2 + 2T$ implies $\hat{\theta}(t,j) = \theta \quad \forall (t,j) \in dom\, \bar{x}$.

This implies that, after two jumps, every jump is to a point in the compact set $X \times X \times \{\theta\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\}$. Furthermore, since flow time is bounded by

T and we have no finite escape time, it follows that $\mathcal{R}^0_{\hat{\mathcal{P}}}(\mathcal{X})$ is bounded [15, Proposition 7].

The next step in applying Proposition 5 to $\hat{\mathcal{P}}$ is to show that $\Omega_{\hat{\mathcal{P}}}(\mathcal{X}) \subset int(\mathcal{X})$. We do this by applying Lemma 1 where $A = \Omega_{\hat{\mathcal{P}}}(\mathcal{X})$, $B = int(\mathcal{X})$ and $x = \bar{x}$. The next several paragraphs will consist of showing that the projection of $\Omega_{\hat{\mathcal{P}}}(\mathcal{X})$ onto each state is a subset of the projection of $int(\mathcal{X})$ onto that state.

To start, x belongs to the compact set X and we have $proj_x(\Omega_{\hat{\mathcal{P}}}(\mathcal{X})) = X \subset int(M_2\mathbb{B}) = proj_x(int(\mathcal{X}))$, for sufficiently large M_2.

For $\hat{x}$, since $\hat{\theta} = \theta$ for hybrid time $t + j \geq 2 + 2T$, and $\hat{x}$ resets periodically to x, we will have the behavior $\dot{\hat{x}} = \dot{x}$ during flows and $\hat{x}^+ = x$ during jumps. This implies $\hat{x} = x$ for $t + j \leq 2 + 2T$ and therefore $proj_{\hat{x}}(\Omega_{\hat{\mathcal{P}}}(\mathcal{X})) = proj_x(\Omega_{\hat{\mathcal{P}}}(\mathcal{X})) = X \subset int(M_2\mathbb{B}) = proj_{\hat{x}}(int(\mathcal{X}))$. This furthermore shows that the projection of $\Omega_{\hat{\mathcal{P}}}(\mathcal{X})$ onto x and $\hat{x}$ is $S = \{x, \hat{x} : x \in X, \hat{x} = x\}$.

For $\hat{\theta}$, we know $proj_{\hat{\theta}}(\Omega_{\hat{\mathcal{P}}}(\mathcal{X})) = \theta$ since we have shown that $\hat{\theta} = \theta$ for $t + j \geq 2 + 2T$. And so, $proj_{\hat{\theta}}(\Omega_{\hat{\mathcal{P}}}(\mathcal{X})) = \theta \subset int((M_3)\mathbb{B}) = proj_{\hat{\theta}}(int(\mathcal{X}))$ for some $M_3 > 0$.

The state w is constrained a priori to the set $M\mathbb{B}$. Therefore $proj_w(\Omega_{\hat{\mathcal{P}}}(\mathcal{X})) \subset int((M + 1)\mathbb{B}) = proj_w(int(\mathcal{X}))$.

For Q, the size of $proj_Q(\Omega_{\hat{\mathcal{P}}}(\mathcal{X}))$ is a function of T and M_1 since Q resets to zero every T seconds, and the growth of Q is bounded by w, which is itself bounded by M_1 and T. But for any T and M_1 there exists M_4 large enough such that $proj_w(\Omega_{\hat{\mathcal{P}}}(\mathcal{X})) \subset \mathcal{SP}^{p \times p}(M_4) = proj_Q(int(\mathcal{X})))$.

For η, note that $\hat{\theta}$ converges to θ for complete solutions, which implies $\dot{\hat{x}} = \dot{x}$, leading to $\hat{x} = x$ (since $\hat{x}$ jumps to x). This implies $h(\bar{x}) = 0$ by Assumption 3, giving us $\dot{\eta} = -k\eta$ during flows, and hence $proj_\eta(\Omega_{\hat{\mathcal{P}}}(\mathcal{X})) = 0 \subset int(M_5\mathbb{B}) = proj_\eta(int(\mathcal{X}))$ for some $M_5 > 0$.

For Γ, the size of $proj_\Gamma(\Omega_{\hat{\mathcal{P}}}(\mathcal{X}))$ is a function of T, M_1, X, and θ since Γ resets to zero every T seconds, and the growth of Γ is bounded by variables whose bounds depend on T, M_1, X and θ. But for any T, M_1, X and θ there exists M_6 large enough such that $S_4 \subset int(M_6\mathbb{B}) = proj_\Gamma(int(\mathcal{X}))$.

For τ, it is clear that $proj_\tau(\Omega_{\hat{\mathcal{P}}}(\mathcal{X})) = [0, T] \subset int([-1, T+1]) = proj_\tau(int(\mathcal{X}))$.

And so, by Lemma 3 we have $\Omega_{\hat{\mathcal{P}}}(\mathcal{X}) \subset int(\mathcal{X})$ which, along with boundedness of $\mathcal{R}^0_{\hat{\mathcal{P}}}(\mathcal{X})$, completes the requirements of Proposition 1. We have that $\Omega_{\hat{\mathcal{P}}}(\mathcal{X})$ is compact and asymptotically stable with basin of attraction containing $\mathcal{X}$. Furthermore, we see from the analysis that $\Omega_{\hat{\mathcal{P}}}(\mathcal{X}) \subset S \times \{\theta\} \times M\mathbb{B} \times \mathcal{SP}^{p\times p}(M_4) \times \{0\} \times M_6\mathbb{B} \times [0, T]$ with $S = \{x, \hat{x} : x \in X, \hat{x} = x\}$ for sufficiently large M_4 and M_6. Moreover, by increasing $M_3 - M_6$ we do not change $\Omega_{\hat{\mathcal{P}}}(\mathcal{X})$ since the states of the system are either bounded or reset to bounded values. Therefore we can take $\mathcal{X}$ to be arbitrarily large, implying that the basin of attraction includes $C \cup D$. And so we conclude that $\hat{\mathcal{P}}$ contains a compact, globally asymptotically stable set $\Omega_{\hat{\mathcal{P}}}(C \cup D) \subset S \times \{\theta\} \times M\mathbb{B} \times \mathcal{SP}^{p\times p} \times \{0\} \times \mathbb{R}^p \times [0, T]$.

Furthermore, noting that u is bounded, we can find M_1 sufficiently large such that solutions to $\mathcal{P}$ are contained in the solution set to $\hat{\mathcal{P}}$. Therefore, our characterization of $\Omega_{\hat{\mathcal{P}}}(C \cup D)$ holds for $\Omega_{\mathcal{P}}(C \cup D)$. ■

A.2 Proof of Theorem 4

We recall two lemmas: one from linear algebra, and another concerning PE signals through LTI systems.

Lemma 4 *[2, Corollary 8.4.10] Given symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, if $A \geq B \geq 0$, then $det(A) \geq det(B)$.*

Lemma 5 *[36, Lemma 2.6.7] Let $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. If u is PE, $u, \dot{u} \in L_\infty$ and $\hat{H}$ is a stable, minimum phase, rational transfer function then $\hat{H}(u)$ is PE.*

We begin the proof by noting that solutions to $\mathcal{P}$ will not end due to the following situation: the w component of the solution is at the boundary of $M\mathbb{B}$ and the flow map of w is directed outside of $M\mathbb{B}$. This situation is excluded by the fact that w resets to zero at jumps, and during the time between jumps it obeys $\dot{w} = \tilde{g} - kw$, which is ISS [38] with respect to $\tilde{g}$. To see this, consider $V(w) = trace(w^\top w)$ as a Lyapunov function. Then

$$\begin{aligned}\dot{V}(w) &= trace(0.5(\dot{w}^\top w + w^\top \dot{w})) = trace(w^\top(\tilde{g} - kw)) \\ &= trace(w^\top \tilde{g} - kw^\top w)) < 0, \quad \forall\, |w| > \frac{1}{k}|\tilde{g}|.\end{aligned}$$

Since x and u belong to compact sets, the inequality

$$M \geq \frac{1}{k} \underset{x\in X\ u\in U}{sup} |\tilde{g}(x,u)|.$$

from Assumption 4 guarantees that solutions to $\mathcal{P}$ will not end in this manner.

Our ensuing method is to consider maximal solutions to $\dot{\bar{x}} = f(\bar{x}, u)$ starting from $\{\bar{x} \in C \cup D : \tau \in [0,1]\}$, and then showing that we can find a T^* where, for all $T \geq T^*$, $\tau(t) = T$ implies $det(Q(t)) \geq \epsilon$.

Proceeding along this path, we see that w is the output of a strictly stable, minimum phase LTI filter with input $\tilde{g}$

$$\tilde{u}(t) = \tilde{g}(x(t), u(t)), \quad \hat{H} = \frac{1}{s+k}, \quad w = \hat{H}(\tilde{u}).$$

We apply Lemma 5 to $\tilde{u}(t)$, and note that $\tilde{u}, \dot{\tilde{u}} \in L_\infty$ since we assume $t \mapsto u(t)$ is globally Lipschitz, x is bounded and $\tilde{g}$ is differentiable. Although this lemma assumes a vector signal, it applies as well for the matrix signal case. It also turns out that the parameters of the PE output are dependent on the initial conditions of the system $\hat{H}$. But $w \in M\mathbb{B}$ means that the initial conditions of $\hat{H}$ lie in a compact set, and therefore we can find parameters independent of these initial conditions.

And so, given that $\tilde{g}$ is PE for system $\mathcal{P}$, w is PE, implying the existence of $\delta, \alpha > 0$ such that

$$\int_0^{\delta} w^\top(s)w(s)ds \geq \alpha I.$$

This means, for any positive integer m,

$$\int_0^{m\delta} w^\top(s)w(s)ds \geq m\alpha I.$$

Choosing m such that $(m\alpha)^p \geq \epsilon$, and then T^* such that $T^* > m\delta + 1$, we have for all $T \geq T^*$

$$Q(T) = \int_0^{T} w^\top(s)w(s)ds + Q(0) \geq Q(T-1)$$

$$= \int_0^{T-1} w^\top(s)w(s)ds + Q(0) \geq \int_0^{m\delta} w^\top(s)w(s)ds \geq m\alpha I$$

which, by Lemma 2, gives us $det(Q(T-1)) \geq det(m\alpha I) = (m\alpha)^p \geq \epsilon$. Given we start from $\{\bar{x} \in C \cup D : \tau \in [0,1]\}$,

$$\begin{aligned} \tau(t) = T \;\Rightarrow\; & t \in [T-1, T] \\ & \Rightarrow\; det(Q(t)) \geq det(Q(T-1)) \;\Rightarrow\; det(Q(t)) \geq \epsilon. \end{aligned}$$

This implies that for system $\mathcal{P}$, solutions starting from $\{\bar{x} \in C \cup D : \tau \in [0,1]\}$ will be in D when $\tau = T$. Since we have ruled out finite escape times in Appendix A, and by Assumption 3, maximal solutions to $\dot{x} = \tilde{f}(x,u) + \tilde{g}(x,u)\theta$ starting from X are complete, we have that maximal solutions starting from $\{\bar{x} \in C \cup D : \tau \in [0,1]\}$ are complete for sufficiently large T.

A.3 Proof of Lemma 1

Let $\epsilon > 0$ and K_x be a compact set satisfying $K_x \subset X$. We introduce the following system

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \\ \dot{\tau} \end{bmatrix} \in \begin{bmatrix} f(x, \alpha(x+\delta_1\mathbb{B}, x_c)) \\ f_c(x+\delta_1\mathbb{B}, x_c) \\ \sigma(\tau) \end{bmatrix},$$
$$(x, x_c, \tau) \in \{(x, x_c, \tau) : ((\{x\}+\delta_1\mathbb{B}) \times \{x_c\}) \cap C_c \neq \emptyset, \tau \in [0,2]\} \cup (X \times \mathbb{R}^c \times [0, \delta_1])$$
$$\begin{bmatrix} x^+ \\ x_c^+ \\ \tau^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_c(x+\delta_1\mathbb{B}, x_c) \\ 0 \end{bmatrix},$$
$$(x, x_c, \tau) \in \{(x, x_c, \tau) : ((\{x\}+\delta_1\mathbb{B}) \times \{x_c\}) \cap D_c \neq \emptyset, \tau \in [0,2]\}, \tag{A.2}$$

which can be thought of as the state feedback system (3.2) subjected to a perturbation which covers temporal regularization perturbations in the manner of (3.3), as well as "inner perturbations".

In the case of (A.2) under Assumption 7 with $\delta_1 = 0$, we have that the set $\mathcal{A} \times [0,2]$ is globally asymptotically stable, which follows from the fact that $\sigma(0) > 0$ and the dynamics affecting the (x, x_c)-component of the state are identical to those of (3.2). Thus, defining $\tilde{x} := (x, x_c, \tau)$ and using Proposition 1, let $\beta \in \mathcal{KL}$ satisfy

$$|\tilde{x}(t,j)|_{\mathcal{A}\times[0,2]} \leq \beta(|\tilde{x}(0,0)|_{\mathcal{A}\times[0,2]}, t+j) \quad \forall (t,j) \in \operatorname{dom} \tilde{x} \tag{A.3}$$

for system (A.2) with $\delta_1 = 0$. Using that K_x and X_c are compact, let $M_1 > 0$ satisfy

$$(K_x + \mathbb{B}) \times X_c \subset \mathcal{A} + M_1\mathbb{B}, \tag{A.4}$$

and define

$$M := \beta(M_1, 0) + 2\epsilon. \tag{A.5}$$

Let $x_{\max} > 0$ and using Proposition 2, let $\delta_1 > 0$ be such that

$$|\tilde{x}(t,j)|_{\mathcal{A}\times[0,2]} \leq \beta(|\tilde{x}(0,0)|_{\mathcal{A}\times[0,2]}, t+j) + \epsilon \quad \forall (t,j) \in \operatorname{dom} \tilde{x} \tag{A.6}$$

is satisfied for solutions to (A.2) starting from $(\mathcal{A} + M_1\mathbb{B}) \times [0,2]$. Define

$$T^* := \left(\max_{(x,x_c,\hat{x})\in(K_x+\mathbb{B})\times X_c\times x_{\max}\mathbb{B}} |f(x, \alpha(\hat{x}, x_c))| \right)^{-1}, \tag{A.7}$$

which is well-defined since X_c is compact, and take $\delta := \min\{\frac{\delta_1}{2}, T^*, 1\}$. Let $T \in (0, \delta]$, $L > 0$, $\bar{x} \in \mathcal{S}_{(3.8)}(K_{K_x,K_{\hat{y}},L})$ and $(\tilde{t}, \tilde{j}) \in \operatorname{dom} \bar{x}$.

From (3.8) it is clear that

$$|f(x, \alpha(\hat{x}(\hat{h}(x,e)), x_c))| \leq \max_{(x,x_c,\hat{x})\in(K_x+\mathbb{B})\times X_c\times x_{\max}\mathbb{B}} |f(x, \alpha(\hat{x}, x_c))|$$

when $x \in (K_x + \mathbb{B})$. Since the x-component of $\bar{x}$ does not change at jumps, if

$$x(t,j) \in (K_x + \mathbb{B}) \qquad (t,j) \in \operatorname{dom} \bar{x} \cap ([0,T] \times \mathbb{N}), \tag{A.8}$$

then

$$|x(t,j)-x(0,0)| \le T \max_{(x,x_c,\hat{x})\in(K_x+\mathbb{B})\times X_c\times x_{\max}\mathbb{B}} |f(x,\alpha(\hat{x},x_c))| \qquad (t,j)\in \mathrm{dom}\bar{x}\cap([0,T]\times\mathbb{N}). \tag{A.9}$$

Thus, by our choice of T and the fact that $x(0,0) \in K_x$ it follows that

$$|x(t,j) - x(0,0)| \le 1 \qquad (t,j) \in \mathrm{dom}\,\bar{x} \cap ([0,T] \times \mathbb{N}), \tag{A.10}$$

which with (A.4) and (A.5) implies that (3.10) is satisfied.

We proceed to analyze the system (A.11) below, which differs from (3.8) in that flow and jump sets include the constraint $|x - \hat{x}(\hat{h}(x,e))| \le \delta$ (due to $\overline{E}_\delta$, defined in (3.9)).

$$\begin{gathered}
\begin{bmatrix} \dot{x} \\ \dot{x}_c \\ \dot{\tau} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} f(x,\alpha(\hat{x}(\hat{h}(x,e)),x_c)) \\ f_c(\hat{x}(\hat{h}(x,e)),x_c) \\ \sigma(\tau) \\ LAe + \phi(x,x_c,\hat{x}(\hat{h}(x,e))) \end{bmatrix}, \ (x,x_c,\tau,e) \in C_{(3.8)} \cap \overline{E}_\delta \\
\begin{bmatrix} x^+ \\ x_c^+ \\ \tau^+ \\ e^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_c(\hat{x}(\hat{h}(x,e)),x_c) \\ 0 \\ e \end{bmatrix}, \ (x,x_c,\tau,e) \in D_{(3.8)} \cap \overline{E}_\delta.
\end{gathered} \tag{A.11}$$

Using (3.11), (A.4), and (A.10), there exists $\bar{x}_1 \in \mathcal{S}_{(A.11)}((\mathcal{A}+M_1\mathbb{B})\times[0,2]\times\mathbb{R}^{p(n_y+1)})$ such that $\bar{x}(t,j) = \bar{x}_1(t-T,j)$ for $(t,j) \in \mathrm{dom}\,\bar{x} \cap ([T,\tilde{t}] \times [0,\tilde{j}])$. Then, since $|x-\hat{x}| \le \delta$

implies $\hat{x} \in (x + \delta\mathbb{B})$, we have $\bar{x}_1 \in \mathcal{S}_{(A.12)}((\mathcal{A} + M_1\mathbb{B}) \times [0,2] \times \mathbb{R}^{p(n_y+1)})$.

$$
\begin{gathered}
\begin{bmatrix} \dot{x} \\ \dot{x}_c \\ \dot{\tau} \\ \dot{e} \end{bmatrix} \in \begin{bmatrix} f(x, \alpha(x + \delta\mathbb{B}, x_c)) \\ f_c(x + \delta\mathbb{B}, x_c) \\ \sigma(\tau) \\ LAe + \phi(x, x_c, \hat{x}(\hat{h}(x, e))) \end{bmatrix}, \\
(x, x_c, \tau, e) \in (\{(x, x_c, \tau) : ((\{x\} + 2\delta\mathbb{B}) \times \{x_c\}) \cap C_c \neq \emptyset, \tau \in [0,2]\} \\
\cup (X \times \mathbb{R}^c \times [0,T])) \times \mathbb{R}^{p(n_y+1)} \\
\begin{bmatrix} x^+ \\ x_c^+ \\ \tau^+ \\ e^+ \end{bmatrix} \in \begin{bmatrix} x \\ G_c(x + \delta\mathbb{B}, x_c) \\ 0 \\ e \end{bmatrix}, \\
(x, x_c, \tau, e) \in \{(x, x_c, \tau) : ((\{x\} + 2\delta\mathbb{B}) \times \{x_c\}) \cap D_c \neq \emptyset, \tau \in [T,2]\} \times \mathbb{R}^{p(n_y+1)}.
\end{gathered}
\tag{A.12}
$$

Furthermore, because $T \leq \delta_1$, $\frac{\delta}{2} \leq \delta_1$ and the dynamics of e do not affect (x, x_c, τ) in (A.12), the (x, x_c, τ)-component of $\bar{x}_1$ is a solution to (A.2) starting from $(\mathcal{A} + M_1\mathbb{B}) \times [0,2]$. Therefore, $\bar{x}_1(t,j)$ satisfies (A.6), and hence $\bar{x}(t,j)$ satisfies

$$
\begin{aligned}
|((x(t,j), x_c(t,j))|_{\mathcal{A}} &= |(x(t,j), x_c(t,j), \tau(t,j))|_{\mathcal{A}\times[0,2]} \\
&\leq \beta(|(x(0,0), x_c(0,0))|_{\mathcal{A}}, t - T + j) + \epsilon \\
&\leq \beta(M_1, 0) + \epsilon \\
&< M \qquad\qquad \forall (t,j) \in \operatorname{dom} \bar{x} \cap ([T, \tilde{t}] \times [0, \tilde{j}]),
\end{aligned}
\tag{A.13}
$$

which implies (3.12).

A.4 Proof of Lemma 2

Let $\epsilon, M, c > 0$, $K_x \subset X$ and $K_{\hat{y}} \subset \mathbb{R}^{p(n_y+1)}$ be compact sets,

$$x^*_{\max} = \max_{(x,x_c)\in\mathcal{A}+M\mathbb{B}} |x| + c,$$

$x_{\max} \geq x^*_{\max}$, $T \in (0,1]$, and $\delta > 0$. We define the following:

$$\begin{aligned}
\tilde{\Phi}_L(x,e) &:= \Phi(\varphi_0(x) - L^{-n_y}e_0, \varphi_1(x) - L^{1-n_y}e_1, \dots, \varphi_{n_y}(x) - e_{n_y}),\\
b &:= \max\nolimits_{x\in K_x, \hat{y}\in K_{\hat{y}}} \left| \begin{bmatrix} \varphi_0(x) - \hat{y}_0 \\ \vdots \\ \varphi_{n_y}(x) - \hat{y}_{n_y} \end{bmatrix} \right|,\\
\Lambda_L &:= \{(x, x_c, \tau, e) \in X \times X_c \times [0,2] \times \mathbb{R}^{p(n_y+1)} : |\tilde{\Phi}_L(x,e)| \leq x_{\max}\},\\
\beta_1 &:= \max\nolimits_{(x,x_c,\hat{x})\in(\mathcal{A}+M\mathbb{B})\times x_{\max}\mathbb{B}} |\phi(x,x_c,\hat{x})|.
\end{aligned} \tag{A.14}$$

Noting that $\tilde{\Phi}_L$ is locally Lipschitz, let $c_1 > 0$ satisfy

$$|\tilde{\Phi}_L(x,e) - \tilde{\Phi}_L(x,0)| \leq \frac{c}{c_1}|e| \quad \forall (x,x_c,e) \in (\mathcal{A}+M\mathbb{B}) \times c_1\mathbb{B}, \tag{A.15}$$

independently of L. Let $P > 0$ satisfy $PA + A^\top P = -I$ and let $L_1^* > 0$ be such that

$$\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} bL^{n_y} \exp\left(\frac{-LT}{2\lambda_{\min}(P)}\right) + \frac{2|P|\beta_1}{L} \leq c_1 \tag{A.16}$$

is satisfied for all $L \geq L_1^*$. Let $L_2^* > 0$ be such that

$$\frac{c}{c_1}\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} bL^{n_y} \exp\left(\frac{-LT}{2\lambda_{\min}(P)}\right) + \frac{2|P|\beta_1}{L}\right) < \delta \tag{A.17}$$

is satisfied for all $L \geq L_2^*$. Let $L^* := \max\{L_1^*, L_2^*\}$, $L \geq L^*$, $\bar{x} \in \mathcal{S}_{(3.8)}(K_{K_x, K_{\hat{y}}, L})$ and $(\tilde{t}, \tilde{j}) \in \operatorname{dom} \bar{x}$. Let $W(e) := \sqrt{e^\top P e}$. Then,

$$\begin{aligned}
\dot{W}(e) &= \frac{1}{2W(e)}(\dot{e}^\top P e + e^\top P \dot{e}) \\
&= \frac{1}{2W(e)}(L e^\top (PA + A^\top P) e + 2 e^\top P \phi(x, x_c, \hat{x}(\hat{h}(x, e)))) \\
&\leq \frac{-L e^\top e}{2W(e)} + \frac{|e||P|\beta_1}{W(e)} \\
&\leq \frac{-L}{2\lambda_{\min}(P)} W(e) + \frac{|P|\beta_1}{\sqrt{\lambda_{\min}(P)}}.
\end{aligned} \tag{A.18}$$

Then by a comparison lemma, noting e does not change at jumps,

$$\begin{aligned}
W(e(t,j)) &\leq W(e(0,0)) \exp\left(\frac{-Lt}{2\lambda_{\min}(P)}\right) + \frac{2|P|\beta_1 \sqrt{\lambda_{\min}(P)}}{L}(1 - \exp\left(\frac{-Lt}{2\lambda_{\min}(P)}\right)) \\
\Rightarrow |e(t,j)| &\leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} |e(0,0)| \exp\left(\frac{-Lt}{2\lambda_{\min}(P)}\right) + \frac{2|P|\beta_1}{L} \\
&\leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} b L^{n_y} \exp\left(\frac{-Lt}{2\lambda_{\min}(P)}\right) + \frac{2|P|\beta_1}{L}.
\end{aligned} \tag{A.19}$$

Therefore, from (A.16) we have

$$|e(t,j)| \leq c_1 \qquad (t,j) \in \operatorname{dom} \bar{x} \cap ([T, \tilde{t}] \times [0, \tilde{j}]).$$

Then using (3.13), (A.14), (A.15), and the fact that $x_{\max} \geq \max_{(x,x_c) \in \mathcal{A} + M\mathbb{B}} |x| + c$, we have

$$\bar{x}(t,j) \in \Lambda_L \qquad (t,j) \in \operatorname{dom} \bar{x} \cap ([T, \tilde{t}] \times [0, \tilde{j}]).$$

And so, using (A.17):

$$\begin{aligned}|x(t,j)-\hat{x}(x(t,j),e(t,j))| &= |\tilde{\Phi}_L(x(t,j),0)-\mathrm{sat}_{x_{\max}}(\tilde{\Phi}_L(x(t,j),e(t,j)))| \\ &= |\tilde{\Phi}_L(x(t,j),0)-\tilde{\Phi}_L(x(t,j),e(t,j))| \\ &\leq \frac{c}{c_1}|e(t,j)| \\ &\leq \frac{c}{c_1}\left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}bL^{n_y}\exp\left(\frac{-Lt}{2\lambda_{\min}(P)}\right)+\frac{2|P|\beta_1}{L}\right) \\ &< \delta \qquad (t,j)\in \mathrm{dom}\,\bar{x}\cap([T,\tilde{t}]\times[0,\tilde{j}]),\end{aligned}$$

which implies (3.14).

A.5 Proof of Proposition 3

This proof is based on [15, Corollary 19]. Let $\epsilon, M, c > 0$, $x^*_{\max} = \max_{(x,x_c)\in\mathcal{A}+M\mathbb{B}} |x|+c$ and $x_{\max} \geq x^*_{\max}$. We introduce the following system, which differs from (3.3) in that the space of (x, x_c) is restricted to $\mathcal{A} + M\mathbb{B}$ when $\tau > T$:

$$\begin{bmatrix}\dot{x}\\ \dot{x}_c\\ \dot{\tau}\end{bmatrix} = \begin{bmatrix}f(x,\alpha(x,x_c))\\ f_c(x,x_c)\\ \sigma(\tau)\end{bmatrix},$$
$$(x,x_c,\tau)\in((C_c\times[0,2])\cup(X\times\mathbb{R}^c\times[0,T]))\cap(\mathcal{A}+M\mathbb{B}\times[0,2]) \tag{A.20}$$

$$\begin{bmatrix}x^+\\ x_c^+\\ \tau^+\end{bmatrix} \in \begin{bmatrix}x\\ G_c(x,x_c)\\ 0\end{bmatrix}, \qquad (x,x_c,\tau)\in(D_c\cap\mathcal{A}+M\mathbb{B})\times[T,2].$$

Using Assumption 7 and $\sigma(0) > 0$, it is clear that $\mathcal{A}\times[0,2]$ is globally asymptotically stable for (A.20) when $T = 0$. Then using Proposition 2, let $\beta \in \mathcal{KL}$ and $T^* \in (0,1]$ such that for all $T \in (0, T^*]$,

$$|\tilde{x}(t,j)|_{\mathcal{A}\times[0,2]} \leq \beta(|\tilde{x}(0,0)|_{\mathcal{A}\times[0,2]}, t+j) + \epsilon \quad \forall (t,j) \in \operatorname{dom} \tilde{x} \tag{A.21}$$

is satisfied for solutions to (A.20) starting from $(\mathcal{A} + M\mathbb{B}) \times [0,2]$. Let $T \in (0, T^*]$ and using (A.14), let $c_1 > 0$ satisfy (A.15) independently of L. Let $P > 0$ satisfy $PA + A^\top P = -I$. Using (A.14), define $L_1^* := \frac{2|P|\beta_1}{c_1} + 1$, let $\rho > 0$ satisfy

$$|\phi(x, x_c, \hat{x}(\hat{h}(x,e)))| \leq \rho |e| \qquad \forall (x, x_c, \tau, e) \in \Lambda_L \tag{A.22}$$

and let $L_2^* > \rho \lambda_{\max}(P)$. Let $L^* := \max\{L_1^*, L_2^*\}$ and $L \geq L^*$.

We begin by showing that there exists $\mathcal{A}_1 \subset (\mathcal{A} + \epsilon\mathbb{B}) \times [0,2]$ such that $\mathcal{A}_1 \times \{0\}$ is globally asymptotically stable for

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{x}_c \\ \dot{\tau} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} f(x, \alpha(\hat{x}(\hat{h}(x,e)), x_c)) \\ f_c(\hat{x}(\hat{h}(x,e)), x_c) \\ \sigma(\tau) \\ LAe + \phi(x, x_c, \hat{x}(\hat{h}(x,e))) \end{bmatrix}, \\
(x, x_c, \tau, e) \in &\quad C_{(3.8)} \cap ((\mathcal{A} + M\mathbb{B}) \times [0,2] \times \{0\}) \\
\begin{bmatrix} x^+ \\ x_c^+ \\ \tau^+ \\ e^+ \end{bmatrix} &\in \begin{bmatrix} x \\ G_c(\hat{x}(\hat{h}(x,e)), x_c) \\ 0 \\ 0 \end{bmatrix}, \\
(x, x_c, \tau, e) \in &\quad D_{(3.8)} \cap ((\mathcal{A} + M\mathbb{B}) \times [0,2] \times \{0\}),
\end{aligned} \tag{A.23}$$

which differs from (3.15) in that the variable e restricted to 0.

First of all, since $e = 0$ for all solutions to (A.23), it follows that the (x, x_c, τ) component of solutions to (A.23) agree with solutions to (A.20), and thus solutions to (A.23) satisfy (A.21). Then since $|(x, x_c, \tau)|_{\mathcal{A}\times[0,2]} = |(x, x_c)|_{\mathcal{A}}$, solutions to (A.23) satisfy

$$|z(t, j)|_{\mathcal{A}} \leq \beta(|z(0, 0)|_{\mathcal{A}}, t + j) + \epsilon \quad \forall (t, j) \in \operatorname{dom} \bar{x}, \tag{A.24}$$

implying that the set $(\mathcal{A} + \epsilon\mathbb{B}) \times [0, 2]$ is uniformly attractive. Then defining $\mathcal{A}_1$ as the Omega-limit set of $\mathcal{A} + M\mathbb{B}$ for (A.20), using results on Omega-limit sets [6], we have that $\mathcal{A}_1$ is globally asymptotically stable for (A.20) and contained in $(\mathcal{A} + \epsilon\mathbb{B}) \times [0, 2]$. Hence $\mathcal{A}_1 \times \{0\}$ is globally asymptotically stable for (A.23).

We proceed to show that $\mathcal{A}_2 := (\mathcal{A} + M\mathbb{B}) \times [0, 2] \times \{0\}$ is globally asymptotically stable for (3.15). Let $V(e) = e^\top P e$. Then,

$$\begin{aligned}
\dot{V}(e) &= (\dot{e}^\top P e + e^\top P \dot{e}) \\
&= (Le^\top (PA + A^\top P)e + 2e^\top P \phi(z, e)) \\
&\leq |\phi(z, e)|\lambda_{max}(P)|e| - L|e|^2.
\end{aligned}$$

Using (A.22) and $L > \rho\lambda_{\max}(P)$, we have that $\dot{V}(e) < 0$ for all $\bar{x} \in \Lambda_L$. Furthermore, $|e| \leq c_1$ implies $\bar{x} \in \Lambda_L$. To see this, we note that the choice of $x_{\max}$ implies $|x| + c \leq x_{\max}$, which, with (A.15) implies $|\tilde{\Phi}_L(x, e)| \leq x_{\max}$, and hence $\bar{x} \in \Lambda_L$.

Furthermore, by defining $W(e) := \sqrt{e^\top P e}$, and using (A.18) and the fact that $L \geq \frac{2|P|\beta_1}{c_1} + 1$, we have that $\dot{W}(e) < 0$ for $|e| \geq c_1$, and hence $\dot{V}(e) < 0$ for $|e| \geq c_1$. Thus V is a global Lyapunov function for $\mathcal{A}_2$, which implies global asymptotic stablity for $\mathcal{A}_2$.

Using the preceding fact, and the fact that $\mathcal{A}_1 \times \{0\}$ is globally asymptotically stable for (A.23), we conclude by [15, Corollary 19] that $\mathcal{A}_1 \times \{0\}$ is globally asymptotically stable for (3.15).

A.6 Proof of Claim 1

Let $T > 0$, v be uniformly distributed on $[0, T]$ constants a, b satisfy $T \geq b > a \geq 0$, $\epsilon = \frac{(b-a)^2}{8}$, $i \in \mathbb{Z}_{\geq 0}$, and $\boldsymbol{\tau} \in \mathcal{S}_r([0, T])$.

The following measurable function rounds a real number down to the nearest integer multiple of a constant:

$$\lfloor \tau \rfloor_\delta := \max\{m \in \{0, \delta, 2\delta, ...\} : m \leq \tau\}.$$

For each $i \in \mathbb{Z}_{\geq 0}$ we define

$$\begin{aligned}
&\boldsymbol{\tau}_{\vdash,i}(\omega) := \boldsymbol{\tau}_\omega(\mathbf{t}_i(\omega), i),\\
&\Omega'_{(a,b),i} := \{\omega \in \Omega : \lfloor \boldsymbol{\tau}_{\vdash,i}(\omega) \rfloor_\delta \in (T - \tfrac{b-a}{4}, T] \Rightarrow (\mathbf{v}_{i+1}(\omega) \in (a + \tfrac{b-a}{4}, b - \tfrac{b-a}{4}) \text{ and}\\
&\qquad\qquad \mathbf{v}_{i+2}(\omega) \in (T - \tfrac{b-a}{4}, T - \tfrac{b-a}{8})),\\
&\qquad \lfloor \boldsymbol{\tau}_{\vdash,i}(\omega) \rfloor_\delta \in [0, T - \tfrac{b-a}{4}] \Rightarrow (\mathbf{v}_{i+2}(\omega) \in (a + \tfrac{b-a}{4}, b - \tfrac{b-a}{4}) \text{ and}\\
&\qquad\qquad \mathbf{v}_{i+1}(\omega) \in (T - \lfloor \boldsymbol{\tau}_{\vdash,i}(\omega) \rfloor_\delta - \tfrac{b-a}{4}, T - \lfloor \boldsymbol{\tau}_{\vdash,i}(\omega) \rfloor_\delta - \tfrac{b-a}{8}))\}\\
&\Omega_{(\mathbf{t}_i+T,i+1)} := \{\omega \in \Omega : \boldsymbol{\tau}_{\vdash,i}(\omega) + \boldsymbol{\tau}_{\vdash,i+1}(\omega) \geq T\}\\
&\Omega_{(\mathbf{t}_i+T,i+2)} := \{\omega \in \Omega : \boldsymbol{\tau}_{\vdash,i}(\omega) + \boldsymbol{\tau}_{\vdash,i+1}(\omega) + \boldsymbol{\tau}_{\vdash,i+2}(\omega) \geq T \geq \boldsymbol{\tau}_{\vdash,i}(\omega) + \boldsymbol{\tau}_{\vdash,i+1}(\omega)\}.
\end{aligned} \tag{A.25}$$

The subsequent identities follow from the dynamics (4.1):

$$
\begin{aligned}
\boldsymbol{\tau}_{\vdash,i}(\omega) &= \mathbf{t}_{i+1}(\omega) - \mathbf{t}_i(\omega) \quad \forall i \in \mathbb{Z}_{\geq 0} \quad \forall \omega \in \Omega \setminus \Omega_{\dashv,i} \\
\boldsymbol{\tau}_{\vdash,i+1}(\omega) &= \mathbf{v}_{i+1}(\omega) \quad \forall i \in \mathbb{Z}_{\geq 0} \quad \forall \omega \in \Omega \setminus \Omega_{\dashv,i} \\
\boldsymbol{\tau}_\omega(\mathbf{t}_i(\omega) + T, i+1) &= \boldsymbol{\tau}_{\vdash,i}(\omega) + \boldsymbol{\tau}_{\vdash,i+1} - T \quad \forall i \in \mathbb{Z}_{\geq 0} \quad \forall \omega \in \Omega_{(\mathbf{t}_i+T,i+1)} \setminus \Omega_{\dashv,i}. \\
\boldsymbol{\tau}_\omega(\mathbf{t}_i(\omega) + T, i+2) &= \boldsymbol{\tau}_{\vdash,i}(\omega) + \boldsymbol{\tau}_{\vdash,i+1} + \boldsymbol{\tau}_{\vdash,i+2} - T \quad \forall i \in \mathbb{Z}_{\geq 0} \quad \forall \omega \in \Omega_{(\mathbf{t}_i+T,i+2)} \setminus \Omega_{\dashv,i}.
\end{aligned}
\tag{A.26}
$$

We proceed to show (4.3a). Let $\delta = \frac{b-a}{8}$ and $\omega \in \Omega'_{(a,b),i}$. If $\omega \in \Omega_{\dashv,i}$, then $\omega \in \Omega_{(a,b),i}$ by definition, otherwise suppose $\omega \notin \Omega_{\dashv,i}$. For case one suppose $\lfloor \boldsymbol{\tau}_{\vdash,i}(\omega) \rfloor_\delta \in [T - \frac{b-a}{4}, T]$. Using (4.2), (A.25), and (A.26) we have

$$
\begin{aligned}
&\boldsymbol{\tau}_{\vdash,i}(\omega) \in [T - \tfrac{b-a}{4}, T] \\
&\boldsymbol{\tau}_{\vdash,i+1}(\omega) = \mathbf{v}_{i+1}(\omega) \in (a + \tfrac{b-a}{4}, b) \\
\Rightarrow\ &\boldsymbol{\tau}_{\vdash,i}(\omega) + \boldsymbol{\tau}_{\vdash,i+1}(\omega) \in [T + a, T + b] \\
\Rightarrow\ &\omega \in \Omega_{(\mathbf{t}_i+T,i+1)} \\
\Rightarrow\ &\boldsymbol{\tau}_\omega(\mathbf{t}_i(\omega) + T, i+1) \in (a, b) \\
\Rightarrow\ &\omega \in \Omega_{(a,b),i}.
\end{aligned}
$$

For case two suppose $\lfloor \boldsymbol{\tau}_{\vdash,i}(\omega) \rfloor_\delta \in [0, T - \frac{b-a}{4})$. Again using (4.2), (A.25), and (A.26) we have

$$
\begin{aligned}
&\boldsymbol{\tau}_{\vdash,i}(\omega) \in [0, T - \tfrac{b-a}{8}) \\
&\boldsymbol{\tau}_{\vdash,i+1}(\omega) = \mathbf{v}_{i+1}(\omega) \in (T - \boldsymbol{\tau}_{\vdash,i}(\omega) - \tfrac{b-a}{4}, T - \boldsymbol{\tau}_{\vdash,i}(\omega)), \\
&\boldsymbol{\tau}_{\vdash,i+2}(\omega) = \mathbf{v}_{i+2}(\omega) \in (a + \tfrac{b-a}{4}, b) \\
\Rightarrow\ &\boldsymbol{\tau}_{\vdash,i}(\omega) + \boldsymbol{\tau}_{\vdash,i+1}(\omega) \in (T - \tfrac{b-a}{4}, T), \\
&\boldsymbol{\tau}_{\vdash,i}(\omega) + \boldsymbol{\tau}_{\vdash,i+1}(\omega) + \boldsymbol{\tau}_{\vdash,i+2}(\omega) \in (T + a, T + b) \\
\Rightarrow\ &\omega \in \Omega_{(\mathbf{t}_i + T, i+2)} \\
\Rightarrow\ &\boldsymbol{\tau}_\omega(\mathbf{t}_i(\omega) + T, i+2) \in (a, b) \\
\Rightarrow\ &\omega \in \Omega_{(a,b),i}.
\end{aligned}
$$

For (4.3b) we note that $\Omega'_{(a,b),i}$ is the result of sigma field operations on measurable sets derived from $\mathcal{F}_{i+2}$-measurable functions.

For (4.3c) we define

$$
\begin{aligned}
A_{\delta,i,m} &:= \{\omega \in \Omega : \lfloor \boldsymbol{\tau}_{\vdash,i}(\omega) \rfloor_\delta = m\delta\} \\
\Omega'_{(a,b),i}|_{\boldsymbol{\tau}_{\vdash,i}=m\delta} &:= \{\omega \in \Omega : m\delta \in (T - \tfrac{b-a}{4}, T] \Rightarrow (\mathbf{v}_{i+1}(\omega) \in (a + \tfrac{b-a}{4}, b - \tfrac{b-a}{4}) \text{ and} \\
&\qquad\qquad \mathbf{v}_{i+2}(\omega) \in (T - \tfrac{b-a}{4}, T - \tfrac{b-a}{8})), \\
&\qquad m\delta \in [0, T - \tfrac{b-a}{4}] \Rightarrow (\mathbf{v}_{i+2}(\omega) \in (a + \tfrac{b-a}{4}, b - \tfrac{b-a}{4}) \text{ and} \\
&\qquad\qquad \mathbf{v}_{i+1}(\omega) \in (T - m\delta - \tfrac{b-a}{4}, T - m\delta - \tfrac{b-a}{8}))\}
\end{aligned}
\tag{A.27}
$$

and note that for all $i, m \in \mathbb{Z}_{\geq 0}$, $\delta > 0$:

$$
\begin{aligned}
&A_{\delta,i,k} \bigcap_{k \neq \ell} A_{\delta,i,\ell} = \varnothing, \\
&\bigcup_{k=0}^{\lfloor \frac{T}{\delta} \rfloor} A_{\delta,i,k} = \Omega \\
&A_{\delta,i,m} \in \mathcal{F}_i \\
&\Omega'_{(a,b),i}|_{\boldsymbol{\tau}_{\vdash,i}=m\delta} \in \sigma(\mathbf{v}_{i+1}, \mathbf{v}_{i+2}) \\
&A_{\delta,i,m} \cap \Omega'_{(a,b),i} = A_{\delta,i,m} \cap \Omega'_{(a,b),i}|_{\boldsymbol{\tau}_{\vdash,i}} \\
&\mathbb{P}(\Omega'_{(a,b),i}|_{\boldsymbol{\tau}_{\vdash,i}}) = \tfrac{(b-a)^2}{16} = \epsilon.
\end{aligned}
\tag{A.28}
$$

Using (A.27) and (A.28), we establish (4.3d):

$$
\begin{aligned}
\mathbb{P}(A_{\delta,i,m} \cap \Omega'_{(a,b),i}) &= \mathbb{P}(A_{\delta,i,m} \cap \Omega'_{(a,b),i}|_{\boldsymbol{\tau}_{\vdash,i}=m\delta}) = \mathbb{P}(A_{\delta,i,m})\mathbb{P}(\Omega'_{(a,b),i}|_{\boldsymbol{\tau}_{\vdash,i}=m\delta}) \\
\Rightarrow \sum_{m=0}^{\lfloor \frac{T}{\delta} \rfloor} \mathbb{P}(A_{\delta,i,m} \cap \Omega'_{(a,b),i}) &= \epsilon \sum_{m=0}^{\lfloor \frac{T}{\delta} \rfloor} \mathbb{P}(A_{\delta,i,m}) \\
\Rightarrow \mathbb{P}(\bigcup_{m=0}^{\lfloor \frac{T}{\delta} \rfloor} A_{\delta,i,m} \cap \Omega'_{(a,b),i}) &= \epsilon \\
\Rightarrow \mathbb{P}(\Omega'_{(a,b),i}) &= \epsilon,
\end{aligned}
$$

Let $B \in \mathcal{F}_i$. Using (4.3d), (A.27), (A.28), the i.i.d. property of the v's and the definition of probability measure, we have

$$\begin{aligned}
\mathbb{P}(B \cap A_{\delta,i,m} \cap \Omega'_{(a,b),i}|_{\tau_{\vdash,i}=m\delta}) &= \mathbb{P}(B \cap A_{\delta,i,m})\mathbb{P}(\Omega'_{(a,b),i}|_{\tau_{\vdash,i}=m\delta}) \\
\Rightarrow \ \mathbb{P}(B \cap A_{\delta,i,m} \cap \Omega'_{(a,b),i}) &= \epsilon\mathbb{P}(B \cap A_{\delta,i,m}) \\
\Rightarrow \ \sum_{m=0}^{\lfloor \frac{T}{\delta} \rfloor} \mathbb{P}(B \cap A_{\delta,i,m} \cap \Omega'_{(a,b),i}) &= \epsilon \sum_{m=0}^{\lfloor \frac{T}{\delta} \rfloor} \mathbb{P}(B \cap A_{\delta,i,m}) \\
\Rightarrow \ \mathbb{P}(\bigcup_{m=0}^{\lfloor \frac{T}{\delta} \rfloor} B \cap A_{\delta,i,m} \cap \Omega'_{(a,b),i}) &= \epsilon\mathbb{P}(\bigcup_{m=0}^{\lfloor \frac{T}{\delta} \rfloor} B \cap A_{\delta,i,m}) \\
\Rightarrow \ \mathbb{P}(B \cap \Omega'_{(a,b),i}) &= \epsilon\mathbb{P}(B) = \mathbb{P}(B)\mathbb{P}(\Omega'_{(a,b),i}),
\end{aligned}$$

which implies (4.3c).

A.7 Proof of Proposition 4

Let $\xi^* \in \mathbb{S}^1$, $\epsilon \in [0, \frac{\pi}{12}]$ and define

$$V(x_i) := \begin{cases} 1 - \mathbb{I}_b(\beta_i), & \xi_i \in R_{\xi^*}([-\epsilon, \epsilon]) \\ 1 - \mathbb{I}_b(\beta_i)\xi^{*\top}R(\epsilon)\xi_i, & \xi_i \in R_{\xi^*}([-\pi, -\epsilon]) \\ 1 - \mathbb{I}_b(\beta_i)\xi^{*\top}R(-\epsilon)\xi_i, & \xi_i \in R_{\xi^*}([\epsilon, \pi]). \end{cases}$$

During flows, $\dot{V}(x_i) \leq -\mathbb{I}_b(\beta_i)(\xi^{*\top}R(\epsilon)J\xi_i)^2 \leq 0$ when $\xi_i \in R_{\xi^*}([-\pi, -\epsilon])$, and $\dot{V}(x_i) \leq -\mathbb{I}_b(\beta_i)(\xi^{*\top}R(-\epsilon)J\xi_i)^2 \leq 0$ when $\xi_i \in R_{\xi^*}([\epsilon, \pi])$. During jumps from mode a to mode b, we have $\xi_i \in R_{\xi^*}\left(\left[-\frac{5\pi}{12}, \frac{5\pi}{12}\right]\right)$ and hence $V^+ - V \leq 0$. During jumps from mode b to mode a, we have $\xi_i \in R_{\xi^*}\left(\left[\frac{7\pi}{12}, \frac{11\pi}{12}\right]\right)$ and hence $V^+ - V \leq 0$. This plus an invariance principle ([34, Theorem 4.7] noting that the 'backend' arc is not invariant, and consecutive jumps are impossible) implies that $R_{\xi^*}([-\epsilon, \epsilon]) \times \{b\}$ is UGAS.

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