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ABSTRACT

Formulas that express in terms of physical scattering functions the discontinuity of any 3-to-3 scattering function across any basic normal threshold cut are derived from field theory. These basic cuts are the cuts in channel energies that start at lowest normal thresholds and extend to plus infinity. The discontinuity across such a cut generally depends on whether it is evaluated above or below each of the remaining basic cuts. Formulas are obtained for all cases. Generalized Steinmann relations are found to hold: the 2282 boundary values from which the discontinuities across basic cuts are formed have a unique extension to a set of $2^{16} = 65,536$ functions, one for each combination of sides of the 16 basic cuts, such that for any pair of overlapping channels the corresponding double discontinuity vanishes. The ordinary Steinmann relations require this property to hold only for the double discontinuities formed from the original 2282 boundary values. The results are derived from the field-theoretic formalism of Bros, Epstein, and Glaser, which is slightly developed and cast into a form suited for calculations of the kind needed here.

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I. INTRODUCTION

The work of Mueller (1) and Tan (2) has demonstrated the usefulness of many-particle generalizations of the optical theorem. Mueller derived important properties of inclusive cross sections from the assumption that certain matrix elements of currents enjoy Regge behavior. Tan showed that Mueller's special assumption about matrix elements of currents can be replaced, with the aid of a many-particle generalization of the optical theorem, by the general Regge hypothesis that the discontinuities of scattering functions across basic cuts enjoy Regge behavior. The generalization of the optical theorem required for this purpose is the inclusive optical theorem.

The ordinary optical theorem relates ordinary cross sections to discontinuities of 2-to-2 scattering functions. Similarly, the inclusive optical theorem, proved in (3), relates inclusive cross sections to discontinuities of n-to-n scattering functions. Mueller's work, augmented by this theorem, illustrates the general fact that information about complicated many-body processes (e.g., high-energy inclusive cross sections) can be derived from information about simpler few-body processes by means of generalizations of the optical theorem.

The most useful generalization of the optical theorem is probably the inclusive optical theorem. For this theorem, like the ordinary optical theorem, refers to measurable cross sections. However, the Regge hypothesis, particularly as formulated by Weis (4), prescribes a form for the discontinuity across any basic cut in appropriate Regge limits. Thus to assess the full content of the Regge hypothesis one needs formulas for the discontinuities across each of these cuts. Some of these discontinuities are directly related to inclusive cross sections. Others are related to cross sections and inclusive cross sections by (Schwartz) inequalities. The rest can have indirect theoretical uses.

Formulas for all of these discontinuities were derived in ref. (5). Those formulas express these discontinuities in terms of various boundary values $\tilde{r}_\lambda$ of the analytically continued scattering functions. However, it appears more useful to have expressions for these discontinuities in terms of the physical scattering functions themselves, instead of their analytic continuations. Formulas that express discontinuities across basic cuts in terms of physical scattering functions will be called generalized optical theorems.

The first aim of the present work is to develop the field-theoretic formalism needed to derive generalized optical theorems. The second aim is to apply this formalism to the 3-to-3 case, and, in particular, to express in terms of physical scattering functions the discontinuity of any 3-to-3 scattering function across any basic cut. The discontinuity across such a cut generally depends on whether it is evaluated above or below each of the other basic cuts. What will be obtained here is a set of formulas that gives the discontinuity across each of the basic cuts evaluated on each possible side of every other

basic cut. Although only normal threshold cuts are explicitly mentioned here there is no neglect of other cuts or singularities: all the formulas are exact.

The discontinuities across the basic cuts are formed from a set of 2282 different boundary values of the analytically continued scattering function. These boundary values are subject to a set of linear relations known as the Steinmann relations, which have played a prominent role in the development of Regge theory. It will be shown here that the exact analog of the Steinmann relations holds for a much larger set of 2^{16} functions. These generalized Steinmann relations will be described presently. First the ordinary Steinmann relations are reviewed.

The ordinary Steinmann relations (6) can be expressed in the following way: Let the off-mass-shell analytically continued scattering function for an arbitrary n-particle process be regarded as a function of the n complex energies k_j^0 $(j = 1,\cdots,n)$, restricted by the conservation law $\Sigma k_j^0 = 0$. The momenta $\vec{k}_j$ are to be held fixed and real. Basic field-theoretic principles ensure that this function of energies is analytic except at points $k^0 \equiv (k_1^0,\cdots,k_n^0)$ that lie on one or more of the planes

$$\sum_{j\varepsilon J} \mathrm{Im}\, k_j^0 = 0 , \qquad (1.1)$$

where the set J can be any nonempty proper subset of the complete set of indices $(1,2,\cdots,n)$. Each such set J defines a <u>channel</u>, and (1.1) is the statement that the energy of channel J is real. The complement of J in $(1,2,\cdots,n)$ is denoted by $\hat{J}$, and it defines the same channel as J.

The various planes (1.1) divide the $n-1$ dimensional complex energy space (restricted by $\Sigma k_j^0 = 0$) into sectors $\Gamma_\lambda^{0'}$, called (energy) cells. Each cell $\Gamma_\lambda^{0'}$ lies on a well-defined side of each of the planes (1.1), and the boundary of each cell $\Gamma_\lambda^{0'}$ lies in the union of these planes.

The analytically continued scattering function is analytic in each cell $\Gamma_\lambda^{0'}$. The boundary value defined by letting k^0 approach the real boundary point p^0 from within the cell $\Gamma_\lambda^{0'}$ is denoted by $\tilde{r}'_\lambda(p)$. These boundary values $\tilde{r}'_\lambda$ are the functions that occur in the ordinary Steinmann relations. The constraints imposed on them by the Steinmann relations are now described.

Two cells $\Gamma_{\lambda_1}^{0'}$ and $\Gamma_{\lambda_2}^{0'}$ are said to be adjacent if and only if they lie on the same side of every plane (1.1) except one. The difference $\tilde{r}'_{\lambda_1} - \tilde{r}'_{\lambda_2}$ between the boundary values associated with two adjacent cells $\Gamma_{\lambda_1}^{0'}$ and $\Gamma_{\lambda_2}^{0'}$ is called the discontinuity across the cut (1.1) that separates these two adjacent cells. The plane (1.1) is usually called the cut (1.1) when the discontinuity across it is being considered.

In general there are many pairs of adjacent cells separated by any given cut (1.1). Each such pair lies on a well-defined side of every other cut (1.1), by virtue of the definition of adjacent cells. Thus each such pair can be identified by specifying the sides of these other cuts upon which it lies. The discontinuity across the given cut (1.1) depends in general on which of these pairs is used. In other words, the discontinuity across any given cut depends in general on upon which sides of the other cuts it is evaluated.

The ordinary Steinmann relations limit this dependence. They assert that the following Steinmann discontinuity property holds: The discontinuity across the cut (1.1) corresponding to a channel J does not depend on whether it is evaluated above or below the cuts (1.1) associated with the channels that overlap J. A channel (or set) J' is said to overlap a channel (or set) J if and only if the four sets $J \cap J'$, $\hat{J} \cap J'$, $J \cap \hat{J}'$, and $\hat{J} \cap \hat{J}'$ are all nonempty. Here $\hat{J}$ and $\hat{J}'$ are the complements of J and J', respectively, relative to the set $(1,2,\cdots,n)$.

This statement of the Steinmann discontinuity property is not manifestly covariant, because the energy cells $\Gamma_\lambda^{0'}$ refer preferentially to energies. A covariant generalization is described in section II. That generalization enlarges each energy cell $\Gamma_\lambda^{0'}$ to a covariantly described cell Γ'_λ that has the same set of real boundary points p. This covariant statement is equivalent to the noncovariant statement given above.

The Steinmann discontinuity property has a general appearance. However, it covers only those discontinuities that can be formed as differences of boundary values from neighboring energy cells $\Gamma_\lambda^{0'}$. This limitation is now discussed in more detail.

Consider, for example, a process with three initial particles and three final particles. The number of nonempty proper subsets J of the complete set $J_6 \equiv (1,2,\cdots,6)$ is $2^6 - 2 = 62$. Only half of these need be considered, since, by virtue of $\sum k_j^0 = 0$, the sets J and $\hat{J} = J_6 - J$ both define the same cut. Moreover, the stability conditions on the one-particle states imply that the discontinuities across 15 of these 31 cuts vanish identically at mass-shell points p.

These 15 cuts, which are called the trivial cuts, are the six cuts corresponding to channels defined by sets $J = \{j\}$ consisting of one single index j, together with the nine cuts corresponding to channels defined by sets $J = \{f,i\}$ consisting of one index f corresponding to a final particle and one index i corresponding to an initial particle. This leaves $31 - 15 = 16$ nontrivial cuts, which correspond to the one total energy, the three initial subenergies, the three final subenergies, and the nine cross energies corresponding to two initial particles combined with one final particle.

These 16 nontrivial cuts divide the five dimensional complex energy space into 2282 regions called zones. Each of these zones contains one or more cells $\Gamma_\lambda^{0'}$ all of which have equal boundary values $\tilde{r}'_\lambda$. This number 2282 of different boundary values is small compared to the number $2^{16} = 65,536$ of boundary values that would occur if the 16 cuts were cuts in 16 independent variables. Thus for most of the 2^{16} combinations of sides of the sixteen nontrivial cuts there is no corresponding cell or zone. This is because the 16 energies are not independent variables; they are linear combinations of the five independent energies k_j^0.

The ordinary Steinmann relations cover only those boundary values that can be obtained as limits from one of the cells $\Gamma_\lambda^{0'}$. This limitation is severe. For example, there is no cell $\Gamma_\lambda^{0'}$ that lies below one single subenergy cut and above the other 15 cuts. Consequently the ordinary Steinmann relations do not apply to any discontinuity involving any such function. This limitation on the Steinmann relations is called the cell limitation.

The generalized Steinmann relations are described next. They are essentially the ordinary Steinmann relations, with the cell limitation removed.

If the 16 channel energies corresponding to nontrivial cuts were indeed independent variables then one could specify independently for each channel whether the limit was to be taken from above or below the corresponding cut. Thus for each set G of nontrivial cuts (1.1) one could define $M^G \equiv M^G(p)$ to be the boundary value of the scattering function obtained by approaching the real limit point p from below every cut g in G, and from above every cut g in $\hat{G} \equiv E - G$, where E represents a set of 16 indices that label the 16 nontrivial cuts. Then for any h in $\hat{G}$ the difference $M^G - M^{Gh} \equiv M_h{}^G$ would be the discontinuity across the cut h, evaluated below all the cuts g in G and above all the cuts g in $E - Gh \equiv E - G\cup\{h\}$. The generalized Steinmann discontinuity property is the property that this discontinuity $M_h{}^G$ across the cut h does not depend on whether it is evaluated above or below any of the cuts $g \varepsilon E$ corresponding to channels that overlap the channel corresponding to h. Symbolically, this property is expressed by the equation

$$M_h{}^{G'} = M_h{}^{G''} \quad \text{if} \quad G'/O_h = G''/O_h \quad , \tag{1.2}$$

where G/O_h represents the set G modulo the set O_h, and O_h is the set of $g \varepsilon E$ such that the channel J_g corresponding to g overlaps the channel J_h corresponding to h.

For each cell $\Gamma_\lambda^{0'}$ there is a unique set of cuts $G(\lambda) \subset E$ such that $\Gamma_\lambda^{0'}$ lies below every cut g in $G(\lambda)$ and above every cut g in $E - G(\lambda)$. A set of functions M^G is said to be an

<u>enlargement</u> of the set of cell functions $\tilde{r}_\lambda$ if and only if

$$M^{G(\lambda)} = \tilde{r}_\lambda \quad \text{for every } \lambda . \tag{1.3}$$

The number of generalized Steinmann discontinuity conditions (1.2) is far greater than the number 2^{16} of functions M^G. Thus it is not clear, a priori, whether any set of functions M^G satisfying (1.2) and (1.3) exists. And if a solution does exist, it is not clear whether it is unique. However, it will be shown that there is a set of 2^{16} functions M^G that satisfy (1.2) and (1.3), and that these two conditions uniquely determine this set.

The functions M^G are, as just stated, uniquely determined by the two algebraic requirements (1.2) and (1.3). Thus no analyticity requirements are needed. However, the identification of M^G with the boundary value taken from below the normal-threshold cuts $g\epsilon G$ and from above the normal-threshold cuts $g\epsilon\hat{G}$ demands that following property hold:

> The function M^G continues analytically into itself around each J-channel normal-threshold singularity by moving into the lower-half plane in the variable
>
> $$k^0(J) = \sum_{j\epsilon J} k_j^{\,0} \tag{1.4}$$
>
> if $J = J_g$ for some g in G, and into the upper-half in this variable if $J = J_g$ for some g in $\hat{G}$.

This analyticity property is not proved in the present work. However, the functions M^G derived here from conditions (1.2) and (1.3) are (when restricted to the mass shell) identical to the

functions M^G derived earlier (7) from S-matrix analyticity requirements that entail this property. Thus the description of M^G as the function evaluated below the cuts $g\epsilon G$ and above the cuts $g\epsilon\hat{G}$ is appropriate. The analytic properties of the M^G with respect to singularities other than normal threshold singularities are also discussed in ref. (7).

The conditions (1.2) and (1.3), together with the fact that there is a unique set of 2^{16} functions M^G that satisfy them, are called the generalized Steinmann relations. These relations are useful because they are not limited to the awkward cell limitation. The 2^{16} functions M^G are linear combinations of the 2282 boundary values. Thus the 2^{16} functions enjoy Regge behavior if the 2282 do. And so likewise do all the single and multiple discontinuities formed from them. Thus in the development of the dynamical consequences of unitarity, Regge behavior, and the Steinmann relations one can use all of the functions M^G instead of merely the 2282 boundary values. This gives a richer set of relations to work with, and it eliminates the problem of having to check always that all of the functions involved in each application of the Steinmann relations are contained among the 2282 special functions covered by the ordinary Steinmann relations.

The generalized Steinmann relations say, in effect, that the nontrivial basic cuts can be treated as if they were cuts in independent variables, insofar as the system of discontinuities across these cuts is concerned, and that the Steinmann discontinuity property continues to hold. A compact formula will be given that expresses all of the 2^{16} functions M^G and all single and multiple discontinuities that can be formed from them in terms of physical scattering functions.

The central part of this work is the calculation of the discontinuities across the basic cuts. These calculations are based on the formulation of field theory developed by Bros, Epstein, and Glaser (8,9). This BEG formalism, which rests very heavily on the earlier work of Ruelle (6), can be regarded as an extension of the formalism of Lehmann, Symanzik, and Zimmermann (10).

This LSZ formalism is based on the use of the advanced and retarded functions introduced by LSZ. These functions are better adapted to the study of analytic properties than the time-ordered functions because their x-space support properties, together with their assumed tempered-distribution character, imply that the corresponding p-space functions have well-defined domains of analyticity. LSZ show that for processes with just two initial particles or just two final particles the S matrix can be expressed directly in terms of these advanced and retarded functions. However, for arbitrary processes, the S matrix cannot be expressed directly in terms of the LSZ advanced and retarded functions alone.

This problem is overcome in the BEG (8,9) formalism by the introduction of the operators that correspond to the boundary values $\tilde{r}'_\lambda(p)$ described earlier. These operators are not linearly independent, but are related by the operator equivalents of the Steinmann relations described above. The BEG formalism is distinguished from the earlier works of ref. (6) by the fact that time-ordered operators are not introduced, and by the development and use of a graphical analysis of the Steinmann relations in terms of tree diagrams. These tree diagrams play a central role in our calculations.

The BEG formalism is described in the published literature only in a short section of a paper (8) dealing principally with other matters. That account is extremely compact, and is couched in an abstract algebraic terminology. Hence much of its implicit content is not set down in the form of explicit equations to which one can refer.

To make our paper more readily understandable to readers unfamiliar with the BEG formalism, and in the hope of making that formalism itself more accessible at a practical level, we shall summarize in section II the basic definitions and results of the BEG formalism. To secure a direct and simple connection to physics this formalism is cast into a form based on the LSZ formalism. This procedure masks some of the generality of the BEG formalism, but allows it to be presented in terms of explicit equations that refer to the field operators themselves, and that can be directly used in calculations of the kind needed here, rather than in terms of abstract descriptions that refer to associated Lie algebras.

With three exceptions the results presented in this summary are merely stated, not proved; the missing proofs are all contained in refs. (8 - 11 and 5), or are simple adaptations of proofs given in these references. The three exceptions are proofs of two results important to our work that are not proved in these references.

Two of the three exceptions are a pair of theorems that establish the connection between the boundary values $\tilde{r}'_\lambda$ and the physical scattering functions. This connection is implicit in the works of Ruelle and Araki (6). However, those works are based on time-ordered functions, which are foreign to the LSZ-BEG formalism.

Our derivation is within the LSZ-BEG framework. The third exception is a proof within the LSZ-BEG framework of the important hermitian-analyticity property of scattering functions. These three proofs serve to make the BEG-LSZ formalism self-contained.*

Bros, Epstein, and Glaser give a precise formulation of the mathematical assumptions needed to derive the k-space analyticity properties and Steinmann relations. The focus of the present work, however, is on applications of the formalism, rather than the mathematical foundations of the theory. Thus the results of BEG will be summarized by theorems that leave unstated the assumptions of BEG field theory itself. Readers interested in these assumptions should consult references (8) and (9).

The BEG assumptions are augmented in the present work by the LSZ assumptions, and in particular by the LSZ asymptotic conditions. Other assumptions could be used to obtain the connection between the BEG functions $\tilde{r}'_\lambda$ and the physical scattering functions. The LSZ assumption has the virtue of being well known and easy to use.

The plan of the work is as follows. The BEG-LSZ formalism is described in section II. The aim is merely to list the basic equations together with brief descriptions of their meanings. The first subsection is a short description of the LSZ formalism, adapted to provide a suitable basis for the BEG formalism. The final subsection is an index that is useful for locating definitions.

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* In recent years Bros, Epstein, Glaser, and Stora have enlarged the BEG framework of refs. (8) and (9) to include time-ordered functions, and have derived in this enlarged framework the general connection between retarded functions and the time-ordered functions (12).

Section III catalogues the 2282 zones. Each zone corresponds to a G such that $G = G(\lambda)$ for some λ. It is the 2282 functions $M^{G(\lambda)}$ that are determined by equation (1.3). Each discontinuity $\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2}$ across a basic cut is connected by (1.3) to a difference $M^{G(\lambda_1)} - M^{G(\lambda_2)}$.

Section IV contains the calculation by means of the BEG formalism of the discontinuity $\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2}$ across each of the basic cuts. The procedure is based on the fact that each cell $\Gamma_\lambda^{0'}$ is associated with a corresponding sum of trees, called the grove ρ_λ. This grove ρ_λ determines both the location of the cell $\Gamma_\lambda^{0'}$ and the explicit form of the operator m_λ associated with the cell. The operator m_λ is a sum of products of in- and out-operators. Hence by inserting appropriate complete sets of in- and out-states one can reduce each term of m_λ to a product of S matrices for various processes. The expressions for the discontinuities obtained in this way are then reduced to formulas that can be directly compared to the formulas for corresponding discontinuities $M^{G(\lambda_1)} - M^{G(\lambda_2)}$ derived in ref. (7) from S-matrix principles.

The generalized Steinmann relations are derived in section V. First the formula given in ref. (7) for the 2^{16} functions M^G and all of their single and multiple discontinuities is presented. It is noted that the discontinuities calculated in section IV coincide with those given by this formula. Equation (1.3) follows directly from this result. Next it is shown that this set of 2^{16} functions M^G satisfies the generalized Steinmann discontinuity property (1.2). Finally it is shown that this property (1.2) allows each of the 2^{16} functions M^G to be expressed as a linear combination of the 2282 functions $M^G = M^{G(\lambda)}$. This ensures that the solution to (1.2) and (1.3) is unique.

II. GENERAL THEORY

A. The LSZ Framework

The LSZ formalism (10) involves products of operators $A_j(x_j)$. The index j is used here to identify a particular operator in some product. Each operator $A_j \equiv A_j(x_j)$ is a local interpolating field associated with a particle of type t_j. The hermitian-adjoint field $A_j^\dagger$ is a local interpolating field associated with the corresponding antiparticle. It will be convenient to label this antiparticle by the type index $-t_j$.

For each $A_j(x_j)$ there is a complete orthonormal set of positive-frequency solutions $f_j^n(x_j)$ of the Klein-Gordon equation. Suppressing an index j one can write

$$f_j^n(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \tilde{f}_j^n(p)\, e^{-ipx} \tag{2.1}$$

where n is a positive integer, $p^0 = (m_j^2 + \underset{\sim}{p}^2)^{\frac{1}{2}}$, m_j is the mass of particles of type t_j, and

$$px = p^0x^0 - \underset{\sim}{p}\cdot\underset{\sim}{x} . \tag{2.2}$$

With the aid of the notation

$$A\overleftrightarrow{\partial}_0 B = A\partial_0 B - (\partial_0 A)B \tag{2.3}$$

the normalization condition can be written

$$-i \int d^3x\, f_j^n(x)\, \overleftrightarrow{\partial}_0 \big(f_j^m(x)\big)^* = \delta_{nm} , \tag{2.4a}$$

or, equivalently,

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \tilde{f}_j^n(p)\, \tilde{f}_j^{m*}(p) = \delta_{nm} . \tag{2.4b}$$

If the negative-frequency solutions to the Klein-Gordon equation are labelled by the negative integers, according to the rule

$$f_j^{-n}(x) \equiv \big(f_j^n(x)\big)^* , \tag{2.5}$$

then $A_j(x)$ can be expanded as

$$A_j(x) = \sum_{n=1}^{\infty} \Big(f_j^n(x)\, A_j^n(t) + f_j^{-n}(x)\, A_j^{-n}(t)\Big) , \tag{2.6}$$

where $t \equiv x^0$, and for all positive and negative integers n

$$A_j^n(t) = -i(\text{sign } n) \int d^3x\, A_j(x)\, \overleftrightarrow{\partial}_0\, f_j^{-n}(x) . \tag{2.7}$$

The LSZ asymptotic condition asserts that for every pair of normalizable states $|\Phi\rangle$ and $|\psi\rangle$ the limit

$$\lim_{t\to\pm\infty} \langle\Phi|A_j^n(t)|\psi\rangle = \langle\Phi|A_j^n(\pm)|\psi\rangle \tag{2.8}$$

defines in-field operators $A_j^n(-)$ and out-field operators $A_j^n(+)$ that are time independent and satisfy the properties that they would have in canonical free-field theory. Thus the operators defined for $n = 1,2\cdots$ as

$$a(n,t_j,\pm) \equiv A_j^n(\pm) \tag{2.9a}$$

and

$$a^\dagger(n,-t_j,\pm) \equiv A_j^{-n}(\pm) \qquad (2.9b)$$

are interpreted as follows:

$a^\dagger(n,-t_j,+)$ creates an outgoing particle of type $-t_j$ and wave function $f_j^n(x)$; (2.10a)

$a(n,t_j,+)$ annihilates an outgoing particle of type t_j and wave function $f_j^n(x)$; (2.10b)

$a^\dagger(n,-t_j,-)$ creates an incoming particle of type $-t_j$ and wave function $f_j^n(x)$; (2.10c)

$a(n,t_j,-)$ annihilates an incoming particle of type t_j and wave function $f_j^n(x)$. (2.10d)

The transition amplitude (S-matrix element) for the scattering from a set of incoming particles i with wave functions $f_i^{n_i}(x_i)$ to a set of outgoing particles f with wave functions $f_f^{n_f}(x_f)$ is

$$\frac{1}{N} \langle 0| \prod_f a(n_f,t_f,+) \prod_i a^\dagger(n_i,t_i,-)|0\rangle$$

$$\equiv S(\{n_f,t_f\};\{n_i,t_i\}) , \qquad (2.11)$$

where N is a normalization factor that is unity if the wave functions of all the incoming particles are orthogonal, and the wave functions of all the outgoing particles are orthogonal.

The momentum-space forms of these equations are obtained by making the following substitutions:

$$(1) \quad n_j \to p_j , \qquad (2.12a)$$

where p_j^0 and n_j have the same sign;

$$(2) \quad f_j^{n_j}(x_j) \to f_j^{p_j}(x_j) \equiv \exp(-ip_jx_j) ; \qquad (2.12b)$$

$$(3) \quad \sum_{n_j} \to \sum_{p_j} \equiv \int \frac{d^3p_j}{(2\pi)^3} \frac{1}{2|p_j^0|} ; \qquad (2.12c)$$

and

$$(4) \quad S(\{n_f,t_f\}; \{n_i,t_i\}) \to S(\{p_f,t_f\}; \{p_i,t_i\})$$

$$\equiv \langle 0| \prod_f a(p_f,t_f,+) \prod_i a^\dagger(p_i,t_i,-)|0\rangle . \qquad (2.13)$$

[The normalization factor N is moved in the continuum case to the many-particle phase space factor, which then becomes continuous.]

For brevity let

$$a_j^\pm \equiv a(p_j,t_j,\pm) \qquad (2.14a)$$

and

$$(a_j^\pm)^\dagger \equiv \left(a(p_j,t_j,\pm)\right)^\dagger . \qquad (2.14b)$$

The normalization of these operators is fixed by the commutation relations

$$[a_i^+,(a_j^+)^\dagger] = [a_i^-,(a_j^-)^\dagger] = (2\pi)^3\, 2|p_i^0|\delta^3(\vec{p}_i - \vec{p}_j)\, \delta_{t_it_j} . \qquad (2.14c)$$

Only spinless bosons are considered here, but it is easy to generalize all results to integral-spin bosons, and we believe that results analogous to those obtained below can be derived also for half-odd integral fermions.

B. The Operators A_i^+ and A_i^-

For brevity let

$$A_j \equiv A_j(x_j) , \quad (2.15)$$

and let θ_{ij} be the θ function in the time variable $x_i^0 - x_j^0$ defined by

$$\theta_{ij} \equiv \begin{cases} 1 & \text{if} \quad x_i^0 > x_j^0 \\ 0 & \text{if} \quad x_i^0 < x_j^0 \\ 1 & \text{if} \quad x_i^0 = x_j^0 \quad i > j \\ 0 & \text{if} \quad x_i^0 = x_j^0 \quad i < j . \end{cases} \quad (2.16)$$

(For notational convenience it is assumed that there is an infinite set $\mathcal{J}$ of indices j, and that for each $j \epsilon \mathcal{J}$ there is a field $A_j \equiv A_j(x_j)$ associated with a particle of type t_j and mass m_j. The set $\mathcal{J}$ is assumed to contain an infinite subset of indices j corresponding to each type of particle, so that any product of operators A_j can be written as a product of A_j's over a set J of indices $j \epsilon \mathcal{J}$ none of which is repeated. Then each operator A_j in any product is unambiguously identified by the single index j.)

Let A represent any ordered product of operators A_j:

$$A \equiv A_{j(1)} A_{j(2)} \cdots A_{j(n)} .$$

Let the retarded product $\theta_{ij}[A_i, A_j]$ of any two operators be represented by

$$\theta_{ij}[A_i, A_j] \equiv (A_i, A_j) . \quad (2.17)$$

Then the two operators A_i^+ and A_i^- acting on A are defined as follows: A_i^+A is the sum of operators obtained by replacing in turn each A_j in A by (A_i, A_j); A_i^-A is the sum of operators obtained by replacing in turn each A_j in A by (A_j, A_i). That is, A_j^+A and A_j^-A are defined as follows:

Definitions

$$A_i^+A \equiv (A_i, A_{j(1)}) A_{j(2)} \cdots A_{j(n)} + A_{j(1)} (A_i, A_{j(2)}) A_{j(3)} \cdots A_{j(n)} + \cdots + A_{j(1)} \cdots A_{j(n-1)} (A_i, A_{j(n)}) \quad (2.18a)$$

and

$$A_i^-A \equiv (A_{j(1)}, A_i) A_{j(2)} \cdots A_{j(n)} + A_{j(1)} (A_{j(2)}, A_i) A_{j(3)} \cdots A_{j(n)} + \cdots + A_{j(1)} \cdots A_{j(n-1)} (A_{j(n)}, A_i) . \quad (2.18b)$$

If f is a c-number function then

$$A^{\pm}(fA) \equiv fA^{\pm}A . \quad (2.18c)$$

The action of A^+ and A^- on sums of products of A_j's and c-number functions is defined by linearity

$$A^{\pm}(x + y) = A^{\pm}x + A^{\pm}y . \quad (2.18d)$$

C. Generalized Reduction Formula

Let F be any sum of products of A_j's and θ-functions. Let K_j be defined by

$$K_j \equiv \Box_j + m_j^2$$
$$\equiv \left(\frac{\partial}{\partial x_j^0}\right)^2 - \nabla_j^2 + m_j^2 \ . \tag{2.19}$$

Then the arguments leading to the LSZ reduction formula give, for $n = 1,2,\cdots$,

$$i\int d^4x_j\ f_j^{-n}(x_j)\ K_j\langle\Phi|A_j^\pm F|\psi\rangle$$
$$= \langle\Phi|[a(n,t_j,\pm),F]|\psi\rangle \tag{2.20a}$$

and

$$i\int d^4x_j\ f_j^{n}(x_j)\ K_j\langle\Phi|A_j^\pm F|\psi\rangle$$
$$= -\langle\Phi|[a^\dagger(n,-t_j,\pm),F]|\psi\rangle \ . \tag{2.20b}$$

These two formulas can be combined into a single formula by introducing

$$c_j^\pm(n) \equiv a(n,t_j,\pm) \quad \text{for } n = 1,2,\cdots, \tag{2.21a}$$

and

$$c_j^\pm(n) \equiv -a^\dagger(-n,-t_j,\pm) \quad \text{for } n = -1,-2,\cdots \ . \tag{2.21b}$$

Then suppressing Φ and ψ one may write (2.20a) and (2.20b) as the single formula

$$i\int d^4x_j\ f_j^{-n}(x_j)\ K_j(A_j^\pm F) = [c_j^\pm(n),F] \ , \tag{2.22}$$

which holds for all positive and negative integers n.

The momentum-space form of (2.22) is

$$i\int d^4x_j\ e^{ip_jx_j}\ K_j(A_j^\pm F) = [c_j^\pm(p_j),F] \ , \tag{2.23a}$$

where

$$c_j^\pm(p_j) \equiv a(p_j,t_j,\pm) \equiv a_j^\pm \qquad \text{for } p^0 > 0 \tag{2.23b}$$

and

$$c_j^\pm(p_j) \equiv -a^\dagger(-p_j,-t_j,\pm) \equiv -\bar{a}_j^\pm \qquad \text{for } p^0 < 0 \ . \tag{2.23c}$$

A closely related equation, which follows from the same argument, is

$$i\int d^4x_j\ e^{ip_jx_j}\ K_j\ A_j = c_j^+(p_j) - c_j^-(p_j) \ , \tag{2.23d}$$

which is essentially eq. (13) of LSZ (12).

D. The Symbol α

Some basic quantities of the BEG formalism are labelled by an index α.

Definition The symbol α represents an ordered set of $n = n(\alpha)$ signs σ, together with an ordered set of $n + 1$ indices $j \in \mathcal{J}$:

$$\alpha \equiv \{\sigma(1,\alpha),\ \sigma(2,\alpha),\ \cdots,\ \sigma(n,\alpha);$$
$$j(0,\alpha),\ j(1,\alpha),\ \cdots,\ j(n,\alpha)\}$$
$$\equiv (\sigma_\alpha;\ J_\alpha) \quad . \tag{2.24}$$

The arguments α appearing in $\sigma(i,\alpha)$ and in $j(i,\alpha)$ indicate that these quantities depend on α. Taken together the $\sigma(i,\alpha)$ and $j(i,\alpha)$ define α.

E. The Steinmann Monomials $A_\alpha(x)$

Definition

$$A_\alpha(x) \equiv A^{\sigma(n,\alpha)}_{j(n,\alpha)}\ A^{\sigma(n-1,\alpha)}_{j(n-1,\alpha)}\ A^{\sigma(1,\alpha)}_{j(1,\alpha)}\ A_{j(0,\alpha)}\ , \tag{2.25}$$

where the symbols A^+_j and A^-_j are defined by (2.18), and where

$$x \equiv \{x_j:\ j\epsilon\ J_\alpha\}\ . \tag{2.26}$$

F. The Operators $M_\alpha(p) \equiv M_\alpha$

Definition

$$M_\alpha(p) \equiv \int \prod_{j\epsilon J_\alpha} \left(id^4x_j\ e^{ip_jx_j}\ K_j \right)\ A_\alpha(x) \tag{2.27}$$

where $A_\alpha(x)$ is defined in (2.25) and p is the set of variables p_j associated with α,

$$p \equiv \{p_j:\ j\epsilon\ J_\alpha\}\ . \tag{2.28}$$

G. The Nested Commutators $m_\alpha(p) \equiv m_\alpha$

Definition

$$m_\alpha(p) \equiv \left[c^{\sigma(n,\alpha)}_{j(n,\alpha)},\ \left[c^{\sigma(n-1,\alpha)}_{j(n-1,\alpha)} \cdots,\ \left[c^{\sigma(1,\alpha)}_{j(1,\alpha)},\ c_{j(0,\alpha)} \right] \cdots \right]\ , \tag{2.29}$$

where p is a set of mass-shell p_j,

$$c^\pm_j \equiv c^\pm_j(p_j)\ , \tag{2.30}$$

and

$$c_j \equiv c^+_j - c^-_j\ . \tag{2.31}$$

H. The Mass-Shell Relation $M_\alpha = m_\alpha$

Repeated application of the generalized reduction formula (2.23) gives, for mass-shell p,

$$M_\alpha(p) = m_\alpha(p)\ . \tag{2.32}$$

I. Commutators of the $A^\pm_i$

The functions θ_{ij} defined in (2.16) satisfy the following identities:

$$\theta_{ij}\,\theta_{kj} - \theta_{ij}\,\theta_{ki} - \theta_{kj}\,\theta_{ik} = 0\ , \tag{2.33a}$$

$$\theta_{ji}\,\theta_{jk} - \theta_{ji}\,\theta_{ik} - \theta_{jk}\,\theta_{ki} = 0\ , \tag{2.33b}$$

and

$$\theta_{ij} + \theta_{ji} = 1\ . \tag{2.33c}$$

These three identities imply the following three identities, respectively:

$$[A_i^+, A_j^+] = 0 \ , \tag{2.34a}$$

$$[A_i^-, A_j^-] = 0 \ , \tag{2.34b}$$

and

$$A_j^+ - A_j^- = \hat{A} \ , \tag{2.34c}$$

where the operator $\hat{A}$ is defined by

$$\hat{A}F \equiv [A,F] \ . \tag{2.34d}$$

J. The Steinmann Relations

The arguments $p \equiv (p_j)$ of the nested commutators $m_\alpha(p)$ can be restricted so that no two p_j add to zero. If the arguments p_j are restricted in this way then the in-operators $c_j^-(p_j)$ commute among themselves and the out-operators $c_j^+(p_j)$ commute among themselves:

$$[c_i^+, c_j^+] = 0 \ , \tag{2.35a}$$

and

$$[c_i^-, c_j^-] = 0 \ . \tag{2.35b}$$

These commutation relations and the definition

$$c_j = c_j^+ - c_j^- \ , \tag{2.35c}$$

together with the Jacobi identity, impose linear relations among the mass-shell operators m_α. Precisely the same linear relations hold among the off-mass-shell quantities $M_\alpha(p)$ and among the Steinmann monomials $A_\alpha(x)$. These latter two sets of linear relations follow from the use of eqs. (2.34) in place of (2.35).

The Steinmann relations are defined by BEG to be the full set of linear relations among the m_α [or among the M_α, or among the A_α] that arise from (2.35) [or (2.34)]. Actually, the restriction on the arguments p_j that led to (2.35a,b) is convenient, but not essential. If they are relaxed then one can use (2.14c) instead of (2.35a,b). The important common feature of (2.35a,b) and (2.14c) is that the right-hand side is a c-number that does not depend on the sign + or - in (2.14c).

K. The Trees t_β and the Groves ρ_α

Bros, Epstein, and Glaser construct a graphical analysis of the Steinmann relations. This analysis is based on a mapping ℓ that takes each nested commutator m_α into a corresponding linear combination ρ_α of trees t_β.

Definition A tree t_β is a simply connected (no loops) graph that consists of:

(i) a collection of vertices v_j^-, which are represented by dots;

(ii) a collection of vertices v_j^+, which are represented by crosses;

and

(iii) a collection of open line segments s, each of which links some dot v_i^- in t_β to some cross v_j^+ in t_β. No line segment s in t_β links two dots or two crosses.

Different trees t_β are regarded as independent basis vectors in a linear vector space of trees.

The index sets J_β^+, J_β^-, and J_β are defined by

$$J^+_\beta \equiv \{j: v^+_j \in t_\beta\}, \quad (2.36a)$$

$$J^-_\beta \equiv \{j: v^-_j \in t_\beta\}, \quad (2.36b)$$

and

$$J_\beta \equiv J^+_\beta \cup J^-_\beta . \quad (2.36c)$$

According to (2.29) the nested commutator m_α is

$$m_\alpha \equiv \left[c^{\sigma(n,\alpha)}_{j(n,\alpha)}, \left[c^{\sigma(n-1,\alpha)}_{j(n-1,\alpha)}, \left[\cdots \left[c^{\sigma(1,\alpha)}_{j(1,\alpha)}, c_{j(0,\alpha)}\right]\cdots\right.\right.\right],$$

where $c_j \equiv c^+_j - c^-_j$.

<u>Definition</u> The <u>grove</u> $\rho_\alpha \equiv \ell(m_\alpha)$ is

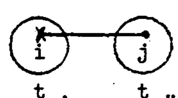
$$\rho_\alpha \equiv \left[v^{\sigma(n,\alpha)}_{j(n,\alpha)}, \left[v^{\sigma(n-1,\alpha)}_{j(n-1,\alpha)}, \left[\cdots \left[v^{\sigma(1,\alpha)}_{j(1,\alpha)}, v_{j(0,\alpha)}\right]\cdots\right.\right.\right], \quad (2.37a)$$

where

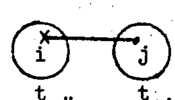
$$v_j \equiv v^+_j - v^-_j . \quad (2.37b)$$

The commutator of two trees is the sum of trees defined by the rule

$$[t_{\beta'},t_{\beta''}] \equiv \sum_{\substack{i\in J^+_{\beta'}\\ j\in J^-_{\beta''}}}$$



$$- \sum_{\substack{i\in J^+_{\beta''}\\ j\in J^-_{\beta'}}}$$



. (2.38)

This rule says that the commutator $[t_{\beta'},t_{\beta''}]$ of the two trees $t_{\beta'}$ and $t_{\beta''}$ is the sum of all trees that can be formed by joining a cross v^+_i in $t_{\beta'}$ to a dot v^-_j in $t_{\beta''}$, minus the sum of all trees that can be formed by joining a cross v^+_i in $t_{\beta''}$ to a dot v^-_j in $t_{\beta'}$. The commutator of two linear combinations of trees is the natural linear combination of the commutators of the component trees.

<u>Remark</u> The commutator symbol $[t_{\beta'},t_{\beta''}]$ is appropriate because this product is linear and both the antisymmetry property

$$[t_{\beta'},t_{\beta''}] = -[t_{\beta''},t_{\beta'}] \quad (2.39a)$$

and the Jacobi identity

$$[t_{\beta'},[t_{\beta''},t_{\beta'''}]] + [t_{\beta''},[t_{\beta'''},t_{\beta'}]] + [t_{\beta'''},[t_{\beta'},t_{\beta''}]] = 0 \quad (2.39b)$$

hold. Note that

$$[v^+_i,v^+_j] = 0 \quad (2.39c)$$

and

$$[v^-_i,v^-_j] = 0 . \quad (2.39d)$$

These two relations are analogous to the commutation relations (2.35) satisfied by the $c^\pm_j$'s.

<u>Definition</u>

$$t'_\beta \equiv (-1)^{n_-(\beta)}\, t_\beta \quad (2.40)$$

where $n_-(\beta)$ is the number of dots in t_β.

<u>Definition</u> $v^\mp_i$ is <u>α-less</u> than $v^\pm_j$ if and only if $v^\mp_i$ stands to the right of $v^\pm_j$ in ρ_α [see (2.37a)].

<u>Definition</u> ρ_α is <u>trivial</u> if and only if it consists of a single $v_j = v^+_j - v^-_j$.

<u>Remark</u> For nontrivial ρ_α

$$\rho_\alpha = \sum_\beta c_{\alpha\beta} t'_\beta , \quad (2.41)$$

where

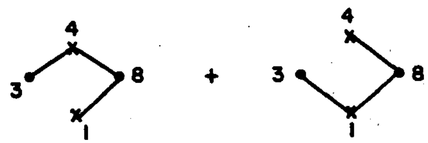
$$c_{\alpha\beta} = 1 \text{ if for each } j\epsilon J_\alpha \text{ except } j(0,\alpha) \text{ the vertex } v_j^{\pm} \text{ lies in } t_\beta, \text{ and is linked by a line segment } s \text{ in } t_\beta \text{ to a } v_i^{\mp} \text{ in } t_\beta \text{ that is } \alpha\text{-less than } v_j^{\pm}.$$

$$c_{\alpha\beta} = 0 \text{ otherwise .}$$

<u>Example</u>

$$\rho_\alpha = [v_3^-,[v_4^+,[v_1^+,v_8]]]$$

$$=$$



These two trees are the only trees t_β that satisfy the condition that for each $j\epsilon J_\alpha$ except $j(0,\alpha)$ the vertex $v_j^{\pm}$ is linked in t_β to a $v_i^{\mp}$ that is α-less than $v_j^{\pm}$. Each of these trees appears in the sum with coefficient $(-1)^{n_-(\beta)} = 1$.

The set of coefficients $c_{\alpha\beta}$ defined in (2.41) play a basic role in the BEG formalism.

L. <u>The x-Space Cones C_β</u>

The assumed local commutation relations of the A_j require each $A_\alpha(x)$ to vanish outside a certain region $\sum_\alpha$. These regions $\sum_\alpha$ are expressed in terms of cones C_β. The cone C_β is associated with the tree t_β.

<u>Definition</u> Let each vertex $v_j^{\pm}$ of t_β be associated with a four-vector x_j. Let $x \equiv \{x_j: j\epsilon J_\beta\}$ be the collection of these x_j. Let $v_j^+(s,+)$ and $v_j^-(s,-)$ be the two vertices of t_β that are joined

by the line segment $s\epsilon t_\beta$. Then

$$C_\beta \equiv \{x: \left(x_{j(s,+)} - x_{j(s,-)}\right) \epsilon \overline{V}^+ \text{ for every } s\epsilon t_\beta\} . \tag{2.42a}$$

Here $\overline{V}^+$ is the closure of the forward light-cone

$$V^+ \equiv \{w: w^2 > 0, \ w^0 > 0\} , \tag{2.42b}$$

where w represents any Minkowski four-vector.

M. <u>The Regions $\sum_\alpha$</u>

<u>Definition</u>

$$\sum_\alpha \equiv \bigcup_{\beta: c_{\alpha\beta}=1} C_\beta . \tag{2.43}$$

<u>Theorem</u>

$$A_\alpha(x) = 0 \quad \text{for } x \notin \sum_\alpha . \tag{2.44}$$

N. <u>The Steinmann Functions r_α, $\tilde{r}_\alpha$, and $\tilde{r}'_\alpha$</u>

<u>Definitions</u>

$$r_\alpha(x) \equiv \langle 0|A_\alpha(x)|0\rangle ,$$
$$\tilde{r}_\alpha(p) \equiv \langle 0|M_\alpha(p)|0\rangle . \tag{2.45}$$

<u>Remark</u> For mass-shell p, eq. (2.32) gives

$$\tilde{r}_\alpha(p) \equiv \langle 0|m_\alpha(p)|0\rangle . \tag{2.46}$$

<u>Remark</u> Translation invariance implies momentum-energy conservation:

$$\tilde{r}_\alpha(p) = 0 \quad \text{if} \quad \sum_{j\epsilon J_\alpha} p_j \neq 0 . \tag{2.47a}$$

Definition of $\tilde{r}'_\alpha(p)$

$$\tilde{r}_\alpha(p) \equiv (2\pi)^4 \delta^4\left(\sum_{j\epsilon J_\alpha} p_j\right) \tilde{r}'_\alpha(p) \quad . \tag{2.47b}$$

The function $\tilde{r}'_\alpha(p)$ is the boundary value of a function $\tilde{r}'_\alpha(k)$ that is analytic in a certain region Γ'_α. This region will be defined presently. Some preliminary definitions are given first.

O. The Complex Momentum Vectors $k = p + iq$

Definitions

$$k \equiv p + iq \quad , \tag{2.48a}$$

$$p \equiv \text{Re } k \quad . \tag{2.48b}$$

$$q \equiv \text{Im } k \quad . \tag{2.48c}$$

P. The Abbreviations $p(J)$, $q(J)$, and $k(J)$

Definitions

$$p(J) \equiv \sum_{j\epsilon J} p_j \quad . \tag{2.49a}$$

$$q(J) \equiv \sum_{j\epsilon J} q_j \quad . \tag{2.49b}$$

$$k(J) \equiv \sum_{j\epsilon J} k_j \quad . \tag{2.49c}$$

Q. The Spaces $P(J)$, $Q(J)$, and $K(J)$

Definitions

$P(J)$ is the space consisting of the points

$$p \equiv \{p_j : j\epsilon J, \ p(J) = 0\} \quad . \tag{2.50a}$$

$Q(J)$ is the space consisting of the points

$$q \equiv \{q_j : j\epsilon J, \ q(J) = 0\} \quad . \tag{2.50b}$$

$K(J)$ is the space consisting of the points

$$k \equiv \{k_j : j\epsilon J, \ k(J) = 0\} \quad . \tag{2.50c}$$

[For example, if the set J has n elements j, then $P(J)$ is the restriction of the $4n$ dimensional space of points $p \equiv \{p_j : j\epsilon J\}$ to the $4n-4$ dimensional subspace on which momentum-energy is conserved: $\sum_{j\epsilon J} p_j = 0$.]

R. The Momentum-Space Cones $\tilde{C}_\beta$, $\tilde{C}'_\beta$, Γ_α, and Γ'_α

Each Steinmann function $\tilde{r}'_\alpha(p)$ is the boundary value of a function $\tilde{r}'_\alpha(k)$ that is analytic in a region Γ'_α. The regions Γ'_α are defined as intersections of certain cones $\tilde{C}'_\beta$, which will now be defined.

Let s be an open line segment contained in the tree t_β:

$$s \in t_\beta \quad . \tag{2.51a}$$

The removal of s from t_β separates t_β into two trees $t^+_{\beta s}$ and $t^-_{\beta s}$, which contain, respectively, the cross and dot linked by s:

$$v^+_{j(s,+)} \in t^+_{\beta s} \subset t_\beta \quad , \tag{2.51b}$$

$$v^-_{j(s,-)} \in t^-_{\beta s} \subset t_\beta \quad . \tag{2.51c}$$

The sets $J^+_{\beta s}$ and $J^-_{\beta s}$ are the sets of indices j that label the vertices of $t^+_{\beta s}$ and $t^-_{\beta s}$, respectively:

$$J^+_{\beta s} \equiv \{j : v^\pm_j \in t^+_{\beta s}\} \quad , \tag{2.51d}$$

$$J^-_{\beta s} \equiv \{j : v^\pm_j \in t^-_{\beta s}\} \quad . \tag{2.51e}$$

It is evident that for any $s \in t_\beta$

$$J^+_{\beta s} \cup J^-_{\beta s} \equiv J_\beta \quad , \tag{2.51f}$$

where $J_\beta \equiv \{j : v^\pm_j \in t_\beta\}$.

Definitions

$$\tilde{C}_\beta \equiv \{q \epsilon Q(J_\beta) : q(J^+_{\beta s}) \epsilon V^+ \quad \text{for every} \quad s \epsilon t_\beta\} \quad . \tag{2.52a}$$

$$\tilde{C}'_\beta \equiv \{k = p + iq : q \epsilon \tilde{C}_\beta, \quad p \epsilon P(J_\beta)\} \quad . \tag{2.52b}$$

Here $Q(J_\beta)$ and $P(J_\beta)$ are the spaces defined in (2.50).

[If the tree t_β is regarded as a diagram representing the flow of conserved complex momentum-energy, where k_j represents the flow out of the diagram at vertex $v^\pm_j$, then the conditions for $\tilde{C}_\beta$ are the conditions that the imaginary part of the momentum-energy flowing from dot to cross along each line s of t_β lies in V^+.]

Remark For any $q \epsilon Q(J_\beta)$

$$\sum_{j \epsilon J_\beta} q_j \, x_j = \sum_{s \epsilon t_\beta} q(J^+_{\beta s}) \left(x_{j(s,+)} - x_{j(s,-)}\right) \quad . \tag{2.52c}$$

Thus for x in C_β and q in $\tilde{C}_\beta$ the definitions (2.52a) and (2.42) yield

$$\sum_{j \epsilon J_\beta} q_j \, x_j > 0 \quad , \tag{2.52d}$$

except at the points x where the x_j for $j \epsilon J_\beta$ are all equal.

Definitions

$$\Gamma_\alpha \equiv \bigcap_{\beta : c_{\alpha\beta = 1}} \tilde{C}_\beta \quad . \tag{2.53a}$$

$$\Gamma'_\alpha \equiv \bigcap_{\beta : c_{\alpha\beta = 1}} \tilde{C}'_\beta \quad . \tag{2.53b}$$

Remark For x in $\sum_\alpha$ and q in Γ_α, the definitions (2.53a) and (2.43) and the result (2.52d) yield

$$\sum_{j \epsilon J_\alpha} q_j \, x_j \equiv q \cdot x > 0 \tag{2.53c}$$

except at points x where the x_j for $j \epsilon J_\alpha$ are all equal. This inequality ensures that if the argument p in the definition (2.27) of $M_\alpha(p)$ is replaced by $k = p + iq$ then the exponential factor $\exp(ik \cdot x)$ will give exponential damping as $\tau \to \infty$ at points $x = \tau x'$ for all x' in the domain $\sum_\alpha$ of integration and all q in Γ_α, except at points where the x'_j are all equal.

S. The Analytic Functions $\tilde{r}'_\alpha(k)$

Definition

$$M'_\alpha(k) \equiv \int \delta^4\left(x_{j(0,\alpha)}\right) \prod_{j\in J_\alpha} \left(id^4x_j\, e^{ik_jx_j}\, K_j\right) A_\alpha(x) \quad . \tag{2.54a}$$

Definition

$$\tilde{r}'_\alpha(k) \equiv \langle 0|M'_\alpha(k)|0\rangle \quad . \tag{2.54b}$$

Remark The delta function $\delta^4\left(x_{j(0,\alpha)}\right)$ in (2.54) suppresses the trivial integration that would otherwise arise from the assumed translational invariance of $\langle 0|A_\alpha(x)|0\rangle$. For $\operatorname{Im} k = 0$ this trivial integration leads to the factor $(2\pi)^4\, \delta^4(\sum p_j)$ that occurs in (2.47b).

Theorem The function $\tilde{r}'_\alpha(k)$ defined by (2.54b) is analytic in Γ'_α and satisfies for all p in the space $P(J_\alpha)$ the condition

$$\lim_{\substack{q\in\Gamma_\alpha \\ q\to 0}} \tilde{r}'_\alpha(p + iq) = \tilde{r}'_\alpha(p) \quad . \tag{2.54c}$$

Remark The functions $\tilde{r}'_\alpha(p)$ and the similar functions $\tilde{r}'_\lambda(p)$ that are introduced later are always, in the BEG formalism, to be interpreted as tempered distributions over the subspace on which momentum-energy conservation holds. The functions $\tilde{r}_\alpha(p)$ and $\tilde{r}_\lambda(p)$ are tempered distributions over the complete p space. Thus (2.54c) is to be interpreted as

$$\lim_{\substack{q\in\Gamma_\alpha \\ q\to 0}} \int \tilde{r}'_\alpha(p + iq)\, \phi(p)\, \delta^4\left[p(J_\alpha)\right] dp = \int \tilde{r}'_\alpha(p)\, \phi(p) \delta^4\left[p(J_\alpha)\right] dp \tag{2.54d}$$

for all appropriate test functions $\phi(p)$ in $P(J_\alpha)$.

T. The S-matrix $S(p)$

The argument p of $S(p)$ is a set $\{p_j\}$ of mass-shell four-vectors $(p_j{}^2 = m_j{}^2)$.

Definitions

$$J_p \equiv \{j : p_j \in p\} \quad ,$$

$$F_p \equiv \{f : p_f{}^0 > 0,\ p_f \in p\} \quad , \tag{2.55a}$$

$$I_p \equiv \{i : p_i{}^0 < 0,\ p_i \in p\} \quad . \tag{2.55b}$$

Definition

$$S(p) \equiv \left\langle 0\left|\left[\prod_{f\in F_p} a(p_f,t_f,+)\right]\left[\prod_{i\in I_p} a^\dagger(-p_i,-t_i,-)\right]\right|0\right\rangle$$

$$= \left\langle 0\left|\left(\prod_{f\in F_p} c_f^+\right) \prod_{i\in I_p} (-c_i^-)\right|0\right\rangle \quad , \tag{2.56a}$$

where (2.23) and (2.30) are used to get the second line.

Remark Comparison with (2.13) gives

$$S(p) = S(\{p_f,t_f\};\ \{-p_i,-t_i\}) \quad , \tag{2.56b}$$

where the indices f and i run over F_p and I_p, respectively.

Remark It will be shown in subsection X below that

$$S(p) = \tilde{r}_\alpha(p)$$

for all mass-shell points p in a certain domain $P_\alpha \subset P(J_\alpha)$. But much of the mass shell lies outside the union of the P_α. However, the set of Steinmann functions $\tilde{r}_\alpha(p)$ is a subset of a larger set of functions $\tilde{r}_\lambda(p)$ called generalized retarded functions. These functions satisfy the relation

$$S(p) = \tilde{r}_\lambda(p)$$

for all mass-shell points in P_λ, where the union of the P_λ covers almost all of the mass shell.

Any point p that lies outside the union of the P_λ lies on a plane

$$p(J') = 0$$

for some nonempty proper subset J' of J_p. Such points are points where energy-momentum conservation is satisfied for some subset J' of the particles $j \in J_p$. Thus all points p lying outside $\bigcup P_\lambda$ are points where the "disconnected parts" of the S matrix can be nonzero. Thus $\tilde{r}_\lambda(p)$ is equated to $S(p)$ only at points p where $S(p)$ equals its connected part $S_c(p)$.

The generalized retarded functions $\tilde{r}'_\lambda(p)$, which are defined by $\tilde{r}_\lambda(p) = (2\pi)^4\, \delta(\sum p)\, \tilde{r}'_\lambda(p)$, are boundary values of functions $\tilde{r}'_\lambda(k)$ that are analytic in domains Γ'_λ called <u>cells</u>. Each cell Γ'_λ is associated with an index set J_λ and is the product of the space $P(J_\lambda)$ with a cone Γ_λ in $Q(J_\lambda)$. The cone Γ_λ is also called a cell.

To describe these cells Γ_λ and the regions $P_\lambda \subset P(J_\lambda)$ it is helpful to consider first the restrictions $\Gamma_\lambda{}^0$ of the cells Γ_λ to energy space.

U. The Energy Cells $\Gamma_\lambda{}^0$ and the Signs $\sigma(J,\lambda)$

<u>Definition</u> For any set $J_\lambda \subset \mathcal{G}$ the space $Q^0(J_\lambda)$ is the restriction of the space $Q(J_\lambda)$ to its energy subspace. Equivalently, $Q^0(J_\lambda)$ is the space consisting of the points

$$q^0 \equiv \{q_j{}^0 : j \in J_\lambda,\ q^0(J_\lambda) = 0\}\ . \tag{2.57}$$

<u>Definition</u> Each energy cell $\Gamma_\lambda{}^0$ is associated with an index set J_λ, and $\Gamma_\lambda{}^0$ lies in $Q^0(J_\lambda)$. Let J represent a nonempty proper subset of J_λ. Each plane $q^0(J)$ divides $Q^0(J_\lambda)$ into two halves. The set of planes $q^0(J)$ divides $Q^0(J_\lambda)$ into several nonempty open cones. Every such open cone is an energy cell $\Gamma_\lambda{}^0$.

<u>Definition</u> The location of $\Gamma_\lambda{}^0$ is determined by a set of signs $\sigma(J,\lambda)$, one for each nonempty proper subset J of J_λ. The sign $\sigma(J,\lambda)$ determines the side of $q^0(J) = 0$ upon which $\Gamma_\lambda{}^0$ lies:

$$\Gamma_\lambda{}^0 \equiv \{q^0 \in Q^0(J_\lambda) : \sigma(J,\lambda)\, q^0(J) > 0 \quad \text{for all } J \subset J_\lambda,\ J \neq J_\lambda,\ J \neq \emptyset\}\ . \tag{2.58}$$

<u>Remark</u> Each energy cell $\Gamma_\lambda{}^0$ corresponds to a definite set of signs $\sigma(J,\lambda)$, one for each nonempty proper subset J of J_λ. However, not every set of signs $\sigma(J)$, one for each such J, corresponds to a cell. For the signs $\sigma(J,\lambda)$ must satisfy conditions such as

$$\sigma(J,\lambda) = -\sigma(J_\lambda - J,\lambda)\ , \tag{2.59a}$$

and

$$\sigma(J \cup J',\lambda) = \sigma(J',\lambda) \tag{2.59b}$$

if $\sigma(J,\lambda) = \sigma(J',\lambda)$ and $J \cap J' = \emptyset$.

V. The q-Space Cells Γ_λ

The q-space cells Γ_λ are defined by a relativistic generalization of the formula that defines Γ_λ^0.

Definition

$$\Gamma_\lambda \equiv \{q \in Q(J_\lambda) : \sigma(J,\lambda)\, q(J) \in V^+ \quad \text{for every } J \subset J_\lambda,\ J \neq J_\lambda,\ J \neq \emptyset\} . \tag{2.60a}$$

Theorem For every α there is a $\lambda \equiv \lambda(\alpha)$ such that

$$\Gamma_\alpha = \Gamma_{\lambda(\alpha)} \ . \tag{2.60b}$$

Remark This theorem implies that the set of indices λ can be regarded as an extension of the set of indices α.

W. The p-Space Regions P_λ

Definition

$$P_\lambda \equiv \{p \in P(J_\lambda) : \sigma(J,\lambda)\, p(J) \in \complement\, \bar{V}^- \quad \text{for every } J \subset J_\lambda,\ J \neq J_\lambda,\ J \neq \emptyset\} , \tag{2.61a}$$

where $\complement\, \bar{V}^-$ is the complement of the closure of the backward light-cone.

Definition

$$P_\alpha \equiv P_{\lambda(\alpha)} \tag{2.61b}$$

where $\lambda(\alpha)$ is defined in (2.60b).

X. $S(p) = \tilde{r}_\alpha(p)$ in P_α

Definition $\mathcal{M}$ is the mass-shell, which is the space consisting of the points

$$p = \{p_j : j \in \mathcal{J},\ p_j^2 = m_j^2\} . \tag{2.62}$$

Theorem

$$S(p) = \tilde{r}_\alpha(p) \quad \text{for } p \text{ in } P_\alpha \cap \mathcal{M} . \tag{2.63}$$

Proof For points p in the mass shell $\mathcal{M}$ the function $\tilde{r}_\alpha(p)$ is given by (2.46),

$$\tilde{r}_\alpha(p) = \langle 0|m_\alpha(p)|0\rangle \ , \tag{2.64}$$

where $m_\alpha(p)$ is the nested commutator defined by (2.29). For $n(\alpha) > 1$, the operator $c_{j(0,\alpha)}$ is replaced by $\sigma(0,\alpha)\, c_{j(0,\alpha)}^{\sigma(0,\alpha)}$ with $\sigma(0,\alpha) = -\sigma(1,\alpha)$, since the other term does not contribute by virtue of (2.14c). The term of m_α in which each c_j^+ stays on the left and each c_j^- moves to the right gives $S(p)$, as defined by (2.56a).

It will now be shown that all remaining terms vanish. Let the multiple commutator m_α be written as

$$m_\alpha \equiv \sum_{\delta=1}^{2^n} b_{\alpha\delta}\, e_\delta \ , \tag{2.65}$$

where the $e_\delta \equiv \Pi(\pm c_j^\pm)$ are the 2^n differently ordered products of $\pm c_j^\pm$'s that occur in the expansion of m_α. For each pair (α,δ) a distinguished tree $t_{\beta(\alpha,\delta)}$ is defined as follows:

Let the ordered sequence of operators $\pm c_j^{\pm}$ in e_δ be mapped into a correspondingly ordered horizontal row of vertices $v_j^{\pm}$. The pair of vertices $v^{\pm}_{j(0,\alpha)}$ and $v^{\mp}_{j(1,\alpha)}$ will be adjacent, and will consist of one cross and one dot. Locate the vertex $v^{\pm}_{j(0,\alpha)}$. Join each other vertex $v_j^{\pm}$ by a line segment s_j to the nearest vertex $v^{\mp}_{j'}$, that lies in the direction of $v^{\pm}_{j(0,\alpha)}$, as indicated by fig. II.1.

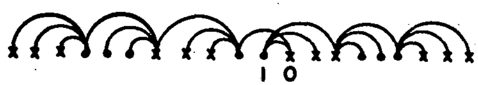


Fig. II.1. A typical tree $t_{\beta(\alpha,\delta)}$. The zero indicates $v^{\pm}_{j(0,\alpha)}$, which in this example is a cross. Thus vertex $v^{\pm}_{j(1,\alpha)}$ is a dot. It is indicated by 1.

In the tree $t_{\beta(\alpha,\delta)}$ each vertex $v_j^{\pm}$ (except $v^{\sigma(0,\alpha)}_{j(0,\alpha)}$) is joined by a line segment s to a vertex $v_i^{\mp}$ that is α-less than $\alpha_j^{\pm}$. Thus (2.41) gives

$$c_{\alpha\beta(\alpha,\delta)} = 1 \quad . \tag{2.66}$$

Let $\delta = 1$ label the e_δ in which all c_j^+'s stand to the left of all $(-c_j^-)$'s. What must be shown is that

$$\left\langle 0 \left| \sum_{\delta=2}^{2^n} b_{\alpha\delta}\, e_\delta \right| 0 \right\rangle = 0 \quad . \tag{2.67}$$

For each $\delta \neq 1$ there must be at least one factor c_i^- in e_δ that stands immediately to the left of some c_j^+ in e_δ. Let the ordered product e_δ be separated into two parts

$$e_\delta = \prod_{j\in J^-} (\pm c_j^{\pm}) \prod_{j\in J^+} (\pm c_j^{\pm}) \tag{2.68}$$

where c_i^- is the rightmost operator in the left-hand set of factors. Then every $s \in t_{\beta(\alpha,\delta)}$ that links a vertex in $\{v_j{}^\sigma : j \in J^-\}$ to a vertex in $\{v_j{}^\sigma : j \in J^+\}$ connects a dot in the first group to a cross in the second as shown in fig. II.2.

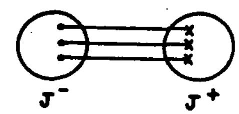


Fig. II.2. For every line segment s in $t_{\beta(\alpha,\delta)}$ that links a $v_j{}^\sigma$ with $j \in J^-$ to a $v^{\sigma'}_{j'}$ with $j' \in J^+$ one has $\sigma = -$ and $\sigma' = +$.

If the tree condition $q(J^+_{\beta s}) \in V^+$, is satisfied for each line s shown in fig. II.2, then the sum $q(J^+)$ of these vectors $q(J^+_{\beta s})$ also lies in V^+. Thus the definition (2.52a) of $\tilde{C}_\beta$ entails that

$$\tilde{C}_{\beta(\alpha,\delta)} \subset \{q : q(J^+) \in V^+\} \quad . \tag{2.69}$$

But then the formula (2.53a),

$$\Gamma_\alpha = \bigcap_{\beta : c_{\alpha\beta=1}} \tilde{C}_\beta \ ,$$

and the result (2.66) that $c_{\alpha\beta(\alpha,\delta)} = 1$ imply that

$$\Gamma_\alpha \subset \{q : q(J^+) \in V^+\} \quad . \tag{2.70}$$

It then follows from the definition (2.60) for Γ_α,

$$\Gamma_\alpha \equiv \{q \in Q(J_\alpha) : \sigma(J,\alpha)\, q(J) \in V^+ \quad \text{for every } J \subset J_\alpha,\ J \neq J_\alpha,\ J \neq \emptyset\} ,$$

that

$$\sigma(J^+,\alpha) = + \quad . \tag{2.71}$$

But then the requirement of the theorem $p \in P_\alpha$ entails [see definition (2.61)] that

$$p(J^+) \in \mathcal{C}\, \bar{V}^- \quad . \tag{2.72a}$$

However, the spectral conditions [see (2.23b,c) and (2.30)] entail that

$$\prod_{j \in J^+} (\pm c_j^{\pm}) |0\rangle = 0 \tag{2.72b}$$

whenever (2.72a) holds. Thus

$$\langle 0 | e_\delta | 0 \rangle = 0 \tag{2.73}$$

for any fixed $\delta \neq 1$, and the corresponding contribution to (2.67) is zero. Hence (2.67) holds, and the theorem is proved.

The similar theorem with α replaced by λ is also true. The proof depends on the fact that many relations that hold for the index α have generalizations that hold for the index λ. These generalizations, which are useful in many contexts, are described in the next few subsections.

Y. The Coefficients $c_{\lambda\beta}$

The coefficients $c_{\alpha\beta}$ were defined in (2.41).

Theorem

$$c_{\alpha\beta} = \begin{cases} 1 & \text{if } \Gamma_\alpha \subset \tilde{C}_\beta \\ 0 & \text{otherwise} \end{cases} \quad . \tag{2.74}$$

Definition

$$c_{\lambda\beta} \equiv \begin{cases} 1 & \text{if } \Gamma_\lambda \subset \tilde{C}_\beta \\ 0 & \text{otherwise} \end{cases} \quad . \tag{2.75}$$

Theorem

$$\Gamma_\lambda = \bigcap_{\beta : c_{\lambda\beta}=1} \tilde{C}_\beta \quad . \tag{2.76}$$

Remark Note the similarity of this equation to (2.53a),

$$\Gamma_\alpha = \bigcap_{\beta : c_{\alpha\beta}=1} \tilde{C}_\beta \quad .$$

Remark BEG use the symbol $k_{\lambda\beta}$ in place of $c_{\lambda\beta}$.

Z. The Groves ρ_λ

The grove ρ_α was shown in (2.41) to be of the form

$$\rho_\alpha = \sum c_{\alpha\beta}\, t'_\beta \quad .$$

Definition

$$\rho_\lambda \equiv \sum c_{\lambda\beta}\, t'_\beta \quad . \tag{2.77}$$

A'. Adjacent Cells Γ_{λ_1} and Γ_{λ_2}

Definition Two cells Γ_{λ_1} and Γ_{λ_2} are said to be adjacent if and only if they differ by just one single condition $\pm q(J) \in V^+$. More precisely, the two sets J_{λ_1} and J_{λ_2} must be equal, and there must be two complementary subsets J' and J'' of $J_\lambda \equiv J_{\lambda_1} = J_{\lambda_2}$ such that

$$\sigma(J',\lambda_1) = -\sigma(J'',\lambda_1) = + , \tag{2.78a}$$

and

$$\sigma(J',\lambda_2) = -\sigma(J'',\lambda_2) = - . \tag{2.78b}$$

But for all other nonempty proper subsets J of J_λ

$$\sigma(J,\lambda_1) = \sigma(J,\lambda_2) . \tag{2.78c}$$

Remark If Γ_{λ_1} and Γ_{λ_2} are adjacent then $\Gamma_{\lambda_1}^0$ and $\Gamma_{\lambda_2}^0$ are also said to be adjacent. The two cells $\Gamma_{\lambda_1}^0$ and $\Gamma_{\lambda_2}^0$ have a common face that lies in the plane

$$q^0(J') = q^0(J'') = 0 . \tag{2.79}$$

This plane separates the two adjacent cells $\Gamma_{\lambda_1}^0$ and $\Gamma_{\lambda_2}^0$.

B'. The Boundary Cells $\Gamma_{\lambda'}$ and $\Gamma_{\lambda''}$

Definition Each pair of adjacent cells Γ_{λ_1} and Γ_{λ_2} is associated with a pair of boundary cells $\Gamma_{\lambda'}$ and $\Gamma_{\lambda''}$, which lie in $Q(J')$ and $Q(J'')$, respectively, and are defined by the following conditions:

$$J_{\lambda'} = J' \tag{2.80a}$$

and

$$\sigma(J,\lambda') = \sigma(J,\lambda_1) = \sigma(J,\lambda_2) \tag{2.80b}$$

$$\text{for all } J \subset J_{\lambda'},\ J \neq J_{\lambda'},\ J \neq \emptyset ;$$

$$J_{\lambda''} = J'' \tag{2.80c}$$

and

$$\sigma(J,\lambda'') = \sigma(J,\lambda_1) = \sigma(J,\lambda_2) \tag{2.80d}$$

$$\text{for all } J \subset J_{\lambda''},\ J \neq J_{\lambda''},\ J \neq \emptyset .$$

Remark

$$J_{\lambda'} \cup J_{\lambda''} = J_{\lambda_1} = J_{\lambda_2} = J_\lambda . \tag{2.81a}$$

$$J_{\lambda'} \cap J_{\lambda''} = \emptyset . \tag{2.81b}$$

C'. The Difference Formula $\rho_{\lambda_1} - \rho_{\lambda_2} = [\rho_{\lambda'},\rho_{\lambda''}]$

Theorem Let Γ_{λ_1} and Γ_{λ_2} be two adjacent cells. Let $\Gamma_{\lambda'}$ and $\Gamma_{\lambda''}$ be the two associated boundary cells defined by (2.80). Let ρ_{λ_1}, ρ_{λ_2}, $\rho_{\lambda'}$, and $\rho_{\lambda''}$ be the groves corresponding to these four cells. Then

$$\rho_{\lambda_1} - \rho_{\lambda_2} = [\rho_{\lambda'},\rho_{\lambda''}] , \tag{2.82}$$

where the commutator product of two groves is defined by (2.38).

This theorem plays a central role in the BEG formalism, and in the applications of that formalism made in this paper.

D'. Formulas for Commutators

The Jacobi identity and the algebraic similarities between the ρ_α, m_α, M_α, and A_α entail the following result.

Theorem For each pair (α',α'') of indices α there is a set of integral coefficients $a(\alpha',\alpha'';\alpha)$ such that

$$[\rho_{\alpha'},\rho_{\alpha''}] = \sum_\alpha a(\alpha',\alpha'';\alpha)\,\rho_\alpha \ , \qquad (2.83a)$$

$$[m_{\alpha'},m_{\alpha''}] = \sum_\alpha a(\alpha',\alpha'';\alpha)\,m_\alpha \ , \qquad (2.83b)$$

$$[M_{\alpha'},M_{\alpha''}] = \sum_\alpha a(\alpha',\alpha'';\alpha)\,M_\alpha \ , \qquad (2.83c)$$

and

$$[A_{\alpha'},A_{\alpha''}] = \sum_\alpha a(\alpha',\alpha'';\alpha)\,A_\alpha \ . \qquad (2.83d)$$

Moreover,

$$\sum_\alpha a(\alpha',\alpha'';\alpha) = 0 \ . \qquad (2.83e)$$

E'. The Expansion $\rho_\lambda = \sum d_{\lambda\alpha}\,\rho_\alpha$

Inductive use of (2.82) and (2.83a) yields the following result.

Theorem For each λ there exists at least one set of coefficients $d_{\lambda\alpha}$ such that

$$\rho_\lambda = \sum_\alpha d_{\lambda\alpha}\,\rho_\alpha \ , \qquad (2.84a)$$

where the ρ_α are defined by (2.37). Furthermore, for each nontrivial ρ_λ (i.e., $\rho_\lambda \neq v_j$) the $d_{\lambda\alpha}$ can (and will) be chosen so that every tree t_β occurring in the expansion (2.84a) with nonzero coefficient has exactly the same set of vertices $v_j^\ddagger$. Moreover, the $d_{\lambda\alpha}$ can be chosen so that

$$\sum_\alpha d_{\lambda\alpha} = 1 \ . \qquad (2.84b)$$

Remark The expansion (2.84a) and the definitions $\rho_\alpha = \sum c_{\alpha\beta}\,t'_\beta$ and $\rho_\lambda = \sum c_{\lambda\beta}\,t'_\beta$ yield the relation

$$\sum_\alpha d_{\lambda\alpha}\,c_{\alpha\beta} = c_{\lambda\beta} \ . \qquad (2.85)$$

F'. The Operators m_λ, M_λ, and A_λ

Definitions For every cell index λ let

$$m_\lambda \equiv \sum_\alpha d_{\lambda\alpha}\,m_\alpha \qquad (2.86a)$$

$$M_\lambda \equiv \sum_\alpha d_{\lambda\alpha}\,M_\alpha \qquad (2.86b)$$

$$A_\lambda \equiv \sum_\alpha d_{\lambda\alpha}\,A_\alpha \ . \qquad (2.86c)$$

where the coefficients $d_{\lambda\alpha}$ are the same as those occurring in (2.84).

Remark Since the $d_{\lambda\alpha}$ occurring in (2.84) are in general not unique, the operators defined above could depend on the particular choice of $d_{\lambda\alpha}$ made in (2.84). However, it follows from the (corollary of the) BEG tree lemma described in the following section that these quantities depend only on λ. This potent BEG tree lemma will also be used later, in the proof that $S(p) = \tilde{r}_\lambda(p)$ for p in $\mathcal{M} \cap P_\lambda$.

G'. The BEG Tree Lemma

Definition A graph γ is a collection of vertices v_j^+ and v_j^- and line segments s such that each $s \in \gamma$ links some $v_j^- \in \gamma$ to some $v_j^+ \in \gamma$. A tree t_β is said to be contained in a graph γ if and only if t_β can be formed from γ by deleting some (possibly empty) subset of the set of $s \in \gamma$. The statement t_β is contained in γ is written $t_\beta \subset \gamma$. The set J_γ labels the vertices of γ.

Definition For any sum of trees

$$t = \sum_\beta c_\beta t_\beta \tag{2.87a}$$

the restriction $(t)_\gamma$ of t to γ is defined by

$$(t)_\gamma \equiv \sum_{\beta\in\bar{\gamma}} c_\beta t_\beta \tag{2.87b}$$

where

$$\bar{\gamma} \equiv \{\beta : t_\beta \subset \gamma\} \ . \tag{2.87c}$$

Remark The linear independence of the trees t_β entails that $(t)_\gamma$ is zero if and only if c_β is zero for every $\beta \in \bar{\gamma}$:

$$[(t)_\gamma = 0] \Longleftrightarrow [c_\beta = 0 \text{ for every } \beta : t_\beta \subset \gamma] \ . \tag{2.88}$$

Definition

$$\hat{\gamma} \equiv \{\alpha : (\rho_\alpha)_\gamma = 0\} \tag{2.89a}$$

$$\equiv \{\alpha : J_\alpha = J_\gamma \text{ and } c_{\alpha\beta} = 0 \text{ for every } \beta\epsilon\bar{\gamma}\} \ . \tag{2.89b}$$

Lemma Let γ be a graph and let the d_α be integers. If

$$\left(\sum_\alpha d_\alpha \rho_\alpha\right)_\gamma = 0 \tag{2.90a}$$

then there is a set of integers d'_α such that

$$\sum_\alpha d_\alpha \rho_\alpha = \sum_{\alpha\epsilon\hat{\gamma}} d'_\alpha \rho_\alpha \ , \tag{2.90b}$$

$$\sum_\alpha d_\alpha m_\alpha = \sum_{\alpha\epsilon\hat{\gamma}} d'_\alpha m_\alpha \ , \tag{2.90c}$$

$$\sum_\alpha d_\alpha M_\alpha = \sum_{\alpha\epsilon\hat{\gamma}} d'_\alpha M_\alpha \ , \tag{2.90d}$$

and

$$\sum_\alpha d_\alpha A_\alpha = \sum_{\alpha\epsilon\hat{\gamma}} d'_\alpha A_\alpha \ . \tag{2.90e}$$

If γ is the maximal graph, in which each dot is joined to each cross, then $(t)_\gamma = t$ and $\hat{\gamma}$ is empty. Thus one obtains the following

Corollary

$$\left[\sum_\alpha d_\alpha \rho_\alpha = 0\right] \Longrightarrow \left[\sum_\alpha d_\alpha m_\alpha = 0\right] \tag{2.91a}$$

$$\Longrightarrow \left[\sum_\alpha d_\alpha M_\alpha = 0\right] \tag{2.91b}$$

$$\left[\sum_{\alpha} d_\alpha \rho_\alpha = 0\right] \Longrightarrow \left[\sum_{\alpha} d_\alpha A_\alpha = 0\right] \quad . \qquad (2.91c)$$

This corollary ensures that the quantities m_λ, M_λ, and A_λ defined in (2.86) depend only on λ: they do not depend on the particular set of $d_{\lambda\alpha}$ chosen in (2.84).

H'. <u>The Generalized Retarded Functions $\tilde{r}_\lambda(p)$</u>

<u>Definition</u>

$$\tilde{r}_\lambda(p) \equiv \langle 0|M_\lambda(p)|0\rangle \quad . \qquad (2.92a)$$

<u>Remark</u> For mass-shell p, (2.32) and (2.86) imply that

$$\tilde{r}_\lambda(p) = \langle 0|m_\lambda(p)|0\rangle \quad . \qquad (2.92b)$$

<u>Definition</u>

$$r_\lambda(x) \equiv \langle 0|A_\lambda(x)|0\rangle \quad . \qquad (2.92c)$$

I'. <u>$S(p) = \tilde{r}_\lambda(p)$ in $P_\lambda \cap \mathcal{M}$</u>

<u>Theorem</u>

$$S(p) = \tilde{r}_\lambda(p) \quad \text{for } p \text{ in } P_\lambda \cap \mathcal{M} \quad . \qquad (2.93)$$

<u>Proof</u> The proof is a generalization of the argument given in section X, which covered the special case in which the index λ was one of the α's.

In that earlier proof the nested commutator m_α was expanded into the sum $\sum b_{\alpha\delta} e_\delta$, where e_δ was an ordered product of $\pm c_j^\pm$'s,

$$e_\delta = \Pi(\pm c_j^\pm) \quad . \qquad (2.94)$$

The only e_δ considered there were the 2^n products that occurred in the expansion of the particular m_α in question. In the present case the quantity $m_\lambda = \sum d_{\lambda\alpha} m_\alpha$ is a sum of different m_α, and cancellations can occur. Thus each m_λ is now written as

$$m_\lambda = \sum_\delta b'_{\lambda\delta} e_\delta \qquad (2.95)$$

where this sum is over a set of basis operators e_δ no two of which are equivalent.

Two e_δ are <u>equivalent</u> if and only if they can be transformed into each other by a sequence of interchanges of adjacent operators $c_j^\pm$ both of which have the same sign. Two equivalent e_δ are equal by virtue of the commutation relations $[c_i^+, c_j^+] = [c_i^-, c_j^-] = 0$, and hence any cell operator m_λ can be expressed in the form (2.95).

As before the e_δ with all c_j^+ standing to the left of all $(-c_i^-)$ is called e_1. The proof consists of showing that $b_{\lambda 1} = 1$, and that for all $\delta \neq 1$ either $\langle 0|e_\delta|0\rangle = 0$ or $b'_{\lambda\delta} = 0$.

For each $\delta \neq 1$ a graph $\gamma(\delta)$ is constructed by first associating each $c_j^\pm$ in e_δ with a $v_j^\pm$, and placing these $v_j^\pm$ on a horizontal line in the same order as the corresponding $c_j^\pm$ in e_δ. The set of vertices $v_j^\pm$ will group themselves into subsets each consisting of a contiguous set of $v_j^\pm$ all of the same sign, and such that adjacent contiguous sets consist of $v_j^\pm$'s of opposite sign. The graph $\gamma(\delta)$ is formed by joining each vertex $v_j^\pm$ of each contiguous set to every vertex $v_j^\mp$ that lies in the adjacent contiguous sets. (see fig. II.3)

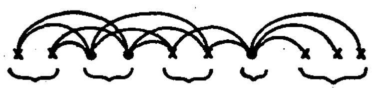


Fig. II.3. A typical graph $\gamma(\delta)$, with the various contiguous sets indicated by brackets.

Consider any fixed $\delta \neq 1$. There are two cases. If $(\rho_\lambda)_{\gamma(\delta)} \neq 0$ then, according to (2.88), there is at least one $t_\beta \subset \gamma(\delta)$ with $c_{\lambda\beta} = 1$. Let any one of these t_β be called $t_{\beta(\lambda,\delta)}$. As before, let the separation J^- and J^+ shown in (2.68) be made. The construction of $\gamma(\delta)$ ensures that the $t_{\beta(\lambda,\delta)}$ satisfies the conditions shown in fig. II.2. But then the earlier argument, with α replaced by λ, gives $\langle 0|e_\delta|0\rangle = 0$ for all p in P_λ.

If $(\rho_\lambda)_{\gamma(\delta)} = 0$ then the BEG tree lemma says that m_λ can be written in the form

$$m_\lambda = \sum d'_{\lambda\alpha} m_\alpha \tag{2.96}$$

$$\{\alpha : J_\alpha = J_{\gamma(\delta)} \text{ and } c_{\alpha\beta} = 0 \text{ for every } \beta\epsilon\bar{\gamma}(\delta)\} .$$

Replacing all the m_α in (2.96) by their expansions (2.95), one obtains

$$m_\lambda = \sum_\delta b'_{\lambda\delta} e_\delta \tag{2.97}$$

where

$$b'_{\lambda\delta} = \sum d'_{\lambda\alpha} b'_{\alpha\delta} \tag{2.98}$$

$$\{\alpha : J_\alpha = J_{\gamma(\delta)} \text{ and } c_{\alpha\beta} = 0 \text{ for every } \beta\epsilon\gamma(\delta)\} .$$

Consider any fixed α. This α contributes to (2.98) only if $b'_{\alpha\delta} \neq 0$. But if $b'_{\alpha\delta} \neq 0$ then some $e_{\delta'}$ equivalent to e_δ must occur in the expansion of m_α. But then the tree $t_{\beta(\alpha,\delta')}$ defined above fig. II.1 satisfies $c_{\alpha\beta(\alpha,\delta')} = 1$, which is (2.66) with δ' in place of δ. But $t_{\beta(\alpha,\delta')}$ is contained in $\gamma(\delta') \equiv \gamma(\delta)$: $t_{\beta(\alpha,\delta')} \subset \gamma(\delta)$. In other words, $\beta(\alpha,\delta') \in \bar{\gamma}(\delta)$. But then α does not belong to $\{\alpha : J_\alpha = J_{\gamma(\delta)}$ and $c_{\alpha\beta} = 0$ for all $\beta\epsilon\bar{\gamma}(\delta)\}$. Hence every term on the right-hand side of (2.98) must vanish. This gives

$$b'_{\lambda\delta} = 0 . \tag{2.99}$$

To complete the proof one must show that

$$b'_{\lambda 1} = 1. \tag{2.100}$$

This follows immediately from (2.84b) and (2.95), together with the fact that $b_{\alpha 1} = 1$ for each α such that $J_\alpha = J_\lambda$.

J'. Difference Formulas

Let Γ_{λ_1} and Γ_{λ_2} be two adjacent cells. Let $\Gamma_{\lambda'}$ and $\Gamma_{\lambda''}$ be the two corresponding boundary cells defined by (2.80). Then (2.82-86) and (2.91) give

$$m_{\lambda_1} - m_{\lambda_2} = [m_{\lambda'}, m_{\lambda''}] , \tag{2.101a}$$

$$M_{\lambda_1} - M_{\lambda_2} = [M_{\lambda'}, M_{\lambda''}] , \tag{2.101b}$$

and

$$A_{\lambda_1} - A_{\lambda_2} = [A_{\lambda'}, A_{\lambda''}] \quad . \tag{2.101c}$$

K'. The Difference Formula $\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2} = \langle 0|[M_{\lambda'}, M_{\lambda''}]|0\rangle$

Let λ_1 and λ_2 label two adjacent cells, and let λ' and λ'' label the associated boundary cells defined in (2.80). Let

$$p' \equiv \{p_j : j\epsilon J'\} \tag{2.102a}$$

and

$$p'' \equiv \{p_j : j\epsilon J''\} \quad . \tag{2.102b}$$

Then (2.92a) and (2.101b) give

$$\tilde{r}_{\lambda_1}(p) - \tilde{r}_{\lambda_2}(p) = \langle 0|[M_{\lambda'}(p'), M_{\lambda''}(p'')]|0\rangle$$

$$= \sum_{p^{int}} \Big[\langle 0|M_{\lambda'}(p')|p^{int}+\rangle\langle p^{int}+|M_{\lambda''}(p'')|0\rangle$$

$$- \langle 0|M_{\lambda''}(p'')|p^{int}+\rangle\langle p^{int}+|M_{\lambda'}(p')|0\rangle\Big] \tag{2.103a}$$

$$= \sum_{p^{int}} \Big[\langle 0|M_{\lambda'}(p')|p^{int}-\rangle\langle p^{int}-|M_{\lambda''}(p'')|0\rangle$$

$$- \langle 0|M_{\lambda''}(p'')|p^{int}-\rangle\langle p^{int}-|M_{\lambda'}(p')|0\rangle\Big] \quad . \tag{2.103b}$$

In these formulas the sum over p^{int} is a sum over the complete set of intermediate in- or out-states. For any p the bras $\langle p\pm|$ and kets $|p\pm\rangle$ are defined by

$$\langle p\pm| \equiv \langle 0| \prod_{j\epsilon J_p} a(p_j, t_j, \pm) \tag{2.104a}$$

$$= \langle 0| \prod_{j\epsilon J_p} c(p_j, t_j, \pm) \tag{2.104b}$$

and

$$|p\pm\rangle = \prod_{j\epsilon J_p} a^{\dagger}(p_j, t_j, \pm)|0\rangle \tag{2.104c}$$

$$= \prod_{j\epsilon J_p} -c(-p_j, -t_j, \pm)|0\rangle \tag{2.104d}$$

where $J_p \equiv \{j : p_j \epsilon p\}$.

Remark The first terms on the right-hand sides of (2.103a) and (2.103b) vanish unless $p(J') = -p(J'')\epsilon\bar{V}^+$, and the second terms vanish unless $p(J'') = -p(J')\epsilon\bar{V}^+$, where $\bar{V}^+$ is the closure of the forward light cone.

L'. The Symbol $\bar{p}$

Let p be any set of arguments p_j. Then $\bar{p}$ is the set generated from p by the substitutions

$$p_j \rightarrow -p_j \tag{2.105a}$$

and

$$t_j \rightarrow -t_j \quad . \tag{2.105b}$$

The substitution (2.105b) means that the particle-type variable t_j is to be replaced by the variable $-t_j$ that specifies the associated antiparticle. This substitution (2.105b) is equivalent to the substitution

$$A_j \rightarrow A_j^\dagger \quad . \tag{2.105c}$$

Comparison of (2.104) with (2.56) shows that $\langle p' + |p'' -\rangle$ is $S(p)$ with the arguments (p) taken to be $(p',\bar{p}'')$:

$$\langle p' + |p'' -\rangle = S(p',\bar{p}'') \equiv S(p) \quad . \tag{2.105d}$$

M'. Formulas for $\langle p^{int} \pm|M_{\lambda'}(p')|0\rangle$ and $\langle 0|M_{\lambda'}(p')|p^{int} \pm\rangle$

Theorem For any cell index λ' and any p' such that $J_{p'} = J_{\lambda'}$

$$\langle p^{int} \pm |M_{\lambda'}(p')|0\rangle = \langle 0|M_{\lambda'\pm}(p^{int},p')|0\rangle = \tilde{r}_{\lambda'\pm}(p^{int},p') \tag{2.106a}$$

and

$$\langle 0|M_{\lambda'}(p')|p^{int} \pm\rangle = \langle 0|M_{\lambda'\pm}(p',\bar{p}^{int})|0\rangle = \tilde{r}_{\lambda'\pm}(p',\bar{p}^{int}) , \tag{2.106b}$$

where λ'^{+} and λ'^{-} label the cells $\Gamma_{\lambda'+}$ and $\Gamma_{\lambda'-}$ defined by the signs

$$\sigma(J \cup Int,\lambda'^{\pm}) = \sigma(J,\lambda'^{\pm}) = \sigma(J,\lambda') = -\sigma(J' - J,\lambda') \tag{2.107a}$$

and

$$\sigma(Int,\lambda'^{\pm}) = -\sigma(J',\lambda'^{\pm}) = \pm \quad . \tag{2.107b}$$

Here J is any nonempty proper subset of $J' \equiv J_{\lambda'} \equiv J_{p'}$, and Int is any nonempty subset of $J_{p^{int}} \equiv J^{int}$. Clearly, $J_{\lambda'\pm} = J' \cup J^{int}$. The primes on λ' and p' are placed there merely to help with the substitution of these formulas into (2.103). These two symbols could be replaced by any others, for example, the λ'' and p'' of (2.103)

N'. The Functions $\tilde{f}'_\beta$ and $\tilde{r}'_\lambda$

Definition of $\tilde{r}'_\lambda(p)$

$$\tilde{r}_\lambda(p) \equiv (2\pi)^4 \delta^4(\sum p_j)\, \tilde{r}'_\lambda(p) \quad . \tag{2.108}$$

This is the λ analog of the definition (2.47b) of $\tilde{r}'_\alpha(p)$.

Theorem For each tree t_β there is a tree function $\tilde{f}'_\beta(k)$ that is analytic in the cone $\tilde{C}'_\beta$, and such that its boundary value

$$\tilde{f}'_\beta(p) = \lim_{\substack{q\in\tilde{C}_\beta \\ q\to 0}} \tilde{f}'_\beta(p + iq) \quad , \tag{2.109a}$$

satisfies for all λ

$$\tilde{r}'_\lambda(p) = \sum_\beta c_{\lambda\beta}\, \tilde{f}'_\beta(p) \quad . \tag{2.109b}$$

Definition

$$\tilde{r}'_\lambda(k) \equiv \sum_\beta c_{\lambda\beta}\, \tilde{f}'_\beta(k) \quad . \tag{2.110}$$

Definition [See (2.76) and (2.52).]

$$\Gamma'_\lambda \equiv \bigcap_{\beta:c_{\lambda\beta}=1} \tilde{C}'_\beta \quad . \tag{2.111}$$

Remark From (2.109-111) it follows that

(1) $\tilde{r}'_\lambda(k)$ is analytic in Γ'_λ (2.112a)

and that

$$(2)\quad \tilde{r}'_\lambda(p) = \lim_{\substack{q\in\Gamma_\lambda \\ q\to 0}} \tilde{r}'_\lambda(p+iq) . \tag{2.112b}$$

Remark The momentum-space properties (2.109) follow from the existence for each t_β of an operator $F_\beta(x)$ satisfying

$$(1)\quad F_\beta(x) = 0 \qquad \text{for } x\notin C_\beta \tag{2.113a}$$

and for all λ

$$(2)\quad A_\lambda(x) = \sum_\beta c_{\lambda\beta} F_\beta(x) . \tag{2.113b}$$

Q'. The Functions $\tilde{r}'(k)$ and $\tilde{r}(k)$

Definition The functions $\tilde{r}'(k)$ and $\tilde{r}(k)$ are defined in $\bigcup_\lambda \Gamma'_\lambda$ by

$$\tilde{r}'(k) \equiv \tilde{r}'_\lambda(k) \qquad \text{for } k \text{ in } \Gamma'_\lambda \tag{2.114}$$

and

$$\tilde{r}(k) \equiv (2\pi)^4 \delta^4\left(\sum_{j:k_j\epsilon k} k_j\right) \tilde{r}'(k) . \tag{2.115}$$

Remark The delta function of complex arguments can be given a well-defined and sensible meaning. The only property of $\delta^4(\sum k_j)$ needed here is that it goes over to $\delta^4(\sum p_j)$ as one takes the limit $k_j \to p_j$. In fact, the function $\tilde{r}(k)$ will be used here only to express in a more compact form equations that can be equally well written in terms of $\tilde{r}'(k)$ by making certain obvious adjustments.

P'. The Basic Discontinuity Formula

From (2.102-115) one obtains for $p\epsilon P(J_{\lambda_1}) = P(J_{\lambda_2})$ and $p(J')\epsilon \mathcal{C}\bar{V}^-$

$$\lim_{\substack{q\epsilon\Gamma_{\lambda_1} \\ q\to 0}} \tilde{r}(p+iq) - \lim_{\substack{q\epsilon\Gamma_{\lambda_2} \\ q\to 0}} \tilde{r}(p+iq)$$

$$= \sum_{p^{int}} \tilde{r}_{\lambda'+}(p',\bar{p}^{int})\, \tilde{r}_{\lambda''+}(p^{int},p'') \tag{2.116a}$$

$$= \sum_{p^{int}} \tilde{r}_{\lambda'-}(p',\bar{p}^{int})\, \tilde{r}_{\lambda''-}(p^{int},p'') , \tag{2.116b}$$

where the sums over p^{int} in (2.116a) and (2.116b) correspond to sums over complete sets of intermediate out- and in-states, respectively. The cells labelled by $\lambda'^{\pm}$ and $\lambda''^{\pm}$ are defined by (2.80) and (2.107).

Q'. The Single Analytic Function $\tilde{r}'(k) \equiv S'_c(k)$

If the masses m_j have a positive lower bound $\bar{m}$, $m_j \geq \bar{m} > 0$, then all the discontinuities (2.116) vanish near the off-mass-shell point $p = 0$. Consequently, the function $\tilde{r}'(k)$ can be extended to a function [also called $\tilde{r}'(k)$] that is analytic in $K(J_k) \cap [(\bigcup \Gamma'_\lambda) \cup n(J_k)]$ where $J_k = \{j : k_j \epsilon k\}$ and $n(J_k)$ is some neighborhood of the origin in the space $K(J_k)$. If $S'(p)$ is defined by $(2\pi)^4 \delta^4(\sum p_j)\, S'(p) = S(p)$ then (2.93), (2.108), and (2.112b) show that for $p\epsilon P_\lambda \cap \mathcal{M}$ the function $S'(p)$ is the limit of $\tilde{r}'(k)$ from points $k\epsilon\Gamma'_\lambda$.

Remark The S-matrix $S'(p)$ generally has disconnected parts, which are terms that contribute only on the surfaces $p(J) = 0$, where J ranges over the nonempty property subsets of the set $J_p \equiv \{j : p_j \in p\}$. The union of the $P_\lambda \subset P(J_p)$ contains no point on any of these surfaces. Thus the connection between $S'(p)$ and $\tilde{r}'(k)$ described above holds only at limit points p where all the disconnected parts of $S'(p)$ vanish. The function $\tilde{r}'(k)$ can thus be regarded as the analytic continuation of the connected part of $S'(p)$. This connected part is denoted by $S'_c(p)$, and is called the physical scattering function. Its connection to $\tilde{r}'(k)$ is recorded by the definition

$$S'_c(k) \equiv \tilde{r}'(k) \quad . \tag{2.117a}$$

The subscript c, which stands for connected part, is essential here; the analytic function $\tilde{r}'(k)$ does not contain any contribution from the disconnected parts of $S'(k)$. This is because any contribution to $A_\alpha(x)$ that is invariant under the translation of some of the variables x_j relative to the others, and satisfies the support condition (2.44), is identically zero, as one sees by considering a large spacelike translation. Alternatively, in momentum space the disconnected parts must have extra delta-function factors if the contributions from the sets $p(J) = 0$, which have measure zero, are to contribute. But delta functions cannot occur in the analytic function $\tilde{r}'(k)$.

The relation (2.117a) can be written in the alternative form

$$S_c(k) \equiv \tilde{r}(k) \quad . \tag{2.117b}$$

R'. Hermitian Analyticity

Definition

$$S(p'; p'') \equiv \langle p' + | p'' - \rangle$$

$$\equiv \langle 0 | (\Pi a'^{+}_j)(\Pi a''^{-}_j)^\dagger | 0 \rangle \tag{2.118}$$

where

$$(\Pi a'^{\pm}_j) \equiv \prod_{j : p'_j \in p'} a(p'_j, t_j, \pm)$$

and

$$(\Pi a''^{\pm}_j)^\dagger \equiv \left[\prod_{j : p''_j \in p''} a(p''_j, t_j, \pm) \right]^\dagger \quad .$$

Definition

$$S^\dagger(p'; p'') \equiv S^*(p''; p') \equiv S(p''; p')^*$$

$$= [\langle 0 | (\Pi a''^{+}_j)(\Pi a'^{-}_j)^\dagger | 0 \rangle]^*$$

$$= \langle 0 | (\Pi a'^{-}_j)(\Pi a''^{+}_j)^\dagger | 0 \rangle$$

$$\equiv \langle p' - | p'' + \rangle \quad . \tag{2.119}$$

Remark Equations (2.118-119) and (2.105d) give

$$S(p) \equiv S(p', \bar{p}'') = S(p'; p'') \quad . \tag{2.120}$$

Definition

$$\Gamma_{-\lambda} \equiv \{q : -q \in \Gamma_\lambda\} \quad . \tag{2.121}$$

Theorem (Hermitian Analyticity) Suppose p lies in $P_\lambda \cap \mathcal{M}$, so that

$$\tilde{r}_\lambda(p) = S_c(p) = S(p'; p'')$$

$$= \langle p' + | p'' - \rangle \ . \tag{2.122}$$

Then

$$\tilde{r}_{-\lambda}(p) = - S^\dagger(p'; p'')$$

$$= - \langle p'' + | p' - \rangle^*$$

$$= - \langle p' - | p'' + \rangle \ . \tag{2.123}$$

Proof It is sufficient, as will be shown later, to prove the more general hermitian-analyticity property

$$i\, S_c(k'; k'') = [i\, S_c(k''^*; k'^*)]^* \equiv [i\, S_c(k'^*; k''^*)]^\dagger, \tag{2.124}$$

where $S_c(k'; k'') \equiv S_c(k', \bar{k}'') \equiv S_c(k) \equiv \tilde{r}(k)$ and where $\bar{k} \equiv \bar{p} + i\bar{q}$ with $\bar{q}_j \equiv -q_j$. [The name "hermitian analyticity" for this property arises from its similarity to the "real analyticity" property

$$f(z) = [f(z^*)]^* \ . \tag{2.124'}$$

The operation of complex conjugation appearing on the right-hand side of (2.124b) is replaced in (2.124a) by hermitian conjugation.]

Note that

$$\tilde{r}(\bar{k}) = S_c(\bar{k}', k'') = S_c(k'', \bar{k}') = S_c(k''; k') \tag{2.125}$$

since the order of the variables of $S_c(k)$ is immaterial (for bosons). Thus equation (2.124) can be written in the more compact form

$$\tilde{r}(k) = - [\tilde{r}(\bar{k}^*)]^* \ . \tag{2.126}$$

Note that

$$\text{Im } k = q = \text{Im } \bar{k}^* \ . \tag{2.127}$$

Thus if Im k lies in Γ_λ, so will Im $\bar{k}^*$. But then, since both sides of (2.126) are analytic functions of k in each cell Γ_λ, it is sufficient to show, for some λ and for all k in Γ'_λ, that

$$\tilde{r}_\lambda(k) = - [\tilde{r}_\lambda(\bar{k}^*)]^* \ . \tag{2.128}$$

For cases in which $\lambda = \lambda(\alpha) = \alpha$, the relation (2.128) follows directly from the identity

$$M_\alpha(k) = - [M_\alpha(\bar{k}^*)]^\dagger \ , \tag{2.129}$$

which holds for all k in Γ'_α, together with the formula (2.54b),

$$\tilde{r}_\alpha(k) = \langle 0 | M_\alpha(k) | 0 \rangle \ . \tag{2.130}$$

The operator $M_\alpha(k)$ is defined for k in Γ'_α by (2.54a),

$$M_\alpha(k) \equiv \int \prod_{j \in J_\alpha} (i\, d^4 x_j\, e^{ik_j x_j} K_j)\, A_\alpha(x) \ . \tag{2.131}$$

The identity (2.129) follows from (2.131) together with the fact that the transformation $t_j \to - t_j$ changes A_j to $A_j^\dagger$ [see (2.105c)]. This change is cancelled by the operation of hermitian conjugation appearing on the right-hand side of (2.129). That operation also reverses the order of the multiple commutator, which brings in (n - 1) sign changes. The complex conjugation of the n factors i in (2.31) brings in n sign changes. The factor (-1) in (2.129) is the result of these (2n - 1) sign changes. The exponents $ik_j x_j$ are unchanged.

By analytic continuation the validity of (2.128) for one value of λ ensures its validity for all values of λ, and also the validity of (2.126).

Letting $\text{Im } k \to 0$ one obtains from (2.128)

$$\tilde{r}_{\lambda'}(p) = -[\tilde{r}_{\lambda'}(\bar{p})]^* , \qquad (2.132a)$$

for every λ'. If p lies in $P_\lambda \cap \mathcal{M}$ (and hence $\bar{p}$ lies in $P_{-\lambda} \cap \mathcal{M}$), then (2.132a), with $\lambda' = -\lambda$, together with (2.93), (2.120b), and (2.125), give the desired result

$$\tilde{r}_{-\lambda}(p) = -[\tilde{r}_{-\lambda}(\bar{p})]^* = -[S(\bar{p})]^* = -S^\dagger(p) . \qquad (2.132b)$$

Remark The above proof makes use of the operations of complex conjugation, hermitian conjugation, and antiparticle conjugation ($t_j \to -t_j$). However, (2.123) can be derived without using any of these operations. Comparison of the final form of (2.123), namely

$$\tilde{r}_{-\lambda}(p) = -\langle p' - | p'' + \rangle ,$$

with

$$\tilde{r}_\lambda(p) = \langle p' + | p'' - \rangle$$

shows that (2.123) is simply the statement that the substitution $c_j^\pm \to c_j^\mp$ reverses all signs $\sigma(J,\lambda)$ and brings in one overall minus sign, and hence converts $r_\lambda(p)$ to $-r_{-\lambda}(p)$. Inspection of (2.29), (2.31), (2.37), (2.52), (2.53), and (2.121) shows that this is the case for $\lambda = \lambda(\alpha) = \alpha$. It is not hard to extend this result to the general case.

S'. The Functions $S^+(p)$ and $S^-(p)$

Definitions

$$S^+(p) \equiv S_c(p'; p'') \equiv$$ p′  p″ (2.133a)

and

$$S^-(p) \equiv -S_c^\dagger(p'; p'') \equiv$$ p′  p″ . (2.133b)

Remark If λ labels the physical cell, then $S^+(p) = \tilde{r}_\lambda(p)$ and $S^-(p) = \tilde{r}_{-\lambda}(p)$. Thus $S^+(p)$ is the physical boundary value of $\tilde{r}(k)$ and $S^-(p)$ is the boundary value from the cell that lies opposite the physical cell.

Remark In what follows the functions $S^+(p)$ and $S^-(p)$ will generally be represented by the plus and minus bubbles shown in (2.133). The lines on the left- and right-hand sides of these bubbles correspond to the variables in p' and p'', respectively.

T'. Two Forms of the Steinmann Relations

The Steinmann relations were defined in subsection J. That abstract definition can be converted into various equivalent concrete statements. Two of these alternative statements are described here.

1. The BEG Form

To represent the symbol α defined in (2.24), let the extra sign $\sigma(0,\alpha) \equiv -\sigma(1,\alpha)$ be added. Let $J_\alpha^\pm \equiv \{ j \in J_\alpha : j = j(i,\alpha)$ and $\sigma(i,\alpha) = \pm$ for some $0 \leq i \leq n(\alpha) \}$. Then BEG show that the Steinmann relations as defined in subsection J, allow any ρ_α to be expressed in the form

$$\rho_\alpha = \sum_{\alpha'} b(\alpha,\alpha')\, \rho_{\alpha'} \qquad (2.134a)$$

where the $b(\alpha,\alpha')$ are positive or negative integers, or zero, and the $\rho_{\alpha'}$ are restricted by the following three conditions:

(1) $J^+_{\alpha'} = J^+_\alpha$ and $J^-_{\alpha'} = J^-_\alpha$;

(2) $j(0,\alpha') = j_0$ for every α' occurring in (2.134a) where j_0 is a fixed but arbitrary element of J_α;

(3) for any prescribed ordering of the indices, $j \epsilon J_\alpha$, one has $j(i+1,\alpha') > j(i,\alpha')$ if $\sigma(i+1,\alpha') = \sigma(i,\alpha')$.

Reference to equations (2.91 a-c) shows that for every instance of (2.134a) one has as well the relations

$$m_\alpha = \sum_{\alpha'} b(\alpha,\alpha')\, m_{\alpha'} \ , \qquad (2.134b)$$

$$M_\alpha = \sum_{\alpha'} b(\alpha,\alpha')\, M_{\alpha'} \ , \qquad (2.134c)$$

and

$$A_\alpha = \sum_{\alpha'} b(\alpha,\alpha')\, A_{\alpha'} \ , \qquad (2.134d)$$

with exactly the same coefficients $b(\alpha,\alpha')$ as in (2.134a).

In what follows, it will be (2.134b) that will be of principal interest. It may be noted that for $n(\alpha) = n(\alpha') > 1$ the first element $c_{j(0,\alpha)}$ in any nested commutator m_α may be replaced by $\sigma(0,\alpha)\, c^{\sigma(0,\alpha)}_{j(0,\alpha)}$ without altering m_α. For this reason, (2.134b) may be expressed as follows: any nested commutator m_α of c^+_i's and c^-_j's can be expanded as a linear combination of nested commutators

$m_{\alpha'}$ built from the same c^+_i's and c^-_j's, but with any specified factor c^+_i or c^-_j occurring first, and with the c^+_i's and c^-_j's that lie in contiguous groups of the same sign ordered in any prescribed fashion.

For each set of $c^\pm_j$ one can prescribe a fixed set of basis elements $m_{\alpha'}$ for the expansion all m_α that are constructed as products, in various orders, of this set of $c^\pm_j$'s. The Steinmann relations impose no relations among these basis elements $m_{\alpha'}$. Thus the expansion of all m_α in terms of these basis elements $m_{\alpha'}$ displays all the Steinmann relations: a linear combination of m_α's is zero by virture of the Steinmann relations if and only if its expansion in terms of the basis elements $m_{\alpha'}$ is identically zero (i.e., if and only if the coefficient of each $m_{\alpha'}$ is zero).

2. Discontinuity Form

Equation (2.116) is a formula for the difference of the boundary values of $\tilde{r}(k)$ taken from two adjacent cells Γ_{λ_1} and Γ_{λ_2}. These two cells, when restricted to energy space, are separated by the plane $q^0(J') = q^0(J'') = 0$. The difference $\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2}$ between these two boundary values will be called the discontinuity across the cut $q^0(J') = 0$.

Equation (2.116) implies the following property:

Steinmann Discontinuity Property (SDP) The discontinuity across the cut $q^0(J') = 0$ is independent of the sign $\sigma(J,\lambda_1) = \sigma(J,\lambda_2)$ if J and J' overlap. The two sets J and J' overlap if and only if $J \cap J'$, $\hat{J} \cap J'$, $J \cap \hat{J}'$, and $\hat{J} \cap \hat{J}'$ are all nonempty. Here $\hat{J} \equiv J_\lambda - J$, $\hat{J}' \equiv J_\lambda - J' = J''$, and $J_\lambda \equiv J_{\lambda_1} \equiv J_{\lambda_2} \equiv J' \cup J''$.

This property follows from (2.116) because the right-hand side of (2.116) depends on a sign $\sigma(J,\lambda_1) = \sigma(J,\lambda_2)$ only if J or $\hat{J}$ is a proper subset of either J' or J''. But the condition that J and J' overlap implies that neither J nor $\hat{J}$ is a proper subset of either J' or J''.

The Steinmann discontinuity property stated here is equivalent to the one stated in the introduction, within the general framework of BEG field theory. The only difference is that the limits $\tilde{r}_\lambda$ are now allowed to be taken from Γ'_λ, instead of from the more restricted region $\Gamma_\lambda^{0'}$.

The following result, which is implicit in the reconstruction theorems of Araki and Ruelle (6), is proved in ref. (11).

Theorem The Steinmann discontinuity property is equivalent to the BEG form of the Steinmann relations.

U'. Trivial Cuts

It was mentioned in the introduction that the discontinuities across certain cuts vanish identically on mass shell, due to stability conditions and spectral conditions. On the mass shell the formulas (2.32) and (2.86) allow the discontinuity formula (2.103) to be written as

$$\tilde{r}_{\lambda_1}(p) - \tilde{r}_{\lambda_2}(p) = \langle 0|[m_{\lambda'}(p'),m_{\lambda''}(p'')]|0\rangle \quad . \tag{2.135}$$

One kind of vanishing cut is the kind labelled by a set $J_{\lambda'}$ that consists of only one element,

$$J_{\lambda'} = \{j\} \quad . \tag{2.136a}$$

In this case (2.135) becomes

$$\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2} = \langle 0|[c_j,m_{\lambda'}(p'')]|0\rangle \quad . \tag{2.136b}$$

This vanishes due to the conditions

$$c_j|0\rangle = 0 = \langle 0|c_j \quad , \tag{2.137}$$

which follow from (2.23d), and the spectral conditions. Condition (2.137) expresses the stability of one-particle states: the in-state is the same as the out-state. The case $J_{\lambda''} = \{j\}$ gives null discontinuity by the same argument.

The second trivial case is the one in which $J_{\lambda'}$ consists of one initial-particle index and one final-particle index,

$$J_{\lambda'} = \{i,f\} \quad . \tag{2.138}$$

There are two possibilities, which correspond to the two signs in

$$m_{\lambda'} = [c_i^{\pm},c_f] \quad . \tag{2.139}$$

The spectral conditions and energy conditions (2.55) give

$$\langle 0|c_i^{\pm} = 0 = c_f^{\pm}|0\rangle \quad . \tag{2.140}$$

The first equation, together with (2.137), gives

$$\langle 0|[c_i^{\pm},c_f] = 0 \tag{2.141}$$

The second equation, together with (2.137) and the identity

$$[c_j^{+},c_{j'}^{+}] = [c_j^{-},c_{j'}^{-}], \tag{2.142}$$

gives

$$[c_i^{\pm},c_f]|0\rangle = 0 \quad . \tag{2.143}$$

Equation (2.142) and the definition $c_j = c_j^{+} - c_j^{-}$ give the important identity

$$[c_j^{\pm},c_{j'}] = [c_{j'}^{\mp},c_j] \quad . \tag{2.144}$$

Equations (2.137) and (2.104) give

$$\langle 0|[c_{f'}^{\pm},c_{f''}] = \langle p_{f'}\ p_{f''}\ +| - \langle p_{f'}\ p_{f''}\ -| \tag{2.145a}$$

and

$$[c_{i'}^{\pm},c_{i''}]|0\rangle = |p_{i'}p_{i''}\ -\rangle - |p_{i'},p_{i''}\ +\rangle \quad . \tag{2.145b}$$

Thus (2.133) gives

$$\langle 0|[c_{f'}^{\pm},c_{f''}]|p\ \pm\rangle = S^{\mp}(p_{f'},p_{f''};\ p) \tag{2.146a}$$

and

$$\langle p \pm |[c^{\pm}_{i'}, c_{i''}]|0\rangle = S^{\pm}(p;\, p_{i'}, p_{i''}) \quad , \tag{2.146b}$$

since for cases with two particles in the initial or final state the connected part of the S matrix is $S_c = S - I$. These formulas are used in section IV.

V'. Index

III. THE 2282 ZONES

The formalism of section II applies to processes with arbitrary numbers of particles. It is now applied to a process of the form $1 + 2 + 3 \to 4 + 5 + 6$. This process is described by an S matrix $S(p)$, where $p = (p_1, \cdots, p_6)$, and, in accord with (2.55),

$$p_j^{\,0} < 0 \quad \text{for} \quad j \epsilon \{1,2,3\} \equiv I \tag{3.1a}$$

and

$$p_j^{\,0} > 0 \quad \text{for} \quad j \epsilon \{4,5,6\} \equiv F \ . \tag{3.1b}$$

Here

$$I \cup F \equiv J_p = \{1, \cdots, 6\} \equiv J_6 \ . \tag{3.1c}$$

For each nontrivial cut g there is a nonempty proper subset J_g of J_6 such that the cut g is

$$q^0(J_g) = q^0(\hat{J}_g) = 0 \ , \tag{3.2a}$$

where

$$\hat{J}_g \equiv J_6 - J_g \ . \tag{3.2b}$$

The set of indices g that label the 16 nontrivial cuts is denoted by E. This set consists of the elements $\overline{t}$, $\overline{i}$, $\overline{f}$, and $\overline{fi}$, where i is 1, 2, or 3, and f is 4, 5, or 6. The corresponding J_g are

$$J_{\overline{t}} \equiv \{4,5,6\} \equiv F \ , \tag{3.3a}$$

$$J_{\overline{i}} \equiv \{4,5,6\} + \{i\} \ , \tag{3.3b}$$

$$J_{\overline{f}} \equiv \{4,5,6\} - \{f\} \ , \tag{3.3c}$$

and

$$J_{\overline{fi}} \equiv \{4,5,6\} - \{f\} + \{i\} \ . \tag{3.3d}$$

With the sixteen sets J_g chosen in this way there is a cell $\Gamma_\lambda^{\,0}$ such that every $\sigma(J_g, \lambda)$ is positive. This cell lies "above" every cut $g \epsilon E$. This cell is equivalent to the physical cell $\Gamma^0_{\lambda(p)}$ in the sense that it has the same boundary value $\tilde{r}_\lambda$ as $\Gamma^0_{\lambda(p)}$. For if $p^0(J_g)$ is nonpositive for any $g \epsilon E$ then the discontinuity across cut g vanishes by virtue of the stability conditions (2.137) and (3.1).

The 16 cuts divide the 5 dimensional space $Q(J_6)$ into 2282 regions called zones. Each zone corresponds to a set of 16 signs $\sigma(J_g)$ such that the 16 conditions

$$\sigma(J_g)\, q^0(J_g) > 0$$

are satisfied for all points q^0 in the zone. In this section the 2282 combinations of sixteen signs $\sigma(J_g)$ that correspond to zones are exhibited.

The sixteen indices $g \epsilon E$ can be arranged in a four-by-four box in the manner shown in fig. III.1.

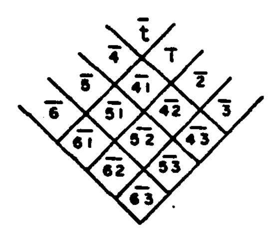


Fig. III.1. Arrangement of the sixteen indices g.

The combinations of signs $\sigma_g \equiv \sigma(J_g)$ that correspond to zones are shown in fig. III.2.

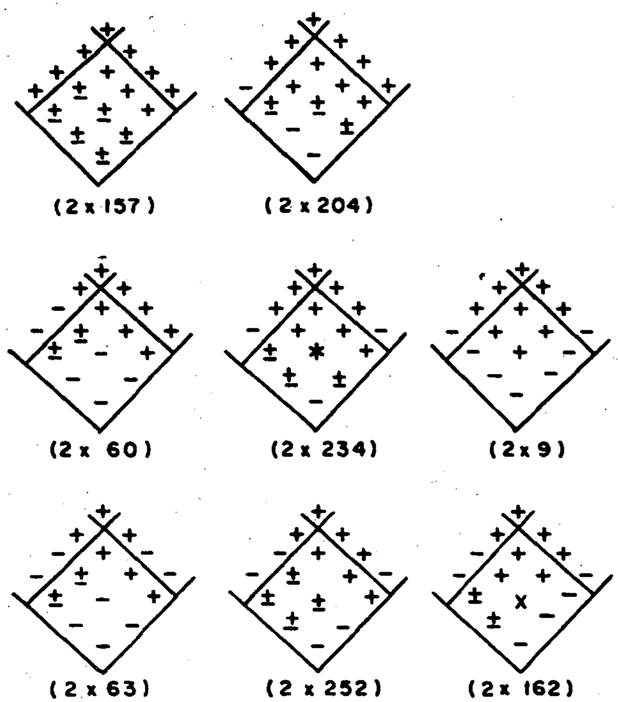


Fig. III.2. The signs σ_g that correspond to zones.

The $\pm$ signs occurring in fig. III.2 are variable signs: they can be chosen to be either plus or minus, subject to the condition that

$$\sigma_{\overline{if}} \leqslant \sigma_{\overline{i'f'}} \qquad \text{for} \quad \bar{i} \geqslant \bar{i}' \quad \text{and} \quad \bar{f} \geqslant \bar{f}' \,. \tag{3.4}$$

That is, if any sign σ represented in fig. III.2 by $\pm$ is chosen to be minus, then all the $\pm$ signs that can be reach by moving downward from σ along the rows and columns must also be minus. For example, if the topmost $\pm$ sign in the top left-hand diagram of fig. III.2 is chosen to be minus, then all the $\pm$ signs in that box must be chosen to be minus.

The two symbols $*$ and $\times$ represent variable signs $\pm$ that are subject, however, to the additional conditions that $*$ cannot be positive if all other variable signs are positive, and $\times$ cannot be negative if all other variable signs are negative.

Each combination of signs σ_g shown in fig. III.2 corresponds to a zone. The other sets of signs σ_g that correspond to zones are those that can be obtained from one of the sets shown by a combination of one or more of the following operations:

permutation of the indices $i \varepsilon I$, (3.5a)

permutation of the indices $f \varepsilon F$, (3.5b)

reflection of the diagram across the vertical axis, (3.5c)

reversal of all the signs . (3.5d)

The number of combinations of signs represented by each diagram is written below it, with the factor two coming from the reversal of all signs explicitly displayed. The total number of allowed combinations is $2 \times 1141 = 2282$.

The results described above were established by first fixing $\sigma(\bar{t}) = +$, then ordering the i and f so that $q_6{}^0 \geqslant q_5{}^0 \geqslant q_4{}^0$ and $-q_3{}^0 \geqslant -q_2{}^0 \geqslant -q_1{}^0$, and then eliminating all the combinations not shown by use of the identities

$$q^0(\bar{i}) + q^0(\bar{f}) = q^0(\bar{t}) + q^0(\overline{if}) \quad , \tag{3.6a}$$

$$q^0(\overline{if}_1) + q^0(\overline{if}_2) + q^0(\bar{f}_3) = 2q^0(\bar{i}) \quad , \tag{3.6b}$$

$$q^0(\overline{i_1f_1}) + q^0(\overline{i_2f_2}) + q^0(\bar{i}_3) + q^0(\bar{f}_3) = 2q^0(\bar{t}) \quad , \tag{3.6c}$$

$$q^0(\overline{i_1f_1}) + q^0(\overline{i_2f_2}) + q^0(\overline{i_3f_3}) = q^0(\bar{t}) \quad , \tag{3.6d}$$

and

$$q^0(\overline{i_1f_2}) + q^0(\overline{i_1f_3}) + q^0(\overline{i_2f_1}) + q^0(\overline{i_3f_1}) = q^0(\bar{i}_1) + q^0(\bar{f}_1) \quad . \tag{3.6e}$$

[In this equation, and here alone, the notation $q^0(g) \equiv q^0(J_g)$ is used.] Then it was checked that each of the remaining combinations of signs were mutually compatible by exhibiting a solution to the set of 16 simultaneous equations $\sigma_g \, q^0(J_g) > 0$.

IV. GENERALIZED OPTICAL THEOREMS

In this section formulas are derived that express in terms of physical scattering functions the discontinuity of any 3-to-3 scattering function across any basic cut. The basic cuts are cuts in the energy space that are confined to the planes (1.1). These planes

divide the space $Q^0(J_6)$ into the 2282 zones catalogued in section III. Two zones are adjacent if they lie on the same side of every plane (1.1) but one. The discontinuities to be derived are the differences of the boundary values corresponding to two adjacent zones. Each zone contains several cells. All the cells $\Gamma_\lambda{}^0$ in a given zone have the same boundary value $\tilde{r}_\lambda$. Thus the discontinuity defined as the boundary value from zone Z_1 minus the boundary value from Z_2 can be expressed as $\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2}$, where $\Gamma_{\lambda_1}{}^0 \subset Z_1$ and $\Gamma_{\lambda_2}{}^0 \subset Z_2$.

The difference $\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2}$ can be expressed in terms of physical scattering functions by taking the formula (2.135)

$$\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2} = \langle 0 | [m_{\lambda'}, m_{\lambda''}] | 0 \rangle$$

and inserting between the operators $c_j^{\pm}$ and c_j that occur in the terms of $m_{\lambda'}$ and $m_{\lambda''}$ appropriate complete sets of in- and out-states. The main task is to determine the explicit forms of the operators $m_{\lambda'}$ and $m_{\lambda''}$ that correspond to a given set of adjacent zones. This will be done by first finding the corresponding groves $\rho_{\lambda'}$ and $\rho_{\lambda''}$. In the 3-to-3 case these groves are groves $\rho_{\alpha'}$ and $\rho_{\alpha''}$, and hence the operators $m_{\lambda'}$ and $m_{\lambda''}$ are the nested commutators $m_{\alpha'}$ and $m_{\alpha''}$.

The discontinuities across final subenergy cuts are considered first. Inspection of the first two diagrams of fig. III.2 reveals a difference in the final subenergy sign $\sigma_{\bar{6}} \equiv \sigma(J_{\bar{6}}) \equiv \sigma(\{4,5\})$. These two signs can be identified with the two signs $\sigma(g,\lambda_1) = +$ and $\sigma(g,\lambda_2) = -$. Then the condition that Z_1 and Z_2 be adjacent requires all other signs in the two diagrams to be pairwise equal.

If $\sigma(J_{\overline{6I}})$ is fixed to be minus, then there are, in view of the sign restriction (3.4), just three remaining possibilities, and these are shown on the left-hand side of fig. IV.1a.

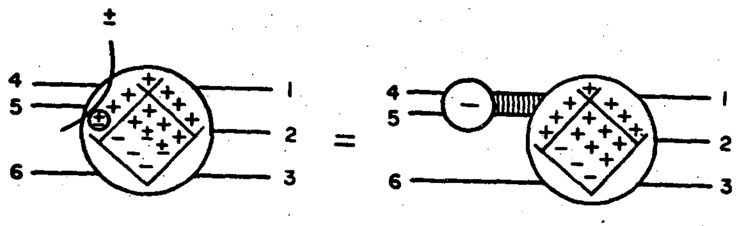


Fig. IV.1a. The basic discontinuity equation for case 1.

On the left-hand side of fig. IV.1a the ± sign on the looping line that cuts off lines 4 and 5 indicates a discontinuity across the cut $q^0(\{4,5\}) \equiv q^0(J_{\bar{6}}) = 0$. This discontinuity is defined to be the function evaluated at $q^0(J_{\bar{6}}) = 0+$ minus the function evaluated at $q^0(J_{\bar{6}}) = 0-$. The circled ± sign occurring in the $\bar{6}$ position of the matrix of signs inside the bubble on the left-hand side is this same ± sign. The other signs in this bubble are the pairwise-equal signs $\sigma_g = \sigma(g,\lambda_1) = \sigma(g,\lambda_2)$. The two ± signs, $\sigma(\{5,2\})$ and $\sigma(\{5,3\})$, can be independently fixed to be either plus or minus, subject to (3.4).

The equation represented by fig. III.1a is an instance of the basic discontinuity equation (2.116a) except that the signs of the trivial cuts are not specified. In this instance the intermediate states are the out or plus states. Thus the functions represented by the small and large bubbles on the right-hand side of fig. IV.1a are

$\tilde{r}_{\lambda'+}$ and $\tilde{r}_{\lambda''+}$, respectively. Equation (2.146a) says that the quantity $\tilde{r}_{\lambda'+}$ is $S^-(p_{f'}p_{f''};\ p^{int})$. This quantity is represented by the small minus bubble on the right-hand side of fig. IV.1a, in accordance with (2.133).

For the large bubble on the right-hand side of fig. IV.1a the ordering conventions of fig. III.1 are used. The set of intermediate lines can be considered to be divided into two nonempty sets, which are associated with lines 4 and 5 of fig. III.1. This representation is adequate because the signs determined by rules (2.107) treat all separations of the set of intermediate lines into two such sets in the same way: the particular content of these two sets is not important.

The signs in the large plus bubble are the signs $\sigma(J_g,\lambda''+)$ associated with $\Gamma_{\lambda''+}$. They are determined by (2.80), (2.107), and (3.3):

$$\sigma(J_{\bar{1}},\lambda''+) = \sigma(J_{\bar{1}},\lambda) = + , \tag{4.1a}$$

$$\sigma(J_{\overline{6I}},\lambda''+) = \sigma(J_{\overline{6I}},\lambda) = - , \tag{4.1b}$$

$$\sigma(J_{\bar{6}},\lambda''+) = + , \tag{4.1c}$$

$$\sigma(J_{\overline{4I}},\lambda''+) = \sigma(J_{\bar{1}},\lambda) = + , \tag{4.1d}$$

$$\sigma(J_{\overline{5I}},\lambda''+) = \sigma(J_{\bar{1}},\lambda) = + , \tag{4.1e}$$

$$\sigma(J_{\bar{4}},\lambda''+) = \sigma(J_{\bar{t}},\lambda) = + , \tag{4.1f}$$

and

$$\sigma(J_{\bar{5}},\lambda''+) = \sigma(J_{\bar{t}},\lambda) = + . \tag{4.1g}$$

Note that the signs on the right-hand side of fig. IV.1a do not depend on the variable signs $\pm$ that occur on the left.

The large bubble on the right-hand side of fig. IV.1a represents a function

$$\langle p^{int} + |m_{\lambda''}(p'')|0\rangle = \langle 0|m_{\lambda''+}(p^{int},p'')|0\rangle \quad , \qquad (4.2)$$

where $m_{\lambda''}$ is an operator

$$m_{\lambda''} = \sum_{\alpha} d_{\lambda''\alpha}\, m_{\alpha} \quad . \qquad (4.3)$$

The main task in this section is to explicitly exhibit this operator $m_{\lambda''}$, and the analogous operators for all the other discontinuities. This task is simplified by the fact that each m_{λ} corresponding to a J_{λ} with less than five elements is a nested commutator m_{α}.

The procedure for determining $m_{\lambda''} = m_{\alpha}$ is as follows:

(1) The given sets of signs $\sigma(J_g,\lambda_1)$ and $\sigma(J_g,\lambda_2)$ determine, via (2.80), a set of signs $\sigma(J_g,\lambda'')$. For each such sign $\sigma(J_g,\lambda'')$ there is, according to (2.58), a condition on $\Gamma^0_{\lambda''}$ of the form

$$\sigma(J_g,\lambda'')\, q^0(J_g) > 0 \quad . \qquad (4.4a)$$

By virtue of the conservation-law condition in the definition (2.57) of $Q^0(J_{\lambda''})$ there is also a condition of the form

$$\sigma(J_g,\lambda'')\, q^0(J_{\lambda''} - J_g) < 0 \quad . \qquad (4.4b)$$

These conditions (4.4a) and (4.4b) define an energy region $\Gamma^*_{\lambda''}$ that contains $\Gamma_{\lambda''}{}^0$.

(2) A set τ of trees t_β is found that satisfies both

$$\bigcap_{\beta:t_\beta\epsilon\tau} \tilde{C}_\beta{}^0 \subset \Gamma^*_{\lambda''} \qquad (4.5a)$$

where $\tilde{C}_\beta{}^0$ is the energy section of $\tilde{C}_\beta$, and, for some α,

$$\sum_{\beta:t_\beta\epsilon\tau} t'_\beta = \rho_\alpha \quad . \qquad (4.5b)$$

(3) The operator $m_{\lambda''}$ is identified as m_α,

$$m_{\lambda''} \equiv m_\alpha \quad , \qquad (4.6)$$

where α is the α parameter in (4.5b).

This procedure is used in cases where the set $J_{\lambda''}$ includes all three initial particle indices i or all three final particles indices f. In these cases the set of signs $\sigma(J_g,\lambda'')$ determines all the signs $\sigma(J,\lambda'')$, and hence $\Gamma^*_{\lambda''} = \Gamma_{\lambda''}{}^0$. Thus in these cases the procedure described above determines the unique $m_{\lambda''} = m_\alpha$. [The same conclusion holds with λ'' replaced by λ'.]

The small bubble in the subenergy discontinuity formula is determined by (2.146). This leaves only the case of the cross-energy discontinuity formula. But in the cross-energy cases the basic discontinuity formula [5] already gives the discontinuity in terms of physical scattering functions, and hence no further work is needed for them.

The procedure outlined above is now carried out in detail for case 1, which is the case shown in fig. IV.1a. In this case, the eqs. (2.80) and (3.3) give

$$\sigma(\{6\},\lambda'') = -\sigma(J'' - \{6\},\lambda'') = \sigma(\bar{\tau},\lambda_1) = + \qquad (4.7a)$$

$$\sigma(\{i\},\lambda") = -\sigma(J" - \{i\},\lambda") = \sigma(\overline{6i},\lambda_1) = - \ , \quad (4.7b)$$

and

$$\sigma(\{6,i\},\lambda") = -\sigma(J" - \{6,i\},\lambda") = \sigma(\bar{i},\lambda_1) = + \ ,$$

$$(4.7c)$$

where i runs over $\{1,2,3\}$, and $J" \equiv J_{\lambda"} \equiv \{1,2,3,6\}$. The corresponding region $\Gamma^*_{\lambda"} = \Gamma_{\lambda"}{}^0$ is then defined by the conditions from (2.58),

$$q^0(\{6\}) > 0 \ , \quad (4.8a)$$

$$q^0(\{i\}) < 0 \ , \quad (4.8b)$$

$$q^0(\{6,i\}) > 0 \ , \quad (4.8c)$$

and

$$q^0(J") = 0 \ . \quad (4.8d)$$

These are the equations (4.4) that define $\Gamma_{\lambda"}{}^0$ for case 1.

The next step is to find a set τ of trees t_β such that

$$\Gamma_{\lambda"}{}^0 \supset \bigcap_{\beta:t_\beta\epsilon\tau} \tilde{C}_\beta{}^0 \neq \emptyset \ . \quad (4.9)$$

The conditions (4.8a,b) and (4.9), together with (2.52), require that

$$v_6{}^{\sigma_6} = v_6^+ \quad (4.10a)$$

and

$$v_i{}^{\sigma_i} = v_i^- \ . \qquad (i = 1,2,3) \quad (4.10b)$$

But there is only one tree t_β that can be formed from one cross v_6^+ and three dots v_i^-, namely the tree shown in fig. IV.1b.

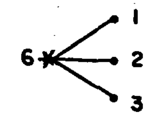


Fig. IV.1b. The tree t_β for case 1.

The tree t_β shown in fig. IV.1b is associated with a cone $\tilde{C}_\beta$ defined by (2.52). Comparison of (2.52) with the conditions (4.8) that define $\Gamma_{\lambda"}{}^0$ shows that $\tilde{C}^0_\beta \subset \Gamma_{\lambda"}{}^0$. Thus (4.9) is satisfied. But the rules (2.38) and (2.41) show that

$$-t_\beta \equiv t'_\beta = (v_1^-,(v_2^-,(v_3^-,v_6))) = \rho_\alpha \ . \quad (4.11)$$

Thus $m_{\lambda"} = m_\alpha$ for case 1 is

$$m_\alpha = [c_1^-,[c_2^-,[c_3^-,c_6]]] \ . \quad (4.12)$$

Having described the procedure in detail for case 1 we now merely list the corresponding results for the remaining final subenergy cases:

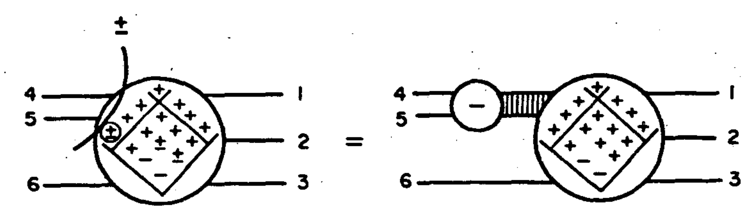


Fig. IV.2a. The basic discontinuity equation for case 2.

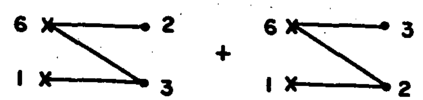


Fig. IV.2b. The sum of trees for case 2.

The m_α for case 2 is

$$m_\alpha = \left[c_1^+,\left[c_2^-,\left[c_3^-,c_6\right]\right]\right] \quad . \tag{4.13}$$

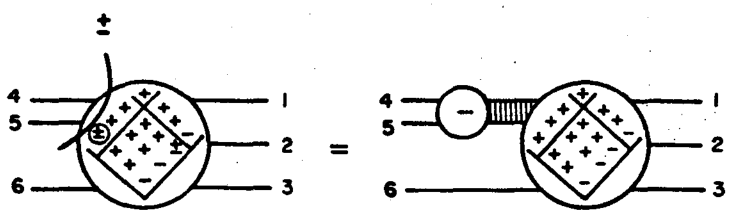


Fig. IV.3a. The basic discontinuity equation for case 3.

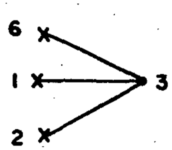


Fig. IV. 3b. The tree t_β for case 3.

The m_α for case 3 is

$$m_\alpha = \left[c_6^+,\left[c_1^+,\left[c_2^+,c_3\right]\right]\right] \quad . \tag{4.14}$$

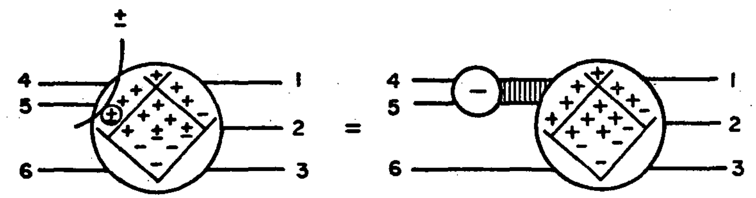


Fig. IV.4a. The basic discontinuity equation for case 4.

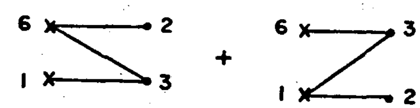


Fig. IV.4b. The sum of trees for case 4.

The m_α for case 4 is

$$m_\alpha = \left[c_2^-,\left[c_1^+,\left[c_6^+,c_3\right]\right]\right] \quad . \tag{4.15}$$

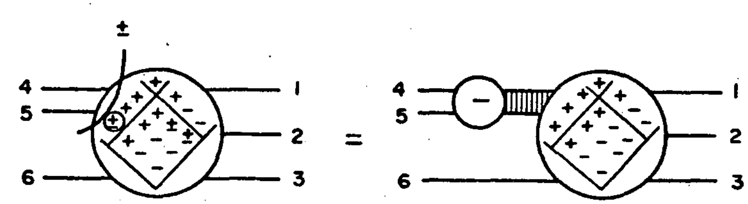


Fig. IV.5a. The basic discontinuity equation for case 5.

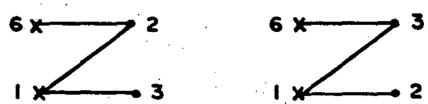


Fig. IV.5b. The sum of trees for case 5.

The m_α for case 5 is

$$m_\alpha = \left[c_6^+,\left[c_2^-,\left[c_3^-,c_1\right]\right]\right] \quad . \tag{4.16}$$

If the sets of signs $\sigma(J_g,\lambda_1)$ and $\sigma(J_g,\lambda_2)$ are restricted to those appearing in fig. III.2, and those obtained from it by the reflection (3.5c), then the five cases listed above exhaust all the pairs of sets satisfying $\sigma(\{4,5\},\lambda_1) = +$; $\sigma(\{4,5\},\lambda_2) = -$, and $\sigma(J',\lambda_1) = \sigma(J',\lambda_2)$ for all $J' \neq \{4,5\}$. The remaining four final-subenergy discontinuities are discontinuities across the cut $q(\{4,6\}) = 0$. The bubble diagram on the right-hand side of each of these remaining four discontinuity equations is the same as the diagram on the right-hand side of one of the $\{4,5\}$ discontinuities already given, except that line five replaces line six. Thus it is enough to give the left-hand side and to identify the case to which it reduces.

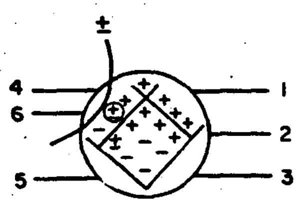


Fig. IV.6. Case 6 reduces to case 2.

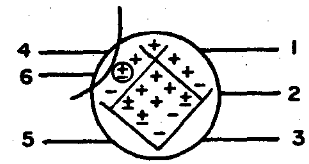


Fig. IV.7. Case 7 reduces to case 3.

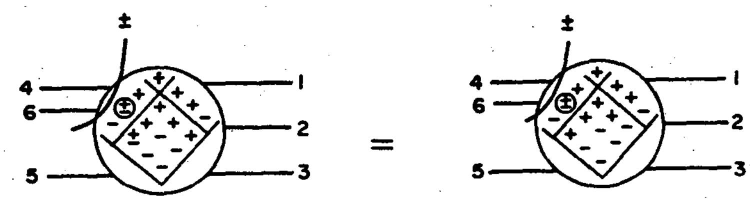


Fig. IV.8. Case 8 reduces to case 4.

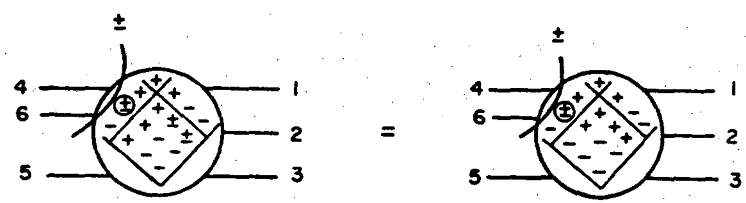


Fig. IV.9. Case 9 reduces to case 5.

The nine cases described above cover all the final-subenergy discontinuities that can arise if the allowed signs are restricted to those either actually shown in fig. III.2, or obtained from fig. III.2 by a reflection across the vertical axis. The remaining final subenergy cases are obtained by relabelling the initial indices, or relabelling the final indices, or reversing all signs, or by applying combinations of these three operations.

The initial-subenergy discontinuities are treated in the same way. The net results are described later.

The total-energy discontinuities are considered next. In these cases there are two nontrivial cells, $\Gamma_{\lambda'}$ and $\Gamma_{\lambda''}$, with $\Gamma_{\lambda'} \subset Q(J_{\lambda'}) \equiv Q(\{4,5,6\})$ and $\Gamma_{\lambda''} \subset Q(J_{\lambda''}) \equiv Q(\{1,2,3\})$. There is a single tree $t_{\beta'}$ corresponding to $\Gamma_{\lambda'}$ and a single tree $t_{\beta''}$ corresponding to $\Gamma_{\lambda''}$. The basic discontinuity equations, the trees $t_{\beta'}$ and $t_{\beta''}$, and the corresponding $m_{\alpha'}$ and $m_{\alpha''}$ for two total-energy discontinuity cases are summarized in fig. IV.10a,b and fig. IV. 11a,b, and in eqs. (4.17) and (4.18). The analogs of (4.7) are $\sigma(\{i\},\lambda'') = +\,\sigma(\bar{i},\lambda_1)$ and $\sigma(\{f\},\lambda') = -\,\sigma(\bar{f},\lambda_1)$.

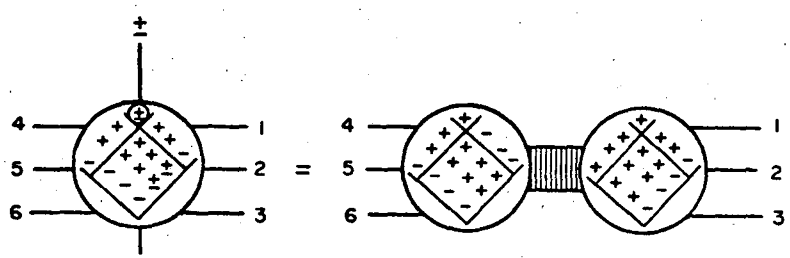


Fig. IV.10a. The basic discontinuity equation for case 10.

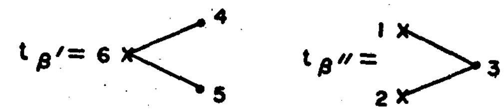


Fig. IV. 10b. The two trees $t_{\beta'}$ and $t_{\beta''}$ for case 10.

The two m_α for case 10

$$m_{\alpha'} = \left[c_4^-,\left[c_5^-,c_6\right]\right] \qquad (4.17a)$$

and

$$m_{\alpha''} = \left[c_1^+,\left[c_2^+,c_3\right]\right] \quad . \qquad (4.17b)$$

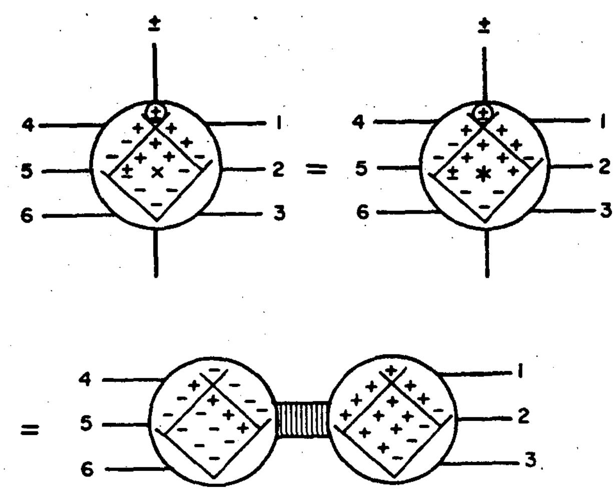


Fig. IV.11a. The basic discontinuity equation for case 11.

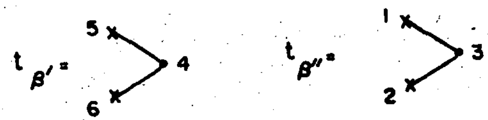


Fig. IV.11b. The two trees $t_{\beta'}$ and $t_{\beta''}$ for case 11.

The two m_α for case 11 are

$$m_{\alpha'} = \left[c_6^+,\left[c_5^+,c_4\right]\right] \tag{4.18a}$$

and

$$m_{\alpha''} = \left[c_1^+,\left[c_2^+,c_3\right]\right] \ . \tag{4.18b}$$

All other total-discontinuity cases follow from cases 10 and 11 by the application of the various operations (3.5).

As already mentioned the cross-energy discontinuities are expressed in terms of physical scattering functions by the basic discontinuity equation itself [5]. These results will be listed later, along with the final results for the subenergy and total-energy discontinuities.

The required expressions in terms of physical scattering functions for the subenergy and total energy discontinuities can be obtained by inserting complete sets of in- or out-states between the factors of $m_{\alpha'}$ and $m_{\alpha''}$. Case 1 is considered first. The second factor $\tilde{r}_{\lambda''+}$ on the right-hand side of the discontinuity equation shown in fig. IV.1a is, by virtue of (4.12), (2.23), (3.1), and (2.31) given by

$$\langle p' + |m_{\lambda''}|0\rangle \qquad \text{(case 1)}$$

$$= \langle p' + |\left[c_1^-,\left[c_2^-,\left[c_3^-,c_6\right]\right]\right]|0\rangle$$

$$= \langle p' + |\left[\left[\left[a_6^+,\bar{a}_3^-\right],\bar{a}_2^-\right],\bar{a}_1^-\right]|0\rangle$$

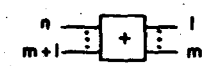
$$= \langle p' + |a_6^+\,\bar{a}_3^-\,\bar{a}_2^-\,\bar{a}_1^-|0\rangle - \langle p' + |\bar{a}_1^-\,a_6^+\,\bar{a}_3^-\,\bar{a}_2^-|0\rangle$$

$$- \langle p' + |\bar{a}_2^-\,a_6^+\,\bar{a}_3^-\,\bar{a}_1^-|0\rangle - \langle p' + |\bar{a}_3^-\,a_6^+\,\bar{a}_2^-\,\bar{a}_1^-|0\rangle$$

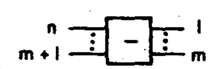
$$+ \langle p' + |\bar{a}_1^-\,\bar{a}_2^-\,a_6^+\,\bar{a}_3^-|0\rangle + \langle p' + |\bar{a}_1^-\,\bar{a}_3^-\,a_6^+\,\bar{a}_2^-|0\rangle$$

$$+ \langle p' + |\bar{a}_2^-\,\bar{a}_3^-\,a_6^+\,\bar{a}_1^-|0\rangle \ . \tag{4.19}$$

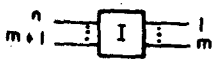
To convert (4.19) and the equations like it that follow into simpler forms a diagrammatic notation is introduced. The basic definitions are as follows:



$$\equiv \langle p_n,\cdots,p_{m+1},+|p_1,\cdots,p_m,-\rangle \ , \tag{4.20a}$$

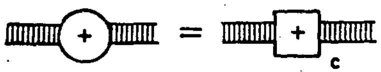


$$\equiv \langle p_n,\cdots,p_{m+1},-|p_1,\cdots,p_m,+\rangle \ , \tag{4.20b}$$

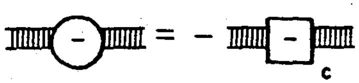


$$\equiv \langle p_n, \cdots, p_{m+1}, + | p_1, \cdots, p_m, + \rangle$$

$$\equiv \langle p_n, \cdots, p_{m+1}, - | p_1, \cdots, p_m, - \rangle \quad ,(4.20c)$$

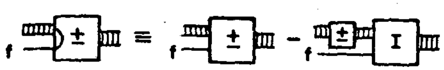


(4.20d)

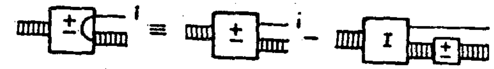


(4.20e)

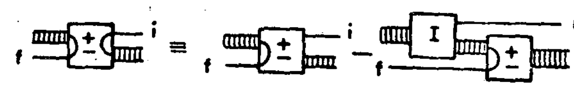
where c denotes connected part. Three other frequently appearing combinations are also represented by special symbols:



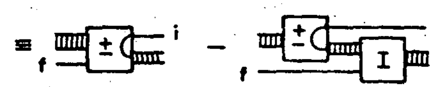
(4.20f)



(4.20g)

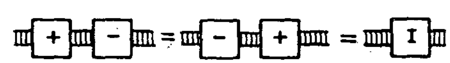


(4.20h)



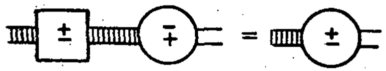
(4.20i)

where the shaded external strips represent fixed sets of lines, and the shaded internal strips represent sums over all numbers of lines, together with an integration of the kind occurring in the unitarity equation, which thus takes the form



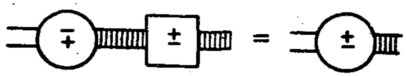
(4.20j)

Unitarity, together with (4.20d) and (4.20e), gives the useful identities



(4.20k)

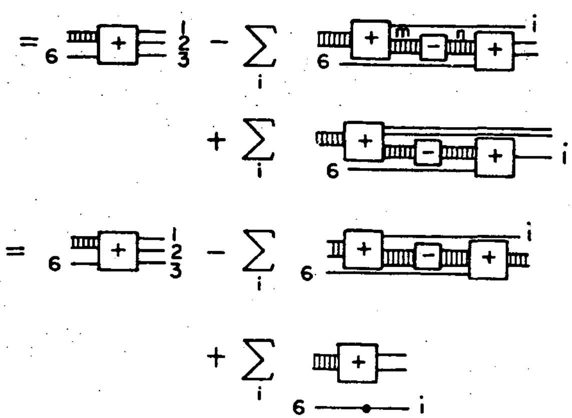
and

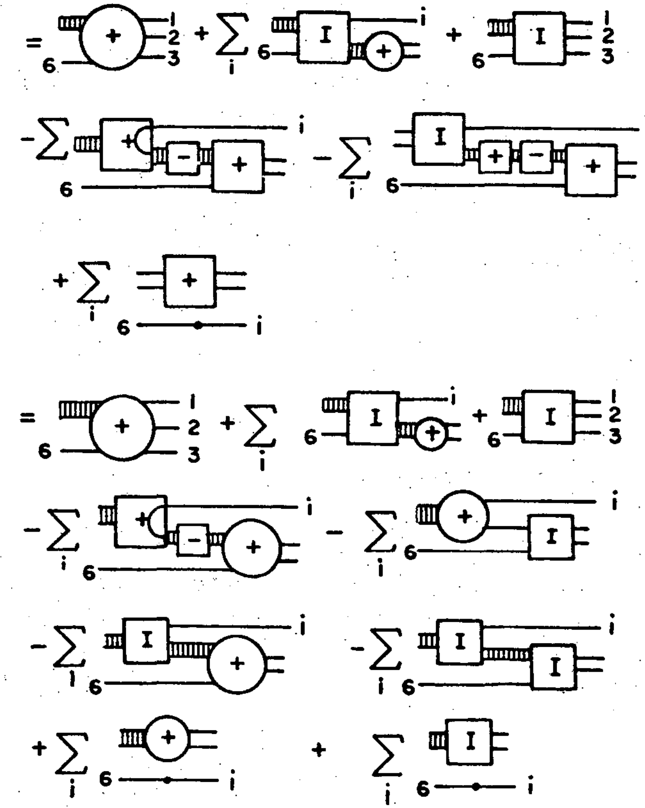


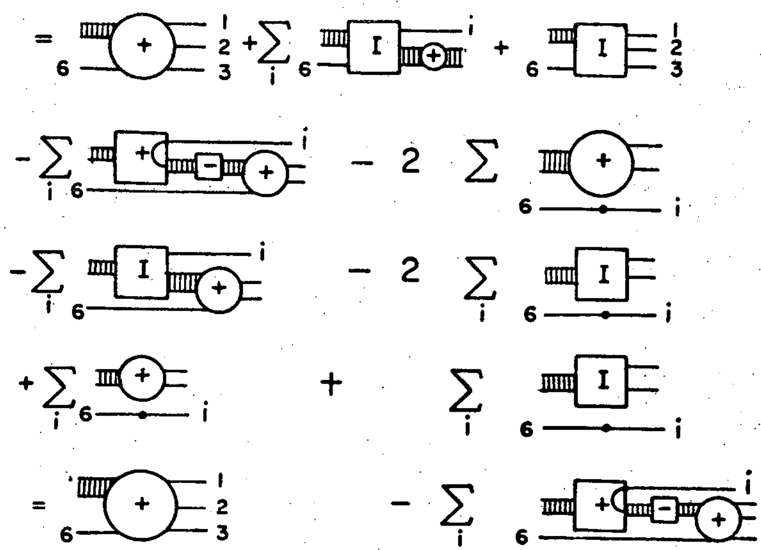
(4.20ℓ)

The first term in (4.19) is just the S matrix for $1 + 2 + 3 \rightarrow 6 + \text{Int}$, where Int represents the set of intermediate particles. The remaining six terms can be converted into expressions involving physical scattering amplitudes by inserting on the left-hand side of a_6^+ the identity operator $I = |m-\rangle\langle m-|\,n+\rangle\langle n+|$. This gives

$\langle p^{int} + | m_{\lambda''} | 0 \rangle$ (Case 1)







For case 2, eq. (4.13) gives

$$\langle p^{int} + |m_{\lambda''}|0\rangle \qquad \text{(Case 2)}$$

$$= \langle p' + |[c_1^+,[c_2^-,[c_3^-,c_6]]]|0\rangle$$

$$= \langle p^{int} + |[[[a_6^+,\bar{a}_3^-],\bar{a}_2^-]\bar{a}_1^+]|0\rangle$$

$$= \langle p^{int} + |a_6^+ \bar{a}_3^- \bar{a}_2^- \bar{a}_1^- |0\rangle - \langle p^{int} + |\bar{a}_1^+ a_6^+ \bar{a}_3^- \bar{a}_2^- |0\rangle$$

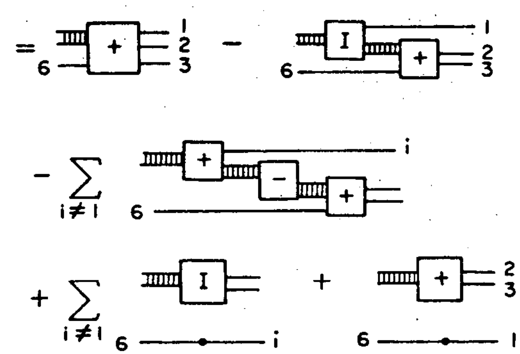
$$- \langle p^{int} + |\bar{a}_2^- a_6^+ \bar{a}_3^- \bar{a}_1^- |0\rangle - \langle p^{int} + |\bar{a}_3^- a_6^+ \bar{a}_2^- \bar{a}_1^- |0\rangle$$

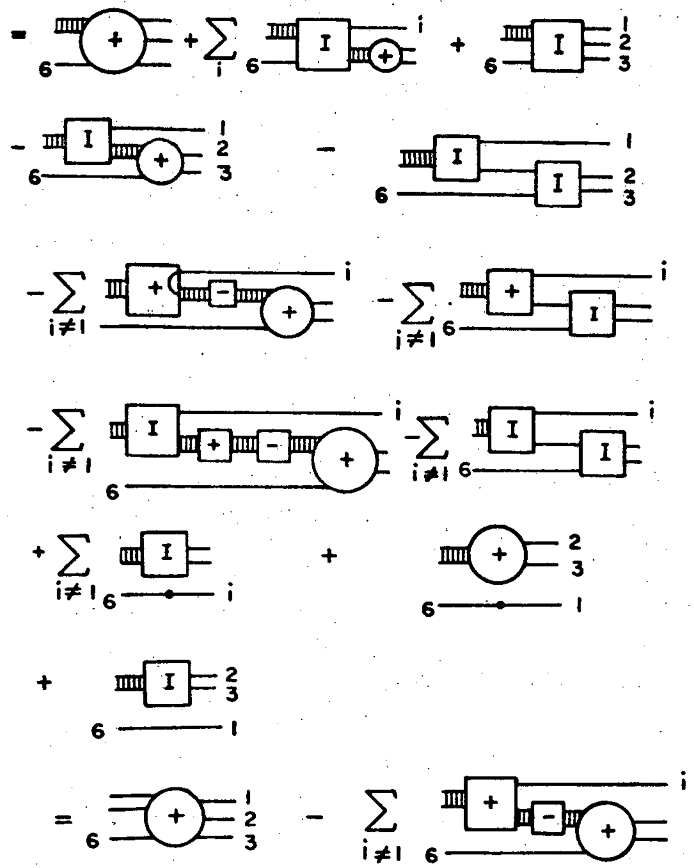
$$+ \langle p^{int} + |\bar{a}_1^+ \bar{a}_2^- a_6^+ \bar{a}_3^- |0\rangle + \langle p^{int} + |\bar{a}_1^+ \bar{a}_3^- a_6^+ \bar{a}_2^- |0\rangle$$

$$+ \langle p^{int} + |\bar{a}_2^- \bar{a}_3^- a_6^+ \bar{a}_1^+ |0\rangle \ , \qquad (4.21a)$$

where $\bar{a}_j^+ |0\rangle = \bar{a}_j^- |0\rangle$ is used. Inserting appropriate complete sets of in- and out- states, one obtains

$$\langle p^{int} + |m_{\lambda''}|0\rangle \qquad \text{(Case 2)}$$





(4.21b)

For case 3, eq. (4.14) gives

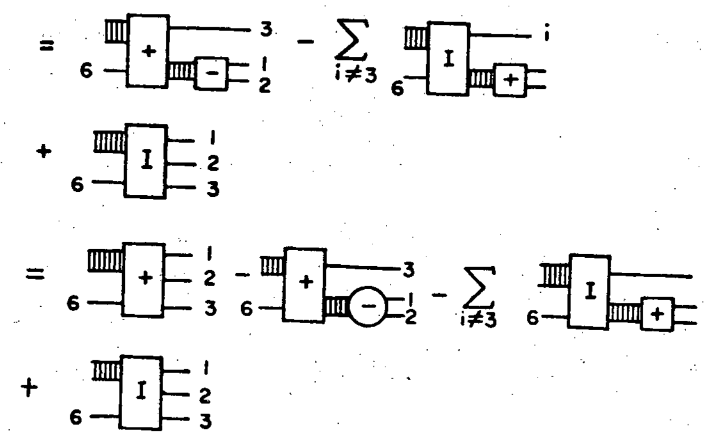
$$\langle p^{int} + |m_{\lambda''}|0\rangle \qquad \text{(Case 3)}$$

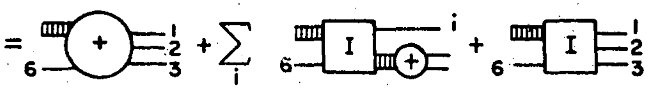
$$= \langle p^{int} + |[c_6^+, [c_1^+, [c_2^+, c_3]]]|0\rangle$$

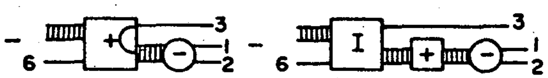
$$= - \langle p^{int} + |[[[\bar{a}_3^-, \bar{a}_2^+], \bar{a}_1^+], a_6^+]|0\rangle$$

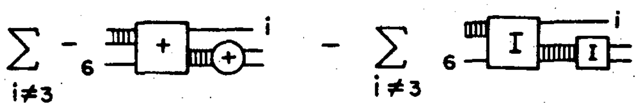
$$= + \langle p^{int} + |a_6^+ \bar{a}_3^- \bar{a}_2^+ \bar{a}_1^+|0\rangle - \langle p^{int} + |a_6^+ \bar{a}_1^+ \bar{a}_3^- \bar{a}_2|0\rangle$$

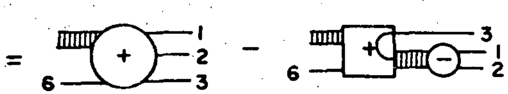
$$- \langle p^{int} + |a_6^+ \bar{a}_2^+ \bar{a}_3^- a_1^-|0\rangle + \langle p^{int} + |a_6^+ \bar{a}_1^+ \bar{a}_2^+ \bar{a}_3|0\rangle$$











(4.22)

For case 4, eq. (4.15) gives

$$\langle p^{int} + |m_{\lambda''}|0\rangle \qquad \text{(Case 4)}$$

$$= \langle p^{int} + |[c_2^-,[c_1^+,[c_6^+,c_3]]]|0\rangle$$

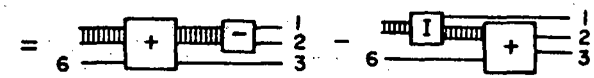
$$= \langle p^{int} + |[[[a_6^+,\bar{a}_3^-],\bar{a}_1^+],\bar{a}_2^-]|0\rangle$$

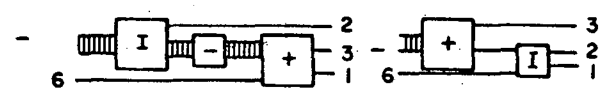
$$= \langle p^{int} + |a_6^+ \bar{a}_3^- \bar{a}_1^+ \bar{a}_2^+|0\rangle - \langle p^{int} + |\bar{a}_1^+ a_6^+ \bar{a}_3^- \bar{a}_2^-|0\rangle$$

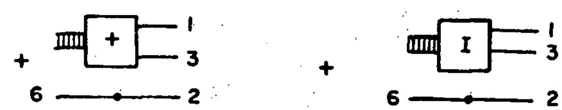
$$- \langle p^{int} + |\bar{a}_2^- a_6^+ \bar{a}_3^- \bar{a}_1^-|0\rangle - \langle p^{int} + |\bar{a}_3^- a_6^+ \bar{a}_1^+ \bar{a}_2^+|0\rangle$$

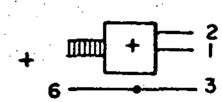
$$+ \langle p^{int} + |\bar{a}_2^- \bar{a}_3^- a_6^+ \bar{a}_1^+|0\rangle + \langle p^{int} + |\bar{a}_1^+ \bar{a}_3^- a_6^+ \bar{a}_2^+|0\rangle$$

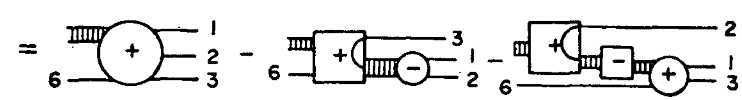
$$+ \langle p^{int} + |\bar{a}_2^- \bar{a}_1^+ a_6^+ \bar{a}_3^+|0\rangle$$











(4.23)

For case 5, eq. (4.16) gives

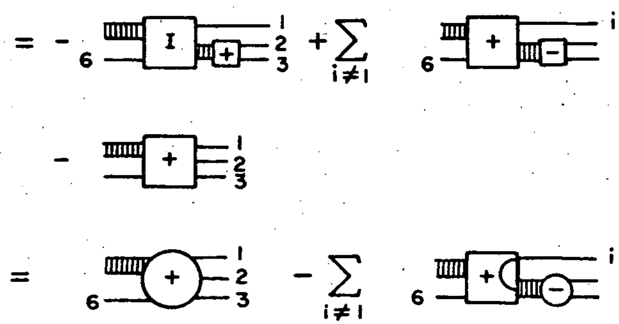
$$\langle p^{int} +|\, m_{\lambda''}|0\rangle \qquad \text{(Case 5)}$$

$$= \langle p^{int} + |\left[c_6^+,\left[c_2^-,\left[c_3^-,c_1\right]\right]\right]|\,0\rangle$$

$$= \langle p^{int} + \ |\left[\left[\left[\bar{a}_1^{\,+},\bar{a}_3^{\,-}\right],\bar{a}_2^{\,-}\right],a_6^+\right]|0\rangle$$

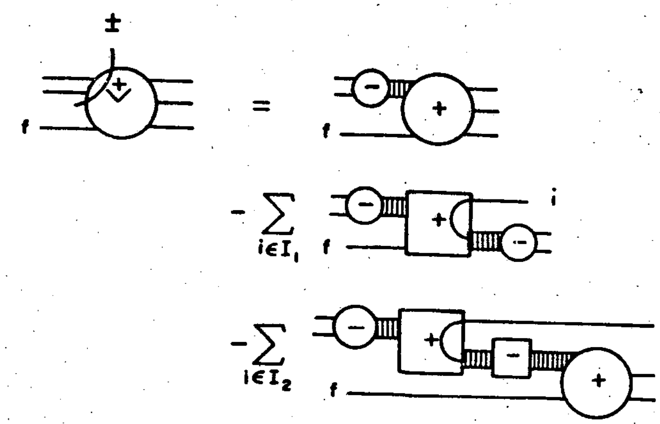
$$= -\langle p^{int} + \ |a_6^+\ \bar{a}_1^{\,+}\ |2,3-\rangle + \langle p^{int} + |\ a_6^+\ \bar{a}_3^{\,-}|\ 1,2+\rangle$$

$$+ \langle p^{int} + \ |a_6^+\ \bar{a}_2^{\,-}|\ 1,3+\rangle - \langle p^{int} + |\ a_6^+|1,2,3-\rangle$$



(4.24)

These five cases, and in fact all final subenergy cases with $\sigma(\bar{t}) = +$, are summarized by the equation



(4.25a)

where

$$I_1 \equiv \{i : \sigma(\bar{I}) = -,\ \sigma(\overline{fI}) = -\} \qquad (4.25b)$$

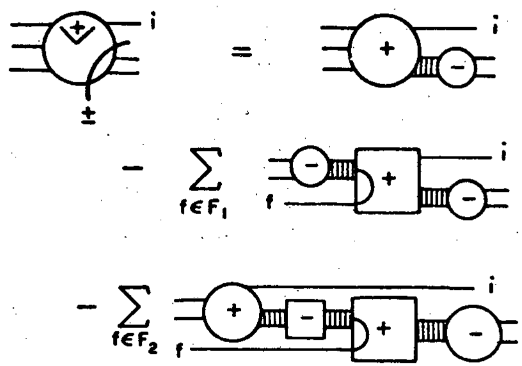
and

$$I_2 \equiv \{i : \sigma(\bar{I}) = +,\ \sigma(\overline{fI}) = -\}\ . \qquad (4.25c)$$

The plus sign inside the bubble on the left-hand side is the sign $\sigma(\bar{t})$. In these equations $\sigma(g) = \sigma(J_g,\lambda_1) = \sigma(J_g,\lambda_2)$.

<u>Remark</u> The case in which $\sigma(\bar{I}) = -$ and $\sigma(\overline{fI}) = +$ does not occur in the BEG framework because of the cell function limitation (3.6a). In section V this limitation is removed and a term corresponding to this case will appear.

The equations analogous to (4.26) for initial subenergies are



(4.26a)

where

$$F_1 \equiv \{f : \sigma(\overline{f}) = -, \ \sigma(\overline{fi}) = -\} \qquad (4.26b)$$

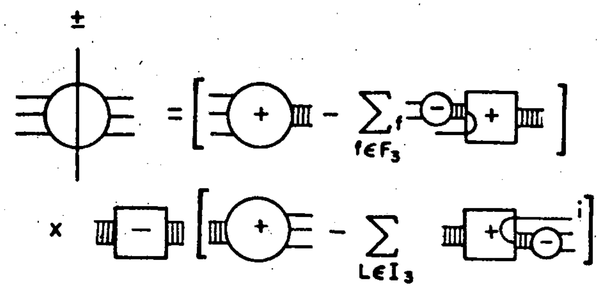
and

$$F_2 \equiv \{f : \sigma(\overline{f}) = +, \ \sigma(\overline{fi}) = -\} \ . \qquad (4.26c)$$

The corresponding equations for the case $\sigma(\overline{t}) = -$ are obtained from the above equations (2.25) and (2.26) by simply reversing all signs inside each box and bubble, except for the circled ± that occurs also outside the bubble on the left-hand side and that says

that the discontinuity is the function for $\sigma(\overline{f}) = +$ minus the function for $\sigma(\overline{f}) = -$ (or for $\sigma(\overline{i}) = +$ minus $\sigma(\overline{i}) = -$).

Starting from (4.17) and (4.18) and proceeding in the same way one obtains the total-energy discontinuity equation



(4.27a)

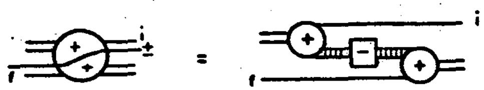
where

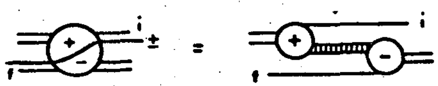
$$I_3 \equiv \{i : \sigma(J_{\overline{I}}, \lambda_1) = \sigma(J_{\overline{I}}, \lambda_2) \equiv \sigma(\overline{I}) = -\} \qquad (4.27b)$$

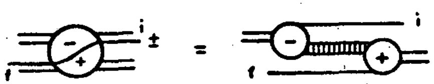
and

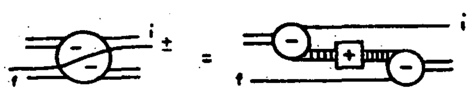
$$F_3 \equiv \{f : \sigma(\overline{f}) = -\} \ . \qquad (4.27c)$$

The cross-energy discontinuities are given by the basic discontinuity equations (5)


(4.28a)


(4.28b)


(4.28c)


(4.28d)

where the upper and lower signs appearing inside the bubbles on the left-hand sides are $\sigma(\bar{f})$ and $\sigma(\bar{i})$, respectively.

Equations (4.25-28), along with their hermitian conjugates, express all the discontinuities $\tilde{r}_{\lambda_1}-\tilde{r}_{\lambda_2}$ at points p lying inside the physical region of the process $1 + 2 + 3 \to 4 + 5 + 6$ in terms of physical scattering amplitudes.

V. GENERALIZED STEINMANN RELATIONS

In this section it is shown that the cell functions $\tilde{r}_\lambda$ have a unique maximal enlargement that satisfies the generalized Steinmann discontinuity property. That is, it is shown that the conditions

$$M_h^{G'} = M_h^{G''} \qquad \text{if} \quad G'/O_h = G''/O_h \tag{5.1}$$

and

$$M^{G(\lambda)} = \tilde{r}_\lambda \qquad \text{for every} \quad \lambda \tag{5.2}$$

determine a unique set of 2^{16} functions M^G.

This unique set of functions is identical to a set of 2^{16} functions M^G derived in ref. 7 from analyticity requirements. It is therefore sufficient simply to verify first that the functions M^G of ref. 7 satisfy (5.1) and (5.2), and then prove uniqueness.

The set of functions M^G is defined in ref. 7 in terms of a certain subset of its multiple discontinuities. As discussed in the introduction, the difference

$$M^G - M^{Gh} \equiv M_h^{\;G} \tag{5.3}$$

for any $h\epsilon\hat{G} \equiv E - G$ is identified as the discontinuity across the cut h evaluated below all the cuts $g\epsilon G$ and above all the cuts $g\epsilon E - Gh$. Similarly, for any $k\epsilon E - Gh$ the function

$$M^G - M^{Gh} - (M^G - M^{Gh})^k$$

$$\equiv M^G - M^{Gh} - M^{Gk} + M^{Ghk} \equiv M_{hk}^{\;\;G} \tag{5.4}$$

is the double discontinuity across the pair of cuts (h,k) evaluated below all the cuts $g \epsilon G$ and above all the cuts $g \epsilon E - Ghk$. More generally, if $(h_1, h_2, \cdots, h_m) = H$ is a set of m distinct elements of $E - G$, and if for any subset X of E the quantity $n(X)$ is the number of distinct elements of X, then

$$M_H^G \equiv \sum_{H' \subset H} (-1)^{n(H')} M^{GH'} \qquad (5.5)$$

defines the m-fold multiple discontinuity across the set of cuts H evaluated below all the cuts $g \epsilon G$ and above all the cuts $g \epsilon E - GH$. The sum in (5.5) runs over all the different subsets H' of H, including the full set H and the empty set ϕ. It is shown in ref. [7] that the set of equations (5.5) can be inverted to give

$$M_H^G = \sum_{G' \subset G} (-1)^{n(G')} M_{HG'} , \qquad (5.6)$$

where the sum runs over all different subsets G' of G, including the full set G and the empty set ϕ.

Equation (5.6) expresses each of the functions M^G and each of the multiple discontinuities formed from them, evaluated on each possible side of every cut, in terms of the multiple discontinuities M_H. Explicit formulas are given in ref. 7 for all of the M_H. This is feasible because most of the M_H vanish.

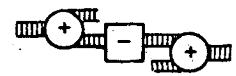
The function $M_\phi \equiv M \equiv M^\phi$ is defined to be the connected part of S,

$$M = S_c = \tilde{r}_{\lambda(p)} , \qquad (5.7)$$

where $\Gamma_{\lambda(p)}$ is the physical cell defined by eq. (2.133c).

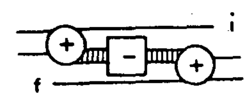
This equation says that M^ϕ, which is the function evaluated above all the cuts, is the physical scattering function $\left[\text{times } (2\pi)^4 \, \delta(\sum_j p_j)\right]$, in accordance with the discussion given below (3.3).

All 16 single discontinuities M_g are given by the single formula

M_g =  (5.8)

where the division of external lines between the two bubbles is defined by g. In particular, if the 16 cuts g are labelled in the way specified in (3.3) then the M_g are

$M_{\bar{t}}$ =  (5.9a)

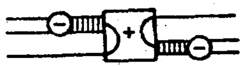
$M_{\overline{fi}}$ =  (5.9b)

$M_{\bar{f}}$ =  (5.9c)

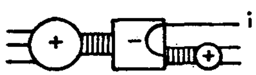
and

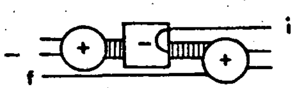
$M_{\bar{i}}$ =  (5.9d)

The nonzero double discontinuities M_{gh} are, in the notation (4.20),

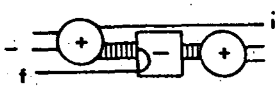
$M_{\bar{f}\,\bar{i}} =$  (5.10a)

$M_{\bar{f}\,\bar{f}} = -$  (5.10b)

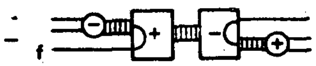
$M_{\bar{f}\,\bar{i}} = -$  (5.10c)

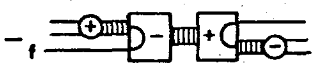
$M_{\bar{f}\,\overline{fi}} = -$  (5.10d)

and

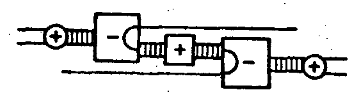
$M_{\overline{fi}\,\bar{i}} = -$  (5.10e)

The nonzero triple discontinuities M_{ghd} are

$M_{\bar{f}\,\bar{f}\,\bar{i}} = -$  (5.11a)

$= -$  (5.11b)

and

$M_{\bar{f}\,\overline{fi}\,\bar{i}} =$  (5.11c)

Finally

$$M_H = 0 \qquad \text{for } n(H) > 3 \,. \tag{5.12}$$

For brevity in what follows two cuts g and h are said to cross, or to be crossed cuts, if and only if $g \varepsilon O_h$, i.e., if and only if J_g overlaps J_h.

The generalized Steinmann discontinuity property (5.1) follows immediately from these formulas. For (5.6) and the fact that M_H vanishes if H contains any pair of crossed cuts means that

$$M_h^{\,G} = M_h^{\,G-O_h} \,, \tag{5.13}$$

where $G - O_h$ consists of the $g \varepsilon G$ that do not lie in O_h. But $G'/O_h = G''/O_h$ is equivalent to $G' - O_h = G'' - O_h$. Hence (5.13) is equivalent to (5.1).

To prove (5.2) it is sufficient to show for some $\lambda = \lambda_0$ that

$$M^{G(\lambda_0)} = \tilde{r}_{\lambda_0} , \qquad (5.14)$$

and for each pair of adjacent cells $\Gamma_{\lambda_1}{}^0$ and $\Gamma_{\lambda_2}{}^0$ that

$$\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2} = M^{G(\lambda_1)} - M^{G(\lambda_2)} \equiv M_h^{G(\lambda_1)} , \qquad (5.15)$$

where h is the cut that separates $\Gamma_{\lambda_1}{}^0$ from $\Gamma_{\lambda_2}{}^0$, and $\Gamma_{\lambda_1}{}^0$ lies above h. Equation (5.2) follows from (5.14), (5.15), and the fact that any $\tilde{r}_\lambda$ can be expressed as $\tilde{r}_{\lambda_0}$ plus a sequence of differences $\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2}$ corresponding to adjacent cells $\Gamma_{\lambda_1}{}^0$ and $\Gamma_{\lambda_2}{}^0$. For (5.14) says that (5.2) holds for $\lambda = \lambda_0$, and this equality can be extended to any other value of λ by a sequence of applications of (5.15).

Equation (5.14) follows from (5.7) for any λ_0 such that Γ_{λ_0} lies above all the cuts $g \epsilon E$. To verify (5.15) one must show that each of the discontinuities $\tilde{r}_{\lambda_1} - \tilde{r}_{\lambda_2}$ given by (4.25-28) is equal to the corresponding discontinuity $M^{G(\lambda_1)} - M^{G(\lambda_2)} = M_h^{G(\lambda_1)}$ given by (5.6). Comparison of (4.25-28) to the formulas summarized in Fig. II.3 of ref. 7 confirms that this is true.

To prove uniqueness it must be shown that if F^G is a set of functions that satisfies

$$F_h^{G'} = F_h^{G''} \qquad \text{if } G'/O_h = G''/O_h \qquad (5.16)$$

and

$$F^{G(\lambda)} = M^{G(\lambda)} \qquad \text{for every } \lambda , \qquad (5.17)$$

then

$$F^G = M^G \qquad \text{for every } G . \qquad (5.18)$$

The functions $F_h{}^G$ in (5.16) are defined for $h \epsilon \hat{G}$ by

$$F_h{}^G \equiv F^G - F^{Gh} . \qquad (5.19)$$

Let $F_H{}^G$ for $H \subset G$ be defined in analogy to the $M_H{}^G$,

$$F_H{}^G \equiv \sum_{H' \subset H} (-1)^{n(H')} F^{GH'} . \qquad (5.20a)$$

These equations can be inverted to give

$$F_H{}^G = \sum_{G' \subset G} (-1)^{n(G')} F_{HG'} , \qquad (5.20b)$$

<u>Lemma A</u> If g crosses h, then $F_{Ggh} = 0$ for all G.

<u>Proof</u> Without loss of generality it can be assumed that $F_{Ggh} = 0$ for all proper subsets G' of G. Then (5.20b) gives

$$F_h{}^{Gg} - F_h{}^G = \sum_{G' \subset G} - (-1)^{n(G')} F_{G'gh}$$

$$= - (-1)^{n(G)} F_{Ggh} . \qquad (5.21)$$

But then (5.16) gives $F_{Ggh} = 0$. QED.

Let C be the set of cross-energy cuts,

$$C \equiv \{\overline{fi} : i = 1,2, \text{ or } 3; \quad f = 4,5, \text{ or } 6\} \; . \tag{5.22a}$$

Any G can be written in the form $G = K \cup H \equiv KH$ where

$$H \subset C \tag{5.22b}$$

and

$$K \subset \hat{C} \equiv E - C \; . \tag{5.22c}$$

Correspondingly, F^G can be written as

$$F^{KH} = \sum_{\substack{K' \subset K \\ K' \subset H}} (-1)^{n(H')} (-1)^{n(K')} F_{K'H'}$$

$$= \sum_{K' \subset K} (-1)^{n(K')} F_{K'}$$

$$+ \sum_{\substack{K' \subset K \\ \phi \neq H' \subset H}} (-1)^{n(K')} (-1)^{n(H')} F_{K'H'}$$

$$\equiv F^K + \Delta(K,H) \; . \tag{5.23}$$

Each cut $\overline{fi} \in C$ crosses every other cut $g \in E$ save $\overline{f}$ and $\overline{i}$. Thus lemma A entails that $F_{K'H'}$ vanishes if $H' \subset C$ has more than one element, and that $F_{K'\overline{fi}}$ vanishes if K' has any element other than $\overline{f}$ or $\overline{i}$. Thus

$$\Delta(K,H) \equiv \sum_{\substack{K' \subset K \\ \phi \neq H' \subset H}} (-1)^{n(K')} (-1)^{n(H')} F_{K'H'}$$

$$= \sum_{\substack{K' \subset K \\ \overline{fi} \in H}} - (-1)^{n(K')} F_{K'\overline{fi}}$$

$$= \sum_{\overline{fi} \in H} \Big[-F_{\overline{fi}} + \sum_{\overline{f}' \in K} \delta_{ff'} F_{\overline{f}'\overline{fi}} + \sum_{\overline{i}' \in K} \delta_{ii'} F_{\overline{fi}\overline{i}'}$$

$$- \sum_{\overline{i}'\overline{f}' \in K} \delta_{ii'} \delta_{ff'} F_{\overline{f}'\overline{fi}\overline{i}'} \Big] \; . \tag{5.24}$$

It will now be shown that each term of $\Delta(K,H)$ is, by virtue of (5.16) and (5.17), a linear combination of the functions $M^{G(\lambda)}$.

A set G such that $G = G(\lambda)$ for some λ is called a BEG set. These are the sets G that correspond to zones. According to fig. III.2 both ϕ and $\{\overline{fi}\}$ are BEG sets. Thus

$$F^{\phi} = M^{\phi} \quad \text{and} \quad F^{\overline{fi}} = M^{\overline{fi}} \; . \tag{5.25a}$$

Therefore

$$F_{\overline{fi}} \equiv F^{\phi} - F^{\overline{fi}} = M^{\phi} - M^{\overline{fi}} \equiv M_{\overline{fi}} \; .$$

The double discontinuity $F_{\overline{f}'\overline{fi}}$ vanishes by virtue of Lemma A unless $f = f'$. In this case

$$F_{\overline{f}\,\overline{fi}} \equiv F - F^{\overline{fi}} - F^{\overline{f}} + F^{\overline{f}\,\overline{fi}}$$

$$= F_{\overline{fi}} - F_{\overline{fi}}^{\overline{f}} = M_{\overline{fi}} - F_{\overline{fi}}^{\overline{f}} \; . \qquad (5.25b)$$

Now all pairs of cuts $\overline{fi}$ and $\overline{fi}'$ are crossed, and by fig. III.2 the sets $\{\overline{f},\overline{fi},\overline{fi}'\}$ and $\{\overline{f},\overline{fi},\overline{fi}',\overline{fi}''\}$ are BEG sets. Thus (5.16) and (5.17) give

$$F_{\overline{fi}}^{\overline{f}} = F_{\overline{fi}}^{\overline{f}\,\overline{fi}'\,\overline{fi}''} = M_{\overline{fi}}^{\overline{f}\,\overline{fi}'\,\overline{fi}''} \; . \qquad (5.25c)$$

Insertion of this result into (5.25b) gives

$$F_{\overline{f}\,\overline{fi}} = M_{\overline{fi}} - M_{\overline{fi}}^{\overline{f}\,\overline{fi}'\,\overline{fi}''} = M_{\overline{f}\,\overline{fi}} \; . \qquad (5.25d)$$

Similarly, $F_{\overline{fi}\,\overline{i}'} = 0$ unless $i = i'$, in which case

$$F_{\overline{fi}\,\overline{i}} = F_{\overline{fi}} - F_{\overline{fi}}^{\overline{i}} = M_{\overline{fi}} - M_{\overline{fi}}^{\overline{i}\,\overline{f'i}\,\overline{f''i}} = M_{\overline{fi}\,\overline{i}} \qquad (5.25e)$$

Finally, $F_{\overline{f}'\overline{fi}\,\overline{i}'} = 0$ unless $f' = f$ and $i' = i$, in which case

$$F_{\overline{f}\,\overline{fi}\,\overline{i}} = F_{\overline{f}\,\overline{fi}} - F^{\overline{i}}_{\overline{f}\,\overline{fi}} = M_{\overline{f}\,\overline{fi}} - F_{\overline{fi}}^{\overline{i}} + F_{\overline{fi}}^{\overline{i}\,\overline{f}}$$

$$= M_{\overline{f}\,\overline{fi}} - M_{\overline{fi}}^{\overline{i}\,\overline{f'i}\,\overline{f''i}} + F_{\overline{fi}}^{\overline{i}\,\overline{f}} \; , \qquad (5.25f)$$

where use was made of an analog of (5.25e). Now since by fig. III.2 both E and $E - \{\overline{fi}\}$ are BEG sets, and since all cuts save $\overline{f}$ and $\overline{i}$ cross $\overline{fi}$, it follows that

$$F_{\overline{fi}}^{\overline{i}\,\overline{f}} = F_{\overline{fi}}^{E-\overline{fi}} = F^{E-\overline{fi}} - F^{E} = M^{E-\overline{fi}} - M^{E} = M_{\overline{fi}}^{E-\overline{fi}} \; . \qquad (5.25g)$$

Equations (5.25a-g) and (5.24) reduce every $\Delta(K,H)$, with $K \subset \hat{C}$ and $H \subset \hat{C}$, to a linear combination of the functions $M^{G(\lambda)}$.

It remains to show that each of the F^{K} with $K \subset \hat{C}$ can be reduced to a linear combination of the $M^{G(\lambda)}$. The construction will depend on whether K belongs to a certain set Z or not. The set Z consists of those sets $K \subset \hat{C}$ such that one (or more) of the following four conditions are satisfied: (1) K contains $\overline{t}$ but none of the $\overline{i}$; (2) K contains $\overline{t}$ but none of the $\overline{f}$; (3) K contains all of the $\overline{i}$ but not $\overline{t}$; (4) K contains all of the $\overline{f}$ but not $\overline{t}$. Symbolically,

$$Z = Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \qquad (5.26a)$$

where

$$Z_1 = \{K \subset \hat{C} \; ; \; \overline{t} \epsilon K \text{ and } \overline{i} \epsilon \hat{K} \text{ for } i = 1, 2, \text{ and } 3\}, \qquad (5.26b)$$

$$Z_2 = \{K \subset \hat{C} : \overline{t} \epsilon K \text{ and } \overline{f} \epsilon \hat{K} \text{ for } f = 4,5, \text{ and } 6\}, \qquad (5.26c)$$

$$Z_3 = \{K \subset \hat{C} : \overline{t} \epsilon \hat{K} \text{ and } \overline{i} \epsilon K \text{ for } i = 1,2, \text{ and } 3\}, \qquad (5.26d)$$

and

$$Z_4 = \{K \subset \hat{C} : \overline{t} \epsilon \hat{K} \text{ and } \overline{f} \epsilon K \text{ for } f = 4,5, \text{ and } 6\}. \qquad (5.26d)$$

If $K \subset \hat{C}$ is not in Z then fig. III.2 shows that there is at least one $H \subset C$ such that $G = KH$ is a BEG set. Let this H be called $H(K)$. Then (5.17) gives

$$F^{KH(K)} = M^{KH(K)} \qquad (5.27a)$$

But then (5.23) gives

$$F^{K} = M^{KH(K)} - \Delta\big(K,H(K)\big) \ , \qquad (5.27b)$$

and hence also, for any $H \subset C$,

$$F^{KH} = F^{K} + \Delta(K,H)$$

$$= M^{KH(K)} - \Delta\big(K,H(K)\big) + \Delta(K,H) \ . \qquad (5.27c)$$

This equation, together with (5.24-5), reduces each $F^{G} = F^{KH}$ with $K \subset \hat{C}$, $K \not\in Z$, and $H \subset C$ to a linear combination of functions $M^{G(\lambda)}$.

The cases $K \varepsilon Z$ are treated by exploiting the fact that any two initial cuts $\bar{i}, \bar{i}'$ are crossed and any two final cuts $\bar{f}, \bar{f}'$ are crossed. Thus (5.20b) and Lemma A give, for each $G \subset \hat{C}$,

$$0 = F^{\bar{i}G}_{\bar{i}'\bar{i}''} = F^{\bar{f}G}_{\bar{f}'\bar{f}''} \qquad (5.28a)$$

and also

$$0 = F^{\bar{t}G}_{\bar{i}\bar{i}'} = F^{\bar{t}G}_{\bar{f}\bar{f}'} \ . \qquad (5.28b)$$

The first of these equations, together with (5.20a), gives, for $G = \phi$,

$$0 = F^{\bar{i}}_{\bar{i}'\bar{i}''} = F^{\bar{i}} - F^{\bar{i}\bar{i}'} - F^{\bar{i}\bar{i}''} + F^{\bar{i}\bar{i}'\bar{i}''} \ . \qquad (5.29a)$$

The first three terms are functions of the form F^{K} with $K \not\in Z$. Thus the relation

$$F^{\bar{i}\bar{i}'\bar{i}''} = F^{\bar{i}\bar{i}'} - F^{\bar{i}}_{\bar{i}''} \qquad (5.29b)$$

expresses the function $F^{\bar{i}\bar{i}'\bar{i}''}$ in terms of functions that have already been reduced, by (5.27c), to linear combinations of the functions $M^{G(\lambda)}$. Similarly, (5.28a) for $G \neq \phi$ leads to the relation

$$F^{\bar{i}\bar{i}'\bar{i}''G} = F^{\bar{i}\bar{i}'G} - F^{\bar{i}G}_{\bar{i}''} \ , \qquad (5.29c)$$

which allows any F^{K} with $K \varepsilon Z_3 - Z_4$ to be expressed in terms of functions that are reduced by (5.27c) to the functions $M^{G(\lambda)}$. The functions F^{K} with $K \varepsilon Z_4 - Z_3$ may be similarly reduced by the equations

$$F^{\bar{f}\bar{f}'\bar{f}''G} = F^{\bar{f}\bar{f}'G} - F^{\bar{f}G}_{\bar{f}''} \qquad (5.29d)$$

which follow from the second part of eq. (5.28a).

The only set $K \varepsilon Z_3 \cap Z_4$ is $K = \{\bar{j} : j = 1,2,\cdots, \text{or } 6\}$. For it equation (5.29c) with $G = \{\bar{f},\bar{f}',\bar{f}''\}$ gives

$$F^{\bar{i}\bar{i}'\bar{i}''\bar{f}\bar{f}'\bar{f}''} = F^{\bar{i}\bar{i}'\bar{f}\bar{f}'\bar{f}''} - F^{\bar{i}\bar{f}\bar{f}'\bar{f}''}_{\bar{i}''} \ . \qquad (5.29e)$$

The terms on the right-hand side of this equation may be reduced by eq. (5.29d).

The first part of eq. (5.28b) with $G = \{f\}$ is

$$0 = F^{\bar{t}\bar{f}}_{\bar{i}\bar{i}'} = F^{\bar{t}\bar{f}} - F^{\bar{t}\bar{f}\bar{i}} - F^{\bar{t}\bar{f}\bar{i}'} + F^{\bar{t}\bar{f}\bar{i}\bar{i}'} \ , \qquad (5.29f)$$

in which each term but $F^{\bar{t}\bar{f}}$ is an F^K with $K \not\in Z$. Thus the relation

$$F^{\bar{t}\bar{f}} = F^{\bar{t}\bar{f}\bar{i}} + F_{\bar{i}}^{\bar{t}\bar{f}\bar{i}'} \qquad (5.29g)$$

expresses $F^{\bar{t}\bar{f}}$ in terms of functions already reduced by (5.27c). Similarly the relation

$$F^{\bar{t}\bar{f}G} = F^{\bar{t}\bar{f}\bar{i}G} + F_{\bar{i}}^{\bar{t}\bar{f}\bar{i}'G} \qquad (5.29h)$$

reduces all F^K with $K \epsilon Z_1 - Z_2$, and

$$F^{\bar{t}\bar{i}G} = F^{\bar{t}\bar{f}\bar{i}G} + F_{\bar{f}}^{\bar{t}\bar{i}\bar{f}'G} \qquad (5.29i)$$

reduces all F^K with $K \epsilon Z_2 - Z_1$.

The intersection $Z_1 \cap Z_2$ contains only the set $K = \{\bar{t}\}$. The function $F^{\bar{t}}$ may be reduced by (5.28b) with $G = \phi$,

$$F^{\bar{t}} = F^{\bar{t}\bar{f}} + F_{\bar{f}}^{\bar{t}\bar{f}'} \quad , \qquad (5.29j)$$

together with (5.29h).

The preceding equations (5.28) and (5.29) reduce every F^K with $K \subset \hat{C}$ and $K \epsilon Z$ to a linear combination of functions $M^{G(\lambda)}$. Thus for any $G = KH$ with $K \subset \hat{C}$, $K \epsilon Z$, and $H \subset C$ these results together with (5.23)

$$F^{KH} = F^K + \Delta(K,H) \qquad (5.30)$$

and (5.25) reduce the function $F^G = F^{KH}$ to a linear combination of $M^{G(\lambda)}$. These results together with those covered by (5.27c)

cover all cases. Thus every F^G has been reduced by means of (5.16) and (5.17) to a linear combination of $M^{G(\lambda)}$.

Since the M^G are a special case of the F^G the functions M^G are equal to these same linear combinations. Thus $F^G = M^G$ for all G. QED.

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