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The Inverse Problem of Linear Lagrangian Dynamics

A comprehensive study is reported herein for the evaluation of Lagrangian functions for linear systems possessing symmetric or nonsymmetric coefficient matrices. Contrary to popular beliefs, it is shown that many coupled linear systems do not admit Lagrangian functions. In addition, Lagrangian functions generally cannot be determined by system decoupling unless further restriction such as classical damping is assumed. However, a scalar function that plays the role of a Lagrangian function can be determined for any linear system by decoupling. This generalized Lagrangian function produces the equations of motion and it contains information on system properties, yet it satisfies a modified version of the Euler–Lagrange equations. Subject to this interpretation, a solution to the inverse problem of linear Lagrangian dynamics is provided. [DOI: 10.1115/1.4038749]

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1 Introduction

The direct problem of Lagrangian dynamics involves the derivation of equations of motion of a system with an assigned Lagrangian function. In contrast, the inverse problem is concerned with finding a scalar function such that the associated Euler–Lagrange equations are equivalent to the assigned equations of motion. This scalar function, termed a Lagrangian, provides a highly compact form of storage of information on system properties; it generates the equations of motion among other things. Owing to utility in several fields, the inverse problem, sometimes referred to as the inverse problem of the calculus of variations, is a well-trodden problem that has attracted the attention of many researchers in the past century.

Darboux [1] demonstrated the existence of Lagrangians for single-degree-of-freedom (SDOF) systems. Leitmann obtained Lagrangians associated with nonpotential forces for which a variational principle exists [2]. Subsequently, Udwadia et al. [3] derived the Lagrangians connected with general nonpotential forces. He [4] used the semi-inverse method to derive Lagrangians of the Korteweg-de Vries and Schrödinger equations. Musielak et al. [5] derived Lagrangians of nonlinear SDOF systems with variable coefficients and presented methods to obtain standard and nonstandard Lagrangians of SDOF systems [6]. A Lagrangian is referred to as standard (or natural) if it can be expressed as the difference between kinetic and potential energy terms; otherwise, the Lagrangian is termed nonstandard (or non-natural). These and other earlier works [7–9] have addressed the inverse problem for SDOF systems.

Solution of the inverse problem for multi-degree-of-freedom (MDOF) systems poses a greater challenge because the equations of motion are usually coupled; it is thus not permissible to focus on individual component equations [10,11]. General conditions for the existence of Lagrangians are provided by the so-called Helmholtz conditions [12,13], an assessment of which requires the solution of certain partial differential equations. Udwadia and Cho [14] obtained Lagrangians for a class of SDOF and MDOF linear systems by invoking the Helmholtz conditions. In general, the Helmholtz conditions offer little assistance in the solution of the inverse problem for MDOF systems. Douglas [15] and Crampin et al. [16] addressed the inverse problem for two-degree-of-freedom systems using Riquier theory with an exhaustive case-by-case examination. Recently, Udwadia [17] obtained Lagrangians for classically damped linear systems using modal analysis. However, damped linear systems are generally not amenable to modal analysis [18].

It will be demonstrated in this paper that system decoupling, used successfully by Udwadia [17] for classically damped systems, cannot be extended to obtain Lagrangians for general linear systems. It will also be shown that many coupled systems do not admit Lagrangian functions, but a scalar function that plays the role of a Lagrangian function can be found for every linear system. The organization of this paper is as follows: The inverse problem of linear Lagrangian dynamics is formulated in Sec. 2, and solutions for SDOF and classically damped MDOF linear systems are reviewed. This is followed in Sec. 3 by a concise exposition of an extension of modal analysis to decouple nondefective MDOF linear systems in real space. The effect of decoupling transformations on the Euler–Lagrange equations is examined in Sec. 4, where generalized Lagrangian functions are determined. Defective linear systems are explored in Sec. 5, and the existence of Lagrangian functions for coupled linear systems is addressed in Sec. 6. Finally, a summary of findings is provided in Sec. 7. Six examples are supplied throughout the paper for illustration.

2 Problem Statement

The equation of motion of an n-degree-of-freedom linear system can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are arbitrary $n \times n$ matrices with $\mathbf{M}$ assumed invertible. These coefficient matrices are real, but they need not possess the familiar properties of symmetry and positive

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definiteness. Thus, Eq. (1) may represent gyroscopic and circulatory systems [11]. The generalized coordinate

$$\mathbf{q} = [q_1 \quad q_2 \quad \cdots \quad q_n]^{\mathrm{T}} \tag{2}$$

is an n-dimensional vector. Equation (1) is one of the most commonly used equations in science and engineering.

When $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are symmetric and positive definite, they are referred to as the mass, damping, and stiffness matrices, respectively. In this case, Eq. (1) is termed a damped linear system. From a practical viewpoint, there is no loss of generality in assuming that $\mathbf{M}$ is invertible. If necessary, static condensation may be applied initially to reduce the number of degrees-of-freedom in order to guarantee that $\mathbf{M}$ is invertible [19]. Because $\mathbf{M}$ is nonsingular, it is convenient to take $\mathbf{M} = \mathbf{I}$ and write Eq. (1) as

$$\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{3}$$

where $\mathbf{C}$ and $\mathbf{K}$ are arbitrary, real $n \times n$ matrices.

Define the derivative of a multivariate scalar function F with respect to an n-dimensional vector such as $\mathbf{q}$ in Eq. (2) by

$$\frac{\partial F}{\partial \mathbf{q}} = \left[\frac{\partial F}{\partial q_1} \quad \frac{\partial F}{\partial q_2} \quad \cdots \quad \frac{\partial F}{\partial q_n}\right]^{\mathrm{T}} \tag{4}$$

Concisely speaking, the inverse problem of linear Lagrangian dynamics amounts to finding a scalar function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ that satisfies the corresponding Euler–Lagrange equation:

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{q}}}\right] - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)(\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q}) = \mathbf{0} \tag{5}$$

where $\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)$ is a nonsingular $n \times n$ matrix multiplier. A general solution to the inverse problem has never been reported in the open literature. However, Lagrangian functions have already been determined for SDOF and classically damped MDOF linear systems. These solutions are now summarized.

2.1 Lagrangians for Single-Degree-of-Freedom Systems.

A linear SDOF system of the form

$$\ddot{p} + d\dot{p} + bp = 0 \tag{6}$$

where d and b are real constants, admits the Lagrangian function [17]

$$L(p, \dot{p}, t) = \frac{1}{2}\left(\dot{p}^2 + d\dot{p}p + \frac{d^2}{2}p^2\right)e^{dt} - \frac{b}{2}p^2 e^{dt} \tag{7}$$

and, alternatively, a more compact Lagrangian function

$$L(p, \dot{p}, t) = \frac{1}{2}\dot{p}^2 e^{dt} - \frac{b}{2}p^2 e^{dt} \tag{8}$$

As direct verification, substitute either Eq. (7) or Eq. (8) into the corresponding Euler–Lagrange equation to yield

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{p}}\right] - \frac{\partial L}{\partial p} = e^{dt}(\ddot{p} + d\dot{p} + bp) = 0 \tag{9}$$

from which the equation of motion (6) can be extracted because $e^{dt} \neq 0$ for all t.

2.2 Lagrangians for Classically Damped Linear Systems.

Suppose the coefficient matrices $\mathbf{C}$ and $\mathbf{K}$ are symmetric and positive definite. Associated with Eq. (3) is the symmetric eigenvalue problem $\mathbf{K}\mathbf{u} = \lambda\mathbf{u}$. Owing to the positive definiteness of $\mathbf{K}$, all eigenvalues λ_j ($j = 1, 2, \ldots, n$) are positive, and the corresponding eigenvectors $\mathbf{u}_j$ are real and orthonormal. Define the modal matrix by

$$\mathbf{U} = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n] \tag{10}$$

If Eq. (3) is classically damped, then it is amenable to modal analysis. Using the modal transformation $\mathbf{q} = \mathbf{U}\mathbf{p}$, the matrices $\mathbf{C}$ and $\mathbf{K}$ are diagonalized simultaneously in real space such that

$$\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}, \quad \mathbf{U}^{\mathrm{T}}\mathbf{C}\mathbf{U} = \mathrm{diag}[d_j], \quad \mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U} = \mathrm{diag}[b_j] \tag{11}$$

In other words, Eq. (3) becomes decoupled in the modal coordinate

$$\mathbf{p} = [p_1 \quad p_2 \quad \cdots \quad p_n]^{\mathrm{T}} \tag{12}$$

Under the assumption of classical damping, Udwadia [17] decoupled Eq. (3) into n independent SDOF systems from which the Lagrangian functions

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2}\left(\dot{\mathbf{q}}^{\mathrm{T}} e^{\mathbf{C}t}\dot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathrm{T}} e^{\mathbf{C}t}\mathbf{C}\mathbf{q} + \frac{1}{2}\mathbf{q}^{\mathrm{T}} e^{\mathbf{C}t}\mathbf{C}^2\mathbf{q}\right) - \frac{1}{2}\mathbf{q}^{\mathrm{T}} e^{\mathbf{C}t}\mathbf{K}\mathbf{q} \tag{13}$$

and

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}} e^{\mathbf{C}t}\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^{\mathrm{T}} e^{\mathbf{C}t}\mathbf{K}\mathbf{q} \tag{14}$$

were constructed by using Eqs. (7) and (8), respectively. Substitute either Eq. (13) or Eq. (14) into Eq. (5) to obtain

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{q}}}\right] - \frac{\partial L}{\partial \mathbf{q}} = e^{\mathbf{C}t}(\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q}) = \mathbf{0} \tag{15}$$

from which the equation of motion (3) is recovered because $\det(e^{\mathbf{C}t}) \neq 0$ for all t. In general, there is no reason why damping in a linear system should be classical. A necessary and sufficient condition [18] under which Eq. (3) is classically damped is given by

$$\mathbf{C}\mathbf{K} = \mathbf{K}\mathbf{C} \tag{16}$$

Practically speaking, classical damping implies that energy dissipation is almost uniformly distributed throughout a system. Experimental modal testing suggests that no physical system is strictly classically damped [20]; damping in linear systems is routinely nonclassical.

3 Generalization of Modal Analysis

The key to successful derivation of Lagrangian functions for classically damped linear systems, as described earlier, is decoupling: the conversion of a given MDOF system into a series of independent SDOF systems. Recently, classical modal analysis has been extended to decouple practically any linear system in real space [21–23]. To be specific, a real and invertible transformation has been determined to convert Eq. (3) into

$$\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{B}\mathbf{p} = \mathbf{0} \tag{17}$$

for which the $n \times n$ coefficient matrices $\mathbf{D}$ and $\mathbf{B}$ are real and diagonal. Unless Eq. (3) represents a classically damped system, the elements of $\mathbf{D}$ and $\mathbf{B}$ are not those specified by Eq. (11). There are no scientific restrictions on this extension of modal analysis, which is termed the method of phase synchronization. All parameters required for decoupling are obtained through the solution of the quadratic eigenvalue problem

$$(\mathbf{I}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{v} = \mathbf{0} \tag{18}$$

which yields $2n$ eigenvalues λ_j $(j = 1, 2, ..., 2n)$ and eigenvectors $\mathbf{v}_j$. System (3) is nondefective if each repeated eigenvalue of Eq. (18) possesses a full complement of independent eigenvectors. In order to provide a concise exposition, it is assumed that all eigenvalues of Eq. (18) are distinct, which guarantees that Eq. (3) is nondefective. If $\mathbf{C}$ and $\mathbf{K}$ are randomly chosen from a uniform distribution, the probability that all eigenvalues of Eq. (18) are distinct is one [21]. Indeed, almost all linear systems are characterized by distinct eigenvalues. Defective systems, which must possess repeated eigenvalues, will be addressed in Sec. 5.

3.1 Methodology for Decoupling Nondefective Linear Systems. To streamline the presentation, an implementation[2] of phase synchronization to decouple nondefective systems with distinct eigenvalues is summarized as a series of tasks.

Task 1. Solve the quadratic eigenvalue problem (18) and index the eigensolutions.

There are $2n$ eigensolutions, and any complex eigensolutions occur in complex conjugate pairs. Let $2c$ eigenvalues be complex and the remaining $2r = 2(n - c)$ be real. Denote the first c eigenvalues λ_j $(j = 1, 2, ..., c)$ as the c complex eigenvalues with positive imaginary parts arranged in order of increasing magnitude of their imaginary parts:

$$S_1 = \{\lambda_1, \lambda_2, ..., \lambda_c : 0 < \mathrm{Im}[\lambda_1] \leq \mathrm{Im}[\lambda_2] \leq \cdots \leq \mathrm{Im}[\lambda_c]\} \quad (19)$$

Enumerate the remaining c complex eigenvalues, which are the complex conjugates of λ_j with negative imaginary parts, in such a way that

$$S_3 = \{\lambda_{n+1} = \overline{\lambda}_1, \lambda_{n+2} = \overline{\lambda}_2, ..., \lambda_{n+c} = \overline{\lambda}_c\} \quad (20)$$

Arrange the $2r$ real eigenvalues in accordance with a primary–secondary pairing scheme [22], where the r largest eigenvalues are referred to as primary eigenvalues and the r smallest eigenvalues are termed secondary eigenvalues. Enumerate the r real secondary eigenvalues in order of increasing magnitude such that

$$S_2 = \{\lambda_{c+1}, \lambda_{c+2}, ..., \lambda_n : \lambda_{c+1} < \lambda_{c+2} < \cdots < \lambda_n\} \quad (21)$$

Also arrange the remaining r real primary eigenvalues in order of increasing magnitude:

$$S_4 = \{\lambda_{n+c+1}, \lambda_{n+c+2}, ..., \lambda_{2n} : \lambda_{n+c+1} < \lambda_{n+c+2} < \cdots < \lambda_{2n}\} \quad (22)$$

Thus, the $2n$ eigenvalues are partitioned into four disjoint subsets. A different indexing scheme for the eigensolutions may be used.

Task 2. Normalize the eigenvectors of Eq. (18).

After the eigensolutions have been indexed, the $2n$ eigenvectors $\mathbf{v}_j$ and $\mathbf{v}_{n+j}$ $(j = 1, 2, ..., n)$ are normalized in accordance with

$$2\lambda_j \mathbf{v}_j^{\mathrm{T}} \mathbf{v}_j + \mathbf{v}_j^{\mathrm{T}} \mathbf{C} \mathbf{v}_j = \lambda_j - \lambda_{n+j} \quad (23)$$

$$2\lambda_{n+j} \mathbf{v}_{n+j}^{\mathrm{T}} \mathbf{v}_{n+j} + \mathbf{v}_{n+j}^{\mathrm{T}} \mathbf{C} \mathbf{v}_{n+j} = \lambda_{n+j} - \lambda_j \quad (24)$$

The normalization scheme represented by Eqs. (23) and (24) reduces to mass-normalization (with $\mathbf{M} = \mathbf{I}$ in this case) for undamped or classically damped systems [19,21]. This is an optional task and a different normalization scheme for the eigenvectors may also be used.

Task 3. Construct the decoupled system (17) from the eigenvalues of Eq. (18).

Using the indexed eigenvalues, assemble the following $n \times n$ matrices:

$$\mathbf{\Lambda}_1 = \mathrm{diag}[\lambda_1, \lambda_2, ..., \lambda_n], \quad \mathbf{\Lambda}_2 = \mathrm{diag}[\lambda_{n+1}, \lambda_{n+2}, ..., \lambda_{2n}] \quad (25)$$

The real and diagonal coefficient matrices of the decoupled system (17) are given by

$$\mathbf{D} = -(\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2), \quad \mathbf{B} = \mathbf{\Lambda}_1 \mathbf{\Lambda}_2 \quad (26)$$

Note that $\mathbf{D}$ and $\mathbf{B}$ are independent of eigenvector normalization because they are constructed from the eigenvalues only.

Task 4. Construct the real decoupling transformation.

Assemble the following $n \times n$ matrices of eigenvectors:

$$\mathbf{V}_1 = [\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \cdots \ \ \mathbf{v}_n], \qquad \mathbf{V}_2 = [\mathbf{v}_{n+1} \ \ \mathbf{v}_{n+2} \ \ \cdots \ \ \mathbf{v}_{2n}] \quad (27)$$

The decoupling transformation, when cast in the state space, is linear and time-invariant such that

$$\begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{S}_1\mathbf{q} + \mathbf{S}_2\dot{\mathbf{q}} \\ \mathbf{S}_3\mathbf{q} + \mathbf{S}_4\dot{\mathbf{q}} \end{bmatrix} \quad (28)$$

where the $2n \times 2n$ real and invertible matrix $\mathbf{S}$ is given by

$$\mathbf{S} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 \\ \mathbf{V}_1\mathbf{\Lambda}_1 & \mathbf{V}_2\mathbf{\Lambda}_2 \end{bmatrix}^{-1} \quad (29)$$

By expansion, the $n \times n$ submatrices $\mathbf{S}_i$ $(i = 1, 2, 3, 4)$ have the representations

$$\mathbf{S}_1 = [(\mathbf{V}_1\mathbf{\Lambda}_1)^{-1} - (\mathbf{V}_2\mathbf{\Lambda}_2)^{-1}] \times [\mathbf{V}_1(\mathbf{V}_1\mathbf{\Lambda}_1)^{-1} - \mathbf{V}_2(\mathbf{V}_2\mathbf{\Lambda}_2)^{-1}]^{-1} \quad (30)$$

$$\mathbf{S}_2 = (\mathbf{V}_1^{-1} - \mathbf{V}_2^{-1})[(\mathbf{V}_1\mathbf{\Lambda}_1)\mathbf{V}_1^{-1} - (\mathbf{V}_2\mathbf{\Lambda}_2)\mathbf{V}_2^{-1}]^{-1} \quad (31)$$

$$\mathbf{S}_3 = (\mathbf{V}_1^{-1} - \mathbf{V}_2^{-1})[\mathbf{V}_1(\mathbf{V}_1\mathbf{\Lambda}_1)^{-1} - \mathbf{V}_2(\mathbf{V}_2\mathbf{\Lambda}_2)^{-1}]^{-1} \quad (32)$$

$$\mathbf{S}_4 = (\mathbf{\Lambda}_1\mathbf{V}_1^{-1} - \mathbf{\Lambda}_2\mathbf{V}_2^{-1})[(\mathbf{V}_1\mathbf{\Lambda}_1)\mathbf{V}_1^{-1} - (\mathbf{V}_2\mathbf{\Lambda}_2)\mathbf{V}_2^{-1}]^{-1} \quad (33)$$

If written in configuration space, the decoupling transformation becomes linear and time-varying [21,23]. To streamline the manipulations, only the state-space decoupling transformation will be used.

3.2 Reduction to Classical Modal Analysis. The decoupling procedure discussed previously is a direct extension of modal analysis. If Eq. (3) is undamped or classically damped, the eigenvectors of Eq. (18) coincide with the normal modes of the system up to arbitrary signs in the columns of the modal matrix $\mathbf{U}$ defined in Eq. (10). It can be shown that Eqs. (30)–(33) simplify to

$$\mathbf{S}_1 = \mathbf{S}_4 = \mathbf{U}^{-1} = \mathbf{U}^{\mathrm{T}}, \quad \mathbf{S}_2 = \mathbf{S}_3 = \mathbf{0} \quad (34)$$

As a consequence, the decoupled coordinate $\mathbf{p}$ and the original coordinate $\mathbf{q}$ are just connected by the modal transformation $\mathbf{q} = \mathbf{U}\mathbf{p}$. With different indexing schemes, phase synchronization generates all possible decoupled forms into which a linear system (with symmetric or nonsymmetric coefficients) can be transformed in real space [22,23].

4 Generalized Lagrangian Functions and Modified Euler–Lagrange Equations

As explained in Sec. 3, Eq. (3) can be decoupled into Eq. (17) using an extension of modal analysis. Upon decoupling, one obtains n independent SDOF systems of the form

$$\ddot{p}_j + d_j\dot{p}_j + b_jp_j = 0 \quad (35)$$

[2]A computer program for decoupling linear systems is available upon request.

where $d_j = -(\lambda_j + \lambda_{n+j})$ $(j = 1, 2, \ldots, n)$ and $b_j = \lambda_j \lambda_{n+j}$ are constants that populate the diagonal of the coefficient matrices $\mathbf{D}$ and $\mathbf{B}$, respectively. Recalling Eq. (7), a Lagrangian function associated with Eq. (35) is

$$L_j(p_j, \dot{p}_j, t) = \frac{1}{2}\left(\dot{p}_j^2 + d_j \dot{p}_j p_j + \frac{d_j^2}{2} p_j^2\right) e^{d_j t} - \frac{b_j}{2} p_j^2 e^{d_j t} \tag{36}$$

It follows that a Lagrangian function for the entire decoupled system (17) is given by [17]

$$\begin{aligned} L(\mathbf{p}, \dot{\mathbf{p}}, t) &= \sum_{j=1}^{n} L_j(p_j, \dot{p}_j, t) \\ &= \frac{1}{2}\left(\dot{\mathbf{p}}^{\mathrm{T}} e^{\mathbf{D}t} \dot{\mathbf{p}} + \dot{\mathbf{p}}^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{D} \mathbf{p} + \frac{1}{2} \mathbf{p}^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{D}^2 \mathbf{p}\right) - \frac{1}{2} \mathbf{p}^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{B} \mathbf{p} \end{aligned} \tag{37}$$

It is straightforward to verify that Eq. (37) is indeed a Lagrangian function for the decoupled system (17) because the equation of motion is recovered from evaluating the associated Euler–Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mathbf{p}}}\right] - \frac{\partial L}{\partial \mathbf{p}} = \mathbf{0} \tag{38}$$

Using Eq. (28), the Lagrangian function $L(\mathbf{p}, \dot{\mathbf{p}}, t)$ for the decoupled system can be expressed in terms of the original coordinate $\mathbf{q}$, resulting in a function

$$\begin{aligned} \hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t) &= \frac{1}{2}\mathbf{q}^{\mathrm{T}}\left(\mathbf{S}_3^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{S}_3 + \mathbf{S}_3^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{D} \mathbf{S}_1 + \frac{1}{2}\mathbf{S}_1^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{D}^2 \mathbf{S}_1 - \mathbf{S}_1^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{B} \mathbf{S}_1\right)\mathbf{q} \\ &+ \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\left(\mathbf{S}_4^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{S}_4 + \mathbf{S}_4^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{D} \mathbf{S}_2 + \frac{1}{2}\mathbf{S}_2^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{D}^2 \mathbf{S}_2 - \mathbf{S}_2^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{B} \mathbf{S}_2\right)\dot{\mathbf{q}} \\ &+ \dot{\mathbf{q}}^{\mathrm{T}}\left[\mathbf{S}_4^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{S}_3 + \frac{1}{2}\left(\mathbf{S}_4^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{D} \mathbf{S}_1 + \mathbf{S}_2^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{D} \mathbf{S}_3 + \mathbf{S}_2^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{D}^2 \mathbf{S}_1\right) - \mathbf{S}_2^{\mathrm{T}} e^{\mathbf{D}t} \mathbf{B} \mathbf{S}_1\right]\mathbf{q} \end{aligned} \tag{39}$$

Under the assumption of classical damping, $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ would be a Lagrangian function for the original system (3). This is precisely the approach adopted by Udwadia [17] in the derivation of Eq. (13). However, $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ as given by Eq. (39) generally does not satisfy the Euler–Lagrange equation in $\mathbf{q}$, i.e.,

$$\frac{d}{dt}\left[\frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}}\right] - \frac{\partial \hat{L}}{\partial \mathbf{q}} \neq \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)(\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q}) \tag{40}$$

for any $\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)$. Thus, $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ is not a Lagrangian function for Eq. (3) even though, as a scalar function, $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ still provides compact storage of system properties. What equation is satisfied by $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$? Can $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ generate the equation of motion?

4.1 Transformation of Euler–Lagrange Equations. The inverse of Eq. (28) can be written as

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \mathbf{S}^{-1}\begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \mathbf{T}\begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{bmatrix}\begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1\mathbf{p} + \mathbf{T}_2\dot{\mathbf{p}} \\ \mathbf{T}_3\mathbf{p} + \mathbf{T}_4\dot{\mathbf{p}} \end{bmatrix} \tag{41}$$

where the $2n \times 2n$ matrix $\mathbf{T}$ is real and invertible with submatrices $\mathbf{T}_i$ $(i = 1, 2, 3, 4)$ given by

$$\mathbf{T}_1 = (\mathbf{V}_1\mathbf{\Lambda}_2 - \mathbf{V}_2\mathbf{\Lambda}_1)(\mathbf{\Lambda}_2 - \mathbf{\Lambda}_1)^{-1} \tag{42}$$

$$\mathbf{T}_2 = (\mathbf{V}_2 - \mathbf{V}_1)(\mathbf{\Lambda}_2 - \mathbf{\Lambda}_1)^{-1} \tag{43}$$

$$\mathbf{T}_3 = (\mathbf{V}_1 - \mathbf{V}_2)(\mathbf{\Lambda}_1\mathbf{\Lambda}_2)(\mathbf{\Lambda}_2 - \mathbf{\Lambda}_1)^{-1} \tag{44}$$

$$\mathbf{T}_4 = (\mathbf{V}_2\mathbf{\Lambda}_2 - \mathbf{V}_1\mathbf{\Lambda}_1)(\mathbf{\Lambda}_2 - \mathbf{\Lambda}_1)^{-1} \tag{45}$$

Denote the elements of $\mathbf{T}_1$ and $\mathbf{T}_3$ by $T_{1,ij}$ $(i, j = 1, 2, \ldots, n)$ and $T_{3,ij}$, respectively. Using Eq. (41),

$$\begin{aligned} \frac{\partial L}{\partial p_j} &= \sum_{i=1}^{n}\left[\frac{\partial \hat{L}}{\partial q_i}\frac{\partial q_i}{\partial p_j} + \frac{\partial \hat{L}}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial p_j} + \frac{\partial \hat{L}}{\partial t}\frac{\partial t}{\partial p_j}\right] \\ &= \sum_{i=1}^{n}\left[\frac{\partial \hat{L}}{\partial q_i}T_{1,ij} + \frac{\partial \hat{L}}{\partial \dot{q}_i}T_{3,ij}\right] \end{aligned} \tag{46}$$

As a consequence,

$$\frac{\partial L}{\partial \mathbf{p}} = \mathbf{T}_1^T\frac{\partial \hat{L}}{\partial \mathbf{q}} + \mathbf{T}_3^T\frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \tag{47}$$

Likewise,

$$\frac{\partial L}{\partial \dot{\mathbf{p}}} = \mathbf{T}_2^T\frac{\partial \hat{L}}{\partial \mathbf{q}} + \mathbf{T}_4^T\frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}} \tag{48}$$

Recall that $L(\mathbf{p}, \dot{\mathbf{p}}, t)$ is a Lagrangian function for the decoupled system (17), satisfying Eq. (38). Substitute Eqs. (47) and (48) into Eq. (38) to obtain

$$\left(\mathbf{T}_4^{\mathrm{T}}\frac{d}{dt}\left[\frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}}\right] - \mathbf{T}_1^{\mathrm{T}}\frac{\partial \hat{L}}{\partial \mathbf{q}}\right) + \left(\mathbf{T}_2^{\mathrm{T}}\frac{d}{dt}\left[\frac{\partial \hat{L}}{\partial \mathbf{q}}\right] - \mathbf{T}_3^{\mathrm{T}}\frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}}\right) = \mathbf{0} \tag{49}$$

This is the equation satisfied by $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$. Moreover, evaluation of this equation yields an equation from which the equation of motion (3) can be extracted. One would consider $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ as a generalized Lagrangian function and Eq. (49) as a modified Euler–Lagrange equation.

Why does $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ satisfy the Euler–Lagrange Eq. (5) when system (3) is classically damped? Why is it necessary to use Eq. (49) in general to extract the equation of motion? If Eq. (3) represents a classically damped system, then Eq. (34) is applicable. In this case, the upper half of the decoupling transformation (28) reduces to the modal transformation $\mathbf{q} = \mathbf{U}\mathbf{p}$ and the lower half reduces to $\dot{\mathbf{q}} = \mathbf{U}\dot{\mathbf{p}}$. The generalized Lagrangian function $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ in Eq. (39) reduces to a traditional Lagrangian function given by Eq. (13). Equivalently, under the assumption that Eq. (3) is classically damped, $\mathbf{T}_1 = \mathbf{T}_4 = \mathbf{U}$ and $\mathbf{T}_2 = \mathbf{T}_3 = \mathbf{0}$. In this case, Eq. (49) simplifies to a traditional Euler–Lagrange equation given by Eq. (5). Essentially, the state-space decoupling transformation (28) or (41) becomes a configuration-space transformation under classical damping. A configuration-space transformation modifies the Euler–Lagrange equation by introducing only a matrix multiplier, and essentially $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ satisfies Eq. (5). In general, Eq. (28) or Eq. (41) is a genuine state-space transformation, which modifies the Euler–Lagrange equation to the form represented by Eq. (49).

In summary, system decoupling in real space, an approach utilized by Udwadia [17], always produces a scalar function $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$, which is either a Lagrangian function or a generalized Lagrangian function. In either case, $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ can be used to generate the equation of motion (3) and it contains information on system properties. Can a Lagrangian function be determined for any linear system? It will be shown in Sec. 6 that many coupled linear systems do not admit Lagrangian functions. In the search of a general solution to the inverse problem of linear Lagrangian dynamics, the generalized Lagrangian functions may be the best one can achieve. Two examples will illustrate the exposition given in this section.

Example 1. Consider a nonclassically damped system specified by

$$\ddot{\mathbf{q}} + \begin{bmatrix} 0.7 & -0.1 \\ -0.1 & 0.2 \end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\mathbf{q} = \mathbf{0} \tag{50}$$

This is a realization of Eq. (3). Solution of the quadratic eigenvalue problem (18) yields

$$\mathbf{\Lambda}_1 = \begin{bmatrix} -0.1792 + 1.0008i & 0 \\ 0 & -0.2708 + 1.6819i \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \overline{\mathbf{\Lambda}}_1 \tag{51}$$

$$\mathbf{V}_1 = \begin{bmatrix} 0.7328 - 0.0949i & 0.7152 + 0.1634i \\ 0.7180 + 0.0945i & -0.7118 + 0.1601i \end{bmatrix}, \quad \mathbf{V}_2 = \overline{\mathbf{V}}_1 \tag{52}$$

The real and diagonal coefficient matrices of the decoupled system (17) are given by

$$\mathbf{D} = \begin{bmatrix} 0.3584 & 0 \\ 0 & 0.5416 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1.0337 & 0 \\ 0 & 2.9022 \end{bmatrix} \tag{53}$$

The real decoupling transformations (28) and (41) are, respectively, defined by the matrices

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix} = \begin{bmatrix} 0.6740 & 0.7294 & -0.0948 & 0.0944 \\ 0.7474 & -0.7282 & 0.0972 & 0.0952 \\ 0.2840 & -0.2836 & 0.7498 & 0.7011 \\ -0.0991 & -0.0932 & 0.6889 & -0.7376 \end{bmatrix} \tag{54}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{bmatrix} = \begin{bmatrix} 0.7158 & 0.7415 & -0.0948 & 0.0972 \\ 0.7349 & -0.6860 & 0.0944 & 0.0952 \\ 0.0980 & -0.2820 & 0.7498 & 0.6889 \\ -0.0976 & -0.2763 & 0.7011 & -0.7376 \end{bmatrix} \tag{55}$$

The scalar function, or generalized Lagrangian function, in Eq. (39) is

$$\begin{aligned} \hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = {} & e^{0.3584t}\left(\begin{aligned} & -0.1455q_1^2 - 0.5543q_1q_2 - 0.2548q_2^2 + 0.2640\dot{q}_1^2 \\ & +0.3607q_1\dot{q}_1 - 0.0427q_2\dot{q}_1 + 0.5351\dot{q}_2\dot{q}_1 + 0.2270q_1\dot{q}_2 \\ & -0.1787q_2\dot{q}_2 + 0.2533\dot{q}_2^2 \end{aligned} \right) \\ & + e^{0.5416t}\left(\begin{aligned} & -0.7847q_1^2 + 1.5096q_1q_2 - 0.7079q_2^2 + 0.2424\dot{q}_1^2 \\ & -0.1316q_1\dot{q}_1 - 0.0075q_2\dot{q}_1 - 0.5352\dot{q}_2\dot{q}_1 - 0.2748q_1\dot{q}_2 \\ & +0.4029q_2\dot{q}_2 + 0.2405\dot{q}_2^2 \end{aligned} \right) \end{aligned} \tag{56}$$

Using Eqs. (55) and (56) to evaluate Eq. (49), one obtains

$$\begin{bmatrix} 0.7328e^{0.3584t} & 0.7180e^{0.3584t} \\ 0.7152e^{0.5416t} & -0.7118e^{0.5416t} \end{bmatrix} \times \left(\ddot{\mathbf{q}} + \begin{bmatrix} 0.7 & -0.1 \\ -0.1 & 0.2 \end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\mathbf{q} \right) = \mathbf{0} \tag{57}$$

Observe that

$$\det\left(\begin{bmatrix} 0.7328e^{0.3584t} & 0.7180e^{0.3584t} \\ 0.7152e^{0.5416t} & -0.7118e^{0.5416t} \end{bmatrix} \right) = -1.0351e^{0.9t} \neq 0 \tag{58}$$

for all t. Therefore, the equation of motion (50) can be extracted from Eq. (57). Indeed, the generalized Lagrangian function $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ generates the equation of motion specified by Eq. (50) from a modified Euler–Lagrange equation.

Example 2. A gyroscopic system is defined by

$$\ddot{\mathbf{q}} + \begin{bmatrix} 0 & -0.2 \\ 0.2 & 0 \end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\mathbf{q} = \mathbf{0} \tag{59}$$

This is a realization of Eq. (3) with a nonsymmetric coefficient matrix. Solution of the quadratic eigenvalue problem (18) yields

$$\mathbf{\Lambda}_1 = \begin{bmatrix} 0.9934i & 0 \\ 0 & 2.0132i \end{bmatrix}, \quad \mathbf{\Lambda}_2 = \overline{\mathbf{\Lambda}}_1 \tag{60}$$

$$\mathbf{V}_1 = \begin{bmatrix} -1.0022 & -0.1330i \\ 0.0661i & 1.0088 \end{bmatrix}, \quad \mathbf{V}_2 = \overline{\mathbf{V}}_1 \tag{61}$$

The real and diagonal coefficient matrices of the decoupled system (17) are given by

$$\mathbf{D} = \mathbf{0}, \quad \mathbf{B} = \begin{bmatrix} 0.9869 & 0 \\ 0 & 4.0531 \end{bmatrix} \tag{62}$$

The real decoupling transformations (28) and (41) are, respectively, defined by the matrices

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix} = \begin{bmatrix} -0.9936 & 0 & 0 & -0.0651 \\ 0 & 0.9741 & 0.0647 & 0 \\ 0 & 0.2603 & -0.9805 & 0 \\ -0.0647 & 0 & 0 & 0.9870 \end{bmatrix} \tag{63}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{bmatrix} = \begin{bmatrix} -1.0022 & 0 & 0 & -0.0661 \\ 0 & 1.0088 & 0.0665 & 0 \\ 0 & 0.2678 & -1.0022 & 0 \\ -0.0657 & 0 & 0 & 1.0088 \end{bmatrix} \tag{64}$$

The scalar function, or generalized Lagrangian function, in Eq. (39) is

$$\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = -0.4850q_1^2 - 1.8890q_2^2 - 0.5106q_2\dot{q}_1 + 0.4723\dot{q}_1^2 - 0.1276q_1\dot{q}_2 + 0.4850\dot{q}_2^2 \tag{65}$$

Using Eqs. (64) and (65) to evaluate Eq. (49), one obtains

$$\begin{bmatrix} -0.9805 & 0 \\ 0 & 0.9870 \end{bmatrix}\left(\ddot{\mathbf{q}} + \begin{bmatrix} 0 & -0.2 \\ 0.2 & 0 \end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\mathbf{q}\right) = \mathbf{0} \tag{66}$$

It follows that the equation of motion (59) can be extracted. This example demonstrates that systems with nonsymmetric coefficients can be readily treated.

5 Defective Linear Systems

Although defective or degenerate systems do not occur routinely, they have been studied by a number of authors [24,25]. If system (3) is defective, there is a repeated eigenvalue of Eq. (18) that does not possess a full set of independent eigenvectors, which must be supplemented with generalized eigenvectors. As demonstrated in Ref. [26], decoupling a defective system (3) is a delicate procedure that can easily vary on a case-by-case basis, but regardless it is always possible to decouple Eq. (3) into Eq. (17).

5.1 Decoupling of Defective Linear Systems. In general, the real and invertible decoupling transformation in the state space that connects systems (3) and (17) is time-varying and has the form [26]

$$\begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p\mathbf{J}_p \end{bmatrix} e^{\mathbf{J}_p t} e^{-\mathbf{J}_q t} \begin{bmatrix} \mathbf{V}_q \\ \mathbf{V}_q\mathbf{J}_q \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \mathbf{S}(t)\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \tag{67}$$

In Eq. (67), $\mathbf{J}_q$ represents a $2n \times 2n$ Jordan matrix of the indexed eigenvalues, and $\mathbf{V}_q$ is an $n \times 2n$ matrix of the corresponding eigenvectors. The $2n \times 2n$ Jordan matrix $\mathbf{J}_p$ is generally a modified form of $\mathbf{J}_q$ whose structure imposes the eigenvalue pairing schemes required for decoupling; the associated $n \times 2n$ matrix $\mathbf{V}_p$ enforces these pairing schemes. The time-varying transformation (67) generally cannot be simplified when system (3) is defective. However, under special circumstances, it is possible to obtain explicit forms for the real $2n \times 2n$ transformation matrix $\mathbf{S}(t)$ in Eq. (67) and its submatrices $\mathbf{S}_i(t)$ $(i = 1, 2, 3, 4)$ [26]. For example, suppose all eigenvalues of system (3) are complex, but only $2N < 2n$ of these eigenvalues are distinct. Assume that each defective eigenvalue has unit geometric multiplicity. Denote the algebraic multiplicity by m_k $(k = 1, 2, \ldots, N)$, and let $\mathbf{J}_k$ be an $m_k \times m_k$ Jordan block associated with one of the N unique eigenvalues λ_k with positive imaginary part:

$$\mathbf{J}_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_k & 1 \\ 0 & \cdots & 0 & 0 & \lambda_k \end{bmatrix} = \lambda_k\mathbf{I}_{m_k} + \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \mathbf{\Lambda}_k + \mathbf{N}_k \tag{68}$$

If an eigenvalue λ_k is not repeated, then $m_k = 1$. With unit geometric multiplicity, λ_k possesses a single eigenvector $\mathbf{v}_1^k$ and $m_k - 1$ generalized eigenvectors that can be arranged in an $n \times m_k$ matrix $\mathbf{V}_k$:

$$\mathbf{V}_k = [\, \mathbf{v}_1^k \quad \mathbf{v}_2^k \quad \cdots \quad \mathbf{v}_{m_k}^k \,] \tag{69}$$

From the matrices $\mathbf{J}_k$, $\mathbf{V}_k$, $\mathbf{\Lambda}_k$, and $\mathbf{N}_k$, construct the $n \times n$ matrices

$$\mathbf{J} = \mathrm{diag}[\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_N], \qquad \mathbf{V} = [\, \mathbf{V}_1 \quad \mathbf{V}_2 \quad \cdots \quad \mathbf{V}_N \,] \tag{70}$$

and

$$\mathbf{\Lambda} = \mathrm{diag}[\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \ldots, \mathbf{\Lambda}_N], \qquad \mathbf{N} = \mathrm{diag}[\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_N] \tag{71}$$

where $\mathbf{\Lambda}$ and $\mathbf{N}$ commute in multiplication. The defective system (3) is decoupled into Eq. (17) with coefficient matrices

$$\mathbf{D} = -(\mathbf{\Lambda} + \overline{\mathbf{\Lambda}}), \quad \mathbf{B} = \mathbf{\Lambda}\overline{\mathbf{\Lambda}} \tag{72}$$

This structure implies that the n independent SDOF systems associated with Eq. (17) consist of N collections of m_k identical systems with generally different initial conditions. This decoupled form is achieved by setting

$$\mathbf{J}_q = \mathrm{diag}[\mathbf{J}, \overline{\mathbf{J}}], \ \mathbf{V}_q = [\mathbf{V} \ \overline{\mathbf{V}}], \ \mathbf{J}_p = \mathrm{diag}[\mathbf{\Lambda}, \overline{\mathbf{\Lambda}}], \ \mathbf{V}_p = [\mathbf{I} \ \mathbf{I}] \tag{73}$$

in Eq. (67), yielding

$$\mathbf{S}(t) = \begin{bmatrix} \mathbf{S}_1(t) & \mathbf{S}_2(t) \\ \mathbf{S}_3(t) & \mathbf{S}_4(t) \end{bmatrix} = \begin{bmatrix} e^{-\mathbf{N}t} & \mathbf{0} \\ \mathbf{0} & e^{-\mathbf{N}t} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{VJ} & \overline{\mathbf{VJ}} \end{bmatrix}^{-1} \tag{74}$$

where

$$\mathbf{S}_1(t) = e^{-\mathbf{N}t}[(\mathbf{VJ})^{-1} - (\overline{\mathbf{VJ}})^{-1}][\mathbf{V}(\mathbf{VJ})^{-1} - \overline{\mathbf{V}}(\overline{\mathbf{VJ}})^{-1}]^{-1} \tag{75}$$

$$\mathbf{S}_2(t) = e^{-\mathbf{N}t}[\mathbf{V}^{-1} - \overline{\mathbf{V}}^{-1}][(\mathbf{VJ})\mathbf{V}^{-1} - (\overline{\mathbf{VJ}})\overline{\mathbf{V}}^{-1}]^{-1} \tag{76}$$

$$\mathbf{S}_3(t) = e^{-\mathbf{N}t}[\mathbf{\Lambda}(\mathbf{VJ})^{-1} - \overline{\mathbf{\Lambda}}(\overline{\mathbf{VJ}})^{-1}][\mathbf{V}(\mathbf{VJ})^{-1} - \overline{\mathbf{V}}(\overline{\mathbf{VJ}})^{-1}]^{-1} \tag{77}$$

$$\mathbf{S}_4(t) = e^{-\mathbf{N}t}[\mathbf{\Lambda V}^{-1} - \overline{\mathbf{\Lambda}}\overline{\mathbf{V}}^{-1}][(\mathbf{VJ})\mathbf{V}^{-1} - (\overline{\mathbf{VJ}})\overline{\mathbf{V}}^{-1}]^{-1} \quad (78)$$

If system (3) is nondefective, then $\mathbf{N} = \mathbf{0}$, $\mathbf{J} = \mathbf{\Lambda}$, and the time-varying Eqs. (75)–(78) reduce, as expected, to their time-invariant counterparts Eqs. (30)–(33) with $\mathbf{\Lambda} = \mathbf{\Lambda}_1$, $\overline{\mathbf{\Lambda}} = \mathbf{\Lambda}_2$, $\mathbf{V} = \mathbf{V}_1$, and $\overline{\mathbf{V}} = \mathbf{V}_2$. When some of the defective eigenvalues are real, concise forms for $\mathbf{S}_i(t)$ are typically not available. It is too laborious to provide an exhaustive summary of decoupling defective systems herein, so the interested reader is referred to Ref. [26] for additional details.

5.2 Transformation of Euler–Lagrange Equations. A Lagrangian function $L(\mathbf{p}, \dot{\mathbf{p}}, t)$ for the decoupled system (17), whether or not Eq. (3) is defective, can always be expressed as Eq. (37). When $L(\mathbf{p}, \dot{\mathbf{p}}, t)$ is expressed in terms of the original system coordinate $\mathbf{q}$, the resulting function $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ still has the form given by Eq. (39), but the submatrices $\mathbf{S}_i$ are time-varying if Eq. (3) is defective. When all eigenvalues are complex, $\mathbf{S}_i = \mathbf{S}_i(t)$ are specified by Eqs. (75)–(78). The inverse of Eq. (67) involves a matrix $\mathbf{T}$, which is also time-varying and is defined as follows:

$$\mathbf{T}(t) = \mathbf{S}^{-1}(t) = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{VJ} & \overline{\mathbf{VJ}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} e^{\mathbf{N}t} & \mathbf{0} \\ \mathbf{0} & e^{\mathbf{N}t} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1(t) & \mathbf{T}_2(t) \\ \mathbf{T}_3(t) & \mathbf{T}_4(t) \end{bmatrix} \quad (79)$$

$$\mathbf{T}_1(t) = (\mathbf{V}\overline{\mathbf{\Lambda}} - \overline{\mathbf{V}}\mathbf{\Lambda})(\overline{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} e^{\mathbf{N}t} \quad (80)$$

$$\mathbf{T}_2(t) = (\overline{\mathbf{V}} - \mathbf{V})(\overline{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} e^{\mathbf{N}t} \quad (81)$$

$$\mathbf{T}_3(t) = [(\mathbf{VJ})\overline{\mathbf{\Lambda}} - (\overline{\mathbf{VJ}})\mathbf{\Lambda}](\overline{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} e^{\mathbf{N}t} \quad (82)$$

$$\mathbf{T}_4(t) = (\overline{\mathbf{VJ}} - \mathbf{VJ})(\overline{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} e^{\mathbf{N}t} \quad (83)$$

Because $\mathbf{T}_i$ are time-varying, it can be shown that the modified Euler–Lagrange equation satisfied by $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ has the form

$$\left(\mathbf{T}_4^{\mathrm{T}} \frac{d}{dt}\left[\frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}}\right] - \left(\mathbf{T}_1^{\mathrm{T}} - \dot{\mathbf{T}}_2^{\mathrm{T}}\right)\frac{\partial \hat{L}}{\partial \mathbf{q}}\right) + \left(\mathbf{T}_2^{\mathrm{T}} \frac{d}{dt}\left[\frac{\partial \hat{L}}{\partial \mathbf{q}}\right] - \left(\mathbf{T}_3^{\mathrm{T}} - \dot{\mathbf{T}}_4^{\mathrm{T}}\right)\frac{\partial \hat{L}}{\partial \dot{\mathbf{q}}}\right) = \mathbf{0} \quad (84)$$

This is a generalization of Eq. (49) when system (3) is defective. As in the nondefective case, Eq. (84) implies that $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ is generally not a Lagrangian function for the defective system (3), but evaluation of Eq. (84) allows Eq. (3) to be unpacked from $\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$.

Example 3. Consider a nonclassically damped system of the form (3) specified by

$$\ddot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}\mathbf{q} = \mathbf{0} \quad (85)$$

Solution of the quadratic eigenvalue problem (18) indicates that the system is defective with a repeated complex eigenvalue such that

$$\mathbf{J} = \begin{bmatrix} -1 + i\sqrt{2} & 1 \\ 0 & -1 + i\sqrt{2} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} -i\sqrt{2} & 3 \\ 1 & 0 \end{bmatrix} \quad (86)$$

$$\mathbf{\Lambda} = (-1 + i\sqrt{2})\mathbf{I}, \quad \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (87)$$

The real and diagonal coefficient matrices of the decoupled system (17) are given by

$$\mathbf{D} = 2\mathbf{I}, \quad \mathbf{B} = 3\mathbf{I} \quad (88)$$

The real decoupling transformations (74) and (79) are, respectively, defined by the matrices

$$\mathbf{S}(t) = \begin{bmatrix} \mathbf{S}_1(t) & \mathbf{S}_2(t) \\ \mathbf{S}_3(t) & \mathbf{S}_4(t) \end{bmatrix} = \begin{bmatrix} -t/4 & 1 - t/4 & 0 & -t/4 \\ 1/4 & 1/4 & 0 & 1/4 \\ -t/4 - 1/4 & 5t/4 - 1/4 & -t/2 & t/4 + 3/4 \\ 1/4 & -5/4 & 1/2 & -1/4 \end{bmatrix} \quad (89)$$

$$\mathbf{T}(t) = \begin{bmatrix} \mathbf{T}_1(t) & \mathbf{T}_2(t) \\ \mathbf{T}_3(t) & \mathbf{T}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 3 - t & -1 & -t \\ 1 & t & 0 & 0 \\ 3 & 3t - 1 & 1 & t + 2 \\ 0 & 1 & 1 & t \end{bmatrix} \quad (90)$$

The scalar function, or generalized Lagrangian function, in Eq. (39) is

$$\hat{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = e^{2t}\left[\begin{aligned} &\left(\frac{3}{32}q_1^2 - \frac{9}{32}q_2^2 + \frac{1}{8}\dot{q}_1^2 + \frac{9}{32}\dot{q}_2^2 - \frac{13}{16}q_1q_2 + \frac{1}{4}q_1\dot{q}_1 - \frac{5}{16}q_1\dot{q}_2 - \frac{1}{2}q_2\dot{q}_1 + \frac{7}{16}q_2\dot{q}_2\right) \\ &+ t\left(\frac{1}{8}q_1^2 + \frac{5}{4}q_2^2 - \frac{1}{8}q_1q_2 + \frac{1}{8}q_1\dot{q}_1 - \frac{3}{8}q_1\dot{q}_2 - \frac{3}{8}q_2\dot{q}_1 + \frac{5}{4}q_2\dot{q}_2 - \frac{3}{8}\dot{q}_1\dot{q}_2\right) \\ &+ t^2\left(\frac{1}{16}q_1^2 + \frac{7}{16}q_2^2 + \frac{1}{8}\dot{q}_1^2 - \frac{1}{16}\dot{q}_2^2 - \frac{5}{8}q_1q_2 + \frac{1}{4}q_1\dot{q}_1 - \frac{1}{8}q_1\dot{q}_2 - \frac{1}{2}q_2\dot{q}_1 - \frac{1}{8}q_2\dot{q}_2\right) \end{aligned}\right] \quad (91)$$

Using Eqs. (90) and (91) to evaluate Eq. (84), one obtains

$$\begin{bmatrix} -te^{2t}/2 & 3e^{2t}/4 \\ e^{2t}/2 & 0 \end{bmatrix}\left(\ddot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}\mathbf{q}\right) = \mathbf{0} \quad (92)$$

Since the matrix multiplier is nonsingular for all t, the equation of motion (85) is extracted from Eq. (92). This example demonstrates that defective systems can indeed be tackled.

6 Existence of Lagrangian Functions for Linear Systems

In this section, it will be shown that some coupled linear systems do not admit Lagrangian functions. Most of the Lagrangian functions for MDOF linear systems reported in the literature [12,14,17] are bilinear of the form

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{A}_1(t)\mathbf{q} + \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{A}_2(t)\dot{\mathbf{q}} + \dot{\mathbf{q}}^{\mathrm{T}}\mathbf{A}_3(t)\mathbf{q} \tag{93}$$

where $\mathbf{A}_i(t)$ $(i = 1, 2, 3)$ are real $n \times n$ matrices that may depend on time. Note that Eq. (93) is a general form that accommodates both standard and nonstandard Lagrangians. In addition, suppose that, for example, it is required that the kinetic energy of Eq. (3) be expressible as a quadratic form of the velocities and the coefficients of the quadratic form can be time-varying. Then a Lagrangian function for this subclass of Eq. (3), if it exists, is reducible to the form given by Eq. (93). Evaluating the Euler–Lagrange equation (5) with Eq. (93) and matching coefficients yields the system of equations

$$\frac{1}{2}\left(\mathbf{A}_2(t) + \mathbf{A}_2^{\mathrm{T}}(t)\right) = \mathbf{Y}(t) \tag{94}$$

$$\frac{1}{2}\left(\dot{\mathbf{A}}_2(t) + \dot{\mathbf{A}}_2^{\mathrm{T}}(t)\right) + \left(\mathbf{A}_3(t) - \mathbf{A}_3^{\mathrm{T}}(t)\right) = \mathbf{Y}(t)\mathbf{C} \tag{95}$$

$$\dot{\mathbf{A}}_3(t) - \frac{1}{2}\left(\mathbf{A}_1(t) + \mathbf{A}_1^{\mathrm{T}}(t)\right) = \mathbf{Y}(t)\mathbf{K} \tag{96}$$

These equations imply that $\mathbf{Y}(t) \neq \mathbf{0}$ is symmetric and satisfies the matrix differential equation

$$\ddot{\mathbf{Y}}(t) - \dot{\mathbf{Y}}(t)\mathbf{C} = \mathbf{K}^{\mathrm{T}}\mathbf{Y}(t) - \mathbf{Y}(t)\mathbf{K} \tag{97}$$

However, due to the symmetry of $\mathbf{Y}(t)$, Eq. (97) constitutes an overdetermined system of differential equations. There are n^2 scalar differential equations associated with Eq. (97), but only $n(n+1)/2$ solutions are needed because $\mathbf{Y}(t)$ is symmetric. With $\mathbf{C}$ and $\mathbf{K}$ arbitrary for Eq. (3), it is generally not possible to make the n^2 scalar differential equations consistent, so an admissible nontrivial solution $\mathbf{Y}(t)$ does not exist. Thus, system (3) will generally not admit Lagrangian functions of the form given by Eq. (93) unless restrictions are placed on the coefficient matrices $\mathbf{C}$ and $\mathbf{K}$. It is important to note that the existence of an acceptable solution $\mathbf{Y}(t)$ does not guarantee the existence of a corresponding Lagrangian function. However, if there is no admissible solution to Eq. (97), then a Lagrangian does not exist for system (3).

To examine Eq. (97) more intimately, assume that Eq. (3) has two degrees-of-freedom, with

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}, \quad \mathbf{Y}(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_2(t) & y_3(t) \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \end{bmatrix} \tag{98}$$

In this case, the four component equations associated with Eq. (97) can be written explicitly as

$$\ddot{y}_1 - c_{11}\dot{y}_1 - c_{21}\dot{y}_2 = 0 \tag{99}$$

$$\ddot{y}_2 - c_{12}\dot{y}_1 - c_{22}\dot{y}_2 + k_{12}y_1 + (k_{22} - k_{11})y_2 - k_{21}y_3 = 0 \tag{100}$$

$$\ddot{y}_2 - c_{11}\dot{y}_2 - c_{21}\dot{y}_3 - k_{12}y_1 - (k_{22} - k_{11})y_2 + k_{21}y_3 = 0 \tag{101}$$

$$\ddot{y}_3 - c_{12}\dot{y}_2 - c_{22}\dot{y}_3 = 0 \tag{102}$$

Notice that y_2 must simultaneously satisfy two differential equations, Eqs. (100) and (101). Subtract Eq. (101) from Eq. (100) to obtain

$$-c_{12}\dot{y}_1 + (c_{11} - c_{22})\dot{y}_2 + c_{21}\dot{y}_3 + 2k_{12}y_1 + 2(k_{22} - k_{11})y_2 - 2k_{21}y_3 = 0 \tag{103}$$

If a Lagrangian function of the form (93) exists, there must be at least one nontrivial solution to Eq. (103) for which y_j $(j = 1, 2, 3)$ are not all zero. Since the elements c_{rs} $(r, s = 1, 2)$ and k_{rs} are arbitrary, the only way Eq. (103) will always be satisfied is for $y_j = 0$ and $\dot{y}_j = 0$, which contradicts the requirement that some y_j be nontrivial. Consequently, $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ defined by Eq. (93) is a Lagrangian function for a subclass of Eq. (3) only, i.e., there are systems of the form given by Eq. (3) that do not admit Lagrangian functions.

6.1 Deductions Using Helmholtz Conditions. If system (3) possesses a Lagrangian function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, the corresponding Euler–Lagrange equation (5) generates the system of equations

$$\mathbf{G}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) = \mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t)(\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q}) \tag{104}$$

Equation (104) must satisfy the Helmholtz conditions, which are necessary and sufficient conditions for the existence of $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ and can be specified in component form [13] as

$$\frac{\partial G_i}{\partial \ddot{q}_j} = \frac{\partial G_j}{\partial \ddot{q}_i}, \quad \frac{\partial G_i}{\partial q_j} - \frac{\partial G_j}{\partial q_i} = \frac{1}{2}\frac{d}{dt}\left[\frac{\partial G_i}{\partial \dot{q}_j} - \frac{\partial G_j}{\partial \dot{q}_i}\right], \quad \frac{\partial G_i}{\partial \dot{q}_j} + \frac{\partial G_j}{\partial \dot{q}_i} = \frac{d}{dt}\left[\frac{\partial G_i}{\partial \ddot{q}_j} + \frac{\partial G_j}{\partial \ddot{q}_i}\right] \tag{105}$$

where $i, j = 1, 2, \ldots, n$. Consider a subclass of Eq. (3) with $\mathbf{Y}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) = \mathbf{Y}(t)$. The ith component of Eq. (104) is given by

$$G_i(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, t) = \sum_{j=1}^{n} Y_{ij}\ddot{q}_j + \sum_{m,j=1}^{n} Y_{im}C_{mj}\dot{q}_j + \sum_{m,j=1}^{n} Y_{im}K_{mj}q_j \tag{106}$$

where $i = 1, 2, \ldots, n$, and $\mathbf{Y} = [Y_{ij}]$, $\mathbf{C} = [C_{ij}]$, and $\mathbf{K} = [K_{ij}]$. Note that

$$\frac{\partial G_i}{\partial \ddot{q}_j} = Y_{ij}, \quad \frac{\partial G_i}{\partial \dot{q}_j} = \sum_{m=1}^{n} Y_{im}C_{mj}, \quad \frac{\partial G_i}{\partial q_j} = \sum_{m=1}^{n} Y_{im}K_{mj} \tag{107}$$

The first part of Eq. (105) implies that $Y_{ij} = Y_{ji}$, and thus $\mathbf{Y}(t)$ must be symmetric. The second part of Eq. (105) yields

$$\sum_{m=1}^{n} Y_{im}K_{mj} - \sum_{m=1}^{n} Y_{jm}K_{mi} = \frac{1}{2}\sum_{m=1}^{n}\left[\dot{Y}_{im}C_{mj} - \dot{Y}_{jm}C_{mi}\right] \tag{108}$$

and therefore

$$\mathbf{Y}\mathbf{K} - \mathbf{K}^{\mathrm{T}}\mathbf{Y} = \frac{1}{2}\left[\dot{\mathbf{Y}}\mathbf{C} - \mathbf{C}^{\mathrm{T}}\dot{\mathbf{Y}}\right] \tag{109}$$

Likewise, the third part of Eq. (105) gives

$$\mathbf{C}^{\mathrm{T}}\mathbf{Y} = 2\dot{\mathbf{Y}} - \mathbf{Y}\mathbf{C} \tag{110}$$

Differentiating Eq. (110) and combining with Eq. (109) results in Eq. (97). Thus, application of the Helmholtz conditions leads to the same conclusion regarding the existence of Lagrangian functions as before.

Three examples are supplied to demonstrate how solution of Eq. (97) could generate a Lagrangian function for Eq. (3), if it exists.

Example 4. Consider a subclass of Eq. (3) consisting of nonclassically damped systems with two degrees-of-freedom such that the symmetric and positive–definite coefficient matrices $\mathbf{C}$ and $\mathbf{K}$ do not commute in multiplication. In this case, $c_{21} = c_{12}$ and $k_{21} = k_{12}$ so that Eq. (103) becomes

$$c_{12}(\dot{y}_3 - \dot{y}_1) + (c_{11} - c_{22})\dot{y}_2 + 2k_{12}(y_1 - y_3) + 2(k_{22} - k_{11})y_2 = 0 \tag{111}$$

The remaining elements of $\mathbf{C}$ and $\mathbf{K}$ are arbitrary, and thus Eq. (111) is satisfied only when

$$\dot{y}_1 = \dot{y}_3, \quad \dot{y}_2 = 0, \quad y_1 = y_3, \quad y_2 = 0 \tag{112}$$

To determine the existence of a Lagrangian function, there are multiple cases to consider. First, assume that $\dot{y}_1 = \dot{y}_3 = 0$ so that $y_1 = y_3 = \alpha =$ constant. In other words, $\mathbf{Y}(t) = \alpha\mathbf{I}$ and $\dot{\mathbf{Y}}(t) = \mathbf{0}$, and Eqs. (99)–(102) are immediately satisfied. Indeed, it can be verified by substitution that $\mathbf{Y}(t) = \alpha\mathbf{I}$ is an admissible solution of Eq. (97). However, as will be deduced by contradiction, a corresponding Lagrangian function does not exist. From Eqs. (94) and (95), $\mathbf{Y}(t) = \alpha\mathbf{I}$ implies that $\mathbf{A}_3(t) - \mathbf{A}_3^{\mathrm{T}}(t) = \alpha\mathbf{C}$ This cannot be true because $\mathbf{A}_3(t) - \mathbf{A}_3^{\mathrm{T}}(t)$ is skew-symmetric and $\mathbf{C}$ is symmetric. If $\mathbf{A}_3(t) = \mathbf{0}$, then either $\alpha = 0$, which is not admissible because it results in the trivial solution $\mathbf{Y}(t) = \mathbf{0}$, or $\mathbf{C} = \mathbf{0}$, which means that the system is undamped.

Next, assume that $\dot{y}_1 = \dot{y}_3 \neq 0$ and $\dot{y}_1 = \dot{y}_3$ is not constant. Because $y_1 = y_3$, Eqs. (99)–(102) imply that $c_{11} = c_{22}$ and $c_{12} = c_{21} = 0$. That means $\mathbf{C} = c_{11}\mathbf{I}$, in which case $\mathbf{CK} = \mathbf{KC}$ and the assumption that $\mathbf{C}$ and $\mathbf{K}$ do not commute in multiplication is contradicted.

Finally, assume that $\dot{y}_1 = \dot{y}_3 = \beta =$ constant $\neq 0$. Invoke Eqs. (99)–(102) to obtain $c_{11} = c_{22} = 0$ and $c_{12} = c_{21} = 0$. That means $\mathbf{C} = \mathbf{0}$ and the system is undamped. Therefore, the only valid solution to Eq. (97) is the trivial solution $\mathbf{Y}(t) = \mathbf{0}$, and a Lagrangian function of the form given by Eq. (93) does not exist as long as $\mathbf{C}$ and $\mathbf{K}$ are arbitrary and do not commute in multiplication.

Example 5. Suppose system (3) is gyroscopic with skew-symmetric $\mathbf{C}$ and symmetric and positive-definite $\mathbf{K}$. Substitute $c_{21} = -c_{12}$ and $k_{21} = k_{12}$ into Eq. (103) to obtain

$$-c_{12}(\dot{y}_3 + \dot{y}_1) + (c_{11} - c_{22})\dot{y}_2 + 2k_{12}(y_1 - y_3) + 2(k_{22} - k_{11})y_2 = 0 \tag{113}$$

The remaining elements of $\mathbf{C}$ and $\mathbf{K}$ are arbitrary, and thus Eq. (113) is satisfied only when

$$y_1 = y_3 = \alpha(t), \quad y_2 = 0, \quad \dot{y}_2 = 0, \quad \dot{y}_1 = -\dot{y}_3 \tag{114}$$

However, $y_1 = y_3 = \alpha(t)$ and $\dot{y}_1 = -\dot{y}_3$ cannot be simultaneously satisfied unless $\alpha(t) = 0$, which is not admissible because this results in the trivial solution $\mathbf{Y}(t) = \mathbf{0}$. Therefore, it must be that $\alpha(t) = \alpha =$ constant $\neq 0$, so an admissible solution to Eq. (97) is $\mathbf{Y}(t) = \alpha\mathbf{I}$, which can be easily verified by direct substitution. Does a Lagrangian function exist if $\mathbf{Y}(t) = \alpha\mathbf{I}$? Let $\alpha = 1$ so $\mathbf{Y}(t) = \mathbf{I}$. Equation (94) implies that $\mathbf{A}_2(t) = \mathbf{I} + \mathbf{R}_1(t)$, where $\mathbf{R}_1(t)$ is a skew-symmetric matrix. Choose $\mathbf{R}_1(t) = \mathbf{0}$ for simplicity. In this case, Eq. (95) reduces to $\mathbf{A}_3(t) - \mathbf{A}_3^{\mathrm{T}}(t) = \mathbf{C}$. But $\mathbf{C}$ is skew-symmetric, and therefore $\mathbf{A}_3(t) = (1/2)\mathbf{C} + \mathbf{R}_2(t)$, where $\mathbf{R}_2(t)$ is a symmetric matrix. Choose $\mathbf{R}_2(t) = \mathbf{0}$ for convenience. It follows from Eq. (96) that $\mathbf{A}_1(t) = -\mathbf{K} + \mathbf{R}_3(t)$, where $\mathbf{R}_3(t)$ is skew-symmetric. Let $\mathbf{R}_3(t) = \mathbf{0}$ for convenience. Then $\mathbf{A}_1(t) = -\mathbf{K}$, $\mathbf{A}_2(t) = \mathbf{I}$, and $\mathbf{A}_3(t) = (1/2)\mathbf{C}$ in Eq. (96). The gyroscopic system (3) admits the Lagrangian function

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{K}\mathbf{q} + \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{C}\mathbf{q} \tag{115}$$

which was reported by Udwadia and Cho [14].

Example 6. Let system (3) be classically damped. Udwadia [17] demonstrated that such a system admits Lagrangians given by Eqs. (13) and (14). To arrive at these results, instead of analyzing condition (103) for general $y_j(t)$ $(j = 1, 2, 3)$ and $\dot{y}_j(t)$, it is simpler in this case to consider initial conditions $y_j(0)$ and $\dot{y}_j(0)$ that satisfy Eq. (103) at $t = 0$, infer the existence of a corresponding solution $\mathbf{Y}(t)$ consistent with these initial conditions, verify that $\mathbf{Y}(t)$ satisfies Eq. (97), and then confirm that associated Lagrangians exist.

Because $\mathbf{C}$ and $\mathbf{K}$ are symmetric in this case, $c_{21} = c_{12}$ and $k_{21} = k_{12}$, and thus Eq. (103) becomes

$$c_{12}(\dot{y}_3 - \dot{y}_1) + (c_{11} - c_{22})\dot{y}_2 + 2k_{12}(y_1 - y_3) + 2(k_{22} - k_{11})y_2 = 0 \tag{116}$$

Note that $\mathbf{C}$ and $\mathbf{K}$ also commute in multiplication, so the remaining components of $\mathbf{C}$ and $\mathbf{K}$ are not arbitrary and satisfy

$$c_{11}k_{12} + c_{12}k_{22} = c_{12}k_{11} + c_{22}k_{12} \tag{117}$$

Does there exist at least one solution to condition (116) under the constraint (117)? Evaluate Eq. (116) at $t = 0$:

$$c_{12}(\dot{y}_{3,0} - \dot{y}_{1,0}) + (c_{11} - c_{22})\dot{y}_{2,0} + 2k_{12}(y_{1,0} - y_{3,0}) + 2(k_{22} - k_{11})y_{2,0} = 0 \tag{118}$$

where $y_{j,0} = y_j(0)$ and $\dot{y}_{j,0} = \dot{y}_j(0)$. Equation (118) is satisfied when, say

$$y_{1,0} = \alpha, \quad y_{2,0} = 0, \quad y_{3,0} = \alpha, \quad \dot{y}_{1,0} = \alpha c_{11}, \quad \dot{y}_{2,0} = \alpha c_{12}, \quad \dot{y}_{3,0} = \alpha c_{22} \tag{119}$$

with $\alpha =$ constant $\neq 0$, which implies

$$\mathbf{Y}(0) = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} = \alpha\mathbf{I}, \quad \dot{\mathbf{Y}}(0) = \begin{bmatrix} \alpha c_{11} & \alpha c_{12} \\ \alpha c_{12} & \alpha c_{22} \end{bmatrix} = \alpha\mathbf{C} \tag{120}$$

Is there a solution $\mathbf{Y}(t)$ associated with these initial conditions? By inspection, the solution $\mathbf{Y}(t) = \alpha e^{\mathbf{C}t}$ satisfies the initial conditions in Eq. (120), and substitution into Eq. (97) verifies that $\mathbf{Y}(t)$ is an admissible solution.

Is there a Lagrangian that corresponds to $\mathbf{Y}(t) = \alpha e^{\mathbf{C}t}$? For convenience, set $\alpha = 1$, and hence $\mathbf{A}_2(t) = e^{\mathbf{C}t} + \mathbf{R}_2(t)$ from Eq. (94), where $\mathbf{R}_2(t)$ is a skew-symmetric matrix that is chosen to be $\mathbf{R}_2(t) = \mathbf{0}$. As a result, and because $\mathbf{C}$ is symmetric, Eq. (95) reveals that $\mathbf{A}_3(t)$ is a symmetric matrix. Equation (96) implies $\mathbf{A}_1(t) = -e^{\mathbf{C}t}\mathbf{K} + \dot{\mathbf{A}}_3(t) + \mathbf{R}_1(t)$, for which $\mathbf{R}_1(t)$ is a skew-symmetric matrix; $\mathbf{R}_1(t) = \mathbf{0}$ is a convenient choice. Thus, with $\mathbf{A}_1(t) = -e^{\mathbf{C}t}\mathbf{K} + \dot{\mathbf{A}}_3(t)$, $\mathbf{A}_2(t) = e^{\mathbf{C}t}$, and $\mathbf{A}_3(t)$ symmetric, Eq. (93) defines a family of Lagrangian functions associated with $\mathbf{Y}(t) = e^{\mathbf{C}t}$. If $\mathbf{A}_3(t) = (1/2)e^{\mathbf{C}t}\mathbf{C}$, then the corresponding Lagrangian is given by Eq. (13); taking $\mathbf{A}_3(t) = \mathbf{0}$ yields the Lagrangian function (14).

7 Conclusions

A comprehensive study of the evaluation of Lagrangian functions for linear systems has been reported. The major results, summarized in the following statements, are applicable to both nondefective and defective linear systems possessing either symmetric or nonsymmetric coefficient matrices.

(1) While Lagrangian functions for decoupled linear systems can be readily found, coupled linear systems may or may not admit Lagrangian functions.
(2) Using an extension of modal analysis, any linear system can be decoupled in real space. Subsequently, a scalar function that plays the role of a Lagrangian function can be

determined. This scalar function is either a traditional Lagrangian function or a generalized Lagrangian function.

(3) A generalized Lagrangian function determined by system decoupling still produces the equation of motion and it still contains information on system properties. However, it satisfies a modified version of the Euler–Lagrange equation.

Given that many coupled linear systems do not admit traditional Lagrangian functions, generalized Lagrangian functions may be the best that one can achieve. Subject to this interpretation, a solution to the inverse problem of linear Lagrangian dynamics has been provided. Six examples have been supplied for illustration.

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