Applications and Extensions of Boij-Söderberg Theory

by

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Abstract

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Boij-Söderberg theory represents a breakthrough in our understanding of free resolutions. In [BS08a], Boij and Söderberg proposed the radical perspective that the numerics of graded free resolutions are best understood "up to scalar multiplication." The conjectures of Boij and Söderberg were then proven in a series of papers [EFW08, ES09a, BS08b, ES09b]. Further, [ES09a] describes the cone of cohomology tables of vector bundles on $\mathbb{P}^n$ by illustrating a duality with the cone of free resolutions. We use the phrase Boij-Söderberg theory to refer to the study of these two cones and the corresponding decomposition theorems.

Applications of graded free resolutions appear throughout algebraic geometry, commutative algebra, topology, combinatorics, and more; since Boij-Söderberg theory provides a structure theorem about the shapes of graded free resolutions, we might hope to apply the theory widely. One such application arose instantly, leading to a proof of the Herzog-Huneke-Srinivasan Multiplicity Conjecture, which had been open for decades.

However, Boij-Söderberg theory is such a radical departure from usual approaches to free resolutions that it is not immediately applicable to many of the situations where free resolutions arise. Boij-Söderberg theory is almost orthogonal to all previous approaches to understanding graded free resolutions, as the theory uses the combinatorial structure of Betti diagrams to group modules into families, whereas more traditional approaches use flatness to understand families of modules. The overarching goal of this thesis is thus to connect Boij-Söderberg theory with some of the previous avenues of research where graded free resolutions have arisen, and we pursue this theme in several directions.

Chapter 2 builds a framework for overcoming the limitation of working "up to scalar multiplication". Namely, we apply Boij-Söderberg theoretic results about the cone of Betti diagrams in order to investigate the integral structure of the semigroup of Betti diagrams. Our main results show that this semigroup is locally finitely generated, but that it can otherwise be quite pathological. In addition, we construct a number of nontrivial obstructions which prevent a diagram in the cone of Betti diagrams from belonging to the semigroup of Betti diagrams.

In Chapter 3, we consider the question of whether the Boij-Söderberg decomposition of a Betti diagram of a module is related to a flat deformation of the module. This is an essential mystery raised by Boij-Söderberg theory, and we provide the first results in this direction by producing large families where the Boij-Söderberg decomposition of a Betti diagram closely reflects a special filtration of the module. These results suggest that Boij-Söderberg theory might have deeper, as yet undiscovered, consequences for free resolutions. In addition, we provide applications to the classification of very singular spaces of matrices. This chapter is based on joint work with David Eisenbud and Frank-Olaf Schreyer.

In Chapter 4, we apply Boij-Söderberg theory to prove a special case of a famous conjecture in commutative algebra: the Buchsbaum-Eisenbud-Horrocks Rank Conjecture. Whereas the Multiplicity Conjecture is about the possible shapes of graded free resolutions, the Buchsbaum-Eisenbud-Horrocks Rank Conjecture is about the possible sizes of graded free resolutions. The conjecture is closely related to a topological conjecture of Carlsson about certain finite group actions on products of spheres. We prove a broad new case of the rank conjecture, and our method of proof–which involves a combination of Boij-Söderberg theory and optimization–is essentially unrelated to any previous work on the conjecture.

Finally, in Chapter 5, we apply Boij-Söderberg theory to the study of the asymptotics of free resolutions. Namely, we provide a lower bound for the Betti numbers of I^t when I is an ideal generated in a single degree. This builds on recent studies of asymptotic Castelnuovo-Mumford regularity, by Cutkosky, Herzog, Kodiyalam, Trung, and Wang.

To Katie,
whose strength is a continual inspiration.

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Chapter 1

Introduction

1.1 Overview

The use of graded free resolutions in algebra is based on a straightforward idea: we replace a complicated object of interest, such as a module or algebra, with a larger object built of simple pieces, namely a complex of free modules. By studying the complex, we hope to discover subtle properties of the original object. Due to their flexibility and power, graded free resolutions have become a fundamental algebraic tool.

Boij and Söderberg recently proposed the radical notion that the numerics of graded free resolutions are easier to understand if one worked "up to scalar multiplication" [BS08a]. The subsequent proof of their conjectures [EFW08, ES09a, BS08b, ES09b] has produced new structure theorems for graded free resolutions. In fact, [ES09a, BS08b] explicitly describe the cone of graded free resolutions, thus providing a complete structure theorem for the numerics of graded free resolutions, up to scalar multiplication. Further, [ES09a] describes the cone of cohomology tables of vector bundles on $\mathbb{P}^n$ by illustrating a duality with the cone of free resolutions. We use the phrase Boij-Söderberg theory to refer to the study of these two cones and the corresponding decomposition theorems.

Graded free resolutions arise throughout mathematics and have been an essential tool in many important results. For instance, Stanley's proof of the Upper Bound Conjecture is a combinatorial result about triangulations of spheres which relies on the use of graded free resolutions: Stanley first attaches a graded algebra to a given triangulation, and then he relates properties of the free resolution of the algebra to properties of the triangulation, via work of Reiner [Sta75, Cor. 5.3]. Sullivan's results on rational homotopy also rely on a variant of free resolutions [Sul77]: to each topological space X, Sullivan first attaches a differential graded algebra, and then he essentially resolves this complicated algebra with a huge differential free graded algebra now called the minimal Sullivan algebra. Graded free resolutions even play an important role in computational algebraic geometry: as illustrated in [BM93], the Castelnuovo-Mumford regularity of a graded module, which is a numerical invariant of the graded free resolution of the module, provides an essential measure of complexity for Gröbner basis computations involving the module. Further, the above examples are just a small sample of the subjects where graded free resolutions have been used.

As a powerful new structure theorem for graded free resolutions, Boij-Söderberg theory offers the potential for exciting applications. One such application arose instantly, and in fact provided the original inspiration for Boij and Söderberg's conjectures. The Herzog-Huneke-Srinivasan Multiplicity Conjecture states that one can use the shape of the minimal free resolution of a graded module to bound the multiplicity (a.k.a. the degree) of that module. Despite an immense amount of work on the conjecture (see [FS07] for an overview of the past literature), it remained open for decades. Then Boij-Söderberg theory instantly led to a proof of the Multiplicity Conjecture (see [ES09a, Cor. 0.3] and [BS08b, Thm. 3]), as well as proof of an analogous bound for the slopes of vector bundles (see [ES09a, Cor. 0.6]).

One might hope to produce similarly powerful applications in other situations where graded free resolutions arise; however, Boij-Söderberg theory is such a radical departure from classical approaches to free resolutions that it is not immediately applicable to most of these situations. The theory is almost orthogonal to all previous approaches to understanding graded free resolutions, as it uses the combinatorial structure of Betti diagrams (defined below) to organize modules into families, whereas more traditional approaches use flatness to understand families of modules. The overarching goal of this thesis is thus to connect Boij-Söderberg theory with some of the previous avenues of research where graded free resolutions have arisen, and to begin exploring and building methods for applying Boij-Söderberg theory.

To illustrate the challenge of applying the theory, let us consider an example from algebraic geometry. We fix notation which will be used throughout. Let $S := k[x_1, \dots, x_n]$ be the polynomial ring over a field k, and fix a finitely generated graded module M. By the graded version of Nakayama's Lemma, we may choose a minimal free resolution of M (c.f. [Eis05, §1B]), i.e. a free resolution

$$0 \longrightarrow F_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0$$

where each free module F_i is chosen to have minimal possible rank. We define the graded Betti numbers $\beta_{i,j}(M)$ by the formula

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}(M)}.$$

The Betti diagram of M, denoted $\beta(M)$, is an element of the infinite dimensional $\mathbb{Q}$-vector space $\mathbb{V} := \oplus_{i=0}^{n} \oplus_{j \in \mathbb{Z}} \mathbb{Q}$, and we write $\beta(M)$ as a matrix

$$\beta(M) = \begin{pmatrix} \vdots & \vdots & \ddots & \vdots \\ \beta_{0,0} & \beta_{1,1} & \dots & \beta_{n,n} \\ \beta_{0,1} & \beta_{1,2} & \dots & \vdots \\ \vdots & \vdots & \ddots & \end{pmatrix}.$$

In order to simplify notation when working with examples, we will follow a few conventions. First, we will assume that the upper left entry of each Betti diagram corresponds to degree 0, unless specifically noted otherwise. Second, we will only write down the finite portion of the Betti diagram which contains nonzero entries. Third, we will write a $-$ for an entry which equals zero.

We now turn to our example. Fix a non-hyperelliptic genus 5 curve C embedded in $\mathbb{P}^4_{\mathbb{C}}$ via the canonical embedding. Let $I_C \subseteq S = \mathbb{C}[x_0, \ldots, x_4]$ be the ideal of functions which vanish on C, and let S_C be the quotient algebra $S_C := S/I_C$.

Green's Conjecture [Eis05, Conj. 9.6], which has been mostly proven [Voi02, Voi05], illustrates that important algebro-geometric information about C can be read off directly from $\beta(S_C)$. In this case, $\beta(S_C)$ has the following form [Eis05, p. 183]:

$$\beta(S_C) = \begin{pmatrix} 1 & - & - & - \\ - & 3 & * & - \\ - & * & 3 & - \\ - & - & - & 1 \end{pmatrix} \tag{1.1.1}$$

for some $* \in \mathbb{N}$. The value of $*$ determines the gonality of C. Namely, $*$ is nonzero if and only the curve C admits a degree 3 map to $\mathbb{P}^1$. The shape of the Betti diagram $\beta(S_C)$ thus captures information about the existence or nonexistence or special line bundles on the curve C. Green's Conjecture states that this holds in general: for a curve C of arbitrary genus, the shape of $\beta(S_C)$ encodes the Clifford index of C (which is closely related to the gonality of C).

To prove even this special case of Green's Conjecture, one must understand the possible values that $*$ can attain. This is actually a somewhat delicate question, and the answer is that $*$ must equal 0 or 2, but it *cannot* equal 1. Geometrically, this means that if C is not a complete intersection then it *must* lie on a rational normal scroll. There are several approaches to proving that $*$ cannot equal 1, but these all require rather specialized algebraic or geometric facts.

By contrast, if we are willing to work "up to scalar multiplication," then Boij-Söderberg provides a pleasantly simple description for the possible values of $*$. Let M be a graded module such that

$$\beta(M) \text{ is a scalar multiple of } \begin{pmatrix} 1 & - & - & \\ - & 3 & * & - \\ - & * & 3 & - \\ - & - & - & 1 \end{pmatrix}.$$

Boij-Söderberg implies that $*$ can attain any rational value between 0 and $\frac{8}{3}$. The simplicity of the Boij-Söderberg theoretic answer stems from the fact that Boij-Söderberg theory provides an explicit description of the cone of graded free resolutions, and we use the boundary of the cone to derive to the desired inequalities. However, this approach would fail to answer the analogous question for a given fixed scalar multiple, since that would require integral information. In the example of our curve of genus 5, we would thus be unable to use Boij-Söderberg to conclude that $*$ cannot equal 1.

This illustrates an essential obstacle to applying Boij-Söderberg theory widely: applications of graded free resolutions often require integral information.

Our goal in Chapter 2 is thus to build a framework for overcoming the limitation of working "up to scalar multiple". We apply Boij-Söderberg theoretic results about the cone of Betti diagrams in order to investigate the integral structure of the semigroup of Betti diagrams. Our main results show that this semigroup is locally finitely generated, but that it can otherwise be quite pathological. In addition, we construct a number of nontrivial obstructions which prevent a diagram in the cone of Betti diagrams from belonging to the semigroup of Betti diagrams. These results originally appeared in [Erm09a].

In Chapter 3, we consider an essential mystery raised by Boij-Söderberg theory: is the Boij-Söderberg decomposition of a Betti diagram of a module related to a flat deformation of the module? We provide the first results in this direction by producing large families where the Boij-Söderberg decomposition of a Betti diagram closely reflects a special filtration of the module. These results have strong implications for the integral investigations considered in Chapter 2, and they also suggest that Boij-Söderberg theory might have deeper, as yet undiscovered, consequences for free resolutions. In addition, we provide applications to the classification of very singular spaces of matrices; this application is related to questions about classifying vector spaces of low rank matrices and torsion-free sheaves on projective spaces, as in [Atk83, AL80, Bea87, EH88]. This chapter is based on joint work with David Eisenbud and Frank-Olaf Schreyer which originally appeared in [EES10].

Chapters 4 and 5 illustrate how the precise numerics of Boij-Söderberg theory, when combined with even a small amount of integrality requirements, can lead to new applications in the study of graded free resolutions. In Chapter 4, we apply Boij-Söderberg theory to prove a special case of a famous conjecture in commutative algebra: the Buchsbaum-Eisenbud-Horrocks Rank Conjecture. Like the Multiplicity Conjecture, there has been a huge amount of research on the Buchsbaum-Eisenbud-Horrocks Rank Conjecture (c.f [Cha97, Cha00, Cha91, CEM90, EG88, Dug00, HR05, HU87, San90]). The conjecture also has implications for topology, as it is closely related to a conjecture of Carlsson about certain finite group actions on products of n-spheres [Car86, Conj. I.3 and II.2]. Our Theorem 1.5.2 covers a broad new case of the conjecture, and our method of proof–which involves a combination of Boij-Söderberg theory and optimization–is essentially unrelated to any previous work on the conjecture.

Finally, in Chapter 5, we apply Boij-Söderberg theory to the study of the asymptotics of free resolutions. Our Theorem 5.2.1 builds on recent studies of asymptotic Castelnuovo-Mumford regularity [Kod00, TW05, CHT99] and is based on the techniques developed in Chapter 4. The results of Chapter 4 and 5 originally appeared in [Erm09b].

1.2 Background on Boij-Söderberg Theory

Boij-Söderberg theory involves a duality between the cone of graded free resolutions over a polynomial ring, and the cone of cohomology tables on projective space. In this thesis, we focus exclusively on the free resolution side of this duality. The main Boij-Söderberg theoretic result about graded free resolutions is a description of the cone of Betti diagrams. This describes the shapes of all possible graded free resolutions, at least if we are willing to work with free resolutions "up to scalar multiple".

Let k be a field, $S := k[x_1, \ldots, x_n]$ with the grading $\deg(x_i) = 1$. Let M be a graded finitely generated module. Recall our notation for Betti diagrams from §1.1.

As an example of our notation, consider the cyclic module $M := k[x_1, x_2]/(x_1^2, x_1x_2, x_2^2)$. The free resolution of M is given by:

$$0 \longrightarrow S(-3)^2 \xrightarrow{\begin{pmatrix} x_2 & 0 \\ -x_1 & x_2 \\ -x_1 & 0 \end{pmatrix}} S(-2)^3 \xrightarrow{\begin{pmatrix} x_1^2 & x_1x_2 & x_2^2 \end{pmatrix}} S^1 \longrightarrow M \longrightarrow 0$$

We write simply:

$$\beta(M) = \begin{pmatrix} 1 & - & - \\ - & 3 & 2 \end{pmatrix}.$$

By observing that $\beta(M \oplus M') = \beta(M) + \beta(M')$, we may think of $\beta(-)$ as a map of semigroups

$$\{\text{ fin. generated graded } S-\text{modules}\} \xrightarrow{\beta} \mathbb{V}.$$

The image of this map is thus a semigroup.

Definition 1.2.1. *The following three definitions will be used throughout:*

1. *The* semigroup of Betti diagrams B_{mod} *is the subsemigroup of* $\mathbb{V}$ *generated by* $\beta(M)$ *for all possible finitely generated, graded* S*-modules* M*.*

2. *The* cone of Betti diagrams $B_{\mathbb{Q}}$ *is defined as the positive cone over* B_{mod} *in* $\mathbb{V}$*.*

3. *The* semigroup of virtual Betti diagrams B_{int} *is the semigroup of lattice points in* $B_{\mathbb{Q}}$*.*

In [ES09a, BS08b], a complete description of this cone is given in terms of extremal rays and facet equations. The extremal rays of $B_{\mathbb{Q}}$ correspond to pure diagrams, which we now define.

Given $d = (d_0, \ldots, d_t) \in \mathbb{Z}^{t+1}$, we say that d is a degree sequence if $d_{i+1} > d_i$ for all i. When $0 \leq t \leq n$, each degree sequence in $\mathbb{Z}^{t+1}$ defines a unique ray in $\mathbb{V}$ where all diagrams D on the ray satisfy

$$\beta_{i,j}(D) \neq 0 \iff j = d_i,$$

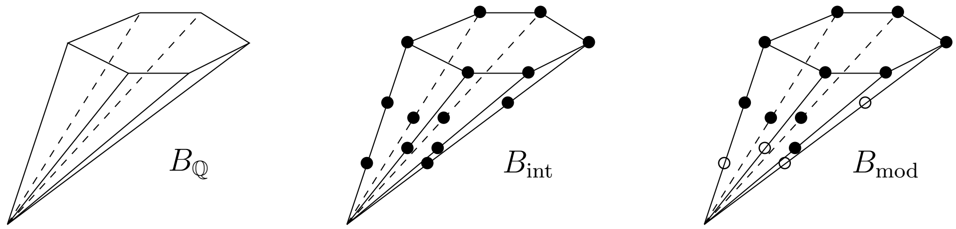


Figure 1.1: The cone of Betti diagrams $B_{\mathbb{Q}}$ is a polyhedral cone which admits a decomposition into a simplicial fan that reflects the partial order on degree sequences; this simplicial fan decomposition is described explicitly in [ES09a] and [BS08b]. This explicit description can be used to understand the integral structure of the semigroup of virtual Betti diagrams B_{int}. The semigroup of Betti diagrams B_{mod} is more mysterious.

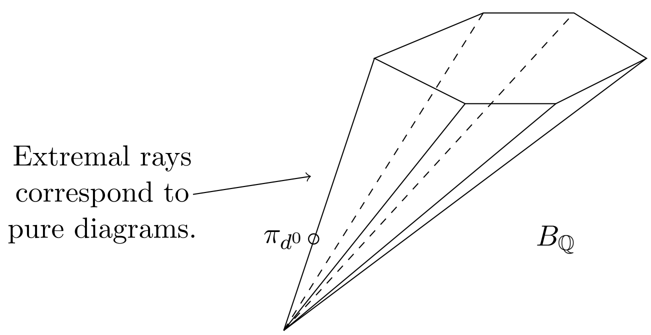


Figure 1.2: The extremal rays of $B_{\mathbb{Q}}$ correspond to pure diagrams.

and where the nonzero entries of D are, up to scalar multiple, given by the formula

$$\beta_{i,d_i}(D) = \prod_{i' \neq i} \frac{1}{|d_i - d_{i'}|}. \tag{1.2.1}$$

We say that $D \in \mathbb{V}$ is a pure diagram of type d if D belongs to this ray. We use the notation π_d to represent the first integral point on the ray, even though π_d may not belong to B_{mod}. However, [EFW08, Thms. 1,2] and [ES09a, Thm. 0.1] imply that some integral multiple of π_d belongs to B_{mod}.

For Cohen-Macaulay modules, the following theorem was first conjectured in [BS08a] and proven in [ES09a]. It was later shown in [BS08b] that Boij-Söderberg's conjecture and Eisenbud-Schreyer's proof extended to the non-Cohen-Macaulay case.

Theorem 1.2.2 (Eisenbud-Schreyer, Boij-Söderberg)**.** *The extremal rays of $B_{\mathbb{Q}}$ are precisely the rays spanned by pure diagrams.*

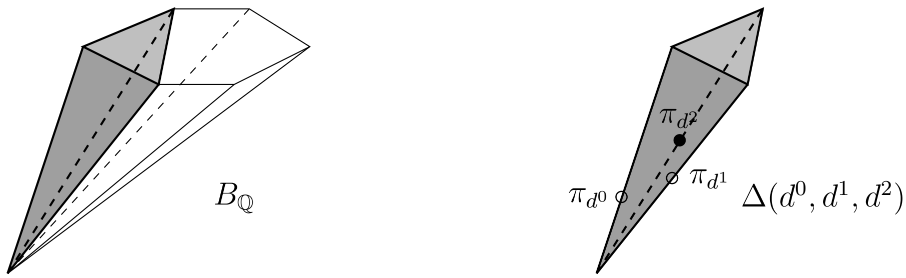


Figure 1.3: The cone $B_{\mathbb{Q}}$ is a simplicial fan. The simplex corresponding to a maximal sequence $d^0 < d^1 < d^2$ is highlighted in gray.

Theorem 1.2.2 thus provides a complete structure theorem for Betti diagrams, up to scalar multiple. An immediate corollary of this theorem is that every Betti diagram may be a written as a positive, rational sum of pure diagrams. Further, if we introduce a particular partial order on the set of degree sequences, then this decomposition is unique.

Definition 1.2.3. *Let $d = (d_0, \ldots, d_t) \in \mathbb{Z}^{t+1}$ and $d' = (d'_0, \ldots, d'_s) \in \mathbb{Z}^{s+1}$ be degree sequences. We say that $d \leq d'$ if $t > s$ or if $t = s$ and $d_i \leq d'_i$ for all i.*

Corollary 1.2.4 (Eisenbud-Schreyer, Boij-Söderberg)**.** *Let M be a graded S-module. There exists a unique chain of degree sequences $d^0 < \cdots < d^s$ and unique positive rational numbers c_i such that*

$$\beta(M) = \sum_{i=0}^{s} c_i \pi_{d^i}.$$

In other words, the pure diagrams are the essential building blocks of all graded free resolutions. In the following subsection, we will see that there is a simple algorithm for decomposing a Betti diagram into pure diagrams.

Another immediate corollary of this theorem is a proof of the Multiplicity Conjecture of Herzog-Huneke-Srinivasan (c.f. [ES09a, Cor. 0.3], [BS08b, Thm. 3]), which had been open for more than 20 years.

1.2.1 Examples of Boij-Söderberg Decomposition

The cone $B_{\mathbb{Q}}$ admits a natural decomposition as a simplicial fan. This simplicial decopmosition is essential to the proof of Corollary 1.2.4. Let $\Delta = (d^0, \ldots, d^s)$ be a chain of degree sequences $d^0 < d^1 < \cdots < d^s$. Consider the simplicial cone $B_{\mathbb{Q}}(\Delta)$ which is the cone spanned by π_{d^i} for $d^i \in \Delta$. The cone $B_{\mathbb{Q}}$ decomposes as the union of all such $B_{\mathbb{Q}}(\Delta)$ by [ES09a, Thm. 0.2] and [BS08b, Prop. 3]. We use the notation $B_{\text{int}}(\Delta)$ and $B_{\text{mod}}(\Delta)$ for the restrictions of B_{int} and B_{mod} to Δ.

The decomposition algorithm of [ES09a, §1] provides a method for writing any $D \in B_{\mathbb{Q}}(\Delta)$ uniquely as

$$D = \sum_{i=0}^{s} c_i \pi_{d^i}$$

with $c_i \in \mathbb{Q}_{>0}$. We refer to this as the Boij-Söderberg decomposition of D. We refer to $c_0\pi_{d^0}$ as the first step of the Boij-Söderberg decomposition, and so on.

We illustrate two examples of this decomposition algorithm. These examples illustrate how the decomposition algorithm enables us to determine when a diagram D belongs to $B_{\mathbb{Q}}$.

Example 1.2.5. *This example illustrates the decomposition algorithm in the case when $D \in B_{\mathbb{Q}}$. Let*

$$D := \begin{pmatrix} 1 & 2 & 1 & - \\ - & 1 & 2 & 1 \end{pmatrix}.$$

Note that D is the Betti diagram of $k[x,y,z]/(x,y,z^2)$, so that D certainly belongs to $B_{\mathbb{Q}}$.

To decompose D as a sum of pure diagrams, we first identify the degree sequence corresponding to the top strand *of D, where the top strand is simply the first nonzero entry in each column:*

$$D := \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} & - \\ - & 1 & 2 & \mathbf{1} \end{pmatrix}.$$

The bold entries correspond to the degree sequence $d^0 = (0,1,2,4)$. By (1.2.1), we compute that

$$\pi_{(0,1,2,4)} = \begin{pmatrix} 3 & 8 & 6 & - \\ - & - & - & 1 \end{pmatrix}.$$

Now, we will substract a scalar multiple of $\pi_{(0,1,2,4)}$ from D. To determine the appropriate scalar multiple, we use a greedy algorithm. Namely, we set $c_{(0,1,2,4)}$ to be the maximal rational number such that the diagram $D - c_{(0,1,2,4)}\pi_{(0,1,2,4)}$ continues to have nonnegative entries. In our example, this will be $c_{(0,1,2,4)} = \frac{1}{6}$ and thus

$$D - \frac{1}{6}\pi_{(0,1,2,4)} = \begin{pmatrix} \frac{1}{2} & \frac{2}{3} & - & - \\ - & 1 & 2 & \frac{5}{6} \end{pmatrix}.$$

We iterate the process for this new diagram, whose top strand corresponds to the degree sequence $(0,1,3,4)$. Since

$$\pi_{(0,1,3,4)} = \begin{pmatrix} 1 & 2 & - & - \\ - & - & 2 & 1 \end{pmatrix},$$

we see that $c_{(0,1,3,4)} = \frac{1}{3}$. We then have:

$$D - \frac{1}{6}\pi_{(0,1,2,4)} - \frac{1}{3}\pi_{(0,1,3,4)} = \begin{pmatrix} \frac{1}{6} & - & - & - \\ - & 1 & \frac{4}{3} & \frac{1}{2} \end{pmatrix}.$$

This equals $\frac{1}{6}\pi_{(0,2,3,4)}$*, and thus we have completed our Boij-Söderberg decomposition of* D*, namely:*

$$D = \frac{1}{6}\pi_{(0,1,2,4)} + \frac{1}{3}\pi_{(0,1,3,4)} + \frac{1}{6}\pi_{(0,2,3,4)}.$$

Example 1.2.6. *We next consider the case when* $D \notin B_{\mathbb{Q}}$*. Let*

$$D := \begin{pmatrix} 1 & 2 & 2 & - \\ - & 2 & 2 & 1 \end{pmatrix}.$$

The top strand of D *corresponds to the degree sequence* $(0,1,2,4)$*. We have*

$$\pi_{(0,1,2,4)} = \begin{pmatrix} 3 & 8 & 6 & - \\ - & - & - & 1 \end{pmatrix}.$$

We compute that $c_{(0,1,2,4)} = \frac{1}{4}$ *in this case. We then have*

$$D - \frac{1}{4}\pi_{(0,1,2,4)} = \begin{pmatrix} \frac{1}{4} & - & \frac{1}{2} & - \\ - & 2 & 2 & \frac{3}{4} \end{pmatrix}.$$

Now consider the top strand of this new diagram

$$\begin{pmatrix} \mathbf{\frac{1}{4}} & - & \mathbf{\frac{1}{2}} & - \\ - & \mathbf{2} & 2 & \mathbf{\frac{3}{4}} \end{pmatrix}.$$

This corresponds to the sequence $(0,2,2,4)$*, which is* not *a degree sequence, since the entries are not strictly increasing. As we cannot continue decomposing* D*, we conclude that* $D \notin B_{\mathbb{Q}}$*. In other words, there does not exist any module* M *such that* $\beta(M)$ *is a scalar multiple of* D*.*

1.3 The semigroup of Betti diagrams

This chapter considers the integral structure of Betti diagrams from the perspective of Boij-Söderberg theory, and begins to survey this new landscape. In particular, we consider several fundamental questions about the structure of the semigroup of Betti diagrams.

Note that Theorem 1.2.2 provides a complete structure theorem for the possible shapes of graded free resolutions, but only up to scalar multiple. Chapter 2 of this thesis focuses on the more classical question of which diagrams actually arise as the Betti diagram of some graded module. Namely, what can we say about B_{mod}?

A naive hope would be that the semigroup B_{mod} is simply equal to the semigroup B_{int}. But a quick search yields virtual Betti diagrams which cannot equal the Betti diagram of a module. Take for example the following pure diagram of type $(0,1,3,4)$

$$D_1 := \pi_{(0,1,3,4)} = \begin{pmatrix} 1 & 2 & - & - \\ - & - & 2 & 1 \end{pmatrix}. \tag{1.3.1}$$

This diagram belongs to the semigroup of virtual Betti diagrams. However, D_1 cannot equal the Betti diagram of an actual module as the two first syzygies would satisfy a linear Koszul relation which does not appear in the diagram D_1.

Since $B_{\text{int}} \neq B_{\text{mod}}$, it is thus natural to compare these two semigroups.

Question 1.3.1. We will consider the following questions about the semigroup of Betti diagrams:

(a) Is B_{mod} locally finitely generated?

(b) Does $B_{\text{mod}} = B_{\text{int}}$ in some special cases?

(c) Is B_{mod} a saturated semigroup?

(d) Is $B_{\text{int}} \setminus B_{\text{mod}}$ a finite set?

(e) On a single ray, can we have consecutive points of B_{int} which fail to belong to B_{mod}? Nonconsecutive points?

In Section 2.2, we answer Question 1.3.1(a) affirmatively:

Theorem 1.3.2. *The semigroup of Betti diagrams B_{mod} is locally finitely generated. Namely, $B_{\text{mod}}(\Delta)$ is finitely generated for any Δ.*

Sections 2.3 and 2.4 develop obstructions which prevent a virtual Betti diagram from being the diagram of some module. This is the technical heart of Chapter 2, as these obstructions are our tools for answering the rest of Question 1.3.1. In Section 2.5, we consider Question 1.3.1(b), and prove the following:

Proposition 1.3.3. *$B_{\text{int}} = B_{\text{mod}}$ for modules of projective dimension 1 and for level modules of projective dimension 2.*

Our proof of Proposition 1.3.3 rests heavily on [Söd06], which shows the existence of level modules of embedding dimension 2 and with a given Hilbert function.

In Section 2.6, we answer Questions 1.3.1(c-e). Here we show that the semigroup of Betti diagrams can have rather complicated behavior (see also Figure 1.3):

Theorem 1.3.4. *Each of the following occurs in the semigroup of Betti diagrams:*

1. *B_{mod} is not necessarily a saturated semigroup.*

2. *The set $|B_{\text{int}} \setminus B_{\text{mod}}|$ is not necessarily finite.*

3. *There exist rays of B_{int} which are missing at least $(\dim S - 2)$ consecutive lattice points.*

4. *There exist rays of B_{int} where the points of B_{mod} are nonconsecutive lattice points.*

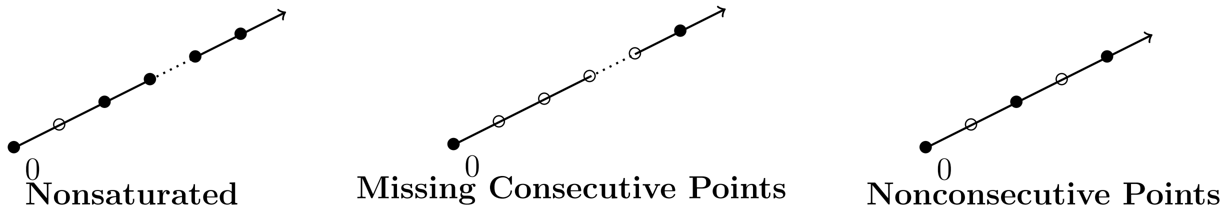


Figure 1.4: There exist rays which exhibit each of the above behaviors.

1.4 Pure filtrations

Boij-Söderberg theory shows that the Betti diagram of a module decomposes as a positive rational linear combination of a pure diagrams. A natural question is whether this numerical decomposition of Betti diagrams is a consequence of some undiscovered structure theorem for graded minimal free resolutions. For instance, we might ask the following rather vague question.

Question 1.4.1. Let M be a graded module. Does the Boij-Söderberg decomposition of $\beta(M)$ correspond to any sort of module-theoretic decomposition of the minimal free resolution of M?

We will show that Question 1.4.1 has an affirmative answer in many surprising cases. Namely, on certain lower-dimensional faces of the cone of Betti diagrams, the decomposition of the Betti diagram $\beta(M)$ can lead to a filtration of the minimal free resolution of M.

We say that a module M admits a pure filtration if there exists a filtration

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_t = 0$$

which extends to a filtration of the minimal free resolution of M, and where each M_i/M_{i+1} has a pure resolution (see below for the definition of a pure resolution). In other words, a pure filtration is a filtration of M which strongly reflects the Boij-Söderberg decomposition of $\beta(M)$. We say that M admits a pure splitting if M admits a pure filtration which also admits a splitting.

Our main result is a proof that, if $\beta(M)$ sits on certain lower-dimensional faces of the cone of Betti diagrams, then M *always* admits a pure filtration (c.f. Figure 1.5). An immediate corollary of this result is that, on these lower-dimensional faces, every Betti diagram of a module decomposes as a positive *integral* linear combination of pure diagrams.

To illustrate this result, we consider an example. Let M be a graded module with Betti diagram

$$\beta(M) = \begin{pmatrix} 6 & - & - & - & - & - \\ - & 60 & 128 & 90 & 32 & - \\ - & 32 & 90 & 128 & 60 & - \\ - & - & - & - & - & 6 \end{pmatrix}. \qquad (1.4.1)$$

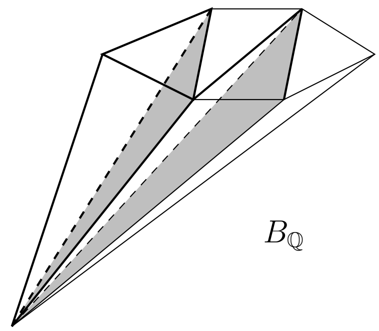


Figure 1.5: On certain lower-dimensional faces of $B_{\mathbb{Q}}$, every module admits a pure filtration. Our examples provide the first indication that Boij-Söderberg decomposition of a module might reflect a decomposition of the module itself.

The decomposition of $\beta(M)$ into pure summands is given by

$$\beta(M) = \begin{pmatrix} 3 & - & - & - & - & - \\ - & 40 & 96 & 90 & 32 & - \\ - & - & - & - & - & - \\ - & - & - & - & - & 1 \end{pmatrix} + \begin{pmatrix} 2 & - & - & - & - & - \\ - & 20 & 32 & - & - & - \\ - & - & - & 32 & 20 & - \\ - & - & - & - & - & 2 \end{pmatrix} + \begin{pmatrix} 1 & - & - & - & - & - \\ - & - & - & - & - & - \\ - & 32 & 90 & 96 & 40 & - \\ - & - & - & - & - & 3 \end{pmatrix}.$$

We will show that any such M admits a pure filtration:

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq M_3 = 0$$

where the Betti diagram of M_i/M_{i+1} corresponds to the $(2-i)$'th pure summand in the above decomposition. We will also deduce that, although all of the entries of $\beta(M)$ are divisible by 2, the existence of pure filtrations rules out the possibility of a module with Betti diagram $\frac{1}{2}\beta(M)$.

The existence of pure filtrations leads to several applications. Our first application illustrates further pathologies of the semigroup of Betti diagrams B_{mod}.

Proposition 1.4.2. *Let p be any prime. Then there exists a diagram $D \in B_{\text{int}}$ such that $cD \in B_{\text{mod}}$ if and only if c is divisible by p. In particular, there is a ray ρ in $B_{\mathbb{Q}}$ where only $\frac{1}{p}$ of the lattice points along ρ correspond to Betti diagrams of modules.*

This result simultaneously strengthens parts (2), (3) and (4) of Theorem 1.3.4

Second, we apply these techniques to classify certain vector spaces of matrices. Namely, we show that certain very singular spaces of matrices have finite moduli. This result is in the spirit of [EH88].

Third, we consider the question of minimal generators for B_{mod}. Although B_{mod} is always locally finitely generated by Theorem 1.3.2, the only examples where explicit generators for

B_{mod} were previously known came from cases where $B_{\text{mod}} = B_{\text{int}}$. We consider one of the simplest cases where $B_{\text{mod}} \neq B_{\text{int}}$, namely the semigroup of Betti diagrams of codimension 3 and regularity 1. In Proposition 3.6.1, we explicitly compute the minimal generators of B_{mod} in this case, thus providing the first example of its kind.

1.5 A special case of the Buchsbaum-Eisenbud-Horrocks Rank Conjecture

The Buchsbaum-Eisenbud-Horrocks Rank Conjecture (herein the BEH Rank Conjecture) says roughly that the Koszul complex is the "smallest" possible minimal free resolution.[1] The conjecture was formulated by Buchsbaum and Eisenbud in [BE77, p. 453] and, independently, the conjecture is implicit in a question of Horrocks [Har79, Problem 24]. Although the conjecture is most commonly phrased for regular local rings, we consider the graded case. Recall that for a finitely generated module M, we let

$$0 \longrightarrow F_p \xrightarrow{\phi_p} F_{p-1} \xrightarrow{\phi_{p-1}} \cdots \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

be the graded minimal free resolution of M. We define $\beta_j(M) := \text{rank}(F_j)$.

Conjecture 1.5.1 (Graded BEH Rank Conjecture)**.** *Let M be a graded Cohen-Macaulay S-module of codimension c. Then:*

$$\beta_j(M) \geq \binom{c}{j}$$

for $j = 0, \ldots, c$.

In Chapter 4, we prove a special case of the graded BEH rank conjecture. We do *not* require that M is Cohen-Macaulay.

Theorem 1.5.2. *Let M be a graded S-module of codimension c, generated in degree ≤ 0, and let $\underline{d}_1(M)$ be the minimal degree of a first syzygy of M. If $\text{reg}(M) \leq 2\underline{d}_1(M) - 2$, then*

$$\beta_j(M) \geq \beta_0(M)\binom{c}{j}$$

for $j = 0, \ldots, c$.

Common generalizations of the BEH rank conjecture include removing the Cohen-Macaulay hypothesis and/or strengthening the conclusion to the statement that $\text{rank}(\phi_j) \geq \binom{c-1}{j-1}$ for

[1] Terminology for this conjecture is inconsistent in the literature. In some places this conjecture is known as Horrocks' Conjecture or as the Syzygy Conjecture.

$$
\begin{array}{c}
\\
0 \\ 1 \\ \vdots \\ \underline{d}_1-2 \\ \underline{d}_1-1 \\ \vdots \\ 2\underline{d}_1-3 \\ 2\underline{d}_1-2
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & \dots & p \end{array} \\
\begin{pmatrix}
* & - & - & \dots & - \\
- & - & - & \dots & - \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
- & - & - & \dots & - \\
- & * & * & \dots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
- & * & * & \dots & * \\
- & * & * & \dots & *
\end{pmatrix}
\end{array}
$$

Figure 1.6: If M has a Betti diagram of the above shape, then it satisfies the Buchsbaum-Eisenbud-Horrocks Rank Conjecture.

$j = 1, \dots, c-1$. A different generalization, suggested in [Car86, Conj II.8], replaces the Betti numbers of a free resolution by the homology ranks of a differential graded module.

The BEH rank conjecture has been shown to hold for all modules of codimension at most 4 [EG88, p. 267]. In codimension at least 5, however, the BEH rank conjecture has only been settled for families of modules with additional structure.

Theorem 1.5.2 applies to modules whose Castelnuovo-Mumford regularity is small relative to the degree of the first syzygies of M. Though the literature on special cases of the BEH rank conjecture is extensive, Theorem 1.5.2 moves in a new direction. The most similar result in the literature is perhaps [Cha97, Thm. 0.1], which shows that the BEH rank conjecture holds when M is a Cohen-Macaulay module annihilated by the square of the maximal ideal $\mathfrak{m}$. Other known cases of the BEH rank conjecture include multigraded modules [Cha91, Thm. 3] and [San90], cyclic modules in the linkage class of a complete intersection [HU87], cyclic quotients by monomial ideals [EG88, Cor 2.5], and several more [Cha00], [CEM90], [Dug00], and [HR05]. See [CE92, pp. 25-27] for an expository account of some of this progress.

The method of proof for Theorem 1.5.2 is quite different from previous work on the BEH rank conjecture, as our proof is an application of Boij-Söderberg theory. At first glance, it might appear that Boij-Söderberg theory would not apply to Conjecture 1.5.1: Boij-Söderberg theory is based on the principle of only considering Betti diagrams up to scalar multiple, whereas the BEH rank conjecture depends on the integral structure of Betti diagrams. However, if the Betti diagram of M has shape as in Figure 1.5, then this imposes conditions on the pure diagrams which can appear in the Boij-Söderberg decomposition of M. This allows us to reduce the proof of Theorem 1.5.2 to a statement about the numerics of pure diagrams. We then use a multivariable calculus argument to degenerate the relevant pure diagrams to a Koszul complex.

1.6 Asymptotic Betti numbers

In our final, brief chapter, we use Boij-Södeberg theory to investigate the asymptotic behavior of $\beta(S/I^t)$ in the situation where I is generated by forms of a single degree. We provide estimates for the growth of the individual Betti numbers in this case. This result is based on asymptotic results about Castelnuovo-Mumford regularity as in [Kod00, TW05, CHT99].

Chapter 2

The semigroup of Betti diagrams

2.1 Overview

The main results of this chapter are a proof of the finite generation of $B_{\mathrm{mod}}(\Delta)$ (Theorem 1.3.2), the construction of obstructions for diagrams to belong to B_{mod} (Propositions 2.3.2 and 2.4.1), and a result illustrating the pathological nature of B_{mod} (Theorem 1.3.4). The results of this chapter originally appeared in [Erm09a].

This chapter is organized as follows. In Section 2.2, we prove that the semigroup of Betti diagrams is finitely generated. Sections 2.3 and 2.4 introduce obstructions for a virtual Betti diagram to be the Betti diagram of some module. The obstructions in Section 2.3 are based on properties of the Buchsbaum-Rim complex; the obstruction in Section 2.4 focuses on the linear strand of a resolution and is based on the properties of Buchsbaum-Eisenbud multiplier ideals. In Section 2.5, we consider the semigroup of Betti diagrams for small projective dimension, and we prove Proposition 1.3.3. In Section 2.6 we prove Theorem 1.3.4 by constructing explicit examples based on our obstructions.

Note that our work on pure filtrations in Chapter 3 has applications to the study of the semigroup of Betti diagrams as well. The relevant results appear in §3.4 and §3.6.

2.2 Finite generation of the semigroup of Betti diagrams

Before proving Theorem 1.3.2, we first prove a simpler analog for the semigroup of virtual Betti diagrams B_{int}.

Lemma 2.2.1. *For any $\Delta = (d^0, \dots, d^s)$, with $d^0 < \cdots < d^s$, the semigroup $B_{\mathrm{int}}(\Delta)$ is finitely generated. There exists an integer m_Δ such that every virtual Betti diagram can be written as a $\frac{1}{m_\Delta}\mathbb{N}$-combination of pure diagrams.*

Proof. The semigroup $B_{\rm int}(\Delta)$ is generated by pure diagrams $\pi_{d^0}, \ldots, \pi_{d^s}$ and by the lattice points inside the fundamental parallelepiped of Δ. This proves the first claim.

For the second claim of the lemma, let $P_1, \ldots, P_N$ be the minimal generators of $B_{\rm int}(\Delta)$. Every generator can be written as a positive rational sum:

$$P_i = \sum_j \frac{p_{ij}}{q_{ij}} \pi_{d^j}, \quad p_{ij}, q_{ij} \in \mathbb{N}.$$

We set m_Δ to be the least common multiple of all the q_{ij}. □

We refer to m_Δ as a universal denominator for $B_{\rm int}(\Delta)$. The existence of this universal denominator is central to our proof of the finite generation of $B_{\rm mod}$. In Remark 2.2.2, we mention how to compute m_Δ explicitly.

Proof of Theorem 1.3.2. Let $\pi_{d^0}, \ldots, \pi_{d^s}$ be the pure diagrams defining Δ, and let m_Δ be the universal denominator for $B_{\rm int}(\Delta)$.

For $i = 0, \ldots, s$, let $c_i \in \mathbb{N}$ be minimal such that $c_i \pi_{d^i}$ belongs to $B_{\rm mod}$. The existence of such a c_i is guaranteed by Theorems 0.1 and 0.2 of [EFW08] and Theorem 5.1 of [ES09a]. Let $\mathcal{S}_1$ be the semigroup generated by the pure diagrams $c_i \pi_{d^i}$. Let $\mathcal{S}_0$ be the semigroup generated by the pure diagrams $\frac{1}{m_\Delta} \pi_{d^i}$. Then we have the following inclusions of semigroups:

$$\mathcal{S}_1 \subseteq B_{\rm mod}(\Delta) \subseteq B_{\rm int}(\Delta) \subseteq \mathcal{S}_0.$$

Passing to semigroup rings gives:

$$k[\mathcal{S}_1] \subseteq k[B_{\rm mod}(\Delta)] \subseteq k[B_{\rm int}(\Delta)] \subseteq k[\mathcal{S}_0].$$

Observe that $k[\mathcal{S}_1]$ and $k[\mathcal{S}_0]$ are both polynomial rings of dimension $s+1$, and that $k[\mathcal{S}_1] \subseteq k[\mathcal{S}_0]$ is a finite extension of rings. This implies that $k[\mathcal{S}_1] \subseteq k[B_{\rm mod}(\Delta)]$ is also a finite extension, and hence $k[B_{\rm mod}(\Delta)]$ is a finitely generated k-algebra. We conclude that $B_{\rm mod}(\Delta)$ is a finitely generated semigroup. □

2.2.1 Computing Generators of the semigroup of virtual Betti diagrams

Minimal generators of $B_{\rm int}(\Delta)$ can be computed explicitly as the generators of the $\mathbb{N}$-solutions to a certain linear $\mathbb{Z}$-system defined by the π_{d^i} and by m_Δ. For an overview of relevant algorithms, see the introduction of [PCVT04]. The following example illustrates the method.

Consider $S = k[x, y], \underline{d} = (0, 1, 4), \overline{d} = (0, 3, 4)$. The corresponding cone of Betti diagrams has several simplices and we choose the simplex Δ spanned by the maximal chain of degree sequences:

$$(0) > (0, 3) > (0, 3, 4) > (0, 2, 4) > (0, 1, 4).$$

The corresponding pure diagrams are:

$$\begin{pmatrix} 1 & - & - \\ - & - & - \\ - & - & - \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 1 & - \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & 2 & - \\ - & - & 1 \end{pmatrix}, \begin{pmatrix} 3 & 4 & - \\ - & - & - \\ - & - & 1 \end{pmatrix} \tag{2.2.1}$$

First we must compute m_Δ. To do this, we consider the square matrix Φ whose columns correspond to the above pure diagrams:

$$\Phi = \begin{pmatrix} 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 \end{pmatrix} \tag{2.2.2}$$

The columns of Φ are indexed by the pure diagrams in (2.2.1) and the rows of Φ are indexed by the Betti numbers $\beta_{0,0}, \beta_{1,1}, \beta_{1,2}, \beta_{1,3}$ and $\beta_{2,4}$ respectively. Since the columns of Φ are $\mathbb{Q}$-linearly independent, it follows that the cokernel of Φ is entirely torsion. Note that each minimal generator of $B_{\text{int}}(\Delta)$ is either a pure diagram or corresponds to a unique nonzero torsion element of $\text{coker}(\Phi)$. The annihilator of $\text{coker}(\Phi)$ is thus the universal denominator for Δ. A computation in [GS] shows that $m_\Delta = 12$ in this case.

We next compute minimal generators of the $\mathbb{N}$-solutions of the following linear $\mathbb{Z}$-system:

$$\mathbb{Z}^{10} \xrightarrow{\begin{pmatrix} -12 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & -12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & -12 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -12 & 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & 0 & 0 & 3 & 1 & 1 \end{pmatrix}} \mathbb{Z}^5.$$

The $\mathbb{N}$-solutions of the above system correspond to elements of $B_{\text{int}}(\Delta)$ under the correspondence:

$$(b_1, b_2, b_3, b_4, b_5, a_1, a_2, a_3, a_4, a_5) \mapsto \frac{a_1}{12}\pi_{(0)} + \frac{a_2}{12}\pi_{(0,3)} + \frac{a_3}{12}\pi_{(0,3,4)} + \frac{a_4}{12}\pi_{(0,2,4)} + \frac{a_5}{12}\pi_{(0,1,4)}.$$

Computation yields that $B_{\text{int}}(\Delta)$ has 14 minimal semigroup generators.[1] These consist of the 5 pure diagrams from line (2.2.1) plus the following 9 diagrams:

$$\begin{pmatrix} 1 & 1 & - \\ - & - & - \\ - & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & - \\ - & 1 & - \\ - & - & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 & - \\ - & - & - \\ - & 1 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 3 & 3 & - \\ - & - & - \\ - & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & - & - \\ - & 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 & - \\ - & 1 & - \\ - & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & - \\ - & 1 & - \\ - & 1 & 1 \end{pmatrix},$$

It is not difficult to verify that each of these generators is the Betti diagram of some module. Thus in this case we have $B_{\text{int}}(\Delta) = B_{\text{mod}}(\Delta)$.

[1] We use Algorithm 2.7.3 of [Stu93] for this computation. Also, see [PCVT04] for other relevant algorithms.

Remark 2.2.2. More generally, for any $\Delta = (d^0, \ldots, d^s)$ we may compute m_Δ as follows. Let $\Phi_\Delta : \mathbb{Z}^{s+1} \to \bigoplus_{i=0}^{n} \bigoplus_{j\in\mathbb{Z}} \mathbb{Z}$, where the i'th generator maps to π_{d^i}. With this notation, m_Δ is simply the generator of the annihilator of $\mathrm{coker}(\Phi_\Delta)$.

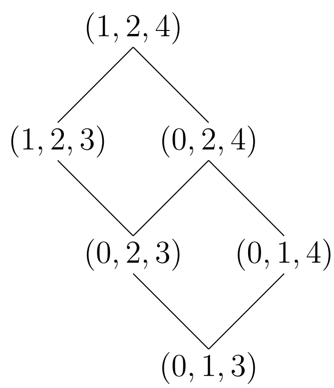
Remark 2.2.3. A better understanding of the combinatorics of B_{int} and $B_{\mathbb{Q}}$ might be useful for computations or applications involving Boij-Söderberg theory. For instance, let $\mathcal{D}$ be any finite collection of degree sequences, and let $B_{\mathbb{Q}}(\mathcal{D})$ be the subcone generated by rays corresponding to elements of $\mathcal{D}$. Define the polytope $P(\mathcal{D})$ as the intersection of $B_{\mathbb{Q}}(\mathcal{D})$ with the hyperplane $\sum_{i=0}^{n} \sum_{j\in\mathbb{Z}} \beta_{i,j} = 1$.

We may view the set $\mathcal{D}$ as a poset via the partial order on degree sequences given in Definition 1.2.3. We may thus define the order polytope $O(\mathcal{D})$ of $\mathcal{D}$ (c.f. [Sta81, Defn. 1.1].) It is natural to wonder whether the polytopes $P(\mathcal{D})$ and $O(\mathcal{D})$ are cominbatorially equivalent. However, in unpublished computations, Sanyal and Sturmfels have each observed that $P(\mathcal{D})$ and $O(\mathcal{D})$ are generally not equivalent.

As an example, let $\mathcal{D}$ contain all pure diagrams of length 3 which fit into the shape

$$\begin{pmatrix} * & * & - \\ * & * & * \\ - & - & * \end{pmatrix}.$$

Thus $\mathcal{D}$ is the following poset of degree sequences:



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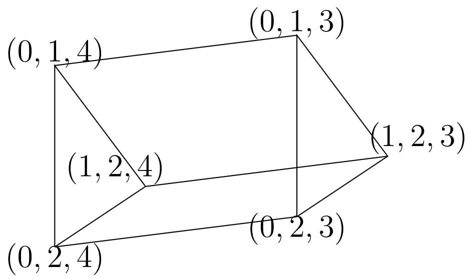
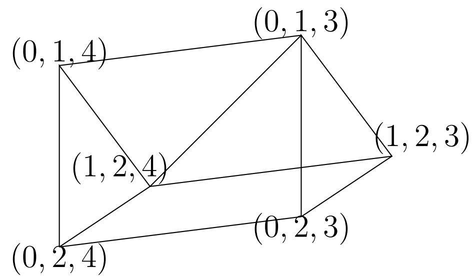
Computation in [GS] yields that the f-vector of $P(\mathcal{D})$ is $(6, 10, 6)$, while the f-vector of $O(\mathcal{D})$ is $(6, 9, 5)$. Further, the boundaries of both $P(\mathcal{D})$ and $O(\mathcal{D})$ are polyhedral 2-spheres, with the boundary for $P(\mathcal{D})$ subdividing the boundary for $O(\mathcal{D})$. This subdivision is illustrated below:

Boundary of $O(\mathcal{D})$



Boundary of $P(\mathcal{D})$



.

It would be particularly interesting to investigate the combinatorics of polytopes of the form $P(\mathcal{D})$, as this might provide more effective tools for working with objects such as $B_{\text{int}}(\mathcal{D})$.

2.3 Buchsbaum-Rim obstructions to existence of Betti diagrams

In Proposition 2.3.2 we illustrate obstructions which prevent a virtual Betti diagram from being the Betti diagram of an actual module. To yield information not contained in the main results of [ES09a] and [BS08b], these obstructions must be sensitive to scalar multiplication of diagrams. For simplicity we restrict to the case that M is generated in degree 0, though all of these obstructions can be extended to the general case.

We say that a diagram D is a Betti diagram if D equals the Betti diagram of some module M, and we say that D is a virtual Betti diagram if D belongs to the semigroup of virtual Betti diagrams B_{int}. Many properties of modules (e.g. codimension, Hilbert function) can be computed directly from the Betti diagram. We extend such properties to virtual diagrams in the obvious way. Proposition 2.3.2 only involves quantities which can be determined entirely from the Betti diagram; thus we may easily test whether an arbitrary virtual Betti diagram is "obstructed" in the sense of this proposition.

Definition 2.3.1. *For any $D \in \mathbb{V}$, we define $\underline{d}_i(D) := \min\{j | \beta_{i,j}(D) \neq 0\}$ and we define $\overline{d}_i(D) : \max\{j | \beta_{i,j}(D) \neq 0\}$.*

Proposition 2.3.2 (Buchsbaum-Rim obstructions)**.** *Let M be a graded module of codimension $e \geq 2$ with minimal presentation:*

$$\bigoplus_{\ell=1}^{b} S(-j_\ell) \xrightarrow{\phi} S^a \to M \to 0.$$

Assume that $j_1 \leq j_2 \leq \cdots \leq j_b$. Then we have the following obstructions:

1. *(Second syzygy obstruction):*

$$\underline{d_2}(M) \leq \sum_{\ell=1}^{a+1} j_\ell$$

2. *(Codimension obstruction)*

$$b = \sum_j \beta_{1,j}(M) \geq e + a - 1$$

If we have equality, then $\beta(M)$ must equal the Betti diagram of the Buchsbaum-Rim complex of ϕ.

3. *(Regularity obstruction in Cohen-Macaulay case): If M is Cohen-Macaulay then we also have that*

$$\operatorname{reg}(M) + e = \overline{d_e}(M) \leq \sum_{\ell=b-e-a+2}^{b} j_\ell.$$

These obstructions are independent of one another, and each obstruction occurs for some virtual Betti diagram.

In addition, note that both the weak and strong versions of the Buchsbaum-Eisenbud-Horrocks rank conjecture about minimal Betti numbers (see [BE77]or [CEM90] for a description) would lead to similar obstructions. Since each Buchsbaum-Eisenbud-Horrocks conjecture imposes a condition on each column of the Betti diagram, the corresponding obstruction would greatly strengthen part (2) of the above proposition.

Remark 2.3.3. For D a diagram, let $D^\vee$ be the diagram obtained by rotating D by 180 degrees. When D is the Betti diagram of a Cohen-Macaulay module M of codimension e, then $D^\vee$ is the Betti diagram of some twist of $M^\vee := Ext_S^e(M, S)$, which is also a Cohen-Macaulay module of codimension e. Thus, in the Cohen-Macaulay case, we may apply these obstructions to D or to $D^\vee$.

Given any map $\widetilde{\phi}$ between free modules F and G, we can construct the Buchsbaum-Rim complex on this map, which we denote as $\mathrm{Buchs}_\bullet(\widetilde{\phi})$. The Betti table of the complex $\mathrm{Buchs}_\bullet(\widetilde{\phi})$ will depend only on the Betti numbers of F and G, and it can be thought of as an approximation of the Betti diagram of the cokernel of $\widetilde{\phi}$.

As in the statement of Proposition 2.3.2, let M be a graded S-module of codimension ≥ 2 with minimal presentation

$$F_1 := \bigoplus_{\ell=1}^{b} S(-j_\ell) \xrightarrow{\phi} S^a \to M \to 0.$$

We will consider free submodules $\widetilde{F_1} \subseteq F_1$, the induced map $\widetilde{\phi} : \widetilde{F_1} \to S^a$, and the Buchsbaum-Rim complex on $\widetilde{\phi}$. By varying $\widetilde{\phi}$ we will produce the obstructions listed in Proposition 2.3.2.

To prove the first obstruction, we introduce some additional notation. Let the first syzygies of M be $\sigma_1, \ldots, \sigma_b$ with degrees $\deg(\sigma_\ell) = j_\ell$. The first stage of the Buchsbaum-Rim complex on ϕ is the complex

$$\bigwedge^{a+1} F_1 \xrightarrow{\epsilon} F_1 \to S^a.$$

A basis of $\bigwedge^{a+1} F_1$ is given by $e_{I'}$ where I' is a subset $I' \subseteq \{1, \ldots, b\}$ with $|I'| = a+1$. Let $\det(\phi_{I' \setminus \{i\}})$ be the maximal minor corresponding to the columns $I' \setminus \{i\}$. Then the map ϵ sends $e_{I'} \mapsto \sum_{i \in I'} e_i \det(\phi_{I' \setminus \{i\}})$. We refer to $\epsilon(e_{I'})$ as a Buchsbaum-Rim second syzygy, and we denote it by $\rho_{I'}$. There are $\binom{b}{a+1}$ Buchsbaum-Rim second syzygies. It may happen that one of these syzygies specializes to 0 in the case of ϕ. But as we now prove, if $\rho_{I'}$ specializes to 0 then we can find another related syzygy in lower degree.

Lemma 2.3.4. *Let $I' = \{i_1, \ldots, i_{a+1}\} \subseteq \{1, \ldots, b\}$, and assume that $\rho_{I'}$ is a trivial second syzygy. Then M has a second syzygy of degree strictly less than $\sum_{i \in I'} j_i$ and supported on a subset of the columns corresponding to I'.*

Proof. Let A be an $a \times b$-matrix representing ϕ. Let $C = \{1, \ldots, b\}$ index the columns of A, and let $W = \{1, \ldots, a\}$ index the rows of A. If $I \subseteq C$ and $J \subseteq W$ then we write $A_{I,J}$ for the corresponding submatrix.

The Buchsbaum-Rim syzygy $\rho_{I'}$ is trivial if and only if all the $a \times a$ minors of $A_{I',W}$ are zero. Let $a' = rank(A_{I',W})$ which by assumption is strictly less than a. We may assume that the upper left $a' \times a'$ minor of $A_{I',W}$ is nonzero. We set $I'' = \{i_1, \ldots, i_{a'+1}\}$ and $J'' = \{1, \ldots, a'\}$. Let τ be the Buchsbaum-Rim syzygy of $A_{I'',J''}$. Then $\tau \neq 0$ because $\det(A_{I'' \setminus \{a'+1\}, J''}) \neq 0$. Also $(A_{I'',J''}) \cdot \tau = 0$. Thus:

$$\left(A_{I'',W}\right) \cdot \tau = \begin{pmatrix} A_{I'',J} \\ A_{I'',W-J''} \end{pmatrix} \cdot \tau = \begin{pmatrix} 0 \\ * \end{pmatrix}.$$

There exists an invertible matrix $B \in GL_a(k(x_1, \ldots, x_n))$ such that:

$$B \cdot A_{I'',W} = \begin{pmatrix} A_{I'',J''} \\ 0 \end{pmatrix}.$$

This gives:

$$0 = (B \cdot A_{I'',W}) \cdot \tau = B \cdot (A_{I'',W} \cdot \tau).$$

Since B is invertible over $k(x_1, \ldots, x_n)$ we conclude that $A_{I'',W} \cdot \tau = 0$. Thus τ is a syzygy on the columns of A indexed by I'', and therefore τ represents a second syzygy of M. The degree of τ is $\sum_{i \in I''} j_i$ which is strictly less than $\sum_{i \in I'} j_i$. $\square$

We may now prove the second syzygy obstruction and the codimension obstruction.

Proof of the second syzygy obstruction in Proposition 2.3.2. Apply Lemma 2.3.4, choosing $I' = \{1, \dots, a+1\}$. □

Proof of codimension obstruction in Proposition 2.3.2. Recall that the module M has minimal presentation:

$$\bigoplus_{\ell=1}^{b} S(-j_\ell) \xrightarrow{\phi} S^a \to M \to 0.$$

Let $\mathrm{Buchs}_\bullet(\phi)$ be the Buchsbaum-Rim complex of ϕ. Then we have

$$\mathrm{codim}(M) \leq \mathrm{pdim}(M) \leq \mathrm{pdim}(\mathrm{Buchs}_\bullet(\phi)) = b - a + 1 = \sum_j \beta_{1,j}(M) - a + 1.$$

Since M has codimension e, we obtain the desired inequality. In the case of equality, the maximal minors of ϕ contain a regular sequence of length e, so we may conclude:

$$\beta(M) = \beta(\mathrm{Buchs}_\bullet(\phi)),$$

as desired. □

Proof of regularity obstruction in Proposition 2.3.2. Since M is Cohen-Macaulay of codimension e, we may assume by Artinian reduction that M is finite length. Recall that $b = \sum_j \beta_{1,j}(M)$ and let ϕ as in the proof of the codimension obstruction. If $b = e + a - 1$ then we have that

$$\mathrm{reg}(M) = \mathrm{reg}(\mathrm{Buchs}_\bullet(\phi)) = \sum_{\ell=1}^{b} j_\ell.$$

We are left with the case that $b > e + a - 1$. Recall that $\sigma_1, \dots, \sigma_b$ is a basis of the syzygies of M. We may change bases on the first syzygies by sending $\sigma_i \mapsto \sum p_{i\ell}\sigma_\ell$ where $\deg(p_{i\ell}) = \deg \sigma_i - \deg \sigma_\ell = j_i - j_\ell$, and where the matrix $(p_{i\ell})$ is invertible over the polynomial ring. We choose a generic $(p_{i\ell})$ which satisfies these conditions. Let $\widetilde{\phi}$ be the map defined by $\sigma_b, \sigma_{b-1}, \dots, \sigma_{b-e-a+2}$. Define $M' := \mathrm{coker}(\widetilde{\phi})$. By construction, M' has finite length, $\beta(M') = \beta(\mathrm{Buchs}_\bullet(\widetilde{\phi}))$, and M' surjects onto M. Thus we have

$$\sum_{\ell=b-e-a+2}^{f} j_\ell = \mathrm{reg}(M') \geq \mathrm{reg}(M) = \overline{d_n}(M).$$

where the inequality follows from Corollary 20.19 of [Eis95]. □

Proof of independence of obstructions in Proposition 2.3.2. To show that the obstructions of Proposition 2.3.2 are independent, we construct an explicit example of a virtual Betti diagram with precisely one of the obstructions.

For Proposition 2.3.2 (1) consider:

$$2\cdot\pi_{(0,1,5,6,7,8)}+\pi_{(0,5,6,7,8,9)}=\begin{pmatrix}3&4&-&-&-&-\\-&-&-&-&-&-\\-&-&-&-&-&-\\-&70&252&336&200&45\end{pmatrix}.$$

Then $\underline{d_2}=5>4$ so this diagram has a Buchsbaum-Rim second syzygy obstruction.

For Proposition 2.3.2 (2) consider:

$$\pi_{(0,1,3,4)}=\begin{pmatrix}1&2&-&-\\-&-&2&1\end{pmatrix}.$$

In this case $\sum\beta_{1,j}(\pi_{(0,1,3,4)})=2<3+1-1=3$. More generally, the pure diagram $\pi_{(0,1,\alpha,\alpha+1)}$ has a codimension obstruction for any $\alpha\geq 3$.

For the case of equality in Proposition 2.3.2 (2), consider:

$$\pi_{(0,1,6,10)}=\begin{pmatrix}6&8&-&-\\-&-&-&-\\-&-&-&-\\-&-&3&-\\-&-&-&-\\-&-&-&-\\-&-&-&-\\-&-&-&1\end{pmatrix}.$$

Since we have $\sum\beta_{1,j}(\pi_{(0,1,6,10)})=8=3+6-1$, the diagram $\pi_{(0,1,6,10)}$ should equal the Betti table of the Buchsbaum-Rim complex on a map: $\phi: R(-1)^8\to R^6$. This is not the case.

For Proposition 2.3.2 (3) consider:

$$2\cdot\pi_{(0,1,4,9,10)}=\begin{pmatrix}6&10&-&-&-\\-&-&-&-&-\\-&-&6&-&-\\-&-&-&-&-\\-&-&-&-&-\\-&-&-&-&-\\-&-&-&6&4\end{pmatrix}.$$

Here we have $\overline{d_4}=10>9=\sum_{j=1}^{9}1$.

□

2.4 A linear strand obstruction in projective dimension 3

In this section we build obstructions based on one of Buchsbaum and Eisenbud's structure theorems about free resolutions in the special case of codimension 3 (see [BE74].) The motivation of this section is to explain why the following virtual Betti diagrams do not belong to B_{mod}:

$$D = \begin{pmatrix} 2 & 4 & 3 & - \\ - & 3 & 4 & 2 \end{pmatrix}, D' = \begin{pmatrix} 3 & 6 & 4 & - \\ - & 4 & 6 & 3 \end{pmatrix}, D'' = \begin{pmatrix} 2 & 3 & 2 & - \\ - & 5 & 7 & 3 \end{pmatrix}. \tag{2.4.1}$$

Note that these diagrams do not have any of the Buchsbaum-Rim obstructions in the sense of Proposition 2.3.2. In fact, there are virtual Betti diagrams similar to each of these which are Betti diagrams of modules. For instance, all of the following variants of D *are* Betti diagrams of modules:

$$\begin{pmatrix} 2 & 4 & 1 & - \\ - & 1 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 2 & - \\ - & 2 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 3 & 1 \\ - & 3 & 5 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 8 & 6 & - \\ - & 6 & 8 & 4 \end{pmatrix}.$$

The problem with D must therefore relate to the fact that it has too many linear second syzygies to *not* contain a Koszul summand. Yet whatever obstruction exists for D must disappear upon scaling from D to $2 \cdot D$. Incidentally, the theory of matrix pencils could be used to show that D and D'' are not Betti diagrams. We do not approach this problem via matrix pencils because we seek an obstruction which does not depend on the fact that $\beta_{0,0} = 2$.

Let $S = k[x, y, z]$ and let M be a graded S-module M of finite length. For the rest of this section, we fix:

$$\Delta := \{(0, 1, 2, 3) < (0, 1, 2, 4) < (0, 1, 3, 4) < (0, 2, 3, 4)\}.$$

Further, let M be generated in degree 0 and with regularity 1, so that

$$\beta(M) = \begin{pmatrix} a & b & c & d \\ - & b' & c' & d' \end{pmatrix}$$

and $\beta(M) \in B_{\text{mod}}(\Delta)$. Let T_i be the maps along the top row of the resolution of M so that we have a complex:

$$0 \to S(-3)^d \to (T_3)S(-2)^c \overset{(T_2)}{\to} S(-1)^b \overset{(T_1)}{\to} S^a \to 0.$$

Similarly, let U_j stand for matrices which give the maps along the bottom row of the resolution of M. Observe that each T_i and U_j consists entirely of linear forms, and that $U_1 = 0$. If $d \neq 0$, then the minimal resolution of M contains a copy of the Koszul complex as a free summand. Since we may split off this summand, we assume that $d = 0$.

We then have the following obstruction:

Proposition 2.4.1 (Maximal minor, codimension 3 obstruction)**.** *Let M as defined above, and continue with the same notation. Then:*

$$b' - a + \mathrm{rank}(T_1) + \mathrm{rank}(U_3) \leq c'.$$

Equivalently $c - d' + \mathrm{rank}(T_1) + \mathrm{rank}(U_3) \leq b$.

Proof. By assumption, M has a minimal free resolution given by:

$$0 \to S(-4)^{d'} \overset{\left(\begin{smallmatrix} Q_3 \\ U_3 \end{smallmatrix}\right)}{\to} S(-2)^c \oplus S(-3)^{c'} \overset{\left(\begin{smallmatrix} T_2 & Q_2 \\ 0 & U_2 \end{smallmatrix}\right)}{\to} S(-1)^b \oplus S(-2)^{b'} \overset{\left(\begin{smallmatrix} T_1 & Q_1 \end{smallmatrix}\right)}{\to} S^a \to M.$$

Each Q_i stands for a matrix of degree 2 polynomials. By [BE74] we know that each maximal minor of the middle matrix is the product of a corresponding maximal minor from the first matrix and a corresponding maximal minor from the third matrix.

Let $\tau = \mathrm{rank}(T_1)$ and $\mu = \mathrm{rank}(U_3)$. Since $\mathrm{codim}(M) \neq 0$, the rank of the matrix $\begin{pmatrix} T_1 & Q_1 \end{pmatrix}$ equals a. By thinking of this matrix over the quotient field $k(x, y, z)$, we may choose a basis of the column space which contains τ columns from T_1 and $a - \tau$ columns from Q_1. Let $\mathcal{D}_1$ be the determinant of the resulting $a \times a$ submatrix, and observe that $\mathcal{D}_1$ is nonzero. Similarly, we may construct a $d' \times d'$ minor $\mathcal{D}_3$ from the last matrix such that $\mathcal{D}_3$ is nonzero and involves μ rows from U_3 and $d' - \mu$ rows from Q_3.

Now consider the middle matrix:

$$\begin{array}{c c} & \begin{array}{cc} c & c' \end{array} \\ \begin{array}{c} b \\ b' \end{array} & \begin{pmatrix} T_2 & Q_2 \\ 0 & U_2 \end{pmatrix} \end{array}.$$

Note that the columns of this matrix are indexed by the rows of the third matrix, and the rows of this matrix are indexed by the columns of the first matrix. Choose the unique maximal submatrix such that the columns repeat none of the choices from $\mathcal{D}_3$ and such that the rows repeat none of the choices from $\mathcal{D}_1$. We obtain a matrix of the following shape:

$$\begin{array}{c c} & \begin{array}{cc} c - d' + \mu & c' - \mu \end{array} \\ \begin{array}{c} b - \tau \\ b' - a + \tau \end{array} & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \end{array}.$$

Since M has finite length, the Herzog-Kühl conditions in [HK84] imply that $c' + c - d' = b + b' - a$, and thus this is a square matrix. If $\mathcal{D}_2$ is the determinant of the matrix constructed above, then $\mathcal{D}_2 = \mathcal{D}_1 \mathcal{D}_3$ by [BE74]. Since $\mathcal{D}_1 \neq 0$ and $\mathcal{D}_3 \neq 0$, this implies that the $(b' - a + \tau \times c - d' + \mu)$ block of zeroes in the lower left corner cannot be too large. In particular,

$$b' - a + \tau + c - d' + \mu \leq b' + b - a.$$

By applying the Herzog-Kühl equality $c' + c - d' = b + b' - a$, we obtain the desired results. $\square$

We now prove a couple of lemmas which will allow us to use this obstruction to rule out the virtual Betti diagrams from line (2.4.1). We continue with the same notation, but without the assumption that $d = 0$.

Definition 2.4.2. *A matrix T is* decomposable *if there exists a change of coordinates on the source and target of T such that T becomes block diagonal or such that T contains a column or row of all zeroes. If T is not decomposable then we say that T is* indecomposable.

Lemma 2.4.3. *If the Betti diagram $\begin{pmatrix} a & b & c & d \\ - & b' & c' & d' \end{pmatrix}$ is Cohen-Macaulay and is a minimal generator of B_{mod}, then T_1 is indecomposable or $b = 0$.*

Proof. If we project the semigroup B_{mod} onto its linear strand via

$$\begin{pmatrix} a & b & c & d \\ - & b' & c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & b & c & d \end{pmatrix},$$

then the image equals the semigroup of linear strands in B_{mod}. By the Herzog-Kühl equations, the linear strand $\begin{pmatrix} a & b & c & d \end{pmatrix}$ of such a Cohen-Macaulay module determines the entire Betti diagram. Hence the projection induces an isomorphism between the subsemigroup of Cohen-Macaulay modules of codimension 3 in B_{mod} and the semigroup of linear strands in B_{mod}. The modules with T_1 decomposable and $b \neq 0$ cannot be minimal generators of the semigroup of linear strands in B_{mod}. □

Lemma 2.4.4. *With notation as above we have:*

1. *If there exists a free submodule $F \subseteq S(-1)^b$ such that $F \cong S(-1)^3$ and such that the restricted map $T_1|_F$ has rank 1, then the minimal resolution of M contains a copy of the Koszul complex as a direct summand.*

2. *If $a = 2, b \geq 3$, and T_1 is indecomposable then T_1 has rank 2.*

Proof. (1) Given the setup of the lemma, we have that $T_1|_F$ is an $a \times 3$ matrix of rank 1 with linearly independent columns over k. All matrices of linear forms of rank 1 are compression spaces by [EH88]. Since the columns of $T_1|_F$ are linearly independent, this means that we may choose bases such that:

$$T_1|_F = \begin{pmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.4.2}$$

The result follows immediately.

(2) Assume that T_1 has rank 1 and apply part (1) with F any free submodule isomorphic to $S(-1)^3$. We may then assume that the first three columns of T_1 look like (2.4.2), and whether $b = 3$ or $b > 3$ it quickly follows that T_1 is decomposable. □

Proposition 2.4.5. *The virtual Betti diagrams*

$$D = \begin{pmatrix} 2 & 4 & 3 & - \\ - & 3 & 4 & 2 \end{pmatrix}, D' = \begin{pmatrix} 3 & 6 & 4 & - \\ - & 4 & 6 & 3 \end{pmatrix}, D'' = \begin{pmatrix} 2 & 3 & 2 & - \\ - & 5 & 7 & 3 \end{pmatrix}$$

do not belong to B_{mod}*.*

Proof. Assuming D were a Betti diagram, Lemma 2.4.3 implies that the corresponding matrices T_1 and U_3 are indecomposable. Lemma 2.4.4 (2) implies that for D as in (2.4.2), we have $\operatorname{rank} T_1 = \operatorname{rank} U_3 = 2$. Observe that D now has a maximal minor obstruction, as $c - d' + \tau + \mu = 5$ while $b = 4$.

Next we consider D'. If D' were a Betti diagram, then the corresponding T_1 and U_3 would both have to be indecomposable. If also T_1 had rank 2, then Theorem 1.1 of [EH88] would imply that it is a compression space. In particular, T_1 would have one of the following forms:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & * \\ * & * & * & * & * & * \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix}.$$

The matrix forms on the left and right fail to be indecomposable. The middle form could not have linearly independent columns, since each $*$ stands for a linear form, and we are working over $k[x, y, z]$. Thus T_1 and U_3 both have rank 3, and it follows that D' has a maximal minor obstruction.

In the case of D'', similar arguments show that the ranks of T_1 and U_3 must equal 2 and 3 respectively. Thus D'' also has a maximal minor obstruction. □

Example 2.4.6. *Note that the diagram* $2 \cdot D$ *belongs to* B_{mod}*. In fact, if* $N = k[x, y, z]/(x, y, z)^2$ *and* $N^\vee = Ext^3(N, S)$ *then:*

$$\beta(N \oplus N^\vee(4)) = \begin{pmatrix} 1 & - & - & - \\ - & 6 & 8 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 8 & 6 & - \\ - & - & - & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 6 & - \\ - & 6 & 8 & 4 \end{pmatrix} = 2 \cdot D.$$

This diagram does not have a maximal minor obstruction as $\operatorname{rank}(T_1) = \operatorname{rank}(U_3) = 3$.

Conversely, up to isomorphism the direct sum $N \oplus N^\vee(4)$ *is the only module* M *whose Betti diagram equals* $2 \cdot D$*. The key observation is that for* M *to avoid having a maximal minor obstruction, we must have that* $\operatorname{rank}(T_1) + \operatorname{rank}(U_3) \leq 6$*. Thus we may assume that* M *is determined by a* 4×8 *matrix of linear forms which has rank* ≤ 3*. Such matrices are completely classified by* [EH88] *and an argument as in Proposition 2.4.5 can rule out all possibilities except that* $M \cong N \oplus N^\vee(4)$*.*

In the proof of Theorem 1.3.4 (4), we will show that $3 \cdot D$ *does not belong to* B_{mod}*.*

2.5 Special cases when all virtual Betti diagrams are actual Betti diagrams

In this section we prove Proposition 1.3.3 in two parts. We first deal with projective dimension 1.

Proposition 2.5.1. *Let $S = k[x]$. Then $B_{\text{int}} = B_{\text{mod}}$ and the semigroup B_{mod} is minimally generated by pure diagrams.*

Proof. Let $D \in B_{\text{int}}$ be a virtual Betti diagram of projective dimension 1. We may assume that D is a Cohen-Macaulay diagram of codimension 1. Then the Herzog-Kühl conditions [HK84] imply that D has the same number of generators and first syzygies. List the degrees of the generators of D in increasing order $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$, and list the degrees of the syzygies of D in increasing order $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_s$. Then $D \in B_{\text{int}}$ if and only if we have:

$$\alpha_i + 1 \leq \gamma_i$$

for $i = 1, \ldots, s$. Choose M to be a direct sum of the modules

$$M_i := \operatorname{coker}(\phi_i : R(-\gamma_i) \to R(-\alpha_i))$$

where ϕ_i is represented by any element of degree $\gamma_i - \alpha_i$ in R. Note that $\beta(M_i)$ equals the pure diagram $\pi_{(\alpha_i,\gamma_i)}$. Thus $D \in B_{\text{mod}}$ and $D = \beta(M) = \sum_i \pi_{(\alpha_i,\gamma_i)}$. □

Recall the definition of a level module [Boi00]:

Definition 2.5.2. *A graded module M is a* level module *if its generators are concentrated in a single degree and its socle is concentrated in a single degree.*

We now show that in the case of projective dimension 2 level modules, the semigroups B_{int} and B_{mod} are equal.

Proposition 2.5.3. *Let $S = k[x, y]$ and fix $\Delta = (d^0, \ldots, d^s)$ such that $d^0_0 = d^s_0$ and $d^0_2 = d^s_2$. Then $B_{\text{int}}(\Delta) = B_{\text{mod}}(\Delta)$.*

Proof. We may assume that $d^0_0 = 0$, and then we are considering the semigroup of level modules of projective dimension 2 with socle degree $(d^0_2 - 2)$. Let $D \in B_{\text{int}}$ and let c be a positive integer such that $cD \in B_{\text{mod}}$. Let $\vec{h}(D) = (h_0, h_1, \ldots)$ be the Hilbert function of D. The main result of [Söd06] shows that $\vec{h}(D)$ is the Hilbert function of some level module of embedding dimension 2 if and only if $h_{i-1} - 2h_i + h_i \leq 0$ for all $i \leq \underline{d_2} - 2$.

Since $cD \in B_{\text{mod}}$, we know that $\vec{h}(cD) = c\vec{h}(D)$ is the Hilbert function of a level module. Thus we have:

$$ch_{i-1} - 2ch_i + ch_i \leq 0.$$

The same holds when we divide by c, and thus $\vec{h}(D)$ is the Hilbert function of some level module M. Since M is also a level module, its Betti diagram must equal D. □

We conjecture that the restriction on Δ is unnecessary:

Conjecture 2.5.4. *In projective dimension* 2*,* $B_{\text{int}}(\Delta) = B_{\text{mod}}(\Delta)$ *for any* Δ*.*

2.6 Pathologies of the semigroup of Betti diagrams

We are now prepared to prove Theorem 1.3.4 and thus show that for projective dimension greater than 2, the semigroups B_{int} and B_{mod} diverge. Recall the statement of Theorem 1.6:

Theorem 1.6: *Each of the following occurs in the semigroup of Betti diagrams:*

1. B_{mod} *is not necessarily a saturated semigroup.*
2. *The set* $|B_{\text{int}} \setminus B_{\text{mod}}|$ *is not necessarily finite.*
3. *There exist rays of* B_{int} *which are missing at least* $(\dim S - 2)$ *consecutive lattice points.*
4. *There exist rays of* B_{int} *where the points of* B_{mod} *are nonconsecutive lattice points.*

The various pieces of the theorem follow from a collection of obstructed virtual Betti diagrams.

Proof of Part (1) of Theorem 1.3.4. We will show that on the ray corresponding to

$$D_1 = \begin{pmatrix} 1 & 2 & - & - \\ - & - & 2 & 1 \end{pmatrix}.$$

every lattice point except D_1 itself belongs to B_{mod}. We have seen in (1.3.1) that $D_1 \notin B_{\text{mod}}$. Certainly $2 \cdot D_1 \in B_{\text{mod}}$ as $2 \cdot D$ is the Buchsbaum-Rim complex on a generic 2×4 matrix of linear forms. We claim that $3 \cdot D_1$ also belongs to B_{mod}. In fact, if we set $S = k[x, y, z]$ and

$$M := coker \begin{pmatrix} x & y & z & 0 & 0 & 0 \\ 0 & 0 & x & y & z & 0 \\ x+y & 0 & 0 & x & y & z \end{pmatrix},$$

then the Betti diagram of M is $3 \cdot D_1$. □

Proof of Part (2) of Theorem 1.3.4. We will show that for all $\alpha \in \mathbb{N}$, the virtual Betti diagram:

$$E_\alpha := \begin{pmatrix} 2+\alpha & 3 & 2 & - \\ - & 5+6\alpha & 7+8\alpha & 3+3\alpha \end{pmatrix}$$

does not belong to B_{mod}.

Note that $E_0 \notin B_{\text{mod}}$ by Proposition 2.4.5. Imagine now that $\beta(M) = E_\alpha$ for some α. Let T_1 be the linear part of the presentation matrix of M so that T_1 is an $(\alpha+2) \times 3$ matrix of linear forms. Let T_2 be the (3×2) matrix of linear second syzygies and write:

$$T_1 \cdot T_2 = \begin{pmatrix} l_{1,1} & l_{1,2} & l_{1,3} \\ l_{2,1} & l_{2,2} & l_{2,3} \\ \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{2,1} & s_{2,2} \\ s_{3,1} & s_{3,2} \end{pmatrix}.$$

By Lemma 2.4.4 (1), the rank of T_1 must be at least 2. Let T_1' be the top two rows of T_1, and by shuffling the rows of T_1, we may assume that the rank of T_1' equals 2. So then may assume that $l_{1,1}$ and $l_{2,2}$ are nonzero. Since each column of T_2 has at least 2 nonzero entries, it follows that the syzygies represented by T_2 remain nontrivial syzygies on the columns of T_1'.

It is possible however that columns of T_1' are not k-linearly independent. But since the rank of T_1' equals 2, we know that at least two of the columns are linearly independent. Let C be the cokernel of T_1', and let $M' := C_{\leq 1}$ be the truncation of C in degrees greater than 1. Then we would have:

$$\beta(M') = \begin{pmatrix} 2 & 3 & 2 & - \\ - & 5 & 7 & 3 \end{pmatrix} \text{ or } = \begin{pmatrix} 2 & 2 & 2 & - \\ - & * & * & * \end{pmatrix}.$$

The first case is impossible by Example 2.4.5, and the second case does not even belong to B_{int}. □

The following definition will be useful in the next part of the proof.

Definition 2.6.1. *For a degree sequence $d = (d_0, \ldots, d_t)$, we define the* normalized pure diagram of type *d, denoted $\overline{\pi}_d$, to be the unique pure diagram of type d where $\beta_{0,d^0}(\overline{\pi}_d) = 1$. Note that formula* (1.2.1) *yields an explicit formula for each Betti number of $\overline{\pi}_d$:*

$$\beta_{i,d_i}(\overline{\pi}_d) = \frac{\prod_{j \neq 0} |d_j - d_0|}{\prod_{j \neq i} |d_j - d_i|}.$$

Proof of Part (3) of Theorem 1.3.4. Fix some prime $P \geq 2$ and let $S = k[x_1, \ldots, x_{P+1}]$. Consider the degree sequence:

$$d := (0, 1, P+1, P+2, ..., 2P).$$

We will show that the first $P-1$ lattice points of the ray r_d have a codimension obstruction.

We claim that:

- $\beta_{1,1}(\overline{\pi}_d) = 2$
- All the entries of $\beta(\overline{\pi}_d)$ are positive integers.

For both claims we use the formula from Definition 2.6.1.

$$\beta_{1,1}(\overline{\pi}_d) = \frac{(P+1)\cdot\dots\cdot(2P-1)\cdot(2P)}{(P\cdot(P+1)\dots(2P-1))} = \frac{2P}{P} = 2.$$

For the other entries of $\overline{\pi}_d$ we compute:

$$\beta_{i,d_i}(\overline{\pi}_d) = \frac{2P\cdot(2P-1)\cdot\dots\cdot(P+1)}{(i-2)!(P-i+1)!}\cdot\frac{1}{P+i-1}\cdot\frac{1}{P+i-2} = \frac{1}{P}\binom{P+i-3}{i-2}\binom{2P}{P-i+1}$$

Note that $\binom{2P}{P-i+1}$ is divisible by P for all $i \geq 2$ and thus $\beta_{i,d_i}(\overline{\pi}_d)$ is an integer as claimed.

Since $\beta_{0,0} = 1$ and $\beta_{1,1} = 2$, the diagram $c \cdot \overline{\pi}_d$ hs a codimension obstruction for $c = 1, \dots, P-1$. Thus the first $P-1$ lattice points of the ray of π_d do not correspond to Betti diagrams. □

Proof of Part (4) of Theorem 1.3.4. Consider the ray corresponding to

$$D_2 = \begin{pmatrix} 2 & 4 & 3 & - \\ - & 3 & 4 & 2 \end{pmatrix}.$$

Proposition 2.4.5 shows that D_2 does not belong to B_{mod}. In Example 2.4.6 we showed that $2 \cdot D_2$ does belong to B_{mod}. Thus, it will be sufficient to show that

$$3 \cdot D_2 = \begin{pmatrix} 6 & 12 & 9 & - \\ - & 9 & 12 & 6 \end{pmatrix}$$

does not belong to B_{mod}.

We assume for contradiction that there exists M such that $\beta(M) = 3 \cdot D_2$. Then the minimal free resolution of M is as below:

$$0 \to R(-4)^6 \xrightarrow{\binom{Q_3}{U_3}} R(-2)^9 \oplus R(-3)^{12} \xrightarrow{\begin{pmatrix} T_2 & Q_2 \\ 0 & U_2 \end{pmatrix}} R(-1)^{12} \oplus R(-2)^9 \xrightarrow{(T_1\ Q_1)} R^6. \tag{2.6.1}$$

where T_1, T_2, U_2 and U_3 are matrices of linear forms. By Proposition 2.4.1 we have that $\text{rank}(T_1) + \text{rank}(U_3) \leq 9$. Since the diagram $3 \cdot D_2$ is Cohen-Macaulay and symmetric, we may use Remark 2.3.3 to assume that $\text{rank}(T_1) \leq 4$.

We next use the fact that, after a change of coordinates, T_2 contains a second syzygy which involves only 2 of the variables of S. This fact is proven in Lemma 2.6.2 below. Change coordinates so that the first column of T_2 represents this second syzygy and equals:

$$\begin{pmatrix} y \\ -x \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since T_1 must be indecomposable, we may put T_1 into the form:

$$T_1 = \begin{pmatrix} x & y & z & 0 & \dots & 0 \\ 0 & 0 & * & * & \dots & * \\ \vdots & & & & & \vdots \\ 0 & 0 & * & * & \dots & * \end{pmatrix}. \tag{2.6.2}$$

Now set $\widetilde{T_1}$ to be the lower right corner of $*$'s in T_1. Since $\operatorname{rank}(T_1) \leq 4$ we have that $\operatorname{rank}(\widetilde{T_1}) \leq 3$. Matrices of rank ≤ 3 are fully classified, and by applying Corollary 1.4 of [EH88] we conclude that $\widetilde{T_1}$ is a compression space. We can rule out the compression spaces cases where $\widetilde{T_1}$ has a column or a row equal to zero, or else T_1 would have been decomposable. Thus $\widetilde{T_1}$ is equivalent to one of the two following forms:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ * & * & * & * & * & * & * & * & * & * \end{pmatrix}.$$

If we subsitute the matrix on the left into the form for T_1 from 2.6.2, then we see that T_1 would have 8 k-linearly indepdendent columns which are supported on only the bottom two rows. Since all entries of T_1 are linear forms in $k[x, y, z]$, this is impossible. We can similarly rule out the possibility of the matrix on the right. □

Lemma 2.6.2. *If there exists a minimal resolution as in Equation (2.6.1), then the matrix T_2 contains a second syzygy involving only 2 variables of S.*

Proof. Assume that this is not the case and quotient by the variable z. Then the quotient matrices $\overline{T_1}$ and $\overline{T_2}$ still multiply to 0. It is possible that after quotienting, some of the columns of T_1 are dependent. However this is not possible for T_2. For if some combination went to 0 after quotienting by z, then there would exist a column of T_2, i.e. a second syzygy of M, which involves only the variable z. This is clearly impossible. Thus the columns of $\overline{T_2}$ are linearly independent.

Nevertheless, we know that the columns of a 6×12 matrix of linear forms over $k[x, y]$ can satisfy at most 6 independent linear syzygies. By changing coordinates we may arrange that 3 of the columns of $\overline{T_2}$ are "trivial" syzygies on $\overline{T_1}$. By a "trivial" syzygy, we mean a column of $\overline{T_2}$ where the nonzero entries of that columns multiply with zero entries of $\overline{T_1}$. For an example of how a nontrivial syzygy over $k[x, y, z]$ can become trivial after quotienting by z, consider:

$$\begin{pmatrix} x & z & 0 \\ y & 0 & z \end{pmatrix} \begin{pmatrix} z \\ -x \\ -y \end{pmatrix} \to \begin{pmatrix} x & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -x \\ -y \end{pmatrix}$$

Change coordinates so that the first 3 columns of $\overline{T_2}$ represent the trivial syzygies and are in Kronecker normal form. By assumption, each column of $\overline{T_2}$ involves both x and y, so these first 3 columns must consist of combinations of the following Kronecker blocks:

$$B_1 = \begin{pmatrix} x \\ y \end{pmatrix}, B_2 = \begin{pmatrix} x & 0 \\ y & x \\ 0 & y \end{pmatrix}, B_3 = \begin{pmatrix} x & 0 & 0 \\ y & x & 0 \\ 0 & y & x \\ 0 & 0 & y \end{pmatrix}.$$

Since each nonzero entry in the trivial part of $\overline{T_2}$ must multiply with a 0 from $\overline{T_1}$, this forces certain columns of $\overline{T_1}$ to equal 0. More precisely, the number of nonzero rows in the trivial part of $\overline{T_2}$ is a lower bound for the number of columns of $\overline{T_1}$ which are identically zero. The block decomposition shows that the trivial part of $\overline{T_2}$ has at least 4 nonzero rows, and thus $\overline{T_1}$ has at least 4 columns which are identically zero.

But now the nonzero part of $\overline{T_1}$ is a 6×8 matrix of linear forms, and this can satisfy at most 4 linear syzygies. This forces two *additional* columns of $\overline{T_2}$ to be trivial syzygies which in turn forces more columns of $\overline{T_1}$ to equal zero, and so on.

Working through this iterative process, we eventually conclude that $\overline{T_1}$ contains 8 columns which are identically zero. This means that T_1 must have contained 8 columns which involved only z. But since T_1 is a 6×12 matrix of linear forms with linearly independent columns, this is impossible. $\square$

Chapter 3

Pure filtrations

3.1 Overview

In this chapter we show that, in certain cases, the Boij-Söderberg decomposition of $\beta(M)$ corresponds to a very special type of filtration of M, namely a *pure filtration*. Recall from Section 1.4 that a pure filtration is a filtration of the module M that closely reflects the Boij-Söderberg decomposition. This chapter consists of joint work with David Eisenbud and Frank-Olaf Schreyer which originally appeared in [EES10].

This chapter is organized as follows. In §3.2, we provide a detailed example which illustrates our method for obtaining pure filtrations. In §3.3, we prove the steps involved in this method and explain the conditions under which our method works. These form the main results of this chapter.

We then turn to applications. In §3.4 we prove Proposition 1.4.2, illustrating certain pathologies of the Semigroup of Betti diagrams; in §3.5, we apply our methods to the study of very singular spaces of matrices; and in §3.6, we compute the minimal generators for a nontrivial example of the semigroup of Betti diagrams. Finally, in §3.7, we touch on situations where pure filtrations do not exist.

3.2 A step-by-step example

In this section, we illustrate our method for obtaining a pure filtration of a module. Throughout, we provide references to the relevant results from §3.3.

Let $n = 5$ so that $S = k[x_1, \ldots, x_5]$. Let $d^0 = (0,2,3,4,5,8), d^1 = (0,2,3,5,6,8)$ and $d^2 = (0,3,4,5,6,8)$. Let M be a module such that $\beta(M) = \pi_{d^0} + 2\pi_{d^1} + \pi_{d^2}$. Then we have:

$$\beta(M) = \begin{pmatrix} 6 & - & - & - & - & - \\ - & 60 & 128 & 90 & 32 & - \\ - & 32 & 90 & 128 & 60 & - \\ - & - & - & - & - & 6 \end{pmatrix}.$$

We will show that there exists a pure filtration of M. Namely, we will produce a filtration:

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq M_3 = 0$$

where $\beta(M_0/M_1) = \pi_{d^2}, \beta(M_1/M_2) = 2\pi_{d^1}$ and $\beta(M_2/M_3) = \pi_{d^0}$.

Let $(F_\bullet, \phi_\bullet)$ be the minimal free resolution of M. Note that $\phi_1 : F_1 \to F_0$ decomposes as $\begin{pmatrix}\psi_1 & \mu_1\end{pmatrix}$ where ψ_1 is a 6×60 matrix with quadratic entries and μ_1 is a 6×32 matrix with cubic entries. Define $N := \operatorname{coker}(\psi_1)$.

Step 1: Computing $\beta(N)$

A priori, we do not know the Betti diagram of N. In this step, we use Boij-Söderberg theory to compute $\beta(N)$. Observe first that

$$\beta(N) = \begin{pmatrix} 6 & - & - & - & - & - \\ - & 60 & 128 & 90 & 32 & - \\ - & - & * & * & * & - \\ - & - & * & * & * & * \\ - & - & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

This is because columns $0, 1, \ldots, 4$ of the top strand of $\beta(M)$ depend only on ψ_1, and hence $\beta(N)$ must have the same top strand in columns $0, 1, \ldots, 4$. (See Proposition 3.3.2 below.)

Although we do not yet know the exact value of $\beta(N)$, we may nevertheless begin computing the Boij-Söderberg decomposition of $\beta(N)$. To eliminate the 90 and the 32 in the top row, we may only use pure diagrams of the form π_d with $d = (0, 2, 3, 4, d_4, d_5)$ for some $d_4 \geq 5, d_5 \geq 8, d_5 \geq d_4$. (We also allow the possibility that the vector d has length 4 or 3, and we think of these as corresponding the cases where $d_5 = \infty$ or $d_4 = d_5 = \infty$.) By applying the formula from equation (1.2.1), we compute that, for all such d,

$$\frac{\beta_{3,4}(\pi_d)}{\beta_{4,5}(\pi_d)} \geq 90/32,$$

with equality if and only if $d_4 = 5$ and $d_5 = 8$. Since the decomposition algorithm implies that we cannot eliminate $\beta_{3,4}(M)$ before we eliminate $\beta_{4,5}(M)$, we must eliminate both entries simultaneously. Thus, the first step of the Boij-Södberg decomposition of $\beta(N)$ is given by $1 \cdot \pi_{d^0} = 1 \cdot \pi_{(0,2,3,4,5,8)}$. (See Corollary 3.3.5 below.)

We now consider the diagram $\beta(N) - \pi_{d^0}$. Essentially the same argument as above shows that the next step of the Boij-Söderberg decomposition must be $2\pi_{d^1} = 2\pi_{(0,2,3,5,6,8)}$. Lastly, we seek to decompose $\beta(N) - \pi_{d^0} - 2\pi_{d^1}$, which equals

$$\beta(N) - \pi_{d^0} - 2\pi_{d^1} = \begin{pmatrix} 1 & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & * & * & * & - \\ - & - & * & * & * & * \\ - & - & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Since the second column consists of all zeroes, this diagram must equal $\pi_{(0)}$. Hence, we have computed that the Boij-Söderberg decomposition of $\beta(N)$ is $\pi_{d^0} + 2\pi_{d^1} + \pi_{(0)}$.

Step 2: Splitting N

Set $N' := H^0_{\mathfrak{m}}(N)$, the 0'th local cohomology module of N, and consider the exact sequence:

$$0 \to N' \to N \to N/N' \to 0.$$

Based on the Boij-Söderberg decompostion of N, we know that the Hilbert polynomial of N is the same as the Hilbert polynomial of $\pi_{(0)}$. Thus N/N' has the same Hilbert polynomial as a free module, is generated in the same degree as S, and has $\operatorname{depth}(N/N') > 0$. This implies that N/N' is actually free, so the above sequence splits. (See Proposition 3.3.7.)

Step 3: Lifting to a filtration of M

Recall that we have written $\phi_1 = \begin{pmatrix} \psi_1 & \mu_1 \end{pmatrix}$. Since N splits, we may further decompose ψ_1 and write:

$$\phi_1 = \begin{pmatrix} \widetilde{\psi}_1 & \widetilde{\mu}_1 \\ 0 & \widetilde{\eta}_1 \end{pmatrix},$$

where $\widetilde{\psi}_1$ is a 5×60 matrix.

We then obtain an exact sequence

$$0 \to \operatorname{coker}(\widetilde{\psi}_1) \to M \to \operatorname{coker}(\widetilde{\eta}_1) \to 0.$$

The map $\operatorname{coker}(\widetilde{\psi}_1) \to M$ induces an isomorphism along the top strands of the resolutions of these modules in columns $1, \ldots, 4$. It then follows from a mapping cone construction that there exists a (non-minimal) free resolution of $\operatorname{coker}(\widetilde{\eta}_1)$ with the Betti table

$$\begin{pmatrix} 1 & - & - & - & - & - & - \\ - & - & - & - & \mathbf{38} & \mathbf{20} & - \\ - & 32 & 90 & 128 & 60 & - & \mathbf{3} \\ - & - & - & - & - & 6 & - \end{pmatrix}.$$

To obtain the minimal free resolution of $\operatorname{coker}(\widetilde{\eta}_1)$, each possible cancellation, indicated in bold above, must actually cancel. Therefore, $\operatorname{coker}(\widetilde{\eta}_1)$ has a pure resolution of type $d^2 = (0, 3, 4, 5, 6, 8)$. (See Proposition 3.3.9.)

We set $M_1 := \operatorname{coker}(\widetilde{\psi}_1)$. By iterating the above argument for $M_1^\vee(8)$, we obtain an exact sequence:

$$0 \to K \to M_1^\vee(8) \to Q \to 0$$

where $\beta(K) = 2\pi_{d^1}$ and $\beta(Q) = \pi_{d^2}$. We set $M_2 := Q^\vee(8)$. This yields the desired result: $\beta(M/M_1) = \pi_{d^2}$, $\beta(M_1/M_2) = \beta(K^\vee(8)) = 2\pi_{d^1}$, and $\beta(M_2) = \pi_{d^0}$.

Remark 3.2.1. The method of this example generalizes to any diagram which is a combination of π_{d^0}, π_{d^1} and π_{d^2}. Namely, let $\Delta = (d^0, d^1, d^2)$ and let E

$$E = a_0\pi_{d^0} + a_1\pi_{d^1} + a_2\pi_{d^2}$$

be an integral diagram $E \in B_{\text{int}}(\Delta)$. The above computations show that, if there exists an M' such that $\beta(M') = E$, then M' admits a pure filtration

$$M' = M'_0 \supsetneq M'_1 \supsetneq M'_2 \supsetneq M'_3 = 0$$

where $\beta(M'_i/M'_{i+1}) = a_{2-i}\pi_{d^{2-i}}$. In particular, if $a_i \notin \mathbb{Z}$ for some i, then this leads to a contradiction, and thus $E \notin B_{\text{mod}}(\Delta)$. For instance, the diagram:

$$\frac{1}{2}\pi_{d^1} + \frac{1}{2}\pi_{d^2} = \begin{pmatrix} 1 & - & - & - & - & - \\ - & 5 & 8 & - & - & - \\ - & 16 & 45 & 56 & 25 & - \\ - & - & - & - & - & 2 \end{pmatrix}$$

is not the Betti diagram of any module.

3.3 Obtaining pure filtrations

Throughout this section, we fix a module M of finite length which has first syzygies in more than one degree. We set $(F_\bullet, \phi_\bullet)$ to be the minimal free resolution of M, and we assume that $\beta(M) \in B_{\text{mod}}(\Delta)$ where $\Delta = (d^0, \ldots, d^s)$. Given this notation, d_1^0 is the minimal degree of a first syzygy of M. Hence, we may write

$$F_1 = S(-d_1^0)^{\beta_{1,d_1^0}(M)} \oplus F_1''.$$

We may then decompose the map ϕ_1 as $\phi_1 = \begin{pmatrix} \psi_1 & \eta_1 \end{pmatrix}$, where $\psi_1 : S(-f_1)^{\beta_{1,d_1^0}(M)} \to F_0$. Under these conditions, we define:

$$\mathcal{L}(M) := \operatorname{coker}(\psi_1).$$

For instance, if M has presentation matrix:

$$\begin{pmatrix} x_1 & x_2 & x_1^3 & 0 & 0x_2^5 \\ -x_2 & x_3 & 0 & x_2^4 & x_3^5 \end{pmatrix}$$

then $\mathcal{L}(M)$ has presentation matrix:

$$\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_3 \end{pmatrix}.$$

We set $N := \mathcal{L}(M)$. Our method for obtaining a pure filtration of M relies on three distinct steps, as outlined in §3.2.

Proof of Step 1: Computing $\beta(N)$

We now show that Boij-Söderberg theoretic conditions on M often enable us to completely determine the Boij-Söderberg decomposition of $N = \mathcal{L}(M)$. This is likely the most technically involved step of our method for obtaining pure filtrations. In most of our applications, we will apply Corollary 3.3.5.

We recursively define the second degree strand, $\sigma_2(D)$, as follows. We set $f_0 := d_0^0$ and $f_1 := \min\{j | j > d_1^0, \beta_{i,j}(D) \neq 0\}$. Then we define $f_i := \min\{j | j > f_{i-1}, \beta_{i,j}(D) \neq 0\}$.

Example 3.3.1. *With D as below, $\sigma_2(D) = (0,3,4,5,6,8)$, as indicated by the strand of bold numbers.*

$$D = \begin{pmatrix} \mathbf{6} & - & - & - & - & - \\ - & 60 & 128 & 90 & 32 & - \\ - & \mathbf{32} & \mathbf{90} & \mathbf{128} & \mathbf{60} & - \\ - & - & - & - & - & \mathbf{6} \end{pmatrix}$$

The idea behind the definition of the second degree strand is that the diagrams $\beta(M)$ and $\beta(N)$ will coincide above $\sigma_2(\beta(M))$. The following proposition makes this idea more precise.

Proposition 3.3.2. *Let $\beta(M) \in B_{\mathrm{mod}}(\Delta)$ with $\Delta = (d^0, \ldots, d^s)$ and let $\sigma_2(\beta(M)) = (f_0, \ldots, f_n)$. If $f_i > d_i^0$ for $i = 1, \ldots \ell$, then the top strands of $\beta(M)$ and $\beta(N)$ agree in columns $0, \ldots, \ell$.*

Proof. We decompose $F_i = F_i' \oplus F_i''$ via

$$F_i' = \oplus_{j<f_i} S(-j)^{\beta_{i,j}(M)} \quad \text{and} \quad F_i'' = \oplus_{j \geq f_i} S(-j)^{\beta_{i,j}(M)}.$$

Our definition of the f_i guarantees that the maps $F_i' \to F_{i-1}''$ equal zero. This yields a block decomposition of each ϕ_i

$$\phi_i = \begin{matrix} & F_i' & F_i'' \\ \begin{matrix} F_{i-1}' \\ F_{i-1}'' \end{matrix} & \begin{pmatrix} A_i & B_i \\ 0 & C_i \end{pmatrix} \end{matrix}.$$

With this notation, $N = \operatorname{coker}(A_1)$. Note that the complex $(F'_\bullet, A_\bullet)$ is a summand of the free resolution of M by definition.

We claim that the complex $(F'_\bullet, A_\bullet)$ is also a summand of the free resolution of N. For F_0 and F_1 this follows by definition. For F_2, observe that any syzygy on the columns of ϕ_1 of degree less than f_2 must be a syzygy on the columns of A_1, and conversely. It follows that A_2 encodes precisely the syzygies of A_1 of degree strictly less than f_2. Iterating this argument yields the claim.

Since $f_i > d_i^0$ for $i = 1, \dots, \ell$, the top strand of $\beta(M)$ is the same as the top strand of $(F'_\bullet, A_\bullet)$ in columns $0, \dots, \ell$. The same holds for $\beta(N)$, which completes the proof. ☐

Remark 3.3.3. With notation as in the previous proposition, the natural map $N \to M$ induces a morphism of free resolutions which is an isomorphism along the subcomplex $(F'_\bullet, A_\bullet)$.

The previous proposition gave us information about the top strand of the free resolution of N. The following proposition provides further information about the Boij-Söderberg decomposition of $\beta(N)$.

Proposition 3.3.4. *Let $D \in B_{\mathbb{Q}}(\Delta)$ where $\Delta = (d^0, \dots, d^s)$.*

1. *For any i, there exists a $d^\ell \in \Delta$ such that*

$$0 < \beta_{i,d_i^0}(\pi_{d^\ell}) \cdot \beta_{i+1,d_{i+1}^0}(D) \leq \beta_{i+1,d_{i+1}^0}(\pi_{d^\ell}) \cdot \beta_{i,d_i^0}(D).$$

2. *Assume further that the above inequality holds only if $d^\ell = d^0$ and assume that in this case, we actually have an equality. Then the first step of the Boij-Söderberg decomposition for D must simultaneously eliminate $\beta_{i,d_i^0}(D)$ and $\beta_{i+1,d_{i+1}^0}(D)$.*

Proof. Proof of (1): Since $D \in B_{\mathbb{Q}}(\Delta)$, we may write $D = \sum c_\ell d^\ell$ uniquely with $c_\ell \in \mathbb{Q}_{\geq 0}$. We thus may write the ordered pair:

$$\left(\beta_{i,d_i^0}(D), \beta_{i+1,d_{i+1}^0}(D)\right) = \sum_\ell c_\ell \left(\beta_{i,d_i^0}(\pi_{d^\ell}), \beta_{i+1,d_{i+1}^0}(\pi_{d^\ell})\right). \tag{3.3.1}$$

By convexity, at least one of the summands on the right hand side must satisfy the claimed inequality.

Proof of (2): Our assumption implies that, for every $\ell \neq 0$, we have either

$$\frac{\beta_{i,d_i^0}(\pi_{d^\ell})}{\beta_{i+1,d_{i+1}^0}(\pi_{d^\ell})} > \frac{\beta_{i,d_i^0}(D)}{\beta_{i+1,d_{i+1}^0}(D)}.$$

or the left hand side is $\frac{0}{0}$. Since we assume that the above ratio is an equality when $\ell = 0$, we see from equation (3.3.1) that $c_\ell \neq 0$ only if $\ell = 0$ or if $\left(\beta_{i,d_i^0}(\pi_{d^\ell}), \beta_{i+1,d_{i+1}^0}(\pi_{d^\ell})\right) = (0,0)$. It follows that $D - c_0\pi_{d^0}$ satisfies

$$\left(\beta_{i,d_i^0}(D - c_0\pi_{d^0}), \beta_{i+1,d_{i+1}^0}(D - c_0\pi_{d^0})\right) = (0,0).$$

The first step $c_0\pi_{d^0}$ of the Boij-Söderberg decomposition of D thus simultaneously eliminates the β_{i,d_i^0} and β_{i+1,d_{i+1}^0} entries of D. $\square$

Corollary 3.3.5. *Let $D \in B_{\mathbb{Q}}(\Delta)$ with $\Delta = (d^0, \ldots, d^s)$. Assume the following:*

1. *We have the equality*
$$\frac{\beta_{i,d_i^0}(\pi_{d^0})}{\beta_{i+1,d_{i+1}^0}(\pi_{d^0})} = \frac{\beta_{i,d_i^0}(D)}{\beta_{i+1,d_{i+1}^0}(D)}.$$

2. *There exists an i such that:*

   (i) For all ℓ where $d_i^\ell = d_i^0$, we have $d_{i'}^\ell = d_{i'}^0$ for all $i' < i$.

   (ii) For all ℓ where $d_i^\ell > d_i^0$, we have $d_{i+1}^\ell > d_{i+1}^0$.

Then the first step of the Boij-Söderberg decomposition of $\beta(N)$ must simultaneously eliminate $\beta_{i,d_i^0}(D)$ and $\beta_{i+1,d_{i+1}^0}(D)$.

Remark 3.3.6. Condition (2) of Corollary 3.3.5 is often satisfied for a module whose resolution contains a linear complex in the top strand. For example, if we have

$$\beta(M) = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & \ldots & t & t+1 & \ldots \end{matrix} \\ \begin{pmatrix} * & - & - & \ldots & - & - & \ldots \\ - & * & * & \ldots & * & - & \ldots \\ - & * & * & \ldots & * & * & \ldots \\ \vdots & & & & & & \ddots \end{pmatrix} \end{array},$$

then condition (2) of Corollary 3.3.5 is satisfied for any $i = 1, \ldots, t-1$.

Proof of Corollary 3.3.5. It suffices to show that we are in the situation of Proposition 3.3.4(2). To prove the desired inequalities, we analyze the functions which define the entries of the pure diagrams π_{d^ℓ}. Namely, we let $\mathfrak{b}_i$ be the rational function $\mathbb{R}^{n+1} \to \mathbb{R}$ where

$$\mathfrak{b}_i(d_0, \ldots, d_n) = \prod_{j \neq i} \frac{1}{|d_j - d_i|}. \tag{3.3.2}$$

Let $\mathbb{D}$ be the domain where $d_{j+1} > d_j$ for all j. Since the entries of any pure diagram of type d^ℓ are unique up to scalar multiple, equation (1.2.1) implies that

$$\frac{\mathfrak{b}_i(d^\ell)}{\mathfrak{b}_{i+1}(d^\ell)} = \frac{\mathfrak{b}_i(d_0^\ell, \ldots, d_n^\ell)}{\mathfrak{b}_{i+1}(d_0^\ell, \ldots, d_n^\ell)} = \frac{\beta_{i,d_i^\ell}(\pi_{d^\ell})}{\beta_{i+1,d_i^\ell}(\pi_{d^\ell})}.$$

To illustrate the hypotheses of Proposition 2, we must show that for $\ell \neq 0$, we either have $\left(\beta_{i,d_i^0}(\pi_{d^\ell}), \beta_{i+1,d_{i+1}^0}(\pi_{d^\ell})\right) = (0,0)$ or we have

$$\frac{\mathfrak{b}_i(d^\ell)}{\mathfrak{b}_{i+1}(d^\ell)} > \frac{\beta_{i,d_i^0}(D)}{\beta_{i+1,d_{i+1}^0}(D)} = \frac{\mathfrak{b}_i(d^0)}{\mathfrak{b}_{i+1}(d^0)}. \tag{3.3.3}$$

By (ii), we see that $\beta_{i,d_i^0}(\pi_{d^\ell}) = 0$ implies that $\beta_{i+1,d_{i+1}^0}(\pi_{d^\ell}) = 0$. Hence, we may restrict attention to the cases where $\beta_{i,d_i^0}(\pi_{d^\ell}) \neq 0$, i.e. where $d_i^\ell = d_i^0$.

Now, assumption (i) implies that the degree sequences d^ℓ and d^0 satisfy $d_{i'}^\ell = d_{i'}^0$ for all $i' \leq i$. Hence, d^ℓ is obtained from d^0 by increasing various values of d_j^0 for $j > i$. If we can then show that

$$\frac{\partial}{\partial d_j}\left(\frac{\mathfrak{b}_i}{\mathfrak{b}_{i+1}}\right) > 0$$

in $\mathbb{D}$ and for all $j > i$, this will imply the inequality in (3.3.3). Using equation (3.3.2), we compute directly that

$$\frac{\partial}{\partial d_j}\left(\frac{\mathfrak{b}_i}{\mathfrak{b}_{i+1}}\right) = \frac{\partial}{\partial d_j}\left(\frac{d_j - d_{i+1}}{d_j - d_i}\right) = \frac{d_{i+1} - d_i}{(d_j - d_i)^2}.$$

The expression on the right is strictly positive inside the domain $\mathbb{D}$, thus completing the proof. □

Proof of Step 2: Splitting N

Many properties of a module M may be computed directly from the Betti diagram $\beta(M)$ (e.g. Hilbert polynomial, projective dimension, depth, etc.). We extend all such notions to arbitrary diagrams D. If D is a diagram whose Hilbert polynomial equals 0, then we say that D is a finite length diagram.

The key result in this step is the following:

Proposition 3.3.7. *Let N be a module generated in a single degree and such that*

$$\beta(N) = D_0 + D_{free}$$

where D_0 is a finite length diagram and D_{free} is a diagram of projective dimension 0. Then $N \cong N_0 \oplus N_{free}$ with $\beta(N_0) = D_0$ and $\beta(N_{free}) = D_{free}$.

Proof. Set $N' := H^0_{\mathfrak{m}}(N)$ and consider the exact sequence

$$0 \to N' \to N \to N/N' \to 0.$$

If N/N' is free, then we have a splitting $N \cong N' \oplus (N/N')$. It would follow that N/N' is a free module with the same Hilbert polynomial as D_{free}, so that $\beta(N/N') = D_{\text{free}}$. It thus suffices to show that N/N' is free.

By shifting degrees, we may assume that N, and hence N/N', is generated entirely in degree 0. Let $Q = k(x_1, \ldots, x_n)$ and let r be the rank of $N/N' \otimes_S Q$. Since a diagram of finite length has Hilbert polynomial 0, it follows that D_{free}, N, and N/N' all have the same Hilbert polynomial as the free module S^r. By choosing generic degree 0 elements $m_1, \ldots, m_r \in N/N'$, we obtain a basis of $N/N' \otimes_S Q$. The submodule L generated by the m_i is a free submodule of N/N'. We claim that $L \cong N/N'$. Note that the Hilbert polynomial of L also equals the Hilbert polynomial of the free module S^r, and thus L is a submodule of N/N' with the same Hilbert polynomial. Thus, if the quotient $(N/N')/L$ is nonzero, then it must have finite length. However, since L is free, the exact sequence:

$$0 \to L \to N/N' \to (N/N')/L \to 0$$

implies that $\operatorname{pdim}(N/N') = \operatorname{pdim}((N/N')/L) = n$ when $(N/N')/L$ is nonzero. This is a contradiction, since $\operatorname{depth}(N/N') > 0$ by definition. □

Proof of Step 3: Lifting to a filtration of M

Recall that $\sigma_2(M) = (f_0, \ldots, f_n)$ is the second degree strand of M.

Lemma 3.3.8. *If N splits as $N' \oplus N''$ with N'' a free module, then there exists an exact sequence:*

$$0 \to N' \to M \to M'' \to 0.$$

Proof. By definition of N, the presentation matrix ϕ_1 of M may be written as $\phi_1 = \begin{pmatrix}\psi_1 & \mu_1\end{pmatrix}$ where ψ_1 is the presentation matrix of N. Since N has a free summand, we may rewrite ψ_1 as a block matrix $\psi_1 = \begin{pmatrix}\widetilde{\psi}_1 \\ 0\end{pmatrix}$, where $\widetilde{\psi}_1$ is the presentation matrix of N'. This enables us to rewrite ϕ_1 in upper triangular form:

$$\phi_1 = \begin{pmatrix}\widetilde{\psi}_1 & \widetilde{\mu}_1 \\ 0 & \widetilde{\eta}_1\end{pmatrix}.$$

Since M is presented by a block triangular matrix, it follows that M is an extension of $\operatorname{coker}(\widetilde{\psi}_1) = N'$ and $\operatorname{coker}(\widetilde{\eta}_1)$. By defining $M'' := \operatorname{coker}(\widetilde{\eta}_1)$, the statement follows immediately. □

The next proposition is concerned with extending the filtration of Lemma 3.3.8 to a filtration of the free resolution of M.

Proposition 3.3.9. *Assume that Proposition 3.3.2 and Lemma 3.3.8 hold. Assume further that $\beta_{i,j}(N') = 0$ whenever $i \leq n-1$ and $j > f_i$. Then*

$$\beta(M) = \beta(N') + \beta(M'').$$

Proof. Let $(G_\bullet, \psi_\bullet)$ be the minimal free resolution of N'. Recalling the notation of the proof of Proposition 3.3.2, we have that $(F'_\bullet, A_\bullet)$ is a summand of the minimal free resolutions of both M and N. Since $N \cong N' \oplus N''$ with N'' free, the minimal free resolutions of N and N' differ only in homological degree 0. The complex $(F'_\bullet, A_\bullet)$ thus induces a summand of the free resolution of N' as well. Namely, we obtain a complex $(G'_\bullet, A'_\bullet)$, which is a summand of $(G_\bullet, \psi_\bullet)$, and where

$$G'_i \cong \begin{cases} F'_i & i > 0 \\ G_0 & i = 0 \end{cases}$$

and the maps A'_i are the natural induced maps. Thus $(G'_\bullet, A'_\bullet)$ is a summand of the free resolutions of both N' and M. Further, as in Remark 3.3.3, the morphism $N' \to M$ induces an isomorphism along the subcomplex $(G'_\bullet, A'_\bullet)$.

We write each $G_i \cong G'_i \oplus G''_i$ for some free module G''_i. Following the notation from Proposition 3.3.2, we have $F_i \cong G'_i \oplus F''_i$ for all $i > 0$. We redefine F''_0 so that $F_0 \cong G'_0 \oplus F''_0$. The mapping cone on the complexes $(G_\bullet, \psi_\bullet) \to (F_\bullet, \phi_\bullet)$ via the map induced by $N' \to M$ thus yields a (non-minimal) free resolution of M''. Since this map induces an isomorphism along $G'_\bullet$, we see that, in fact, the induced mapping cone of

$$((G_\bullet, \psi_\bullet)/(G'_\bullet, A'_\bullet))) \cong (G''_\bullet, \psi''_\bullet) \to (F''_\bullet, \phi''_\bullet) \cong ((F_\bullet, \phi_\bullet)/(G'_\bullet, A'_\bullet)) \tag{3.3.4}$$

also yields a (non-minimal) free resolution of M''. If we let $(H''_\bullet, \pi_\bullet)$ be the minimal free resolution of M'', then this minimal free resolution of M'' may be obtained by minimizing the mapping cone free resolution obtained from (3.3.4)

We claim that H''_i is a summand of F''_i for all i, and we proceed by induction. The beginning of the mapping cone free resolution is given by

$$\begin{matrix} G''_0 \\ \oplus \\ F''_1 \end{matrix} \to F''_0.$$

However, since $G_0 \cong G'_0$, it follows that $G''_0 = 0$. This yields the cases $i = 0$ and $i = 1$.

For the inductive case, we assume that H''_i is a summnad of F''_i. Consider the mapping cone resolution:

$$\begin{matrix} G''_i \\ \oplus \\ F''_{i+1} \end{matrix} \to \begin{matrix} G''_{i-1} \\ \oplus \\ F''_i \end{matrix}.$$

By the inudction hypothesis, H''_i is a summand of F''_i, which implies that H''_i is generated in degree $\geq f_i$. Since $\beta_{i,j}(N') = 0$ for $j > f_i$, this implies that G''_i is generated in degree

$\leq f_i$. Thus, G''_i must also cancel when passing from the mapping cone resolution of M'' to the minimal free resolution of M''. This implies that H''_{i+1} is a summand of F''_{i+1}, as desired, thus proving the claim.

Since H''_i is a summand of F''_i for all $i = 0, \ldots, n$, this implies that, for $i = 0, \ldots, n-1$, G''_i must have totally cancelled in the process of minimizing the mapping cone resolution. Further, G''_n must also have cancelled in the minimzation process, since the projective dimension of M'' is n. Using $\beta(-)$ for the Betti diagram of a (possibly non-minimal) free resolution, we have thus shown that

$$\begin{aligned}\beta(M'') := \beta(H''_\bullet) &= \beta(F''_\bullet) - \beta(G''_\bullet)\\ &= (\beta(F_\bullet) - \beta(G'_\bullet)) - (\beta(G_\bullet) - \beta(G'_\bullet))\\ &= \beta(F_\bullet) - \beta(G_\bullet)\\ &= \beta(M) - \beta(N').\end{aligned}$$

□

Remark 3.3.10. Although our main goal in Proposition 3.3.9 is to produce pure filrations, there are many cases where the hypotheses of Proposition 3.3.9 are satisfied but where neither N' nor M'' have pure resolutions. See Example 3.7.1.

3.4 Application 1: Further Pathologies of the semigroup of Betti diagrams

Let us reconsider the example from Remark 3.2.1. Let $\Delta = (d^0, d^1, d^2)$ where $d^0 = (0, 2, 3, 4, 5, 8), d^1 = (0, 2, 3, 5, 6, 8)$, and $d^2 = (0, 3, 4, 5, 6, 8)$. We may parametrize the integral diagrams $E \in B_{\text{int}}(\Delta)$ by

$$E = \frac{b_0}{2}\pi_{d^0} + \frac{b_1}{2}\pi_{d^0} + \frac{b_2}{2}\pi_{d^2}$$

where $b_0 + b_1 + b_2 \equiv 0 \mod 2$. (See [Erm09a, pp. 347–9] for details on computing this parametrization.) Our work in §3.2 implies that any $\beta(M) \in B_{\text{mod}}(\Delta)$ admits a pure filtration, which would imply that each b_i is divisible by 2. Thus, in some sense, only half of the elements of $B_{\text{int}}(\Delta)$ correspond to Betti diagrams of modules.

Proposition 1.4.2 states that, in fact, there are rays where far less than half of the integral diagrams correspond to Betti diagrams of modules. Further, the proof will show that such pathologies already arise in codimension 3.

Proof of Proposition 1.4.2. Let $S = k[x_1, x_2, x_3]$ and let $p \geq 5$ prime. Set $d^0 = (0, 1, 2, p), d^1 = (0, \lfloor p/2 \rfloor, \lceil p/2 \rceil, p)$ and $d^2 = (0, p-2, p-1, p)$, and set $\Delta = (d^0, d^1, d^2)$. Consider the diagram

$$D = \frac{1}{p}\pi_{d^0} + \frac{\alpha}{p}\pi_{d^1} + \frac{1}{p}\pi_{d^2}$$

where α is any positive integer such $\alpha + 1 + \binom{p-1}{2} \equiv 0 \mod p$. We claim that $D \in B_{\text{int}}(\Delta)$ but that $cD \in B_{\text{mod}}(\Delta)$ if and only if c is divisible by p.

We first check the integrality of D. Observe that each Betti number of π_{d^0} is divisible by p except for the 0'th Betti number; each Betti number of π_{d^2} is divisible by p except for the 3'rd Betti number; and the Betti numbers of π_{d^1} are $(1, p, p, 1)$. Hence, we only need to check that $\beta_{0,0}(D)$ and $\beta_{3,p}(D)$ are integral. We compute

$$\beta_{0,0}(D) = \frac{1}{p}\beta_{0,0}(\pi_{d^0}) + \frac{\alpha}{p}\beta_{0,0}(\pi_{d^1}) + \frac{1}{p}\beta_{0,0}(\pi_{d^2}) = \frac{1}{p} + \frac{\alpha}{p} + \frac{\binom{p-1}{2}}{p}.$$

Our assumption on α then implies that $\beta_{0,0}(D)$ is integral. A symmetric computation works for $\beta_{3,p}(D)$.

We next prove that $cD \in B_{\text{mod}}(\Delta)$ only if p divides c by showing that any M such that $\beta(M) = cD$ admits a pure filtration. To obtain this pure filtration, consider $N := \mathcal{L}(M)$. Since N satisfies the conditions of Proposition 3.3.2, it follows that N has the same top strand as M in columns $0, 1$ and 2. Then, since N satisfies the conditions of Corollary 3.3.5 with $i = 1$, the first step of the Boij-Söderberg decomposition of N is given by $\frac{c}{p}\pi_{d^0}$. Since $\beta(N) - \frac{c}{p}\pi_{d^0}$ has no nonzero entries in column 1, it follows that $\beta(N) - \frac{c}{p}\pi_{d^0}$ must have projective dimension 0. Applying Proposition 3.3.7, Lemma 3.3.8, and Proposition 3.3.9, we obtain a short exact sequence:

$$0 \to N' \to M \to M'' \to 0$$

where $\beta(N') = \frac{c}{p}\pi_{d^0}$. We could continue by applying a similar argument to M''– thus obtaining a pure filtration of M– but we have already shown that $cD \in B_{\text{mod}}(\Delta)$ only if p divides c. Namely, since $\beta(N') = \frac{c}{p}\pi_{d^0}$, it follows that c must be divisible by p.

Finally, we show that $cD \in B_{\text{mod}}(\Delta)$ if c divides p. This is because $\pi_{d^i} \in B_{\text{mod}}(\Delta)$ for all i. In particular, $\pi_{d^2} = \beta(R)$ where $R := S/(x_1, x_2, x_3)^{p-2}$, and $\pi_{d^0} = \beta(R^\vee(p))$. To see that $\pi_{d^1} \in B_{\text{mod}}(\Delta)$, let A be a $p \times p$ skew-symmetric matrix of generic linear forms. By [BE77], the principal Pfaffians of A define an ideal $I \subseteq S$ such that $\beta(S/I) = \pi_{d^1}$.

This completes the proof when $p \geq 5$. For the cases $p = 2$ (respectively 3), we may choose the diagram $D = \frac{1}{2}\pi_{(0,1,2,4)} + \frac{1}{2}\pi_{(0,2,3,4)}$ (respectively $D = \frac{1}{3}\pi_{(0,1,2,5)} + \frac{2}{3}\pi_{(0,3,4,5)}$) and apply similar arguments as above. □

3.5 Application 2: Very singular spaces of matrices

Let $S = k[x_1, \ldots, x_n]$ and fix some $e > 1$. Fix $d^0 := (0, e, e+1, \ldots, e+n-1)$ and let $d^1 := (0, 1, \ldots, n-1, e+n-1)$. Let $R := S/(x_1, \ldots, x_n)^e, \omega_R := R^\vee$, and $\widetilde{\omega}_R := \omega_R(e+n-1)$. Note that $\beta(R) = \pi_{d^1}$ and $\beta(\widetilde{\omega}_R) = \pi_{d^0}$.

Let M be any module such that $\beta(M) \in B_{\mathrm{mod}}(\Delta)$ where $\Delta = (d^0, d^1)$. Then $\beta(M)$ has the shape

$$\beta(M) = \begin{pmatrix} * & * & \dots & * & - \\ - & - & \dots & - & - \\ \vdots & & & & \vdots \\ - & - & \dots & - & - \\ - & * & \dots & * & * \end{pmatrix}.$$

Proposition 3.5.1. *With notation as above, M admits a pure splitting. Namely, if $\beta(M) = c_0\pi_{d^0} + c_1\pi_{d^1}$, then $M \cong \widetilde{\omega}_R^{c_0} \oplus R^{c_1}$.*

Proof. To obtain the pure filtration, we first define $N := \mathcal{L}(M)$ (see §3.3). Since N satisfies the conditions of Proposition 3.3.2, it follows that N has the same top strand as M in columns $0, 1, \dots, n-1$. Further, since N satisfies the conditions of Corollary 3.3.5 with $i = n-2$, we see that the first step of the Boij-Söderberg decomposition of N is given by $c_0\pi_{d^0}$. Now, since $\beta(M) - c_0\pi_{d^0}$ has no nonzero entries in column 1, it follows that $\beta(M) - c_0\pi_{d^0}$ equals the Betti diagram of the free module S^{c_1}. Applying Proposition 3.3.7 and Proposition 3.3.9, we then obtain a short exact sequence:

$$0 \to N' \to M \to M'' \to 0$$

where $\beta(N') = c_0\pi_{d^0}$ and $\beta(M'') = c_1\pi_{d^1}$. Every such N' is isomorphic to $\widetilde{\omega}_R^{c_0}$ and every such M'' is isomorphic to R^{c_1}. Our exact sequence is thus

$$0 \to \widetilde{\omega}_R^{c_0} \to M \to R^{c_1} \to 0.$$

Considering the above sequence as a sequence of R-modules, we obtain a splitting of M. This lifts to a splitting of M as S-modules, and thus we obtain our pure splitting of M. □

We next apply Proposition 3.5.1 to classify certain vector spaces of matrices. We define an n-dimensional space of $a \times b$ matrices to be a vector space V of $a \times b$ matrices over a field k, with basis $V_1, \dots, V_n$. We say that two spaces of matrices V and W are isomorphic if there exists a linear change of coordinates of the source and target of V, and a change of basis of V, such that $V_i = W_i$ for all i.

We may use V to define a graded finite length S-module $M(V)$ of regularity one as follows. As a graded vector space, we set

$$M(V)_i := \begin{cases} k^a & \text{if } i = 0 \\ k^b & \text{if } i = 1 \\ 0 & \text{if } i \notin \{0, 1\} \end{cases}.$$

To give $M(V)$ a module structure, we must only define the action of x_j on $M(V)_0$ for each j, and we let $x_j : M(V)_0 \to M(V)_1$ be given by the matrix V_j. If V and W are two spaces of matrices, then $M(V) \cong M(W)$ if and only if $V \cong W$. Note also that every graded module N which satisfies $N_i = 0$ for $i \notin \{0,1\}$ may be written as $N = M(V)$ for some space of matrices V.

Definition 3.5.2. *Let V be an n-dimensional space of $a \times b$ matrices. We say that V is* very singular *if the Betti diagram of $M(V)$ only involves the pure summands $\pi_{(0,1,\dots,n-1,n)}$, $\pi_{(0,1,\dots,n-1,n+1)}$, $\pi_{(0,2,3,\dots,n+1)}$ and $\pi_{(1,2,\dots,n+1)}$.*

Corollary 3.5.3. *Up to isomorphism, there are only finitely many n-dimensional very singular spaces of $a \times b$ matrices for any a, b, and n. The isomorphism classes of these spaces are parametrized by non-negative integral solutions (c_0, c_1, c_2, c_3) to the system of equations:*

$$\begin{cases} c_0 + nc_1 + c_2 = a \\ c_1 + nc_2 + c_3 = b \end{cases}.$$

Proof. We will show that if V is very singular, then $M(V)$ admits a pure splitting. We write:

$$\beta\left(M(V)\right) = c_0\pi_{(0,1,\dots,n-1,n)} + c_1\pi_{(0,1,\dots,n-1,n+1)} + c_2\pi_{(0,2,3,\dots,n+1)} + c_3\pi_{(1,2,\dots,n+1)}. \tag{3.5.1}$$

If $\beta(M(V))$ involves any generator e of degree 1, then since $M(V)_2 = 0$, we have $x_je = 0$ for all x_j. It follows that we can split off precisely c_3 summands of $k(-1)$ from $M(V)$. Applying a similar argument to $M(V)^\vee$, we may split off c_0 copies of k from $M(V)$ whenever $M(V)$ has a socle element in degree 0. We may thus write $M(V) \cong k^{c_0} \oplus N \oplus k(-1)^{c_3}$ where

$$\beta(N) = c_1\pi_{(0,1,\dots,n-1,n+1)} + c_2\pi_{(0,2,3,\dots,n+1)}.$$

By setting $R := S/(x_1,\dots,x_n)^2$ and applying Proposition 3.5.1 to N, we then obtain that:

$$M(V) \cong k^{c_0} \oplus \widetilde{\omega}_R^{c_1} \oplus R^{c_2} \oplus k(-1)^{c_3}.$$

Thus, every n-dimensional space of very singular $a \times b$ matrices is classified, up to isomorphism, by its splitting type as above.

The Hilbert function of $M(V)$ is (a, b), and thus the Betti diagram of $M(V)$ has the form:

$$\begin{pmatrix} a & * & \dots & * & * \\ * & * & \dots & * & b \end{pmatrix}.$$

Combining this fact with equation (3.5.1) yields the desired parametrization. □

Remark 3.5.4. The κ-vector is an invariant of a vector space of matrices introduced in [EV09] that provides a generalized notion of rank for a vector space of matrices. The κ-vector of V encodes the same information as the Betti diagram of $M(V)$, and thus the κ-vector may be used to explicitly determine whether a space of matrices is very singular. For instance, let $V = \langle V_1, V_2, V_3\rangle$ be a 3-dimensional vector space of $d \times d$ matrices which contains a matrix of full rank. Then V is very singular if and only if $\kappa_1(V) = \frac{3}{2}d$, where $\kappa_1(V)$ is the rank of $3d \times 3d$ block matrix:

$$\begin{pmatrix} 0 & V_1 & -V_2 \\ -V_1 & 0 & V_3 \\ V_2 & -V_3 & 0 \end{pmatrix}.$$

3.6 Application 3: Computing generators of the semigroup of Betti diagrams

In this section, we compute all minimal generators of $B_{\mathrm{mod}}(\Delta)$ in a situation where $B_{\mathrm{int}}(\Delta) \neq B_{\mathrm{mod}}(\Delta)$. This provides the first detailed and nontrivial example of the generators of $B_{\mathrm{mod}}(\Delta)$. Further, this computation illustrates that the techniques introduced in §3.3 can be extended to more situations, but at the cost of wrestling with integrality conditions and precise numerics.

Throughout this section, we set $d^0 = (0,1,2,3), d^1 = (0,1,2,4), d^2 = (0,1,3,4), d^3 = (0,2,3,4)$ and $d^4 = (1,2,3,4)$ and we let $\widetilde{\Delta} = (d^0, d^1, d^2, d^3, d^4)$. Our goal is to compute the minimal generators of $B_{\mathrm{mod}}(\widetilde{\Delta})$. If $\beta(M) \in B_{\mathrm{mod}}(\Delta)$, then $\beta(M)$ has the shape:

$$\beta(M) = \begin{pmatrix} \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \beta_{3,3} \\ \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \beta_{3,4} \end{pmatrix}.$$

However, if $\beta_{3,3}(M)$ (or $\beta_{0,1}(M)$) is nonzero, then a copy of the residue field k (or $k(-1)$) splits from M. It is therefore equivalent to restrict to the case where $\beta_{3,3} = \beta_{0,1} = 0$ and to compute the generators for $B_{\mathrm{mod}}(\Delta)$ where $\Delta = (d^1, d^2, d^3)$. The result of this computation is summarized in the following proposition.

Proposition 3.6.1. *The semigroup $B_{\mathrm{mod}}(\Delta)$ has 10 minimal generators. These consist of the following ten Betti diagrams:*

$$\begin{pmatrix} 3 & 8 & 6 & - \\ - & - & - & 1 \end{pmatrix}, \begin{pmatrix} 1 & - & - & - \\ - & 6 & 8 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 1 & - \\ - & 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & - & - \\ - & 3 & 5 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 5 & 3 & - \\ - & - & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 4 & 1 & - \\ - & 1 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 7 & 3 & - \\ - & - & 3 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 3 & - & - \\ - & 3 & 7 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 4 & - & - \\ - & - & 4 & 2 \end{pmatrix} \begin{pmatrix} 3 & 6 & - & - \\ - & - & 6 & 3 \end{pmatrix}.$$

Before proving this proposition, we introduce some simplifying notation. Every element of $B_{\mathrm{int}}(\Delta)$ can be represented as:

$$D = \frac{r}{6}\pi_{(0,1,2,4)} + \frac{s}{6}\pi_{(0,1,3,4)} + \frac{t}{6}\pi_{(0,2,3,4)}$$

with $(r,s,t) \in \mathbb{Z}^3_{\geq 0}$ (c.f. [Erm09a, pp. 347–9].) However, an arbitrary $(r,s,t) \in \mathbb{Z}^3_{\geq 0}$ will not induce an element of $B_{\mathrm{int}}(\Delta)$. The necessary and sufficient conditions to obtain such an integral point are:

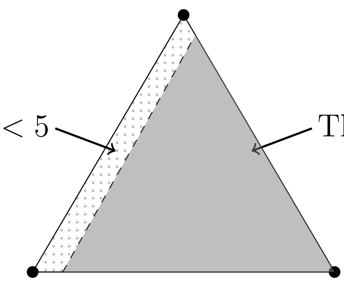


Figure 3.1: Proposition 3.6.1 can be illustrated by considering a slice of the cone $B_{\mathbb{Q}}(\Delta)$ where $r + s + t = c$ for some $c \gg 0$. In the region where $s < 5$, roughly half of the points of B_{int} belong to B_{mod}. In the region where $s \geq 5$, every point of B_{int} belongs to B_{mod}.

- $r + s \equiv 0 \mod 3$
- $r + t \equiv 0 \mod 3$
- $r + s + t \equiv 0 \mod 2$.

For the rest of this section, we use triplets (r, s, t) to refer to diagrams in $B_{\text{int}}(\Delta)$, and we only consider triplets (r, s, t) that satisfy the above congruency conditions. In this notation, Proposition 3.6.1 amounts to the claim that the following ten (r, s, t) triplets are the generators of B_{mod}:

$$(6,0,0), (0,0,6), (1,2,1), (3,3,0), (0,3,3),$$
$$(1,8,1), (3,9,0), (0,9,3), (0,12,0), (0,18,0).$$

Proof of Proposition 3.6.1. We first note that each of the ten diagrams listed in Proposition 3.6.1 is the Betti diagram of an actual module. When $\beta_{0,0} = 1$ or $\beta_{3,4} = 1$, such examples are straightforward to construct. Next, we have

$$\beta\left(\operatorname{coker}\begin{pmatrix} x & y & y+z & 0 & z^2 \\ 0 & z & x & y & 0 \end{pmatrix}\right) = \begin{pmatrix} 2 & 4 & 1 & - \\ - & 1 & 4 & 2 \end{pmatrix}.$$

Let L be any 2×3 matrix of linear forms whose columns satisfy no linear syzygies, and let $N := \operatorname{coker}(L)$. Then

$$\beta(N/\mathfrak{m}^2 N) = \begin{pmatrix} 2 & 3 & - & - \\ - & 3 & 7 & 3 \end{pmatrix}$$

The Betti diagram of $(N/\mathfrak{m}^2 N)^\vee$ then yields the dual diagram. Finally, examples corresponding to $(0, 12, 0)$ and $(0, 18, 0)$ are given in [Erm09a, Proof of Thm. 1.6(1)].

We must now show that every diagram in $B_{\mathrm{mod}}(\Delta)$ may be written as a sum of our ten generators. We proceed by analyzing cases based on the different possible values of s in our (r,s,t) representation of diagrams.

The case $s = 0$

Based on Proposition 3.5.1, we conclude that $(r,0,t)$ corresponds to an element of $B_{\mathrm{mod}}(\Delta)$ if and only if both r and t are divisible by 6.

The case $s = 1$

There are two families of triplets $(r,1,t)$ satisfying the congruency conditions. The first family is parametrized by $(2+6\gamma, 1, 5+6\alpha)$ for some $\gamma, \alpha \in \mathbb{Z}_{\geq 0}$, and the second family is parametrized by $(5+6\gamma, 1, 2+6\alpha)$. To prove that none of these diagrams belongs to $B_{\mathrm{mod}}(\Delta)$, it suffices (by symmetry under $M \mapsto M^\vee$) to rule out the first family.

We thus assume, for contradiction, that there exists M such that $\beta(M)$ corresponds to the triplet $(2+6\gamma, 1, 5+6\alpha)$ for some $\alpha, \gamma \in \mathbb{Z}_{\geq 0}$. We apply Proposition 3.3.2 to $N := \mathcal{L}(M)$ and obtain

$$\beta(N) = \begin{pmatrix} 2+\alpha+3\gamma & 3+8\gamma & 2+6\gamma & - \\ - & - & \beta_{2,3}(N) & \beta_{3,4}(N) \\ - & - & \beta_{2,4}(N) & \beta_{3,5}(N) \\ - & - & \vdots & \vdots \end{pmatrix}.$$

To produce the Boij-Söderberg decomposition, we begin by subtracting $c_1\pi_{d^1}$ for some $c_1 \geq 0$. If $c_1 < \gamma$, then $\beta(N) - c_1\pi_{d^1}$ would violate Proposition 3.3.4(1). Hence, we must have $c_1 \geq \gamma$.

If now $c_1 = \gamma$, then Proposition 3.3.4(2) applied to $\beta(N) - \gamma\pi_{d^1}$ would imply that the next step of the Boij-Söderberg decomposition must be $\frac{1}{5}\pi_{(0,1,2,5)}$. This would leave nothing left in column 1, and thus $\beta(N) - \gamma\pi_{d^1} - \frac{1}{5}\pi_{(0,1,2,5)}$ would be a diagram of projective dimension 0. But this would contradict the integrality of $\beta(N)$, since it would imply that $\beta_{3,5}(N) = \frac{1}{5}$.

The final possibility is that $c_1 > \gamma$, in which case c_1 must equal $\frac{1}{3} + \gamma$. After subtracting $(\frac{1}{3} + \gamma)\pi_{d^1}$, we are left with:

$$\beta(N) - \left(\frac{1}{3} + \gamma\right)\pi_{(d^1)} = \begin{pmatrix} 1+\alpha & \frac{1}{3} & - & - \\ - & - & \beta_{2,3}(N) & \beta_{3,4}(N) - (\frac{1}{3}+\gamma) \\ - & - & \vdots & \vdots \end{pmatrix}.$$

Since $\beta_{3,4}(N) - (\frac{1}{3} + \gamma)$ is nonzero (it is not an integer), the next step of the Boij-Söderberg decomposition must eliminate this entry. This means that the next step of the decomposition must be $\frac{1}{6}\pi_{d^2}$. However, this would leave a 0 in column 1 and a nonzero entry in column 2, which is impossible.

The case $s = 2$

There are two families of triplets $(r, 2, t)$ satisfying the congruency conditions. The first family has the form $(1+6\gamma, 2, 1+6\alpha)$ and the second family has the form $(4+6\gamma, 2, 4+6\alpha)$, where $\gamma, \alpha \in \mathbb{Z}_{\geq 0}$. Every element of the first family is a sum of our proposed generators, so we must show that no element of the second family belongs to $B_{\text{mod}}(\Delta)$. We obtain a contradiction by essentially the same analysis as in the case $s = 1$.

The case $s = 4$

There are two families of triplets $(r, 4, t)$ satisfying the congruency conditions, namely $2(+6\gamma, 4, 2+6\alpha)$ and $(5+6\gamma, 4, 5+6\alpha)$. Since every element of the first family is a sum of our proposed generators, we must show that no element of the second family belongs to $B_{\text{mod}}(\Delta)$. A similar, though more involved, analysis as in the case $s = 1$ then illustrates that there are no such diagrams.

The cases $s = 3, 5, 6$

We claim that if $D \in B_{\text{int}}(\Delta)$ corresponds to an (r, s, t)-triplet where $s = 3, 5$, or 6, then $D \in B_{\text{mod}}(\Delta)$, with the exception of $(0, 6, 0)$. There are six families to consider in total: $(3+6\gamma, 3, 6\alpha), (6\gamma, 3, 3+6\alpha), (4+6\gamma, 5, 1+6\alpha), (1+6\gamma, 5, 4+6\alpha), (3+6\gamma, 6, 3+6\alpha)$, and $(6\gamma, 6, 6\alpha)$. Any element from any of these families may be written as a sum of our proposed generators, except for $(0, 6, 0)$. The diagram corrresponding to $(0, 6, 0)$ does not belong to B_{mod} by [Erm09a, Proof of Thm. 1.6(1)].

The cases $s > 6$

One may directly check that all elements of $B_{\text{int}}(\Delta)$ with $s > 6$ can be written as an integral sum of the proposed generators. □

3.7 Nonexistence of pure filtrations

Since the Boij-Söderberg decomposition of a module may involve non-integral coefficients, it is clear that there exist many graded modules which do not admit pure filtrations. For instance, let $n = 2$, $R = k[x, y]/(x, y)^2$, and $M = k[x, y]/(x, y^2)$. Then:

$$\beta(M) = \begin{pmatrix} 1 & 1 & - \\ - & 1 & 1 \end{pmatrix} = \frac{1}{3}\beta(R) + \frac{1}{3}\beta(\omega_R(4)).$$

Clearly M cannot admit a pure filtration; however, we might hope that $M^{\oplus 3}$ admits such a filtration. Unfortunately, this is not the case either [SW09, Ex. 4.5].

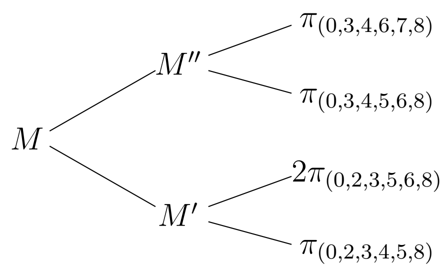


Figure 3.2: A diagram of the "branched" pure filtration from Example 3.7.1.

There does, however, exist a flat deformation M' of $M^{\oplus 3}$ such that M' admits a pure filtration:

$$0 \to R \to M' \to \omega_R(4) \to 0.$$

Namely, we may set $M' = (S/(x, y^2)) \oplus (S/(x^2, y)) \oplus (S/(x+y, (x^2 - 2y + y^2))$. This suggests a more subtle possible affirmative answer to our Question 1.4.1.

Another possibility is the existence of "branched" pure filtrations.

Example 3.7.1. *Let* $E := \pi_{(0,2,3,4,5,8)} + 2\pi_{(0,2,3,5,6,8)} + \pi_{(0,3,4,5,6,8)} + \pi_{(0,3,4,6,7,8)}$. *Note that this is the diagram* D *from §3.2 plus an extra* $\pi_{(0,3,4,6,7,8)}$. *Let* M *be a module such that* $\beta(M) = E$. *Running exactly the same argument as in §3.2, we obtain a filtration*

$$0 \to M' \to M \to M'' \to 0$$

where $\beta(M') = \pi_{(0,2,3,4,5,8)} + 2\pi_{(0,2,3,5,6,8)}$ *and* $\beta(M'') = \pi_{(0,3,4,5,6,8)} + \pi_{(0,3,4,6,7,8)}$. *Both* M' *and* M'' *admit pure filtrations. This example is illustrated in Figure 3.2. It would be interesting to know whether the pure filtrations of* M' *and* M'' *lift to a pure filtration of* M.

Chapter 4

A special case of the Buchsbaum-Eisenbud-Horrocks Rank Conjecture

4.1 Overview

The main result of this chapter is Theorem 1.5.2, where we prove a special case of the Buchsbaum-Eisenbud-Horrocks Rank Conjecture about the minimal size of a graded free resolution. Our proof is based on Boij-Söderberg theory (namely, Corollary 1.2.4) together with a detailed analysis of the numerics of pure diagrams. The results of this chapter originally appeared in [Erm09b].

This chapter is organized as follows. In §4.2, we investigate the numerics of pure diagrams. This analysis of pure diagrams is the foundation for the proof of Theorem 1.5.2, which appears in §4.3. In §4.4, we consider applications of Theorem 1.5.2 to geometric examples.

4.2 Ranks of pure diagrams

We now investigate the numerics of normalized pure diagrams. For a degree sequence d, recall the definition of the normalized pure diagram of type d, denoted $\overline{\pi}_d$, from Definition 2.6.1. We have that $\overline{\pi}_d$ is the unique pure diagram of type d such that $\beta_{0,d_0}(\overline{\pi}_d) = 1$. The diagram $\overline{\pi}_d$ may have non-integral entries. For instance

$$\overline{\pi}(0,1,2,4) = \begin{pmatrix} 1 & \frac{8}{3} & 2 & - \\ - & - & - & \frac{1}{3} \end{pmatrix}.$$

Further, Definition 2.6.1 provides precise formulas for the Betti numbers of $\overline{\pi}_d$. We will study the numerics of these Betti numbers, but restrict attention to the pure diagrams that satisfy the condition of Theorem 1.5.2.

We introduce auxiliary functions to simplify the notation. For $\mathbf{e} = (e_1, \ldots, e_s) \in \mathbb{R}^s$, we define linear functions, $T_i, U_{i,j}$, and $V_{i,j}$ from $\mathbb{R}^s$ to $\mathbb{R}$ by the following formulas:

$$\begin{aligned} T_i(\mathbf{e}) &:= i + e_1 + e_2 + \cdots + e_i, \text{ for } i = 1, \ldots, s \\ U_{i,j}(\mathbf{e}) &:= (j - i + 1) + e_i + e_{i+1} + \ldots e_j, \text{ whenever } i < j \\ V_{i,j}(\mathbf{e}) &:= (i - j) + e_{j+1} + \cdots + e_i, \text{ whenever } i > j \end{aligned}$$

Let $\mathfrak{d} : \mathbb{R}^s \to \mathbb{R}^{s+1}$ be the linear map:

$$\mathfrak{d}_j(\mathbf{e}) = \begin{cases} 0 & \text{for } j = 0 \\ j + \sum_{i=0}^{j} e_i, & \text{for } j = 1, \ldots, s. \end{cases}$$

Note that $\mathbf{e} \in \mathbb{Z}^s_{\geq 0}$ if and only if $\mathfrak{d}(\mathbf{e})$ is a degree sequence with first entry equal to 0.

We define the rational function $\mathfrak{b}_j : \mathbb{R}^s \to \mathbb{R}$ by:

$$\mathfrak{b}_j(\mathbf{e}) := \frac{\left(\prod_{i=1, i\neq j}^{s} T_i\right)}{\left(\prod_{i=2}^{j-1} U_{i,j}\right)\left(\prod_{i=j+1}^{s} V_{i,j}\right)} \tag{4.2.1}$$

for $j = 1, \ldots, s$. The rational function $\mathfrak{b}_j$ has no poles on $\mathbb{R}^s_{\geq 0}$. The purpose of these definitions is summarized in the following lemma:

Lemma 4.2.1. *Let $\mathbf{e} \in \mathbb{Z}^s_{\geq 0}$. Then we have:*

$$\mathfrak{b}_j(\mathbf{e}) = \beta_j\left(\overline{\pi}_{\mathfrak{d}(\mathbf{e})}\right)$$

Proof. For any degree sequence d of length s, a result of [HK84] can be used to give the explicit formulas

$$\beta_j\left(\overline{\pi}_d\right) = \prod_{\substack{1 \leq i \leq s \\ i \neq j}} \frac{|d_i - d_0|}{|d_i - d_j|}$$

(c.f. [BS08a, Defn 2.3]). Now let $d = \mathfrak{d}(\mathbf{e})$ and fix some $i \neq j$. Observe that $|d_i - d_0| = T_i$; further, $|d_i - d_j| = U_{i,j}$ if $i < j$ and $|d_i - d_j| = V_{i,j}$ if $i > j$. This proves the lemma. ☐

Lemma 4.2.2. *On the domain $\mathbf{e} \in \mathbb{R}^s_{\geq 0}$, we have $\frac{\partial}{\partial e_1}\mathfrak{b}_j \geq 0$.*

Proof. Consider the expression for $\mathfrak{b}_j$ given in (4.2.1), and observe that the denominator is not a function of e_1. Hence it is sufficient to show that $\frac{\partial}{\partial e_1}\left(\prod_{i=1, i\neq j}^{s} T_i\right) \geq 0$ when $\mathbf{e} \in \mathbb{R}^s_{\geq 0}$, and this is immediately verified. ☐

Lemma 4.2.3. *Let $\mathbf{e} \in \mathbb{R}^s_{\geq 0}$ and fix some $j, k \in \{1, \ldots, s\}$.*

1. *If* $k < j$ *then* $\left(\frac{\partial}{\partial e_j} - \frac{\partial}{\partial e_k}\right) \mathfrak{b}_j \leq 0.$

2. *If* $k > j+1$ *then* $\left(\frac{\partial}{\partial e_{j+1}} - \frac{\partial}{\partial e_k}\right) \mathfrak{b}_j \leq 0.$

Proof. Throughout this proof, we restrict all functions to the domain $\mathbf{e} \in \mathbb{R}^s_{\geq 0}$. It is sufficient to prove the statements for $\log \mathfrak{b}_j$. We may write:

$$\log \mathfrak{b}_j = \sum_{\substack{1 \leq i \leq s \\ i \neq j}} \log T_i - \sum_{i=1}^{j-1} \log U_{i,j} - \sum_{i=j+1}^{s} \log V_{i,j}. \tag{4.2.2}$$

To prove part (1) of the lemma, we assume that $k < j$ and we fix some $i \in \{1, \ldots, s\}$ where $i \neq j$. Observe that

$$\left(\frac{\partial}{\partial e_j} - \frac{\partial}{\partial e_k}\right) \log T_i = \left(\frac{\partial}{\partial e_j} - \frac{\partial}{\partial e_k}\right) \log(i + e_1 + e_2 + \cdots + e_i) \leq 0$$

with equality if and only if $i < k$ or $i > j$. Similarly, if $i \in \{1, \ldots, j-1\}$, then

$$\left(\frac{\partial}{\partial e_j} - \frac{\partial}{\partial e_k}\right) \log U_{i,j} = \left(\frac{\partial}{\partial e_j} - \frac{\partial}{\partial e_k}\right) \log(j - i + 1 + e_i + e_{i+1} + \cdots + e_j) \geq 0$$

with equality if and only if $k < i$. Since $k < j$, we also have that

$$\left(\frac{\partial}{\partial e_j} - \frac{\partial}{\partial e_k}\right) \log V_{i,j} = \left(\frac{\partial}{\partial e_j} - \frac{\partial}{\partial e_k}\right) \log(i - j + e_{j+1} + \cdots + e_i) = 0$$

for all $i \in \{j+1, \ldots, s\}$. By combining equation (4.2.2) with the results of these three computations, we conclude that $(\frac{\partial}{\partial e_j} - \frac{\partial}{\partial e_k})(\log \mathfrak{b}_j) \leq 0$ as desired.

To prove part (2) of the lemma, we now assume that $k > j+1$ and we fix some $i \in \{1, \ldots, s\}$ with $i \neq j$. Observe first that $\left(\frac{\partial}{\partial e_{j+1}} - \frac{\partial}{\partial e_k}\right) \log U_{i,j} = 0$ for all i; second, that if $i < j+1$ or $i \geq k$ then $\left(\frac{\partial}{\partial e_{j+1}} - \frac{\partial}{\partial e_k}\right) \log T_i = 0$; and third, that if $i \geq k$ then $\left(\frac{\partial}{\partial e_{j+1}} - \frac{\partial}{\partial e_k}\right) \log V_{i,j} = 0$. It remains to show that

$$\left(\frac{\partial}{\partial e_{j+1}} - \frac{\partial}{\partial e_k}\right) \sum_{i=j+1}^{k-1} \log T_i - \log V_{i,j} \leq 0.$$

This follows from the computation:

$$\left(\frac{\partial}{\partial e_{j+1}} - \frac{\partial}{\partial e_k}\right) \sum_{i=j+1}^{k-1} \log T_i - \log V_{i,j} = \sum_{i=j+1}^{k-1} \frac{1}{i + e_1 + \cdots + e_i} - \frac{1}{i - j + e_{j+1} + \cdots + e_i}$$
$$= \frac{(i - j + e_{j+1} + \cdots + e_i) - (i + e_1 + \cdots + e_i)}{(i + e_1 + \cdots + e_i)\,(i - j + e_{j+1} + \cdots + e_i)}$$
$$= \frac{-j - e_1 - \cdots - e_j}{(i + e_1 + \cdots + e_i)\,(j - i + e_{j+1} + \cdots + e_i)} < 0$$

which completes the proof. □

Lemma 4.2.4. *Let* $\mathbf{e} \in \mathbb{R}^s_{\geq 0}$ *with* $e_1 \geq \sum_{j=1}^s e_j$. *Then*

$$\mathfrak{b}_j(\mathbf{e}) \geq \binom{s}{j}.$$

Proof. By Lemma 4.2.2, it is sufficient to prove the lemma in the case $e_1 = \sum_{i=2}^n e_i$. Furthermore, by Lemma 4.2.3, we may assume that $e_i = 0$ for $i \notin \{1, j, j+1\}$.

Assume for the moment that $j \notin \{1, s\}$ and let $\widetilde{\mathbf{e}} = (e_1, e_j, e_{j+1})$. Under these assumptions we may write $\mathfrak{b}_j$ as a function of $\widetilde{\mathbf{e}}$. Our goal is to show that $\mathfrak{b}_j(\widetilde{\mathbf{e}}) \geq \binom{s}{j}$ given the constraint $e_1 = e_j + e_{j+1}$ and the domain $\widetilde{\mathbf{e}} \in \mathbb{R}^3_{\geq 0}$.

We introduce a new variable t and write $e_j = te_1$ and $e_{j+1} = (1-t)e_1$. Under this change of coordinates, our constrained minimization problem is now equivalent to minimizing the function:

$$\mathfrak{c}_j := \frac{(1 + e_1)(2 + e_1) \cdots (j - 1 + e_1) \cdot (j + 1 + 2e_1) \cdots (n + 2e_1)}{(j - 1 + te_1) \cdots (1 + te_1)(1 + (1-t)e_1) \cdots ((n-j) + (1-t)e_1)}$$

over the domain $(t, e_1) \in [0, 1] \times [0, \infty)$.

We claim that the minimum of $\log \mathfrak{c}_j$ on the domain $[0, 1] \times [0, \infty)$ occurs when $e_1 = 0$. The partial derivative $\frac{\partial \log \mathfrak{c}_j}{\partial e_1}$ is the sum of the following 4 functions:

- $f_1 := \dfrac{1}{1 + e_1} + \cdots + \dfrac{1}{j - 1 + e_1}$
- $f_2 := \dfrac{2}{j + 1 + 2e_1} + \cdots + \dfrac{2}{n + 2e_1}$
- $f_3 := -\dfrac{t}{1 + te_1} - \cdots - \dfrac{t}{j - 1 + te_1}$
- $f_4 := -\dfrac{1-t}{1 + (1-t)e_1} - \cdots - \dfrac{1-t}{(n-j) + (1-t)e_1}$

We observe first that:

$$-f_1 - f_3 = \sum_{i=1}^{j-1} \frac{-1}{i+e_1} + \frac{t}{i+te_1}$$
$$= \sum_{i=1}^{j-1} \frac{-(i+te_1)+t(i+e_1)}{(i+e_1)(i+te_1)}$$
$$= \sum_{i=1}^{j-1} \frac{(-1+t)i}{(i+e_1)(i+te_1)}.$$

Hence $-f_1 - f_3 \le 0$ whenever $(t, e_1) \in [0,1] \times [0, \infty)$. We next observe that:

$$-f_2 - f_4 = \sum_{i=j+1}^{n} \frac{-2}{i+2e_1} + \frac{1-t}{i+(1-t)e_1}$$
$$= \sum_{i=j+1}^{n} \frac{(-2i-2(1-t)e_1)+((1-t)i+2(1-t)e_1)}{(i+2e_1)(i+(1-t)e_1)}$$
$$= \sum_{i=j+1}^{n} \frac{-i-it}{(i+2e_1)(i+(1-t)e_1)}.$$

Hence $-f_2 - f_4 \le 0$ whenever $(t, e_1) \in [0,1] \times [0, \infty)$. Combining these two observations, we have that:

$$-\frac{\partial \log \mathfrak{c}_j}{\partial e_1} = -f_1 - f_2 - f_3 - f_4 \le 0$$

on the domain $[0,1] \times [0, \infty)$. A minimum of the function $\log \mathfrak{c}_j$ on the domain $[0,1] \times [0,\infty)$ thus occurs when $e_1 = 0$, and it follows that the same statement holds for the function $\mathfrak{c}_j$. Direct computation yields that $\mathfrak{c}_j(t, 0) = \binom{s}{j}$, which completes the proof when $j \notin \{1, s\}$.

If $j = 1$, then we may still apply Lemma 4.2.3 and reduce to the case that $e_i = 0$ for $i \neq 1$. Then we have:

$$\mathfrak{b}_1(e_1) = \frac{(2+e_1)(3+e_1)\cdots(s+e_1)}{(s-1)!}$$

which is at least than $\binom{s}{1}$ whenever $e_1 \ge 0$. If $j = s$, we reduce to the case that $e_s = e_1$ and we have:

$$\mathfrak{b}_s(e_1, e_s) = \frac{(1+e_1)\cdots(s-1+e_1)}{(s-1)!}$$

which is at least $\binom{s}{s}$ whenever $e_1 \ge 0$. □

Corollary 4.2.5. *Let $d \in \mathbb{Z}^{s+1}$ such that $d_0 \le 0$ and such that $d_s - s \le 2d_1 - 2$. Then:*

$$\beta_j(\overline{\pi}_d) \ge \binom{s}{j}$$

Proof. Let $\mathbf{e} = (e_1, \ldots, e_s)$ where $e_i = d_i - d_{i-1} - 1$, so that $\mathfrak{d}(\mathbf{e}) = d$. Since $d_s = d_0 + s + \sum_{i=1}^{s} e_i$ and $d_1 = d_0 + e_1 + 1$, we have that:

$$\sum_{i=2}^{s} e_i = (d_s - s) - d_0 - e_1 \leq (2d_1 - 2) - d_0 - e_1 = d_1 - 1 \leq e_1.$$

The corollary now follows from Lemmas 4.2.1 and 4.2.4. □

4.3 Proof of Theorem 1.5.2

We now prove our main result.

Proof of Theorem 1.5.2. By Corollary 1.2.4, we may write the Betti diagram of M as a positive rational sum of pure diagrams:

$$\beta(M) = \sum_{i=0}^{t} c_i \pi_{d^i} \tag{4.3.1}$$

By linearity, it is sufficient to show that:

$$\beta_j(\overline{\pi}_d) \geq \binom{c}{j} \tag{4.3.2}$$

for every pure diagram appearing with nonzero coefficient in (4.3.1) and for every $j \in \{1, \ldots, c\}$. Let $d = (d_0, \ldots, d_t)$ be a degree sequence corresponding to such a pure diagram, and let $\mathbf{e} = (e_1, \ldots, e_t)$ defined by $e_i := d_i - d_{i-1} - 1$. By Hilbert polynomial considerations, we see that $t \geq c$. Since $\overline{\pi}_d$ appears with positive coefficient in equation (4.3.1), it must contribute to the Betti diagram $\beta(M)$. It follows that $d_0 \leq 0$ and that

$$d_t - t \leq \operatorname{reg}(M) \leq 2\underline{d}_1(M) - 2 \leq 2d_1 - 2.$$

Hence d satisfies the hypotheses of Corollary 4.2.5, and $\beta_j(\overline{\pi}_d) \geq \binom{t}{j}$. Since $t \geq c$, it follows that $\binom{t}{j} \geq \binom{c}{j}$, and we obtain inequality (4.3.2). □

Remark 4.3.1. With more care, one could show that equality in Theorem 1.5.2 may only occur in cases where $\operatorname{codim}(M) \leq 2$ or where there exists $m \in \mathbb{N}$ such that $M \cong k^m$ as a graded S-module.

4.4 Examples

In this section, we consider several applications of Theorem 1.5.2, and we remark on the necessity of the hypothesis that $\operatorname{reg}(M) \leq 2\underline{d}_1(M) - 2$.

Example 4.4.1. *Let $V \subseteq S_c$ be any vector space of forms of degree c with $c > 1$, and let $I \subseteq S$ be the ideal $V + \mathfrak{m}^{c+1}$. Then S/I satisfies the hypotheses of Theorem 1.5.2. More generally, if $c' \leq 2c - 1$ and J is the ideal generated by $V + \mathfrak{m}^{c'}$, then S/J satisfies the hypotheses of Theorem 1.5.2.*

Example 4.4.2 (Curves of High Degree)**.** *Let $C \subseteq \mathbb{P}^{n-1}$ be a smooth curve of genus embedded by a complete linear system of degree at least $2g + 1$. Let $I_C \subseteq k[x_1, \ldots, x_n]$ be the ideal defining C. Then $\operatorname{reg}(S/I_C) \leq 2$ by* [Eis05, Corollary 8.2]*, and hence S/I_C satisifes the hypotheses of Theorem 1.5.2.*

Example 4.4.3 (Toric Surfaces)**.** *Let $X \subseteq \mathbb{P}^n$ be a toric surface embedded by a complete linear system $|A|$. Let $I_X \subseteq S = k[x_0, \ldots, x_n]$ be the defining ideal of X. We claim that S/I_X satisfies the hypotheses of Theorem 1.5.2, and hence that $\beta_i(S/I_X) \geq \binom{n-2}{i}$. Since I_X has no generators in degree* 1*, we must show that* $\operatorname{reg}(S/I_X) \leq 2$*. It is equivalent to show that the sheaf $\mathcal{I}_X := \widetilde{I_X}$ is* 3*-regular* [Eis05, Prop 4.16].

We first check that $H^1(\mathbb{P}^n, \mathcal{I}_X(2)) = 0$. Since X is a toric surface and A is an ample divisor, the corresponding lattice polygon has at least 3 *lattice points on its boundary. From* [Sch04, Cor 2.1]*, we see that X satisfies condition N_0, and hence that X is projectively normal. The surjectivity of the map*

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)) \to H^0(X, \mathcal{O}_X(2))$$

then implies that $H^1(\mathbb{P}^n, \mathcal{I}_X(2)) = 0$.

Next, we check that $H^2(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$. This follows from the exact sequence:

$$H^1(X, \mathcal{O}_X(1)) \to H^2(\mathbb{P}^n, \mathcal{I}_X(1)) \to H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$$

and the fact that higher cohomology of ample line bundles vanishes on toric varieties. It now follows that $\mathcal{I}_X$ is 3*-regular, which implies that S/I_X satisfies the hypotheses of Theorem 1.5.2.*

Example 4.4.4. *Let I be any ideal with minimal degree generator in degree $\underline{d}_1$ and maximal degree generator in degree $\overline{d}_1$. Assume that $\overline{d}_1(I) < 2\underline{d}_1(I)$. Then*

$$\operatorname{reg}(S/I^t) \leq t\overline{d}_1 + b$$

for some b and for all $t \geq 1$ [CHT99, Thm 1.1(i)]*. Since $\overline{d}_1(I) < 2\underline{d}_1(I)$, it follows that, for all $t \gg 0$, $t\overline{d}_1(I) + b < 2t\underline{d}_1(I) - 2$. Hence, for all $t \gg 0$, the module S/I^t satisfies the hyoptheses of Theorem 1.5.2.*

The method of proof for Theorem 1.5.2 breaks down if one removes the hypothesis that $\operatorname{reg}(M) \leq 2\underline{d}_1(M) - 2$. One issue is that the statement:

$$\beta_j(M) \geq \beta_0(M)\binom{\operatorname{codim}(M)}{j}$$

is *not* true in general. For example, if $S = k[x, y]$ and N is the cokernel of a generic 2×3 matrix of linear forms, then

$$\beta_1(N) = 3 < 4 = \beta_0(N)\binom{\operatorname{codim}(N)}{1}.$$

There also exist pure diagrams with integral entries which do not satisfy the graded BEH rank conjecture. For instance, the diagram:

$$\overline{\pi}(0, 1, 2, 3, 5, 6) = \begin{pmatrix} 1 & \frac{9}{2} & \frac{15}{2} & 5 & - & - \\ - & - & - & - & \frac{3}{2} & \frac{1}{2} \end{pmatrix}$$

does not satisfy any version of Conjecture 1.5.1.

Chapter 5

Asymptotic Betti numbers

5.1 Overview

The purpose of this brief chapter is to illustrate a technique for applying Boij-Söderberg theory to obtain numerical information about free resolutions. Our main result is an asymptotic lower bound for the Betti numbers of S/I^t in the case where I is a graded ideal generated in a single degree. The basic idea behind our technique was developed in Chapter 4: if we can bound the shape of the Betti diagram of a graded module M, then, by studying pure diagrams of the relevant shape, we can obtain numerical information about $\beta(M)$.

To illustrate this technique, we focus on a case where we can obtain meaningful bounds on the shape of $\beta(M)$. Namely, we let I be an ideal generated in a single degree δ, and we consider $\beta(S/I^t)$ for $t \gg 0$. The bounds on the shape of $\beta(S/I^t)$ are derived from [TW05, Thm 3.2] and [Kod00, Cor 3], which imply that $\operatorname{reg}(I^t) = \delta t + b$ for $t \gg 0$ and some $b \in \mathbb{N}$. Since I^t is generated in degree δt, $\beta(S/I^t)$ thus has the following shape:

$$\beta(S/I^t) = \begin{matrix} \\ 0 \\ 1 \\ \\ \vdots \\ \\ \delta t-2 \\ \delta t-1 \\ \delta t \\ \\ \vdots \\ \\ \delta t+b-2 \\ \delta t+b-1 \end{matrix} \begin{matrix} 0 & 1 & 2 & \dots & p \end{matrix} \atop \begin{pmatrix} 1 & - & - & \dots & - \\ - & - & - & \dots & - \\ & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & \\ - & - & - & \dots & - \\ - & * & * & \dots & * \\ - & - & * & \dots & * \\ & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & \\ - & - & * & \dots & * \\ - & - & * & \dots & * \end{pmatrix}.$$

As in Chapter 4, we then consider pure diagrams which fit into this shape, and we use optimization techniques to provide lower bounds for the Betti numbers of S/I^t. In particular, we show that, for all $j \in \{1, \ldots, c\}$, the function $\beta_j(S/I^t)$ is bounded below by a function of order t^{c-1}. The main result of this chapter originally appeared in [Erm09b].

5.2 Proof of asymptotic formula

Let I be an ideal generated in a single degree δ. The regularity of I^t becomes a linear function $\mathrm{reg}(I^t) = \delta t + b$ for $t \gg 0$ (c.f. [TW05, Thm 3.2], [Kod00, Cor 3]). We define b to be the asymptotic regularity defect of I. The following theorem gives lower bounds for the Betti numbers of S/I^t.

Theorem 5.2.1. *Let I be an ideal of codimension c generated in a single degree δ and with asymptotic regularity defect b. We have the following lower bound on the Betti numbers of S/I^t:*

$$\beta_j(S/I^t) \geq \frac{(b!)^2\delta^{c-1}}{(j-1+b)!(c-j+b)!}t^{c-1} + O(t^{c-2})$$

for all $j = 1, \ldots, c$ and for all $t \gg 0$.

Proof. By Corollary 1.2.4 we may write $\beta(S/I^t)$ as a sum of pure diagrams as in equation (4.3.1). Let $d = (d_0, \ldots, d_s)$ be some degree sequence such that $\overline{\pi}_d$ appears with nonzero coefficient in this sum. The equality $\mathrm{codim}(I^t) = \mathrm{codim}(I) = c$, implies that $s \geq c$. Let $\mathbf{e} = (e_1, \ldots, e_s)$ defined by $e_i = d_i - d_{i-1} - 1$. Since I^t is generated in degree $t\delta$, we have $e_1 = t\delta$. Let $t \gg 0$ so that $\mathrm{reg}(S/I^t) = t\delta + b$. Since $\mathrm{reg}(S/I^t) = \overline{d}_p(S/I^t) - p$ we have that $\sum_{i=2}^s e_i \leq b$.

It is sufficient to prove the lower bound for the Betti numbers of the pure diagram $\overline{\pi}_d$. In fact, it is sufficient to prove the lower bound for the functions $\mathfrak{b}_j(\mathbf{e})$ under the constraints $e_1 = t\delta$ and $\sum_{i=2}^s e_i \leq b$. Let $j \in \{1, \ldots, c\}$. By Lemma 4.2.3, we may assume that $e_i = 0$ unless $i \in \{1, j, j+1\}$. Hence we reduce to the case that $e_j + e_{j+1} \leq b$. We now seek to compute $\mathfrak{b}_j$.

$$\mathfrak{b}_j(\mathbf{e}) = \frac{(1+t\delta)(2+t\delta)\cdots(j-1+t\delta)(j+1+t\delta+b)\cdots(s+t\delta+b)}{(j-1+e_j)\cdots(1+e_j)(1+e_{j+1})\cdots(s-j+e_{j+1})}$$

Note that e_j and e_{j+1} only appear in the denominator, and both are positive numbers less than b. Hence setting $e_j = e_{j+1} = b$ only decreases the right-hand side. This yields:

$$\mathfrak{b}_j(\mathbf{e}) \geq \frac{(1+t\delta)(2+t\delta)\cdots(j-1+t\delta)(j+1+t\delta+b)\cdots(s+t\delta+b)}{(j-1+b)\cdots(1+b)(1+b)\cdots(s-j+b)} \tag{5.2.1}$$

Since $s \geq c$ we may rewrite the right-hand side of (5.2.1) as

$$\left(\frac{(1+t\delta)(2+t\delta)\cdots(j-1+t\delta)(j+1+t\delta+b)\cdots(c+t\delta+b)}{(1+b)\cdots(j-1+b)(1+b)\cdots(c-j+b)}\right)\left(\prod_{i=1}^{s-c}\frac{(c+i+t\delta+b)}{(c+i-j+b)}\right).$$

Each term of the product on the right is greater than 1, so by deleting this product and substituting back into (5.2.1), we obtain the inequality:

$$\begin{aligned}\mathfrak{b}_j(\mathbf{e}) &\geq \frac{(1+t\delta)(2+t\delta)\cdots(j-1+t\delta)(j+1+t\delta+b)\cdots(c+t\delta+b)}{(1+b)\cdots(j-1+b)(1+b)\cdots(c-j+b)} \\ &= \frac{(b!)^2\delta^{c-1}}{(j-1+b)!(c-j+b)!}t^{c-1}+O(t^{c-2}).\end{aligned}$$

This completes the proof. □

Example 5.2.2. *Let $I = \langle x^6, y^6, z^6, x^3y^3 + y^3z^3 + z^3x^3\rangle \subseteq \mathbb{Q}[x, y, z]$. The asymptotic regularity defect equals 7, and the codimension of I is 3. We thus obtain the following quadratic bound for $\beta_3(S/I^t)$:*

$$\beta_3(S/I^t) \geq \frac{(7!)^2 6^2}{(9)!(7)!}t^2 + O(t^{c-2}) = \frac{9}{14}t^2 + O(t).$$

Computation in [GS] *for $t \leq 10$ yields that $\beta_3(S/I^t) = 2t^2-2t+2$; since $\frac{9}{14} < 2$, this confirms our lower bound.*

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