Title
Combinatorics of Crystal Folding

Author
Dusel, John M.

Publication Date
2015

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
RIVERSIDE

Combinatorics of Crystal Folding

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

John Michael Dusel

August 2015

Dissertation Committee:

Dr Jacob Greenstein, Chairperson
Dr Vyjayanthi Chari
Dr Wee Liang Gan

The Dissertation of John Michael Dusel is approved:

Committee Chairperson

University of California, Riverside

Acknowledgments

I thank Dr Jacob Greenstein for his professional and personal mentorship throughout the preparation of this dissertation. He has been extremely generous with his time and energy, and his unrelenting faith in my abilities and better nature are an inspiration. I also thank Drs Vyjayanthi Chari and Wee Liang Gan, my dissertation committee, and for providing valuable feedback on my work.

The UC Riverside Mathematics Department has been a stimulating and supportive work environment, I thank all of its faculty and staff members. I would like to especially thank Drs Gerhard Gierz and Vyjayanthi Chari, who served as chair during my graduate education and granted me appointments well beyond their contractual obligation.

I gratefully acknowledge the financial support of the UC Riverside Graduate Division, the Gordon E Hein Foundation, and the JW and Ida M Jameson Foundation.

On a personal level, I thank Molly Moreland for her partnership and support of my professional advancement. Credit is also due to my late father Daniel Dusel, who played a role in developing my early interest in mathematics. In some small way this work is an attempt to honor the bitter profit I made from his sacrifice. The last word goes to August Moreland Dusel, whose arrival during the preparation of this report completely upset my work-life balance and immeasurably benefitted of my humanity and sense of purpose in life.

To Molly and August.
There but for the grace of you go I.

The man who believes that the secrets of the world are forever hidden lives in mystery and fear. Superstition will drag him down. The rain will erode the deeds of his life. But that man who sets himself the task of singling out the thread of order from the tapestry will by the decision alone have taken charge of the world and it is only by such taking charge that he will effect a way to dictate the terms of his own fate.

Cormac McCarthy, *Blood Meridian*

ABSTRACT OF THE DISSERTATION

Combinatorics of Crystal Folding

by

John Michael Dusel

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, August 2015
Dr Jacob Greenstein, Chairperson

We study the structure of a Kashiwara crystal of simply-laced Cartan type $\mathscr{C}$ under an automorphism σ, via a process known as folding. We define the category of σ-foldable crystals, which is a monoidal category and admits a monoidal functor into the category of $\mathscr{C}^{\sigma\vee}$-crystals. Various properties of a foldable crystal—normality, Weyl group action, *etc.*—can be transferred to its quotient modulo the σ-action. On the other hand, the quotient of a foldable highest-weight crystal contains a new type of crystal, which we call multi-highest-weight.

We consider multi-highest-weight crystals B obtained as foldings of Kashiwara Littelmann crystals $B(\lambda)$. The structure of the highest-weight set of B is explained by certain subsets of the Weyl group we call the balanced parabolic quotients; in many cases the latter parameterizes a generating set for the highest-weight elements. A balanced parabolic quotient relates the branching rules, Demazure crystals, and σ-action on $B(\lambda)$, and is enumerated by a forest graph with self-similar components.

Contents

Introduction

Some of the most celebrated means of passing from the algebraic subject of representation theory into the domain of combinatorics are Kashiwara's theory of crystals [Kas90a, Kas90b, Kas91, Kas93], and Lusztig's theory of canonical bases [Lus90a, Lus93]. Independently introduced in the early 1990s, these theories were shown by Lusztig to coincide whenever the crystal base and canonical base are both defined [Lus90b]. Crystals are combinatorial analogues of a Lie algebra (or more generally, a Kac-Moody algebra), its associated quantum group, and their representations, which are important for a wide variety of research programs in mathematics and physics. This dissertation is a study of the structure of a crystal under an automorphism σ, via a process known as *folding*. Folding allows us to use a portion of a crystal for one type of Lie algebra to study the structure of another, and has applications to the fixed-point resolution problem in conformal field theory [FSS96].

This dissertation establishes foundational results on the structure of a crystal modulo the action of the automorphism σ, with a focus on the crystal $B(\infty)$ associated to a quantum group of simply-laced type. Remarkably, the quotient of $B(\infty)$ by the σ-action contains a new type of multiply-laced crystal, which we call multi-highest-weight. This crystal parameterizes a basis of a natural subalgebra of the quantum group, and is related to the representation theory of certain foldings of the latter [BG11]. A principal goal of this dissertation is to understand the structure of this crystal, with a view towards providing a representation-theoretic interpretation.

Owing to their representation-theoretic origin, crystals are primarily studied via their algebraic counterparts; however, little is known about the algebraic objects to which a multi-highest-weight crystal corresponds. In this dissertation we invert that paradigm by using a purely combinatorial approach to 'reverse engineer' results.

Foldable crystals

Crystal graphs

The essential combinatorial datum for our purposes is a symmetric matrix $A = (a_{ij})_{i,j \in I}$ such that $a_{ii} = 2$ and $a_{ij} \in \{-1, 0\}$ if $i \neq j$. That is to say, A is the Cartan matrix of a Kac-Moody Lie algebra $\mathfrak{g}(A)$ of simply-laced type. An A-crystal graph B is a countable digraph with I-colored edges such that

1. Removing all edges, except those with a particular label, results in a disjoint union of linear graphs.

2. There is a weight function $\mathrm{wt} : B \to \mathbb{Z}^{\oplus I}$ with the property that $\mathrm{wt}(b) - \mathrm{wt}(b') = i$ if $b \xrightarrow{i} b'$. This function is monotonic on each linear graph from (1).

The dependency of this construction on A comes from the pairing $\langle \cdot, \cdot \rangle : \mathbb{Z}^{\oplus I} \otimes_{\mathbb{Z}} \mathbb{Z}^{\oplus I} \to \mathbb{Z}$ given by extending $\langle i, j \rangle := a_{ij}$. Note that this definition, taken from [Jos09], is equivalent to the original definitions of Kashiwara and Lusztig (see section 1.3).

As an example we consider the matrix $A_3 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, which is the Cartan matrix of the Lie algebra $\mathfrak{g}(A_3) = \mathfrak{sl}_4(\mathbb{C})$ of traceless 4×4 matrices with complex entries. Figure 0.1 shows a crystal graph associated to this matrix; more precisely this is the crystal graph of the simple finite-dimensional $\mathfrak{sl}_4(\mathbb{C})$ module of highest weight $\varpi_1 + \varpi_3$.

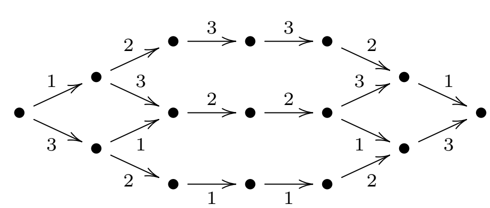


Figure 0.1: An A_3-crystal graph

Foldability

In the folding situation we consider permutations σ of I satisfying $\sigma[A] = A$, with the additional technical requirement that $a_{ij} = 0$ whenever i, j belong to the same σ-orbit. Note that σ

acts on $\mathbb{Z}^{\oplus I}$ by permuting its free generators, and $\langle i, j\rangle = \langle \sigma(i), \sigma(j)\rangle$. Such permutations are called admissible automorphisms of $\mathscr{C}$, after Lusztig [Lus93].

In section 4.1 we introduce the category of σ-foldable crystal graphs with σ-foldable crystal morphisms (that is, digraph homomorphisms commuting with the σ-action). Roughly speaking, these are the crystal graphs B equipped with an action $\sigma : B \to B$ that is compatible with the grading and satisfies

- $b \xrightarrow{i} b'$ if and only if $\sigma(b) \xrightarrow{\sigma(i)} \sigma(b')$
- If $i, j \in I$ belong to the same σ-orbit and b has outgoing edges labeled i and j, then there is a diamond as shown:

$$\begin{tikzcd}
 & \bullet \arrow[dr, "j"] & \\
b \arrow[ur, "i"] \arrow[dr, "j"'] & & \bullet \\
 & \bullet \arrow[ur, "i"'] &
\end{tikzcd}$$

It turns out that this category inherits several desirable properties from the category of A-crystals, in particular we have the following.

Theorem (Theorem 4.3.1)**.** *The set of σ-foldable A-crystals is a tensor category.*

In type A_3 we have $\sigma = (1, 3)$, and the crystal graph shown in Figure 0.1 is $(1, 3)$-foldable (along the horizontal axis of symmetry). The foldability of a crystal graph B ensures that the set B_σ of σ-orbits in B is also a crystal graph, but for a different matrix: the *Langlands dual folded matrix* (see section 2.2 for more details.).

Theorem (Theorem 4.2.6)**.** *Let A be the Cartan matrix of a simply-laced semisimple Lie algebra $\mathfrak{g}$ with automorphism σ. Suppose B is a σ-foldable A-crystal, and let $\widehat{B}_\sigma$ denote the set of all $b \in B$ with σ-invariant weight. Then the collection $B_\sigma := \widehat{B}_\sigma/\sigma$ of σ-orbits has an $A^{\sigma\vee}$-crystal graph structure, where $A^{\sigma\vee}$ is the Langlands dual folded matrix.*

Highest-weight and multi-highest-weight crystals

An important family of crystal graphs are the so-called *highest weight* crystals. This type of crystal graph is central to the representation theory of the quantum group $U_q(A)$, in that the most important category of modules (integrable weight modules, also known as the category $\mathcal{O}_q^{int}$) is semisimple, and the simples admit highest-weight crystal graphs (see p.5). This means simply that

B is connected and contains a unique element with no incoming edges, called the highest-weight element. The crystal graph shown in Figure 0.1 is highest-weight.

Suppose B is a highest-weight crystal with highest-weight element b. An admissible automorphism σ operates on B, and the latter is σ-stable if and only if $\sigma(\mathrm{wt}(b)) = \mathrm{wt}(b)$. When applied to a highest-weight crystal, the folding procedure yields a crystal which is 'multi-highest-weight' in the following sense.

Theorem (Theorem 4.4.9)**.** *Let B be a highest-weight crystal. The subset of σ-invariant elements in B is a highest-weight subcrystal of B_σ, and each of the remaining connected components of B_σ is generated by a* set *of highest-weight elements.*

An example is shown in Figure 0.2. Theorem 4.4.9 motivates several natural questions about the graph structure of B_σ, which are described next.

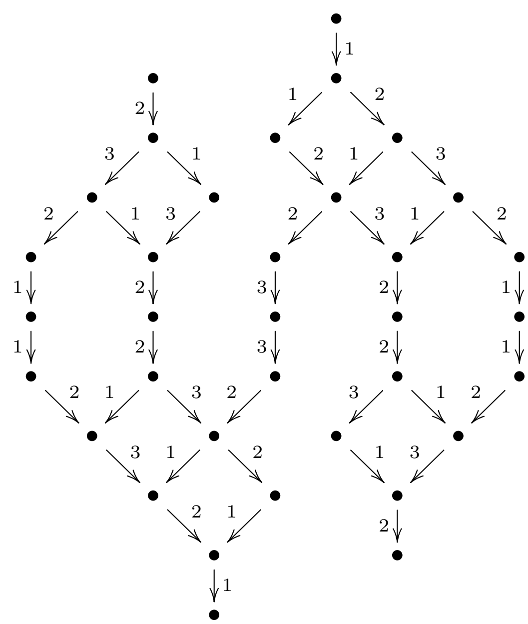


Figure 0.2: A multi-highest-weight crystal graph

The set of highest-weight elements is known to be σ-stable (section 4.4), and in a precise sense contains most of the structure of B_σ.

Problem 1. Characterize the highest-weight elements of B_σ.

The multi-highest-weight crystals have not appeared in the literature, thus:

Problem 2. Describe the isoclasses of the connected components of B_σ.

Problem 2 is related to an open problem in crystal graph theory. There are conditions, due to Joseph [Jos95, 6.4.21] and Stembridge [Ste03, Propositions 1.3, 1.4], under which a highest-weight crystal is a crystal graph of a $U_q(A)$-module. To the best of the author's knowledge every known example of a highest-weight crystal graph is the crystal graph of a $U_q(A)$-module, yet there seems to be no proof of the contrary in the literature. Under certain conditions B_σ contains highest-weight subcrystals which are truncations of multi-highest-weight graphs in a precise sense; these are natural candidates for examples of highest-weight crystal graphs not arising from highest weight integrable modules for the folded type (or any other type). Thus far all such graphs we have considered are in fact isomorphic to crystal graphs of $U_q(A)$-modules; it remains to be seen whether this is true in general.

The connectedness of B_σ determines a decomposition of its corresponding quantum group representation. Towards the long-term goal of classifying this representation, we have considered the following.

Problem 3. Find necessary and sufficient conditions for two highest-weight elements of B_σ to lie in the same component. Determine the number of highest-weight elements contained in a given component.

Problem 4. Determine the number of connected components in B_σ.

Problem 5. Determine the number of highest weight elements of B_σ of a given weight.

Representation theory and the crystal $B(\infty)$

Crystal bases

The quantum group $U_q(A)$ is a can be viewed as a deformation of $\mathfrak{g}(A)$, according to the quantum parameter q. Kashiwara used the $q \to 0$ limit, called crystallization, to construct the crystal graph $B(V)$ of a simple finite-dimensional $U_q(A)$-module V; the actions of the Chevalley generators of $U_q(A)$ on V induce operators e_i, f_i $(i \in I)$ on $B(V)$ that satisfy certain properties [Kas91]. Declaring $b \xrightarrow{i} b'$ if and only if $f_i b = b'$ gives $B(V)$ a crystal graph structure, with a grading by the weight lattice of A. A crystal graph of a certain $U_q(A_3)$-module is shown in Figure 0.1. The collection of all $B(V)$, with V a simple finite-dimensional $U_q(A)$-module, is a directed set. The limit of this set, denoted $B(\infty)$, is a highest-weight crystal with highest weight vector b_∞. This crystal has a striking property: it parameterizes a basis of an algebra U, and a basis of every simple U-module V is obtained by multiplying a particular $v \in V$ by the elements of $B(\infty)$, discarding those that became zero.

Theorem (Theorem 5.2.1)**.** *The crystal $B(\infty)$ associated to a quantum group $U_q(A)$ of simply-laced type under the automorphism σ is foldable.*

It follows from this result and the tensor category structure that each $B(V)$ with $\sigma(V) = V$ is also foldable. This dissertation is primarily concerned with the structure of $B(\infty)_\sigma$, in that this crystal is naturally related to the folding of quantum groups [BG11].

Balanced parabolic quotients

Let W denote the Weyl group of $\mathfrak{g}$. A subset $J \subset I$ determines a parabolic quotient ${}^JW \subset W$, which is the set of minimal-length coset representatives in $W_J \backslash W$, where W_J is the J-th parabolic subgroup. Stembridge [Ste96] showed that, for certain J, the reduced decomposition of $w \in {}^JW$ is unique up to an exchange of commuting simple reflections (such elements are called fully commutative). The *balanced parabolic quotient* is the subset ${}^JW_\sigma$ of all $w \in {}^JW$ such that the number of occurrences of s_i and $s_{\sigma(i)}$ in a reduced expression for w are equal.

The highest-weight set of $B(\infty)_\sigma$ can be understood in terms of the balanced parabolic quotient. Given a reduced expression $s_{i_1} \cdots s_{i_\ell}$ of $w \in W$, the Demazure crystal $B_w(\infty) \subset B(\infty)$ coincides with the set of all $f_{i_\ell}^{k_\ell} \cdots f_{i_1}^{k_1} b_\infty$ with $k_j \geq 0$. These subsets give a combinatorial filtration of $B(\infty)$, and parameterize bases of the Demazure modules of $U_q(A)$. In particular, when W is finite the subset $B_{w_\circ}(\lambda)$ corresponding to the longest element coincides with $B(\lambda)$. A Demazure crystal is a combinatorial analogue of a Demazure module for an extremal vector in a simple finite-dimensional $\mathfrak{g}$-module of highest-weight λ.

Theorem (Theorem 5.3.1)**.** *A Demazure crystal $B_w(\infty)$ contains a (representative of a) highest-weight element of $B(\infty)_\sigma$ if and only if $w \in {}^JW_\sigma$.*

That is, this result indicates the location of highest-weight elements in the Demazure filtration of $B(\infty)$, and exhibits them on the level of the monoid generated by the f_i. On the level of $\mathfrak{g}$-modules, a balanced quotient describes the interaction between the branching rule for a Levi subalgebra, Demazure modules, and the σ-action on weight spaces in a σ-stable simple module. Note that although our results are applicable to representation theory, our methods use only the combinatorial properties of the Kashiwara-Littelmann crystals along with the Kashiwara $\star$ operation [Kas93].

Foldings of quantum groups

The permutation σ induces an automorphism of negative (or positive) half $U_q^-(A)$ of the quantum group, and the subalgebra $U_q^-(A)^\sigma$ of σ-invariants is properly contained in the subalgebra $U_q^-(A)_\sigma$ spanned by elements of σ-invariant weight. This subalgebra has a crystal basis, namely $B(\infty)_\sigma$, but its representation theory has never been studied. The combinatorial properties of $B(\infty)_\sigma$ provide insight into the structure of this algebra and its modules. As is the case for $U_q^-(A)$, a representation of $U_q^-(A)_\sigma$ admits a decomposition into irreducibles, and the latter are in one-to-one correspondence with the connected components of $\widehat{B}(\infty)_\sigma$, viewed as a subgraph of $B(\infty)$. Characterizing these irreducible representations using $\widehat{B}(\infty)_\sigma$ is different from Problems 3 and 4: some—but not all—components of the former are σ-stable, and this information is lost when passing to $B(\infty)_\sigma$.

In section 5.5 we handle type A_3, and show that $B(\infty)_\sigma$ is isomorphic to the direct sum of countably many copies of the B_2-crystal $B(\infty)$. From this it follows that $U_q(A_3)_\sigma \cong U_q(B_2)^{\oplus \mathbb{N}}$. On the other hand, the subset $V_\sigma \subset V$ spanned by weight-invariant vectors is a $U_q^-(A)^\sigma$-module, and decomposes into a direct sum of simple $U_q(B_2)$-modules.

In type D_4 the algebra $U_q^-(A)_\sigma$ is isomorphic to the direct sum of $U_q^-(C_3)$ and an unidentified quantum algebra, and V_σ has a direct summand that is simple and multi-highest-weight. The crystal graph shown in Figure is one such example. The quotient $B(\infty)_\sigma$ of the limiting crystal contains a component with infinitely many highest-weight elements (section 5.4).

Geometry and polyhedral combinatorics

We apply methods from convex geometry and polyhedral combinatorics to Problems 3 and 4. We can identify the elements of $B(\infty)$ with the lattice points of a certain polyhedral convex cone Σ in a finite-dimensional vector space [Kas93, NZ97]. This cone's semigroup and geometric structures allow one to treat the highest-weight elements of $B(\infty)_\sigma$ as an abelian semigroup. In fact, something much stronger is true.

Theorem (section 5.3.2)**.** *The abelian semigroup HW Σ_σ of highest-weight elements of Σ_σ admits a unique finite $\subset$-minimal generating set $\mathcal{H}$. The balanced parabolic quotient identifies with a subset of $\mathcal{H}$.*

This result reduces Problem 1 to describing a finite subset of $B(\infty)_\sigma$, and leads to the following conjectural answer to Problem 3, known to be true in many special cases.

Conjecture. *The set $\mathcal{H}$ is parameterized by the balanced parabolic quotient. Two elements of $\mathcal{H}$ lie in the same component if and only if their corresponding elements lie in the same component of the normal form forest of the balanced parabolic quotient.*

Each generator $\mathbf{x} \in \mathcal{H}$ lies on a face $F_{\mathbf{x}} \subset \Sigma$, and the set of all such faces is partially ordered by inclusion. Based on a wealth of computational evidence the following conjecture is made.

Conjecture. *The Hasse diagram of $\{F_{\mathbf{x}} \,|\, \mathbf{x} \in \mathcal{H}\}$ under inclusion is isomorphic to the normal form forest of the balanced parabolic quotient.*

These conjectures would indicate that the geometric, algebraic, and graph-theoretic structures of HW Σ_σ are related to one another via the balanced parabolic quotients.

Chapter 1

Preliminaries

This chapter contains a summary of the background material needed in the sequel. In section 1.2 we introduce symmetrizable Cartan data. A symmetrizable Cartan datum is the fundamental combinatorial ingredient in the definition of essentially every object appearing in this work (root systems, Weyl groups, Lie algebras, quantized enveloping algebras, and crystals). Section 1.3 begins with Kashiwara's definition of the category of crystals associated with a symmetrizable Cartan datum. After presenting some basic examples that will appear in the sequel, we review highest-weight crystals and Kashiwara's tensor product operation. Section 1.4 covers the Kashiwara-Littelmann family of crystals, which are the focus of Chapter 5. Finally, we review two aspects of these crystals that are essential for our main results: the branching rule for parabolic subdata and the notion of a Demazure crystal. Section 1.5 is devoted to the polyhedral crystal of Kashiwara-Nakashima-Zelevinsky, which is our main computational tool.

1.1 Monoidal categories and monoidal functors

Following MacLane [ML98], a monoidal (tensor) category $\mathcal{C} = (\mathcal{C}, \boxtimes, \mathbf{1}, \alpha, \lambda, \varrho)$ comprises a category $\mathcal{C}$, a bifunctor $\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an identity object $\mathbf{1} \in \mathcal{C}$, and three natural isomorphisms α, λ, ϱ. Explicitly,

$$\alpha = \alpha_{X,Y,Z} : (X \boxtimes Y) \boxtimes Z \cong X \boxtimes (Y \boxtimes Z) \tag{1.1.1}$$

is natural for all objects X, Y, Z in $\mathcal{C}$, and the following pentagonal diagram

$$
\begin{tikzcd}
& (X \boxtimes (Y \boxtimes Z)) \boxtimes W \arrow[dr, "\alpha_{X,Y\boxtimes Z,W}"] & \\
((X \boxtimes Y) \boxtimes Z) \boxtimes W \arrow[ur, "\alpha_{X,Y,Z}\boxtimes 1_W"] \arrow[d, "\alpha_{X\boxtimes Y,Z,W}"'] & & X \boxtimes ((Y \boxtimes Z) \boxtimes W) \arrow[d, "1_X\boxtimes \alpha_{Y,Z,W}"] \\
(X \boxtimes Y) \boxtimes (Z \boxtimes W) \arrow[rr, "\alpha_{X,Y,Z\boxtimes W}"'] & & X \boxtimes (Y \boxtimes (Z \boxtimes W))
\end{tikzcd}
\tag{1.1.2}
$$

commutes for all objects X, Y, Z, W in $\mathcal{C}$. (This axiom guarantees that an n-fold tensor product is well-defined and independent—up to a canonical natural isomorphism—of the order of the tensor factors.) The natural isomorphisms λ, ϱ satisfy

$$\lambda_X : \mathbf{1} \boxtimes X \cong X, \qquad \varrho_X : X \boxtimes \mathbf{1} \cong X, \qquad \lambda_{\mathbf{1}} = \varrho_{\mathbf{1}} : \mathbf{1} \boxtimes \mathbf{1} \cong \mathbf{1} \tag{1.1.3}$$

and the triangular diagram

$$
\begin{tikzcd}
(X \boxtimes \mathbf{1}) \boxtimes Y \arrow[r, "\alpha_{X,e,Y}"] \arrow[d, "\varrho\boxtimes 1_Y"'] & X \boxtimes (\mathbf{1} \boxtimes Y) \arrow[d, "1_X\boxtimes\lambda_Y"] \\
X \boxtimes Y \arrow[r, equal] & X \boxtimes Y
\end{tikzcd}
\tag{1.1.4}
$$

commutes for all objects X, Y in $\mathcal{C}$.

A monoidal (tensor) functor $(F, F_2, F_0) : \mathcal{C} \to \mathcal{C}'$ between monoidal categories $\mathcal{C}$ and $\mathcal{C}'$ consists of the following

1. A functor $F : \mathcal{C} \to \mathcal{C}'$ between categories

2. For objects X, Y in $\mathcal{C}$ a morphism

$$F_2(X, Y) : F(X) \boxtimes F(Y) \to F(X \boxtimes Y) \tag{1.1.5}$$

in $\mathcal{C}'$ which is natural in X and Y.

3. For the units $\mathbf{1}, \mathbf{1'}$, a morphism in $\mathcal{C}'$

$$F_0 : \mathbf{1'} \to F(\mathbf{1}). \tag{1.1.6}$$

Together, these must make the following three diagrams in $\mathcal{C}'$ commute

$$\begin{tikzcd}
(F(X) \boxtimes F(Y)) \boxtimes F(Z) \arrow[rr, "\alpha'_{F(X),F(Y),F(Z)}"] \arrow[d, "F_2(X,Y)\boxtimes 1_Z"'] & & F(X) \boxtimes (F(Y) \boxtimes F(Z)) \arrow[d, "1_X \boxtimes F_2(Y,Z)"] \\
F(X \boxtimes Y) \boxtimes F(Z) \arrow[d, "F_2(X\boxtimes Y,Z)"'] & & F(X) \boxtimes F(Y \boxtimes Z) \arrow[d, "F_2(X,Y\boxtimes Z)"] \\
F((X \boxtimes Y) \boxtimes Z) \arrow[rr, "F(\alpha_{X,Y,Z})"'] & & F(X \boxtimes (Y \boxtimes Z))
\end{tikzcd} \tag{1.1.7}$$

$$\begin{tikzcd}
F(X) \boxtimes \mathbf{1}' \arrow[r, "\varrho'_{F(X)}"] \arrow[d, "1_{F(X)}\boxtimes F_0"'] & F(X) \\
F(X) \boxtimes F(\mathbf{1}) \arrow[r, "F_2(X,\mathbf{1})"'] & F(X \boxtimes \mathbf{1}) \arrow[u, "F(\varrho)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1}' \boxtimes F(X) \arrow[r, "\lambda'_{F(X)}"] \arrow[d, "F_0\boxtimes 1_{F(X)}"'] & F(X) \\
F(\mathbf{1}) \boxtimes F(X) \arrow[r, "F_2(\mathbf{1},X)"'] & F(\mathbf{1} \boxtimes X) \arrow[u, "F(\lambda)"']
\end{tikzcd} \tag{1.1.8}$$

1.2 Cartan data

Let I be a finite set, and $A = (a_{ij})_{i,j\in I}$ be an integral matrix such that $a_{ii} = 2$, $a_{ij} \leq 0$ and $a_{ij} = 0$ if and only if $a_{ji} = 0$. Select a vector $\mathbf{d}$ of r positive integers such that $(d_i a_{ij})_{i,j\in I}$ is symmetric. Such a matrix is referred to as a symmetrizable generalized Cartan matrix.

Let $\Lambda^\vee$ be a free abelian group of rank $2|I| - \mathrm{rank}(A)$ with $\mathbb{Z}$-basis $\{\alpha_i^\vee \,|\, i \in I\} \cup \{t_k \,|\, 1 \leq k \leq |I| - \mathrm{rank}(A)\}$, and let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \Lambda^\vee$. We call $\Lambda^\vee$ the dual weight lattice and $\mathfrak{h}$ the Cartan subalgebra. Define the weight lattice to be $\Lambda = \{\lambda \in \mathfrak{h}^* \,|\, \lambda(\Lambda^\vee) \subset \mathbb{Z}\}$. Let $\langle\cdot,\cdot\rangle : \Lambda^\vee \otimes_{\mathbb{Z}} \Lambda \to \mathbb{Z}$ denote the evaluation pairing; note that this extends to a bilinear pairing $\mathfrak{h}^* \otimes_{\mathbb{C}} \mathfrak{h} \to \mathbb{C}$.

Set $\Pi^\vee = \{\alpha_i^\vee \,|\, i \in I\}$ and choose a linearly independent subset $\Pi = \{\alpha_i \,|\, i \in I\} \subset \mathfrak{h}^*$, called the set of *simple roots*, satisfying

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}, \qquad \langle t_k, \alpha_j \rangle \in \{0, 1\}.$$

When working with the symmetrized version of A, declare $\langle \alpha_i^\vee, \alpha_j \rangle = d_i a_{ij}$ instead. The span $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset \Lambda$ of Π is called the *root lattice*. Finally, we define the fundamental weights $\varpi_i \in \mathfrak{h}^*$ $(i \in I)$ to be the linear functionals on $\mathfrak{h}$ given by

$$\langle \alpha_i^\vee, \varpi_j \rangle = \delta_{ij}, \qquad \langle t_i, \varpi_j \rangle = 0.$$

The sextuple $\mathscr{C} = (A, \mathbf{d}, \Lambda^\vee, \Lambda, \Pi^\vee, \Pi)$ is called a symmetrizable Cartan datum. This is the combinatorial datum on which the crystal theory is based, it is an abstraction of the datum of a root system from the Lie theory. Our level of generality corresponds to the symmetrizable Kac-Moody Lie algebras (see below).

1.2.1 Weyl groups

The Weyl group is a Coxeter group, generated by the set $\{s_i \,|\, i \in I\}$ of simple reflections modulo the relations

$$(s_i s_j)^{m_{ij}} = 1, \qquad \text{where } m_{ij} = \begin{cases} 1 & i = j \\ 2 & a_{ij} = 0 \\ 3 & a_{ij}a_{ji} = 1 \\ 4 & a_{ij}a_{ji} = 2 \\ 6 & a_{ij}a_{ji} = 3 \end{cases} \tag{1.2.1}$$

and acts on Λ via

$$s_i\lambda := \lambda - \langle \alpha_i^\vee, \lambda\rangle \alpha_i. \tag{1.2.2}$$

An expression $w = s_{i_1} \cdots s_{i_t}$ is *reduced* when t takes its minimal value amongst all possible expressions for w as products of simple reflections, and the tuple $(i_1, \cdots, i_t) \in I^t$ is called a *reduced word* for 2. The *length* of w is $\ell(w) := t$, and the set of all reduced words for w is denoted $\mathcal{R}(w)$.

Recall that the Bruhat order is the transitive closure of the relation $u < us_i$, where $\ell(us_i) = \ell(u) + 1$.

A subset $J \subsetneq I$ determines a subdatum $\mathscr{C}_J \subset \mathscr{C}$ with simple roots $\Pi_J := \{\alpha_j \,|\, j \in J\}$, weight lattice $\Lambda_J := \sum_{j\in J} \mathbb{Z}\varpi_j$, root system $\Phi_J = \Phi \cap \mathbb{Z}\Pi_J$ and Weyl group $W_J := \langle s_j \,|\, j \in J\rangle$. The latter is called the Jth *parabolic subgroup* of W. As before, we have a decomposition $\Phi_J = \Phi_J^+ \sqcup \Phi_J^-$, where $\Phi_J^\pm := \Phi^\pm \cap \Phi_J$. It is well-known (see, *e.g.*, [Jos95, A.1.19]) that

$$W^J := \{w \in W \,|\, w\Pi_J \subset \Phi^+\} \tag{1.2.3}$$

coincides with the set of minimal-length left coset representatives for W/W_J. The image ${}^JW$ of W^J under the antiautomorphism of W defined by $w \mapsto w^{-1}$ is the set of minimal-length right coset representatives for $W_J\backslash W$. The sets ${}^JW, W^J$ are known as *parabolic quotients* of W modulo J.

1.2.2 Kac-Moody Lie algebras

Let $\mathscr{C}$ be a Cartan datum. The Kac-Moody Lie algebra $\mathfrak{g}(\mathscr{C})$ is the Lie algebra generated by the elements x_i, y_i, h_i $(i \in I)$ modulo the defining relations

$$\begin{aligned}
[h_i, h_j] &= 0 \\
[x_i, y_i] &= h_i, \quad [x_i, y_j] = 0 \text{ if } i \neq j \\
[h_i, x_j] &= a_{ij} x_j, \quad [h_i, y_j] = -a_{ij} y_j \\
(\text{ad } x_i)^{1-a_{ij}} x_j &= 0 \\
(\text{ad } y_i)^{1-a_{ij}} y_j &= 0
\end{aligned}$$

1.2.3 Root systems

Let $\mathfrak{g} = \mathfrak{g}(\mathscr{C})$ be the Kac-Moody Lie algebra associated with $\mathscr{C}$. For each $\alpha \in Q$ let

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h_i, x] = \alpha(h_i)x \text{ for all } i\}.$$

Then we have the root space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha, \text{ with } \dim \mathfrak{g}_\alpha < \infty \text{ for all } \alpha \in Q.$$

If $\alpha \neq 0, \mathfrak{g}_\alpha \neq 0$, then α is called a root of $\mathfrak{g}$. The dimension of $\mathfrak{g}_\alpha$ is called the root multiplicity of α; it is well-known that when $\mathscr{C}$ is of finite type—in other words, when A has full rank—all root multiplicities equal 1.

The set of all roots of $\mathfrak{g}$ is denoted Φ, and called the root system. The choice of simple roots $\Pi = \{\alpha_i \mid i \in I\}$ determines a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$ of the root system into subsets of positive and negative roots. To express $\beta \in \Phi$ in terms of the simple roots, we write $\beta = \sum_{i \in I} \beta(i)\alpha_i$.

The root system of a simply-laced Cartan datum $\mathscr{C}$ is of finite type admits a simple combinatorial description: $\Pi \subset \Phi$, and given $\alpha, \beta \in \Phi$ we have $\alpha + \beta \in \Phi$ if and only if $\langle \alpha^\vee, \beta \rangle = -1$, where $\left(\sum_{i \in I} n_i \alpha_i\right)^\vee = \sum_{i \in I} n_i \alpha_i^\vee$. In fact, in the sequel we will only need to use root systems of these types.

1.3 Abstract crystals

Let $\mathscr{C}$ be a symmetrizable Cartan datum. This section recalls the relevant definitions, examples, and results about $\mathscr{C}$-crystals needed in the sequel. Unless specified otherwise, all definitions, examples, and results in this section are due to Kashiwara.

Definition 1.3.1. A *$\mathscr{C}$-crystal* is a set B equipped with structural maps $\mathrm{wt} : B \to \Lambda$ and $\varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}$ and operators $e_i, f_i : B \sqcup \{0\} \to B \sqcup \{0\}$ for $i \in I$. Adjoined to B is a special ghost element 0, for which we declare $e_i 0 = f_i 0 = 0$. The structural maps and operators obey the following axioms: for all $b, b' \in B$

1. $b' = f_i b$ if and only if $e_i b' = b$
2. If $e_i b \neq 0$ then
   (a) $\mathrm{wt}(e_i b) = \mathrm{wt}(b) + \alpha_i$
   (b) $\varepsilon_i(e_i b) = \varepsilon_i(b) - 1$
3. If $f_i b \neq 0$ then
   (a) $\mathrm{wt}(f_i b) = \mathrm{wt}(b) - \alpha_i$
   (b) $\varepsilon_i(f_i b) = \varepsilon_i(f) + 1$
4. $\varepsilon_i(b) = -\infty$ implies $e_i b = f_i b = 0$

A $\mathscr{C}$-crystal is graded by Λ, and we write

$$B = \bigsqcup_{\mu \in \Lambda} B_\mu, \qquad B_\mu := \mathrm{wt}^{-1}(\mu). \tag{1.3.1}$$

The subset B_μ is called the μ-weight-subset of B.

Denote by $\mathcal{F}$, respectively $\mathcal{E}$, the monoids generated by all f_i, respectively e_i with $i \in I$. The monoid generated by all operators f_i, e_i is denoted $\mathcal{A}$. Given a sequence $\underline{i} = (i_1, \ldots, i_n) \in I^n$, we denote $x(\underline{i}) := x_{i_1} \cdots x_{i_n}$ for $x \in \{e, f\}$.

Let $\mathcal{G}B$ be the colored digraph having as vertices the elements of B. An arrow $b \xrightarrow{i} b'$ of color i exists if and only if $f_i b = b'$ (or, equivalently, $e_i b' = b$). This is the *crystal graph* of B. From this perspective a $\mathscr{C}$-crystal morphism is (essentially) a morphism of directed graphs $\mathcal{G}B \to \mathcal{G}B'$ that commutes with $\mathrm{wt}, \varepsilon_i$.

Definition 1.3.2. A *$\mathscr{C}$-crystal morphism* is a function $\psi : B \sqcup \{0\} \to B' \sqcup \{0\}$ such that $\psi(0) = 0$ and, for all $i \in I$ and all $b \in B$ such that $\psi(b) \neq 0$

1. (a) $\mathrm{wt} \circ \psi(b) = \mathrm{wt}(b)$

   (b) $\varepsilon_i \circ \psi(b) = \varepsilon_i(b)$

2. For $x \in \{e, f\}$, if $x_i b \neq 0$ and $\psi(x_i b) ne 0$ then $\psi(x_i b) = x_i \psi(b)$.

A morphism is *strict* if it commutes with the action of $\mathcal{A}$. An *embedding* is a morphism with injective underlying function.

The identity 1_B is a $\mathscr{C}$-crystal morphism, the composite of two $\mathscr{C}$-crystal morphisms is again a $\mathscr{C}$-crystal morphism, and composition of $\mathscr{C}$-crystal morphisms is associative. Thus we have a category.

Definition 1.3.3. The category of $\mathscr{C}$-crystals with $\mathscr{C}$-crystal morphisms is denoted $\mathrm{Crys}(\mathscr{C})$.

Example 1.3.4. Fix $i \in I$. The *ith elementary crystal* C_i has underlying set

$$C_i := \{c_i(n) \mid n \in \mathbb{Z}\} \tag{1.3.2}$$

and $\mathscr{C}$-crystal structure

$$\mathrm{wt}(c_i(n)) := n\alpha_i \tag{1.3.3}$$

$$\varepsilon_j(c_i(n)) := \begin{cases} -n, & j = i \\ -\infty, & j \neq i \end{cases} \tag{1.3.4}$$

$$e_j c_i(n) := \begin{cases} c_i(n+1), & j = i \\ 0, & j \neq i \end{cases} \tag{1.3.5}$$

$$f_j c_i(n) := \begin{cases} c_i(n-1), & j = i \\ 0, & j \neq i \end{cases} \tag{1.3.6}$$

Example 1.3.5. T_λ is the singleton crystal $T_\lambda = \{t_\lambda\}$ with $\mathrm{wt}(t_\lambda) = \lambda$ and $\varepsilon_i(t_\lambda) = -\infty$ for all i.

Example 1.3.6 ([Jos95]). S_λ is the singleton crystal $S_\lambda = \{s_\lambda\}$ with $\mathrm{wt}(s_\lambda) = \lambda$ and $\varepsilon_i(s_\lambda) = -\langle \lambda, \alpha_i^\vee \rangle$ and $e_i s_\lambda = f_i s_\lambda = 0$.

Definition 1.3.7. An element $b \in B$ of a $\mathscr{C}$-crystal is called a *highest-weight element* in case that $e_i(b) = 0$ for all i. The set of highest-weight elements of B is denoted HW B. Following [Jos95], we refer to a $\mathscr{C}$-crystal B as a *highest-weight crystal of highest weight* $\lambda \in \Lambda$ when

1. There exists a highest-weight element $b \in B$ such that $\mathrm{wt}(b) = \lambda$.

2. $B = \mathcal{F}b$, which is to say that B is generated as a crystal by b over the monoid $\mathcal{F}$.

From ε_i and wt an auxiliary structural map is defined by

$$\varphi_i(b) := \varepsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b) \rangle. \tag{1.3.7}$$

Definition 1.3.8. A $\mathscr{C}$-crystal B is *upper normal* if

$$\varepsilon_i(b) = \max\{k \geq 0 \,|\, e_i^k b \neq 0\} \text{ for all } b \in B. \tag{1.3.8}$$

On the other hand, B is *lower normal* when

$$\varphi_i(b) = \max\{k \geq 0 \,|\, f_i^k b \neq 0\} \text{ for all } b \in B. \tag{1.3.9}$$

A $\mathscr{C}$-crystal satisfying (1.3.8) and (1.3.9) is said to be *normal*.

Definition 1.3.9. The direct sum $B \oplus B'$ of two crystals is the crystal with underlying set $B \sqcup B'$ and the natural structure: if $b \in B \sqcup B'$ is an element of B, respectively B', then we use the crystal structure of B, respectively B' to determine $\varepsilon_i(b), \mathrm{wt}(b), f_i b, e_i b$.

Observe that (1) $\mathcal{G}(B \oplus B') = \mathcal{G}B \oplus \mathcal{G}B'$ (direct sum of graphs), and (2) for any morphism $\psi : B \to B'$ we have $B = \psi^{-1}B \oplus \psi^{-1}0$.

In the direct sum construction B and B' identify with subcrystals of $B \oplus B'$, and inclusion $B, B' \hookrightarrow B \oplus B'$ is a strict embedding. Conversely, if there exists a strict embedding $\psi : B \to B'$ then ψB is a subcrystal of B' that is isomorphic to B and $B' = \psi[B] \oplus (B' \smallsetminus \psi[B])$. In the language of crystal graphs that conclusion means $\mathcal{G}\psi[B]$ is a component of $\mathcal{G}B'$.

1.3.1 Tensor products

Let B, B' be $\mathscr{C}$-crystals. The underlying set of $B \otimes B'$ is the Cartesian product $B \times B'$ with $b \otimes b' := (b, b')$ and $0 \otimes b' := 0 =: b \otimes 0$. The structural maps are defined by

$$\mathrm{wt}(b \otimes b') := \mathrm{wt}(b) + \mathrm{wt}(b') \tag{1.3.10}$$

$$\varepsilon_i(b \otimes b') := \max\{\varepsilon_i(b), \varepsilon_i(b') - \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle\} \tag{1.3.11}$$

and the operators by

$$f_i(b \otimes b') := \begin{cases} f_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') \\ b \otimes f_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b') \end{cases} \tag{1.3.12}$$

$$e_i(b \otimes b') := \begin{cases} e_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b') \\ b \otimes e_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'). \end{cases} \tag{1.3.13}$$

Kashiwara considered the monoidal category structure of $\mathrm{Crys}(\mathscr{C})$ in [Kas93, Kas95]. It was shown in the latter paper that for any $\mathscr{C}$-crystals B, B', B'' there is an isomorphism

$$\alpha_{B,B',B''} : (B \otimes B') \otimes B'' \cong B \otimes (B' \otimes B'') \tag{1.3.14}$$

and isomorphisms

$$\lambda_B : T_0 \otimes B \cong B, \qquad \varrho_B : B \otimes T_0 \cong B \tag{1.3.15}$$

(these results are the contents of [Kas95, Lemma 7.1] and [Kas95, Example 7.3], respectively).

Lemma 1.3.10 ([Kas95, section 7.3])**.** *($Crys(\mathscr{C}), \otimes, T_0, \alpha, \lambda, \varrho$) is a monoidal category.*

1.4 The Kashiwara-Littelmann crystals

Following [Jos95], we say that a collection $\{C(\lambda) \mid \lambda \in \Lambda^+\}$ of normal highest-weight crystals is *closed under tensor products* when the subcrystal of $C(\lambda) \otimes C(\mu)$ generated by $c_\lambda \otimes c_\mu$ over $\mathcal{A}$ is isomorphic to $C(\lambda + \mu)$. According to [Jos95, Proposition 6.4.21] there is exactly one such collection up to isomorphism, the *Kashiwara-Littelmann* (KL) family $\{B(\lambda) \mid \lambda \in \Lambda^+\}$ introduced in [Kas91] and independently constructed in [Lit94].

There exists an upper normal crystal $B(\infty)$ which is a limit of the $B(\lambda), \lambda \in \Lambda^+$ in a certain precise sense. The original construction [Kas91] of $B(\infty)$ uses the $q \to 0$ theory of quantized enveloping algebras, whereas a purely combinatorial construction is given in [Jos02]. As a crystal $B(\infty)$ is highest-weight of highest weight 0 and the unique element of weight 0 is denoted b_∞.

For each $\lambda \in \Lambda^+$ there exists an injection $\bar{\iota}_\lambda : B(\lambda) \hookrightarrow B(\infty)$ carrying $b_\lambda \longmapsto b_\infty$ which commutes with the $\mathcal{E}$-action but not the $\mathcal{F}$-action. Tensoring $B(\infty)$ with S_λ removes this obstacle:

Lemma 1.4.1 ([Jos95, Lemma 5.3.13])**.** *The map $\iota_\lambda : B(\lambda) \to B(\infty) \otimes S_\lambda$ whereby $b \longmapsto \bar{\iota}_\lambda(b) \otimes s_\lambda$ is a strict embedding of $\mathscr{C}$-crystals.*

1.4.1 Kashiwara's involution and Demazure crystals

Let $\Lambda^{++} \subset \Lambda^+$ be the semigroup of regular weights, that is $\Lambda^{++} = \sum_{i\in I} \mathbb{Z}_{>0}\varpi_i$, and suppose $\lambda \in \Lambda^{++} \cup \{\infty\}$. For each $w \in W$, the Kashiwara-Littelmann crystal $B(\lambda)$ contains a unique subset $B_w(\lambda)$, stable under all $\tilde{e}_i$, called a Demazure crystal [Kas93]. Several important properties of the Demazure crystals are given in [Kas93, Propositions 3.2.4, 3.2.5]. The relevant ones for our purposes are the following.

1. If $v \leq w$ in the Bruhat order, then $B_v(\lambda) \subset B_w(\lambda)$.

2. Given $\mathbf{i} = (i_\ell, \dots, i_1) \in I^n$ denote $\tilde{f}_{\mathbf{i}} := \tilde{f}_{i_\ell} \cdots \tilde{f}_{i_1}$; for $\mathbf{k} = (k_\ell, \dots, k_1) \in \mathbb{N}^\ell$ denote $\tilde{f}_{\mathbf{i}}^{\mathbf{k}} := \tilde{f}_{i_\ell}^{k_\ell} \cdots \tilde{f}_{i_1}^{k_1}$. Then
$$B_w(\infty) = \bigcup_{\mathbf{k}\in\mathbb{N}^n} \tilde{f}_{\mathbf{i}}^{\mathbf{k}} b_\infty \tag{1.4.1}$$
for a reduced expression $s_{\mathbf{i}} = s_{i_\ell} \cdots s_{i_1}$ of w

Let $\star : B(\infty) \to B(\infty)$ denote the Kashiwara involution of $B(\infty)$ and define operators $x_i^\star : B(\infty) \to B(\infty)$ for $x \in \{e, f\}$ by $x_i^\star := \star x_i \star$ [Kas93].

Theorem ([Kas93, Theorem 2.2.1])**.** *For any $i \in I$ there exists a unique strict $\mathscr{C}$-crystal embedding $\Psi_i : B(\infty) \hookrightarrow B_i$ sending b_∞ to $b_\infty \otimes b_i(0)$ with the following properties:*

1. *If $\Psi_i(b) = b_0 \otimes b_i(-m)$ then $\Psi_i(f_i^* b) = b_0 \otimes b_i(-m-1)$ and $\varepsilon_i(b^*) = m$.*

2. *im $\Psi_i = \{b \otimes b_i(-m) \,|\, \varepsilon_i(b^*) = 0, m \geq 0\}$.*

Let $\mathbf{i}^{-1} := (i_1, \dots, i_\ell)$ and $\mathbf{k}^{-1} = (k_1, \dots, k_\ell)$ denote the reversals of $\mathbf{i}$ and $\mathbf{k}$. It is shown in [RH02] that
$$B_w(\infty) = \bigcup_{\mathbf{k}\in\mathbb{N}^n} \tilde{f}_{\mathbf{i}^{-1}}^{\star\mathbf{k}^{-1}} b_\infty. \tag{1.4.2}$$

Now we have the following easy consequence of (1.4.1) and (1.4.2).

Lemma 1.4.2. *Take $u \in W$ and $i \in I$ such that $us_i < u$. Then $B_u(\infty) = \bigcup_{k\geq 0} \tilde{f}_i^{*k} B_{us_i}(\infty)$*

1.4.2 Branching rules

Given a subset $J \subset I$ we can regard the $\mathscr{C}$-crystal $B(\lambda)$ as a $\mathscr{C}_J$-crystal. From this perspective $\mathrm{HW}_J B(\lambda)$ comprises the elements of $B(\lambda)$ that are highest-weight with respect to its $\mathscr{C}_J$-crystal structure. Let $\pi_J : \Lambda \twoheadrightarrow \Lambda_J$ be the canonical projection homomorphism, whereby $\varpi_i \mapsto \varpi_i$ if $i \in J$ and $\varpi_i \mapsto 0$ otherwise. The Jth branching rule (*cf.* [Kas95, 4.6]) states that $B(\lambda)$ is isomorphic, as a $\mathscr{C}_J$-crystal, to a direct sum of the $\mathscr{C}_J$-crystals $B_J(\mu)$, for $\mu \in \Lambda_J^+ \cup \{\infty\}$, as follows:

$$B(\lambda) \cong \bigoplus_{b \in \mathrm{HW}_J B(\lambda)} B_J(\pi_J(\mathrm{wt}\ b)) \text{ for } \lambda \in \Lambda^+, \qquad B(\infty) \cong B_J(\infty)^{\bigoplus \mathrm{HW}_J B(\infty)}.$$

Note that for $\lambda \in \Lambda^+$, this is a combinatorial analogue of the decomposition of a simple finite-dimensional $\mathfrak{g}(\mathscr{C})$-module $V(\lambda)$ of highest weight λ into a direct sum of simple finite dimensional $\mathfrak{g}(\mathscr{C}_J)$-modules.

1.4.3 Littelmann path crystals

When working with the Littelmann path crystals $\mathscr{P}_\lambda \cong B(\lambda)$, $\lambda \in \Lambda^+$ we use the notation of [Lit94] and assume $\sigma(\lambda) = \lambda$. Recall that $\pi \in \mathscr{P}_\lambda$ has the form $\pi = (\underline{\tau}, \underline{a})$ where $\underline{\tau} = \tau_1 > \cdots > \tau_r$ is a sequence of linearly (Bruhat) ordered elements of W and $\underline{a} = a_0 := 0 < a_1 \cdots a_r := 1$ is a sequence of rational numbers satisfying certain conditions [Lit94, Sections 2.1, 2.2]. We regard π as the concatenation of the straight line paths $\pi_k(t) := \tau_k \lambda t$ living in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, modulo reparameterization.

Recall the surjection $\phi : \mathscr{P}_\lambda \to W/\mathrm{Stab}_W \lambda$ [Lit94, 5.2] whereby a path $\pi = (\underline{\tau}, \underline{a})$ is mapped to its "first direction" τ_1; we may assume $\mathrm{im}\ \phi \subset W$ by taking $\lambda \in \Lambda^{++}$. In the same paper path analogues $\mathscr{P}_{\lambda,w} := \{\pi \in \mathscr{P}_\lambda \,|\, \phi(\pi) \leq w\}$ of the Demazure crystal are defined. For each $w \in W$ there is a bijection $\mathscr{P}_{\lambda,w} \cong B_w(\lambda)$ commuting with the $\tilde{e}_i$-action, where $\mathscr{P}_{\lambda,w} := \{\pi \in \mathscr{P}_\lambda \,|\, \phi(\pi) \leq w\}$. Under this bijection $\{\pi \in \mathscr{P}_\lambda \,|\, \phi(\pi) = w\} \cong \bar{B}_w(\lambda)$.

1.5 The polyhedral crystal

The polyhedral crystal $\mathbb{Z}_\iota^\infty$—introduced in [Kas93] and refined in [NZ97]—plays an important role in this dissertation. The crystal $B(\infty)$ can be embeddded into $\mathbb{Z}_\iota^\infty$, and thus so can $B(\lambda)$ according to Lemma 1.4.1. We use $\mathbb{Z}_\iota^\infty$ to facilitate calculations involving $B(\infty)$, especially in section 5.2. Furthermore, the polyhedral crystal has an algebraic and a geometric structure that we use to describe certain subsets of $B(\infty)$ in section 5.3.

We begin by reviewing the polyhedral realization as presented in [NZ97], with some minor notational variation. Then we explain the connection between $B(\infty)$ and $\mathbb{Z}_\iota^\infty$ via the Kashiwara embedding theorem [Kas93] and present the geometric description of [NZ97]. Lastly, we record some lemmata to be used in the sequel.

1.5.1 The crystal $\mathbb{Z}_\iota^\sigma$

Let $\iota = (\cdots, \iota_k, \cdots, \iota_2, \iota_1)$ be an infinite sequence of indices from I such that

$$\iota_k \neq \iota_k + 1 \text{ for all } k, \text{ and } \#\{k \mid \iota_k = i\} = \infty \text{ for all } i. \tag{1.5.1}$$

The underlying set of the polyhedral realization is the abelian group

$$\mathbb{Z}^\infty = \{x = (\ldots, x_k, \ldots, x_2, x_1) : x_i \in \mathbb{Z}, x_k = 0 \text{ for } k \gg 0\},$$

on which we place a $\mathscr{C}$-crystal structure according to ι. The weight of $x \in \mathbb{Z}^\infty$ is

$$\mathrm{wt}(x) := -\sum_{k \geq 1} x_k \alpha_{\iota_k}. \tag{1.5.2}$$

For each positive integer k define the Kashiwara functions $\gamma_k : \mathbb{Z}^\infty \to \mathbb{Z}$ by

$$\gamma_k(x) := x_k + \sum_{l > k} a_{\iota_l \iota_k} x_l. \tag{1.5.3}$$

For each $i \in I$ define

$$\varepsilon_i(x) := \max\{\gamma_k(x) \mid k \geq 1 \text{ and } \iota_k = i\}, \tag{1.5.4}$$

$$M_i(x) := \{k \geq 1 \mid \iota_k = i \text{ and } \gamma_k(x) = \varepsilon_i(x)\}, \tag{1.5.5}$$

observe $|M_i(x)| < \infty$ if and only if $\varepsilon_i(x) > 0$. The operators e_i, f_i are given by $e_i 0 = 0 = f_i$, $e_i x = 0$ if $\varepsilon_i(x) = 0$, and otherwise

$$e_i x = x - \mathbf{e}(\max M_i(x)), \tag{1.5.6}$$

$$f_i x = x + \mathbf{e}(\min M_i(x)). \tag{1.5.7}$$

where $\mathbf{e}(k)$ is the kth standard basis element of $\mathbb{Z}^\infty$. Finally, define $\varphi_i(x) := \varepsilon_i(x) + \langle \alpha_i^\vee, \mathrm{wt}(x)\rangle$. The crystal thus obtained is denoted $\mathbb{Z}_\iota^\infty$.

Remark 1.5.1. The bijection $\mathbb{Z}^\infty_\iota \xrightarrow{\sim} \cdots \otimes B_{\iota_k} \otimes \cdots \otimes B_{\iota_2} \otimes B_{\iota_1}$ whereby

$$(\cdots, x_k, \cdots, x_2, x_1) \longmapsto \cdots \otimes b_{\iota_k}(-x_k) \cdots \otimes b_{\iota_1}(-x_2) \otimes b_{\iota_1}(-x_1)$$

can be shown to be a $\mathscr{C}$-crystal isomorphism, as alluded to in [Jos95, 5.2], for example (the reason for the sign change will become clear later). In fact, the crystal structure (1.5.2), (1.5.4), (1.5.6), (1.5.7) is nothing but a reformulation of the crystal structure of $\cdots \otimes B_{\iota_k} \otimes \cdots \otimes B_{\iota_2} \otimes B_{\iota_1}$.

1.5.2 Relationship with $B(\infty)$

The Kashiwara embedding theorem [Kas93, Theorem 2.2.1] gives a procedure to describe $B(\infty)$: Given $i_1, \cdots, i_N \in I$ such that no $i_k = i_{k+1}$ define $\Psi_{i_N,\cdots,i_1} : B(\infty) \hookrightarrow B(\infty) \otimes B_{i_N} \cdots B_{i_1}$ by $\Psi_{i_N,\cdots,i_1} := \Psi_{i_N} \cdots \Psi_{i_1}$. For each $b \in B(\infty)$ one can choose $i_1, \cdots, i_N$ so that $\Psi_{i_N,\cdots,i_1}(b) \in b_\infty \otimes B_{i_N} \cdots \otimes B_{i_1}$. In this manner $B(\infty)$ is isomorphic to a subcrystal of the limit $\cdots \otimes B_{i_k} \cdots \otimes \otimes B_{i_2} B_{i_1}$, provided that (1.5.1) is satisfied.

Theorem ([Kas93], see also [NZ97] Theorem 2.5)**.** *For ι satisfying* (1.5.1) *there is a unique strict $\mathscr{C}$-crystal embedding*

$$\Psi_\iota : B(\infty) \hookrightarrow \cdots \otimes B_{\iota_k} \cdots \otimes B_{\iota_1} \tag{1.5.8}$$

such that $\Psi_\iota(b_\infty) = (\cdots, 0, \cdots, 0, 0)$.

According to remark 1.5.1 we have the following.

Theorem 1.5.2 ([NZ97] Theorem 2.5)**.** *For ι satisfying* (1.5.1) *there exists a unique strict embedding of crystals $B(\infty) \hookrightarrow \mathbb{N}^\infty_\iota \subset \mathbb{Z}^\infty_\iota$ whereby $b_\infty \mapsto (\cdots, 0, \cdots, 0, 0)$.*

Definition 1.5.3. The *Nakashima-Zelevinsky polyhedral realization* of $B(\infty)$ is $\Sigma_\iota := \mathrm{im}\ \Psi_\iota$.

Remark 1.5.4. It is well-known (see, for example, [Nak99, Section 4.2]) that when $W(\mathscr{C})$ is finite the crystal appearing on RHS(1.5.8) can be taken to have $\ell(w_\circ)$ tensor factors. Thus $B(\infty)$ for finite type embeds into the finite rank lattice $\mathbb{Z}^{\ell(w_\circ)}$, which has given a crystal structure based on the finite sequence $(\iota_{\ell(w_\circ)}, \cdots, \iota_1)$.

Remark 1.5.5. Given $w \in W$, the image $\Sigma^\iota_w := \Psi_\iota[B_w(\infty)]$ of the w-th Demazure crystal is described by the inequalities of [NZ97, Theorem 3.1] with the additional requirement that $x_k = 0$ for $k > \ell$ [Nak02, (2.21)].

1.5.3 Geometric description of Σ_ι

Regard $\mathbb{Z}_\iota^\infty$ as a lattice in the vector space $\mathbb{Q}^\infty$. Beginning with the coordinate forms $x_k^* \in (\mathbb{Q}^\infty)^*$ a recursive procedure is described in [NZ97, 3.1, 3.2] for generating a set of forms $\Xi_\iota \subset (\mathbb{Q}^\infty)^*$ that presents Σ_ι as the dual of a rational polyhedral convex cone. It turns out that the forms of Ξ_ι have integer coordinates with respect to the standard basis of coordinate forms. The following result depends on the technical assumption

$$\text{if } k^{(-)} = 0 \text{ then } \varphi_k \geq 0 \text{ for any } \varphi = \sum_{k \geq 1} \varphi_k x_k^* \in \Xi_\iota \tag{1.5.9}$$

where $k^{(-)} := \min\{0 \leq j < k \,|\, \iota_j = \iota_k\}$, which is called the *positivity assumption* by Nakashima and Zelevinsky. This condition is verified to be true when $\mathfrak{g}(\mathscr{C})$ is of finite or affine type.

Theorem ([NZ97] Theorem 3.1)**.** *If ι satisfies* (1.5.1) *and* (1.5.9)*. Then*

$$\Sigma_\iota = \{x \in \mathbb{Z}^\infty \,|\, \varphi(x) \geq 0 \textit{ for all } \varphi \in \Xi_\iota\}. \tag{1.5.10}$$

That is, Σ_ι is the set of lattice points of the dual of the rational polyhedral convex cone generated by Ξ_ι.

Chapter 2

Folding

In this short chapter we describe the folding of a simply-laced symmetrizable Cartan datum $\mathscr{C}$ by a special type of automorphism. Passing to the set of fixed points under this automorphism yields a different, but related, Cartan datum called the *Langlands dual folded datum* which plays a central role in the rest of this dissertation.

2.1 Admissible automorphisms

We call a permutation $\sigma : I \to I$ an automorphism of $\mathscr{C}$ if

$$\sigma[A] = A. \tag{2.1.1}$$

A σ-orbit in I is variously denoted $\mathbf{i}$, or $\langle\sigma\rangle i$ for some $i \in I$. Following [Lus93, 12.1.1] we refer to σ as *admissible* when

$$a_{ij} = 0 \text{ if } j \in \langle\sigma\rangle i. \tag{2.1.2}$$

In the sequel, I/σ denotes the set of σ-orbits in I and I^σ denotes the set of $i \in I$ fixed by σ.

An automorphism of $\mathscr{C}$ permutes any set indexed by I; when such indexing reflects the structure of the Cartan matrix, the σ-action is compatible with said structure by virtue of (2.1.1). For example, σ permutes the sets $\Pi^\vee$ of simple coroots and $\{\varpi_i \,|\, i \in I\}$ of fundamental weights, and induces an action

$$\sigma \in \mathrm{End}_{\mathrm{Ab}}(\Lambda) \tag{2.1.3}$$

satisfying

$$\langle \alpha_i^\vee, \lambda \rangle = \langle \alpha_{\sigma(i)}^\vee, \sigma(\mu) \rangle \tag{2.1.4}$$

Admissible automorphisms of $\mathscr{C}$ are optimal for folding: condition (2.1.2) means that vertices in the same σ-orbit are unconnected in the Dynkin diagram. On the level of the root system, an admissible automorphism of $\mathscr{C}$ only folds together simple roots that are orthogonal.

2.2 Folding and the Langlands dual operation

Associated to $\Pi, \Lambda, \Pi^\vee$ are subobjects $\widehat{\Pi}^\sigma \subset \widehat{Q}^\sigma \subset \widehat{\Lambda}^\sigma \subset \Lambda$ and $\widehat{\Pi}^{\vee\sigma} \subset \widehat{\Lambda}^{\vee\sigma} \subset \Lambda^\vee$, indexed by I/σ and obtained as follows: Given $\mathbf{i} \in I/\sigma$, put

$$\widehat{\alpha}_{\mathbf{i}} := \sum_{i\in\mathbf{i}} \alpha_i \qquad \widehat{\Pi}^\sigma := \{\widehat{\alpha}_{\mathbf{i}} \,|\, \mathbf{i} \in I/\sigma\} \qquad \widehat{Q}^\sigma := \bigoplus_{\mathbf{i}\in I/\sigma} \mathbb{Z}\alpha_{\mathbf{i}} \tag{2.2.1}$$

$$\widehat{\alpha}_{\mathbf{i}}^\vee := \sum_{i\in\mathbf{i}} \alpha_i^\vee \qquad \widehat{\Pi}^{\vee\sigma} := \{\widehat{\alpha}_{\mathbf{i}}^\vee \,|\, \mathbf{i} \in I/\sigma\} \qquad \widehat{\Lambda}^{\vee\sigma} := \bigoplus_{\mathbf{i}\in I/\sigma} \mathbb{Z}\widehat{\alpha}_{\mathbf{i}}^\vee \tag{2.2.2}$$

$$\widehat{\varpi}_{\mathbf{i}} := \sum_{i\in\mathbf{i}} \varpi_i \qquad \widehat{\Lambda}^\sigma := \bigoplus_{\mathbf{i}\in I/\sigma} \mathbb{Z}\hat{\varpi}_{\mathbf{i}} \tag{2.2.3}$$

There are two different ways to fold the Cartan matrix.

1. In the classical case A is folded to $C^\sigma = (c_{\mathbf{ij}})_{\mathbf{i},\mathbf{j}\in I/\sigma}$, where

   $$c_{\mathbf{i},\mathbf{j}} := \sum_{i\in\mathbf{i}} a_{ij}, \quad j \in \mathbf{j}.$$

   It is easy to see that $c_{\mathbf{i},\mathbf{j}}$ is independent of the choice of $j \in \mathbf{j}$ made in its definition. Admissibility (2.1.2) ensures that C^σ is a symmetrizable Cartan matrix. Corresponding to C^σ is a symmetrizable Cartan datum $(C^\sigma, \bar{\mathbf{d}}^\sigma, \Pi^\sigma, \Lambda^\sigma, \Pi^{\vee\sigma}, \Lambda^{\vee\sigma})$. There are isomorphisms in the corresponding categories

   $$\widehat{\Pi}^\sigma \xrightarrow{\sim} \Pi^\sigma : \widehat{\alpha}_{\mathbf{i}} \longmapsto \alpha_{\mathbf{i}}$$

   $$\widehat{\Lambda}^\sigma \xrightarrow{\sim} \Lambda^\sigma : \widehat{\varpi}_{\mathbf{i}} \longmapsto \varpi_{\mathbf{i}}$$

   $$\widehat{\Pi}^{\vee\sigma} \xrightarrow{\sim} \Pi^{\vee\sigma} : \widehat{\alpha}_{\mathbf{i}}^\vee \longmapsto \alpha_{\mathbf{i}}^\vee$$

   which also provide an isomorphism $\widehat{Q}^\sigma \xrightarrow{\sim} Q^{\sigma\vee} = \Lambda^{\sigma\vee}$.

2. The relations defining the quantized enveloping algebra $U_q(\mathscr{C})$ require a symmetric matrix. Accordingly, the quantum case uses $\operatorname{diag}(d_i)A$, which is folded to $A^\sigma = (a^\sigma_{\mathbf{ij}})_{\mathbf{i},\mathbf{j}\in I/\sigma}$, where

$$a^\sigma_{\mathbf{ij}} := \sum_{(i,j)\in\mathbf{i}\times\mathbf{j}} a_{ij}. \tag{2.2.4}$$

Let $\mathscr{C} = (A, \Lambda, \Lambda^\vee, \Pi^\vee, \Pi)$ be a Cartan datum. The *Langlands dual* datum to $\mathscr{C}$ is defined to be

$$\mathscr{C}^\vee := (A^T, \Lambda^\vee, \Lambda, \Pi, \Pi^\vee).$$

That is to say, to obtain $\mathscr{C}^\vee$ from $\mathscr{C}$ one transposes the Cartan matrix and interchanges the roles of the weight lattice and dual weight lattice.

Lemma 2.2.1 ([Lus93])**.** *A^σ is the symmetrized Cartan matrix diag$(\mathbf{d}^\sigma)C$ of the Langlands dual $\mathscr{C}^{\sigma\vee}$ of the folded Cartan datum*

$$\mathscr{C}^{\sigma\vee} = (C^{\sigma T}, \mathbf{d}^\sigma, \Pi^{\sigma\vee}, \Lambda^{\sigma\vee}, \Pi^\sigma, \Lambda^\sigma) \tag{2.2.5}$$

for $\mathbf{d}^\sigma = (|\mathbf{i}_1|, \cdots, |\mathbf{i}_{|I/\sigma|}|)$. In other words $a^\sigma_{\mathbf{ij}} = d_{\mathbf{j}} c_{\mathbf{ji}}$.

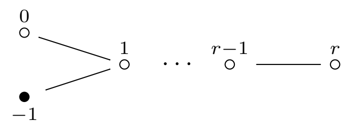
Remark 2.2.2. To allow the folded simple roots $\widehat{\alpha}_{\mathbf{i}}$ to function as simple roots, we combine the Langlands dual operation with the folding procedure and pass from $\mathscr{C}$ to $\mathscr{C}^{\sigma\vee}$: The orbital sum defining $\widehat{\alpha}_{\mathbf{i}} \in \widehat{\Pi}^\sigma$ is manifeslty not a root vector, although these sums are linearly independent. Replacing $\widehat{\Pi}^\sigma$ with $\widehat{\Pi}^{\vee\sigma}$ provides a collection of linearly independent vectors functioning as simple coroots for $\mathscr{C}^\sigma$—that is to say, as simple roots for $\mathscr{C}^{\sigma\vee}$.

2.3 Parabolic subdata adapted to folding

The parabolic data (*cf.* p.12) used in this dissertation are the *highest-weight configurations* (see Definition 4.4.1), and can be obtained from the *maximal proper* such subdata (shown below) by pairwise intersection. In the following diagrams the vertex of $I \setminus J$ is indicated by $\bullet$.

The D-series

$$\begin{aligned} \mathscr{C} &= D_{r+2} \qquad I_r := \{-1, 0, 1, \cdots, r\} \\ \sigma &:= (-1, 0) \qquad J_r := I_r \setminus \{-1\} \end{aligned} \tag{2.3.1}$$

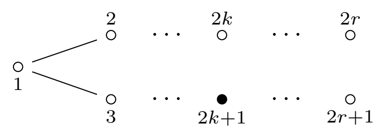


folds to the C series

$$\mathscr{C}^{\sigma\vee} = C_{r+1} \qquad I_r/\sigma = \{\mathbf{0}, \mathbf{1}, \cdots, \mathbf{r}\} \tag{2.3.2}$$

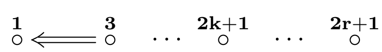
The odd-ranked[1] A-series

$$\begin{aligned} \mathscr{C} &= A_{2r+1} \qquad I_r := \{1, \cdots, 2r+1\} \\ \sigma &= \prod_{i=1}^{r}(2i, 2i+1) \qquad J_{r,k} := I_r \smallsetminus \{2k+1\} \end{aligned} \tag{2.3.3}$$



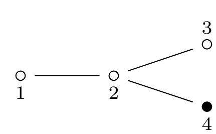
for some $1 \leq k \leq r$ folds to the B series

$$\mathscr{C}^{\sigma\vee} = B_{r+1} \qquad I_r/\sigma = \{\mathbf{1}, \mathbf{3}, \cdots, \mathbf{2r+1}\} \tag{2.3.4}$$



The triality of D_4

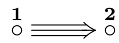
$$\begin{aligned} \mathscr{C} &= D_4 \qquad I := \{1, 2, 3, 4\} \\ \sigma &:= (1, 3, 4) \qquad J = \{1, 2, 3\} \end{aligned} \tag{2.3.5}$$



folds to type G_2

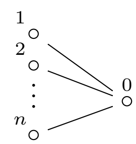
$$\mathscr{C}^{\sigma\vee} = G_2 \qquad I/\sigma = \{\mathbf{1}, \mathbf{2}\} \tag{2.3.6}$$

[1] In even rank, the automorphism of $\mathscr{C}$ effecting a reversal of the Dynkin diagram is not admissible.



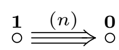
Cases $(D_3, C_2) \cong (A_3, B_2)$ and the triality are special cases of the most general rank two folding

$$\begin{aligned} \mathscr{C} &= T_n & I_n &:= \{0, 1, \cdots, n\} \\ \sigma &:= (1, \cdots, n) & J &:= I_n \smallsetminus \{n\} \end{aligned} \tag{2.3.7}$$



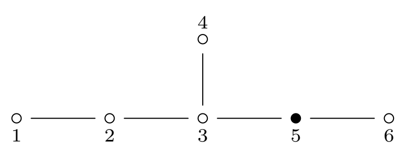
which folds to a type we call T_2^n

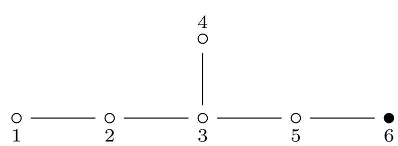
$$\mathscr{C}^{\sigma\vee} = T_2^n \qquad I/\sigma = \{\mathbf{1}, \mathbf{0}\} \tag{2.3.8}$$



The exceptional type

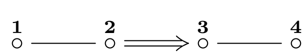
$$\begin{aligned} \mathscr{C} &= E_6 & I &= \{1, 2, 3, 4, 5, 6\} \\ \sigma &:= (1, 6)(2, 5) & J &= \{1, 2, 3, 4, 5\} \text{ or } \{1, 2, 3, 4, 6\} \end{aligned} \tag{2.3.9}$$





folds to type F_4

$$\mathscr{C}^{\sigma\vee} = F_4 \qquad I/\sigma = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\} \tag{2.3.10}$$



2.4 Remarks on generality

Other foldings of simply laced symmetrizable Cartan data exist, see [Kas96] for more examples. The theory developed in Chapter 2 works for any equivalence relation $\sim$ on I such that $a_{ij} = a_{i'j'}$ and $a_{ii'} = 0$ when $i \sim i', j \sim j'$. Our farthest-reaching results are for the D-series foldings, as is the case for quantized enveloping algebra folding [BG11].

2.5 Applications of folding

Folding allows simple finite-dimensional Lie algebras can be used to describe the nontwisted affine Lie algebras [Kac90, pp.105–110]. The former can be constructed directly from the root lattice Q. That is, $Q(ADE)$ are explicitly defined. A form $\langle \cdot, \cdot \rangle$ is chosen to distinguish the foot system Φ as $\{\beta \in A \,|\, \langle \beta, \beta \rangle = 2\}$. An asymmetry function is introduced which "breaks the symmetry" of $\langle \cdot, \cdot \rangle$. Now, the Cartan subalgebra $\mathfrak{h} := Q \otimes_{\mathbb{Z}} \mathbb{C}$ gives the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\beta \in \Phi} \mathbb{C} e_\beta \right)$, and a bracket is put on the latter using combinatorial properties of Φ.

The non-simply-laced simple finite-dimensional Lie algebras are then constructed using admissible automorphisms of the simply-laced data. The automorphism σ indices an automorphism of $\mathfrak{g}$, which decomposes as $\mathfrak{g} = \bigoplus_{k=0}^{|\sigma|-1} \mathfrak{g}^{(k)}$, where $g^(k)$ is the $\exp(2\pi i k/|\sigma|)$-eigenspace of σ. After changing notation by passing to σ-invariants (orbital sums), it can be easily checked that $g^{(0)}$ is one of the non-simply-laced simple finite-dimensional Lie algebras.

To construct the canonical basis for the upper-triangular half of a quantum group of simply-laced type, Lusztig [Lus93] employed the machinery of perverse sheaves on quiver varieties. Through this he defined the geometric realization of the algebra, and using the geometric action of the algebra on itself he constructed the canonical basis. For a multiply-laced Dynkin diagram, one needs to consider valued quivers in order to have a quiver variety. This requires extensions of the base field of finite degree, and so one can not work over an algebraically closed field. However, in this situation the geometry becomes intractable.

To construct the canonical basis for an algebra of multiply-laced type, Lusztig realized a multiply laced $\tilde{\mathscr{C}}$ as a folding of a simply-laced $\mathscr{C}$ by an admissible automorphism σ. He showed that σ induces an automorphism of the algebra, which acts on the canonical basis. The fixed points under this action turned out to function as a canonical basis for the original multiply-laced algebra.

There is a short set of axioms, due to Stembridge [Ste03], which allows one to determine from local structural conditions whether a crystal graph is the crystal of a representation in the simply-laced case. (Only a partial characterization is available for doubly-laced crystals, and the at present the literature contains no account of the other multiply-laced cases.) In the introduction of this paper, the author remarks that

> "These simply-laced crystals are arguably the most important, since all highest weight crystals of finite or affine type—the ones of widest interest—are either simply laced or may be obtained from such crystals by a standard technique of 'folding' by diagram automorphisms."

Indeed, since Lusztig's construction of canonical bases the fixed point set of a crystal under a (not necessarily admissible!) automorphism has received a great deal of attention. Naito and Sagaki proved analogues of Lusztig's result for various realizations of Kashiwara's $B(\infty)$ crystal in a series of papers [NS01, NS02, NS03, NS04, NS05].

Remark 2.5.1. In each of the situations described above, the fixed-point set of σ plays the crucial role. Because the automorphism's action on the fixed-point set is trivial by definition, there has been no motivation to fold the *structure* of the objects under consideration. In this report we work with the largest proper subset of a crystal on which σ-acts, a consider the crystal structure of this set modulo the automorphism's action.

Chapter 3

Balanced parabolic quotients

Many aspects of the representation and structural theory of a semisimple Lie algebra $\mathfrak{g}$ and its quantum analogue are closely related to and connected by the combinatorial properties of the Weyl group W of $\mathfrak{g}$ (see [Kac90, Lus93, Kas93, Jos95] to name but a few). In this chapter we study a subset of the Weyl group W, comprising elements of a parabolic quotient ${}^J W \subset W$ whose reduced decompositions are balanced with respect to a diagram automorphism σ. We call this set the *balanced quotient* and denote it by ${}^J W_\sigma$.

A canonical set of reduced expressions for ${}^J W_\sigma$ in type AD is enumerated by a forest digraph with self-similar components. (This is different than the normal forest of [BB05, 3.4] in that we are considering a subset of W.) By design a balanced quotient is a proper subset of set of fully commutative elements in a Coxeter group. As such, our normal forms can be viewed as refinements of results of Stembridge [Ste97].

We give a characterization of an element of ${}^J W_\sigma$ by a property of its inversion set. In [Fan95] it is shown—in the context of a simply-laced Coxeter group—that w is a fully commutative element if and only if w yields an abelian subalgebra of the Borel. Combining our result on inversion sets with this property, we find that an element of the balanced quotient gives an abelian subalgebra of the Borel that decomposes into the direct sum of two subalgebras: one contained in the Borel for the Jth Levi subalgebra, and one consisting of σ-invariants. Formulae for the dimensions of these subalgebras are given, based on the structure of a reduced decomposition of w.

3.1 The balanced parabolic quotient

A well-known theorem of Tits [Tit69, Theorème 3] states that two reduced expressions for $w \in W$ can be obtained from one another by applying a sequence of commutation/braid relations. We refer to w as *fully commutative* if any two reduced decompositions are obtained from one another using only commutation relations. A subset of W is called *fully commutative* when each of its elements is fully commutative.

In this chapter we focus our attention on those Cartan data that correspond to the simple finite-dimensional Lie algebras of simply-laced type. Thus, the matrix A is assumed to be positive definite and cannot be written in a block-diagonal form, and W is finite. The subdata we use are related to the action of an admissible automorphism of $\mathscr{C}$; they are maximal proper subsets $J \subsetneq I$ that meet each σ-orbit in I nontrivially.

- Type A_{2r+1} with $|J| = |I| - 1$.
- Type D_r with $\mathscr{C}_J \cong A_{r-1}$.
- Type E_6 with $\mathscr{C}_J \cong D_5$.
- Type D_4 under the triality with $\mathscr{C}_J \cong A_3$.

More details on these data, including indexing schemes and the corresponding automorphisms are shown in displays (2.3.1), (2.3.3), (2.3.9), and (2.3.5).

Theorem 3.1.1 ([Ste96, Theorem 6.1])**.** *In the cases listed above the parabolic quotient ${}^JW$ is fully commutative.*

Given $\mathbf{i} = (i_1, \ldots, i_t) \in I^t$, let $\ell_i(\mathbf{i}) := |\{1 \le r \le t \,|\, i_r = i\}|$. A reduced decomposition of a *fully commutative* w admits no braid relation. Accordingly, $\ell_i(\mathbf{i})$ is independent of $\mathbf{i} \in \mathcal{R}(w)$ and will be denoted by $\ell_i(w)$. Following Theorem 3.1.1 we make the following definition.

Definition 3.1.2. The *σ-balanced quotient* ${}^JW_\sigma$ is the set of all $w \in {}^JW$ such that $\ell_i(w) = \ell_{\sigma(i)}(w)$ for all $i \in I$.

The reference to σ is omitted when the context is clear.

3.2 Normal forms for ${}^JW_\sigma$ in types A and D

Recall the weak right ordering $\leq_R$ on W, defined by $u \leq_R uv$ if and only if $\ell(uv) = \ell(u)+\ell(v)$. This is a partial-order relation on W, which restricts to a partial ordering on the balanced quotient ${}^JW_\sigma$. Recall that a subset O of a poset $(X, \leq)$ is an *order ideal* if $\{x \in O \,|\, x \leq a\} \subset O$ for all $a \in O$.

Lemma 3.2.1 ([Ste96, Proposition 2.5]). *For any $J \subset I$ the set ${}^JW$ is an order ideal of W under the right weak ordering. In particular, we can regard ${}^JW = [1, {}^Jw_\circ]_R$.*

In section 3.2 we give a set of normal forms for the elements of ${}^JW_\sigma$. This set is enumerated by a forest digraph with self-similar components; in section 3.3 we present these graphs, which have a remarkable self-similarity property, and obtain from them an enumeration of ${}^JW_\sigma$.

The key ingredient in our proofs of Theorems 3.4.1 and 5.3.1 is a set of normal forms for the balanced quotient. This set can be viewed as a refinement of the one given in [Ste97] for the fully commutative elements of W.

Remark 3.2.2. The cases covered in this paper correspond to the cases in [Ste96, Theorem 6.1] such that W^J is minuscule, $\mathscr{C}$ admits an admissible automorphism, and J is adapted to the σ-action as described above. Up to an application of σ, (2.3.1)–(2.3.5) exhaust these cases.

In types D and A it is most convenient to state our normal forms in the injective limits $W(D_\infty)$ and $W(A_\infty)$, which are also partially ordered by $\leq_R$. That is, we specify a set of normal forms for ${}^{J_\infty}W(D_\infty)_\sigma$ and ${}^{J_{\infty,k}}W(A_\infty)_\sigma$ ($k \geq 1$) from which the normal forms for ${}^{J_r}W(D_r)_\sigma$ and ${}^{J_{r,k}}W(A_{2r+1})_\sigma$ are obtained by an appropriate restriction of parameters.

In type D (2.3.1), given $1 \leq j \leq k$ let

$$s_k^{(j)} := \prod_{t=0}^{j-1} s_{k-t} \in W^\sigma, \qquad \ell(s_k^{(j)}) = j.$$

Then our set of normal forms is as follows.

Proposition 3.2.3. *In the injective limit ${}^{J_\infty}W(D_\infty)$ of type D as in* (2.3.1) *we have ${}^{J_1}w_\circ = s_{-1}s_1s_0$ and ${}^{J_m}w_\circ$ is σ-balanced if and only if m is odd, in which case*

$$ {}^{J_m}w_\circ = {}^{J_{m-2}}w_\circ s_{m-1}s_m \cdots s_2s_3s_1s_2{}^{J_1}w_\circ \tag{3.2.1}$$

Every $w \in {}^{J_\infty}W(D_\infty)_\sigma$ *not of the form* ${}^{J_m}w_\circ$ *with odd* m *has a reduced expression*

$$w = {}^{J_m}w_\circ s_{m+1}^{(j_1)} \cdots s_{m+n}^{(j_n)} \tag{3.2.2}$$

with $2\ell_{-1}(w) - 1 = m \geq j_1 \geq \cdots \geq j_n \geq 1$. *Taking all elements of the form* (3.2.2) *such that* $m + n \leq r$ *along with* ${}^{J_m}w_\circ$ *for odd* $m \leq r$ *gives a set of normal forms for* ${}^{J_r}W(D_r)_\sigma$.

In type A (2.3.3), given $0 \leq j < k$ let

$$\hat{s}_k^{(j)} := \prod_{i=0}^{j} s_{2(k-j)}s_{2(k-j)+1}, \qquad \ell(\hat{s}_k^{(j)}) = 2j,$$

and declare $\hat{s}_k := s_k^{(0)}$. Then our set of normal forms is as follows.

Proposition 3.2.4. *In the injective limit* ${}^{J_{\infty,k}}W(A_\infty)$ *of type* A *as in* (2.3.3)*, the elements* ${}^{J_{r,k}}w_\circ, r \geq k$ *are all* σ*-balanced and have reduced expressions*

$$ {}^{J_{k,k}}w_\circ = s_{2k+1} \cdots s_3 s_1 s_2 \cdots s_{2k} \tag{3.2.3}$$

$$ {}^{J_{m+1,k}}w_\circ = {}^{J_{m,k}}w_\circ \hat{s}_{m+1} \cdots \hat{s}_{k+1} {}^{J_{k,k}}w_\circ, m \geq k. \tag{3.2.4}$$

Every $w \in {}^{J_{\infty,k}}W(A_\infty)_\sigma$ *not of the form* ${}^{J_{m,k}}w_\circ$ *with* $m \geq k$ *has a reduced decomposition*

$$w = {}^{J_{m,k}}w_\circ \hat{s}_{m+1}^{(j_1)} \cdots \hat{s}_{m+n}^{(j_n)} \tag{3.2.5}$$

with $m - k \geq j_1 \geq \cdots \geq j_n \geq 1$. *Taking all elements of the form* (3.2.5) *such that* $r \geq m + n$ *along with* ${}^{J_{m,k}}w_\circ$ *for* $k \leq m \leq r$ *gives a set of normal forms for* ${}^{J_{r,k}}W_(A_{2r+1})$.

The longest element $w_\circ \in W$ factors uniquely as $w_\circ = w_\circ(J) \cdot {}^J w_\circ$, with $w_\circ(J)$ being the longest element of W_J. Accordingly

$$\ell({}^J w_\circ) = |\Phi^+| - |\Phi_J^+|. \tag{3.2.6}$$

As ${}^J w_\circ$ is the unique element of ${}^J W$ with this length, ${}^J w_\circ = w_\circ^J$ and (1.2.3) gives

$$\Phi^+({}^J w_\circ) = \Phi^+ \setminus \Phi_J^+. \tag{3.2.7}$$

We begin with type D, and use the notation (2.3.1). Equation (3.2.6) indicates

$$\ell({}^{J_r}w_\circ) = \frac{(r+1)(r+2)}{2}$$

hence $\ell({}^{J_r}w_\circ) - \ell({}^{J_{r-1}}w_\circ) = r+1$. That is to say, for each $r \geq 0$ there is a reduced expression

$${}^{J_r}w_\circ = {}^{J_{r-1}}w_\circ\tau_r, \quad \ell(\tau_r) = r+1.$$

More precisely:

Lemma 3.2.5. *We have $\tau_0 = s_{-1}$ and, for all $r \geq 1$,*

$$\tau_r = \begin{cases} s_r \cdots s_1 s_0 & r \text{ odd} \\ s_r \cdots s_1 s_{-1} & r \text{ even.} \end{cases} \tag{3.2.8}$$

In particular the longest element of ${}^{J_r}W_\sigma$ is σ-balanced if and only if r is odd, in which case it has a reduced expression

$${}^{J_r}w_\circ = {}^{J_{r-2}}w_\circ s_{r-1}s_r \cdots s_2 s_3 s_1 s_2 {}^{J_1}w_\circ, \text{ odd } r \tag{3.2.9}$$

and if r is even then

$$w = {}^{J_{r-1}}w_\circ s_r \cdots s_{-1}, \text{ even } r. \tag{3.2.10}$$

Proof. We argue by induction on r. Since ${}^{J_1}w_\circ = s_{-1}s_1s_0$ and ${}^{J_2}w_\circ = s_{-1}s_1s_0s_2s_1s_{-1}$, induction begins. Take an even $n \geq 1$. By the induction hypothesis there is a reduced expression

$${}^{J_{n-1}}w_\circ s_n \cdots s_1 s_{-1} = {}^{J_{n-2}}w_\circ s_{n-1} \cdots s_1 s_0 s_r \cdots s_1 s_{-1}.$$

Then

$$\ell({}^{J_{n-1}}w_\circ s_n \cdots s_1 s_{-1}) = \frac{1}{2}n(n+1) + (n+1) = \ell({}^{J_n}w_\circ).$$

Both ${}^{J_{n-2}}w_\circ$ and $s_{n-1} \cdots s_1 s_0 s_r \cdots s_1 s_{-1}$ are fully commutative, hence

$${}^{J_{n-1}}w_\circ s_n \cdots s_1 s_{-1} \in {}^{J_{n-1}}W.$$

Therefore ${}^{J_{n-1}}w_\circ s_n \cdots s_1 s_{-1}$ coincides with ${}^{J_n}w_\circ$. The case for odd n is proved similarly. □

Proof of Proposition 3.2.3. We argue by induction on r. The induction begins because ${}^{J_2}W_\sigma =$

$\{{}^{J_1}w_\circ s_2, {}^{J_1}w_\circ s_2 s_1\}$. For $1 < m \le r$ let

$$\overline{{}^{J_m}W} := {}^{J_m}W \smallsetminus {}^{J_{m-1}}W$$

and $\overline{{}^{J_1}W} := {}^{J_1}W$ so that ${}^{J_r}W = \bigsqcup_{1\le m\le r} \overline{{}^{J_m}W}$. A reduced expression for $w \in \overline{{}^{J_m}W}$ contains s_r and does not contain any s_p with $p > m$. Accordingly $w = vu$ with $v \in {}^{J_{m-1}}W$ of maximal length, $\ell(w) = \ell(v) + \ell(u)$, and $u \le_R \tau_r$ by Lemma 3.2.5. That is, $u \le_R s_r \cdots s_1 s_i$ for $i \in \{0, -1\}$ depending on the parity of r.

If $\ell(us_{-1}) < \ell(u)$ then w is balanced if and only if v is balanced. In this case, using the induction hypothesis we obtain

$$w = {}^{J_{2t-1}}w_\circ s_{2t}^{(j_1)} \cdots s_{2t+n}^{(j_n)} s_m^{(j)}$$

where $m - 1 = 2t + n$. If $j > j_n$ then a reduced expression for w contains $s_{m-j_n-1} s_{m-j_n} s_{m-j_n-1}$ as a subword, contradicting full commutativity. Therefore in this case w has the desired form.

If $u = \tau_r$ and $v <_R {}^{J_{r-1}}w_\circ$ then $\ell(vs_j) < \ell(v)$ for some $1 \le v \le r-1$. But then a reduced expression for w contains the expression $s_j s_{j+1} s_j$, contradicting full commutativity. And so if $u = \tau_r$ then $w = {}^{J_r}w_\circ$. □

Remark 3.2.6. The alternating occurrences of $-1, 0$ in a reduced expression of $w \in {}^{J_r}W$ is one of several equivalent conditions for an element to be A_r-stable in the sense of [Ste97]. In fact ${}^{J_r}W$ is a proper subset of the set of A_r-stable elements of W. Besides the central role played by σ, our perspective differs in that we focus on the subdatum $A_{r+1} \subset D_{r+2}$, while $D_{r+1} \subset D_{r+2}$ is studied in [Ste97].

Next we treat type A, using the notation (2.3.3). Put $J''_{r,k} := \{2i+1 \,|\, k < i \le r\}$ and $J'_{r,k} := J_{r,k} \smallsetminus J''_{r,k}$ so that

$$\mathscr{C}_{J_{r,k}} = \mathscr{C}_{J'_{r,k}} \times \mathscr{C}_{J''_{r,k}} \cong A_{r+k} \times A_{r-k}. \tag{3.2.11}$$

Now $w_\circ(J_{r,k}) \cong w_\circ(A_{r+k}) \cdot w_\circ(A_{r-k})$ and equation (3.2.6) indicates

$$\ell({}^{J_{r,k}}w_\circ) = (r+1)^2 - k^2$$

and also that $\ell({}^{J_{r+1,k}}w_\circ) - \ell({}^{J_{r,k}}w_\circ) = 2r + 3$.

Lemma 3.2.7. *For all* $1 \leq k \leq r$

$$^{J_{k,k}}w_\circ = s_{2k+1} \cdots s_3 s_1 s_2 \cdots s_{2k} \tag{3.2.12}$$

$$^{J_{r+1,k}}w_\circ = {}^{J_{r,k}}w_\circ \hat{s}_{r+1} \cdots \hat{s}_{k+1} {}^{J_{k,k}}w_\circ. \tag{3.2.13}$$

Proof. This proof is similar to the proof of Lemma 3.2.5, and is omitted. □

Proof of Proposition 3.2.4. Each $^{J_{m,k}}w_\circ$ with $k \leq m \leq r$ is balanced, so we need not consider these elements in this proof. Note that

$$^{J_{k,k}}W_\sigma = \{^{J_{k,k}}w_\circ\} \tag{3.2.14}$$

by (3.2.12). For $k < m \leq r$ let

$$\overline{^{J_{m,k}}W} := {}^{J_{m,k}}W \smallsetminus {}^{J_{m-1,k}}W$$

and $\overline{^{J_{k,k}}W} = {}^{J_{k,k}}W$ so that $^{r,k}W = \bigsqcup_{k \leq m \leq r} \overline{^{J_{m,k}}W}$. A reduced expression for $w \in \overline{^{J_{m,k}}W}$ contains s_{2m} or s_{2m+1} and contains no s_p with $p > 2m+1$. Accordingly $w = v s_{2m} \cdots s_{2(m-j)} s_{2m+1} \cdots s_{2(m-j')+1}$ with $v \in {}^{J_{m-1,k}}W$ of maximal length.

Now w is balanced if and only if $j = j'$ and v is balanced. By induction $v = {}^{J_{t,k}}w_\circ \hat{s}_{t+1}^{(j_1)} \cdots \hat{s}_{t+n}^{(j_n)} \hat{s}_m^{(j)}$ (where $m = t + n + 1$). If $j > j_n$ then a reduced expression for w contains the expression $s_{t+n-j_n} s_{t+n-j_n+1} s_{t+n-j_n}$, contradicting full commutativity. □

3.3 Enumeration of $^J W_\sigma$

For each positive integer k define an infinite digraph $T_k := (V_k, E_k)$ as follows

$$V_k := \{(k; \varnothing)\} \cup \{(k; j_1, \ldots, j_n) \mid k \geq j_1 \geq \cdots \geq j_n\}$$

$$(k; j_1, \ldots, j_n) \to (k; j'_1, \ldots, j'_m) \iff m = n + 1 \text{ and } j'_t = j_t \ \forall\ t = 1, \ldots, n.$$

It is easy to see T_k is an infinite tree. Indeed, first note that there is a unique path between any vertex $(k; j_1, \ldots, j_n)$ and $(k; \varnothing)$. Now suppose $(k; j_1, \ldots, j_n)$ and $(k; l_1, \ldots, l_m)$ are arbitrary. Then the unique path between these two vertices is given as follows: Let $i := \max\{p \geq 1 \mid j_t = l_t \text{ for all } 1 \leq t \leq p\}$. If no such i exists, then take the unique path from $(k; j_1, \ldots, j_n)$ to $(k; \varnothing)$ followed by the unique path from $(k; \varnothing)$ to $(k; l_1, \ldots, l_m)$. Otherwise, we have

$$(k; j_1, \ldots, j_n) \leftarrow \cdots \leftarrow (k; j_1, \ldots, j_p) = (k; l_1, \ldots, l_p) \to \cdots \to (k; l_1, \ldots, l_m)$$

uniquely.

Observe that T_k is isomorphic to a subgraph of each $T_r, r > k$ via the natural inclusions of vertices and edges. Furthermore, T_k is isomorphic to infinitely many subgraphs of itself via $(k; j_1, \dots, j_n) \longmapsto (k; \underbrace{k, \dots, k}_{m}, j_1, \dots, j_n), m \geq 0$.

Declare $T_k^{(0)} := \{(k; \varnothing)\}$, and for a fixed positive integer N define the Nth truncation $T_k^{(N)} := \left(V_k^{(N)}, E_k^{(N)}\right)$ by

$$V_k^{(N)} := \{(k; \varnothing), (k; j_1, \dots, j_n) \,|\, k \geq j_1 \geq \cdots \geq j_n \text{ and } n \leq N\}$$
$$E_k^{(N)} := \{x \to y \,|\, x, y \in V_k^{(N)}\},$$

and let $T_k^{(N)}$ be the empty graph when $N < 0$. Figures 3.1, 3.2 show respectively $T_2^{(3)}$, $T_3^{(3)}$.

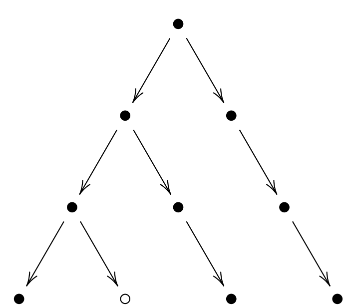


Figure 3.1: $T_2^{(3)}$ with vertex $(2; 2, 2, 1)$ indicated by $\circ$.

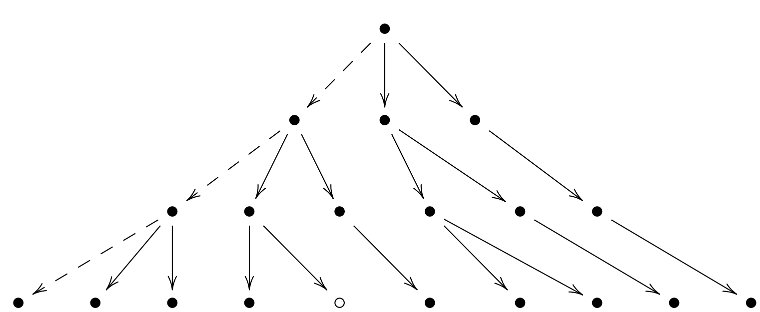


Figure 3.2: $T_3^{(3)}$ with vertex $(3; 3, 2, 1)$ indicated by $\circ$. The subgraph obtained by deleting the dashed arrows is isomorphic to $T_2^{(3)} \oplus T_2^{(2)} \oplus T_2^{(1)} \oplus T_2^{(0)}$, as indicated in the proof of Lemma 3.3.1

Lemma 3.3.1. *For all $r \geq 1, k \geq 0$ we have $|T_k^{(r)}| = \binom{r+k}{k}$.*

Proof. We prove that

$$|V_k^{(r)}| = \binom{r+k}{k} \text{ for all } r \geq 0 \tag{3.3.1}$$

by induction on k. When $k = 1$, (3.3.1) is true by the definition of $V_1^{(r)}$, thus induction begins. Now let $k > 1$ and assume for induction that (3.3.1) holds for $k-1$. Observe that there exists a bijection $V_k \cong \bigoplus_{n\in\mathbb{N}} V_{k-1}$ whereby

$$V_{k-1} \ni (k-1; j_1, \ldots, j_m) \longmapsto (k; \underbrace{k, \ldots, k}_{n}, k-1, j_1, \ldots, j_m) \in V_k, \quad n \geq 0$$

Accordingly, for all $r \geq 0$ there exists a bijection

$$V_k^{(r)} \cong \bigoplus_{n=0}^{r} V_{k-1}^{(n)}.$$

Now, by the induction hypothesis $|V_k^{(r)}| = \sum_{n=0}^{r} \binom{n+k-1}{k-1}$.

To complete the inductive step it remains to prove that for all $a \geq 0$, $b \geq 1$ we have $\sum_{n=0}^{a} \binom{n+b-1}{b-1} = \binom{a+b}{b}$ This claim is clear for $a = 0$ and for all $b \geq 1$. Assuming $\sum_{n=0}^{s} \binom{n+b-1}{b-1} = \binom{s+b}{b}$, for all $s \leq a$ and $b \geq 1$, we have

$$\sum_{n=0}^{a+1} \binom{n+b-1}{b-1} = \binom{a+b}{b} + \binom{a+b}{b-1} = \binom{a+b+1}{b}.$$

Thus,

$$|V_k^{(r)}| = \sum_{n=0}^{r} \binom{n+k-1}{k-1} = \binom{r+k}{k},$$

which completes the inductive step. □

Corollary 3.3.2. *The normal form forest $\mathcal{H}({}^{J_\infty}W(D_\infty), \leq_R)$ is isomorphic as a graph to the directed forest $\bigoplus_{k>0} T_{2k}$, and thus*

$$\mathcal{H}({}^{J_r}W(D_{r+2}), \leq_R) \cong \bigoplus_{k\geq 1} T_{2k}^{(r-2k+1)}.$$

Therefore, in type D we have $|{}^{J_r}W(D_{r+2})_\sigma| = 2^r - 1$.

Proof. By Proposition 3.2.3, the assignments ${}^{J_m}w_\circ \mapsto (m+1; \varnothing)$ and

$${}^{J_m}w_\circ s_{m+1}^{(j_1)} \cdots s_{m+n}^{(j_n)} \mapsto (m+1; j_1, \ldots, j_n)$$

place the elements of ${}^{J_\infty}W(D_\infty)$ in one-to-one correspondence with the vertices of $\bigoplus_{k>0} T_{2k}$.

Using the convention that $\binom{a}{b} = 0$ if $b > a$, we have $|\bigoplus_{k\geq 1} T_{2k}^{(r-2k+1)}| = \sum_{k\geq 1}\binom{r+1}{2k}$ by Lemma 3.3.1. It is well-known that $\sum_{j\geq 0}\binom{r+1}{j} = 2^{r+1}$, and also that $\sum_{j\geq 0}(-1)^j\binom{r+1}{j} = 0$. Adding these identities gives $\sum_{k\geq 0} 2\binom{r+1}{2k} = 2^{r+1}$, whence $\sum_{k\geq 1}\binom{r+1}{2k} = 2^r - 1$. $\square$

There is a map ${}^{J_{\infty,k}}W(A_\infty)_\sigma \to \bigoplus_{k>0} T_k$ given by ${}^{J_{m,k}}w_\circ \longmapsto (m-k;\varnothing)$ and $w \longmapsto (m-k; j_1, \ldots, j_n)$ for w of the form (3.2.5). However, owing to the value $m-k$ dominating the sequence $j_1 \geq \cdots \geq j_n$, the preimage of a vertex of $\bigoplus_{k>0} T_k$ is infinite. Indeed, for a fixed $(t; j_1, \ldots, j_n)$ we have

$$\{w \in {}^{J_{\infty,k}}W(A_\infty)_\sigma \,|\, w \mapsto (t; j_1, \ldots, j_n)\} = \{{}^{J_{m,k}}w_\circ \hat{s}_{m+1}^{(j_1)} \cdots \hat{s}_{m+n}^{(j_n)} \,|\, m-k = t\}$$

However, for fixed (r,k) the normal form forest of $({}^{J_{r,k}}W(A_{2r+1})_\sigma, \leq_R)$ is as follows.

Corollary 3.3.3. *In type A, suppose $r \geq k \geq 1$. Then we have an isomorphism of digraphs*

$$\mathcal{H}({}^{J_{r,k}}W(A_{2r+1})_\sigma, \leq_R) \cong \bigoplus_{m\geq 1} T_m^{(r-k-m)}$$

Hence

$$\left|{}^{J_{r,k}}W(A_{2r+1})_\sigma\right| = 2^{r-k} - 1. \tag{3.3.2}$$

Remark 3.3.4. The graphs $T_k^{(r)}$ may be combined in different ways to make forests with interesting combinatorial properties. Let

$$\mathcal{F}_l^{(n)} := \bigoplus_{k=1}^{\left\lfloor \frac{n+l-1}{l} \right\rfloor} T_k^{(n+l-1-lk)}.$$

Then, for example, $|\mathcal{F}_2^{(n)}| = f_n - 1$, where f_n are the Fibonacci numbers.

On the other hand, let $\mathcal{G}^{(n)} := \bigoplus_{k=2}^{\left\lfloor \frac{n+l-1}{l} \right\rfloor} T_k^{(n+l-1-lk)}$; that is, $\mathcal{G}^{(n)}$ equals $\mathcal{F}_2^{(n)}$ with the 1-branching component deleted. Then $|\mathcal{G}^{(n)}| = f_{n+5} - n - 4$. Now, define sequences $(d_t)_{t=0}^\infty$ and $(b_t)_{t=0}^\infty$ by

$$d_t := |{}^{J_t}W(D_{t+2})_\sigma|$$

and $b_t :=$ the tth nonnegative integer having no consecutive zeros or no consecutive ones in its binary representation. For example, $b_t = t$ for $0 \leq t \leq 11$, but $b_{12} = 13$ because $(12)_2 = 1100$. Then one can show (see, *e.g.*, [OEIS, A107909]) that

$$b_{|\mathcal{G}^{(t-1)}|} = d_t.$$

3.4 Inversion sets and the balanced quotient

Given $w \in W$, $\Phi^+(w) := \Phi^+ \cap w^{-1}\Phi^-$ is called *the inversion set* of w. It is well-known (*cf.* [Hum72, Section 10]) that $\ell(w) = |\Phi^+(w)|$ and W contains a unique element $w_\circ$, called *the longest element*, with $\ell(w_\circ) = |\Phi^+|$. The map $w \mapsto \Phi^+(w)$ is injective ([Jos95, A.1.1]) and provides a way to recognize w on the level of the root system (that is, without appealing to a reduced decomposition). Our first main result is the following, which we prove in section 3.4.

Theorem 3.4.1. *In types D_n (2.3.1), E_6 (2.3.9) and D_4 under the triality (2.3.5), for all $w \neq {}^J w_\circ$ we have $w \in {}^J W_\sigma$ if and only if*

$$\Phi^+(w) \smallsetminus \Phi_J^+ \subset \Phi^\sigma \text{ and } w^{-1}\Pi_J \subset \Phi^+. \tag{3.4.1}$$

In type A_{2r+1} (2.3.3) we have $w \in {}^J W_\sigma$ if and only if

$$\Phi^+(w) = \{\beta \in \Phi^+(I_r) \,|\, \beta \text{ is supported on } \alpha_{\sigma(i)}\} \tag{3.4.2}$$

or (3.4.1) *holds.*

Associated to $\mathscr{C}$ is the simple complex Lie algebra $\mathfrak{g}(\mathscr{C})$, (*cf.* [Hum72, Section 18]), which has a presentation depending only on the matrix A. The choice of Cartan subalgebra $\mathfrak{h} := \mathbb{C}\Pi^\vee$ provides a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta\in\Phi} \mathfrak{g}_\beta$ with Borel subalgebra $\mathfrak{b} := \bigoplus_{\beta\in\Phi^+} \mathfrak{g}_\beta$; recall that $\dim_{\mathbb{C}} \mathfrak{g}_\beta = 1$ for all $\beta \in \Phi$. A result of Fan [Fan95, Proposition 7] implies that the inversion set $\Phi^+(w)$ of a fully commutative element gives a commutative subalgebra $\mathfrak{a}_w := \bigoplus_{\beta\in\Phi^+(w)} \mathfrak{g}_\beta$ of the Borel; these are known (*cf.* [Kos98]) to be related to abelian ideals in the Borel subalgebra.

Let $\mathfrak{g}_J$ be the Levi subalgebra of $\mathfrak{g}$ associated with J; thus, $\mathfrak{g}_J$ is generated as a Lie algebra by $\mathfrak{g}_\alpha$, $\pm\alpha \in \Pi_J$. Let $\mathfrak{b}_J := \mathfrak{b} \cap \mathfrak{g}_J$ be the Borel subalgebra of $\mathfrak{g}_J$.

Corollary 3.4.2. *Assume $\mathscr{C}$ is of type D_{r+2} (2.3.1), A_{2r+1} (2.3.3), E_6 (2.3.9) and D_4 under the triality (2.3.5), and let $w \neq {}^J w_\circ$. Then the abelian subalgebra $\mathfrak{a}_w \subset \mathfrak{b}$ decomposes as*

$$\mathfrak{a}_w = \mathfrak{a}_{w,J} \oplus \mathfrak{a}_w^\sigma$$

where $\mathfrak{a}_{w,J} \subset \mathfrak{b}_J$ and $\mathfrak{a}_w^\sigma$ consists of σ-invariants. The dimensions of these algebras are as follows, *where m and k are uniquely determined by w per Propositions 3.2.3 and 3.2.4*

$$\dim_{\mathbb{C}} \mathfrak{a}_{w,J} = \begin{cases} m+1 & \mathscr{C} = D_{r+2} \\ m(m+1) - k(k-1) + 1 & \mathscr{C} = A_{2r+1} \end{cases}$$

and $\dim_{\mathbb{C}} \mathfrak{a}_w^\sigma = \ell(w) - \dim_{\mathbb{C}} \mathfrak{a}_{w,J}$.

In [Fan95], the following characterization of fully commutative elements in the simply-laced case was given.

Theorem 3.4.3 ([Fan95, Proposition 7])**.** *Suppose W is a simply-laced Weyl group. Then an element $w \in W$ is fully commutative if and only if there do not exist three roots $\alpha, \beta, \alpha+\beta \in \Phi^+(w)$.*

Recall the following standard facts. See [Jos95, A.1.1], for example, for proofs.

Lemma 3.4.4. *Take a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ and define $\beta_j := s_{i_\ell} \cdots s_{i_{j+1}} \alpha_{i_j}$ for $1 \leq j \leq \ell$. Then $\Phi^+(w) = \{\beta_j \mid 1 \leq j \leq \ell\}$.*

Lemma 3.4.5. *For all $w \in W$ and $i \in I$ we have*

$$\Phi^+(s_i w) = \Phi^+(w) \sqcup \{w^{-1}\alpha_i\} \text{ in case } \ell(s_i w) > \ell(w) \tag{3.4.3}$$

$$\Phi^+(w^{-1}) = -w\Phi^+(w). \tag{3.4.4}$$

These yield a right-insertion analogue that we failed to find in the literature.

Lemma 3.4.6. *For all $v, w \in W$ such that $\ell(vw) = \ell(v) + \ell(w)$, we have $\Phi^+(vw) = w^{-1}\Phi^+(v) \sqcup \Phi^+(w)$.*

Proof. We use induction on the length of w. If $\ell(v) = 1$ then (3.4.3) and (3.4.4) give

$$\begin{aligned} \Phi^+(vs_i) &= -s_i v^{-1} \Phi^+(s_i v^{-1}) \\ &= -s_i v^{-1}[\Phi^+(v^{-1}) \sqcup \{v\alpha_i\}] \\ &= s_i(-v^{-1}\Phi^+(v^{-1})) \sqcup \{-s_i v^{-1} v \alpha_i\} \\ &= s_i \Phi^+(v) \sqcup \{\alpha_i\} \end{aligned}$$

hence the induction begins.

Assume $\Phi^+(vw) = w^{-1}\Phi^+(v) \sqcup \Phi^+(w)$ for all w of length $\ell(w) \le n$ and take i such that $\ell(ws_i) = \ell(w) + 1$. By the above, we have

$$\Phi^+(vws_i) = s_i\Phi^+(vw) \sqcup \{\alpha_i\}.$$

Then, using the induction hypothesis and Lemma 3.4.4, we obtain

$$\begin{aligned}\Phi^+(vws_i) &= s_i(w^{-1}\Phi^+(v) \sqcup \Phi^+(w)) \sqcup \{\alpha_i\} \\ &= (ws_i)^{-1}\Phi^+(v) \sqcup (s_i\Phi^+(w) \sqcup \{\alpha_i\}) \\ &= (ws_i)^{-1}\Phi^+(v) \sqcup \Phi^+(ws_i).\end{aligned}$$

□

3.4.1 Inversion set of a long element's coset

Suppose $\mathscr{C}$ is of type D_{r+2} (2.3.1) or A_{2r+1} (2.3.3). A given $w \in {}^JW_\sigma$ has the form $w = {}^{J'}w_\circ\hat{s}$, with $\hat{s} \in W^\sigma$ and

$${}^{J'}w_\circ = \begin{cases} {}^{J_m}w_\circ, \text{ odd } 1 \le m \le r & \text{if } \mathscr{C} = D_{r+2} \\ {}^{J_{m,k}}w_\circ, \ k \le m \le r & \text{if } \mathscr{C} = A_{2r+1} \end{cases}$$

by Propositions 3.2.3 and 3.2.4. The inclusion of Cartan data $\iota : \ D_{m+2} \hookrightarrow D_{r+2}$ (respectively $\iota : \ A_{2m+1} \hookrightarrow A_{2r+1}$) induces a monomorphism $\iota : \ W(D_{m+2}) \hookrightarrow W(D_{r+2})$ and an inclusion $\iota : \ \Phi^+(D_{m+2}) \hookrightarrow \Phi^+(D_{r+2})$ (respectively $\iota : \ W(A_{2m+1}) \hookrightarrow W(A_{2r+1})$ and $\iota : \ \Phi^+(A_{2m+1}) \hookrightarrow \Phi^+(A_{2r+1})$) that commute with σ and the map $w \longmapsto \Phi^+(w)$. Denote $\mathscr{C}' := D_{m+2}, A_{2m+1}$ when $\mathscr{C} = D_{r+2}, A_{2r+1}$ respectively.

Now, by Lemma 3.4.6 we have

$$\begin{aligned}\Phi^+(w) &= \hat{s}^{-1}\iota[\Phi^+({}^{J'}w_\circ)] \sqcup \Phi^+(\hat{s}) \\ &= (\hat{s}^{-1}\iota[\Phi^+({}^{J'}w_\circ)] \setminus \Phi(\mathscr{C})^\sigma) \sqcup (\hat{s}^{-1}\iota[\Phi^+({}^{J'}w_\circ)] \cap \Phi(\mathscr{C})^\sigma) \sqcup \Phi^+(\hat{s}),\end{aligned}$$

hence

$$|\Phi^+(w)| = |\hat{s}^{-1}\iota[\Phi^+({}^{J'}w_\circ)] \setminus \Phi(\mathscr{C})^\sigma| + |\hat{s}^{-1}\iota[\Phi^+({}^{J'}w_\circ)] \cap \Phi(\mathscr{C})^\sigma| + \ell(\hat{s}).$$

Because $\hat{s} \in W^\sigma$ we have $\hat{s}^{-1}\beta \in \Phi^\sigma$ if and only if $\beta \in \Phi^\sigma$; hence

$$|\hat{s}^{-1}\iota[\Phi^+({}^{J'}w_\circ)] \setminus \Phi(\mathscr{C})^\sigma| = |\iota[\Phi^+({}^{J'}w_\circ)] \setminus \Phi(\mathscr{C})^\sigma|$$
$$|\hat{s}^{-1}\iota[\Phi^+({}^{J'}w_\circ)] \cap \Phi(\mathscr{C})^\sigma| = |\iota[\Phi^+({}^{J'}w_\circ)] \cap \Phi(\mathscr{C})^\sigma| + \ell(\hat{s}).$$

In $W(\mathscr{C}')$ we have $\sigma({}^{J'}w_\circ) = w_\circ^{J'}$, thus by (3.2.7) we have

$$\iota[\Phi^+({}^{J'}w_\circ)] = \iota[\Phi^+(\sigma(w_\circ^{J'}))] = \sigma \circ \iota[\Phi^+(w_\circ^{J'})] = \sigma \circ \iota[\Phi^+(\mathscr{C}') \setminus \Phi^+_{J'}]. \tag{3.4.5}$$

This relationship will be used to handle the inversion sets of the elements ${}^{J_m}w_\circ$ and ${}^{J_{m,k}}w_\circ$ in sections 3.4.2, 3.4.3, and 3.4.4.

3.4.2 Type D

Proof of Theorem 3.4.1. We use induction on r. Lemma 3.4.4 gives $\Phi^+({}^{J_1}w_\circ) = \{\alpha_0, \alpha_0+\alpha_1, \alpha_{-1}+\alpha_0+\alpha_1\}$, and so induction begins.

Suppose $w \in {}^{J_r}W \setminus \{{}^{J_m}w_\circ \mid \text{odd } m \leq r\}$. Then w is balanced if and only if $w = {}^{J_m}w_\circ s_{m+1}^{(j_1)} \cdots s_r^{(j_n)}$ for some odd $m < r$ by Lemma 3.2.3. Lemma 3.4.6 gives

$$\Phi^+(w) = \left(s_r^{(j_n)}\right)^{-1}\Phi^+({}^{J_m}w_\circ s_{m+1}^{(j_1)} \cdots s_{r-1}^{(j_{n-1})}) \sqcup \Phi^+(s_r^{(j_n)})$$

but $\Phi^+(s_r^{(j_n)}) \subset \Phi^\sigma$ since $s_r^{(j_n)} \in W^\sigma$. If $\beta \in \Phi^+(w) \setminus \Phi^+_J$ then $\beta = \left(s_r^{(j_n)}\right)^{-1}\gamma$ for some $\gamma \in \Phi^+({}^{J_m}w_\circ s_{t+1}^{(j_1)} \cdots s_{r-1}^{(j_{n-1})})$. As $s_r^{(j_n)} \in W^\sigma$, it must be that $\gamma \notin \Phi^+_J$ as well. By induction $\gamma \in \Phi^\sigma$ and it follows that $\beta \in \Phi^\sigma$.

Suppose instead that $w = {}^{J_m}w_\circ$ for odd $1 \leq m < r$. Recalling (3.4.5), we have

$$\begin{aligned}\Phi^+(w) &= \sigma \circ \iota[\Phi^+(D_{m+2}) \setminus \Phi^+_{J_m}] \\ &= \sigma[\{\beta \in \iota[\Phi^+(D_{m+2})] \mid \beta \text{ is supported on } \alpha_{-1}\}] \\ &= \{\beta \in \iota[\Phi^+(D_{m+2})] \mid \beta \text{ is supported on } \alpha_0\}.\end{aligned}$$

Now, if $\gamma \in \Phi^+(w)$ is supported on α_{-1} then it must be that $\gamma \in \Phi^\sigma$. □

Example 3.4.7. In type D_4 we have $I_2 = \{-1, 0, 1, 2\}$, $J_2 = \{0, 1, 2\}$, $J_1 = \{0, 1\}$ and $\sigma = (-1, 0)$. To illustrate the case $w = {}^{J_t}w_\circ$ (odd t) of the above proof, let us examine the inversion set of the coset of the long element ${}^{J_1}w_\circ$ from type D_3 in the context of $\Phi^+(D_4)$.

Figure 3.3 shows the Hasse diagram of the root poset $\Phi^+(D_4)$, decorated as follows:

- To visually separate the parabolic subsystem $\Phi^+_{J_2} \cong \Phi^+(A_3)$ from the set of roots supported on α_{-1}, edges corresponding to adding α_{-1} to $\beta \in \Phi^+_{J_2}$ are indicated by $\dashrightarrow$.
- Elements of Φ^σ (those supported on $\alpha_1 + \alpha_0 + \alpha_{-1}$) are indicated by $\bigcirc$.
- Elements of $\Phi^+({}^{J_1}w_\circ)$ are indicated by $*$.
- Elements of $\Phi^+({}^{J_1}w_\circ) \cap \Phi^\sigma$ are indicated by $\circledast$.
- All other elements of $\Phi^+(w)$ are indicated by $\bullet$.

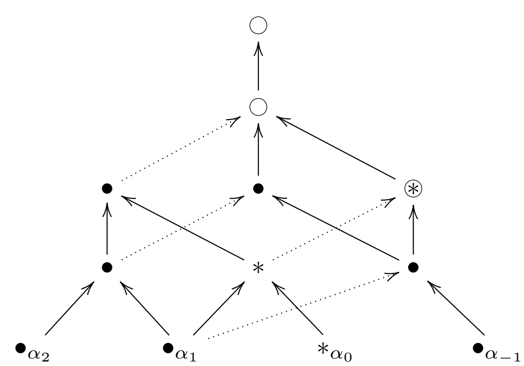


Figure 3.3: Root poset of type D_4, decorated as described in Example 3.4.7.

In accordance with Theorem 3.4.1, each element of $\Phi^+({}^{J_1}w_\circ)$ that is supported on α_{-1} is symmetric. Regarding the conclusion of Corollary 3.4.2, this calculation indicates that $\mathfrak{a}_{{}^{J_1}w_\circ, J_2}$ has dimension 2 and $\mathfrak{a}^\sigma_{{}^{J_1}w_\circ}$ has dimension 1.

Example 3.4.8. In type D_6 we have $I_4 = \{-1, 0, 1, 2, 3, 4\}$, $J_4 = \{0, 1, 2, 3, 4\}$, $J_3 = \{0, 1, 2, 3\}$, and $\sigma = (-1, 0)$. Let us now examine the inversion set of the coset of the long element ${}^{J_3}w_\circ$ from type D_5 in the context of $\Phi^+(D_6)$.

Figure 3.4 shows the Hasse diagram of the root poset $\Phi^+(D_6)$, decorated in the same manner as Example 3.4.7.

In accordance with Theorem 3.4.1, each element of $\Phi^+({}^{J_3}w_\circ)$ that is supported on α_{-1} is symmetric. Regarding the conclusion of Corollary 3.4.2, this calculation indicates that $\mathfrak{a}_{{}^{J_3}w_\circ, J_4}$ has dimension 3 and $\mathfrak{a}^\sigma_{{}^{J_3}w_\circ}$ has dimension 7.

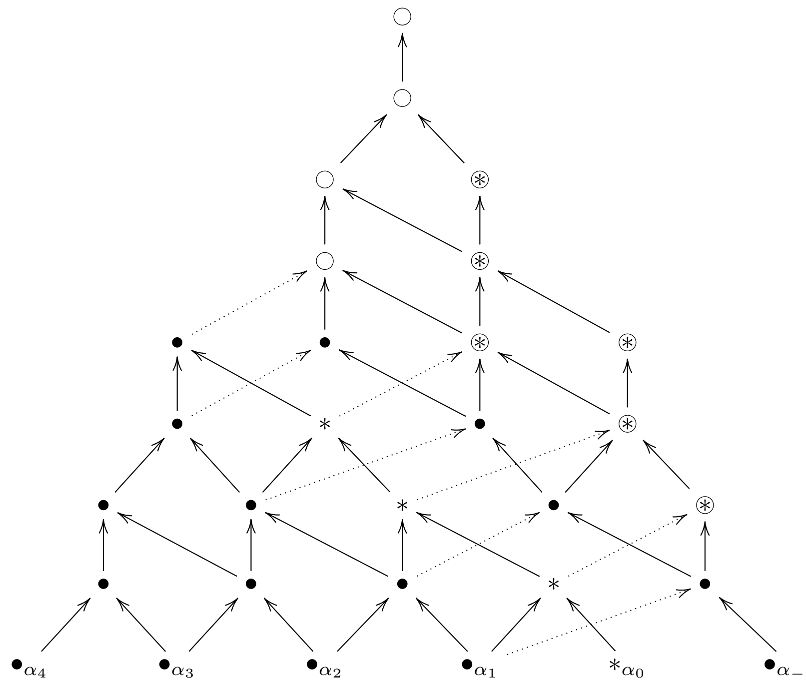


Figure 3.4: Root poset of type D_6, decorated as described in Example 3.4.7.

3.4.3 Type A

Proof of Theorem 3.4.1. First we verify (3.4.2) holds if and only if $w = {}^{J_{m,k}}w_\circ$ for $k \le m \le r$. But this is an immediate consequence of (3.4.5), indeed, in this case

$$\begin{aligned}\Phi^+(w) &= \sigma \circ \iota[\Phi^+(A_{2m+1}) \smallsetminus \Phi^+_{J_{m,k}}] \\ &= \sigma[\{\beta \in \iota[\Phi^+(A_{2m+1})] \,|\, \beta \text{ is supported on } \alpha_{2k+1}\}] \\ &= \{\beta \in \iota[\Phi^+(A_{2m+1})] \,|\, \beta \text{ is supported on } \alpha_{2k+2}\}.\end{aligned}$$

In case $r = k$ condition (3.4.1) holds (vacuously) by (3.2.14). Assume that condition (3.4.1) holds for some pair (r, k) with $r \ge k$. We only need to check the elements of $\overline{{}^{J_{r+1,k}}W}$ excluding ${}^{J_{r+1,k}}w_\circ$. Now $w \in \overline{{}^{J_{r+1,k}}W}_\sigma$ if and only if it has the form provided by Theorem 3.2.4 with $m + n = r + 1$.

Lemma 3.4.4 gives

$$\Phi^+(w) = (\hat{s}^{(j_n)}_{r+1})^{-1}\Phi^+(w') \sqcup \Phi^+((\hat{s}^{(j_n)}_{r+1})^{-1})$$

for $w' = w(\hat{s}_{r+1}^{(j_n)})^{-1} \in {}^{J_{r,k}}W$. Because $m - k \geq j_n$ it follows that $\Phi^+((\hat{s}_{r+1}^{(j_n)})^{-1}) \subset \Phi_J^+$. Accordingly if $\beta \in \Phi^+(w) \smallsetminus \Phi_J^+$ then $\beta = (\hat{s}_{r+1}^{(j_n)})^{-1}\gamma$ for some $\gamma \in \Phi^+(w')$. Since $\ell_{2k+1}((\hat{s}_{r+1}^{(j_n)})^{-1}) = 0$ it follows that $\beta \notin \Phi_J^+$ if and only if $\gamma \notin \Phi_J^+$. By induction, this happens if and only if $\gamma \in \Phi^\sigma$. Therefore $\beta \in W^\sigma\Phi^\sigma \subset \Phi^\sigma$. □

3.4.4 Proof of Corollary 3.4.2

Retain the notation of section 3.4.1. Recalling (3.4.5), since σ, ι are injective it follows that

$$\dim_{\mathbb{C}} \mathfrak{a}_{w,J} = |(\Phi^+(\mathscr{C}') \smallsetminus \Phi_{J'}^+) \smallsetminus \Phi(\mathscr{C}')^\sigma|$$

$$\dim_{\mathbb{C}} \mathfrak{a}_w^\sigma = |(\Phi^+(\mathscr{C}') \smallsetminus \Phi_{J'}^+) \cap \Phi(\mathscr{C}')^\sigma| + \ell(\hat{s}).$$

The cardinalities of these sets are readily determined. Because $\dim_{\mathbb{C}} \mathfrak{a}_w = \ell(w)$, which is known by virtue of Propositions 3.2.3 and 3.2.4, it suffices to determine one of them.

Type D

Suppose $w = {}^{J_m}w_\circ s_{m+1}^{(j_1)} \cdots s_{m+n}^{(j_n)}$; then

$$(\Phi^+(D_{m+2}) \smallsetminus \Phi_{J_m}^+) \smallsetminus \Phi(D_{m+2})^\sigma = \{\alpha_{-1}, \alpha_{-1} + \alpha_1, \alpha_{-1} + \alpha_1 + \cdots + \alpha_m\},$$

whence

$$\dim_{\mathbb{C}} \mathfrak{a}_{w,J_r} = m + 1.$$

Now, since $\ell({}^{J_m}w_\circ) = \frac{1}{2}(m+1)(m+1)$ it follows that

$$\dim_{\mathbb{C}} \mathfrak{a}_w^\sigma = \frac{1}{2}m(m+1) + \sum_{i=1}^n j_i.$$

Type A

Suppose $w = {}^{J_{m,k}}w_\circ \hat{s}_{m_1}^{(j_1)} \cdots \hat{s}_{m+n}^{(j_n)}$; then

$$(\Phi^+(A_{2m+1}) \smallsetminus \Phi^+(J_{m,k})) \cap \Phi(A_{2m+1})^\sigma = \left\{\alpha_1 + \sum_{i=1}^t (\alpha_{2i} + \alpha_{2i+1}) \,\middle|\, k \leq t \leq m\right\},$$

whence

$$\dim_{\mathbb{C}} \mathfrak{a}_w^\sigma = k - m + 1 + 2\sum_{i=1}^n j_i.$$

Now, since $\ell({}^{J_{m,k}}w_\circ) = (m+1)^2 - k^2$, it follows that

$$\dim_{\mathbb{C}} \mathfrak{a}_{w,J_{r,k}} = m(m+1) - k(k-1) + 1.$$

3.5 Exceptional types

The exceptional cases E_6 (2.3.9) and D_4 under the triality (2.3.5) are treated by hand in in section 3.5, where we exhibit the normal forms, directly check that Theorem 3.4.1 holds, and calculate the dimensions of $\mathfrak{a}_{w,J}$ and $\mathfrak{a}_w^\sigma$ for each $w \in {}^JW_\sigma$.

3.5.1 Type D_4 with the triality

In this case ${}^JW_\sigma = \{s_4s_2s_3s_1, s_4s_2s_3s_1s_2\}$ and

$$\Phi^+(s_4s_2s_3s_1) \smallsetminus \Phi_J^+ = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$$

$$\Phi^+(s_4s_2s_3s_1s_2) \smallsetminus \Phi_J^+ = \{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}$$

while ${}^JW \smallsetminus {}^JW_\sigma = \{s_4, s_4s_2, s_4s_2s_1, s_4s_2s_3, s_4s_2s_3s_1s_2s_4\}$ and

$$\Phi^+(s_4) \smallsetminus \Phi_J^+ = \{\alpha_4\}$$

$$\Phi^+(s_4s_2) \smallsetminus \Phi_J^+ = \{\alpha_2 + \alpha_4\}$$

$$\Phi^+(s_4s_2s_1) \smallsetminus \Phi_J^+ = \{\alpha_1 + \alpha_2 + \alpha_4\}$$

$$\Phi^+(s_4s_2s_3) \smallsetminus \Phi_J^+ = \{\alpha_2 + \alpha_3 + \alpha_4\}$$

$$\Phi^+(s_4s_2s_3s_1s_2) \smallsetminus \Phi_J^+ = \{\alpha_2 + \alpha_3 + \alpha_4\}$$

thus condition (3.4.1) holds.

3.5.2 Type E_6

In this case ${}^J W_\sigma = \{s_6s_5s_4s_2s_3s_1, s_6s_5s_4s_2s_3s_1s_4, s_6s_5s_4s_2s_3s_1s_4s_3, s_6s_5s_4s_2s_3s_1s_4s_3s_5s_4s_2\}$ and

$$\Phi^+(s_6s_5s_4s_2s_3s_1) \smallsetminus \Phi_J^+ = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi^+(s_6s_5s_4s_2s_3s_1s_4) \smallsetminus \Phi_J^+ = \{\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi^+(s_6s_5s_4s_2s_3s_1s_4s_3) \smallsetminus \Phi_J^+ = \{\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6\}$$

$$\Phi^+(s_6s_5s_4s_2s_3s_1s_4s_3s_5s_4s_2) \smallsetminus \Phi_J^+ = \{\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6\}$$

while $\Phi^+(w) \smallsetminus \Phi_J^+$ for each of the 22 remaining w contains an asymmetric root supported on α_6. Thus condition (3.4.1) holds.

Chapter 4

Crystal folding

In this chapter $\mathscr{C}$ denotes a simply-laced Cartan datum with admissible automorphism σ. In section 4.1 we introduce the categories $\widehat{\mathrm{Crys}}(\mathscr{C},\sigma)$ of structurally σ-foldable crystals and $\mathrm{Crys}(\mathscr{C},\sigma)$ of σ-foldable crystals. Our first main results are the construction of functors $\widehat{F}_\sigma : \widehat{\mathrm{Crys}}(\mathscr{C},\sigma) \to \mathrm{Crys}(\mathscr{C}^{\sigma\vee})$ and $F_\sigma : \mathrm{Crys}(\mathscr{C},\sigma) \to \mathrm{Crys}(\mathscr{C}^{\sigma\vee})$ (Theorems 4.2.1 and 4.2.6). Our second main result (Theorem 4.3.1) is that $\widehat{\mathrm{Crys}}(\mathscr{C},\sigma)$ and $\mathrm{Crys}(\mathscr{C},\sigma)$ are monoidal categories under the tensor product, and furthermore that the functor $\widehat{F}_\sigma$ is monoidal (Theorem 4.3.2).

We consider the highest-weight elements of a foldable crystal in section 4.4, and show how folding a highest-weight crystal leads to a new type of crystal we call *multi-highest-weight*.

4.1 Category of σ-foldable crystals

Definition 4.1.1. The category of *structurally σ-foldable* $\mathscr{C}$-crystals, denoted $\widehat{\mathrm{Crys}}(\mathscr{C},\sigma)$, is the full subcategory of $\mathrm{Crys}(\mathscr{C}$ whose objects satisfy:

1. The pairs f_i, f_j and e_i, e_j commute, as operators on B, when $j \in \langle\sigma\rangle i$
2. $\varepsilon_i(e_j^n b) = \varepsilon_i(b)$ for all distinct $i, j \in I$ with $j \in \langle\sigma\rangle i$.

Definition 4.1.2. Let $(B, f_i, e_i, \mathrm{wt}, \varepsilon_i)$ be a $\mathscr{C}$-crystal. We say that B is *σ-foldable* if it is structurally foldable there is an action $\sigma : B \to B, \sigma(0) = 0$ such that the following diagrams commute

$$\begin{tikzcd} B \arrow[r, "\sigma"] \arrow[d, "\mathrm{wt}"'] & B \arrow[d, "\mathrm{wt}"] \\ \Lambda \arrow[r, "\sigma"'] & \Lambda \end{tikzcd} \tag{4.1.1}$$

$$\begin{tikzcd} B \arrow[rr, "\sigma"] \arrow[dr, "\varepsilon_i"'] & & B \arrow[dl, "\varepsilon_{\sigma(i)}"] \\ & \mathbb{Z} \cup \{-\infty\} & \end{tikzcd} \tag{4.1.2}$$

$$\begin{tikzcd} B \sqcup \{0\} \arrow[r, "\sigma"] \arrow[d, "x_i"'] & B \sqcup \{0\} \arrow[d, "x_{\sigma(i)}"] \\ B \sqcup \{0\} \arrow[r, "\sigma"'] & B \sqcup \{0\} \end{tikzcd} \tag{4.1.3}$$

for $x \in \{e, f\}$

A *σ-foldable morphism* between σ-foldable $\mathscr{C}$-crystals is a $\mathscr{C}$-crystal morphism $\psi : B \to B'$ commuting with the σ-actions on B and B'.

When B is σ-foldable it follows from Definition 1.3.1(4) and (4.1.2) that $\sigma \circ \varphi_i = \varphi_{\sigma(i)} \circ \sigma$. It is clear that the identity morphism is σ-foldable, and that the composite of two σ-foldable $\mathscr{C}$-crystal morphisms is a σ-foldable $\mathscr{C}$-crystal morphism. The associativity of the composition operation applied to σ-foldable $\mathscr{C}$-crystal morphisms is inherited from Crys($\mathscr{C}$). Hence the following definition.

Definition 4.1.3. The *category of σ-foldable crystals* and σ-foldable crystal morphisms is denoted Crys($\mathscr{C}, \sigma$).

The following proposition is evident.

Proposition 4.1.4. *Crys($\mathscr{C}, \sigma$) is closed under direct sums.*

Example 4.1.5. Crystals for representations of $U_q(\mathfrak{so}_6)$ (type D_3). Here $\sigma = (2, 3)$.

1. Spin crystals are structurally foldable but not foldable; operators 2 and 3 commute (trivially), but the graphs are not stable under the automorphism. In fact, they are conjugate to one another under σ.

$$\mathbf{B}_{\mathrm{sp}}^{+} : \ (+,+,+) \xrightarrow{3} (+,-,-) \xrightarrow{1} (-,+,-) \xrightarrow{2} (-,-,+)$$

$$\mathbf{B}_{\mathrm{sp}}^{-} : \ (+,+,-) \xrightarrow{2} (+,-,+) \xrightarrow{1} (-,+,+) \xrightarrow{3} (-,-,-)$$

2. Vector representation crystal (also known as the crystal of letters) is foldable. The action is given by $\sigma(\boxed{3}) = \boxed{\overline{3}}$ otherwise $\sigma(\boxed{i}) = \boxed{i}$.

$$\begin{tikzcd}
& & \boxed{3} \arrow[dr, "3"] & & \\
\boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[ur, "2"] \arrow[dr, "3"'] & & \boxed{\overline{2}} \arrow[r, "1"] & \boxed{\overline{1}} \\
& & \boxed{\overline{3}} \arrow[ur, "2"'] & &
\end{tikzcd}$$

4.2 The folded crystal

The functor $\widehat{F}_\sigma$ folds the crystal structure of B, and yields a $\mathscr{C}^{\sigma\vee}$ structure on a certain subset—which is the largest subset on which such a structure can be placed.

When B is structurally σ-foldable the orbital products

$$\widehat{y}_{\mathbf{i}} := \prod_{i \in \mathbf{i}} y_i, \qquad y \in \{e, f\}, \mathbf{i} \in I/\sigma \tag{4.2.1}$$

are well-defined by virtue of Definition 4.1.2(1). Denote

$$\widehat{B}_\sigma := \mathrm{wt}^{-1}[\widehat{\Lambda}^\sigma].$$

and let $\widehat{\mathcal{F}}^\sigma, \widehat{\mathcal{E}}^\sigma, \widehat{\mathcal{A}}^\sigma$ denote the monoids generated respectively by $\{\hat{f}_{\mathbf{i}} \,|\, i \in I/\sigma\}, \{\hat{e}_{\mathbf{i}} \,|\, i \in I/\sigma\}, \{\hat{f}_{\mathbf{i}}, \hat{e}_{\mathbf{i}} \,|\, i \in I/\sigma\}$. It is clear that $\widehat{\mathcal{A}}^\sigma \widehat{B}_\sigma \subset \widehat{B}_\sigma \sqcup \{0\}$. Given $b \in \widehat{B}_\sigma$ and $\mathbf{i} \in I/\sigma$ define

$$\widehat{\varepsilon}_{\mathbf{i}}(b) := \min\{\varepsilon_i(b) \,|\, i \in \mathbf{i}\}. \tag{4.2.2}$$

Let ω_σ denote the canonical $\mathbb{Z}$-lattice isomorphism $\omega_\sigma : \widehat{\Lambda}^\sigma \cong \Lambda^{\vee\sigma}$, given by extending $\widehat{\varpi}_{\mathbf{i}} \longmapsto \alpha_{\mathbf{i}}^\vee$.

Theorem 4.2.1. *Let $(B, f_i, e_i, wt, \varepsilon_i)$ be a structurally σ-foldable $\mathscr{C}$-crystal. Then the set $\widehat{B}_\sigma$ is a $\mathscr{C}^{\sigma\vee}$-crystal under*

$$y_{\mathbf{i}} := \widehat{y}_{\mathbf{i}} \text{ for } y \in \{e, f\} \tag{4.2.3}$$

$$wt^\sigma := \omega_\sigma \circ wt \tag{4.2.4}$$

$$\varepsilon_{\mathbf{i}} := \widehat{\varepsilon}_{\mathbf{i}} \tag{4.2.5}$$

where $\mathbf{i} \in I/\sigma$. Suppose that B, B' are σ-foldable $\mathscr{C}$-crystals and $\psi : B \to B'$ is a $\mathscr{C}$-crystal morphism. Then, the restriction $\widehat{\psi}_\sigma$ of ψ to $\widehat{B}_\sigma$ is a $\mathscr{C}^{\sigma\vee}$-crystal morphism $\widehat{\psi}_\sigma : \widehat{B}_\sigma \to \widehat{B}'_\sigma$.

Proof. We verify the conditions of Definition 1.3.1.

1. That $\widehat{f}_{\mathbf{i}}b = b'$ if and only if $\widehat{e}_{\mathbf{i}}b' = b$ follows from repeated application of 1.3.1(1) for B.

2. (a) By definition, $\mathrm{wt}^{\sigma}(\widehat{e}_{\mathbf{i}}b) = \omega_{\sigma} \circ \mathrm{wt}((\prod_{i \in \mathbf{i}} e_i)b)$. Repeated application of Definition 1.3.1(2a) for B gives $\mathrm{wt}^{\sigma}(\widehat{e}_{\mathbf{i}}b) = \omega_{\sigma}(\mathrm{wt}(b) + \widehat{\alpha}_{\mathbf{i}}) = \mathrm{wt}^{\sigma}(b) + \alpha_{\mathbf{i}}$.

   (b) By definition, $\widehat{\varepsilon}_{\mathbf{i}}(\widehat{e}_{\mathbf{i}}b) = \min\{\varepsilon_i(\widehat{e}_{\mathbf{i}}b) \,|\, i \in \mathbf{i}\}$. For a fixed $i \in \mathbf{i}$ we have $\varepsilon_i(\widehat{e}_{\mathbf{i}}b) = \varepsilon_i(e_i b) = \varepsilon_i(b) + 1$ by Definitions 4.1.2(2) and 1.3.1(2b) for B. Thus, $\widehat{\varepsilon}_{\mathbf{i}}(\widehat{e}_{\mathbf{i}}b) = \min\{\varepsilon_i(b) + 1 \,|\, i \in \mathbf{i}\} = \widehat{\varepsilon}_{\mathbf{i}}(b) + 1$.

3. The proofs of conditions (2) and (3) are the same, *mutatis mutandis*.

4. If $\widehat{\varepsilon}_{\mathbf{i}}(b) = -\infty$, then $\varepsilon_i(b) = -\infty$ for some $i \in \mathbf{i}$. Therefore $\widehat{e}_{\mathbf{i}}b = 0$ by Definition 1.3.1(4) for B.

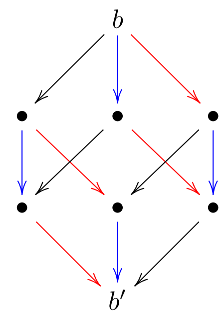
In total, $\widehat{B}_{\sigma}$ is a $\mathscr{C}^{\sigma\vee}$-crystal. $\square$

Proposition 4.2.2. *The functions $B \longmapsto \widehat{B}_{\sigma}$ and $\psi \longmapsto \widehat{\psi}_{\sigma}$ define an essentially surjective, full functor $\widehat{F}_{\sigma} : \widehat{Crys}(\mathscr{C}, \sigma) \to Crys(\mathscr{C}^{\sigma\vee})$.*

Proof. First we prove that $\widehat{F}_{\sigma}$ is essentially surjective. Select an arbitrary $B \in \mathrm{Crys}(\mathscr{C}, \sigma)$; we shall construct a structurally foldable $\mathscr{C}$-crystal $\overline{B}_{\sigma}$ such that $\widehat{F}_{\sigma}(\overline{B}_{\sigma}) \cong B$. Let $\mathcal{G}(B)$ denote the underlying graph of B; that is, the image of B under the forgetful functor $\mathrm{Crys}(\mathscr{C}^{\sigma\vee}) \to \mathrm{DiGraph}_I$ in the category of directed graphs with I-colored edges. The underlying graph of $\overline{B}_{\sigma}$ is obtained by applying the following procedure to each edge $b \xrightarrow{\mathbf{i}} b'$ in $\mathcal{G}(B)$: Insert $(\#\mathbf{i})!$ vertices between b and b' in such a way that all $n!$ paths from b to b' are realized, with any two pair of edges commuting. For example, if $\mathbf{i} = \{i, j\}$ then that edge replaced by:

$$\begin{tikzcd}
 & b \arrow[dl, "i"'] \arrow[dr, "j"] & \\
\bullet \arrow[dr, "i"'] & & \bullet \arrow[dl, "j"] \\
 & b' &
\end{tikzcd}$$

or if $\mathbf{i} = \{i, j, k\}$ then that edge is replaced by:



This yields a directed graph with I-colored edges, with the property that $f_i f_j = f_j f_i$ when $j \in \langle\sigma\rangle i$. For each $b \in B$ we have $\omega_\sigma^{-1} \circ \mathrm{wt}(b) \in \Lambda(\mathscr{C})$, which gives a notion of weight for some vertices of $\overline{B}_\sigma$. Extend this to a grading of $\overline{B}_\sigma$ using the rule $x \xrightarrow{i} y \implies \mathrm{wt}(x) - \mathrm{wt}(y) = \alpha_i$. An arbitrary vertex in the orbit expansion of $b \xrightarrow{\mathbf{i}} b'$ has the form $x = f_{i_1} \cdots f_{i_n} b$ for some $\{i_1 \ldots, i_n\} \subset \mathbf{i}$. For each $k \in I$ put $\varepsilon_k(x) := \varepsilon_{\mathbf{i}}(b) - 1$ if $j \in \{i_1, \ldots, i_n\}$ or $\varepsilon_k(x) := \varepsilon_{\mathbf{i}}(b)$ otherwise. It is clear that these definitions make $(\overline{B}_\sigma, \mathrm{wt}, \varepsilon_i, e_i, f_i)$ a $\mathscr{C}$-crystal, and that $\widehat{F}_\sigma(\overline{B}_\sigma) \cong B$.

Now we prove that $\widehat{F}_\sigma$ is full. Suppose that B and B' are structurally foldable, and select an arbitrary $\mathscr{C}^{\sigma\vee}$-crystal morphism $\widehat{\psi} : \widehat{B}_\sigma \to \widehat{B}'_\sigma$; we shall construct a $\mathscr{C}$-crystal morphism $\psi : B \to B'$ such that $\widehat{F}_\sigma(\psi) = \widehat{\psi}$. That is to say, we will indicate how $\widehat{\psi}$ can be extended to a $\mathscr{C}$-crystal morphism. If $\widehat{\psi}(b) \neq 0$, then each edge $b \xrightarrow{\mathbf{i}} b'$ in is carried by $\widehat{\psi}$ to an edge $\widehat{\psi}(b) \xrightarrow{\mathbf{i}} \widehat{\psi}(b')$ in B'. Define an extension $\overline{\psi} : \overline{B}_\sigma \to \overline{B}'_\sigma$ of $\widehat{\psi}$ as follows: Select an arbitrary vertex $x = f_{i_1} \cdots f_{i_n} b$ in the orbit expansion of $b \xrightarrow{\mathbf{i}} b'$, and declare $\overline{\psi}(x) := f_{i_1} \cdots f_{i_n} \widehat{\psi}(b)$ (this is well-defined by the structural foldability of $\overline{B}_\sigma$). It is clear that $\overline{\psi}$ is a $\mathscr{C}$-crystal morphism. Now extend $\overline{\psi}$ to a $\mathscr{C}$-crystal morphism $\psi : B \to B'$ by declaring $\psi[B \smallsetminus \overline{B}_\sigma] := \{0\}$. $\square$

Remark 4.2.3. Suppose that B and B' are structurally σ-foldable, and $\psi : B \to B'$ is a $\mathscr{C}$-crystal morphism. The $\mathscr{C}$-crystal $\overline{B}_\sigma$ obtained by applying the Proposition to $\widehat{B}_\sigma$ is in general different from B. Similarly, the morphism obtained by applying the Proposition to $\widehat{\psi}_\sigma : \widehat{B}_\sigma \to \widehat{B}'_\sigma$ is in general different from ψ. This fact precludes $\widehat{F}_\sigma$ from being faithful.

The functor F_σ folds the $\mathscr{C}$-crystal structure of B, and transfers this structure to the quotient of $\widehat{B}_\sigma$ by the σ-action. The following lemma ensures that the root operators and structural maps are well defined on the latter.

Lemma 4.2.4. *Suppose B is a σ-foldable $\mathscr{C}$-crystal. Then $\widehat{y}_{\mathbf{i}} \circ \sigma = \sigma \circ \widehat{y}_{\mathbf{i}}$ and $wt \circ \sigma = wt$ and $\widehat{\varepsilon}_{\mathbf{i}} \circ \sigma(b) = \widehat{\varepsilon}_{\mathbf{i}}$ on $\widehat{B}_\sigma$.*

Proof. Suppose $b, b' \in \widehat{B}_\sigma$ and $b' = \sigma(b)$. By definition wt and σ commute on $\widehat{B}_\sigma$. Condition (4.1.2) and the definition of $\widehat{\varepsilon}_{\mathbf{i}}$ imply that $\widehat{\varepsilon}_{\mathbf{i}}(b') = \widehat{\varepsilon}_{\mathbf{i}}(b)$. Assuming $\widehat{y}b \neq 0$, observe that $\sigma \circ \widehat{y}_{\mathbf{i}}(b) = \sigma \circ \prod_{i \in \mathbf{i}} y_i(b) = \prod_{i \in \mathbf{i}} y_{\sigma(i)} \circ \sigma(b)$ by (4.1.3). □

Definition 4.2.5. Let B_σ denote the set of σ-orbits in $\widehat{B}_\sigma$. An element of B_σ is variously expressed as $\mathbf{b}$ or $[b] := \langle \sigma \rangle b$ for $b \in \widehat{B}_\sigma$, and $\Phi_B^\sigma : \widehat{B}_\sigma \to B_\sigma$ denotes the projection map $\Phi_B^\sigma : b \longmapsto [b]$.

The compatibility of B's crystal structure with the σ-action allows the root operators and structural maps to descend to the quotient $\widehat{B}_\sigma \twoheadrightarrow B_\sigma$.

$$\begin{tikzcd}
\widehat{B}_\sigma \sqcup \{0\} \arrow[r, "\widehat{x}_{\mathbf{i}}"] \arrow[d, "\Phi_B^\sigma"'] & \widehat{B}_\sigma \sqcup \{0\} \arrow[d, "\Phi_B^\sigma"] \\
B_\sigma \sqcup \{0\} \arrow[r, dashed, "x_{\mathbf{i}}"'] & B_\sigma \sqcup \{0\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{B}_\sigma \arrow[r, "\mathrm{wt}"] \arrow[d, "\Phi_B^\sigma"'] & \widehat{\Lambda}^\sigma \arrow[d, "\omega_\sigma"] \\
B_\sigma \arrow[r, dashed, "\mathrm{wt}^\sigma"'] & \Lambda^{\vee\sigma}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{B}_\sigma \arrow[r, "\widehat{\varepsilon}_{\mathbf{i}}"] \arrow[d, "\Phi_B^\sigma"'] & \mathbb{Z} \sqcup \{-\infty\} \\
B_\sigma \arrow[ur, dashed, "\varepsilon_{\mathbf{i}}"'] &
\end{tikzcd}$$

This yields operators and structural maps which turn out to be a $\mathscr{C}^{\sigma\vee}$-crystal structure for B_σ.

Theorem 4.2.6. *The set B_σ is a $\mathscr{C}^{\sigma\vee}$-crystal under*

$$x_{\mathbf{i}} := \Phi_B^\sigma \circ \widehat{x}_{\mathbf{i}} \circ (\Phi_B^\sigma)^{-1} \text{ for } x \in \{e, f\} \tag{4.2.6}$$

$$wt^\sigma := \omega_\sigma \circ wt \circ (\Phi_B^\sigma)^{-1} \tag{4.2.7}$$

$$\varepsilon_{\mathbf{i}} := \widehat{\varepsilon}_{\mathbf{i}} \circ (\Phi_B^\sigma)^{-1} \tag{4.2.8}$$

where $\mathbf{i} \in I/\sigma$. A foldable $\mathscr{C}$-crystal morphism $\psi : B \to B'$ induces a $\mathscr{C}^{\sigma\vee}$-crystal morphism $\psi_\sigma : B_\sigma \to B'_\sigma$.

Proof. That the proposed definitions of $a_{\mathbf{i}}, \mathrm{wt}^\sigma, \varepsilon_{\mathbf{i}}$ are independent of the choice of representative $b \in (\Phi_B^\sigma)^{-1}(\mathbf{b})$ follows from Lemma 4.2.4.

Next the axioms of Definition 1.3.1 must be verified. Let $\mathbf{b}, \mathbf{b}' \in B_\sigma$ and $\mathbf{i} \in I/\sigma$.

1. First note that $\sigma \circ \widehat{x}_{\mathbf{i}} = \widehat{x}_{\mathbf{i}} \circ \sigma$ on $\widehat{B}_\sigma$. Now $f_{\mathbf{i}}[b] = [b']$ if and only if $[\widehat{f}_{\mathbf{i}} b] = [b']$. For the moment, take for granted that $[\widehat{f}_{\mathbf{i}} b] = [b']$ if and only if $[b] = [\widehat{e}_{\mathbf{i}} b']$. From this it follows immediately that $[b] = e_{\mathbf{i}}[b']$.

To check the claim, observe first that we are assuming $\{\sigma^n(\widehat{f}_{\mathbf{i}}b) \mid n \in \mathbb{Z}\} = \{\sigma^m(b') \mid m \in \mathbb{Z}\}$. But then for all n there exists m such that $\sigma^n(\widehat{f}_{\mathbf{i}}b) = \widehat{f}_{\mathbf{i}}(\sigma^n(b)) = \sigma^m(b')$. This is true if and only if $\sigma^n(b) = \widehat{e}_{\mathbf{i}}(\sigma^m(b'))$ by Definition 1.3.1(3), thus $\sigma^n(b) = \sigma^m(\widehat{e}_{\mathbf{i}}b)$. Therefore $\{\sigma^n(b) \mid n \in \mathbb{Z}\} = \{\sigma^m(\widehat{e}_{\mathbf{i}}b') \mid m \in \mathbb{Z}\}$.

2. First observe that (4.2.6) gives $e_{\mathbf{i}}[b] = [\widehat{e}_{\mathbf{i}}b]$. Now $\mathrm{wt}^\sigma(e_{\mathbf{i}}[b]) = \mathrm{wt}^\sigma(\widehat{e}_{\mathbf{i}}b)$ by (4.2.7). Next $\mathrm{wt}^\sigma(\widehat{e}_{\mathbf{i}}b) = \mathrm{wt}^\sigma b + \alpha_{\mathbf{i}}$ by Definition 1.3.1(1). Finally $\mathrm{wt}^\sigma(e_{\mathbf{i}}[b]) = \mathrm{wt}^\sigma[b] + \alpha_{\mathbf{i}}$ by (4.2.7) again. Therefore $\mathrm{wt}^\sigma(e_{\mathbf{i}}[b]) = \mathrm{wt}^\sigma[b] + \alpha_{\mathbf{i}}$. Using (4.2.8) gives $\varepsilon_{\mathbf{i}}(e_{\mathbf{i}}[b]) = \widehat{\varepsilon}_{\mathbf{i}}(\widehat{e}_{\mathbf{i}}b)$ which in turn equals $\min\{\widehat{\varepsilon}_i(\widehat{e}_{\mathbf{i}}b) \mid i \in \mathbf{i}\}$ by definition. Definition 4.1.2(2) now gives $\varepsilon_{\mathbf{i}}(e_{\mathbf{i}}[b]) = \min\{\varepsilon_i(e_ib) \mid i \in \mathbf{i}\}$, which equals $\min\{\varepsilon_i(b) \mid i \in \mathbf{i}\} - 1$ according to Definition 1.3.1(1). Applying (4.2.8) again yields $\varepsilon_{\mathbf{i}}(e_{\mathbf{i}}[b]) = \varepsilon_{\mathbf{i}}([b]) - 1$ as desired.

3. The proofs of conditions (2) and (3) are identical, *mutatis mutandis.*

4. Follows immediately from Definition 1.3.1(4) and (4.2.6).

This completes the proof that $(B_\sigma, \mathrm{wt}^\sigma, \varepsilon_{\mathbf{i}}, e_{\mathbf{i}}, f_{\mathbf{i}})$ is a $\mathscr{C}^{\sigma\vee}$-crystal.

Let $\psi : B \to B'$ be a σ-foldable $\mathscr{C}$-crystal morphism. To begin, observe ψ maps $\widehat{B}_\sigma$ into $\widehat{B}'_\sigma$ because ψ commutes with wt. Note also that $\psi([b]) = [\psi(b)]$. Now define $\psi_\sigma : B_\sigma \to B'_\sigma$ by

$$\psi_\sigma := \Phi_{B'} \circ \psi \circ (\Phi^\sigma_B)^{-1}. \tag{4.2.9}$$

The conditions of Definition 1.3.2 must be verified.

(1a) Consider the following diagram

$$\begin{tikzcd}
 & & \Lambda^\sigma \arrow[ddd, "\omega"] \\
\widehat{B}_\sigma \arrow[urr, bend left, "\mathrm{wt}"] \arrow[r, "\psi"] \arrow[d, two heads, "\Phi^\sigma_B"'] & \widehat{B}'_\sigma \arrow[ur, "\mathrm{wt}"'] \arrow[d, two heads, "\Phi^\sigma_{B'}"] & \\
B_\sigma \arrow[r, "\psi_\sigma"'] \arrow[drr, bend right, "\mathrm{wt}^\sigma"'] & B'_\sigma \arrow[dr, "\mathrm{wt}^\sigma"] & \\
 & & \Lambda^{\sigma\vee}
\end{tikzcd}$$

We claim that $\mathrm{wt}^\sigma \circ \psi_\sigma = \mathrm{wt}^\sigma$. Calculating gives

$$\mathrm{wt}^\sigma \circ \psi_\sigma = \mathrm{wt}^\sigma \circ \Phi^\sigma_{B'} \circ \psi \circ (\Phi^\sigma_B)^{-1} = \omega \circ \mathrm{wt} \circ \psi \circ (\Phi^\sigma_B)^{-1} = \omega \circ \mathrm{wt} \circ (\Phi^\sigma_B)^{-1} = \mathrm{wt}^\sigma$$

by (4.2.9), (4.2.7), Definition 1.3.2(1a), (4.2.7) respectively, as desired.

(1b) Consider the following diagram

$$\begin{tikzcd}
\widehat{B}_\sigma \arrow[rrrr, "\psi"] \arrow[dr, "\Phi^\sigma_B"] \arrow[ddrr, bend right, "\widehat{\varepsilon}_{\mathbf{i}}"'] & & & & \widehat{B}'_\sigma \arrow[dl, "\Phi^\sigma_{B'}"'] \arrow[ddll, bend left, "\widehat{\varepsilon}'_{\mathbf{i}}"] \\
& B_\sigma \arrow[rr, "\psi_\sigma"'] \arrow[dr, "\varepsilon_{\mathbf{i}}"'] & & B'_\sigma \arrow[dl, "\varepsilon'_{\mathbf{i}}"] & \\
& & \mathbb{Z} \sqcup \{-\infty\} & &
\end{tikzcd}$$

We claim that $\varepsilon'_{\mathbf{i}} \circ \psi_\sigma = \varepsilon_{\mathbf{i}}$. Calculating gives

$$\varepsilon'_{\mathbf{i}} \circ \psi_\sigma = \varepsilon'_{\mathbf{i}} \circ \Phi^\sigma_{B'} \circ \psi \circ (\Phi^\sigma_B)^{-1} = \widehat{\varepsilon}'_{\mathbf{i}} \circ \psi \circ (\Phi^\sigma_B)^{-1} = \widehat{\varepsilon}_{\mathbf{i}} \circ (\Phi^\sigma_B)^{-1} = \varepsilon_{\mathbf{i}}$$

by (4.2.9), (4.2.8), and Definition 1.3.2(1b) respectively, as desired.

(2) Let $y \in \{e, f\}$, we claim that $\psi_\sigma \circ a_{\mathbf{i}} = a_{\mathbf{i}} \circ \psi_\sigma$ on $\{\mathbf{b} \in B_\sigma \mid a_{\mathbf{i}}\mathbf{b} \neq 0\}$. Consider the following diagram

$$\begin{tikzcd}
\widehat{B}_\sigma \arrow[rr, "\psi"] \arrow[dr, "\widehat{a}_{\mathbf{i}}"] \arrow[dd, two heads, "\Phi^\sigma_B"'] & & \widehat{B}'_\sigma \arrow[dr, "\widehat{a}_{\mathbf{i}}"] & \\
& \widehat{B}_\sigma \arrow[rr, "\psi"] & & \widehat{B}'_\sigma \arrow[dd, two heads, "\Phi^\sigma_{B'}"] \\
B_\sigma \arrow[rr, "\psi_\sigma"] \arrow[dr, "a_{\mathbf{i}}"'] & & B'_\sigma \arrow[dr, "a_{\mathbf{i}}"'] & \\
& B_\sigma \arrow[rr, "\psi_\sigma"'] & & B'_\sigma
\end{tikzcd}$$

Calculating gives

$$\psi_\sigma \circ a_{\mathbf{i}} = \Phi^\sigma_{B'} \circ \psi \circ \widehat{a}_{\mathbf{i}} \circ (\Phi^\sigma_B)^{-1} = \Phi^\sigma_{B'} \circ \widehat{a}_{\mathbf{i}} \circ \psi \circ (\Phi^\sigma_B)^{-1} = a_{\mathbf{i}} \circ \psi_\sigma$$

by (4.2.9) and (4.2.6), Definition 1.3.2(2,3), (4.2.9) and (4.2.6) respectively, as desired.

To prove that we have a functor we must show the following

1. For all σ-foldable $\mathscr{C}$-crystals B, we have $(1_B)_\sigma = 1_{B_\sigma}$.

   By (4.2.9), $(1_B)_\sigma = \Phi^\sigma_B \circ 1_B \circ (\Phi^\sigma_B)^{-1}$. Given $\mathbf{b} \in B_\sigma$ and $b \in \mathbf{b}$ arbitrary, we have $(1_B)_\sigma(\mathbf{b}) = \Phi^\sigma_B \circ 1_B \circ (\Phi^\sigma_B)^{-1}(\mathbf{b}) = \Phi^\sigma_B \circ 1_B(b) = \Phi^\sigma_B(b) = \mathbf{b}$. Therefore $(1_B)_\sigma = 1_{B_\sigma}$.

2. For all σ-foldable $\mathscr{C}$-crystal morphisms $\psi : B \to B'$ and $\psi' : B' \to B''$, we have $(\psi' \circ \psi)_\sigma = \psi'_\sigma \circ \psi_\sigma$.

   Observe that $\psi'_\sigma \circ \psi_\sigma = \Phi^\sigma_{B''} \circ \psi' \circ \Phi^{-1}_{B'} \circ \Phi^\sigma_{B'} \circ \psi \circ (\Phi^\sigma_B)^{-1} = \Phi^\sigma_{B''} \circ \psi' \circ \psi \circ (\Phi^\sigma_B)^{-1}$. By (4.2.9), the latter coincides with $(\psi' \circ \psi)_\sigma$. $\square$

Proposition 4.2.7. *The functions $B \longmapsto B_\sigma$ and $\psi \longmapsto \psi_\sigma$ define an essentially surjective, full functor $F_\sigma : Crys(\mathscr{C}, \sigma) \to Crys(\mathscr{C}^{\sigma\vee})$.*

Proof. Equip the structurally foldable $\mathscr{C}$-crystal $\overline{B}_\sigma$ constructed in the proof of Proposition 4.2.2 with the trivial σ-action. $\square$

4.3 Monoidal category structure

Theorem 4.3.1. *The categories $\widehat{Crys}(\mathscr{C}, \sigma)$ of structurally σ-foldable crystals and $Crys(\mathscr{C}, \sigma)$ of σ-foldable crystals are monoidal categories under the tensor product.*

Proof. Because $\mathrm{Crys}(\mathscr{C})$ is a monoidal category, it suffices to prove that the tensor product $B \otimes B'$ of two σ-foldable $\mathscr{C}$-crystals B, B' is σ-foldable. It will follor from this that $\widehat{\mathrm{Crys}}(\mathscr{C}, \sigma)$ is a monoidal category under $\otimes$ as well.

Define an action $\sigma : B \otimes B' \to B \otimes B'$ by

$$\sigma(b \otimes b') := \sigma(b) \otimes \sigma(b'). \tag{4.3.1}$$

The diagrams and conditions of Definition 4.1.2 must be verified.

Condition (4.1.1) for $B \otimes B'$ follows from (4.1.1) and (2.1.3) for B, B', and (1.3.10). The calculation is as follows

$$\begin{aligned} \mathrm{wt} \circ \sigma(b \otimes b') &= \mathrm{wt}(\sigma(b) \otimes \sigma(b')) \\ &= \mathrm{wt} \circ \sigma(b) + \mathrm{wt} \circ \sigma(b') \\ &= \sigma \circ \mathrm{wt}(b) + \sigma \circ \mathrm{wt}(b') \\ &= \sigma(\mathrm{wt}(b) + \mathrm{wt}(b')) \\ &= \sigma \circ \mathrm{wt}(b \otimes b'). \end{aligned}$$

Condition (4.1.2) for $B \otimes B'$ follows from (1.3.11), (4.1.2) for B, B', (4.1.1) for B, and (2.1.4). The calculation is as follows

$$\begin{aligned}
\varepsilon_{\sigma(i)} \circ \sigma(b \otimes b') &= \max\{\varepsilon_{\sigma(i)} \circ \sigma(b), \varepsilon_{\sigma(i)} \circ \sigma(b') - \langle \alpha^\vee_{\sigma(i)}, \mathrm{wt} \circ \sigma(b)\rangle\} \\
&= \max\{\varepsilon_i(b), \varepsilon_i(b') - \langle \alpha^\vee_i, \mathrm{wt}(b)\rangle\} \\
&= \varepsilon_i(b \otimes b').
\end{aligned}$$

It suffices to check condition (4.1.3) for $B \otimes B'$ with $x = f$. This follows from (1.3.12), (4.1.3) for B, B', and (4.1.2) along with the corresponding equation for φ for B, B'. The calculation is as follows

$$\begin{aligned}
f_{\sigma(i)} \circ \sigma(b \otimes b') &= \begin{cases} f_{\sigma(i)} \circ \sigma(b) \otimes \sigma(b'), & \varphi_{\sigma(i)} \circ \sigma(b) > \varepsilon_{\sigma(i)} \circ \sigma(b') \\ \sigma(b) \otimes f_{\sigma(i)} \circ \sigma(b'), & \varphi_{\sigma(i)} \circ \sigma(b) \leq \varepsilon_{\sigma(i)} \circ \sigma(b') \end{cases} \\
&= \begin{cases} \sigma \circ f_i(b) \otimes \sigma(b'), & \varphi_i(b) > \varepsilon_i(b') \\ \sigma(b) \otimes \sigma \circ f_i(b'), & \varphi_i(b) \leq \varepsilon_i(b') \end{cases} \\
&= \sigma \circ f_i(b \otimes b').
\end{aligned}$$

The calculation for $x = e$ is similar.

For condition (1) select distinct $i, j \in \mathbf{i} \in I/\sigma$. Calculating gives

$$e_j e_i(b \otimes b') = \begin{cases} e_j e_i b \otimes b', & \varphi_j(e_i b) \geq \varepsilon_j(b') \text{ and } \varphi_i(b) \geq \varepsilon_i(b') \\ e_i b \otimes e_j b', & \varphi_j(e_i b) < \varepsilon_j(b') \text{ and } \varphi_i(b) \geq \varepsilon_i(b') \\ e_j b \otimes e_i b', & \varphi_j(b) \geq \varepsilon_j(e_i b') \text{ and } \varphi_i(b) < \varepsilon_i(b') \\ b \otimes e_i e_j b', & \varphi_j(b) < \varepsilon_j(e_i b') \text{ and } \varphi_i(b) < \varepsilon_i(b') \end{cases}$$

$$e_i e_j(b \otimes b') = \begin{cases} e_i e_j b \otimes b', & \varphi_i(e_j b) \geq \varepsilon_i(b') \text{ and } \varphi_j(b) \geq \varepsilon_j(b') \\ e_j b \otimes e_i b', & \varphi_i(e_j b) < \varepsilon_i(b') \text{ and } \varphi_j(b) \geq \varepsilon_j(b') \\ e_i b \otimes e_j b', & \varphi_i(b) \geq \varepsilon_i(e_j b') \text{ and } \varphi_j(b) < \varepsilon_j(b') \\ b \otimes e_j e_i b', & \varphi_i(b) < \varepsilon_i(e_j b') \text{ and } \varphi_j(b) < \varepsilon_j(b') \end{cases}$$

To conclude as desired we verify the following four statements

$$\varphi_j(e_ib) \geq \varepsilon_j(b') \text{ and } \varphi_i(b) \geq \varepsilon_i(b') \iff \varphi_i(e_jb) \geq \varepsilon_i(b') \text{ and } \varphi_j(b) \geq \varepsilon_j(b') \tag{4.3.2}$$

$$\varphi_j(e_ib) < \varepsilon_j(b') \text{ and } \varphi_i(b) \geq \varepsilon_i(b') \iff \varphi_i(b) \geq \varepsilon_i(e_jb') \text{ and } \varphi_j(b) < \varepsilon_j(b') \tag{4.3.3}$$

$$\varphi_j(b) \geq \varepsilon_j(e_ib') \text{ and } \varphi_i(b) < \varepsilon_i(b') \iff \varphi_i(e_jb) < \varepsilon_i(b') \text{ and } \varphi_j(b) \geq \varepsilon_j(b') \tag{4.3.4}$$

$$\varphi_j(b) < \varepsilon_j(e_ib') \text{ and } \varphi_i(b) < \varepsilon_i(b') \iff \varphi_i(b) < \varepsilon_i(e_jb') \text{ and } \varphi_j(b) < \varepsilon_j(b') \tag{4.3.5}$$

and appeal to condition (1) for B, B'. Note that

$$\varphi_i(e_jb) = \varphi_i(b) \tag{4.3.6}$$

by the definition of φ, Definition 4.1.2(2), Definition 1.3.1(2a), and (2.1.2). Statement (4.3.2) is a tautology. Statements (4.3.3), (4.3.4), (4.3.5) all follow from (4.3.6) and Definition 4.1.2(2). The calculation for $x = f$ is similar, and omitted.

For condition (2) it suffices to show that $\varepsilon_i \circ e_j(b \otimes b') = \varepsilon_i(b \otimes b')$. This follows from (1.3.13), (1.3.11), Definition (4.1.2)(2) for B, B', and (2.1.2). The calculation is as follows

$$\begin{aligned}
\varepsilon_i \circ e_j(b \otimes b') &= \begin{cases} \varepsilon_i(e_jb \otimes b'), & \varphi_j(b) \geq \varepsilon_j(b) \\ \varepsilon_i(b \otimes e_jb'), & \varphi_j(b) < \varepsilon_j(b) \end{cases} \\
&= \begin{cases} \max\{\varepsilon_i \circ e_j(b), \varepsilon_i(b') - \langle \alpha_i^\vee, \mathrm{wt} \circ e_j(b)\rangle\}, & \varphi_j(b) \geq \varepsilon_j(b) \\ \max\{\varepsilon_i(b), \varepsilon_i \circ e_j(b') - \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle\}, & \varphi_j(b) < \varepsilon_j(b) \end{cases} \\
&= \begin{cases} \max\{\varepsilon_i(b), \varepsilon_i(b') - \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle\}, & \varphi_j(b) \geq \varepsilon_j(b) \\ \max\{\varepsilon_i(b), \varepsilon_i(b') - \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle\}, & \varphi_j(b) < \varepsilon_j(b) \end{cases} \\
&= \varepsilon_i(b \otimes b')
\end{aligned}$$

This completes the proof. □

Theorem 4.3.2. *The functor $\widehat{F}_\sigma : Crys(\mathscr{C}, \sigma) \to Crys(\mathscr{C}^{\sigma\vee})$ of Theorem 4.2.1, together with*

$$F_2^\sigma(B, B') : \widehat{B}_\sigma \otimes \widehat{B}'_\sigma \to (\widehat{B \otimes B'})_\sigma, \quad b \otimes b' \longmapsto b \otimes b' \tag{4.3.7}$$

$$F_0^\sigma : T_0^{\sigma\vee} \to (\widehat{T}_0)_\sigma. \quad t_0^{\sigma\vee} \longmapsto t_0 \tag{4.3.8}$$

forms a monoidal functor.

Proof. We must show that three diagrams in $\mathrm{Crys}(\mathscr{C}^{\sigma\vee})$ commute. For diagram (1.1.7) with $X = B, Y = B', Z = B'', \square = \otimes, \alpha' = \alpha^{\sigma\vee}$ we must check that

$$F_2^\sigma(B, B' \otimes B'') \circ (1_B \otimes F_2^\sigma(B', B'')) \circ \alpha^{\sigma\vee} = \widehat{F}_\sigma(\alpha) \circ F_2^\sigma(B \otimes B', B'') \circ (F_2^\sigma(B, B') \otimes 1_{B''}).$$

We will omit the parenthetic indication of the dependence of F_2^σ for readability's sake. Letting $(b \otimes b') \otimes b'' \in (\widehat{B}_\sigma \otimes \widehat{B''}_\sigma) \otimes \widehat{B''}_\sigma$, we have

$$\begin{aligned}
F_2^\sigma \circ (1_B \otimes F_2^\sigma) \circ \alpha^{\sigma\vee}((b \otimes b') \otimes b'') &= F_2^\sigma \circ (1_B \otimes F_2^\sigma)(b \otimes (b' \otimes b'')) \\
&= F_2^\sigma(b \otimes (b' \otimes b'')) \\
&= (b \otimes (b' \otimes b''))
\end{aligned}$$

$$\begin{aligned}
\widehat{F}_\sigma(\alpha) \circ F_2^\sigma \circ (F_2^\sigma \otimes 1_{B''})((b \otimes b') \otimes b'') &= \widehat{F}_\sigma(\alpha) \circ F_2^\sigma((b \otimes b') \otimes b'') \\
&= \widehat{F}_\sigma(\alpha)((b \otimes b') \otimes b'') \\
&= (b \otimes (b' \otimes b'')).
\end{aligned}$$

For diagram (1.1.8) with $X = B, e = T_0, e' = T_0^{\sigma\vee}, \square = \otimes, \varrho' = \varrho^{\sigma\vee}$ we must verify

$$\rho^{\sigma\vee} = \widehat{\rho}_\sigma \circ F_\sigma^2(B, T_0) \circ (1 \otimes F_\sigma^0)$$

Letting $b \otimes t_0^{\sigma\vee} \in \widehat{B}_\sigma \otimes T_0^{\sigma\vee}$. we have $\rho^{\sigma\vee}(b \otimes t_0) = b$, and

$$\begin{aligned}
\widehat{\rho}_\sigma \circ F_\sigma^2(B, T_0) \circ (1 \otimes F_\sigma^0)(b \otimes t_0) &= \widehat{\rho}_\sigma \circ F_\sigma^2(B, T_0)(b \otimes t_0^{\sigma\vee}) \\
&= \widehat{\rho}_\sigma(b \otimes t_0^{\sigma\vee}) = b.
\end{aligned}$$

The calculation for the diagram involving $\lambda^{\sigma\vee}$ is nearly identical, and omitted. $\square$

4.4 Highest-weight elements of the folded crystal

Throughout this section we assume B is a σ-foldable $\mathscr{C}$-crystal with $\widehat{B}_\sigma$ and B_σ as in section 4.2. Recall that

$$\mathrm{HW}\ B_\sigma = \{\mathbf{b} \mid e_{\mathbf{i}}\mathbf{b} = 0 \text{ for all } \mathbf{i} \in I/\sigma\}.$$

Suppose that $\mathbf{b} \in \mathrm{HW}\ B_\sigma$, and choose a representative $b \in \Phi_B^{-1}(\mathbf{b})$. According to the definition (4.2.6) of the $e_{\mathbf{i}}$-action, for all $\mathbf{i} \in I/\sigma$ there exists $j = j(b, \mathbf{i}) \in \mathbf{i}$ such that $e_j b = 0$. Denote

$$J(b) := \{j \in I \mid e_j b = 0\}.$$

Definition 4.4.1. Let $\mathscr{C}$ be a Cartan datum with admissible automorphism $\sigma : I \to I$. The set of *highest-weight configurations* of I with respect to σ is

$$\mathrm{HW}_\sigma I := \{J \subset I \mid J \cap \mathbf{i} \neq \varnothing \text{ for all } \mathbf{i} \in I/\sigma\}.$$

A highest-weight configuration J is *minimal* in case $|J \cap \mathbf{i}| = 1$ for all $\mathbf{i}$, and *maximal proper* if $|J| = |I| - 1$. To each highest-weight configuration corresponds a *highest-weight subset* of $\widehat{B}_\sigma$

$$\mathrm{HW}_J \widehat{B}_\sigma := \{b \in \widehat{B}_\sigma \mid J(b) = J\}.$$

The following is evident.

Proposition 4.4.2. *The set of highest-weight configurations of I is partially ordered by set inclusion, as is the set of highest-weight subsets of $\widehat{B}_\sigma$, and the map $J \to HW_J \widehat{B}_\sigma$ is an order anti-isomorphism. Furthermore, $(HW_\sigma I, \subset)$ is a bounded join-semilattice under union and, dually, $(\{HW_J \widehat{B}_\sigma \mid J \in HW_\sigma I\}, \subset)$ is a bounded meet-semilattice under intersection with*

$$HW_J \widehat{B}_\sigma \cap HW_J \widehat{B}_\sigma = HW_{J \cup J'} \widehat{B}_\sigma.$$

Example 4.4.3. Highest-weight configurations for some cases of the D, A, T-series foldings.

1. The D series, labeled as in (2.3.1): For all r there are three highest-weight configurations for (D_{r+2}, C_{r+1}), namely $J_r, \sigma[J_r], I_r$. The Hasse diagram is

$$\begin{tikzcd} & I_r & \\ J_r \arrow[ur] & & \sigma[J_r] \arrow[ul] \end{tikzcd}$$

2. The A series, labeled as in (2.3.3): For all r there are $2r$ distinct maximal proper highest-weight

configurations, namely $J_{r,k}, \sigma[J_{r,k}] : 1 \leq k \leq r$ For (A_5, B_3) $(r = 2)$ the Hasse diagram is

$$\begin{tikzcd}
 & & & I_2 & & & \\
J_{2,2} \arrow[urrr] & & J_{2,1} \arrow[ur] & & \sigma[J_{2,1}] \arrow[ul] & & \sigma[J_{2,2}] \arrow[ulll] \\
J_{2,1} \cap J_{2,2} \arrow[u] \arrow[urr] & & \sigma[J_{2,1}] \cap J_{2,2} \arrow[ull] \arrow[urr] & & J_{2,1} \cap \sigma[J_{2,2}] \arrow[ull] \arrow[urr] & & \sigma[J_{2,1}] \cap \sigma[J_{2,2}] \arrow[ull] \arrow[u]
\end{tikzcd}$$

3. The Hasse diagram for (D_4, G_2) with I, σ as in (2.3.5) is

$$\begin{tikzcd}
 & \{1,2,3,4\} & \\
\{1,2,3\} \arrow[ur] & \{1,2,4\} \arrow[u] & \{2,3,4\} \arrow[ul] \\
\{1,2\} \arrow[u] \arrow[ur] & \{2,3\} \arrow[ul] \arrow[ur] & \{2,4\} \arrow[ul] \arrow[u]
\end{tikzcd}$$

Remark 4.4.4. It follows from foldability axiom (4.1.3) that $\sigma[\mathrm{HW}_J \widehat{B}_\sigma] = \mathrm{HW}_\sigma^{\sigma[J]} \widehat{B}_\sigma$. Let $\{J_1, \ldots, J_n\}$ be a complete set of representatives of the σ-orbits in $\mathrm{HW}_\sigma I$; we have

$$\mathrm{HW}\ \widehat{B}_\sigma = \bigcup_{i=1}^{n} \langle\sigma\rangle \mathrm{HW}_J \widehat{B}_\sigma.$$

To study an element $\mathbf{b} \in \mathrm{HW}\ B_\sigma$ it suffices to work with a single representative $b \in \Phi_B^{-1}(\mathbf{b})$. Now because $\{J(x) \,|\, x \in \mathbf{b}\} = \langle\sigma\rangle J(b)$ we need only consider highest-weight configurations modulo the σ-action. In summary, to analyze an arbitrary highest-weight element of B_σ it is enough to choose a representative J of each σ-orbit in $\mathrm{HW}_\sigma I$ and describe an arbitrary element of $\mathrm{HW}_J \widehat{B}_\sigma$.

Example 4.4.5. The poset $\left(\mathrm{HW}_J \widehat{B}_\sigma\right)/\sigma$ for some cases of the D, A, T-series foldings.

1. The D series, labeled as in (2.3.1):

$$\begin{tikzcd}
I_r \\
J_r \arrow[u]
\end{tikzcd}$$

2. For (A_5, B_3)

$$\begin{tikzcd}
& I_2 & \\
J_{2,2} \arrow[ur] & & J_{2,1} \arrow[ul] \\
J_{2,1} \cap J_{2,2} \arrow[u] \arrow[urr] & & \sigma[J_{2,1}] \cap J_{2,2} \arrow[ull]
\end{tikzcd}$$

3. The Hasse diagram for (D_4, G_2) with I, σ as in (2.3.5) is

$$\begin{tikzcd}
\{1,2,3,4\} \\
\{1,2,3\} \arrow[u] \\
\{1,2\} \arrow[u]
\end{tikzcd}$$

Remark 4.4.6. The highest-weight elements of $\mathrm{HW}_J\widehat{B}(\infty)_\sigma$ for $(\mathscr{C}, J) = (D_r, J_r), (A_{2r+1}, I_{r,k})$ are completely classified by the balanced parabolic quotient, see Theorem 5.3.1. In section 5.3.3 we provide a set of $\{1,2\}$-highest-weight elements for type D_4 under the triality which in a precise sense generates $\mathrm{HW}_{1,2}\widehat{B}(\infty)_\sigma$.

The following lemma severely restricts the weight of a highest-weight element in a normal crystal.

Lemma 4.4.7. *Let B be an upper-normal σ-foldable crystal with all φ_i non-negative and take $b \in HW\ \widehat{B}_\sigma$. Denoting $J(b)^\sigma := \{j \in J(b) \,|\, \langle\sigma\rangle j \subset J(b)\}$, we have*

$$wt(b) \subset \Lambda^+(J(b)^\sigma) \tag{4.4.1}$$

Proof. That

$$\langle \mathrm{wt}(b), \alpha_i^\vee \rangle \quad \text{is} \quad \begin{cases} \geq 0, & \langle\sigma\rangle i \subset J(b) \\ = 0, & \langle\sigma\rangle i \not\subset J(b) \end{cases}$$

follows immediately from (1.3.7), (1.3.8), and the fact that φ_i is non-negative. $\square$

Consider the D-seires folding. The subdiagram indexed by $I_r^\sigma = \{1, \cdots, r\}$ is isomorphic to the Dynkin diagram of type A_r. Identifying the type A_r weight lattice with its image under the natural $\mathbb{Z}$-module embedding $\Lambda(A_r) \hookrightarrow \Lambda(D_{r+2})$, we have the following:

Corollary 4.4.8. *Let B be a σ-foldable D_{r+2}-crystal. Then*

$$wt[HW_{J_r}\widehat{B}_\sigma] \subset \Lambda^+(A_r)$$

The cyclicity phenomenon for foldable highest-weight crystals

Take B to be a σ-foldable highest-weight $\mathscr{C}$-crystal of highest weight $\hat{\lambda} \in \widehat{\Lambda}^{\sigma+} \cup \infty$. In particular $B = \mathcal{F}b_{\hat{\lambda}}$ where $b_{\hat{\lambda}}$ denotes the unique element of weight $\hat{\lambda}$. That is to say *the $\mathscr{C}$-crystal $B(\hat{\lambda})$ is generated by $b_{\hat{\lambda}}$ over the monoid $\mathcal{F}$*; in this sense B is cyclic on b_λ. Remark $\mathrm{wt}[B] \subset \hat{\lambda} - \mathbb{N}\Pi$.

Define the weight $\mathrm{wt} : \mathcal{F} \to Q$ by $f = f_{i_1} \cdots f_{i_n} \longmapsto -\sum_{k=1}^{n} \alpha_{i_k}$, and put $\widehat{\mathcal{F}}_\sigma := \mathrm{wt}^{-1}[Q^\sigma]$. Now let $\mathcal{F}_\sigma := \widehat{\mathcal{F}}_\sigma / \langle \sigma \rangle$. It is clear that $\widehat{B}_\sigma = \widehat{\mathcal{F}}_\sigma b_\lambda$, and hence

$$B_\sigma = \mathcal{F}_\sigma[b_{\hat{\lambda}}],$$

$$\mathrm{wt}[B_\sigma] \subset \lambda - \mathbb{N}\Pi^{\sigma\vee}.$$

where $\lambda = \omega(\hat{\lambda}) \in \Lambda^{\sigma\vee}$. Given $\xi = \sum_{\mathbf{i} \in I/\sigma} \xi(\mathbf{i})\alpha_{\mathbf{i}} \in Q^{\sigma\vee}$, define the height $|\xi| := \sum_{\mathbf{i}} |\xi(\mathbf{i})|$. For a non-negative integer k let $B_{\sigma,k} := \{\mathbf{b} \in B_\sigma \,|\, |\mathrm{wt}(\mathbf{b}) - \lambda| = k\}$. Definition 1.3.1(2a) indicates

$$e_{\mathbf{i}} B_{\sigma,k} \subset B_{\sigma,k-1} \sqcup \{0\} \tag{4.4.2}$$

for each $\mathbf{i}$.

Let C be a connected component of B_σ (more precisely, of $\mathcal{G}B_\sigma$), and put HW $C = C \cap$ HW B_σ. For a non-negative integer k put $C_k := C \cap B_{\sigma,k}$, and let $k_0 = k_0(C) := \min\{k \geq 0 \,|\, C_k \neq \varnothing\}$. By design C is the disjoint union of $\{C_k \,|\, k \geq k_0\}$.

Theorem 4.4.9. *Let B to be a σ-foldable highest-weight $\mathscr{C}$-crystal. Then each connected component of B_σ is generated by the set of its highest-weight elements over the monoid $\mathcal{F}^{\sigma\vee}$.*

Proof. We must show that for all $\mathbf{b} \in C$ there exist $\mathbf{x} \in$ HW C and $\mathbf{f} \in \mathcal{F}^{\sigma\vee}$ such that $\mathbf{b} = \mathbf{f}\mathbf{x}$. Proceed by induction on $k \in \{k_0, k_0+1, \cdots\}$.

Arbitrarily select $\mathbf{b} \in C_{k_0}$. If $\mathbf{b}$ is highest-weight then there is nothing to prove. If there exists $\mathbf{i}$ such that $e_{\mathbf{i}}\mathbf{b} \neq \varnothing$, then this element belongs to C by connectedness. But then $e_{\mathbf{i}}\mathbf{b} \in C_{k_0-1}$, in contradiction to the definition of k_0. Hence $C_{k_0} \subset$ HW C and, taking $\mathbf{x} = \mathbf{b}$ and $\mathbf{f} = \mathbf{1}$, induction begins.

For induction we may assume that a given $\mathbf{b}$ in a given C_k with $k > k_0$ is not highest-weight. If $e_{\mathbf{i}}\mathbf{b} \neq 0$ then, as before, $\mathbf{b} \in C_{k-1}$. By induction there exist $\mathbf{x} \in \mathrm{HW}\ C, \mathbf{f} \in \mathcal{F}^{\sigma\vee}$ such that $e_{\mathbf{i}}\mathbf{b} = \mathbf{fx}$. Now $\mathbf{b} = \mathbf{f_i fx}$ by Definition 1.3.1(1). $\square$

A σ-foldable highest-weight $\mathscr{C}$-crystal B folds to the $\mathscr{C}^{\sigma\vee}$-crystal B_σ, which decomposes as

$$B_\sigma = \bigoplus_{C \leq B} C, \quad C = \mathcal{F}^{\sigma\vee}(\mathrm{HW}\ C). \tag{4.4.3}$$

The task of describing the crystal structure of B_σ reduces to describing the crystal structure of B_σ's subcrystals. For a KL crystal $B = B(\lambda)$ with $\sigma(\lambda) = \lambda$ when $\lambda \in \Lambda^+$, by a result of Lusztig [Lus93] (see also section 4.5.3) the subcrystal $B(\lambda)^\sigma$ is known to be isomorphic with the $\mathscr{C}^{\sigma\vee}$-crystal $B(\omega(\lambda))$. The other subcrystals of $B(\lambda)_\sigma$ have never been considered in the literature.

Example 4.4.10. For the pair $(\mathscr{C}, \mathscr{C}^{\sigma\vee}) = (D_4, C_3)$, consider the Kashiwara-Littelmann crystal $B(\widehat{\varpi}_1 + \widehat{\varpi}_3)$. We calculate that $\widehat{B}(\widehat{\varpi}_1 + \widehat{\varpi}_3)_\sigma$ contains five highest-weight elements $b_\infty, x_1, x_2, x_3, x_4$, and further that $x_2 = \sigma(x_1), x_4 = \sigma(x_3)$. Thus

$$\widehat{B}(\widehat{\varpi}_1 + \widehat{\varpi}_3)_\sigma = \widehat{\mathcal{F}}^\sigma\{x_1, x_3\} \sqcup \widehat{\mathcal{F}}^\sigma\{x_2, x_4\} \sqcup \widehat{B}(\widehat{\varpi}_1 + \widehat{\varpi}_3)^\sigma$$

as a set. Furthermore, denoting $J = \{1, 2, 3\}$ we have $\mathrm{HW}_J \widehat{B}(\widehat{\varpi}_1 + \widehat{\varpi}_3)_\sigma = \{x_1, x_3\}$. This of course implies that $\sigma[\widehat{\mathcal{F}}^\sigma\{x_1, x_3\}] = \widehat{\mathcal{F}}^\sigma\{x_2, x_4\}$. Passing to the quotient, we have

$$B(\widehat{\varpi}_1 + \widehat{\varpi}_3)_\sigma \cong \mathcal{F}^{\sigma\vee}\{[x_1], [x_3]\} \oplus B(\varpi_1 + \varpi_3) \in \mathrm{Crys}(C_3)$$

by Lusztig's characterization of the invariant subcrystal (section 4.5.3). The crystal graph of $\mathcal{F}^{\sigma\vee}\{[x_1], [x_3]\}$ is shown in Figure 0.2.

4.5 Miscellaneous results

4.5.1 Transfer of normality

Proposition 4.5.1. *If B is an upper (lower) normal σ-foldable $\mathscr{C}$-crystal, then B_σ is an upper (lower) normal $\mathscr{C}^{\sigma\vee}$-crystal.*

Proof. Assume B is upper-normal and arbitrarily select $\mathbf{b} \in B_\sigma$ and $\mathbf{i} \in I/\sigma$. If $k \leq \varepsilon_{\mathbf{i}}(\mathbf{b})$ then $k \leq \varepsilon_i(b)$ for all $(i, b) \in \mathbf{i} \times \mathbf{b}$ by (4.2.2) and (4.2.8). Now $e_i^k b \neq 0$ for all $(i, b) \in \mathbf{i} \times \mathbf{b}$ by (5.1.2) for

B, and it follows that $e_{\mathbf{i}}^k \mathbf{b} \neq 0$. If instead $k > \varepsilon_{\mathbf{i}}(\mathbf{b})$ then $k > \varepsilon_i(b)$ for some $(i, b) \in \mathbf{i} \times \mathbf{b}$, again by (4.2.2) and (4.2.8). But now $e_i^k b = 0$, and it follows that $e_{\mathbf{i}}^k \mathbf{b} = 0$. The proof for lower normality is the same, *mutatis mutandis*, and is omitted. $\square$

4.5.2 Weyl group action

When B is normal it admits a unique action $W \to \mathrm{Aut}_{\mathrm{Set}}(B)$ of the Weyl group $W(\mathscr{C})$ given by [Kas94]

$$s_i.b := \begin{cases} f_i^{\langle \mathrm{wt}\ b, \alpha_i^\vee \rangle} b & \langle \mathrm{wt}\ b, \alpha_i^\vee \rangle \geq 0 \\ e_i^{-\langle \mathrm{wt}\ b, \alpha_i^\vee \rangle} b & \langle \mathrm{wt}\ b, \alpha_i^\vee \rangle \leq 0 \end{cases}$$

and satisfying

$$\mathrm{wt}(w.b) = w(\mathrm{wt}\ b), \text{ all } w \in W, b \in B.$$

In effect the action of s_i on b reverses the i-string through b.

Given $\mathbf{i} \in I/\sigma$ define $\widehat{s}_{\mathbf{i}} := \prod_{i \in \mathbf{i}} s_i$. Let $W^\sigma := \langle \widehat{s}_{\mathbf{i}} \,|\, \mathbf{i} \in I/\sigma \rangle$. The following lemma is evident.

Lemma 4.5.2. *For all $b \in \widehat{B}_\sigma$ we have*

$$\widehat{s}_{\mathbf{i}}.b = \begin{cases} \widehat{f}_{\mathbf{i}}^{\langle wt\ b, \widehat{\alpha}_{\mathbf{i}}^\vee \rangle} b & \langle wt\ b, \widehat{\alpha}_{\mathbf{i}}^\vee \rangle \geq 0 \\ \widehat{e}_{\mathbf{i}}^{-\langle wt\ b, \widehat{\alpha}_{\mathbf{i}}^\vee \rangle} b & \langle wt\ b, \widehat{\alpha}_{\mathbf{i}}^\vee \rangle \leq 0 \end{cases}$$

We know, from [BG11, Proposition 2.4], for example there exists an isomorphism $W(\mathscr{C}^{\sigma\vee}) \cong W(\mathscr{C})^\sigma$.

Proposition 4.5.3. *Let B be a normal σ-foldable $\mathscr{C}$-crystal. Then B_σ admits a unique action $W(\mathscr{C}^{\sigma\vee}) \to Aut_{Set}(B_\sigma)$ of the Weyl group $W(\mathscr{C}^{\sigma\vee})$ via*

$$s_{\mathbf{i}}.[b] := [\widehat{s}_{\mathbf{i}} b]. \tag{4.5.1}$$

Each subcrystal of B_σ is stable under this action.

Proof. Use the previous Lemma and the properties of the folded root operators. Because $W(\mathscr{C}^{\sigma\vee})$ acts by folded root operators it preserves each subcrystal of B_σ. $\square$

4.5.3 The symmetric portion of a crystal

The subset of a crystal B fixed by the action of a diagram automorphism σ was first considered by Lusztig and described further by Naito and Sagaki. Before discussing these results we introduce this object in the context of our theory.

Definition 4.5.4. For a σ-foldable $\mathscr{C}$-crystal B let $\widehat{B}^\sigma := \{b \in B \mid \sigma(b) = b\}$ and $B^\sigma := \Phi_B[\widehat{B}^\sigma]$.

The following proposition is evident.

Proposition 4.5.5. *If B is a σ-foldable $\mathscr{C}$-crystal then B^σ is a sub-$\mathscr{C}^{\sigma\vee}$-crystal of B_σ. That is to say, $\mathcal{G}B_\sigma = \mathcal{G}B^\sigma \oplus \mathcal{G}(B_\sigma \setminus B^\sigma)$.*

The machinery of σ-foldability is unnecessary for studying B^σ. Indeed, the σ-action on $\widehat{B}^\sigma$ is trivial by definition, which allowed other authors to work with $\widehat{B}^\sigma$ and $\widehat{\mathcal{A}}^\sigma$ and obviated the need for a theory of crystal folding. For the Kashiwara-Littelmann crystals there is the following well-known result of Lusztig.

Theorem ([Lus93, Theorem 14.4.9])**.** *Let σ be an admissible automorphism of $\mathscr{C}$, and let $\lambda \in \Lambda^+ \cup \{\infty\}$. Then σ acts on $B(\lambda) \in Crys(\mathscr{C})$ and $\widehat{B}(\lambda)^\sigma$ is naturally isomorphic to $B(\lambda) \in Crys(\mathscr{C}^{\sigma\vee})$.*

The polyhedral crystal $B = \mathbb{Z}_\iota^\infty$ may be given a $\mathscr{C}^{\sigma\vee}$-crystal structure in the manner of subsection 1.5.1, denote this crystal $\mathbb{Z}_{\boldsymbol{\iota}}^\infty(\mathscr{C}^{\sigma\vee})$. Naito and Sagake define the operators $\widehat{a}_{\mathbf{i}}$ as in (4.2.1), without explicitly noting the condition of Definition 4.1.2(1), and obtain the following result.

Theorem ([NS04] Theorem 2.4.1)**.** *We have $\mathcal{A}^\sigma(\mathbb{Z}_\iota^\infty)^\sigma \subset (\mathbb{Z}_\iota^\infty)^\sigma \sqcup \{0\}$. Furthermore, there exists a canonical bijection $\Phi : (\mathbb{Z}_\iota^\infty)^\sigma \to \mathbb{Z}_{\boldsymbol{\iota}}^\infty(\mathscr{C}^{\sigma\vee})$ such that $\Phi \circ \widehat{x}_{\mathbf{i}} = x_{\mathbf{i}} \circ \Phi$ for all $\mathbf{i} \in I/\sigma$, $x \in \{e, f\}$.*

No crystal structure is placed on $(\mathbb{Z}_\iota^\infty)^\sigma$ in [NS04] and it is not claimed that Φ is an isomorphism of crystals. A similar result is obtained for the path model $P(\lambda)$.

Theorem ([NS02, Theorem 4.2], [NS01, Theorem 3.2.4])**.** *Suppose $\lambda \in \Lambda^+$ satisfies $\sigma(\lambda) = \lambda$. Then there exists a canonical bijection $\Phi : P(\lambda) \to P(\omega_\sigma(\lambda))$ such that $\Phi \circ \widehat{x}_{\mathbf{i}} = x_{\mathbf{i}} \circ \Phi$ for all $\mathbf{i} \in I/\sigma, x \in \{e, f\}$.*

Chapter 5

Folded structure of the Kashiwara-Littelmann crystals

For the Cartan data described in section 2, the σ-foldability of $B(\infty)$ is follows from that of the polyhedral crystal. Theorem 4.3.1 then implies $B(\lambda) : \lambda \in \Lambda^+$ is σ-foldable. The complete structure of B_σ for type $(D_3, C_2) \cong (A_3, B_2)$ is given in subsection 5.5, and examples showing that HW C can have more than one element are given for type.

After discussing the σ-action on the polyhedral crystal in section 5.1, we prove that the polyhedral crystal is σ-foldable (Theorem 5.2.1). It follows immediately that $B(\infty)$ is σ-foldable, and being a highest-weight $\mathscr{C}$-crystal it is subject to the cyclicity phenomenon of section. In types A, D, E_6 we give a complete characterization of the highest-weight elements of $B(\infty)_\sigma$ in terms of the balanced parabolic quotient (Theorem 5.3.1) via Demazure crystals. We show that the J_r-highest-weight subset of $\widehat{\Sigma}_{\iota,\sigma}(D_{r+2})$ is a semigroup, and admits a unique finite $\subset$-minimal generating set. The balanced parabolic quotient ${}^{J_r}W_\sigma$ embeds into the generating set (Proposition 5.3.11), and there is a bijection between the two for $1 \leq r \leq 7$. In section 5.3.3 we exhibit a convex rational polytope whose discrete volume gives the number of J_r-highest-weight elements of a given weight, and relate this volume to the kernel of a certain integral matrix. Next, we show that there exists a component of $B(\infty)_\sigma$ in $D_n, n \geq 4$ containing infinitely many highest-weight elements. A table providing connectedness data for several $B(\lambda)_\sigma$ is given, and we conjecture a decomposition of $B(\infty)_\sigma$ as a C_3-crystal. Finally, in section 5.5 we compute the structure of the C_2-crystal $B(\infty)_\sigma$.

5.1 The σ-action on the polyhedral crystal

The following result—valid in far greater generality but only needed in the present form—is essential to constructing a σ-action on the polyhedral crystal.

Lemma ([Nak99, Proposition 4.1])**.** *Suppose $j \in \langle\sigma\rangle i$. Then there exists a $\mathscr{C}$-crystal isomorphism $\phi_{ij} : B_i \otimes B_j \cong B_j \otimes B_i$ whereby $b_i(x) \otimes b_j(y) \longmapsto b_j(y) \otimes b_i(x)$.*

For our purposes we choose a sequence ι that is adapted for the folding of $\mathscr{C}$ by an admissible automorphisim σ. Repeat a Coexeter word[1] $\boldsymbol{\iota}_{|I/\sigma|}, \cdots, \boldsymbol{\iota}_2, \boldsymbol{\iota}_1$ from I/σ infinitely many times, producing a sequence $\boldsymbol{\iota}$ from I/σ. Lift $\boldsymbol{\iota}$ to a sequence ι from I by replacing each orbit $\boldsymbol{\iota}_k$ by its elements. Besides its organization based on the structure of I/σ, we have ensured that (1.5.1) is satisfied.

Definition 5.1.1. A sequence ι from I constructed in the manner described above is called *σ-adapted.*

A priori there are multiple lifts of $\boldsymbol{\iota}$ to ι, each corresponding to an ordering of $\boldsymbol{\iota}_k$'s elements. The crystals $\cdots \otimes B_{\iota_k} \otimes \cdots \otimes B_{\iota_2} \otimes B_{\iota_1}$ and $\cdots \otimes B_{\iota'_k} \otimes \cdots \otimes B_{\iota'_2} \otimes B_{\iota'_1}$ corresponding to different lifts ι', ι of $\boldsymbol{\iota}$ are isomorphic by [Nak99, Proposition 4.1]. Identifying these crystals, we assume there is a unique lift ι, consisting of 'blocks' that are σ-orbits in I.

Use [Nak99, Proposition 4.1] to define an isomorphism $\phi_\sigma : \cdots \otimes B_{\iota_k} \otimes \cdots \otimes B_{\iota_2} \otimes B_{\iota_1} \cong \cdots \otimes B_{\sigma(\iota_k)} \otimes \cdots \otimes B_{\sigma(\iota_2)} \otimes B_{\sigma(\iota_1)}$, where $\sigma(\iota) := (\sigma(\iota_k) \,|\, k \geq 1)$. Let $1_\sigma : \cdots \otimes B_{\sigma(\iota_k)} \otimes \cdots \otimes B_{\sigma(\iota_2)} \otimes B_{\sigma(\iota_1)} \to \cdots \otimes B_{\iota_k} \otimes \cdots \otimes B_{\iota_2} \otimes B_{\iota_1}$ be the bijection that is the identity function on its domain. Abusing notation, remark 1.5.1 yields

$$\mathbb{Z}^\infty_\iota \xrightarrow{\phi_\sigma} \mathbb{Z}^\infty_{\sigma(\iota)} \xrightarrow{1_\sigma} \mathbb{Z}^\infty_\iota$$

A σ-adapted sequence ι affords a convenient indexing scheme. In this case $\iota_{k+|I|} = \iota_k$ for all k. Now a given $k \geq 1$ has the form $k = n|I| + r$ with $n \geq 0$ and $1 \leq r \leq |I|$, which gives a bijection $\mathbb{N} \xrightarrow{\sim} \mathbb{N} \times I$ whereby $k \longmapsto (n+1, r)$. Expressing $\iota_k = \iota_{n+1,\iota_r}$ gives

$$\iota = (\iota_{n,i} \,|\, n \geq 1, i \in I) \tag{5.1.1}$$

$$\mathbf{x} = (x_{n,i} \,|\, n \geq 1, i \in I). \tag{5.1.2}$$

[1] That is, a sequence in which each index appears exactly once.

Example 5.1.2. Consider the pair $(\mathscr{C}, \mathscr{C}^{\sigma\vee}) = (D_3, C_2)$ as in (2.3.1) with $\boldsymbol{\iota} = (\cdots, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1})$ lifted to $\iota = (\cdots, -1, 0, 1, -1, 0, 1)$. Expression (5.1.2) is

$$\mathbf{x} = (\cdots, x_{2,-1}, x_{2,0}, x_{2,1}, x_{1,-1}, x_{1,0}, x_{1,1}) \tag{5.1.3}$$

Lemma ([NS04, 2.3])**.** *The automorphism σ yields a bijection $\sigma \in Bij(\mathbb{Z}_\iota^\infty)$ which can be described on elements as follows: for all $\mathbf{x} = (x_{n,i}) \in \mathbb{Z}_\iota^\infty$ we have $\sigma(\mathbf{x}) = \mathbf{y}$, where*

$$\mathbf{y} = (y_{n,i})_{(n,i)\in\mathbb{N}\times I} \text{ with } y_{n,i} = x_{n,\sigma^{-1}(i)}. \tag{5.1.4}$$

Write $(n, j) > (m, i)$ if $n|I| + j > m|I| + r$. The doubly-indexed Kashiwara functions are given by

$$\gamma_{m,i}(x) = x_{m,i} + \sum_{(n,j)>(m,i)} a_{ij} x_{n,j}. \tag{5.1.5}$$

In this notation, the definition of the Kashiwara operators is formulated as follows. Let

$$m_i(x) := \min\{m \geq 1 \mid \gamma_{m,i}(x) = \varepsilon_i(x)\} \tag{5.1.6}$$

$$m^i(x) := \max\{m \geq 1 \mid \gamma_{m,i}(x) = \varepsilon_i(x)\}. \tag{5.1.7}$$

Then, when the respective images are nonzero, we have

$$e_i x = x - \mathbf{e}(m^i(x), i) \tag{5.1.8}$$

$$f_i x = x + \mathbf{e}(m_i(x), i) \tag{5.1.9}$$

(There is a unique such index because each element of I appears once per block when ι is σ-adapted.)

We shall need two technical results on the relationship between the σ-action and the Kashiwara functions.

Lemma 5.1.3. *If $a_{ij} = 0$ then*

$$\gamma_k(y_j \mathbf{x}) = \gamma_k(\mathbf{x}) \text{ for all } \iota_k = i, y \in \{e, f\}, \mathbf{x} \in \mathbb{Z}_\iota^\infty. \tag{5.1.10}$$

Proof. For all $k,m \geq 1$ and all $x \in \mathbb{Z}_\iota^\infty$

$$\gamma_k(\mathbf{e}(m)) = \begin{cases} a_{ij}, & k < m \\ 1, & k = m \\ 0, & k > m \end{cases} \tag{5.1.11}$$

by (1.5.3). The claim now follows from (1.5.6) and (1.5.7). $\square$

Lemma 5.1.4. *For all $x \in \mathbb{Z}_\iota^\infty$ and all $(m,i) \in \mathbb{N} \times I$ the following diagram commutes*

$$\begin{tikzcd} \mathbb{Z}_\iota^\infty \arrow[rr, "\sigma"] \arrow[dr, "\gamma_{m,i}"'] & & \mathbb{Z}_\iota^\infty \arrow[dl, "\gamma_{m,\sigma(i)}"] \\ & \mathbb{Z} & \end{tikzcd} \tag{5.1.12}$$

Proof. By (5.1.5) and (5.1.4)

$$\begin{aligned} \gamma_{m,\sigma(i)} \circ \sigma(x) &= y_{m,\sigma(i)} + \sum_{(n,j)>(m,\sigma(i))} a_{\sigma(i),j} y_{n,j} \\ &= x_{m,i} + \sum_{(m,j)>(n,\sigma(i))} a_{\sigma(i),j} x_{n,\sigma^{-1}(j)} \\ \gamma_{m,i}(x) &= x_{m,i} + \sum_{(n,j)>(m,i)} a_{i,j} x_{n,j}. \end{aligned}$$

Now for all $(n,j) > (m,\sigma(i))$ there exists a unique $(n',j') > (m,i)$ such that $a_{\sigma(i),j} x_{n,\sigma^{-1}(j)} = a_{i,j'} x_{n',j'}$. Indeed, letting $(n',j') = n, \sigma^{-1}(j)$ we have

$$a_{i,j'} x_{n',j'} = a_{i,\sigma^{-1}(j)} x_{n,\sigma^{-1}(j)} = a_{\sigma(i),j} x_{n,\sigma^{-1}(j)}$$

by (2.1.1). $\square$

5.2 Foldability

Theorem 5.2.1. *The polyhedral crystal $\mathbb{Z}_\iota^\infty$ with σ-adapted sequence ι is σ-foldable.*

Proof. The σ-action is given by (5.1.4), we must verify the conditions of Definition 4.1.2.

Equation (4.1.1): According to (5.1.2) and (1.5.2)

$$\mathrm{wt} \circ \sigma(x) = \sum_{i \in I} \left(\sum_{n \geq 1} x_{n,\sigma^{-1}(i)} \right) \alpha_i$$

whereas

$$\sigma \circ \mathrm{wt}(\mathbf{x}) = \sum_{i \in I} \left(\sum_{n \geq 1} x_{n,i} \right) \alpha_{\sigma(i)}.$$

Equation (4.1.2): According to (1.5.4) and (5.1.4)

$$\varepsilon_i \circ \sigma(\mathbf{x}) = \max\{\gamma_{n,i} \circ \sigma(\mathbf{x}) \mid n \geq 1\} = \max\{\gamma_{n,\sigma^{-1}(i)}(\mathbf{x}) \mid n \geq 1\} = \varepsilon_{\sigma^{-1}(i)}(\mathbf{x}).$$

It suffices to check equation (4.1.3) for the f operator. According to (5.1.8) we have

$$f_i \circ \sigma(\mathbf{x}) = f_i \mathbf{y} = \mathbf{y} + \mathbf{e}(m_i(\mathbf{y}), i).$$

But (5.1.4) and (4.1.2) give

$$m_i(\mathbf{y}) = m_{\sigma^{-1}(i)}(\mathbf{x}).$$

Now

$$\begin{aligned} f_i \circ \sigma(\mathbf{x}) &= \sigma(\mathbf{x}) + \mathbf{e}(m_{\sigma^{-1}(i)}(\mathbf{x}), i) \\ &= \sigma(\mathbf{x} + \mathbf{e}(m_{\sigma^{-1}(i)}(\mathbf{x}), \sigma^{-1}(i))) = \sigma \circ f_{\sigma^{-1}(i)}(\mathbf{x}) \end{aligned}$$

by (5.1.8) again.

It is enough to check condition (1) for the f operator. According to (5.1.9) we have

$$\begin{aligned} f_i f_j \mathbf{x} &= \mathbf{x} + \mathbf{e}(m_j(\mathbf{x}), j) + \mathbf{e}(m_i(f_j \mathbf{x}), i) \\ f_j f_i \mathbf{x} &= \mathbf{x} + \mathbf{e}(m_i(\mathbf{x}), i) + \mathbf{e}(m_j(f_i \mathbf{x}), j). \end{aligned}$$

Thus it suffices to prove that $m_i(f_j \mathbf{x}) = m_i(\mathbf{x})$ when $j \in \langle\sigma\rangle i$. This follows, indeed it holds in even greater generality by Lemma 5.1.3.

Condition (2) is an immediate consequence of Lemma 5.1.3. □

It is clear from the definitions that a subcrystal of a σ-foldable crystal is σ-foldable. Hence [NZ97, Theorem 2.5] gives

Corollary 5.2.2. *The limit Kashiwara-Littelmann crystal $B(\infty)$ is σ-foldable.*

Corollary 5.2.3. *For each $\lambda \in \Lambda^+$ satisfying $\sigma(\lambda) = \lambda$ the KL crystal $B(\lambda)$ is σ-foldable.*

Proof. Recall that $B(\lambda) : \lambda \in \Lambda^+$ is isomorphic to the subcrystal of $B(\infty) \otimes S_\lambda$ generated by $b_\infty \otimes s_\lambda$. Define $\sigma : S_\lambda \to S_\lambda$ by $\sigma(s_\lambda) = s_\lambda$, it is easily checked that S_λ is σ-foldable. (The σ-invariance of λ is required for condition (4.1.1) to hold.) Therefore $B(\lambda)$ is σ-foldable by Theorem 4.3.1. $\square$

5.3 Highest-weight elements of the KL crystals

5.3.1 Demazure crystals, branching rules, and σ-invariant weight spaces

Fix a simply-laced symmetrizable Cartan datum $\mathscr{C}$ with admissible automorphism σ. It turns out that the balanced quotients characterize the Demazure crystals in which weight-invariant J-highest-weight elements appear.

Theorem 5.3.1. *Let $\lambda \in \Lambda^{++} \cup \{\infty\}$. For $(\mathscr{C}, J)$ of type D_n (2.3.1), A_{2r+1} (2.3.3), E_6 (2.3.9), and D_4 under the triality (2.3.5) an element $w \in {}^JW$ is σ-balanced if and only if $\bar{B}_w(\lambda) := B_w(\lambda) \smallsetminus \bigcup_{v \leq w} B_v(\lambda)$ contains a weight-invariant element of $HW_J B(\lambda)$.*

Elements of $\mathrm{HW}_J B(\lambda)$—that is, the elements of $B(\lambda)$ that are highest-weight for the Jth branching rule—are readily seen to be related to ${}^JW$ by the following, which we failed to find in the literature.

Lemma 5.3.2. *For any $J \subset I$, we have $HW_J B(\lambda) \subset \bigcup_{w \in {}^JW} B_w(\lambda)$, $\lambda \in \Lambda^{++} \cup \{\infty\}$*

Proof. It suffices to prove the claim using the path crystal. Immediately from [Lit94, Proposition 1.5], we have that

$$\tilde{e}_j \pi = 0 \iff \langle \alpha_j^\vee, \mathrm{im}\ \pi \rangle \subset [0, \infty). \tag{5.3.1}$$

We have $\pi \in \mathrm{HW}_J \mathscr{P}_\lambda$ if and only if (5.3.1) holds for all $j \in J$. In particular $\langle \alpha_j^\vee, \tau_1 \lambda \rangle > 0$ for all $j \in J$. Now $\langle \tau_1^{-1} \alpha_j^\vee, \lambda \rangle > 0$ for all $j \in J$ implies that $\tau_1^{-1} \in W^J$ by (1.2.3).

Observe that, $B_w(\lambda) = \bigoplus_{\nu \in \mathbb{N}\Pi} B_w(\lambda)_{\lambda - \nu}$, where

$$B_w(\lambda)_{\lambda-\nu} := \{ f_{i_\ell}^{n_\ell} \cdots f_{i_1}^{n_1} b_\lambda \mid \sum_{1 \leq k \leq \ell} n_k \alpha_{i_k} = \nu \}.$$

Now $|B_w(\lambda)_{\lambda-\nu}|$ are increasing in λ for the ordering $\prec$ of Section 1.2 and uniformly bounded by the number of monomials $f_{i_\ell}^{n_\ell} \cdots f_{i_1}^{n_1}$ with $\sum_{1 \leq k \leq \ell} n_k \alpha_{i_k} = \nu$. Thus there exists $\lambda \in \Lambda^{++}$ such that $|B_w(\lambda)_{\lambda-\nu}|$ is maximal. The set $B_w(\infty)$ is defined to be the disjoint union of these $B_w(\lambda)_{\lambda-\nu}$, hence the results of $B_w(\lambda)$ imply the result on $B_w(\infty)$. $\square$

The proof of Theorem 5.3.1 is technical and uses a combination of the polyhedral realization, the combinatorics of the reduced expressions of w, and Kashiwara's $\star$ involution. We begin by outlining the steps.

Proof outline of Theorem 5.3.1. We must show that

$$\bar{B}_w(\infty) \cap \mathrm{HW}_J B(\infty) \text{ contains a weight-invariant element } \iff w \in {}^J W_\sigma \tag{5.3.2}$$

We restrict our attention to type D as in (2.3.1), type A is proved similarly and the exceptional cases are checked by hand.

First, we shall prove that if a representative $w \in {}^J W$ is σ-balanced, then $\bar{B}_w(\infty)$ contains a weight-invariant element of $\mathrm{HW}_J B(\infty)$, by constructing such an element. Given a reduced word $(i_\ell, \ldots, i_1) \in \mathcal{R}(w)$ let $b := \tilde{f}_{i_\ell} \cdots \tilde{f}_{i_1} b_\infty$. It follows immediately that b is weight-invariant, since $\mathrm{wt}(b) = -\sum_{k=1}^{\ell} \alpha_{i_k}$ and w is σ-balanced. Note also that $b \in \bar{B}_w(\infty)$, by (2) above.

Proposition 5.3.3. *$b \in HW_J B(\infty)$*

Proposition 5.3.3 is proved using the polyhedral realization $\Psi_\iota : B_w(\infty) \cong \Sigma_{\iota,w}$ [Nak02]. Using a relationship between the operators $\tilde{f}_i, \tilde{f}_j$ when $a_{ij} = -1$ (Lemma 5.3.6 below), we compute $\Psi_\iota(b)$. Next, we apply a result of Littelmann (Lemma 5.3.7 below), which describes the structure of a reduced expression of a fully commutative element, to conclude that $\tilde{e}_j b = 0$ for all $j \in J$. This shows that b is an element of $\bar{B}_w(\infty) \cap \mathrm{HW}_J B(\infty)$ having σ-invariant weight, which proves the "only if" direction of (5.3.2).

Remark 5.3.4. It does not follow immediately from the definitions of $(B(\infty), \tilde{e}_i, \tilde{f}_i)$ alone that b is J-highest-weight.

The following Lemma completes the proof of (5.3.2).

Lemma 5.3.5. *If $w \in {}^J W$ is not σ-balanced, then $\bar{B}_w(\infty)$ does not contain a weight-invariant element of $HW_J B(\infty)$.*

To prove Lemma 5.3.5 we first observe that by Lemma 3.2.5, if $w \in {}^J W$ is not σ-balanced then it has a unique reduced factorization $w = u s_{-1} s_1 s_2 \cdots s_k$ with $u \in {}^J W_\sigma$ of maximal length and $\ell(w) = \ell(u) + k + 1$. If $w = u s_{-1}$ we check that Lemma 5.3.5 holds by a direct calculation in $\Sigma_{\iota,w}$. Otherwise, if $w = u s_{-1} s_1 \cdots s_i$ with $i \geq 1$ we can use Kashiwara's $\star$-operation along with the combinatorial properties of Demazure crystals from section 1.4.1. □

Proof of Proposition 5.3.3. First, we need a lemma about the interaction of the operators $\tilde{f}_i, \tilde{f}_j$ when $a_{ij} = -1$. The notations appearing below are defined in section 1.5

Lemma 5.3.6. *Let $w \in {}^J W_\sigma$ and $\mathbf{x} \in \Sigma_w^\iota$ and i, j be such that $a_{ij} = -1$. If $\tilde{f}_j$ acts on $\mathbf{x}$ at position k, then $\tilde{f}_j$ acts on $\tilde{f}_i\mathbf{x}$ at position $k' \geq k$.*

Proof. We show that $\min M^{(j)}(\tilde{f}_i\mathbf{x}) \geq \min M^{(j)}(\mathbf{x})$. Let $\ell = \ell(w)$. By definition (see section 1.5), $\tilde{f}_i\mathbf{x} = \mathbf{x} + \mathbf{e}_{\min M^{(i)}(\mathbf{x})}$. Take $k \geq 1$ such that $\iota_k = j$; then

$$\begin{aligned}\gamma_k(\tilde{f}_i\mathbf{x}) &= \gamma_k(\mathbf{x}) + \gamma_k(\delta_{k,\min M^{(i)}(\mathbf{x})}) \\ &= \begin{cases} \gamma_k(\mathbf{x}), & k > \min M^{(i)}(\mathbf{x}) \\ \gamma_k(\mathbf{x}) - 1, & k < \min M^{(i)}(\mathbf{x}) \end{cases}\end{aligned}$$

by [Nak02, 2.8]. There are three cases to consider:

1. If $M^{(j)}(\mathbf{x}) \subset (\min M^{(i)}(\mathbf{x}), \ell]$ then $\varepsilon_j(\tilde{f}_i\mathbf{x}) = \varepsilon_j(\mathbf{x})$ and $\min M^{(j)}(\tilde{f}_i\mathbf{x}) = \min M^{(j)}(\mathbf{x})$ by the above display.

2. If $M^{(j)}(\mathbf{x}) \subset [1, \min M^{(i)}(\mathbf{x}))$ then, similarly as before, $\varepsilon_j(\tilde{f}_i\mathbf{x}) = \varepsilon_j(\mathbf{x}) - 1$ and $\min M^{(j)}(\tilde{f}_i\mathbf{x}) = \min M^{(j)}(\mathbf{x})$.

3. If $M^{(j)}(\mathbf{x}) \cap [1, M^{(i)}(\mathbf{x}))$ and $M^{(j)}(\mathbf{x}) \cap (M^{(i)}(\mathbf{x}), \ell]$ are nonempty then there exists $k \in M^{(j)}(\mathbf{x})$ such that $k > \min M^{(i)}(\mathbf{x})$ and $\gamma_k(\mathbf{x}) = \varepsilon_j(\mathbf{x})$. Thus $\varepsilon_j(\tilde{f}_i\mathbf{x}) = \varepsilon_j(\mathbf{x})$ and $\min M^{(j)}(\tilde{f}_i\mathbf{x}) > \min M^{(j)}(\mathbf{x})$.

These calculations complete the proof. □

It follows from this Lemma and Theorems 3.2.3, 3.2.4, and the definition of $\tilde{f}_i, \tilde{f}_j$ (see section 1.5) that $\Psi_\iota(b)$ is the sequence $(\cdots, 0, \underbrace{1, \ldots, 1}_{\ell(w)})$.

To prove that $\Psi_\iota(b) \in \mathrm{HW}_J \Sigma_\iota$ we use combinatorial information about the parameterizing sequence ι, which is based on $(i_\ell, \ldots, i_1) \in \mathcal{R}(w)$, via the following.

Lemma 5.3.7 ([Lit98, Lemma 3.4])**.** *Let $(i_r, \ldots, i_1)$ index a reduced expression of a fully commutative $w \in W$. Then*

1. *If $j \geq 2$ is such that $i_k \neq i_j$ for all $k < j$ then there exists exactly one $l \in \{1, \ldots, j-1\}$ such that $a_{jl} \neq 0$.*

2. *Suppose $i = i_p = i_q$ but $i_j \neq i$ for all $p < j < q$. Then there exist two (not necessarily different) i_j, i_k with $p < j < k < q$ such that $a_{i,i_j}, a_{i,i_k} = -1$.* $\square$

We are now in a position to show that $\tilde{e}_j b = 0$ for all $j \in J$. It suffices to show $\varepsilon_j(b) = 0$ for all $j \in J$, by to the upper-normality of $B(\infty)$. Let $\mathbf{x} := \Psi_\iota(b)$. By [Nak02, (2.13)] it suffices to show $\gamma_l(\mathbf{x}) \leq 0$ for all l such that $\iota_l = j$, where $j \in J$ is arbitrary. Recall

$$\gamma_l(\mathbf{x}) = 1 + \sum_{l < t \leq \ell} a_{\iota_t, j}$$

and note that only those t with $\iota_t \in \{2, -1\}$ contribute to the sum.

If there exists a unique l such that $\iota_l = j$ then $\varepsilon_j(\mathbf{x}) = \max\{0, \gamma_l(\mathbf{x})\}$. By Lemma 5.3.7(1) there exists a unique $t > l$ such that $a_{\iota_t, j} = -1$. Thus $\gamma_l(\mathbf{x}) = x_l + (-1)x_t = 0$.

It there exist more than one l such that $\iota_l = j$, say there are n such indices $l_n > \cdots > l_1$. Then Lemma 5.3.7(2) provides two indices $l_{k+1} > i_r(k) > i_s(k) > l_k$ such that $a_{\iota_r, j} = a_{\iota_s, j} = -1$. By Lemma 5.3.7(1) there exists a unique $l > l_n$ such that $a_{\iota_l, j} = -1$. Thus

$$\begin{aligned} \gamma_{l_k}(\mathbf{x}) &= x_{l_k} - x_{l_n} + \sum_{k < t \leq n} 2x_{l_t} - (x_{i_r}(k) + x_{i_s}(k)) \\ &= 1 - 1 + \sum_t 2 - (1 + 1) = 0. \end{aligned}$$

Hence in either case $\gamma_l(\mathbf{x}) \leq 0$, which completes the proof of Proposition 5.3.3.

Proof of Lemma 5.3.5. Recall that we use the indexing scheme (2.3.1) To begin, observe that Lemma 3.2.5 implies that $w \in {}^J W$ that is not σ-balanced has a unique reduced factorization $w = us_{-1}s_1s_2 \cdots s_k$ with $u \in {}^J W_\sigma$ of maximal length.

In the case $w = us_{-1}$ it follows from Lemma 3.2.5 that $w = {}^{2k}w_\circ$. We check that $\bar{B}_w(\infty)$ does not contain a weight-invariant element of $\mathrm{HW}_J B(\infty)$ by a direct calculation. Fix $n = 2k > 0$ and take the reduced decomposition of $w = {}^{J_n}w_\circ = \tau_0 \cdots \tau_n$ afforded by Lemma 3.2.5. Let $(i_\ell, \cdots, i_1)$ be the reduced word for this decomposition, and let $\iota = (\cdots, \iota_k, \cdots, \iota_2, \iota_1)$ be an infinite sequence from I such that $\iota_k = i_k$ for all $1 \leq k \leq \ell$; combinatorial properties of ι will be used frequently and without reference in the sequel.

We claim that $\mathrm{HW}_J(\Sigma_w^\iota)_\sigma = \varnothing$. Indeed, let $\mathbf{x} \in \mathrm{HW}_J(\Sigma_W^\iota)_\sigma$. Given $i \in I$, set $\mathcal{K}_i := \{k \geq 1 \mid \iota_k = i\}$, $k_i := \min \mathcal{K}_i$, $K_i := \max \mathcal{K}_i$. Then

$$\mathbf{x} \in \mathrm{HW}_J \Sigma^\iota \iff \gamma_k(\mathbf{x}) \leq 0 \text{ for all } k \in \bigcup_{j \in J} \mathcal{K}_j.$$

and

$$\mathbf{x} \in \Sigma^{\iota}_{\sigma} \iff \sum_{k \in \mathcal{K}_0} x_k = \sum_{k \in \mathcal{K}_{-1}} x_k \tag{5.3.3}$$

We are considering only those $\mathbf{x}$ lying in the interior of the cone Σ^{ι}_{w}, which means all $x_k > 0$. We will show that

$$\sum_{k \in \mathcal{K}_0} x_k \leq \sum_{k \in \mathcal{K}_{-1} \setminus \{k_{-1}\}} x_k, \tag{5.3.4}$$

which contradicts equation (5.3.3). It will be useful to use the following form of the definition of γ_k, which is adapted to our simply-laced case

$$\gamma_k(\mathbf{x}) = x_k + 2 \sum_{m \in \mathcal{K}_{\iota_k} : m > k} x_m - \sum_{m \in \mathcal{K}_{(\iota_k)} : m > k} x_m, \quad \mathcal{K}_{(i)} := \bigcup_{j : a_{ij} < 0} \mathcal{K}_j$$

Let $\mathcal{K}^* := \{k \geq 1 \,|\, k_0 \leq k \leq K_{n-1}\}$, this range of indices corresponds to τ_{n-1} in our preferred reduced decomposition of w. In the following we describe how to systematically reduce the set of inequalities $\{\gamma_k(\mathbf{x}) \leq 0 \,|\, k \in \mathcal{K}^*\}$ to the desired contradictory inequality. Consider two inequalities at a time, beginning with the largest index: In case $k = K_{n-1}$ we have

$$x_{K_{n-1}} \leq x_{K_{n-2}} \tag{5.3.5}$$

whereas in case $k = K_{n-1} - 1$, because $\iota_{K_{n-1}-1} = n-2$ and $\mathcal{K}_{(n-2)} = \mathcal{K}_{n-1} \cup \mathcal{K}_{n-3}$ we have

$$x_{\max(\mathcal{K}_{n-2} \setminus \{K_{n-2}\})} + 2x_{K_{n-2}} \leq x_{K_{n-1}} + x_{\max(\mathcal{K}_{n-3} \setminus \{K_{n-3}\})} + x_{K_{n-3}} \tag{5.3.6}$$

Combining (5.3.5) and (5.3.6) gives

$$x_{\max(\mathcal{K}_{n-2} \setminus \{K_{n-2}\})} + x_{K_{n-2}} \leq x_{\max(\mathcal{K}_{n-3} \setminus \{K_{n-3}\})} + x_{K_{n-3}} \tag{5.3.7}$$

In case $k = K_{n-1} - 2$, because $\iota_k = n-3$ and $\mathcal{K}_{(n-3)} = \mathcal{K}_{n-2} \cup \mathcal{K}_{n-1}$ (go to the final steps if $k \leq 4$) we have

$$\begin{aligned} &x_{\max(\mathcal{K}_{n-3} \setminus \{K_{n-3}, \max(\mathcal{K}_{n-3} \setminus \{K_{n-3}\})\})} + 2x_{\max(\mathcal{K}_{n-3} \setminus \{K_{n-3}\})} + 2x_{K_{n-3}} \\ &\leq x_{n_1} + x_{n_2} + x_{n_3} + x_{n_4} + x_{n_5} \end{aligned} \tag{5.3.8}$$

where

$$n_1 = \max(\mathcal{K}_{n-2} \smallsetminus \{K_{n-2}\})$$

$$n_2 = \max(\mathcal{K}_{n-4} \smallsetminus \{K_{n-4}, \max(\mathcal{K}_{n-4} \smallsetminus \{K_{n-4}\})\})$$

$$n_3 = K_{n-2}$$

$$n_4 = \max(\mathcal{K}_{n-4} \smallsetminus \{K_{n-4}\})$$

$$n_5 = K_{n-4}$$

Combining (5.3.7) and (5.3.8) gives

$$x_{\max(K_{n-3} \smallsetminus \{\max K_{n-3}, \max(K_{n-3} \smallsetminus \{\max K_{n-3}\})\})} + x_{\max(K_{n-3} \smallsetminus \{\max K_{n-3}\})} + x_{\max K_{n-3}}$$
$$\leq x_{k_2} + x_{k_4} + x_{k_5}$$

This process eventually terminates; the final step is the case $k = \min K_0$ in which we have

$$x_{k_0} + 2 \sum_{m \in \mathcal{K}_0 \smallsetminus \{k_0\}} x_m \leq \sum_{n \in \mathcal{K}_1 \smallsetminus \{k_1\}} x_n.$$

In case $k+1$ we found that

$$\sum_{n \in \mathcal{K}_1 \smallsetminus \{k_1\}} x_n \leq \sum_{r \in \mathcal{K}_0 \smallsetminus \{k_0\}} x_r + \sum_{s \in \mathcal{K}_{-1} \smallsetminus \{k_{-1}\}} x_s.$$

Combining these inequalities yields (5.3.4). Thus, $\mathrm{HW}_J(\Sigma_w^\iota)_\sigma = \varnothing$.

In case $w = us_{-1}s_1 \cdots s_i$ with $i \geq 1$ we use Kashiwara's $\star$-operation. We will show that $b \in \mathrm{HW}_J \bar{B}_w(\infty)_\sigma = \varnothing$. Assume that b is an element of that set, using the notation of (1.4.1) we can express $b = \tilde{f}_{\mathbf{i}}^{\mathbf{k}} \tilde{f}_i^k b_\infty$, with $k, \min k_j > 0$, where $\mathbf{i} \in \mathcal{R}(ws_i)$. Then Lemma 4.1 provides $b' \in B_{ws_i}(\infty)$ such that $b = \tilde{f}_i^{\star k} b'$. Now $\mathrm{wt}\, b = \mathrm{wt}\, b' - k\alpha_i$ and $\varepsilon_j(b) = \varepsilon_i(b')$ for all $j \neq i$ by [Kas93, Corollary 2.2.2].

In case $\sigma(i) = i$ we have $i \in J$. Since w is not σ-balanced, we see that ws_i is also not σ-balanced. Furthermore, $b' \in B_{ws_i}(\infty)_\sigma$. By induction there exists $j \in J$ such that $\varepsilon_j(b') > 0$. If $j \neq i$ then $b \notin \mathrm{HW}_J B_w(\infty)$, contradicting the choice of b. Therefore it must be that $\varepsilon_j(b') = 0$ for all $j \in J \smallsetminus \{i\}$. Writing $\Psi_i(b') = b_0 \otimes c_i(-k)$, it follows from (1.4.1) that $\Psi_i(b) = b_0 \otimes c_i(-m-k)$.

Now because Ψ_i is an isomorphism the tensor product axioms give

$$\begin{aligned} 0 < \varepsilon_i(b') &= \max\{\varepsilon_i(b_0), \varepsilon_i(c_i(-m)) - \langle \alpha_i^\vee, \mathrm{wt}(b_0)\rangle\} \\ &= \max\{\varepsilon_i(b_0), -(\langle \alpha_i^\vee, \mathrm{wt}(b')\rangle + m)\} \end{aligned}$$

and

$$\begin{aligned} 0 = \varepsilon_i(b) &= \max\{\varepsilon_i(b_0), \varepsilon_i(c_i(-m-k)) - \langle \alpha_i^\vee, \mathrm{wt}(b_0)\rangle\} \\ &= \max\{\varepsilon_i(b_0), k - (\langle \alpha_i^\vee, \mathrm{wt}(b')\rangle + m)\}. \end{aligned}$$

Accordingly $\varepsilon_i(b_0) = 0$, forcing $-(\mathrm{wt}_i b' + m) > 0$. But now $k - (\mathrm{wt}_i b' + m) \leq 0$, which is impossible.

$\square$

The map η

Define a function

$$\eta : \bigcup_{n \geq 1} I^{\times n} \to B(\infty) \text{ whereby } (i_n, \ldots, i_1) \longmapsto f_{i_n} \cdots f_{i_1} b_\infty. \tag{5.3.9}$$

When ι is as in section 1.5.1, we denote $\eta_\iota := \Psi_\iota \circ \eta$, whose image lives in Σ_ι. Recall (section 5.2) that if $a_{ij} = 0$ then $f_i f_j = f_j f_i$ as operators on $B(\infty)$. On the other hand, it is easy to show that if the simple reflections s_i, s_j commute if and only if $a_{ij} = 0$. Accordingly, if $w \in W$ is fully commutative them $\eta[\mathcal{R}(w)]$ is a singleton; let $\eta(w)$ denote this element. In particular, for $\mathscr{C}$ of types D_n (2.3.1), A_{2r+1} (2.3.3), E_6 (2.3.9) we have $\eta_\iota : {}^J W_\sigma \to \Sigma_\iota$.

5.3.2 Free generators in the polyhedral crystal

Suppose A is positive definite, so that $\mathfrak{g}(\mathscr{C})$ is of finite type and the polyhedral crystal lives in $\mathbb{Q}^{\ell(w_\circ)}$. Recalling the geometric presentation of the Kashiwara-Nakashima-Zelevinsky crystal from section 1.5.3, denote $C_\iota := \mathbb{R}_{\geq 0} \Xi_\iota$. Theorem 3.1 of [NZ97] presents Σ_ι as the associated semigroup of the dual cone $C_\iota^\vee$, which is to say $\Sigma_\iota = C_\iota^\vee \cap \mathbb{Z}^\ell$. Owing to the fact that the coordinate forms $x_k^*, k \geq 1$ belong to Ξ_ι, the dual cone C_ι is strongly convex, in the sense that it contains no nontrivial subspace of $\mathbb{R}^\ell$.

Let C be a convex rational polyhedral cone in $\mathbb{R}^d$. We say that a vector $\mathbf{x} \in C^\vee \cap \mathbb{Z}^d$ is *irreducible* if $\mathbf{x} = \mathbf{y} + \mathbf{z}$ with $\mathbf{y}, \mathbf{z} \in C^\vee \cap \mathbb{Z}^d$ implies $\mathbf{y} = 0$ or $\mathbf{z} = 0$.

Lemma 5.3.8 ([CLS11, Proposition 1.2.23]). *The set* $\mathcal{H}_\iota := \{\mathbf{x} \in \varSigma_\iota \,|\, \mathbf{x}$ *is irreducible*$\}$ *has the following properties:*

1. $\mathcal{H}_\iota$ *is finite and generates* $\varSigma_\iota$ *as a semigroup.*

2. $\mathcal{H}_\iota$ *contains the ray generators of the edges of* $C_\iota^\vee$.

3. $\mathcal{H}_\iota$ *is the* $\subset$*-minimal generating set of the semigroup* $\varSigma_\iota$.

Remark 5.3.9. According to [Jos09, 4.1], there is no hope of explicitly describing $\mathcal{H}_\iota$ in the general case, because it is extremely sensitive to the choice of parameterizing sequence ι.

Let $\mathscr{C}$ be of type D_{r+2}, (2.3.1), and given $\mathbf{x} \in \mathbb{Z}_\iota^\infty$, write $\sum_{i \in I} \mathrm{wt}_i(\mathbf{x})\alpha_i := \mathrm{wt}(x)$. We can repeat the previous line of reasoning, beginning from the set $\varXi_{\iota,\sigma}^r := \varXi_{\iota,\sigma} \cup \Gamma_r$, where

$$\varXi_{\iota,\sigma} := \varXi_\iota \cup \{\pm(\mathrm{wt}_{-1} - \mathrm{wt}_0)\} \tag{5.3.10}$$

$$\Gamma_r := \{-\gamma_k \,|\, \iota_k \in J_r\}. \tag{5.3.11}$$

These data yield a presentation of the J_r-highest-weight subset of $\widehat{\varSigma}_{\iota,\sigma}$ as the associated semigroup of the cone $C_{\iota,\sigma}^r := \mathbb{R}_{\geq 0}\varXi_{\iota,\sigma}^r$

$$\mathrm{HW}_{J_r}\widehat{\varSigma}_{\iota,\sigma} = \left(C_{\iota,\sigma}^r\right)^\vee \cap \mathbb{Z}^\ell. \tag{5.3.12}$$

The obvious variant of Lemma [CLS11, Proposition 1.2.23] provides a unique finite $\subset$-minimal generating set $\mathcal{H}_{\iota,\sigma}^r$ of $\mathrm{HW}_{J_r}\varSigma_{\iota,\sigma}$.

Fact 5.3.10. *For* $\mathscr{C}$ *of type* D_{r+2} *as in* (2.3.1), *with* $1 \leq r \leq 7$ *and* ι *obtained from the Coxeter word* $(-1, 0, 1, \ldots, r)$, *we have* $\mathcal{H}_{\iota,\sigma}^{J_r} = \eta_\iota\left[{}^{J_r}W_\sigma\right]$.

Fact 5.3.10 should be compared to Remark 5.3.9.

Conjecture 1. *Fact 5.3.10 holds for all* $r \geq 1$.

Proposition 5.3.11. *For* $\mathscr{C}$ *of type* D_{r+2} *as in* (2.3.1), *and* ι *obtained from the Coxeter word* $(-1, 0, 1, \ldots, r)$, *we have* $\eta_\iota[{}^{J_r}W_\sigma] \subset \mathcal{H}_{\iota,\sigma}^r$.

The proof of Proposition 5.3.11 relies on the following technical lemmata. The first is a reformulation of Lemma 5.3.6 in the double-indexing scheme, and the second concerns Demazure crystals in the polyhedral realization.

Lemma 5.3.12. *Let ι be a sequence from I satisfying* (1.5.1). *Then for all i and all $\mathbf{x}$ we have* $\min M_i(f_i\mathbf{x}) \leq \min M_i(\mathbf{x})$.

Given $\mathbf{i} := (i_n, \ldots, i_1), \mathbf{p} := (p_n, \ldots, p_1) \in \mathbb{N}^n$, denote $f_{\mathbf{i}}^{\mathbf{p}} := f_{i_n}^{p_n} \cdots f_{i_1}^{p_n} \in \mathcal{F}$. Thus, if $\mathbf{i} \in \mathcal{R}(w)$ then $B_w(\infty) = \left\{ f_{\mathbf{i}}^{\mathbf{p}} b_\infty \,|\, \mathbf{p} \in \mathbb{N}^{\ell(w)} \right\}$ (*cf. condition (2) on p.18*).

Lemma 5.3.13. *Suppose that $\mathscr{C}$ is of type D_{r+2}, 2.3.1, and let ι be the adapted sequence obtained from the Coxeter word $(-1, 0, 1, \cdots, r)$. For all $n \geq 1$ odd and all $\mathbf{x} \in (\varSigma_\iota)_{{}^{J_n}w_\circ}$ we have*

$$\max\{m \geq 1 \,|\, (\exists i \in I)\, x_{m,i} > 0\} \leq n+1$$

That is, an element of $(\varSigma_\iota)_{{}^{J_n}w_\circ}$ is only supported up to block $n+1$.

Proof. We may assume that $n = r+2$ without loss of generality. Lemma 3.2.5 indicates that

$$\mathcal{R}({}^{J_n}w_\circ) \ni \mathbf{i} := (\underbrace{-1}_{\text{block } n+1}, \underbrace{1, 0}_{\text{block } n}, \underbrace{2, 1, -1}_{\text{block } n-1}, \ldots, \underbrace{n, n-1, \ldots, 1, 0}_{\text{block } 1}).$$

Accordingly, writing $\ell := \ell({}^{J_n}w_\circ)$ we must show that the claim holds for $x = f_{\mathbf{i}}^{\mathbf{p}}\mathbf{0}$, where $\mathbf{p} := (p_\ell, \ldots, p_1) \in \mathbb{N}^\ell$ is arbitrary.

First consider block 1. Obviously $m_0(\mathbf{0}) = m_0(f_0^t\mathbf{0}) = 1$ for all $1 \leq t \leq p_1$. Now $\gamma_{1,1}(f_0^{p_1}\mathbf{0}) < 0$, while $\gamma_{2,1}(f_0^{p_1}\mathbf{0}) = 0$, so $m_1(f_0^{p_1}\mathbf{0}) = 2$. If $t < p_1/2$ then $\gamma_{1,1}(f_1^t f_0^{p_0}\mathbf{0}) = -p_1 + 2t$ and $m_1(f_1^t f_0^{p_0}\mathbf{0}) = 2$; otherwise $m_1(f_1^t f_0^{p_0}\mathbf{0}) = 1$. Similarly, $m_2(f_1^{p_2} f_0^{p_1}\mathbf{0}) = 3$ and $m_2(f_2^t f_1^{p_2} f_0^{p_1}\mathbf{0}) \leq 3$ for $t \leq p_3$ by Lemma 5.3.12 Continuing in this fashion, we find that $m_n(f_{n-1}^{p_n} \cdots f_1^{p_2} f_0^{p_1}\mathbf{0}) = n+1$ and $m_n(f_n^t f_{n-1}^{p_n} \cdots f_1^{p_2} f_0^{p_1}\mathbf{0}) \leq n+1$ for $t \leq p_{n+1}$ by Lemma 5.3.12

Repeat this argument for block 2. Letting $\mathbf{y} := f_n^{p_{n+1}} f_{n-1}^{p_n} \cdots f_1^{p_2} f_0^{p_1}\mathbf{0}$ We have $m_{-1}(\mathbf{y}) = m_{-1}(f_1^{p_2} f_0^{p_1}\mathbf{0})$ by Lemma 5.1.3. Now $\gamma_{1,-1}(f_1^{p_2} f_0^{p_1}\mathbf{0}) = -t$ for $t \leq p_2$ and $\gamma_{2,-1}(f_1^{p_2} f_0^{p_1}\mathbf{0}) = 0$, so $m_{-1}(f_{-1}^t\mathbf{y}) = 2$ and $m_{-1}(f_{-1}^t f_n^{p_{n+1}} f_{n-1}^{p_n} \cdots f_1^{p_2} f_0^{p_1}\mathbf{0}) \leq 2$ for $t \leq p_{n+2}$ by Lemma 5.3.12. Letting $\mathbf{y} := f_{-1}^{p_{n+2}} \cdots f_0^{p_1}\mathbf{0}$, it is clear from the previous steps and the structure of $\mathbf{i}$ that $\max\{(m,i) \,|\, y_{m,i} > 0\} \leq (3,2)$; thus $\gamma_{m,1}(\mathbf{y}) = 0$ for $m \geq 3$, which implies that $m_1(f_1^t\mathbf{y}) \leq 3$ for $0 \leq t \leq p_{n+2}$. As before, letting $\mathbf{y} := f_1^{p_{n+2}} \cdots f_0^{p_1}\mathbf{0}$, it is clear from the previous steps and the structure of $\mathbf{i}$ that $\gamma_{m,2}(\mathbf{y}) \leq 0$ for $m \geq 4$; thus $m_2(f_2^t\mathbf{y}) \leq 4$ for all $0 \leq t \leq p_{n+3}$. Continuing in this fashion we arrive at $m_n(f_n^t f_{n-1}^{p_{2n}} \cdots f_0^{p_1}\mathbf{0}) \leq n+1$ for $0 \leq t \leq p_{2n+1}$.

Repeating this argument for each of the $n+1$ blocks of indices in $\mathbf{i}$, for the leftmost index i of each block we find that $m_i(f_i^t f_{i-1}^{p_k} \cdots f_0^{p_1}\mathbf{0}) \leq n+1$. The claim follows. $\square$

For each $w \in {}^{J_r}W_\sigma$ there exists an odd integer $n \geq 1$ such that $w \leq {}^{J_n}w_\circ$. If $\mathbf{j}, \mathbf{i}$ denote reduced words for $w, {}^{J_n}w_\circ$ respectively then $\mathbf{j}$ is a subword of $\mathbf{i}$, and all elements of $(\Sigma_\iota)_w$ have the form $f_{\mathbf{i}}^{\mathbf{p}}\mathbf{0}$, where $p_k = 0$ if i_k does not appear in $\mathbf{j}$. Thus

Corollary 5.3.14. *Suppose that $\mathscr{C}$ is of type D_{r+2}* (2.3.1)*, and let ι be the adapted sequence obtained from the Coxeter word $(-1, 0, 1, \cdots, r)$. For all $w \in {}^{J_r}W_\sigma$ and all $\mathbf{x} \in (\Sigma_\iota)_w$ we have*

$$\max\{m \geq 1 \,|\, (\exists i \in I)\, x_{m,i} > 0\} \leq r+1$$

Proof of Proposition 5.3.11. Given $w \in {}^{J_r}W_\sigma$, consider $\eta_\iota(w) \in \Sigma_\iota$. Because w is fully commutative and its reduced expressions are σ-balanced, we have $\eta_\iota(w) \in \Sigma_{\iota,\sigma}$. Let ι' be a sequence from I satisfying (1.5.1) and such that $(\iota_{\ell(w)}, \ldots, \iota_1) \in \mathcal{R}(w)$. The proof of Proposition 5.3.3 shows that $\eta_{\iota'}(w) \in \mathrm{HW}_{J_r}\Sigma_{\iota'}$. Letting $\psi : \Sigma_{\iota'} \to \Sigma_\iota$ denote the braid-type isomorphism [?Nak99], we have $\eta_\iota(w) = \psi \circ \eta_{\iota'}(w) \in \mathrm{HW}_{J_r}\Sigma_{\iota,\sigma}$.

If $\eta_\iota(w)$ is not irreducible in $\mathrm{HW}_{J_r}\Sigma_{\iota,\sigma}$, then there exist nonzero $\mathbf{x}, \mathbf{y} \in \Sigma_{\iota,\sigma}$ such that $\eta_\iota(w) = \mathbf{x} + \mathbf{y}$. By Theorem 5.3.1 there exist $u, v \in {}^{J_r}W_\sigma$ such that $\mathbf{x} \in (\Sigma_{\iota,\sigma})_u$ and $\mathbf{y} \in (\Sigma_{\iota,\sigma})_v$. Since $\mathrm{wt}(\mathbf{x}) + \mathrm{wt}(\mathbf{y}) = \mathrm{wt}(\eta_\iota(w))$ it must be that $\ell_0(u), \ell_0(v) < \ell_0(w)$. Therefore, if $u \in T_s, v \in T_t, w \in T_k$ then $s, t < k$. According to Corollary 5.3.14, there exists an index (m, i) on which $\eta_\iota(w)$ is supported but neither $\mathbf{x}$ nor $\mathbf{y}$ is supported. This contradiction indicates that $\eta_\iota(w)$ is irreducible. $\square$

5.3.3 Multiplicities of highest-weight elements

Assume $\mathscr{C}$ is of finite type and ι is adapted. A finite set $\mathcal{H} \subset \mathrm{HW}\,\widehat{\Sigma}_{\iota,\sigma}$ generates a subset $S(\mathcal{H}) := \mathbb{N}\mathcal{H}$ of highest-weight elements. The latter inherits a grading $S(\mathcal{H}) = \bigsqcup_{\beta \in Q^+(\mathscr{C}^{\sigma\vee})} S(\mathcal{H})_{-\beta}$ from the structurally folded crystal $\widehat{\Sigma}_{\iota\sigma}$.

We describe an elementary procedure for computing $\#S(\mathcal{H})_{-\beta}$. Identify $Q^+(\mathscr{C}^{\sigma\vee})$ with $\mathbb{Z}^{\oplus|I/\sigma|}$ via $\alpha_{\mathbf{i}} \leftrightarrow \mathbf{e}(\mathbf{i})$, where $\{\mathbf{e}(\mathbf{i}) \,|\, \mathbf{i} \in I/\sigma\}$ denotes the standard basis. Form the matrix $H \in \mathrm{Mat}_{|I/\sigma| \times |\mathcal{H}|}(\mathbb{Z})$ whose columns are $\mathrm{wt}_\sigma(\mathbf{h}), \mathbf{h} \in \mathcal{H}$. Now

$$\#S(\mathcal{H})_{-\beta} = \#\{\mathbf{x} \in \mathbb{N}^{|\mathcal{H}|} \,|\, H\mathbf{x} = \beta\}$$

that is to say, $\#S(\mathcal{H})_{-\beta}$ equals the vector partition function of H. Given a particular solution $H\mathbf{t}_\beta = \beta$, we have

$$\#S(\mathcal{H})_{-\beta} = \#\left(\{\mathbf{z} \in \ker H \,|\, \mathbf{t}_\beta + \mathbf{z} \geq 0\} \cap \mathbb{Z}^{|\mathcal{H}|}\right)$$

which is the discrete volume of a certain lattice polytope; expressing $F \in \ker H$ in terms of the basis afforded by Gauss-Jordan elimination we get an explicit presentation of this polytope.

DC series

Recall the setup of Proposition 5.3.11, and consider $\mathcal{H}^r_{\iota,\sigma} := \Psi_\iota \circ \eta[{}^{J_r}W_\sigma]$. The proof of Proposition 5.3.3 indicates that $S(\mathcal{H}^r_{\iota,\sigma}) \subset \mathrm{HW}_{J_r}\widehat{B}(\infty)_\sigma$; indeed, equality holds for $1 \leq r \leq 7$ according to Fact 5.3.10.

Given $\beta \in \widehat{Q}^\sigma$, we have

$$S(\mathcal{H}^r_{\iota,\sigma})_{-\beta} \neq \varnothing \iff \beta = \sum_{w \in {}^{J_r}W_\sigma} c_w \mathrm{wt}(w), \quad c_w \in \mathbb{N}.$$

Define an ordering ${}^{J_r}W_\sigma = \{w_1, \ldots, w_{2^r-1}\}$ by $w_1 := {}^{J_1}w_\circ$, and arrange elements of $\overline{{}^{J_r}W_\sigma}$ (p.35) according to their length. Remark that $\mathrm{wt}(w_{2^k}) = (1, \ldots, 1)$ for all k. This yields a matrix of nonnegative integers whose columns are $\mathrm{wt}(w_k)$

$$H^r_{\iota,\sigma} := \begin{pmatrix} \mathrm{wt}(w_1) & \mathrm{wt}(w_2) & \ldots & \mathrm{wt}(w_{2^r-1}) \end{pmatrix}_{(r+1)\times(2^r-1)}.$$

Note that

$$\operatorname{rank} H^r_{\iota,\sigma} = r+1, \qquad \operatorname{nullity} H^r_{\iota,\sigma} = 2^{r-1} - r - 1.$$

Let $\mathbf{e}(i) : 1 \leq i \leq r+1$ denote the standard basis of $\mathbb{R}^{r+1}$, and identify $Q^+(C_{r+1})$ with $\mathbb{N}^{r+1}$ via $\alpha_i \leftrightarrow \mathbf{e}(i+1)$.

Example 5.3.15. The first four matrices are as follows

$$H^1_{\iota,\sigma} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad H^2_{\iota,\sigma} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \qquad H^3_{\iota,\sigma} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$H^4_{\iota,\sigma} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\ 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The columns in these matrices are obtained from the following reduced expressions

$$\overline{{}^{J_1}W_\sigma}\begin{cases}\mathrm{wt}^\sigma({}^{J_1}w_\circ) = \alpha_0+\alpha_1\end{cases}$$

$$\overline{{}^{J_2}W_\sigma}\begin{cases}\mathrm{wt}^\sigma({}^{J_1}w_\circ s_2) = \alpha_0+\alpha_1+\alpha_2\\ \mathrm{wt}^\sigma({}^{J_1}w_\circ s_2 s_1) = \alpha_0+2\alpha_1+\alpha_2\end{cases}$$

$$\overline{{}^{J_3}W_\sigma}\begin{cases}\mathrm{wt}^\sigma({}^{J_1}w_\circ s_2 s_3) = \alpha_0+\alpha_1+\alpha_2+\alpha_3\\ \mathrm{wt}^\sigma({}^{J_1}w_\circ s_2 s_1 s_3) = \alpha_0+2\alpha_1+\alpha_2+\alpha_3\\ \mathrm{wt}^\sigma({}^{J_1}w_\circ s_2 s_1 s_3 s_2) = \alpha_0+2\alpha_1+2\alpha_2+\alpha_3\\ \mathrm{wt}^\sigma({}^{J_3}w_\circ) = 2\alpha_0+3\alpha_1+2\alpha_2+\alpha_3\end{cases}$$

Let $\pi^r_{\iota,\sigma} : Q^+(C_{r+1}) \to \mathbb{N}$ be the vector partition function associated to $H^r_{\iota,\sigma}$

$$\pi^r_{\iota,\sigma}(\beta) := \#\left\{\mathbf{x}\in\mathbb{N}^{r+1} \mid H^r_{\iota,\sigma}\mathbf{x} = \beta\right\}.$$

By virtue of the setup,

$$\#\mathrm{HW}_{J_r}\widehat{B}(\infty)_{-\beta} = \pi^r_{\iota,\sigma}(\beta).$$

Remark 5.3.16. Let $\mathrm{incl} : Q^+(C_{r+1}) \hookrightarrow \widehat{Q}^{\sigma+}(C_{r+2})$ denote the natural inclusion monomorphism. If for $\beta = \sum_{i=0}^r b_i\alpha_i$ it happens that $b_r = 0$, then $\beta = \mathrm{incl}(\gamma)$ for some $\gamma \in Q^+(C_{r+2})$. In this case $\#\mathrm{HW}_{J_{r+1}}B(\infty)_{-\beta} = \#\mathrm{HW}_{J_r}B(\infty)_{-\gamma}$.

Proposition 5.3.17. *Given $\beta = \sum_{i=0}^r b_i\alpha_i \in Q^+(C_{r+1})$, the vector*

$$\mathbf{t}_2 := (b_0-b_2)\mathbf{e}(1) + (b_0-b_1+b_2)\mathbf{e}(2) + (-b_0+b_1)\mathbf{e}(3)$$

is a solution to $H^2_{\iota,\sigma}\mathbf{x} = \beta$, and for $r \geq 3$ the vector

$$\mathbf{t}_r := \mathbf{t}_{r-1} + (-b_r)\mathbf{e}(2^{r-2}) + b_r\mathbf{e}(2^{r-1})$$

is a solution to $H^r_{\iota,\sigma}\mathbf{x} = \beta$.

Proof. We proceed by induction on r. The claim is easily checked for $r = 2$. Indeed

$$H^2_{\iota,\sigma} = \begin{pmatrix}1&1&1\\1&1&2\\0&1&1\end{pmatrix}\begin{pmatrix}b_0-b_2\\b_0-b_1+b_2\\-b_0+b_1\end{pmatrix} = \begin{pmatrix}b_0\\b_1\\b_2\end{pmatrix}$$

and so the induction begins. Now, observe that $\mathbf{t}_{r-1}$ is not supported on any $\mathbf{e}(k) : k \geq 2^{r-1}$. Regarding the former as the image of $\mathbf{t}_{r-1}$ under the inclusion homomorphism, w have

$$H^r_{\iota,\sigma}\mathbf{t}_{r-1} = \text{incl. } H^{r-1}_{\iota,\sigma}\mathbf{t}_{r-1} = (b_0, \ldots, b_r, 0)^T. \tag{5.3.13}$$

Now

$$\begin{aligned}(-b_r)H^r_{\iota,\sigma}\mathbf{e}(2^{r-2}) &= (-b_r)\mathrm{wt}(w_{2^{r-2}}) \\ &= (-b_r)(1, 1, \ldots, 1, 0)^T \qquad (5.3.14)\\ b_r H^r_{\iota,\sigma}\mathbf{e}(2^{r-1}) &= b_r\mathrm{wt}(w_{2^{r-2}}) \\ &= b_r(1, 1, \ldots, 1, 1)^T. \qquad (5.3.15)\end{aligned}$$

Combining (5.3.13), (5.3.14), (5.3.15) yields

$$\begin{aligned}H^r_{\iota,\sigma}\mathbf{t}_r &= H^r_{\iota,\sigma}\mathbf{t}_{r-1} + (-b_r)H^r_{\iota,\sigma}\mathbf{e}(2^{r-2}) + b_r H^r_{\iota,\sigma}\mathbf{e}(2^{r-1}) \\ &= \begin{pmatrix} b_0 \\ \vdots \\ b_{r-1} \\ 0 \end{pmatrix} + \begin{pmatrix} -b_r \\ \vdots \\ -b_r \\ 0 \end{pmatrix} + \begin{pmatrix} b_r \\ \vdots \\ b_r \\ b_r \end{pmatrix} = \begin{pmatrix} b_0 \\ \vdots \\ b_{r-1} \\ b_r \end{pmatrix}\end{aligned} \qquad \square$$

It follows from Gauss-Jordan elimination we see that any solution $\mathbf{x} \in \mathbb{R}^{n+1}$ to $H^r_{\iota,\sigma}\mathbf{x} = \mathbf{b}$ has the form $\mathbf{x} = \mathbf{t}_r + \mathbf{z}$, where $\mathbf{z} \in \ker H^r_{\iota,\sigma}$. Therefore

$$|\mathrm{HW}_{J_r}\widehat{B}(\infty)_{-\beta}| = \left|\left\{\mathbf{z} \in \ker H^r_{\iota,\sigma} \,|\, \mathbf{t}_r + \mathbf{z} \geq 0\right\} \cap \mathbb{Z}^{r+1}\right|$$

meaning that the number of highest-weight elements of weight $\beta \in Q^{+\sigma}$ equals the discrete volume of the lattice polytope described by the inequalities $\mathbf{t}_r + \mathbf{z} \geq 0, \mathbf{z} \in \ker H^r_{\iota,\sigma}$.

Taking advantage of the facts that $H^r_{\iota,\sigma} : r = 1, 2$ have trivial kernels, we get the following.

Fact 5.3.18. *In type (D_{r+2}, C_{r+1}) (2.3.1) for $1 \leq r \leq 7$, suppose $\beta = \sum_{i=0}^r b_i\alpha_i \in Q^+(C_{r+1})$. Then*

1. *For $r = 1$ we have*

$$|HW\ B(\infty)_{-\beta}| = \begin{cases} 1, & b_0 = b_1 \\ 0, & \textit{otherwise} \end{cases}$$

2. For $r = 2$ we have

$$|HW\,B(\infty)_{-\beta}| = \begin{cases} 1, & \begin{cases} b_0 - b_2 \geq 0 \\ b_0 - b_1 + b_2 \geq 0 \\ -b_0 + b_1 \geq 0 \end{cases} \\ 0, & \textit{otherwise} \end{cases}$$

The situation in type (D_5, C_4) $(r = 3)$ is already very different. Indeed, we have

$$|\mathrm{HW}\ B(\infty)_{-\beta}| = \#(z_1, z_2, z_3) \in \mathbb{N}^3 \text{ such that } \begin{cases} b_0 - b_2 \geq z_1 - z_2 \\ b_0 - b_1 + b_2 - b_3 \geq -z_1 - z_3 \\ -b_0 + b_1 \geq z_1 + z_2 + z_3 \\ b_3 \geq z_1 + z_2 + z_3 \end{cases}$$

Observe that if $b_3 = 0$ then there is either a unique solution $\mathbf{z} = \mathbf{0}$ or no solution, in accordance with Remark 5.3.16. It would be interesting to find a closed form, or at least a good estimate, for the number of lattice points in this polytope.

Let us describe $\ker H^r_{\iota,\sigma}$ more precisely. Given a matrix $M = (m_{x,y})_{x \in X, y \in Y}$ and subsets $X' \subset X, Y' \subset Y$, let $M_{X',Y'}$ denote the submatrix on the index set $X' \times Y'$. For each $r \geq 1$ we have $H^r_{\iota,\sigma} = (H^{r+1}_{\iota,\sigma})_{[1,r]\times[1,2^r-1]}$. Accordingly, for $r > 1$ we put

$$\overline{H}^r_{\iota,\sigma} := (H^r_{\iota,\sigma})_{[1,r]\times[2^{r-1},2^r-1]} = (\mathrm{wt}(w))_{w \in \overline{{}^{J_r}W_\sigma}}$$

and letting $\mathrm{incl} : \mathbb{R}^{2^{r-1}-1} \hookrightarrow \mathbb{R}^{2^r-1}$ it is clear that $\mathrm{incl}[\ker H^{r-1}_{\iota,\sigma}]$ is a direct summand of $\ker H^r_{\iota,\sigma}$. Write

$$\ker H^r_{\iota,\sigma} = \mathrm{incl}[\ker H^{r-1}_{\iota,\sigma}] \oplus \overline{\ker H^r_{\iota,\sigma}}, \qquad \dim \overline{\ker H^r_{\iota,\sigma}} = 2^{r-1} - 1$$

where $\overline{\ker H^r_{\iota,\sigma}} := \ker H_{\iota,\sigma} \setminus \mathrm{incl}[\ker H^{r-1}_{\iota,\sigma}]$

We describe a sequence of bases $X^r_{\iota,\sigma}$ for $\ker H^r_{\iota,\sigma}$ with the property that $X^r_{\iota,\sigma} \hookrightarrow X^{r+1}_{\iota,\sigma}$. Of course $X^r_{\iota,\sigma} = \varnothing$ for $r = 1, 2$; for $r = 3, 4$ Gaussian elimination gives

$$X^3_{\iota,\sigma} = \left(\begin{array}{ccc} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \\ \hline -1 & -1 & -1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \qquad X^4_{\iota,\sigma} = \left(\begin{array}{ccc|ccccccc} 0 & 1 & -1 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 1 & 2 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 \\ \hline -1 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ \cline{4-10} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Observe that $X^3_{\iota,\sigma} \hookrightarrow X^4_{\iota,\sigma}$; indeed the former coincides with the $[1, 7] \times [1, 3]$-submatrix of the latter. Denote

$$X^3_{\iota,\sigma} = \left(\begin{array}{c} \overline{X}^3_{\iota,\sigma} \\ \hline \mathbf{v}^{(1)}_3 \\ \hline I_3 \end{array}\right) \qquad X^4_{\iota,\sigma} = \left(\begin{array}{c|c} & \overline{X}^4_{\iota,\sigma} \\ \cline{2-2} X^3_{\iota,\sigma} & \mathbf{v}^{(2)}_3 \\ \cline{2-2} & 0_{3\times 7} \\ \hline 0_{8\times 3} & \mathbf{v}^{(1)}_4 \\ \cline{2-2} & I_7 \end{array}\right) = \left(\begin{array}{c|c} \overline{X}^3_{\iota,\sigma} & \overline{X}^4_{\iota,\sigma} \\ \hline \mathbf{v}^{(1)}_3 & \mathbf{v}^{(2)}_3 \\ \hline I_3 & 0_{3\times 7} \\ \hline 0_{8\times 3} & \mathbf{v}^{(1)}_4 \\ \cline{2-2} & I_7 \end{array}\right)$$

For $r \geq 3$ let $\mathbf{v}^{(1)}_r := [-1, \ldots, -1]^T \in \mathbb{Z}^{2^{r-1}-1}$. Continuing this pattern, we can show by induction on r that for $r \geq 3$ there exists a uniquely determined $3 \times (2^{r-1} - 1)$ integer matrix $\overline{X}^r_{\iota,\sigma}$ and integer vectors $\mathbf{v}^{(t)}_k \in \mathbb{R}^{2^{r-1+(t-1)}-1}, 3 \leq k \leq r, 1 \leq t \leq k$ such that

$$\left(\begin{array}{c|c} & \overline{X}^r_{\iota,\sigma} \\ \cline{2-2} & \mathbf{v}^{(r-2)}_3 \\ \cline{2-2} & 0_{3\times(2^{r-1}-1)} \\ \cline{2-2} X^{r-1}_{\iota,\sigma} & \vdots \\ \cline{2-2} & \mathbf{v}^{(2)}_{r-1} \\ \cline{2-2} & 0_{(2^{r-2}-1)\times(2^{r-1}-1)} \\ \hline 0_{2^{r-1}\times 2^{r-2}-r} & \mathbf{v}^{(1)}_r \\ & I_{2^{r-1}-1} \end{array}\right)$$

Corollary 5.3.19 (Corollary to Proposition 5.3.17)**.** *Expressing* $\mathbf{z} \in \ker H^r_{\iota,\sigma}$ *using* $X^r_{\iota,\sigma}$*, the set of inequalities* $\mathbf{t}_r + \mathbf{z} \geq 0$ *comprises* $r+1$ *inequalities along with* $z_k \geq 0$ *for all* k*. The* $r+1$*st inequality is* $z_{2^{r-1}} + \cdots + z_{2^r-1} \leq b_r$

A better understanding of the structures of $\overline{X}^{r-1}_{\iota,\sigma}$ and the various $\mathbf{v}^{(t)}_k$ will yield a more precise description of $\ker H^r_{\iota,\sigma}$ and perhaps lead to an estimate for $|\mathrm{HW}_{J_r}\widehat{B}(\infty)_{-\beta}|$.

Triality of D_4

Recall from section 4.4 that $\mathrm{HW}_\sigma I = \langle\sigma\rangle\{1,2\} \cup \langle\sigma\rangle\{1,2,3\}$.

Fact 5.3.20. *Express* $\beta \in Q^+(G_2)$ *as* $\beta = b_1\alpha_1 + b_2\alpha_2$*. Then*

$$\#HW_J(\widehat{B}(\infty))_{-\beta} = \begin{cases} \begin{cases} \left(\left\lfloor \frac{2b_1-b_2}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{2b_1-b_2}{2} \right\rceil + 1\right), & \#J = 2 \\ 1, & \#J = 3 \end{cases} & \text{if } \begin{cases} 2b_1 \geq b_2 \\ b_2 \geq b_1 \end{cases} \\ 0, & \text{otherwise} \end{cases}$$

Proof. The weight matrix H of the Hilbert basis for $\mathrm{HW}_{1,2}\widehat{B}(\infty)_\sigma$ is

$$H = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}.$$

The equation $H\mathbf{x} = \beta$ has particular solution $\mathbf{t}_\beta = (2b_1 - b_2, 0, -b_1 + b_2, 0)^T$, and a basis for $\ker H$ is $\{(-2,0,0,1)^T, (-1,1,0,0)^T\}$. Accordingly,

$$\#\mathrm{HW}_{1,2}(\widehat{B}(\infty)_\sigma)_{-\beta} = \#\left\{(x_1,x_2) \in \mathbb{N}^2 \mid 2b_1 - b_2 - 2x_1 - y \geq 0, -b_1 + b_2 \geq 0\right\}.$$

Now, given $n \geq 0$, observe that

$$\begin{aligned} \#\{(x_1,x_2) \in \mathbb{N}^2 \mid 2x_1 - x_2 \leq n\} &= \sum_{k=0}^{n} \left\lfloor \frac{k}{2} \right\rfloor + 1 \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} (n+1-2k) \\ &= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)(n+1) - \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \\ &= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left[n - \left\lfloor \frac{n}{2} \right\rfloor + 1\right] \\ &= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \end{aligned}$$

On the other hand, it is clear that

$$\mathrm{HW}_{1,2,3}\widehat{B}(\infty)_{(134)} \subset \mathrm{HW}_{1,2,3}\widehat{B}(\infty)_{(34)}.$$

Applying Fact 5.3.18(2), the claim follows at once. $\square$

5.4 Connectedness

Lemma 5.4.1. *Let ι be an admissible sequence in the sense of* [NZ97]. *Then for all $\mathbf{x} \in \Sigma_\iota$ and any $(n, i) \in \mathbb{N} \times I$ we have $nf_i\mathbf{x} = f_i^n(n\mathbf{x})$.*

This has been proved for $\lambda \in \Lambda^+$ by Littelmann [Lit94] and Kashiwara [Kas96] independently.

Proposition 5.4.2. *In type (D_r, C_{r-1}) for $r \geq 4$, there exists a connected component of $B(\infty)_\sigma$ with infinitely many highest-weight elements.*

Proof. It suffices to prove this statement for $r = 4$, because $B(\infty) \in \mathrm{Crys}(D_4)$ identifies with a subgraph of $B(\infty) \in \mathrm{Crys}(D_r)$ by the branching rule [Kas95, 4.6].

We work in the polyhedral realization $\Psi_\iota : B(\infty) \cong \Sigma_\iota$ with adapted sequence ι obtained from the Coxeter word $(\ldots, -1, 0, 1, 2)$. Denote $x := f_{-1}f_1f_0b_\infty$. We know that $n\Psi_\iota(x) \in \mathrm{HW}_{-1,0,1,2}\widehat{\Sigma}_{\iota,\sigma}$ for all $n \geq 0$ and using Lemma 5.4.1 we calculate

$$n\Psi_\iota(x) = n(\mathbf{e}(3) + \mathbf{e}(6) + \mathbf{e}(8)).$$

It is straightforward to show that

$$\begin{aligned}\widehat{f}_{\mathbf{0}}\widehat{f}_{\mathbf{1}}^3\widehat{f}_{\mathbf{2}}\widehat{f}_{\mathbf{0}}(n\Psi_\iota(x)) &= \widehat{f}_{\mathbf{1}}\widehat{f}_{\mathbf{0}}\widehat{f}_{\mathbf{1}}\widehat{f}_{\mathbf{2}}((n+1)\Psi_\iota(x)) \\ &= (n+1)\mathbf{e}(3) + \mathbf{e}(4) + (n+2)\mathbf{e}(6) + (n+1)\mathbf{e}(8) + \mathbf{e}(9) + \mathbf{e}(10) + \mathbf{e}(11)\end{aligned}$$

for all $n \geq 1$. Passing to $B(\infty)_\sigma$, this calculation shows that there exist finite sequences $\underline{i}, \underline{j}$ from I/σ such that

$$[nx] = \left(e(\underline{j})f(\underline{i})\right)^{n-1}[x] \text{ for all } n \geq 1.$$

The claim follows. $\square$

Remark 5.4.3. We have many other examples of pairs of highest-weight elements and monomials connecting them. At the time of this writing, a pattern is not yet clear.

5.4.1 Examples of connectedness in type D_4

Let $G = (V, E)$ be a graph. It is well-known that the relation on V given by $v \sim v'$ if and only if v, v' belong to the same connected component of G is an equivalence relation. Naturally, the set $V/\sim$ coincides with the set of connected components of G.

Replacing the edges between elements of $\widehat{B}(\lambda)_\sigma$ from $\mathcal{F}$ by edges from $\widehat{\mathcal{F}}^\sigma$, we can view $\widehat{B}(\lambda)_\sigma$ as a subgraph of $B(\lambda)$. Thus an element of HW $\widehat{B}(\lambda)_\sigma/\widehat{\mathcal{A}}^\sigma$ is the set of $\widehat{\mathcal{A}}^\sigma$-highest-weight elements of a particular connected component of $\widehat{B}(\lambda)_\sigma$. Similarly, an element of HW $B(\lambda)_\sigma/\widehat{A}^{\sigma\vee}$ is the set of $\mathcal{A}^{\sigma\vee}$-highest-weight elements of a particular connected component of $B(\lambda)_\sigma$. Note that the number of connected components of $\widehat{B}(\lambda)_\sigma$ (respectively, $B(\lambda)_\sigma$) equals $|\mathrm{HW}\ \widehat{B}(\lambda)_\sigma/\widehat{\mathcal{A}}^\sigma|$ (respectively, HW $B(\lambda)_\sigma/\widehat{A}^{\sigma\vee}$). In the following table we consider certain $\lambda \in \widehat{\Lambda}^{\sigma+}$, expressed as $\lambda = \sum_{i\in I} l_i\alpha_i$ and identified with $(l_0, l_1, \ldots, l_r)$.

λ	$\lvert\mathrm{HW}\ \widehat{B}(\lambda)_\sigma\rvert$	$\lvert\mathrm{HW}\ \widehat{B}(\lambda)_\sigma/\widehat{\mathcal{A}}^\sigma\rvert$ (#HW $C \mid C \le \widehat{B}(\lambda)_\sigma$)	$\lvert\mathrm{HW}\ B(\lambda)_\sigma\rvert$	$\lvert\mathrm{HW}\ B(\lambda)_\sigma/\mathcal{A}^{\sigma\vee}\rvert$ (#HW $C \mid C \le B(\lambda)_\sigma$)
(1,1,1,1)	11	3 (1,5,5)	6	2 (1,5)
(2,1,1,1)	11	3 (1,5,5)	6	2 (1,5)
(1,2,1,1)	17	3 (1,8,8)	9	2 (1,8)
(1,3,1,1)	23	3 (1,11,11)	12	2 (1,11)
(1,1,2,2)	19	2 (1,18)	10	2 (1,9)
(1,1,3,3)	27	2 (1,26)	14	2 (1,13)
(2,2,1,1)	17	3 (1,8,8)	9	2 (1,8)
(2,1,2,2)	23	2 (1,22)	12	2 (1,22)
(1,2,2,2)	29	2 (1,28)	15	2 (1,14)
(1,3,2,2)	20	2 (1,19)	39	2 (1,38)
(2,2,2,2)	35	2 (1,34)	18	2 (1,17)
(1,2,3,3)	41	2 (1,40)	21	2 (1,20)

Based on these data, we make the following conjecture.

Conjecture 2. *In the category of C_3-crystals, we have $B(\infty)_\sigma \cong B(\infty) \oplus B$, where B is a connected upper normal mult-highest-weight crystal whose highest-weight set coincides with the semigroup generated by $\eta({}^{J_2}W_\sigma)$.*

5.5 Type (D_3, C_2)

In this section we work in type D_3 with notation as in (2.3.1). Let $\iota = (\iota_k \mid k \ge 1)$ be the σ-adapted sequence obtained from the Coxeter word $(1, 0, -1)$. Because $|\Phi^+(D_3)| = 6$ the polyhedral crystal Σ_ι identifies with a subset of $\mathbb{N}^6$. More precisely, we regard Σ_ι as the lattice points of a certain rational polyhedral convex cone in $\mathbb{R}^6$. The generating set Ξ_ι (*cf.* (1.5.3)) of the dual cone is as follows

$$\Xi_\iota = \{x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_2^* + x_3^* - x_4^*, x_3^* - x_5^*, x_4^* - x_5^*, x_2^* - x_6^*, x_4^* - x_6^*, x_4^* - x_5^* - x_6^*\}$$

According to Fact 5.3.10, a J_1-highest-weight element $\mathbf{x} \in \widehat{\Sigma}_{\iota,\sigma}$ has the form

$$\mathbf{x} = n(f_{-1} f_1 f_0 \mathbf{0}) = (0, 0, n, n, n, 0), \quad n \in \mathbb{N}.$$

The subset generated by $\mathbf{x}$ over the monoid $\widehat{F}^\sigma$ has the following form.

Lemma 5.5.1. *For all* $\mathbf{x} \in HW_{J_1} \widehat{\Sigma}_{\iota,\sigma}$ *we have*

$$\widehat{\mathcal{F}}^\sigma \mathbf{x} = \mathbf{x} + \{(d, c, c, b, a, a) \mid a \geq 0, b \geq c \geq d \geq 0\} \tag{5.5.1}$$

Proof. Each $\mathbf{y} \in \widehat{\mathcal{F}}^\sigma \mathbf{x}$ has the form $\widehat{f}_{\mathbf{i}_m} \cdots \widehat{f}_{\mathbf{i}_1} \mathbf{x}$ for some $m \geq 1$. We prove the Lemma by induction on m. The claim is clear for $m = 0$, and so induction begins.

Suppose that $\mathbf{y} = \widehat{f}_{\mathbf{i}_m} \cdots \widehat{f}_{\mathbf{i}_1} \mathbf{x}$ has the form prescribed by (5.5.1). By (1.5.7),

$$\widehat{f}_{\mathbf{1}} \mathbf{y} = \begin{cases} \mathbf{x} + (d, c, c, b+1, a, a), & b + d \geq 2c \\ \mathbf{x} + (d+1, c, c, b, a, a), & 2c > b + d \end{cases}$$

It is clear that $(b+1) \geq c \geq d$; on the other hand, if $2c - b > d$, then it follows that $c > d$ because $b - c \geq 0$ from the polyhedral inequalities. Thus in either case $\widehat{f}_{\mathbf{1}} \mathbf{y}$ has the desired form. By (1.5.3) we have $\gamma_1(\mathbf{y}) = \gamma_2(\mathbf{y}) + n$, and $\gamma_4(\mathbf{y}) = \gamma_5(\mathbf{y}) + n$, and it follows that $\min M_0(\mathbf{y}) = \min M_{-1}(\mathbf{y}) + 1$. In other words, f_{-1}, f_0 act on $\mathbf{y}$ at the same block of ι (*cf.* section 5.1). Therefore by (1.5.7)

$$\widehat{f}_{\mathbf{0}} \mathbf{y} = \begin{cases} \mathbf{x} + (d, c, c, b, a+1, a+1), & a + c \geq b \\ \mathbf{x} + (d, c+1, c+1, b, a, a), & b > a + c \end{cases}$$

The inequalities of (5.5.1) are clearly satisfied in the first case; on the other hand, if $b > a + c$ then also $b \geq c + 1$, and so $\widehat{f}_{\mathbf{0}} \mathbf{y}$ has the desired form. $\square$

Let $\boldsymbol{\iota}$ denote the sequence obtained from the Coxeter word $(\mathbf{1}, \mathbf{0})$ in type C_2, and let $\Sigma_{\boldsymbol{\iota}}$ denote the corresponding polyhedral realization of $B(\infty) \in \mathrm{Crys}(C_2)$. Nakashima and Zelevinsky have shown [NZ97, Theorem 4.1] that

$$\Sigma_{\boldsymbol{\iota}} = \{(d, c, b, a) \,|\, a \geq 0, b \geq c \geq d \geq 0\}$$

Denote $\mathbf{x} := f_{-1}f_1f_0\mathbf{0}$, and for each $n \geq 0$ define a map $\Phi_n : \widehat{F}^{\sigma}(n\mathbf{x}) \to \Sigma_{\boldsymbol{\iota}}$ by

$$\Phi_n\left(n\mathbf{x} + (d, c, c, b, a, a)\right) := (d, c, b, a).$$

Lemma 5.5.2. *For all $n \in \mathbb{N}$ the map Φ_n is a bijection satisfying $\Phi_n \circ \widehat{f}_{\mathbf{i}} = f_{\mathbf{i}} \circ \Phi_n$.*

Given $\mathbf{y} \in \widehat{F}^{\sigma}(n\mathbf{x})$ consider $\Phi_n(\mathbf{y}) \in \Sigma_{\boldsymbol{\iota}}$. If $\mathbf{i} \in I/\sigma$ satisfies $e_{\mathbf{i}}\Phi_n(\mathbf{y}) \neq 0$ then there exists a unique $\mathbf{z} \in \widehat{F}^{\sigma}(n\mathbf{x})$ such that $e_{\mathbf{i}}\Phi_n(\mathbf{y}) = \Phi_n(\mathbf{z})$. Axiom 1.3.1(1) gives $\Phi_n(\mathbf{y}) = f_{\mathbf{i}}\Phi_n(\mathbf{z}) = \Phi_n(\widehat{f}_{\mathbf{i}}\mathbf{z})$, and so it must be that $\mathbf{y} = \widehat{f}_{\mathbf{i}}\mathbf{z}$, which is to say $\mathbf{z} = \widehat{e}_{\mathbf{i}}\mathbf{y}$. Thus we have proved

Proposition 5.5.3. *Let $\mathbf{x} = f_{-1}f_1f_0\mathbf{0}$. For all $n \geq 0$, $\mathcal{F}^{\sigma\vee}[n\mathbf{x}]$ is a subcrystal of $\Sigma_{\iota,\sigma}$. Furthermore, each $\mathcal{F}^{\sigma\vee}[n\mathbf{x}]$ is isomorphic to the C_2-crystal $\Sigma_{\boldsymbol{\iota}}$.*

Corollary 5.5.4. *In type D_3, $B(\infty)_\sigma$ is isomorphic to $B(\infty)_{C_2}^{\oplus HWB(\infty)_\sigma}$ in the category of C_2-crystals.*

Bibliography

[BB05] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.

[BG11] A. Berenstein and J. Greenstein, *Quantum folding*, Int. Math. Res. Not. IMRN **21** (2011), 4821–4883.

[CLS11] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011.

[Fan95] C. K. Fan, *A Hecke algebra quotient and properties of commutative elements of a Weyl group*, ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[FSS96] J. Fuchs, B. Schellekens, and C. Schweigert, *From Dynkin diagram symmetries to fixed point structures*, Comm. Math. Phys. **180** (1996), no. 1, 39–97.

[Hum72] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Berlin, 1972.

[Jos95] A. Joseph, *Quantum groups and their primitive ideals*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 29, Springer-Verlag, Berlin, 1995.

[Jos02] ———, *Combinatoire de Crystaux, Cours de troisiéte cycle*, Université P. et M. Curie (2002).

[Jos09] ———, *A pentagonal crystal, the golden section, alcove packing and aperiodic tilings*, Transform. Groups **14** (2009), no. 3, 557–612.

[Kac90] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990.

[Kas90a] M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. **133** (1990), no. 2, 249–260.

[Kas90b] ———, *Bases cristallines*, C. R. Acad. Sci. Paris Sér. I Math. **311** (1990), no. 6, 277–280 (French, with English summary).

[Kas91] ———, *On crystal bases of the Q-analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), no. 2, 465–516.

[Kas93] ———, *The crystal base and Littelmann's refined Demazure character formula*, Duke Math. J. **71** (1993), no. 3, 839–858.

[Kas94] ———, *Crystal bases of modified quantized enveloping algebra*, Duke Math. J. **73** (1994), no. 2, 383–413.

[Kas95] ———, *On crystal bases*, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 155–197.

[Kas96] ———, *Similarity of crystal bases*, Lie algebras and their representations (Seoul, 1995), Contemp. Math., vol. 194, Amer. Math. Soc., Providence, RI, 1996, pp. 177–186.

[Kos98] B. Kostant, *The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations*, Internat. Math. Res. Notices **5** (1998), 225–252.

[Lit94] P. Littelmann, *A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras*, Invent. Math. **116** (1994), no. 1-3, 329–346.

[Lit98] ———, *Cones, crystals, and patterns*, Transform. Groups **3** (1998), no. 2, 145–179.

[Lus90a] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), no. 2, 447–498.

[Lus90b] ———, *Canonical bases arising from quantized enveloping algebras. II*, Progr. Theoret. Phys. Suppl. **102** (1990), 175–201 (1991). Common trends in mathematics and quantum field theories (Kyoto, 1990).

[Lus93] ———, *Introduction to quantum groups*, Progress in Mathematics, vol. 110, Birkhäuser Boston, Inc., Boston, MA, 1993.

[ML98] S. Mac Lane, *Categories for the working mathematician*, 2nd ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[NS01] S. Naito and D. Sagaki, *Lakshmibai-Seshadri paths fixed by a diagram automorphism*, J. Algebra **245** (2001), no. 1, 395–412, DOI 10.1006/jabr.2001.8904. MR1868198 (2002j:17026)

[NS02] ———, *Standard paths and standard monomials fixed by a diagram automorphism*, J. Algebra **251** (2002), no. 1, 461–474.

[NS04] ———, *A rationalization of the crystal $\mathbb{Z}^\infty$ and a diagram automorphism*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 279–295.

[Nak99] T. Nakashima, *Polyhedral realizations of crystal bases and braid-type isomorphisms*, Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), Contemp. Math., vol. 248, Amer. Math. Soc., Providence, RI, 1999, pp. 419–435.

[Nak02] ———, *Polytopes for crystallized Demazure modules and extremal vectors*, Comm. Algebra **30** (2002), no. 3, 1349–1367.

[NS01] Satoshi Naito and Daisuke Sagaki, *Lakshmibai-Seshadri paths fixed by a diagram automorphism*, J. Algebra **245** (2001), no. 1, 395–412.

[NS02] ———, *Standard paths and standard monomials fixed by a diagram automorphism*, J. Algebra **251** (2002), no. 1, 461–474.

[NS03] ———, *Three kinds of extremal weight vectors fixed by a diagram automorphism*, J. Algebra **268** (2003), no. 1, 343–365.

[NS04] ———, *A rationalization of the crystal $\mathbb{Z}^\infty$ and a diagram automorphism*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 279–295.

[NS05] ———, *Crystal base elements of an extremal weight module fixed by a diagram automorphism*, Algebr. Represent. Theory **8** (2005), no. 5, 689–707.

[NZ97] T. Nakashima and A. Zelevinsky, *Polyhedral realizations of crystal bases for quantized Kac-Moody algebras*, Adv. Math. **131** (1997), no. 1, 253–278.

[Slo] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences.*

[RH02] S. Ryom-Hansen, *Littelmann's refined Demazure character formula revisited*, Sém. Lothar. Combin. **49** (2002/04), Art. B49d, 10 pp. (electronic).

[Ste96] J. R. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), no. 4, 353–385.

[Ste97] ———, *Some combinatorial aspects of reduced words in finite Coxeter groups*, Trans. Amer. Math. Soc. **349** (1997), no. 4, 1285–1332.

[Ste03] ———, *A local characterization of simply-laced crystals*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 4807–4823 (electronic).

[Tit69] J. Tits, *Le problème des mots dans les groupes de Coxeter*, Symposia Mathematica (INDAM, Rome, 1967/68), Academic Press, London, 1969, pp. 175–185 (French).