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Options and Expectations

Hayne E. Leland

"Perfection of means, and confusion of goals, seem in my opinion to characterize the age."
—Albert Einstein

Since the pioneering work of Black and Scholes [1973], much has been written on the pricing and hedging of options. Less attention has been paid to an equally important question: Who should buy (and who should sell) options?

Because derivatives have a zero net supply — for every long position there is a short — the average or consensus investor will neither buy nor sell derivatives. The average investor must hold the market portfolio, which includes no net derivative positions. Thus derivatives will be purchased *only by investors who differ from the average investor.*

Investors may differ from average on three dimensions: their risk aversion, their expectations, or their hedging needs (resulting from initial positions that differ from average). While hedging is an important source of derivatives demand, we shall not focus here on investors with special hedging needs.

Leland [1980] examines the demand for options that may result from differences in risk aversion. In an economy with a single risky asset (the "market portfolio") and a risk-free asset in zero net supply, Leland shows that the demand for long options positions (generating a payoff that is strictly convex in the market return) depends on the *rate of change* of an investor's risk aversion with wealth, relative to the average investor's

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rate of change. An investor whose risk aversion declines more rapidly with wealth than average will demand a convex payoff, which can be implemented by the purchase of long option positions.[1]

This article considers investors who have average risk aversion, but who hold option positions on the market portfolio because their expectations differ from consensus or average expectations. Such investors are appropriately termed "speculators." The question we pose is: *What characterizes the expectations, relative to average, of a speculator who optimally buys or sells options?*[2]

Our methodology might be considered reverse-engineering. Portfolio theory typically specifies investor expectations and risk aversion (utility function), and derives the optimal portfolio from these specifications. We assume an optimal portfolio including options, and average risk aversion, and derive the expectations the investor must have to warrant purchasing these option positions.

Our results suggest that those who buy or sell "plain vanilla" options have "reasonable" expectations. Expectations are path independent and exhibit an important martingale property: The expectation of the market's future expected return equals its current expected return. That is, investors who buy or sell ordinary options do not expect that the market is going to be more or less favorable in the future, in common with the average investor.

Speculators who sell ordinary options must believe that the market is mean-reverting. Speculators who buy options must have expectations that are non-mean-reverting, i.e., exhibit momentum: As the market rises, expected market return rises.[3] Note that non-mean reversion means that the variance of the market grows more than proportionately with time. Therefore buyers of options believe that the variance of the market is greater than the consensus investor believes.

Speculators who purchase exotic (path-dependent) options must have somewhat bizarre expectations. Their probabilities are path dependent. Those optimally buying average-rate options must expect that the expected market return will fall through time. Look-back purchasers must believe the opposite.

While we have formulated the discussion in terms of options, our results hold equally for dynamic strategies. Recall that the average investor cannot engage in a dynamic strategy, assuming the supply of securities remains fixed. Thus differences from the average investor (in any or all of the three dimensions outlined above) are required for dynamic strategies to be optimal.

Dynamic strategies that raise the level of risk (relative to wealth) as the market rises will be optimal only for investors who believe expected market rates of return go up as the market rises — non-mean reversion. Investors whose dynamic strategies reduce risk relative to wealth as the market rises must have expectations that exhibit mean reversion. And investors who rebalance to keep asset proportions constant must have expected market returns that are independent both of time and of market level.

A SIMPLE BINOMIAL MODEL OF MARKET PRICE MOVEMENT

We consider a financial market with two assets: a risky asset ("the market portfolio"), and a riskless asset in zero net supply whose return we normalize to zero.[4] The market portfolio is assumed to follow a simple binomial process. In each period t, the market value S_t either moves up to a value $S_{t+1} = uS_t$, or down to a value $S_{t+1} = S_t/u$, where u is a constant. For simplicity, we assume that dividends are zero.[5]

Let π_u denote the consensus (i.e., the average investor's) probability of an up move, and $\pi_d = 1 - \pi_u$ be the consensus probability of a down move. In the examples that follow, we assume that these consensus probabilities are constant, and therefore independent of time and the level of the market.

Following Cox and Rubinstein [1985], the risk-neutral probability P_u of an up move is

$$P_u = (ru - 1)/(u^2 - 1)$$

$$P_d = 1 - P_u$$

where r is one plus the risk-free interest rate. Note r = 1 in our framework, in which case $P_u = 1/(u + 1)$.

Our examples are based on the assumptions:

u = 1.2 (a gain of 20% relative to the risk-free asset);
d = 1/1.2 = 0.833 (a loss of 16.7% relative to the risk-free asset);
π_u = 0.667; and
π_d = 0.333;

implying $P_u = 0.455$, and $P_d = 0.545$.

Exhibit 1 shows the tree of market values over a three-period horizon, starting from a value of $S_0 = 100$. At each node, the consensus probability of an up move is 0.667.

There are a total of 2^T paths (or future "states") through a binomial tree with T periods. In our examples with T = 3, there are eight possible paths, each associated with a different sequence of up and down moves. Let s represent an arbitrary path, and U(s) the number of up moves along that path. The number of down moves will be D(s) = T − U(s). Note that any path ending with the same final market value will have the same number of up (and down) moves.

Two statistics are important for each state or path. The first is its consensus probability. This is given by

$$\pi_s = \pi_u^{U(s)}\, \pi_d^{D(s)}$$

The second is its risk-neutral probability or price:

$$P_s = P_u^{U(s)}\, P_d^{D(s)}$$

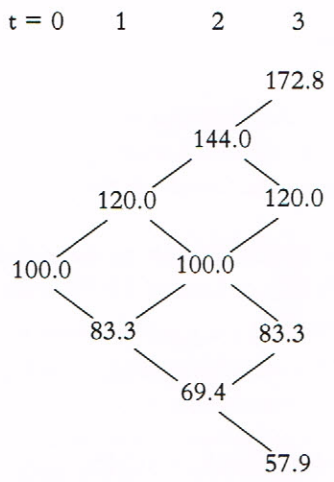
For example, the path s = [up, down, down] in our example has probability

$$(0.667)^1\ (0.333)^2 = 0.074$$

and price

$$(0.455)^1\ (0.545)^2 = 0.135$$

EXHIBIT 1
BINOMIAL MOVES OF THE MARKET VALUE



Note that both the consensus and risk-neutral probabilities of the paths sum to one.

CONSENSUS RISK AVERSION

The average or consensus investor maximizes expected utility of terminal wealth (at t = T) subject to a budget constraint:

$$\text{Maximize}_{[W_{Ts}]}\ \Sigma_s \pi_s\, U(W_{Ts})$$

$$\text{subject to}\ \ \Sigma_s P_s W_{Ts} = W_0$$

where W_{Ts} is wealth at time T, given state (or path) s, and π_s is the consensus probability of state s occurring. Without loss of generality, we can standardize $W_0 = S_0 = 100$.

First-order conditions are

$$\pi_s U'(W_{Ts}) = \lambda P_s \qquad \text{for } s = 1, \ldots, S \qquad (1)$$

$$\Sigma_s P_s W_{Ts} = 100$$

where $U'(\cdot)$ is the marginal utility of wealth, and λ is a positive constant.

The average or representative investor's utility will be maximized when consuming the market portfolio. Thus the optimal choice of wealth across states will satisfy the first-order conditions (1) when $W_{Ts} = S_{Ts}$, the value of the market portfolio at time T in state s.

It is known from Brennan [1979] that the consensus utility function that satisfies these conditions for the lognormal distribution of market returns belongs to the class of power functions:

$$U(W) = W^{1-\alpha}/(1 - \alpha)$$

for $\alpha > 0$, with marginal utility

$$U'(W) = W^{-\alpha}$$

The power function also works for the binomial model with arbitrary horizon T, with

$$\alpha = [\ln(u\pi_u/\pi_d)]/[2\ln(u)]$$

$$= 2.40 \qquad (2)$$

given the example values for u, π_u, and π_d. The proof of Equation (2) is given in the appendix.

We have thus determined the consensus risk aversion by identifying the utility function of the representative investor.

OPTIONS AND EXPECTATIONS: STRATEGIES USING EUROPEAN OPTIONS

Now consider investors with consensus risk aversion who choose to hold a portfolio that differs from the market portfolio. How will their probabilities differ from the consensus probabilities?

We must calculate the probability of each path π_s^i of an investor i who chooses an arbitrary portfolio (with initial cost 100) that gives terminal wealth W_{Ts}^i in each state. This wealth will of course depend on the investor's chosen assets, including derivatives.

From the first-order conditions (1), we can solve for π_s^i:

$$\pi_s^i = \lambda P_s / U'(W_{Ts}^i) \qquad (3)$$

$$= \lambda P_s (W_{Ts}^i)^{2.40}$$

where the second line uses the power utility function for our example.

Because W_{Ts}^i is known from the chosen portfolio, and prices P_s are known for each path (and are independent of the investor we are examining), Equation (3) determines the investor's probability for each path s up to a multiple λ. λ is chosen so that the investor's state probabilities sum to one.

Exhibit 2 derives the expectations of an investor who buys at-the-money European call options on the market portfolio at their fair (binomially determined) price of $13.60, and places the balance of initial wealth in cash. To allow comparability of initial expectations (an up move probability of 0.67), we consider an investor who buys 1.5 call options at $13.60, and holds the remaining $79.60 in cash.

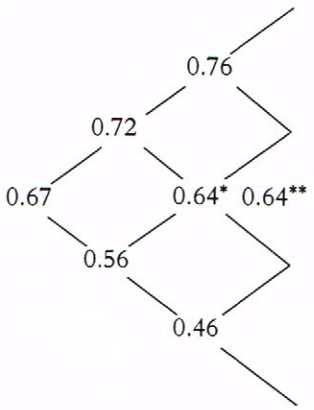
Column (1) of Exhibit 2 delineates the alternative states, the eight paths of differing up (u) or down (d) sequences of market moves. Column (2) shows the value of the market portfolio at the terminal date (T = 3), given the state. Column (3) shows the payoff of a call option with strike 100, and column (4) (W_{Ts}) shows the final payoff of the portfolio with 1.5 call options plus cash (recall the riskless interest rate is zero).

EXHIBIT 2
AT-THE-MONEY CALL OPTION

1.5 Call Options (K = 100) @ $13.60 + $79.60 Cash

(1) State (Path)	(2) Market Value	(3) Option Value	(4) Portfolio Payoff	(5) State Price	(6) Average Probability	(7) Investor Probability
(u, u, u)	172.8	72.8	188.8	0.094	0.296	0.366
(u, u, d)	120.0	20.0	109.6	0.113	0.148	0.119
(u, d, u)	120.0	20.0	109.6	0.113	0.148	0.119
(d, u, u)	120.0	20.0	109.6	0.113	0.148	0.119
(u, d, d)	83.3	0.0	79.6	0.135	0.074	0.066
(d, u, d)	83.3	0.0	79.6	0.135	0.074	0.066
(d, d, u)	83.3	0.0	79.6	0.135	0.074	0.066
(d, d, d)	57.9	0.0	79.6	0.162	0.037	0.079

Implied Nodal Probabilities $\pi_u^i (S_t, t)$ of an Up Move:



*Probability of up move when preceding market value higher.
**Probability of up move when preceding market value lower.

Column (5) lists the prices P_s of paths, and column (6) the consensus or average probabilities π_s of paths. There is now sufficient information to estimate the π_s^i from Equation (3); these are the state or path probabilities that the speculator must have, reported in column (7).

A more instructive way of presenting the investor's expectations is to show the *probabilities of an up move at each node* that are consistent with the path probabilities in column (7). These probabilities $\pi_u^i (S_t, t)$ are shown in the tree in Exhibit 2. Along any path s, the product of the up (or down) probabilities consistent with that path must equal π_s^i. For example, the path [up, down, down] has probability $\pi_s^i = 0.066 = 0.670 (1 - 0.724) (1 - 0.643)$.

Also note that at the middle node of T = 2, there are two numbers listed (0.64, 0.64). This is

because there are two possible prior paths leading to this node. The upper of the two numbers at the node is the probability when the previous market level was higher than the nodal value. The lower number is the probability when the previous market level was lower.

The probability of an up move following from this node will in general depend on which of the paths preceded.[6] The fact that the subsequent up probabilities are identical here implies that they are independent of the prior path.

Note that, at all relevant t, the option buyer's probabilities of an up move increase with the level of the market. Equivalently, the expected rate of return on the market increases with the level of the market. The variance of the option buyer's market return at T, as seen from S_0, exceeds the average investor's. This is reflected in the higher probabilities of the call purchaser's extreme returns (the first and last states).

Finally, consider the investor's expected probability of an up move in future periods, as seen from period 0. The probability of an up move in Period 1 will either be 0.72 (if an up move occurs, which happens with probability 0.67), or 0.56 (with probability 0.33). We observe that $0.72 \times 0.67 + 0.56 \times 0.33 = 0.67$, which equals the initial probability of an up move. As seen from Period 0, the expected probability of an up move in Period 2 also equals 0.67.

Indeed, the expected market return has a martingale property: It is invariant to time, as seen from any time and market level. This indicates that option buying is not motivated by more optimistic expectations.[7] Non-mean reversion of buyers' expectations, and the resulting higher volatility, creates the speculative demand for options.

Exhibit 3 presents similar data for an investor who chooses to buy out-of-the-money calls with strike 110. Again, to keep the initial probability of an up move equal to 0.67, the investor purchases 1.8 options (at cost \$9.27 each), and puts the remaining \$83.30 in cash.

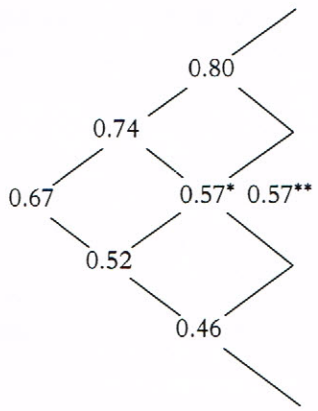
Compared to the investor purchasing the at-the-money calls, this investor has *more pronounced non-mean-reverting expectations* in the neighborhood of the initial market value. The investor's expectations imply greater variance of market returns, as reflected by larger extreme value probabilities than those of the purchaser of the at-the-money calls.[8] The investor's probabilities also reflect the path independence and martingale properties seen in the first example.

EXHIBIT 3
OUT-OF-THE-MONEY CALL OPTION

1.8 Call Options (K = 110) @ \$9.27 + \$83.30 Cash

State (Path)	Market Value	Option Value	Portfolio Payoff	Average Probability	Investor Probability
(u, u, u)	172.8	62.8	196.3	0.296	0.400
(u, u, d)	120.0	10.0	101.3	0.148	0.098
(u, d, u)	120.0	10.0	101.3	0.148	0.098
(d, u, u)	120.0	10.0	101.3	0.148	0.098
(u, d, d)	83.3	0.0	83.3	0.074	0.073
(d, u, d)	83.3	0.0	83.3	0.074	0.073
(d, d, u)	83.3	0.0	83.3	0.074	0.073
(d, d, d)	57.9	0.0	83.3	0.037	0.088

Implied Nodal Probabilities $\pi_u^i (S_t, t)$ of an Up Move:



*Probability of up move when preceding market value higher.
**Probability of up move when preceding market value lower.

Our observations suggest that the expectations of purchasers of call options:

- Reflect non-mean reversion in the neighborhood of the initial market level S_0; the probability of an up move increases, and therefore *the expected rate of return of the market increases* as S_t increases.[9]
- Imply a greater variance of the terminal-period market return than the consensus variance, as a consequence of non-mean reversion.
- Exhibit path independence.
- Imply a martingale property; the expected return to the market at any future time t, as seen from the initial node (S_0 = 100), is constant.

While these conclusions are drawn from only two examples, we extend them later to the expectations

of *any investor who optimally holds long option positions* in addition to keeping a constant fraction of investments in the market and in cash.[10] These results also describe the expectations of any investor who does not purchase options, but follows a path-independent "momentum" dynamic strategy.[11]

Symmetrically, any investor who sells options in addition to keeping a constant fraction of investments in the market and cash, or who follows a "reversal" dynamic strategy, must have expectations that are the inverse.[12] These sellers must believe that there is local mean reversion of market returns.

EXOTIC DERIVATIVES AND EXPECTATIONS

Exhibit 4 reflects the payoffs and probabilities associated with holding cash plus average-price futures. An average-price future is a derivative that for each path pays upon maturity the difference between the average market price along that path and the initial futures price (100 at t = 0, if interest rates are zero).[13]

Column (3) lists the payoff of the average-price future at maturity (t = 3), given the state, and column (4) gives the total portfolio payoff in each state. We observe that an investor who purchases such a derivative has expectations that

- Exhibit minimal mean aversion or mean reversion.
- Are (slightly) path dependent.
- Imply an expected return to the market that decreases with time.

The second property is typical of path-dependent derivative positions or dynamic strategies. The choice of path-dependent returns requires that investors have path-dependent beliefs, if they share (path-independent) consensus risk preferences.

The third property is perhaps surprising. The expected probability of an up move, as seen from the initial node, is 0.67 initially, 0.60 in the next period, and 0.53 in the final period. This decreasing expected market return (relative to the average investor) is shared by buyers of any average or Asian derivative.

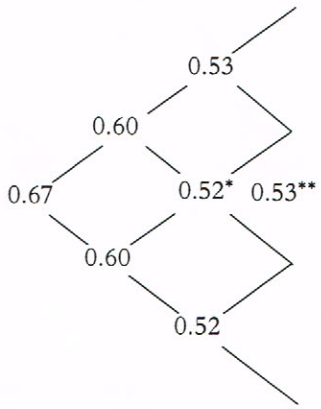
This is because an averaging contract's value becomes less sensitive to changes in the underlying asset value through time, as the average is over an ever-greater number of observations. The buyer of an Asian option bears less expected risk through time, and requires less expected return.

EXHIBIT 4
AVERAGE-PRICE ("ASIAN") FUTURE

1.33 Average-Price Futures + $100 Cash

(1) State (Path)	(2) Market Value	(3) Futures Value	(4) Portfolio Payoff	(5) Average Probability	(6) Investor Probability
(u, u, u)	172.8	34.2	145.5	0.296	0.212
(u, u, d)	120.0	21.0	127.9	0.148	0.187
(u, d, u)	120.0	10.0	113.3	0.148	0.140
(d, u, u)	120.0	0.8	101.1	0.148	0.106
(u, d, d)	83.3	0.8	101.1	0.074	0.127
(d, u, d)	83.3	–8.3	88.9	0.074	0.094
(d, d, u)	83.3	–16.0	78.8	0.074	0.070
(d, d, d)	57.9	–22.3	70.3	0.037	0.064

Implied Nodal Probabilities $\pi_u^i(S_t, t)$ of an Up Move:



*Probability of up move when preceding market value higher.
**Probability of up move when preceding market value lower.

Average-price (Asian) call options are examined in Exhibit 5. We assume a speculator buys 2.25 average-price calls with at-the-money strikes at $6.91 each. The remaining $84.45 is held in cash. The results reflect a combination of the previous results.

Like an ordinary option, the average-price option portfolio implies that the purchaser has probabilities that reflect local mean non-reversion. As with an Asian future, the investor must have state-dependent probabilities and must believe that (as seen from the initial node) the expected return of the market portfolio falls through time.

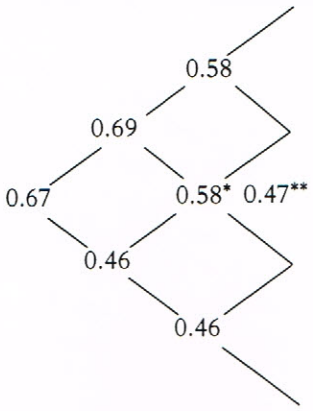
Look-back options are considered in Exhibit 6. The investor purchases 2.2 look-back calls at $17.35 each, and holds the remaining $61.83 in cash. Again reflecting the general behavior of call option

EXHIBIT 5
AVERAGE-PRICE ("ASIAN") CALL OPTION

2.25 Average-Price Calls (K = 100) @ $6.91 + $84.45 Cash

State (Path)	Market Value	Option Value	Portfolio Payoff	Average Probability	Investor Probability
(u, u, u)	172.8	34.2	161.4	0.296	0.267
(u, u, d)	120.0	21.0	131.7	0.148	0.197
(u, d, u)	120.0	10.0	107.0	0.148	0.119
(d, u, u)	120.0	0.8	86.3	0.148	0.071
(u, d, d)	83.3	0.8	86.3	0.074	0.086
(d, u, d)	83.3	0.0	84.5	0.074	0.081
(d, d, u)	83.3	0.0	84.5	0.074	0.081
(d, d, d)	57.9	0.0	84.5	0.037	0.097

Implied Nodal Probabilities π_u^i (S_t, t) of an Up Move:



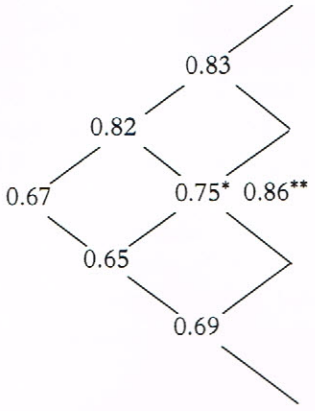
*Probability of up move when preceding market value higher.
**Probability of up move when preceding market value lower.

EXHIBIT 6
LOOK-BACK CALL OPTION

2.2 Look-Back Call Options @ $17.35 + $61.83 Cash

State (Path)	Market Value	Option Value	Portfolio Payoff	Average Probability	Investor Probability
(u, u, u)	172.8	72.8	222.0	0.296	0.454
(u, u, d)	120.0	20.0	105.8	0.148	0.092
(u, d, u)	120.0	20.0	105.8	0.148	0.092
(d, u, u)	120.0	36.7	142.5	0.148	0.187
(u, d, d)	83.3	0.0	61.8	0.074	0.030
(d, u, d)	83.3	0.0	61.8	0.074	0.030
(d, d, u)	83.3	0.0	92.4	0.074	0.079
(d, d, d)	57.9	0.0	61.8	0.037	0.036

Implied Nodal Probabilities π_u^i (S_t, t) of an Up Move:



*Probability of up move when preceding market value higher.
**Probability of up move when preceding market value lower.

buyers, the investor's probabilities will be locally non-mean-reverting.

There is quite pronounced path dependence. At the middle node of the next-to-last period, the market will have a higher probability of going up if in the prior period it rose than if in the prior period it fell. This indicates a serial trending property of the investor's beliefs.

Finally, in contrast to the buyer of average-price options, the buyer of look-backs expects that the market's expected return will increase through time. The probability of an up market move, as seen from the initial node, is 0.67 initially, 0.76 in the next period, and 0.81 in the final period. A look-back provides more expected exposure to the underlying asset through time; purchasers must expect to receive a higher expected return to compensate for the greater risk.

GENERALIZATIONS

Our results are derived from several simplifying assumptions. We have assumed that the consensus investor has a constant probability of an up move, implying that the market follows a random (binomial) walk with an expected rate of return that is invariant to time and market level. An increase in this constant consensus probability of an up move will increase the α of the power utility function through Equation (3), but will not change the nature of our results in any way.

We also standardize the investor's initial up move probability (and initial-period expected market return) to equal the consensus. What if the investor has, say, a lower initial expected market return? An investor whose up probabilities are constant at all nodes, but lower than the consensus up probability, will want to

hold a constant positive fraction of wealth in cash (and the complementary fraction in the market portfolio). Any strategy that increases the proportional market exposure as the market rises will imply up probabilities that rise with the market, and therefore exhibit mean non-reversion (about a lower mean).[14]

If the consensus up move probabilities are not constant, the consensus utility function will not be a power function.[15] Our results will continue to hold for investors who share the market's consensus return (up probability) in the first period.

For example, assume the average investor expects the market to be mean-reverting. Then, to warrant purchasing a long option position (or following a "momentum" dynamic investment strategy), the investor must believe the market is *less mean-reverting* than the average investor does. Similarly, the investor writing options against a market position (or following a "reversal" dynamic strategy) must believe the market is *more mean-reverting* than the average investor does. Simply knowing an investor believes the market is mean-reverting does not justify a reversal strategy or selling options; the investor must believe the market is even more mean-reverting than the average investor does.

The result that the investor who purchases derivatives with path-independent payoffs (such as ordinary options) will have path-independent probabilities also can be generalized to arbitrary consensus utility functions.[16] In He and Leland [1993] it is shown that path independence puts substantial restrictions on allowable stochastic processes. Stringent conditions must be satisfied by investors' beliefs about how the market return evolves.

For example, if market volatility is constant, the martingale property must hold. The expected market return in any future period will equal the current expected market return.

CONCLUSIONS

We have provided a straightforward technique to determine the expectations an investor must have in order to justify purchasing various kinds of derivatives if that investor has consensus risk preferences. These are "speculative" beliefs, as by assumption we have ruled out the other two reasons that one might purchase or sell derivatives: differing risk reversion or hedging abnormal exposures.

We have characterized the way speculative beliefs must differ from average to warrant holding various kinds of derivatives positions. Sellers of ordinary calls believe asset returns are more mean-reverting than average; buyers believe the opposite. Speculative buyers of Asian or average-price options in addition must expect that future expected market returns will fall through time. They must have path-dependent probabilities. Buyers of look-backs must expect the future expected market return to rise through time; they too must have path-dependent probabilities.

That there are rather bizarre expectations required for speculation in exotic derivatives suggests that their increased use most likely results from hedging or other non-speculative reasons. Of course, there is always the possibility that speculators in these complex instruments simply are not fully aware of the implications of their choices. This work is a first step in trying to clarify those implications.

APPENDIX

From the work of Mossin [1968], it is known that the power utility function with independent and identical asset returns through time exhibits a myopic property. At each period, the indirect utility function is identical to the terminal utility function.

Consider a single period with an up or a down move. Let S_t be the wealth of the average investor at time t. Recalling that the average investor must demand the market portfolio, wealth at time t + 1 will be uS_t or $(1/u)S_t$.

Local first-order conditions for this to be optimal are

$$\pi_u (uS_t)^{-\alpha} = \lambda P_u \qquad \text{(A-1)}$$

$$\pi_d (S_t/u)^{-\alpha} = \lambda P_d \qquad \text{(A-2)}$$

Taking the ratio of (A-1) to (A-2) and recalling $P_d = 1 - P_u$ gives

$$(1 - P_u)\pi_u u^{-2\alpha} = \pi_d P_u \qquad \text{(A-3)}$$

(Observe that the solution to (A-3) for α is independent of S_t and t.)

Substituting $P_u = 1/(u + 1)$ into (A-3) and simplifying yields

$$u\pi_u/\pi_d = u^{2\alpha} \qquad \text{(A-4)}$$

Taking natural logarithms of each side and solving for α gives

$$\alpha = [\ln(u\pi_u/\pi_d)]/[2 \ln(u)] \qquad \text{(A-5)}$$

Substituting the example values of u, π_u, and π_d gives $\alpha = 2.40$.

ENDNOTES

[1]Note that it is not the *level* of risk reversion that determines the demand for options or for portfolio insurance, but rather the *rate of change of that level* with wealth (i.e., the first derivative). Investors with risk aversion lower than average will take an aggressive market position, but will also choose to protect that position with put options if their risk aversion decreases more rapidly with wealth than average.

[2]Shimko [1994] addresses this question for ordinary options, using quite a different formulation. Shimko postulates an ad hoc objective function that implies that derivatives demand is related to the difference between investor probabilities and risk-neutral probabilities (rather than consensus probabilities). As the latter two probabilities can differ substantially, it would appear that the average investor could demand a non-zero derivatives position, although this cannot be consistent with equilibrium.

[3]The general statement is that option sellers (buyers) have expectations that are mean-reverting (non-mean-reverting) *relative to the average investor.* Our examples assume an average investor whose expectations are neither mean-reverting nor non-mean-reverting, allowing us to omit the "relative to" condition in this special case.

[4]Equivalently, this assumption implies that returns on the market portfolio are denominated in units of a risk-free bond growing at a constant interest rate r.

[5]As the number of binomial moves becomes large, the market return becomes lognormally distributed under these assumptions; see Cox, Ross, and Rubinstein [1979]. Thus we have assumed the discrete-time equivalent of the Black-Scholes price process.

[6]For large T, there will be many nodes with multiple paths leading to them. This poses no conceptual or technical problems.

[7]A constant but higher probability of an up move (and therefore greater expected market return) will induce an investor to buy a greater market position, but keep the ratio of market to cash investment constant. Since the average investor is assumed to hold the market portfolio (and no additional cash position), greater expected market returns will lead to leverage.

[8]More exactly, the distribution represents a mean-preserving spread of the consensus distribution.

[9]From Exhibit 3, the probability of an up move is increasing in S_t, for values of S_t near S_0. But as S_t becomes very large (as is possible when binomial branching is frequent), the probability of an up move must approach the consensus probability of 0.667. This is because a (very) deep in-the-money option plus cash behaves like the market portfolio as $S_t \rightarrow \infty$, and holding the market portfolio implies $\pi_u^i = 0.667$.

[10]It follows directly from put-call parity that the expectations of a buyer of a call option plus cash will be the same as the expectations of a buyer of a same-strike put option plus the market portfolio ("portfolio insurance").

[11]A "momentum" investment strategy is one whose fraction of wealth invested in the market increases as the market value rises. For path independence of dynamic strategies, see Cox and Leland [1982].

[12]"Reversal" strategies are dynamic investment policies that invest a smaller fraction of investor wealth in the market as the market value increases.

[13]Markets have not seen fit to introduce average-price futures, although average-price options ("Asian" options) do trade over the counter. We assume the average price of a path is the average of market value at each node on the path, t = 0, 1, 2, 3.

[14]Such strategies needn't involve options. For example, a buy-and-hold strategy that *begins* with a fifty-fifty mix of stock and cash will have greater market exposure relative to wealth, as the market rises. Therefore, this static strategy will imply that the investor has probabilities reflecting non-mean reversion.

[15]He and Leland [1993] show how the stochastic process is related to the consensus utility function in a continuous-time model.

[16]If in Equation (2) the investor wealth and state prices are path independent, the probabilities must also be. This result requires the usual assumption that utility functions (consensus and individual) depend only on final wealth, and not additionally on the path that brought wealth to that value. Absent this assumption, consensus probabilities themselves will generally be path dependent, and the individual investor buying a path-independent payoff will also have path-dependent probabilities.

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