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Essays in Microeconomic Theory

By

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Abstract

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Since the 2000s, matching theory has seen an increasing number of applications, from school assignments to organ donation. This dissertation collects three papers contributing to the theory of one-sided matching with endowments.

In the first chapter, I study the testable implications of the core in an exchange economy with unit demand when agents' preferences are unobserved. To do so, I develop a model of *aggregate matchings* in which the core is testable; the identifying assumption is that agents' preferences are solely determined by observable characteristics. I give conditions that characterize when observed economies are compatible with the core. These conditions are meaningful, intuitive, and tractable; they provide a nonparametric test for the core in the style of revealed preferences. I also develop a parametric method to estimate preference parameters from multiple observations of exchange economies. An allocation being in the core implies necessary moment inequalities, which I leverage to obtain partial identification.

The second chapter is coauthored with Will Sandholtz. We study the classic house-swapping problem of Shapley and Scarf (1974) in a setting where agents may have "objective" indifferences, i.e., indifferences that are shared by all agents. In other words, if any one agent is indifferent between two houses, then all agents are indifferent between those two houses. The most direct interpretation is the presence of multiple copies of the same object. Our setting is a special case of the house-swapping problem with general indifferences. We derive a simple, easily interpretable algorithm that produces the unique strict core allocation of the house-swapping market, if it exists. Our algorithm runs in $O(n^2)$ time, where n is the number of agents and houses. This is an improvement over the $O(n^3)$ time methods for the more general problem.

The third chapter is also coauthored with Will Sandholtz. We note that the proof of Bird (1984), the first to show group strategy-proofness of top trading cycles (TTC), requires a correction. We provide a counter-example to a critical claim, then present a corrected proof in the spirit of the original.

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Part I

Revealed Preferences of One-Sided Matching

Preface

In the first chapter, I study the testable implications of the core in an exchange economy with unit demand when agents' preferences are unobserved. To do so, I develop a model of *aggregate matchings* in which the core is testable; the identifying assumption is that agents' preferences are solely determined by observable characteristics. I give conditions that characterize when observed economies are compatible with the core. These conditions are meaningful, intuitive, and tractable; they provide a nonparametric test for the core in the style of revealed preferences. I also develop a parametric method to estimate preference parameters from multiple observations of exchange economies. An allocation being in the core implies necessary moment inequalities, which I leverage to obtain partial identification.

1 Introduction

This paper studies the testable implications of the core in exchange economies with indivisible goods and unit demand. The setting coincides with the house-swapping matching model of Shapley and Scarf (1974). As in classical revealed preference theory, I take agents, endowments, and allocations to be observable, but preferences to be unobserved. Given such data, I investigate the testable implications of the core in exchange economies. This paper also develops a parametric method to estimate preference parameters from multiple observations of such data. In both models with and without monetary transfers, I find conditions that characterize when the observables are consistent with the core ("rationalizability"). Conversely, these conditions can falsify the market being in the equilibrium.

The exchange economy is a foundational model in economics for situations without explicit production. With unit demand and indivisible goods, these models correspond to exchanges or allocations of "large" objects. Furthermore, the process of allocation may be unknown or ambiguous – even in the setting with monetary transfers, competitive prices are not inherent to the model. Shapley and Scarf refer to these large indivisible goods as "houses"; indeed, this is interpretable as a stylized model of housing allocation. The model is also applied to settings such as living donor organ exchange, school assignment, and course allocation. The allocation processes can be decentralized trade, as in a Walrasian market; or via a centralized mechanism.

An example of a market with many of these attributes is the Singaporean public housing market.[1] The government allocates new public housing quarterly via a centralized build-to-order mechanism. Applicant households submit interest in a new development and are awarded a subsidized 99 year lease, which they have the right to sell after five years. This setting incorporates many of the features described above; housing is a large good, initial prices are restricted, and owners have trading rights afterward. It is also plausible that households with the same observable characteristics share the same preferences over developments.

The core is a game theoretic solution concept and a natural equilibrium notion for this setting. Informally, it captures group stability by requiring that no coalition would prefer to break off and re-trade their endowments among themselves. Alternatively, any beneficial trades have already been made. Implicitly, these coalitions can plausibly find each other to form. In this way, it is the right equilibrium notion for a market that is "small" relative to the "large" goods. It is also Pareto efficient. Importantly, the core does not require prices, which are not inherent to this model. However, I also present equivalence results for the core and competitive equilibrium in this model.

The conditions I present characterize restrictions on the observable data of core allocations. An analyst may wish to check for the core for a few reasons. Equilibrium itself may be the object of interest – an economy which satisfies the conditions is plausibly stable and Pareto optimal. Other analysis may also require equilibrium, such as study of the preferences. The conditions for rationalizability also provide ex ante predictions for equilibrium market outcomes. Finally, even in settings with centralized mechanisms, we may wonder whether decentralized markets would select similar outcomes; the restrictions provide a way to test such outcomes.

[1] Population Trends, Department of Statistics, Ministry of Trade & Industry, Republic of Singapore – This is also an important market; 79% of Singaporeans live in public housing.

There are two other ways to interpret this paper. Observers may deal with settings where the centralized mechanism is unknown and therefore cannot be directly evaluated. In practice, many mechanisms are hidden, or no particular mechanism is used at all, such as administrators exercising personal judgment to determine allocations. But we nevertheless want to determine whether these unknown mechanisms might be stable. Grigoryan and Möller (2023) develop a theory of *auditability*, where mechanism implementers may deviate for various reasons; auditability measures how much information the participants need to detect a deviation. This paper offers a way to evaluate mechanisms when essentially nothing is known about the matching process, but the analyst still wants to determine whether the allocation is may be stable. Alternatively, there may be no centralized mechanism at all. In this interpretation, I develop a theory to test stability when there is no particular matching process.

To rationalize a market, it is sufficient to find a preference profile such that it is in the core. In classical consumer demand revealed preference theory, we infer that the chosen option is the best among affordable options. Afriat (1967) then proceeds from here to construct utility values. However, in an exchange economy, the available options are not exogenously determined by some budget. Stability in an exchange market is determined by all other agents' preferences. Further, the core is not equivalent to maximizing social utility, even when we allow monetary transfers.

To gain traction in this setting, I deal with aggregate matchings, akin to Choo and Siow (2006)'s empirical work in marriage markets. Objects are grouped into types, equivalent within type and distinct across types. For instance, these may be apartments in the same development or houses in the same neighborhood, which can be regarded as essentially the same. I also assume that agents can be binned into "types" with the same preference, analogous to the assumption of Echenique, Lee, Shum, and Yenmez (2013). Stated another way, agents with the same observable characteristics (such as age, wealth, and socioeconomics) have the same preferences. This is a strong assumption as it rules out individual heterogeneity.[2] However, allowing for enough individual heterogeneity also allows any observed market to be rationalized.[3] In exchange, the resulting test for rationalizability is nonparametric, in the spirit of revealed preferences.

To show the main results, I introduce a graph representation of exchange economies and develop graph theoretic results around it. This construction is extremely tractable, and it gives rise to intuitively appealing conditions for rationalizability. Through the graph representation, I am able to prove related results about the underlying exchange economies; I find a partition of any exchange economy into *market segments* that only interact within themselves. I also prove a previously informal result that any house-swapping economy can be partitioned into trading cycles.[4] The graph construction's tractability also suggests ways to develop "smoother" definitions and statistical tests of rationalizability. In the setting without transfers, rationalizability is equivalent to equal treatment within each type in each market segment. In the setting with monetary transfers, there are two equivalent conditions: the existence of a price vector rationalizing the allocation as a competitive equilibrium, and a cyclic monotonicity condition similar to many in the revealed preferences literature.

[2] In the model without transfers, rankings are purely ordinal, so small cardinal heterogeneity is allowed.

[3] Simply declare all agents' allocations to be their favorite things.

[4] Not necessarily Gale's top trading cycles – no claim on optimality is made here.

I also develop a parametric method to estimate utility parameters if the data consist of multiple aggregate matchings without transfers. The setting is similar to Fox (2010) and Echenique, Lee, and Shum (2013). Heterogeneity across aggregate matchings is allowed. Each aggregate matching can first be checked for stability by applying the conditions in the first part of the paper. Stability of the matching implies necessary moment inequalities, which I leverage to obtain partial identification. I illustrate the method using data simulated from the experiment of Chen and Sönmez (2006) and applying the method of Chernozhukov, Chetverikov, and Kato (2019).

This paper contributes to the study of the testable implications of equilibria. The Sonnenschein–Mantel–Debreu theorem (1972; 1974; 1974) gives a famous "anything goes" result on the excess demand function in competitive equilibrium. In the same vein, Mas-Colell (1977) shows that there are essentially no restrictions on rationalizable prices in competitive equilibria. Brown and Matzkin (1996) apply revealed preference theory to obtain restrictions on competitive equilibrium outcomes when a series of markets is observed. Bossert and Sprumont (2002) find conditions for core rationalizability in a two agent economy with divisible commodities. I study a distinct setting – exchange economies with indivisible goods and unit demand – and find tractable and intuitive restrictions on core allocations.

Additionally, I contribute to the growing literature on the revealed preferences of matching. Echenique, Lee, Shum, and Yenmez study the revealed preferences of matching in marriage markets with aggregate matching type data. Echenique (2008) finds testable implications for two-sided matching when individuals participate in a series of markets.

This paper provides a partial identification result for a one-sided matching model without transfers. Given allocations presumed to be stable, I find a set of possible utility parameters. In a model with transferable utility, Choo and Siow study aggregate matchings in the marriage market. In the non-transferable utility case, analysts can use intermediate matching data to recover the agents' preferences; Hitsch, Hortaçsu, and Ariely (2010) use rejections in online dating. Recent work by Galichon, Kominers, and Weber (2019) develops an intermediate case, where utility is imperfectly transferable. Echenique, Lee, and Shum develop moment conditions for aggregate two-sided matching data. I direct the reader to Chiappori and Salanié (2016) for a survey of the econometrics of matching.

2 Model

I will first present the model and notation for the case without monetary transfers. Then I will present the additions for the case of monetary transfers.

2.1 Without transfers

The basis of the model is the Shapley and Scarf house-swapping model with the addition that objects and agents are grouped into types. This will also turn out to be a pure exchange economy with unit demand. Agents of the same type share the same (unobserved to the analyst) preference. Let the set of agent types as $A = \{1, 2, ..., A\}$, where A denotes both the set and its cardinality at minimal risk of confusion; let $|A| < \infty$. Denote the set of individual agents as $\mathcal{A} = \{1a, 1b, ...; 2a, 2b, ...; Aa, Ab, ...\}$, and let $|\mathcal{A}| < \infty$. Implicitly, $\mathcal{A}$

also encodes the types of each individual; e.g., $1a$ and $1b$ are two individuals of the same type 1. I will refer to $i \in A$ as a "type", and $ik \in \mathcal{A}$ as an "individual" or "agent".

Denote the set of object types H, also with cardinality H. I denote each object as a unit vector in $\mathbb{R}^H$; that is,

$$H = \{\underbrace{(1,0,...,0)}_{:=h_1}, \underbrace{(0,1,0,...,0)}_{:=h_2}, \underbrace{(0,...,0,1)}_{:=h_H}\} \subset \mathbb{R}^H$$

I will not refer to individual objects – i.e., there is no object analogue of $\mathcal{A}$.

Each agent is endowed with an object, denoted $e_{ik} \in H$. An endowment vector is $e = (e_{ik})_{ik\in\mathcal{A}}$. An allocation is $x = (x_{ik})_{ik\in\mathcal{A}}$ such that $\sum_{ik\in\mathcal{A}} x_{ik} = \sum_{ik\in\mathcal{A}} e_{ik}$. That is, the number of allocated objects of each type is equal to the number supplied.

A feasible sub-allocation for a coalition $A \subseteq \mathcal{A}'$ is $x' = (x'_{ik})_{ik\in A'}$ such that $\sum_{ik\in A'} x'_{ik} = \sum_{ik\in A'} e_{ik}$.

Each type i has a *strict* preference $\succsim_i$ over H; all ik of type i have the same preference. I will discuss this more in Section 2.3. Denote $\succsim = (\succsim_i)_{i\in A}$ be the preference profile. With minimal risk (or consequence) of confusion, this could also be the profile of agents $\succsim = (\succsim_{ik})_{ik\in\mathcal{A}}$.

The equilibrium concept used in this paper is the core.

Definition 1. A weak blocking coalition is $A' \subseteq \mathcal{A}$ with feasible sub-allocation x' such that $x'_{ik} \succsim_i x_{ik}$ for all $ik \in A'$, and $x'_{ik} \succ_i x_{ik}$ for at least one $ik \in A'$. An allocation x is in the **strict core** for a preference profile $\succsim$ if there is no weak blocking coalition.

By convention, when a blocking coalition A' is one individual, I say x is not individually rational.[5]

I can now state the main objective of the paper. If we observe individuals, types, endowments, and allocations, could the market be in the core? Formally, is there a preference profile such that x is in the strict core?

Definition 2. A tuple $(A, \mathcal{A}, H, e, x)$ is an **NT-economy** (non-transfers-economy). An economy is **NT-rationalizable** if there exists a preference profile $\succsim$ such that x is in the strict core.

2.2 With transfers

I now introduce monetary transfers. The notation for types, agents, and objects remains the same. Endowments are now an object and amount of money, $(e, \omega) = (e_{ik}, \omega_{ik})_{ik\in\mathcal{A}}$, where $e_{ik} \in H$ and $\omega_{ik} \in \mathbb{R}_{++}$. Likewise, an allocation is an object and amount of money $(x, m) = (x_{ik}, m_{ik})_{ik\in\mathcal{A}}$, such that $m_{ik} \in \mathbb{R}_{++}$, $\sum_{ik\in\mathcal{A}} x_{ik} = \sum_{ik\in\mathcal{A}} e_{ik}$, and $\sum_{ik\in\mathcal{A}} m_{ik} \leq \sum_{ik\in\mathcal{A}} \omega_{ik}$. Note that endowed and allocated money are restricted to be strictly positive. Analogously, a feasible sub-allocation for a coalition A' is $(x', m') = (x'_{ik}, m'_{ik})_{ik\in A'}$ such that $\sum_{ik\in A'} x'_{ik} = \sum_{ik\in A'} e_{ik}$ and $\sum_{ik\in A'} m'_{ik} \leq \sum_{ik\in A'} \omega_{ik}$.

Let utility $V_i : H \times \mathbb{R}_+ \to \mathbb{R}$ be quasilinear, given by $V_i(h, m) = v_i(h) + m$. Notice that the subscript is on types. The $v_i(\cdot)$ can be interpreted as a utility index over H; that is,

[5] A blocking coalition of one individual ik means $e_{ik} \succ_i x_{ik}$.

Table 1: Notation

Object	Without transfers	With transfers	Generic member
Agent types	A		i
Individuals/agents	$\mathcal{A}$		ik
Objects	H		h
Endowment	e	(e, ω)	
Allocation	x	(x, m)	
Preferences	$\succsim$	$V_i(h, m) = v_i(h) + m$	

it is an H-dimensional vector of real numbers representing cardinal utility for objects. We can regard this model as a partial equilibrium analysis, where all other goods are grouped into money. This is also a common assumption in market design and matching (e.g. Gul, Pesendorfer, and Zhang, 2018).

The equilibrium concept in the transfers model is the weak core.

Definition 3. For an allocation (x, m), a strong blocking coalition is $A' \subseteq \mathcal{A}$ with feasible sub-allocation $(x', m')|_{A'}$ such that $V_i(x'_{ik}, m'_{ik}) > V_i(x_{ik}, m_{ik})$ for all $ik \in A'$. An allocation (x, m) is in the **weak core** for (v_i) if there is no strong blocking coalition.

The weak core and strict core coincide in most cases, as any strictly better off members can give ε payments to any indifferent members. The exception is when all strictly better off members exhaust their money in a candidate blocking coalition. The assumption that $\omega_{ik}, m_{ik} > 0$ ensures that money truly enters the model and that the weak core and strict core coincide for rationalizable allocations.[6]

The definition of rationalizability is completely analogous. The analyst observes individuals, types, endowments, and allocations (the latter two including money). I seek a preference profile such that (x, m) is in the core.

Definition 4. A tuple $(A, \mathcal{A}, H, (e, \omega), (x, m))$ is a **T-economy** (transfers-economy). An economy is **T-rationalizable** (transfers-rationalizable) if there exists utility indexes (v_i) such that (x, m) is in the weak core. It is **strictly T-rationalizable** if it is T-rationalizable with some strict utility indexes; that is, $v_i(h) = v_i(h')$ if and only if $h = h'$ for all i.

The main result for T-economies will deal with T-rationalizability, so I will not impose that the utility indexes (v_i) are strict over H. I will discuss afterwards how strict T-rationalizability is a corollary of the main result.

2.3 Discussion

This paper derives necessary and sufficient conditions for an economy to be rationalizable. Stated another way, I characterize allocations which are compatible with the core. As mentioned earlier, this characterization can be used to check for equilibrium; this may be of interest in and of itself or be necessary for further analysis.

[6] It can be argued as in Kaneko (1982) and Quinzii (1984) that money is a bundle of goods outside the model, and it is not "normal" to consume only one indivisible good.

Mechanically, this economy is the "reverse direction" of the classic house-swapping economy. That is, we have a house-swapping market as in Shapley and Scarf (1974) where there are potentially multiple copies of each object. Given an allocation, we are seeking preferences generating it.

The key identifying assumption is common preferences within agent type. This introduces discipline to the problem. As noted above, this gives the economy testable content; with enough individual heterogeneity, any economy is rationalizable.[7] While not explicitly modeled, this is akin to an assumption that preferences are solely functions of agents' observable characteristics. If there are observable traits of agents X_a and of objects X_h, rankings are generated by some utility function $u(X_a, X_h)$. For the non-transfers case, the resulting characterizations are completely nonparametric. For transfers case, I impose quasilinear utility; but the utility for objects $v_i(h)$ is otherwise nonparametric. Since the non-transfers preferences are purely ordinal, some *cardinal* heterogeneity is allowed, as long as the same *ordinal* rankings are generated.

If types are constructed from binned variables, the analyst has some degree of choice. Coarser bins result in stronger implications on the allocation, and finer bins result in weaker implications. The "correct" tradeoff is outside of the model of this paper, but the analyst can decide on the most reasonable choice.

Finally, rationalizability is a meaningful concept; it is not hard to construct economies that are not rationalizable. Indeed, the formal results characterize such economies.

2.4 Graphs

I will represent economies in graph-theoretic terms. This will allow me to take advantage of results from graph theory and to parsimoniously present the main results. I first introduce some standard definitions for directed graphs that will be useful. Familiar readers can skip this subsection.

Definition 5. A **directed graph (digraph)** is $D = (V, E)$, where V is the set of vertices, and E is the set of arcs. An **arc** is an sequence of two vertices (v_i, v_j); here I allow for arcs of the form (v_i, v_i), called a self-loop.[8] A (v_1, v_k)**-path** is sequence of vertices $(v_1, v_2, ..., v_k)$ where each v_i is distinct, and $(v_{i-1}, v_i) \in E$ for each $i \in \{2, ..., k\}$. A **cycle** is a sequence of vertices $(v_1, v_2, ..., v_k, v_1)$, where each v_i is distinct except for the first and last, and $(v_{i-1}, v_i) \in E$ for each $i \in \{2, ..., k\}$. I will also include self-loops (v_1, v_1) as cycles. Equivalently, a path is a sequence of arcs $((v_1, v_2), ..., (v_{k-1}, v_k))$, and analogously for cycles. The **indegree** of a vertex $d^-(v_i) = |v_j : (v_j, v_i) \in E|$ is the number of arcs pointing at v_i. Likewise, the **outdegree** of a vertex $d^+(v_i) = |v_j : (v_i, v_j) \in E|$ is the number of arcs pointing from v_i.

Definition 6. A **weighted directed graph** is $D = (V, E, \ell(\cdot))$, where (V, E) is a directed graph, and $\ell : E \to \mathbb{R}$ is the **length** (or **weight**) function over arcs. The length of a path or cycle $(v_1, v_2, ..., v_k)$ is $\sum_{i=1}^{k-1} \ell(v_i, v_{i+1})$.

The next definition is used in the main result and its discussion.

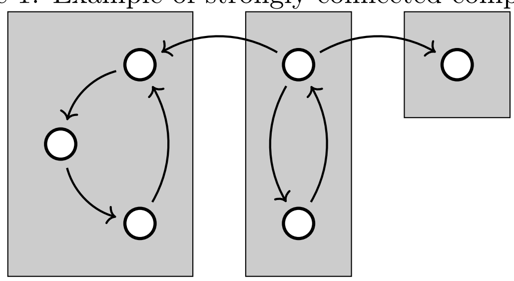
[7]There are alternatives, such as repeated re-matchings as in Echenique (2008).

[8]This is more formally called a **directed pseudograph**.

Definition 7. A **strongly connected component (SCC)** of a digraph $D = (V, E)$ is a maximal set of vertices $S \subseteq V$ such that for all distinct vertices $v_i, v_j \in S$, there is a (v_i, v_j)-path and a (v_j, v_i)-path. By convention, there is always a path from v_i to itself, even if $(v_i, v_i) \notin E$; an isolated vertex is an SCC.

Informally, an SCC is a maximal set of vertices such that there is a path from any vertex to any vertex.

Figure 1: Example of strongly connected components



The vertices of any digraph can be uniquely partitioned into SCCs. An algorithm by Tarjan (1972) finds a partition in linear time, $O(|V| + |E|)$. Figure 1 illustrates such a partition; the SCCs are shaded. For example, in the left-most SCC, there is a path from any vertex to any other vertex. It is also maximal, since including other vertices destroys this property.

3 Rationalizability

I now give necessary and sufficient conditions for an economy to be rationalizable. I will first present the graph representation of economies, which I use to show the result for NT-economies. I will then present the analogous results for T-economies.

3.1 Without transfers

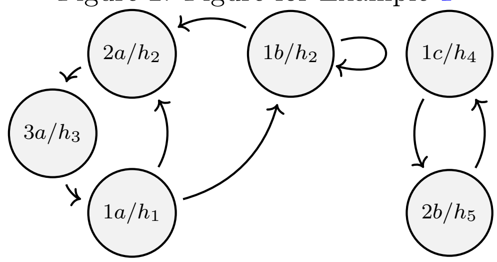
First, I introduce a graph construction that is important for the main results. Construct the digraph $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x) = (\mathcal{A}, E)$ as follows: each individual is a vertex. Draw arcs from ik to *all* vertices $i'k'$ that are endowed with x_{ik}. That is, let $(ik, i'k') \in E$ if $x_{ik} = e_{i'k'}$.

Example 1. Consider the economy described below.

ik	e_{ik}	x_{ik}
$1a$	h_1	h_2
$1b$	h_2	h_2
$1c$	h_4	h_5
$2a$	h_2	h_3
$2b$	h_5	h_4
$3a$	h_3	h_1

That is, $e_{1b} = e_{2a}$, and other endowments are unique. The graph $\mathcal{G}_{NT}$ is given below in Figure 2.

Figure 2: Figure for Example 1



The SCCs of $\mathcal{G}_{NT}$ are the focus of the main result. In the context of this paper's setting, the SCCs are interpretable as partitioning the market into segments that trade among themselves. I will refer to these informally as *market segments*. Readers familiar with matching may recognize that any allocation can be partitioned into trading cycles.[9] In the setting of Shapley and Scarf without indifferences, this partition is unique. In the present setting, these cycles may not be unique; however, $\mathcal{G}_{NT}$ superimposes all such trading cycles onto one graph.

I now present the main result for the NT-economy.

Theorem 1. *Fix an NT-economy $(A, \mathcal{A}, H, e, x)$, and consider $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x)$. The economy is NT-rationalizable if and only if: for agents of the same type ik, ik' in the same SCC S, $x_{ik} = x_{ik'}$. That is, if $ik, ik' \in S$ are the same type and in the same SCC, they receive the same object type.*

Proof. Appendix. □

The full proof is contained in the appendix. I give a sketch of the proof below.

Proof sketch of Theorem 1. A key feature of $\mathcal{G}_{NT}$ is that all objects of the same type are contained in the same SCC. The proof of this claim is under 7.

To prove "if": First, find the decomposition of $\mathcal{G}_{NT}$ into SCCs. Then assign an arbitrary order to the SCCs, and assign preferences in this order. In the first SCC S_1, set all types' top rank to be the objects they receive. By assumption, all agents of the same type in the same SCC receive the same object, so this is a well defined procedure. In the second SCC S_2, there may be types who were not present in S_1; let these types' top rank be the objects they receive. For types who were present in S_1, set their second ranked object to be what they receive. Then continue through the remaining SCCs in this way. Since all objects of the same type are in the same SCC, the procedure never attempts to "re-assign" a preference in a later step. That is, objects never "re-appear" after being assigned to a preference rank the first time. The argument that this creates no blocking coalitions is similar to the argument behind Gale's proof for TTC.

To prove "only if", I show that when the condition is violated, there is a blocking coalition for all preference profiles. One of the two agents of the same type must be worse off; this one can form a blocking coalition with a subset of other members of the SCC. □

[9]Not necessarily Gale's TTC cycles – no claim on optimality is made here yet.

Example (Example 1 continued.)**.** The $\mathcal{G}_{NT}$ has two SCCs: the left component and the right component. To apply the theorem, select one order arbitrarily. Let the left component be S_1, and the right be S_2. Let $\succsim_i (k)$ denote type i's k^{th} favorite object.

1. In S_1, assign all agents' $\succsim_i (1) = \mu(i)$, so

i	$\succsim_i (1)$
1	h_2
2	h_3
3	h_1

2. In S_2, assign all agents' $\succsim_i (1) = \mu(i)$ for any i who were not in S_1. (Here, both types 1 and 2 were present in S_1.) Otherwise, let $\succsim_i (2) = \mu(i)$.

i	$\succsim_i (2)$
1	h_5
2	h_4

3. Assign remaining preferences arbitrarily (omitted).

To check for a blocking coalition, observe that all agents in S_1 all receive their favorite objects. Only agents in S_2 are unsated. Then in any candidate blocking coalition (A', μ'), we require $\mu'(1c) = h_2$ or $\mu'(2b) = h_3$. This requires at least one agent in $A' \cap S_1$ to receive either h_4 or h_5, which are strictly dispreferred.

The condition required in Theorem 1 is easy to check; Tarjan's algorithm finds the partition into SCCs in linear time. Within each SCC, checking for a non-repeated agent type-object type pair is linear in the number of agents.

3.2 Discussion and related results

The most direct interpretation of Theorem 1 is this: whenever agents with the same preferences are in the same market segment, they receive the same object type. Informally, agents in the same market segment have similar market power; if there are multiple agents of the same type in an SCC, one should not be worse off. Within a market segment, any agent can make a series trades to receive any object in this segment; the formal proof of Theorem 1 uses this idea this to find a blocking coalition.

More formally, a second interpretation is in the context of a competitive equilibrium market.[10] Roth and Postlewaite (1977) show that any strict core allocation is a competitive equilibrium allocation in the typical object exchange setting with no indifferences. Wako (1984) establishes that a strict core allocation is also a competitive equilibrium allocation in the setting with indifferences. If x is in the core for some preference profile $\succsim$, it is also a competitive equilibrium allocation for some price vector. A supporting price vector is descending in the (arbitrarily selected) order of SCCs. Thus if two agents are in the same SCC, their endowments are worth the same in competitive equilibrium. The necessity of the

[10]This is the usual definition. I give the formal definition for in Definition 9 in the appendix.

condition becomes immediate; two agents with the same budget and same strict preferences should purchase the same object type.

I now present some related results. First, an immediate implication of Theorem 1 is the following corollary:

Corollary 1. *Fix an economy* $(A, \mathcal{A}, H, e, x)$. *The economy is rationalizable only if: whenever agents* ik, ik' *are the same type and* $e_{ik} = e_{ik'}$, $x_{ik} = x_{ik'}$.

Proof. Appendix. □

That is, identical agents (of same type and same endowment) must receive the same object type. Briefly, the theorem requires equal treatment of equals. When types determine both preferences and endowments, this corollary gives us the condition for rationalizability.

Corollary 2. *Suppose* $e_{ik} = e_{ik'}$ *for all* k, k' *and for all* $i \in A$. *That is, all agents of the same type have the same endowment. Then the economy* $(A, \mathcal{A}, H, e, x)$ *is rationalizable if and only if* $x_{ik} = x_{ik'}$ *for all* k, k' *and for all* $i \in A$. *That is, if and only if all agents of the same type receive the same object.*

Proof. "Only if" is a consequence of Corollary 1. To prove "if", note that everyone of the same type receives the same object, so we can let everyone's favorite object be their allocated object. □

This resembles the Debreu and Scarf (1963) theorems for general equilibrium. Their model is an endowment economy with a finite number of goods, agent types, k copies of each type, and types determining both endowment and preferences. Only allocations assigning the same bundle to all agents of the same type are in the core. While neither the Debreu and Scarf model nor my model contains the other, it would be interesting future work to investigate a whether deeper connection exists.

Theorem 1 characterizes which observed economies are consistent with the core. That is, the condition offers a restriction on the kinds of allocations that can be seen in equilibrium. Many allocations can be ruled out ex ante. On the positive side, Corollary 1 gives a clear prediction for markets in the core.

Another related question is: what is the minimum number of agent types necessary to rationalize an allocation? That is, suppose we are free to choose agent types. What is the minimum preference type heterogeneity required to put x in the core? This question is sensible, since allowing every individual to be his own type always rationalizes an allocation.

Let $\tilde{\mathcal{A}}$ be the set of individual agents, without encoding information on types. With this data, we can construct a graph $\tilde{\mathcal{G}}_{NT}\left(\tilde{\mathcal{A}}, H, e, x\right)$ in the same way as $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x)$.

Corollary 3. *Consider* $\tilde{\mathcal{G}}_{NT}(\mathcal{A}, H, e, x)$, *and decompose this into SCCs,* $\{S_1, ..., S_M\}$. *Let* α_m *be the number of distinct object types in* S_m. *The minimum number of types necessary to construct* $\succsim$ *such that* x *is in the core is* $\alpha = \max\{\alpha_1, ..., \alpha_m\}$.

Proof. This is a corollary of Theorem 1. Individuals in the same SCC_m who receive different objects must be different agent types. There is no other restriction on agent types.

Within S_m, there are α_m distinct objects; order them arbitrarily. Let everyone who receives object 1 be type 1, and so on. By Theorem 1, this will be rationalizable. It is also clear that having fewer than α types will make the economy not NT-rationalizable. □

The result also solves the analogous economy for two-sided matching in the strict core. That is, it solves a strict stability analogue of Echenique, Lee, Shum, and Yenmez with non-transferable utility. There are types of men and women, and each type has a strict preference over types of the other side. The result follows from transforming house-swapping into two-sided matching in the usual way. To do this, let each agent type have a unique endowment (him- or her- self), and restrict preferences to find only endowments of the other side acceptable. The condition for rationalizability is given by Corollary 1; an observed market is rationalizable if and only if all men of the same type are assigned the same type of women, and vice versa.

3.3 With transfers and related results

I derive necessary and sufficient conditions for a T-economy to be T-rationalizable. First, I introduce a new weighted digraph $\mathcal{G}_T(A, \mathcal{A}, H, (e, \omega), (x, m)) = (\mathcal{A}, E, \ell(\cdot))$. Draw vertices and arcs as in $\mathcal{G}_{NT}$; let each agent be a node, and draw arcs from ik to *all* vertices $i'k'$ that are endowed with x_{ik}. In addition, define the lengths arcs by $\ell(ik, i'k') = \omega_{ik} - m_{ik}$. Note $\ell(ik, i'k')$ depends only on the first vertex, not the second.

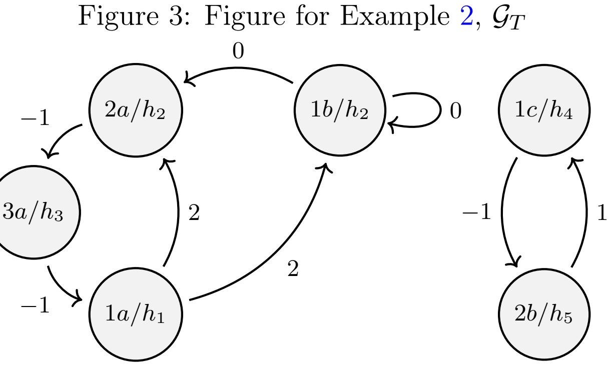
The following example adds to 1.

Example 2. Consider the economy described in Example 1, adding the following payments:

$\mathcal{A}$	e_{ik}	x_{ik}	$\omega_{ik} - m_{ik}$
$1a$	h_1	h_2	2
$1b$	h_2	h_2	0
$1c$	h_4	h_5	1
$2a$	h_2	h_3	-1
$2b$	h_5	h_4	-1
$3a$	h_3	h_1	-1

The following figure illustrates $\mathcal{G}_T$.

Figure 3: Figure for Example 2, $\mathcal{G}_T$



I now give the main result for T-rationalizability.

Theorem 2. *Fix a T-economy $(A, \mathcal{A}, H, (e, \omega), (x, m))$. The following are equivalent:*

1. *The economy is T-rationalizable.*

2. *There exists a vector* $p \in \mathbb{R}_+^{|H|}$ *such that*

$$(x_{ik} - e_{ik}) \cdot p = \omega_{ik} - m_{ik} \quad \forall ik \in \mathcal{A} \tag{P}$$

3. *The graph* $\mathcal{G}_T(A, \mathcal{A}, H, (e, \omega), (x, m))$ *has no cycles with length* > 0.

Proof. Appendix. □

The vector p in (P) (suggestively denoted) is interpretable is a price vector for objects. Indeed, the left side is the difference in price between the allocated and endowed object, and the right side is the net payment from ik. This suggests an easy interpretation of the theorem: an economy is TU-rationalizable if and only if everyone who "buys" an object type pays the same price for it.

A direct interpretation of (3) is that no cycles "export" money. This is clearly a necessary condition – any cycle that pays money outward can implement their object allocation while keeping more money. In the full proof, I show that (3) also allows construction of a price vector p

I now present a sketch of the proof.

Proof sketch of Theorem 2. I first show (2) $\Longrightarrow$ (1). Given p, I seek v_i such that (x, m) is a competitive equilibrium. By the usual arguments, a competitive equilibrium allocation is in the weak core.[11] We are looking for utility indexes v_i such that all agents ik are maximizing subject to their budget constraints, given by $e_{ik} \cdot p + \omega_{ik}$. Then this becomes a classic consumer demand revealed preference problem. To see this, reinterpret a type i as a single consumer, and each individual ik as a demand data point:

$$\left\{ \underbrace{(x_{ik}, m_{ik})}_{\text{consumed good and money}}, \underbrace{(e_{ik} \cdot p + \omega_{ik})}_{\text{budget}}, \underbrace{p}_{\text{price}} \right\}_{ik}$$

In this structure, such demand data are always rationalizable (in the consumer demand revealed preference sense). The easiest way to show this is to let $v_i(x_{ik}) = x_{ik} \cdot p$ for all i, ik, though I show in the full proof this knife-edge construction is not the only one. Any utility indexes satisfying Afriat's inequalities work. Then (x, m) is a competitive equilibrium supported by p, and thus (x, m) is in the weak core.

I now show (1) $\Longrightarrow$ (3). To see this, note that a cycle C's length $\sum_{ik \in C} \omega_{ik} - m_{ik}$ is its members' total net payment of money. If this is greater than 0, then this cycle net spends money. Its members can form a blocking coalition – they can allocate objects the same way as in (x, m), but keep their full endowed money for themselves.

Finally, to show (3) $\Longrightarrow$ (2), I use the shortest path length on $\mathcal{G}_T$ between two objects to construct the price difference between those objects. The construction is similar to that in Quinzii (1984). (We can choose an arbitrary base price high enough so that $p \geq 0$.) In the full proof, I show that this construction is consistent – the minimum path length between objects of the same type is always 0. This completes the proof. □

[11] For example, Mas-Colell, Whinston, and Green (1995), pg. 653.

I give an example to illustrate T-rationalizability.

Example (Example 2 continued.). For simplicity, let $\omega_{ik} = 3$ for all ik. It can be seen that all cycles have length 0, so this is rationalizable. Figure 3 shows $\mathcal{G}_T$, with $\omega_{ik} - m_{ik}$ as arc lengths.

Example 3. To construct utilities, set p as follows. In the left SCC, let $p_{h_1} = 3$ arbitrarily, and set the prices of other objects in this SCC by the minimum path length from h_2 plus 3, giving $p_{h_2} = 5, p_{h_3} = 4$. Notice that the path length between the two copies of h_2 is 0. In the right SCC, let $p_{h_4} = 1$ arbitrarily, and set $p_{h_5} = 2$ since the path length from h_4 to h_5 is 1. Altogether,

$$p_{h_1} = 3$$
$$p_{h_2} = 5$$
$$p_{h_3} = 4$$
$$p_{h_4} = 1$$
$$p_{h_5} = 2$$

The easiest way to construct T-rationalizing preferences is to let $v_i(x_{ik}) = x_{ik} \cdot p$ for all i. Though as mentioned above (and demonstrated in the full proof), this is not the only construction.

The theorem establishes a connection between T-rationalizability, competitive equilibrium, and consumer demand rationalizability. The question of T-rationalizability is equivalent to consumer demand rationalizability, à la Samuelson and Afriat. That is, an allocation is rationalizable if and only if each agent type, interpreted as demand data, is consumer demand rationalizable. Thus, we are looking for utility indexes such that every agent type is optimizing in their demand. Competitive equilibrium follows.

This yields the theorem's two equivalent and intuitive conditions for T-rationalizability. The first condition is the existence of a price vector supporting the allocation as a competitive equilibrium. That is, an allocation is T-rationalizable if and only if it can be supported as a competitive equilibrium. The second condition is reminiscent of cyclic monotonicity results common in revealed preference literature. It is readily interpretable directly; a cycle having positive length means it net transfers money outwards. Then its members can implement the same object allocation while retaining its money, establishing a blocking coalition.

I now give some corollaries of Theorem 2. First, I give conditions for strict T-rationalizability.

Corollary 4. *Fix a T-economy* $(A, \mathcal{A}, H, x, m, e, \omega)$. *The economy is strictly T-rationalizable if and only if **both** of the following are true:*

1. *The economy is T-rationalizable.*

2. *If* $ik, ik' \in S$ *are the same type and in the same SCC in* $\mathcal{G}_T$*, then either* $x_{ik} = x_{ik'}$ *OR the shortest path length from* x_{ik} *to* $x_{ik'} \neq 0$.

Proof. Appendix. □

This is the T-rationalizability analogue to Theorem 1. The additional condition says that two individuals of the same type, in the same SCC, should either be allocated the same object or pay different amounts. Having a zero path length between x_{ik} and $x_{ik'}$ means their prices must be the same. Then if two different individuals of type i purchase each one in competitive equilibrium, they must have the same utility. Conversely, having a nonzero path length allows us to construct different prices, and thus different utilities.

The following examples illustrate the corollary.

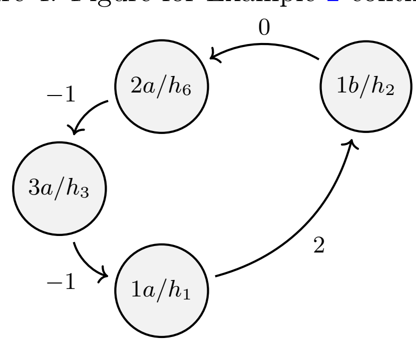
Example (Example 2 continued.)**.** This example is strictly T-rationalizable. The only thing to check is x_{1a} and x_{1b}. Since $x_{1a} = x_{1b}$, the economy is strictly TU-rationalizable – indeed, the utility given in the original example suffices.

Example 4. Suppose instead $x_{1b} = e_{2a} = h_6$, a new object type, with no other changes. Focusing on the left SCC:

ik	e_{ik}	x_{ik}	$\omega_{ik} - m_{ik}$
$1a$	h_1	h_2	2
$1b$	h_2	h_6	0
$2a$	h_6	h_3	-1
$3a$	h_3	h_1	-1

This economy is T-rationalizable, but not strictly T-rationalizable. The minimum path

Figure 4: Figure for Example 2 continued.



length from $x_{1a} = h_2$ to $x_{1b} = h_6$ is 0, forcing $p_{h_2} = p_{h_6}$. If $v_1(h_2) > v_1(h_6)$, then $1b$ is not maximizing subject to his budget, so the allocation is not a competitive equilibrium and not in the weak core.

The next corollary characterizes possible utility indexes $v_i(\cdot)$ that a T-economy.

Corollary 5. *Fix a T-economy $(A, \mathcal{A}, H, (e, \omega), (x, m))$. A T-rationalizable economy's solutions $v_i(\cdot)$ are characterized by solutions to the following linear system.*

for p s.t. $(x_{ik} - e_{ik}) \cdot p = \omega_{ik} - m_{ik} \quad \forall ik \in \mathcal{A}:$

1. $v_i(x_{ik}) \leq v_i(x_{ik'}) + p \cdot (x_{ik} - x_{ik'}) - w_{ik'} \quad \forall i, \forall ik, ik'$
2. *for any h such that $h \neq x_{ik}$ $\forall x_{ik}$, and for any ik such that $h \cdot p \leq e_{ik} \cdot p + \omega_{ik}$:*

   $v_i(h) - h \cdot p \leq v_i(x_{ik}) - x_{ik} \cdot p$

Proof. Appendix. □

The first line characterizes valid price vectors p. The inequalities define the utility indexes (v_i) given a valid p. Inequality 1. is the Afriat inequality for quasilinear utility (with marginal utility of money equal to one). Inequality 2. gives restrictions on utilities for any objects that are never consumed by type i. If an object h is never consumed but is affordable under some budget $e_{ik} \cdot p + \omega_{ik} := I_{ik}$, its consumption bundle $(h, I_k - h \cdot p)$ must be dispreferred to the actually consumed bundle $(x_{ik}, I_k - x_{ik} \cdot p)$.

This linear system gives possible values of (v_i) from the observed data. As is the case in consumer demand, these are joint restrictions rather than valid ranges for each $v_i(h)$. For example, there are many possible price vectors, each leading to a range of possible utility indexes v_i's. I also show in the proof of Theorem 2 that relative prices are determined within an SCC but not across SCCs.[12] Nevertheless, Corollary 5 characterizes the joint restrictions for valid v_is.

4 Estimating utility parameters from aggregate matching data

I turn to the task of estimating preferences from aggregate matchings without transfers. In the original setting, it is hard to determine rationalizing preference profiles. The proof Theorem 1 shows that many preference profiles rationalize an economy, and they are "dissimilar" due to the arbitrary order of SCCs. However, with a series of observations involving the same agent types and object types; and if we assume a parametric form of utility; it is possible to estimate utility parameters.

In this section, I derive an econometric method to estimate a confidence region for utility parameters from multiple stable matchings. The setup is similar to Fox (2010), though the resulting method is distinct. In the absence of perfectly transferable utility, we cannot assume utility maximizing choices. Thus, my objective is to estimate utility parameters from revealed preferences-type data. I will derive necessary moment inequalities for stability, then follow method suggested by Canay, Gaston, and Velez (2023).

4.1 Setup

The basic setup is the same as the exchange economy without transfers in Section 2.1.

Definition 8. The **aggregate matching matrix** X is the matrix with $A \times H$ rows, representing agent type-endowed object pairs; and H columns, representing allocated objects. Entry $X_{ie,h}$ is the number of type i endowed with e allocated h.

Now we observe $t = 1, ..., T$ rationalizable economies with the same types, each represented by X_t. Given a series of aggregate matchings, we can first apply the condition in Theorem 1 to check for rationalizability. Additionally, let preferences be given by utility $u_i(h; \beta, \varepsilon_{iht})$, a function of observable characteristics of the object, unknown parameter β,

[12] For this reason I conjecture it is not possible to write a linear system without the existential statement of (P).

and heterogeneity ε_{iht} with known distribution. This heterogeneity term is allows *types* to have heterogenous utility for objects across aggregate matchings t.

4.2 Moments and identification

An aggregate matching being in the core implies moment inequalities we can use to estimate the parameter β. First, the allocations must respect individual rationality. For $e, h \in H$ and $e \neq h$, an individual of type i must prefer his allocation to his endowment

$$\mathbb{1}\left(X_{ie,h} > 0\right) \Longrightarrow \left[\mathbb{1}\left(h \succ_i e\right) = 1\right]$$

Giving

$$\mathbb{P}\left(X_{ie,h} > 0\right) \leq \mathbb{P}\left[\mathbb{1}\left(h \succ_i e; \beta\right) = 1\right]$$

and moment inequality

$$\mathbb{E}[\underbrace{\mathbb{1}\left(X_{ie,h} > 0\right) - \mathbb{P}\left[\mathbb{1}\left(h \succ_i e; \beta\right) = 1\right]}_{:=m_1(X,i,e,h;\beta)}] \leq 0 \tag{1}$$

Likewise, the core implies no blocking coalitions of size 2. For $e \neq h', e' \neq h, e \neq e'$, and $i \neq i'$, we have

$$\mathbb{1}\left(X_{ie,h} > 0, X_{i'e',h'} > 0\right) \Longrightarrow \left[\mathbb{1}\left(e \succ_{i'} h'\right)\mathbb{1}\left(e' \succ_i h\right) = 0\right]$$

That is, it cannot be that there is an individual of type i and one of type i' that prefer each other's endowments. Then

$$\mathbb{P}\left[X_{ie,h} > 0, X_{i'e',h'} > 0\right] \leq \mathbb{P}\left[\mathbb{1}\left(e \succ_{i'} h'\right)\mathbb{1}\left(e' \succ_i h\right) = 0; \beta\right]$$

This gives the analogous moment inequality

$$\mathbb{E}[\underbrace{\mathbb{1}\left(X_{ie,h} > 0, X_{i'e',h'} > 0\right) - \mathbb{P}\left[\mathbb{1}\left(e \succ_{i'} h'\right)\mathbb{1}\left(e' \succ_i h\right) = 0; \beta\right]}_{:=m_2(X,i,i',e,e',h,h';\beta)}] \leq 0 \tag{2}$$

I use inequalities in 1 and 2 to estimate β. The identified set is given by parameters consistent with 1 and 2.

$$\left\{\beta : 1\ \forall i, e \neq h', \quad 2\ \forall i \neq i', e \neq h', e' \neq h, e \neq e'\right\}$$

These are necessary conditions for the core; they form an outer bound for the true β. It is also possible to add analogous inequalities for coalitions of size ≥ 3. However, the number of inequalities grows combinatorially, so the trade-off in tractability is unlikely to be favorable.

These conditions do not come from utility maximization in a choice set, as in Choo and Siow (2006) or Fox (2010). When utility is not transferable, the allocation is not utility maximizing allocation in general. Additionally, agents' choice sets are functions of the matching process, rather than exogenously given.

There is now substantial econometric literature on estimating confidence sets from moment inequalities; e.g. Chernozhukov, Hong, and Tamer (2007); Chernozhukov, Chetverikov, and Kato (2019); Canay, Gaston, and Velez (2023). A number of methods are possible to estimate the given model. I follow the suggestion of Canay, Gaston, and Velez (2023) and use Chernozhukov, Chetverikov, and Kato (2019) to construct a test for the hypothesis

$$\begin{aligned} H_0 : \{&\mathbb{E}[m_1(X, i, e, h; \beta)] \leq 0 \;\forall i, e \neq h'; \\ &\mathbb{E}[m_2(X, i, i', e, e', h, h'; \beta)] \leq 0 \;\forall i \neq i', e \neq h', e' \neq h, e \neq e'\} \end{aligned}$$

then invert the test to find β which do not reject the hypothesis. This also highlights a feature of the model – when the model fits better (that is, when the aggregate matchings are more "stable"), the confidence sets will be wider.

5 Conclusion

I present testable implications of the core in exchange economies with and without monetary transfers. The key identifying assumption is on agent types – that preferences are solely a function of observable characteristics of the agents. The analyst observes these types, endowments, and allocations, but not the preferences. Given this, I derive tractable and intuitive conditions for the core to be rationalizable.

The conditions in Theorems 1 and 2 characterize markets that are compatible with the core. That is, they can falsify a market being in the core; they also serve as ex ante predictions for market outcomes. The results can also be applied to audit mechanisms when the matching procedure is unknown.

I also develop a parametric method to estimate parameters of utility generating core allocations. Given a set of aggregate matchings over the same types, the core implies a series of moment conditions, which I use to obtain partial identification.

The work here suggests paths for future research. One takeaway is that other information must be observed (such as partial data on preferences or some structure of the allocation process) to further distinguish outcomes. Analogously to the development of GARP, one can implement smoother measures of rationalizability or construct statistical tests for rationalizability. The tractability of the graph construction $\mathcal{G}_{NT}$ and $\mathcal{G}_T$ may be useful in such work.

6 Proofs

6.1 Results for $\mathcal{G}_{NT}$

First, I introduce another graph construction. Given a NT-economy[13], draw $\mathcal{G}_{small}(A, \mathcal{A}, H, e, x) = (\mathcal{A}, E')$ as follows:

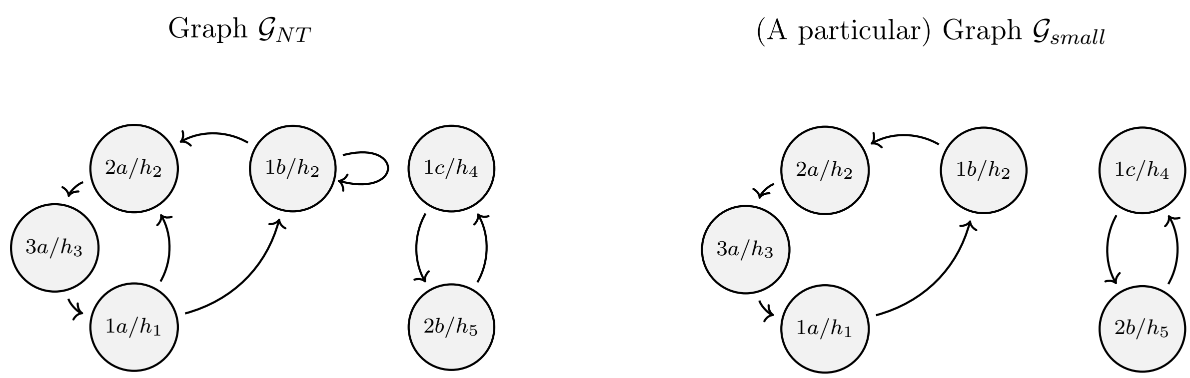
Step 0. Draw all agents $\mathcal{A}$ as vertices.

[13] Or, if given a T-economy, discard ω and m.

Step m. Consider all agents receiving h_m, that is all ik such that $x_{ik} = h_m$. Order them according to their index; refer to these as the "left" side. Similarly, order agents endowed with h_m according to their index; these are the "right" side. By construction, these two sets are the same cardinality. Draw one arc from the first agent on the left side to the first agent on the right side, and so on. If $m < \eta$, continue to step $m + 1$.

The graph produced after $|H|$ steps represents the allocation μ. Note that each agent has one out-arc and one in-arc. Recall the construction of $\mathcal{G}_{NT} = (\mathcal{A}, E)$. Note also that $E \supseteq E'$; that is, $\mathcal{G}_{NT}$ can be obtained by adding arcs to $\mathcal{G}_{small}$. Figure 5 shows both constructions for Example 1.

Figure 5: Figure for Example 1



I now provide some intermediate results related to the constructed graphs $\mathcal{G}_{small}$ and $\mathcal{G}_{NT}$. These will be key for the proof of Theorem 1.

Proposition 1. *Consider $\mathcal{G}_{small}(A, \mathcal{A}, H, e, x) = (\mathcal{A}, E')$. $\mathcal{G}_{small}$ has a subgraph partition into cycles. That is, there are disjoint subgraphs $C_1, ..., C_N$ such that $\mathcal{G}_{small} = \cup_{n=1}^{N} C_n$, $C_m \cap C_n = \emptyset$ for all m, n, and each C_n is a cycle.*

Proof. Note each vertex i has $d^-(ik) = d^+(ik) = 1$. We can invoke a version of Veblen's theorem:

> (Veblen's theorem) A directed graph $D = (V, E)$ admits a partition of arcs into cycles if and only if $d^-(v) = d^+(v)$ for all vertices $v \in V$. (Veblen, 1912; Bondy and Murty, 2008)

Since $d^-(ik) = d^+(ik)$, $\mathcal{G}_{small}$ has a partition of arcs into cycles. There are no isolated vertices, so every vertex is in at least one cycle. Further, since $d^-(ik) = d^+(ik) = 1$ each vertex must be in at most one cycle. Thus the arc partition into cycles also partitions the vertices into cycles. □

Proposition 2. *Consider $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x)$. For every strongly connected component S of $\mathcal{G}_{NT}$, there is a cycle including all vertices in S.*

Proof. By Proposition 1, $\mathcal{G}_{small}$ admits a partition of vertices into cycles. Recall $\mathcal{G}_{NT} = (\mathcal{A}, E)$ and $\mathcal{G}_{small} = (\mathcal{A}, E')$, where $E \supseteq E'$. Then these cycles also partition $\mathcal{G}_{NT}$'s vertices. The SCC S in $\mathcal{G}_{NT}$ is composed of the vertices in a number of $\mathcal{G}_{small}$-cycles. It cannot include a strict subset of vertices in a $\mathcal{G}_{small}$-cycle since there is always a path between any two vertices in a cycle.

The remaining argument is by strong induction on the number K of $\mathcal{G}_{small}$-cycles contained in S. Assign an order to these cycles in the following way. Let the first cycle be any of these. Choose the k^{th} cycle such that it has the same object type as one of the first $k-1$ cycles. It is always possible to do this – suppose at some point none of the remaining cycles has the same object type as the first k cycles. Then there are no paths in $\mathcal{G}_{NT}$ between the first k cycles and the remaining cycles (recall arcs are drawn from an agent to all agents whose endowment he receives), so they are not in the same SCC.

Claim. There is a cycle in $\mathcal{G}_{NT}$ covering all vertices in the first k $\mathcal{G}_{small}$-cycles in S. As shorthand, I will call this the "big-cycle", and the $\mathcal{G}_{small}$-cycles will be "small-cycles".

Base. For $k = 1$, the claim is trivial.

k^{th}. Suppose the claim is true for the first $k-1$ cycles. That is, there is a $k-1^{th}$ big-cycle in $\mathcal{G}_{NT}$ covering all the vertices in the first $k-1$ small-cycles. I show that there is a cycle covering all vertices in the $k-1^{th}$ big-cycle and the k^{th} small-cycle. The following argument is illustrated in Figure 6. There are three cases, depending on whether either cycle is a self-loop.

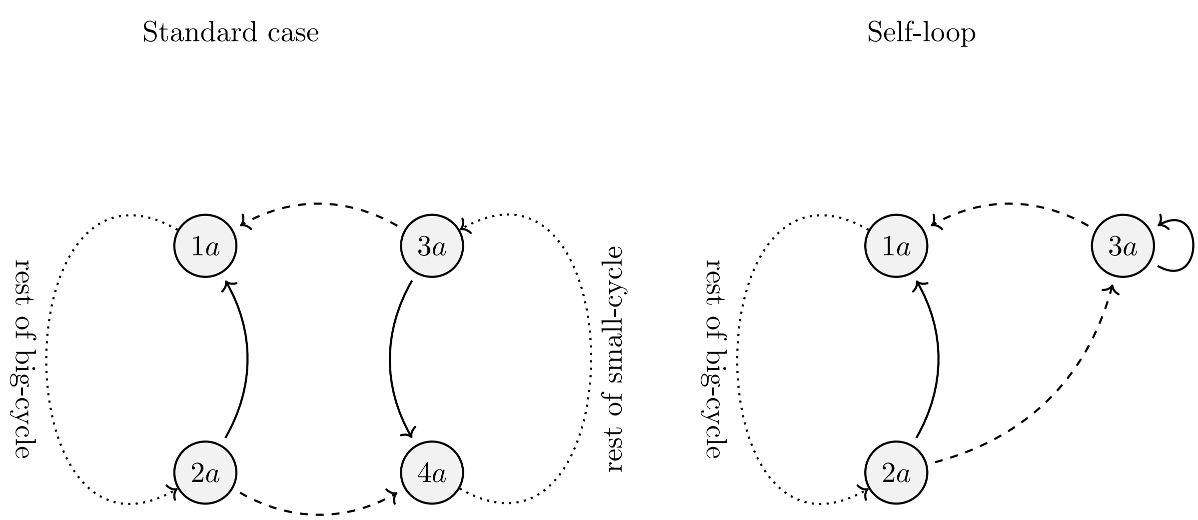
Case 1. Suppose neither is a self-loop. Let the big-cycle be $(1a, ..., 2a, 1a)$, and the k^{th} small-cycle be $(3a, 4a, ..., 3a)$. That is, $x_{2a} = e_{1a}$ and so on. I do not require that the denoted agents are all different types; e.g. $2a$ can be $1b$. By the ordering of the cycles, the k^{th} small-cycle and the $k-1^{th}$ big-cycle have at least one of the same object type. Without loss of generality let $e_{1a} = e_{4a}$. This gives $x_{2a} = e_{1a} = e_{4a}$, so we have the arc $(2a, 4a) \in E$. Similarly, $x_{3a} = e_{4a} = e_{1a}$, so we have the arc $(3a, 1a) \in E$. This gives us a new big-cycle across all the vertices in the first k small-cycles: $(\underbrace{1a, ..., 2a}_{\text{big-cycle } k-1}, \underbrace{4a, ..., 3a}_{k^{th} \text{ cycle}}, 1a)$.

Case 2. Suppose the k^{th} small-cycle is a self-loop, but the $k-1^{th}$ big-cycle is not. Then let the big-cycle be $(1a, ..., 2a, 1a)$, and the k^{th} small-cycle be $(3a, 3a)$. Again, let $e_{1a} = e_{3a}$ without loss of generality. Then $x_{2a} = e_{1a} = e_{3a}$ implies $(2a, 3a) \in E$. Likewise, $x_{3a} = e_{3a} = e_{1a}$ implies $(3a, 1a) \in E$. So we have a new big-cycle $(\underbrace{1a, ..., 2a}_{\text{big-cycle } k-1}, 3a, 1a)$. The case if the big-cycle is a self-loop is the same (this may occur in the $k = 2$ claim).

Case 3. Suppose both are self-loops. Then let the big-cycle be $(1a, 1a)$ and the k^{th} small-cycle be $(3a, 3a)$. Again, we suppose $e_{1a} = e_{3a}$. Then $x_{1a} = e_{1a} = e_{3a}$ implies $(1a, 3a) \in E$, and likewise $(3a, 1a) \in E$. So we have a new big-cycle $(1a, 3a, 1a)$.

This completes the proof. □

Figure 6: Illustration of Proposition 2



The following lemma is derived from Proposition 2 and its proof.

Lemma 1. *Consider $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x)$. Every strongly connected component S has no in- or out- arcs. That is, if $ik \in S$ and $(ik, i'k') \in E$ or $(i'k', ik) \in E$, then $i'k' \in S$. Alternatively, the strongly connected components and (weak) components coincide.*

Proof. There is a cycle covering all vertices of S by Proposition 2. Suppose there is an out-arc from S pointing to a vertex in a different SCC S'. S' also has a cycle covering all its vertices. The same argument as in the induction part of the proof of Proposition 2 establishes an arc from S' to S. Thus there are paths from between any vertices in S and S', and they are in the same SCC, a contradiction. The case for no in-arcs is a relabeling of S and S'. □

The following is a corollary of Lemma 1.

Corollary 6. *Consider $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x)$. Let ik and $i'k'$ be distinct vertices. There exists a $(ik, i'k')$-path if and only if ik and $i'k'$ are in the same SCC. Equivalently, there exists a $(ik, i'k')$-path if and only if there exists a $(i'k', ik)$-path.*

Proof. If ik and $i'k'$ are in the same SCC, there exists a $(ik, i'k')$-path by definition. Suppose there exists a $(ik, i'k')$-path. By Lemma 1, there are no paths between different SCCs, so ik and $i'k'$ must be in the same SCC. □

Corollary 7. *Consider $\mathcal{G}_{NT}(A, \mathcal{A}, H, e, x)$. All copies of the same object type are in the same SCC. That is, if $e_{ik} = e_{i'k'}$ and $ik \in S$, then $i'k' \in S$.*

Proof. Let $e_{ik} = e_{i'k'}$. There is at least one agent pointing to ik, so $\exists a \in \mathcal{A}$ such that $(a, ik) \in E$. Then $(a, i'k') \in E$ as well by construction. By Corollary 6, there are (ik, a)-

and $(i'k', a)$- paths. Then there are $(ik, i'k')$- and $(i'k', ik)$- paths (through a), so ik and $i'k'$ are in the same SCC. □

The above results give us significant information about the SCCs of $\mathcal{G}_{NT}$. The following is a summary of these results. From Proposition 2, each SCC contains a cycle covering all its vertices. From Lemma 1 and Corollary 6, $\mathcal{G}_{NT}$ can be vertex- *and* arc- partitioned into its SCCs. That is, $\mathcal{G}_{NT}$ consists of SCCs with no links between them. Finally, Corollary 7 tells us all copies of a given object type are in the same SCC.

If we take Theorem 1 as given for now, we can use the above result to prove Corollary 1.

Proof of Corollary 1. If if $e_{ik} = e_{ik'}$, then ik and ik' are in the same SCC. Then apply Theorem 1 to get the desired result. □

6.2 Proof of Theorem 1

Proof of Theorem 1. ("If") Let the supposition be true: whenever agents of the same type are in the same SCC, they receive the same object type. I find a preference profile $\succsim$ that such that x is in the core. First find the partition of vertices into SCCs. Then assign an arbitrary order to the SCCs, and denote them $S_1, ...S_M$. Construct the preferences by the following procedure. Let $\succsim_i (n)$ denote type i's n^{th} favorite object.

Step 1. In S_1, for all $i \in S_1$, let $\succsim_i (1) = x_i$. This is well defined since if there are multiple agents of the same type in S_1, they all receive the same object type.

2. In S_2, for all $i \in S_2$, let $\succsim_i (1) = x_i$ if possible. This is possible if there were no type i's in S_1. Otherwise, let $\succsim_i (2) = x_i$. By Corollary 7, an object never reappears in a later step, so this never assigns an object to two places in the same preference.

m. In S_m for $m = 2, ..., M$, for all $i \in S_k$, let $\succsim_i (m') = x_i$ for the lowest unassigned $m' = 2, ..., m$. Again by the same argument above, this never assigns two objects to the same type; it also never assigns the same object type to multiple places in the same preference.

$M + 1$. Assign remaining preferences in any order, if necessary.

I now show this preference profile admits no blocking coalition. Suppose that there is a coalition of agents $A' \subseteq \mathcal{A}$ and feasible sub-allocation μ' such that for all $ik \in A' : x'_{ik} \succsim_i x_{ik}$. The argument is by induction on the number of SCCs M. In each SCC S_m, the claim to demonstrate is that $x'_{ik} = x_{ik}$ for all $ik \in A' \cap S_m$.

Base. In S_1, all agents receive their favorite object. Then $x'_{ik} \sim x_{ik}$ for all $i \in A' \cap S_1$. The only indifferences are between copies of the same object type, so this gives $x'_{ik} = x_{ik}$.

m^{th}. Suppose the claim is true for all agents in $A' \cap (S_1 \cup \cdots \cup S_{m-1})$. This implies that x' allocates all agents in $A' \cap (S_1 \cup \cdots \cup S_{m-1})$ objects in their own SCC. That is, $x'_{ik} \in \cup_{ik \in A' \cap S_m} e_{ik}$.

Toward a contradiction, suppose that $\exists ik \in S_m$ such that $x'_{ik} := h \succ_i x_{ik}$. Then it must be $h \in \cup_{ik \in S_1 \cup \cdots \cup S_{m-1}} e_{ik}$, since all strictly preferred objects are in earlier SCCs.

Further, since x' reallocates within A', it must be $h \in \cup_{ik \in A' \cap (S_1 \cup \cdots \cup S_{m-1})} e_{ik}$. Then it must be that an agent in $A' \cap (S_1 \cup \cdots \cup S_{m-1})$ receives an object in $\cup_{ik \in A' \cap (S_1 \cup \cdots \cup S_{m-1})} e_{ik}$. This contradicts the supposition, so it must be that $x'_{ik} \sim x_{ik}$ for $ik \in A' \cap S_m$, giving $x'_{ik} = x_{ik}$.

Thus $x'_{ik} = x_{ik}$ for all $ik \in A'$, and A' is not a blocking coalition.

("Only if") Toward the contrapositive, suppose there is an SCC S with two agents of the same type who receive different objects. By Proposition 2, there is a cycle covering all vertices in S. I now construct a blocking coalition using this cycle. Note that two of these vertices represent agents of the same type who receive different objects. Let these two agents be $1a$ and $1b$; I consider cases based on their relative positions in the cycle.

1. Suppose the cycle is $1a \to \underbrace{2a \to \cdots \to 1b}_{:=c} \to 3a \to \cdots \to 1a$, and $e_{2a} \neq e_{3a}$. Suppose $e_{2a} \succ_1 e_{3a}$. Then $1b \to \underbrace{2a \to \cdots \to 1b}_{c}$ represents a blocking coalition. Note that this is a feasible sub-allocation; it contains its own endowment, and $1b$ is strictly better off. The case $e_{2a} \prec_1 e_{3a}$ is a rotation and relabeling of the cycle.

2. Suppose the cycle is $1a \to 1b \to \underbrace{2a \to \cdots \to 1a}_{:=c}$. If $e_{2a} \succ_1 e_{1b}$, then $1a \to \underbrace{2a \to \cdots \to 1a}_{c}$ is a blocking coalition. If instead $e_{1b} \succ_1 e_{2a}$, then x is not individually rational for $1b$.

3. If the cycle is $1a \to 1b \to 1a$ and $e_{1a} \neq e_{1b}$, then μ is not individually rational.

This completes the proof. $\square$

Remark. For readers familiar with the result in Quint and Wako (2004), it suffices to show that executing their "$\mathcal{STRICTCORE}$" algorithm on the above constructed preferences results in the allocation μ. This is readily apparent, and a formal proof is omitted.

6.3 Proof of Theorem 2 and related results

I first give a formal definition of competitive equilibrium in an exchange economy setting.

Definition 9. Let $E = \{(\omega_{ik}, e_{ik}), (u_{ik})\}_{ik \in \mathcal{A}}$ be an exchange economy, where $u_{ik}(\cdot) : H \times \mathbb{R}_+ \to \mathbb{R}$ are utility functions. A **competitive equilibrium** is a price vector $p \in \mathbb{R}^H$ and a feasible allocation $(x_{ik}, m_{ik})_{ik \in \mathcal{A}}$ such that for all $ik \in \mathcal{A}$:

- $m_{ik} + p \cdot x_{ik} \leq \omega_{ik} + p \cdot e_{ik}$
- $(u_{ik}(h, m) > u_{ik}(x_{ik}, m_{ik})) \implies (m + p \cdot h > \omega_{ik} + p \cdot e_{ik})$

That is, all agents' allocations are affordable for them, and any better allocation is unaffordable. A **competitive equilibrium allocation** is $(x_{ik}, m_{ik})_{ik \in \mathcal{A}}$ for which there exists a price vector supporting it as a competitive equilibrium.

Definition 10. Let $\{(x^r, p^r, I^r)\}_{r=1}^N$ be observed demand, price, and budget data, where $x^r \in \mathbb{R}^H_+; p^r, I^r \in \mathbb{R}^H_{++}$. The data is **quasilinear rationalizable** if for all r, x^r solves

$$\max_{x \in \mathbb{R}^n_{++}} v(x) + m$$
$$\text{s.t. } p^r \cdot x + m = I^r$$

for some concave v.

I also give Afriat's theorem for quasilinear rationalizability. (These are the usual Afriat inequalities with $\lambda = 1$.)

Theorem 3 (Afriat's theorem)**.** *Data* $(x^r, p^r, I^r), r = 1, ..., N$ *are quasilinear rationalizable if and only if there exist numbers* $v_j \in \mathbb{R}$ *such that*

$$v^k \leq v^j + \left(p^j \cdot x^r - I^r\right) \tag{A}$$

I now give the full proof for Theorem 2.

Proof of Theorem 2. I show (2) ⇒ (1) ⇒ (3) ⇒ (2).

I now show (2) $\implies$ (1). Suppose there exists a vector p satisfying equation (P). I first seek to show that this p supports (x, m) as a competitive equilibrium for some utility indexes (v_i). That is, I want to construct v_i such that all agents ik are maximizing utility subject to their budget constraints $e'_{ik} \cdot p + \omega_{ik}$.[14] This becomes a classic consumer demand revealed preference problem. To see this, reinterpret an agent type i as a single consumer, and each individual agent ik as a demand data point from this consumer:

$$\left(\underbrace{(x_{ik}, m_{ik})}_{\text{consumed good and money}}, \underbrace{(e'_{ik} \cdot p + \omega_{ik}) := I_{ik}}_{\text{budget}}, \underbrace{p}_{\text{price}} \right)_{k \in \{1, ..., K_i\}}$$

That is, i is a consumer, and each ik is a single observation of demand at a particular budget. There are $|A|$ consumers and K_i demand points for each consumer i. We seek to rationalize the demand data in a consumer revealed demand sense by constructing (v_i) such that each consumer i is maximizing utility $V_i(h, m) = v_i(h) + m$ in each consumption bundle-budget pair.

The easiest way to do this is to let $v_i(x_{ik}) = x'_{ik} \cdot p$, making all agents indifferent to any possible consumption bundle while still satisfying assumption $(A2)$. However, these data are rationalizable more broadly; any indexes fulfilling Afriat's inequalities (A) will also suffice for (v_i).

I now show (1) $\implies$ (3). I show the contrapositive; suppose $\mathcal{G}_T$ has a cycle C with positive length; i.e. $\sum_{ik \in C} \omega_{ik} - m_{ik} > 0$. The members of C can form a blocking coalition for (x, m) by allocating to each $ik \in C$

$$\left(x_{ik}, m + \frac{\sum_{ik \in C} \omega_{ik} - m_{ik}}{|C|} \right)$$

[14] Agent ik sells his endowment e'_{ik} at price p and is additionally endowed with ω_{ik} money.

That is, each agent receives the same object and receives more money from the excess endowment. This is feasible for C and strictly preferred by all $ik \in C$.

Finally, I prove (3) $\implies$ (2). Suppose $\mathcal{G}_T$ has no cycles with length > 0. I construct a price p satisfying (P) via path lengths on $\mathcal{G}_T$. Note that Proposition 2, Lemma 1, and Corollary 7 still apply to $\mathcal{G}_T$. Every SCC has a cycle covering all its vertices; there are no paths between two SCCs; and all objects of the same type are in the same SCC. Denote p_h as the price of object type $h \in H$. Construct p as follows:

1. For each SCC, choose any object type h in this SCC and set p_h to be any number.

2. For all objects h' in this SCC, set $p_{h'} - p_h$ to be length of the shortest path from h to h'. That is, the shortest path between an agent endowed with h to an agent endowed with h' determines the price difference.

3. Repeat steps 1 and 2 for all SCCs.

4. Add a constant to p to ensure $p \geq 0$.

I will show that all paths between two vertices are the same length, then that the path length between an object type h and itself is always 0, so that the construction is consistent, i.e. $p_h - p_{h'} = 0$ when $h = h'$. The rest of the proof will immediately follow.

Note the whole economy is budget balanced; we have $\sum_{ik \in \mathcal{A}} \omega_{ik} = \sum_{ik \in A} m_{ik}$. For any cycles that form a vertex-partition of $\mathcal{G}_T$: these cycles must have length 0. A negative length cycle that is in a partition of the overall economy implies a positive length cycle elsewhere by budget balancedness, a contradiction.

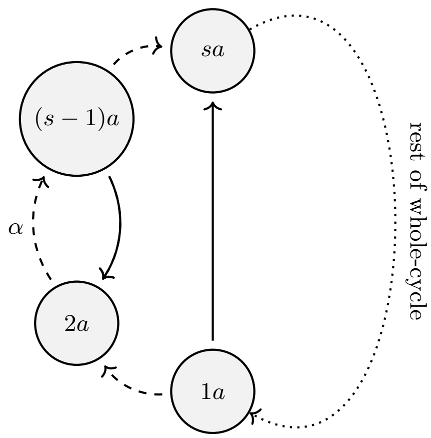
In particular, by Proposition 2, each SCC has a cycle containing all its vertices; call this the "whole-cycle" as shorthand. These partition the whole economy, so each whole-cycle must have length 0. For the following claims, assume the SCC has at least three vertices. I will show the cases for one or two vertices separately. Enumerate the whole-cycle as $(1a, 2a, ..., sa, ...(S-1)a, Sa, 1a)$. (Allowing any of these agents to be of the same type – this is unimportant.) Now consider $1a$ and sa distinct and in the same SCC (recall there are no paths between SCCs), and consider the path $(1a, ..., sa)$ via the whole-cycle. Denote this path $(1a, \underbrace{2a, ..., (s-1)\,a}_{:=\alpha}, sa)$, and call it the "whole-cycle path" as shorthand.

Claim 1. If the arc $(1a, sa)$ exists, it is the same length as the whole-cycle path. That is, $\ell(1a, sa) = \ell(1a, 2a, ..., (s-1)\,a, sa)$.

Figure 7 illustrates the following argument. If the arc $(1a, sa)$ exists, then $e_{2a} = e_{sa}$, so there is an arc $((s-1)a, 2a)$. Then $(\underbrace{2a, ..., (s-1)a}_{:=\alpha}, 2a)$ forms a cycle, and

$(1a, sa, \underbrace{...}_{\text{rest of whole-cycle}}, 1a)$ also forms a cycle. Since the two cycles partition the SCC, they are part of a partition of the overall economy; thus both cycles must have length 0. If $\ell(1a, sa) > \ell(1a, 2a, ..., (s-1)\,a, sa)$, then the latter cycle has positive length, a contradiction. This is because the whole-cycle has length 0 as established, and we have found a cycle with shorter length. If instead $\ell(1a, sa) < \ell(1a, 2a, ..., (s-1)\,a, sa)$, then the latter cycle has negative length, also a contradiction. Note the same argument carries through if $2a = (s-1)a$ – the first cycle is a self-loop, and $1a = (s-1)a$ is symmetric.

Figure 7: Illustration of Claim 1



Claim 2. If the arc $(sa, 1a)$ exists, it has length negative of the whole-cycle path from $1a$ to sa. That is, $\ell(sa, 1a) = -\ell(1a, 2a, ..., (s-1)\,a, sa)$.

From Claim 1, $\ell(sa, 1a) = \ell(sa, (s+1)a, ..., Sa, 1a)$. Notice that $(sa, (s+1)a, ..., Sa, 1a)$ and $(1a, 2a, ..., (s-1)a, sa)$ form the whole cycle, so their lengths sum to 0. That is, $\ell(sa, 1a) + \ell(1a, 2a, ..., (s-1)a, sa) = 0$, and the claim follows.

Remark 1. The indexing of $1a$ and sa in Claims 1 and 2 is not important. Since the whole-cycle is a cycle, $1a$ can be any vertex. (It is convenient to have $1 \leq s \leq S$.)

Claim 3. Any $(1a, sa)$-path is the same length as the whole-cycle path $(1a, \underbrace{2a, ..., (s-1)\,a}_{:=\alpha}, sa)$.

The $(1a, sa)$-path is some permutation of a subset of vertices of the SCC. Denote this $(\underbrace{\sigma_1 a}_{=1a}, \sigma_2 a, ..., \sigma_{j-1} a, \underbrace{\sigma_j a}_{=sa})$, where $j \leq S$. I will show

$$\ell(\sigma_1 a, ..., \sigma_{j-1} a, \sigma_j a) = \underbrace{\ell(1a, 2a) + \cdots + \ell((\sigma_j - 1)a, \sigma_j a)}_{\text{whole-cycle path}} \equiv \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a)$$

Note that $\sigma_{j-1} \neq \sigma_j - 1$ in general.

I will show the claim by strong induction on the length of j. The base case of $j = 1$ is Claim 1. Now suppose the claim is true for j; that is, $\ell(1a, ..., \sigma_{j-1} a, \sigma_j a) = \sum_{i=1}^{\sigma_{j-1}} \ell(ia, (i+1)a)$. Now consider $j+1$. We have $\ell(1a, \sigma_{j+1} a) = \ell(1a, \sigma_j a) + \ell(\sigma_j a, \sigma_{j+1} a)$. If $\sigma_{j+1} > \sigma_j$, then by Claim 1 write

$$\ell(\sigma_j a, \sigma_{j+1} a) = \sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)$$

So

$$\begin{aligned}
\ell(1a, ..., \sigma_j a, \sigma_{j+1} a) &= \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a) + \ell(\sigma_j a, \sigma_{j+1} a) \\
&= \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a) + \sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a) \\
&= \sum_{i=1}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)
\end{aligned}$$

If $\sigma_{j+1} < \sigma_j$, then by Claim 2 write

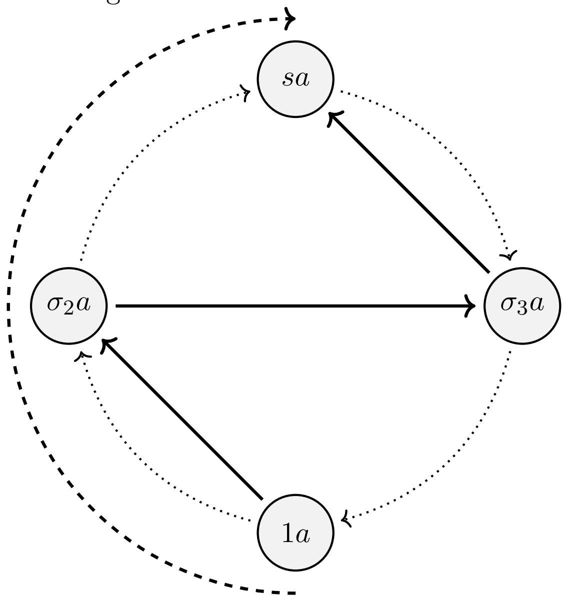
$$\ell(\sigma_j a, \sigma_{j+1} a) = -\sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)$$

So

$$\begin{aligned}
\ell(1a, ..., \sigma_j a, \sigma_{j+1} a) &= \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a) + \ell(\sigma_j a, \sigma_{j+1} a) \\
&= \sum_{i=1}^{\sigma_{j+1}-1} \ell(ia, (i+1)a) + \sum_{i=1}^{\sigma_j - 1} \ell(ia, (i+1)a) - \sum_{i=\sigma_j}^{\sigma_{j+1}-1} \ell(ia, (i+1)a) \\
&= \sum_{i=1}^{\sigma_{j+1}-1} \ell(ia, (i+1)a)
\end{aligned}$$

as desired.

Figure 8: Illustration of Claim 3



Claim 4. The length of any path between an object type h and itself is 0.

Figure 9 illustrates the following argument. Note that two vertices (agents) may be endowed with the same object type, so these can be distinct nodes. Recall that all copies of the same object type are contained in the same SCC. The path length from a vertex to itself is 0 since the whole-cycle has length 0, and any other path is the same length. Now suppose h is contained in two distinct vertices, $1a$ and $2a$. Consider a node sa such that $x_{sa} = h$. (This may be $1a$ or $2a$.) Then the arcs $(sa, 1a)$ and $(sa, 2a)$ exist. These have the same length, $\omega_{sa} - m_{sa}$, by construction of $\mathcal{G}_T$. Denote $\ell(sa, 1a) = \ell(sa, 2a) = \ell_1$. I show the length of the path from $1a$ to $2a$ is 0. Denote this path $(1a, ..., 2a)$, and let $\ell(1a, ..., 2a) = \ell_2$. Both $(sa, 1a, ..., 2a)$ and $(sa, 2a)$ are paths from sa to $2a$, so must have the same length. Then $\ell_1 = \ell_1 + \ell_2$, giving us $\ell_2 = 0$ as desired.

Figure 9: Illustration of Claim 4

I have shown the above claims for SCCs of size at least three. Now consider an SCC of only one vertex. The only arc must be $(1a, 1a)$, which constitutes the whole-cycle and must have length 0, and the path length from this object type to itself is 0.

Now consider an SCC of two vertices, $1a$ and $2a$. If they are endowed with distinct object types, the arcs $(1a, 2a)$ and $(2a, 1a)$ are the only arcs, and the claims are true trivially. If they are endowed with the same object type, the self loops are also present. The two self-loops partition the SCC, so have length 0. We have $\ell(1a, 1a) = \ell(1a, 2a)$ by construction, so $\ell(1a, 2a) = 0$, and similarly $\ell(2a, 1a) = 0$. Then all arcs have length 0 in this SCC, so the claims are again true.

The rest of the proof follows easily. The path length between any object type h and itself is 0 (so the minimum path length is 0), ensuring it is possible to construct prices this way. Next, for any $ik \in \mathcal{A}$, the path length from $e_{ik} := h$ to $x_{ik} := h'$ is $m_{ik} - \omega_{ik}$, so that $p_{h'} - p_h = m_{ik} - \omega_{ik}$. This gives

$$(x_{ik} - e_{ik}) \cdot p = p_{h'} - p_h = m_{ik} - \omega_{ik}$$

as desired.

This completes the proof of the theorem. □

Proof of Corollary 4. As argued in the proof of Theorem 2, any price must satisfy $(x_{ik} - e_{ik}) \cdot p = \omega_{ik} - m_{ik}$ for all $ik \in \mathcal{A}$. By the construction of $\mathcal{G}_T$, $x_{ik} - e_{ik}$ is an arc from e_{ik} to x_{ik} with length $\omega_{ik} - m_{ik}$, which is also the price difference between these objects. Inductively (I will omit the full formality), a path from x_{ik} to $x_{ik'}$ has path length 0 if and only if the price difference between them is 0. (Note that by Claim 2, there also must be a path from $x_{ik'}$ to x_{ik}, and it has length 0 as well.)

("If") Let both conditions be true. As in the main theorem, it is sufficient to set $v_i(x_{ik}) = p \cdot x_{ik}$. Since prices can be set arbitrarily across SCCs, we can ensure no two objects in different SCCs have the same price.

("Only if") Toward a contradiction, suppose the economy is not T-rationalizable. Then it is of course not strictly T-rationalizable. Now suppose the second condition is false. That is, there are ik, ik' in the same SCC such that $x_{ik} \neq x_{ik'}$, but the shortest path length between them is 0. Then $p_{x_{ik}} = p_{x_{ik'}}$. Suppose $v_i(x_{ik}) > v_i(x_{ik'})$ without loss of generality. Then ik' can afford $(x_{ik}, m_{ik'})$, which is preferable to $(x_{ik'}, m_{ik'})$. Thus (x, m) is not a competitive equilibrium, so is not strictly T-rationalizable.

In particular, x_{ik} can purchase $x_{ik'}$ instead. Since $m_{ik} > 0$ by assumption, ik can form a blocking coalition by compensating other members of the blocking coalition. □

Proof of Corollary 5. This comes from the proof of Theorem 2.

The first line is conditions for valid vectors p, which comes from Theorem 2 and its proof.

The first inequality is (A). This is exactly Afriat's inequalities when the marginal utility of money is 1. These give joint restrictions on any the utility for objects actually consumed by agent type i given some p. Necessity and sufficiency are from Afriat's theorem.

The second inequality gives restrictions on the utility for objects not consumed by type i. An object h that is affordable under some ik's budget must have $V(h, e_{ik} \cdot p + \omega_{ik} - p \cdot h) \leq V(x_{ik}, e_{ik} \cdot p + \omega_{ik} - p \cdot x_{ik})$, else (x, m) is not a competitive equilibrium. This gives the inequality in the corollary:

$$v_i(h) + (e_{ik} \cdot p + \omega_{ik} - h \cdot p) \leq v_i(x_{ik}) + (e_{ik} \cdot p + \omega_{ik} - x_{ik} \cdot p)$$
$$v_i(h) - h \cdot p \leq v_i(x_{ik}) - x_{ik} \cdot p$$

That is, if h is affordable to ik, then its utility (including leftover money) must be less than that of x_{ik}. Note that an object that is too expensive for all ik is allowed to have any utility. Again, necessity and sufficiency are immediate. □

Part II
House-Swapping with Objective Indifferences

Preface

The second chapter is coauthored with Will Sandholtz. We study the classic house-swapping problem of Shapley and Scarf (1974) in a setting where agents may have "objective" indifferences, i.e., indifferences that are shared by all agents. In other words, if any one agent is indifferent between two houses, then all agents are indifferent between those two houses. The most direct interpretation is the presence of multiple copies of the same object. Our setting is a special case of the house-swapping problem with general indifferences. We derive a simple, easily interpretable algorithm that produces the unique strict core allocation of the house-swapping market, if it exists. Our algorithm runs in $O(n^2)$ time, where n is the number of agents and houses. This is an improvement over the $O(n^3)$ time methods for the more general problem.

7 Introduction

The house-swapping problem originally studied by Shapley and Scarf (1974) assumes that agents have a strict preference ordering over the set of the agents' houses. Implicitly, all houses are distinct. As Roth and Postlewaite (1977) show, in this setting the strict core is always non-empty and consists of a single allocation, which can identified using the Top Trading Cycles algorithm (TTC).

In the more general setting where agents' preference rankings may contain indifferences, the strict core may be empty. Moreover, when the strict core is non-empty, it may contain multiple allocations. Quint and Wako (2004) devised an algorithm, Top Trading Segmentation (TTS), that finds a strict core allocation, when it exists. Alcalde-Unzu and Molis (2011) devise Top Trading Absorbing Sets (TTAS) which finds the strict core when it exists and the weak core otherwise. They leave computational complexity of their algorithm as an open question. Jaramillo and Manjunath (2012) also solve the general indifference problem with Top Cycle Rules (TCR), which has complexity $O(n^6)$. Aziz and Keijzer (2012) present Generalized Absorbing Top Trading Cycle (GATTC), generalizing TTAS and TCR and show that TTAS has exponential time complexity. Plaxton (2013) develops a different mechanism to produce a strict core allocation with time complexity $O(n^3)$.

We study a more structured problem, where any indifferences are shared across all agents. We use the phrase "objective indifferences" to describe this setting. Conversely, we use the phrase "subjective indifferences" to describe indifferences that are not necessarily shared by all agents. Objective indifferences are the leading case of indifferences, since many objects we encounter in daily life are commodified. This additional structure enables us to develop a simple algorithm to find the strict core, when it exists, with time complexity $O(n^2)$.

Our setting can be thought of as an intermediate case between the original Shapley and Scarf setting and the general setting studied first by Quint and Wako. With objective indifferences, as in the house-swapping problem with subjective indifferences, the strict core may be empty. However, when the strict core is non-empty it contains a unique allocation. We propose a simple algorithm that finds the strict core allocation of a house-swapping market with objective indifferences in square-polynomial time. This algorithm is faster than the polynomial time algorithms that are needed for house-swapping markets with subjective indifferences.

8 Model

Let $I = \{1, 2, ..., I\}$ be a set of agents, each of whom is endowed with a house. Let $H = \{1, 2, ..., H\}$ be the set of possible house types in the market. Note that $H < I$ implies that some agents were endowed with houses of the same type. The endowment function $E : I \to H$ maps each agent to the house type he was endowed with.

Each agent $i \in I$ has strict preferences $\succsim_i$ over H. Implicitly, all agents are indifferent between two houses of the same type. We use $\succsim = \{\succsim_1, \succsim_2, ..., \succsim_I\}$ to denote the preference profile of all agents.

An allocation μ is a function $\mu : I \to H$ such that $|\mu^{-1}(h)| = |E^{-1}(h)|$ for all $h \in H$. That is, $\mu(i) = h$ means agent i is assigned a house of type h, and the number of agents who are allocated to a house type is equal to the supply of it.

The house-swapping market is summarized as the tuple $(I, H, E, \succsim)$. We are interested in whether the strict core exists.

Definition 11. An (sub-)allocation μ is **feasible** for a coalition of agents $I' \subseteq I$ if $\left|\mu^{-1}(h)\right| = \left|E^{-1}(h) \cap I'\right|$ for any $h \in E(I')$. That is, the quantity of each house type required in the (sub-)allocation is the same as the quantity in the coalition's endowment.

Definition 12. A (feasible) allocation μ is in the **strict core** of the house-swapping market $(I, H, E, \succsim)$ if there is no coalition $I' \subseteq I$ and no sub-allocation μ' such that:

1. μ' is feasible for I'
2. $\mu'(i) \succsim_i \mu(i)$ for all $i \in I'$
3. $\mu'(i) \succ_i \mu(i)$ for at least one $i \in I'$

We derive an algorithm that finds the strict core of a house-swapping market $(I, H, E, \succsim)$ when it exists.

9 Directed Graphs

Before proceeding to our main results and the algorithm, we review some useful concepts related to directed graphs. The definitions are standard, and a familiar reader may skim this section.

A **directed graph** is given by $D(V, E)$ where V is the set of vertices and E is the set of arcs. An **arc** is a sequence of two vertices (v, v'). We allow for arcs of the form (v, v), which we call **self-loops**. A (v_1, v_k)**-path** is a sequence of vertices $(v_1, v_2, ..., v_k)$ where each v_i is distinct and $(v_{i-1}, v_i) \in E$ for all $i = 2, 3, ..., k$. A **cycle** is a path where $v_1 = v_k$ is the only repeated vertex. A **sink** of a directed graph is a vertex v such that $(v, v') \notin E$ for all $v' \in V$.

A **strongly connected component (SCC)** of a directed graph $D(V, E)$ is a maximal set of vertices $S \subseteq V$ such that for all distinct vertices $v, v' \in S$, there is both a (v, v')-path and a (v', v)-path. By convention, there is always a path from v to itself, regardless of whether $(v, v) \in E$. The collection of strongly connected components of a directed graph forms a partition of V. (To see this, note that the definition of an SCC implies that a vertex can be in exactly one SCC.)

The **condensation** of a directed graph $D(V, E)$ is the directed graph $D(V^{SCC}, E^{SCC})$ where V^{SCC} is the set of SCCs of $D(V, E)$ and $(S, S') \in E^{SCC}$ if and only if there exist $v \in S$ and $v' \in S'$ such that $(v, v') \in E$. In other words, it is the arc-contraction of D on each SCC – replace each SCC with a single vertex, and keep any arcs between SCCs. Condensations of directed graphs are always acyclic.

A **topological ordering** of a directed acyclic graph $D(V, E)$ is a total order $\leq$ of the elements of V such that if $(v, v') \in E$, then $v \leq v'$. A directed graph has a topological ordering if and only if it is acyclic.[15] It is immediate that the vertex with the highest topological ordering is a sink.

[15]See Korte and Vygen (2008), Section 2.2.

10 Results

In this section, we give our algorithm to determine whether a strict core of a market $(I, H, E, \succsim)$ exists and to find it when it does. First, we define a function B_i that denotes the i's most preferred house type among a subset of house types. Let $B_i : I \times \mathcal{P}(H) \to H$ be given by $B_i(H') = h$ if $h \succsim_i h'$ for all $h' \in H'$.

We now give our algorithm.

Algorithm 1. *House Top Trading Segments (HTTS)*

Step 1. Let $R_1 = H$. Construct the directed graph $D_1 = D(R_1, E_1)$ where $(h, h') \in E_1$ if $B_i(R_1) = h'$ for some $i \in E^{-1}(h)$. That is, draw an arc (h, h') exists if an owner of h top-ranks h' among all house types $R_1 = H$. Find an SCC H_1 of D_1 with no outgoing arcs; i.e., for any $h \in H_1$ and $h' \notin H_1$, $(h, h') \notin E_1$.[16] *We call H_1 a "house top trading segment".*

Let $I_1 = E^{-1}(H_1)$. For all $i \in I_1$, set $\mu(i) = B_i(R_1)$. That is, assign every agent endowed with a house in H_1 to his favorite house (also in H_1).

Check that μ is feasible for I_1. If so, proceed to part c. Otherwise, stop.

Let $R_2 = R_1 \setminus H_1$. If $R_2 = \emptyset$, stop; otherwise, proceed to Step 2.

Step d. Construct the directed graph $D_d = D(R_d, E_d)$ where $(h, h') \in E_d$ if $B_i(R_d) = h'$ for some $i \in E^{-1}(h)$. Find an SCC H_d of D_d with no outgoing arcs.

Let $I_d = E^{-1}(H_d)$. For all $i \in I_d$, set $\mu(i) = B_i(R_d)$. That is, assign each agent in I_d to his favorite remaining house. Since H_d has no outgoing arcs, this house is also in H_d.

Check that μ is feasible for I_d. If so, proceed to part c. Otherwise, stop.

Let $R_{d+1} = R_d \setminus H_d$. If $R_{d+1} = \emptyset$, stop; otherwise, proceed to Step $d + 1$.

Remark 2. Note that at each step, house types are removed, and thus agents owning them are also removed. Since there are finitely many house types H, the algorithm terminates in finite time.

Remark 3. At part b of each step, μ is feasible for I_d if and only if for each $h \in H_d$, $\left|E^{-1}(h) \cap I_d\right| = |\{i : B_i H_d) = h, i \in I_d\}|$. That is, the number of copies of h available in I_d is equal to the number of agents who top-rank h among the remaining houses. Informally, "supply equals demand."

The house top trading segments we find in each step are analogous to TTC trading cycles. At each step, agents "point" from their owned house to their favorite house. We then find the trading segment and execute the trades, if possible ("feasible"). For readers familiar with Quint and Wako (2004), these are modified versions of top trading segments.

Theorem 4. *Let $(I, H, E, \succsim)$ be a market.*

1. *The strict core exists if and only if Algorithm 1 terminates in part c of a step. That is, each step's HTTS gives a feasible allocation, and the algorithm did not terminate in part b of a step.*

2. *Algorithm 1 finds a strict core allocation, when one exists.*

[16] There always exists an SCC with no outgoing arcs. To see this, consider the condensation (contract each SCC to a single vertex). The result is a directed acyclic graph, which has at least one sink. The sink is the (contracted) desired SCC with no outgoing arcs. Note that there may be multiple SCCs with no outgoing arcs. If so, pick any arbitrarily.

3. *The strict core allocation is unique, when it exists.*[17]

4. *Algorithm 1 has time complexity* $O(|H|^2 + |H||I|)$.

Before the proof of Theorem 4, we give the following example to illustrate it and Algorithm 1.

Example 5. Consider the house-swapping market $(I, H, E, \succsim)$ where

$$\begin{aligned} I &= \{1,2,3,4,5\} \\ H &= \{h_1,h_2,h_3,h_4\} \\ E(1) &= h_1,\ E(2) = E(3) = h_2,\ E(4) = h_3,\ E(5) = h_4 \end{aligned}$$

and $\succsim = \{\succsim_1, \succsim_2, \succsim_3, \succsim_4, \succsim_5\}$ is given by

$$\begin{aligned} &h_2 \succ_1 ... \\ &h_1 \succ_2 ... \\ &h_3 \succ_3 h_2 \succ_3 ... \\ &h_4 \succ_4 ... \\ &h_3 \succ_5 ... \end{aligned}$$

1. *Step 1*: Set $R_1 = H$. Construct the directed graph $D(R_1, E_1)$ where $(h, h') \in E_1$ if $B_i(R_1) = h'$ for some $i \in E^{-1}(h)$. That is, some owner of h top ranks h'. There are two SCCs in $D(R_1, E_1)$: $\{h_1, h_2\}$ and $\{h_3, h_4\}$. Only $S = \{h_3, h_4\}$ has no outgoing arcs. Then set $H_1 = \{h_3, h_4\}$ and $I_1 = \{4, 5\}$.
   (a) Assign $\mu(4) = h_4; \mu(5) = h_3$.
   (b) Check that this is feasible for I_1. We have
   $$\begin{aligned} \left|E^{-1}(h_3) \cap I_1\right| &= |\{4\}| = 1 \\ |\{i : B_i(H_1) = h_3, i \in I_1\}| &= |\{5\}| = 1 \end{aligned}$$
   and likewise for h_4, so this is feasible.
   (c) Set $R_2 = R_1 \setminus H_1 = \{h_1, h_2\}$ and continue to Step 2.

2. *Step 2*: Construct the directed graph $D(R_2, E_2)$ where $(h, h') \in E_2$ if $B_i(R_2) = h'$ for some $i \in E^{-1}(h)$. That is, some owner of h top ranks h' among the remaining houses $R_2 = \{h_1, h_2\}$. The entire graph forms an SCC, so set $H_2 = \{h_1, h_2\}$ and $I_2 = \{1, 2, 3\}$.
   (a) Assign $\mu(1) = h_2; \mu(2) = h_1; \mu(3) = h_2$.
   (b) Check that this is feasible for I_2 (it is).
   (c) Set $R_3 = R_2 \backslash H_2 = \emptyset$. So the algorithm terminates.

[17] Recall the definition of an allocation is a matching between agents and house types. The individual identities of the houses do not matter.

Therefore, a House Top Trading Segmentation of H is given by

$$\mathcal{H} = \{H_1 = \{h_3, h_4\}, H_2 = \{h_1, h_2\}\}.$$

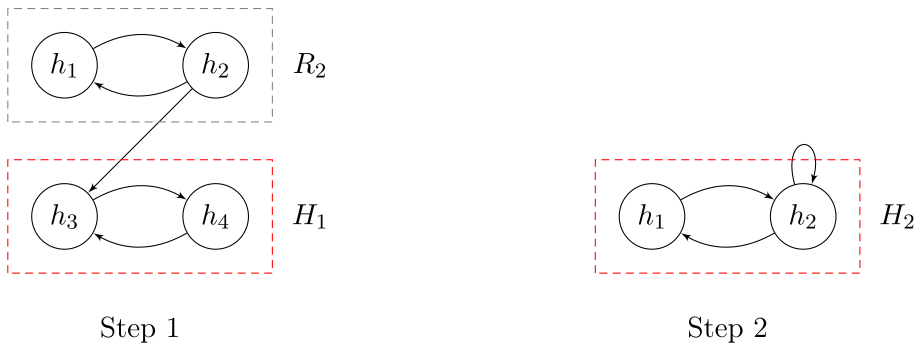


Figure 10: Applying Algorithm 1 to Example 5.

By Theorem 4, the unique strict core of this market is given by

$$\mu(1) = h_2$$
$$\mu(2) = h_1$$
$$\mu(3) = h_2$$
$$\mu(4) = h_4$$
$$\mu(5) = h_3$$

10.1 Proof of Theorem 4

The proofs for the strict core claims unsurprisingly follow Gale's proof for TTC. The first key insight is that by focusing on house types as nodes (instead of agents), we ensure that we remove all copies of a house at the same time. This lets us easily deal with objective indifferences. The second key insight is that when we assign houses within an SCC without outgoing arcs, we assign a set of houses and their owners at the same time.

Proof of Claim 2. Let $(I, H, E, \succsim)$ be a market, and let μ^{HTTS} be the allocation produced by Algorithm 1. That is, the algorithm terminated in part c of some step.

We first argue that μ^{HTTS} is indeed a feasible allocation. At each step, we arrive at a house trading segment H_d. Note that H_d has no outgoing arcs in D_d. Thus all agents endowed with a house $h \in H_d$ (denoted I_d) top-rank a house in H_d from among the remaining houses. By our assumption that Algorithm 1 terminated in part c (and not part b) of some step, we know that μ^{HTTS} is feasible for I_d. Part c of this step removes H_d and thus I_d from further consideration. Thus $\{H_1, ..., H_d, ..., H_K\}$ and $\{I_1, ..., I_d, ..., I_K\}$ partition the house types and agents, respectively. If μ is feasible for each I_d, then it is feasible for I.

Toward a contradiction, suppose there is a blocking coalition I' and sub-allocation μ'.

For at least one agent $i \in I$, $\mu'(i) \succ \mu^{HTTS}(i)$. Consider the step d at which i was assigned in Algorithm 1. By construction, $\mu^{HTTS}(i) = B_i(H_d) = B_i\left(\cup_{d' \geq d} H_{d'}\right)$. So it must be that $\mu'(i) \in \cup_{d'<d} H_{d'}$. Feasibility of μ' implies that there is some $i' \in I_k$ for $k < d$ such that $\mu'(i') \in \cup_{k'>k} H_{k'}$. But then $\mu'(i') \prec \mu^{HTTS}(i')$, so this is not a blocking coalition. In other words, for $\mu'(i) \succ \mu^{HTTS}(i)$, i must be assigned to a house from an earlier segment. But then an agent from an earlier segment must be assigned to a house from a later segment, which is strictly dispreferred. □

Proof of Claim 3. Let $(I, H, E, \succsim)$ be a market, and let μ^{HTTS} be the allocation produced by Algorithm 1. That is, the algorithm terminated in a part c. Let μ' be another strict core allocation. We again show $\mu' = \mu^{HTTS}$ by strong induction on the number of steps in HTTS.

Base claim. Consider H_1 and I_1. We have $\mu^{HTTS}(i) \succsim_i \mu'(i)$ for all $i \in I_1$, since every $i \in I_1$ receives his favorite house. Since μ^{HTTS} is feasible for I_1 and μ' is in the strict core, we must also have $\mu'(i) \succsim_i \mu^{HTTS}(i)$ for all $i \in I_1$. (Otherwise I_1 can form a blocking coalition with sub-allocation $\mu^{HTTS}|_{I_1}$.) But then $\mu^{HTTS}(i) = \mu'(i)$ for $i \in I_1$.

Claim d**.** Assume $\mu^{HTTS}(i) = \mu'(i)$ for all $i \in I_1 \cup \cdots \cup I_{d-1}$. Then $\mu'(I_d) \subseteq \cup_{d' \geq d} H_{d'}$. That is, the houses assigned to agents in I_d are drawn from the houses that remain after step $d-1$. By construction, we have $\mu^{HTTS}(i) \succsim_i h$ for any $h \in \cup_{d' \geq d} H_{d'}$ for all $i \in I_d$, so we have $\mu^{HTTS}(i) \succsim_i \mu'(i)$ for $i \in I_d$. Since μ^{HTTS} is feasible for I_k and μ' is in the strict core, we must have $\mu'(i) \succsim_i \mu^{HTTS}(i)$ for all $i \in I_d$. But then $\mu^{HTTS}(i) = \mu'(i)$ for $i \in I_d$.

□

Proof of Claim 1. We now have that μ^{HTTS} is the unique strict core allocation, when it exists. Thus, if μ^{HTTS} is not feasible, there is no strict core allocation. □

Proof of Claim 4. We apply Tarjan's algorithm (Tarjan, 1972). For any directed graph $G = D(V, E)$, the order in which Tarjan's algorithm returns the SCCs of G is a reverse topological ordering of the condensation $G^{SCC} = D(V^{SCC}, E^{SCC})$ of G.[18]. Concretely, suppose $\mathcal{S} = \{S_1, S_2, ..., S_\ell\}$ is the set of SCCs of G in the order in which they were returned by Tarjan's algorithm (i.e., S_1 is the first SCC returned, S_2 is the second, etc.). Then S_1 must be a sink of G^{SCC}. Therefore, S_1 is an SCC of G with no outgoing arcs.

At each step d of Algorithm 1, we perform two computations. First, we use Tarjan's algorithm to identify an SCC H_d with no outgoing arcs.[19] Tarjan's algorithm has time complexity $O(|H| + |I|)$. Second, we check whether the strict core allocation is feasible for $I_d = E^{-1}(H_d)$. That is, for each $h \in H_d$, we check $\left|E^{-1}(h) \cap I_d\right| = |\{i : B_i(H_d) = h, i \in I_d\}|$. This has time complexity $O(|H|)$. Therefore, each step of Algorithm 1 has time complexity $O(|H| + |I|)$.

Since Algorithm 1 terminates in at most $|H|$ steps, it has time complexity $O(|H|^2 + |H||I|)$. □

[18] See **?**, Section 2.3

[19] We need not find all SCCs. The first SCC returned by Tarjan's algorithm will suffice.

11 Conclusion

In this paper, we study the house-swapping problem in a setting where agents' preferences may contain "objective indifferences." We assume that agents have strict preferences over a set of house types and that multiple agents may be endowed with copies of the same house type. We derive a square-polynomial time algorithm that finds the unique strict core allocation of a house-swapping market, if it exists. This is faster than the methods that are needed to find strict core allocations in the setting where agents are allowed to have subjective indifferences. Moreover, our algorithm is interpretable as a series of "house top trading segments", which are analogous to top trading cycles. The condition for the non-emptiness of the strict core is readily interpretable – within each house top trading segment, supply and demand for each house type are equal.

Part III

Group Incentive Compatibility in a Market with Indivisible Goods: A Comment

Preface

The third chapter is also coauthored with Will Sandholtz. We note that the proof of Bird (1984), the first to show group strategy-proofness of top trading cycles (TTC), requires a correction. We provide a counter-example to a critical claim, then present a corrected proof in the spirit of the original.

12 Introduction

In the classic house swapping problem, we have a finite set of agents, each of whom owns an indivisible object. Agents have strict preferences over the objects. As shown in Shapley and Scarf (1974), the top trading cycles (TTC) algorithm finds an allocation in the strong core. Roth and Postlewaite (1977) show that this is also the unique competitive equilibrium allocation. The house swapping model is applied to situations like organ exchange, school assignment, and (indeed) housing assignments. TTC forms the basis for other algorithms in these settings when efficiency is most desired.

Roth (1982) shows that TTC is strategy-proof. Bird (1984) presents a proof that TTC is group strategy-proof. In this note, we show that Lemma 1, which is critical to the main result, requires modification. To our knowledge, we are the first to do so. While other authors have since provided more proofs for TTC's group strategy-proofness[20], we present a new proof in the spirit of the original in Bird (1984).

13 Notation

We retain the notation in Bird (1984) and recount it briefly here. Let $N = \{1, ..., n\}$ be the set of agents, and $w = (w_1, ..., w_n)$ be the endowment, where i is endowed with w_i. Each agent i has a strict preference profile P_i over the houses; let $P = (P_1, ..., P_n)$. An allocation is a vector $x = (x_1, ..., x_n)$ where each x_i can be mapped 1-1 into a corresponding w_j.

Denote $T(N, P)$ be the allocation resulting from TTC applied to (N, P). Let $S_k(P) \subseteq N$ be the agents in the k^{th} trading cycle produced by TTC under preference profile Pgma, and let $S_0 = \emptyset$. Also define $R_k(P) = \cup_{i=1}^{k} S_i(P)$. [21] For convenience, let K be the total number of cycles resulting from TTC applied to (N, P).

We seek to show that TTC is group strategy-proof. That is, no subset $S \subseteq N$ can misreport preferences and make all members of S strictly better off.

14 Lemma 1, counterexample, and corrected version

Bird (1984) states the following lemma, which is critical to the main result.

Lemma 2 (Bird (1984), Lemma 1)**.** *Let $x = T(N, P)$ and let $x' = T(N, P')$. If there is an $i \in S_k(P)$ such that $x'_i P_i x_i$, then there exists a $j \in R_{k-1}(P)$ and $h \in N - R_{k-1}(P)$ such that $w_h P'_j x_j$.*

He gives the following intuition (verbatim):

> [I]f any trader wants to get a more preferred good, he needs to get a trader in an earlier cycle to change his preference to a good that went in a later trading cycle. From this result, the group incentive compatibility follows easily.

The lemma as stated requires correction. We first give a counterexample.

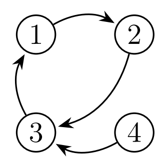
[20] See Moulin (1995) Lemma 3.3 and Pápai (2000)

[21] The order of cycles generated by TTC is not generally unique, since there can be two cycles formed at once. However, the results carry through under any order.

Example (Counterexample to Bird (1984), Lemma 1)**.** Let $N = \{1, 2, 3, 4\}$ with the following preferences; the figure shows the first step of TTC.

P_1	P_2	P_3	P_4
w_2	w_3	w_1	w_3
w_1	w_2	w_4	w_4
		w_3	

Figure 11: First step of $T(N, P)$

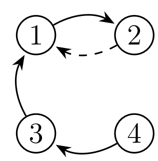


The TTC allocation is $x = T(N, P) = (w_2, w_3, w_1, w_4)$. The figure shows the first step of TTC, where the dotted line is 2's changed report.

Now consider a misreport P':

P_1	P'_2	P_3	P_4
w_2	w_1	w_1	w_3
w_1	w_3	w_4	w_4
	w_2	w_3	

Figure 12: First step of $T(N, P')$



Then $x' = T(N, P') = (w_2, w_1, w_4, w_3)$.

In the notation of Lemma 1, we have $i = 4$ and $k = 2$. That is, $i = 4 \in S_2(P)$ and $x'_4 P_4 x_4$. However, $\nexists j \in R_{k-1}(P) = S_1(P)$ such that $\exists h \in N - R_{k-1}(P) = S_2(P)$ such that $w_h P'_j x_j$. The only candidate for j is $j = 2 \in R_1(P) = S_1(P)$. But she does not rank any objects from $N - R_{k-1}(P) = S_2(P)$ above $x_2 = w_3$.

In the the proof of Lemma 1, the following erroneous claim is made.

Claim 5. Let $x_m P'_m w_n$ for all $m \in R_{k-1}(P)$ and $n \in N - R_{k-1}(P)$. That is, all members of the first $k-1$ cycles continue to rank these objects above objects in the later cycles (though perhaps in a different order). Then $R_{k-1}(P') = R_{k-1}(P)$. That is, the set of agents assigned in the first $k-1$ cycles under P' is the same as the set assigned in the first $k-1$ cycles under P.

This claim is false; the above counterexample also serves as a counterexample to this claim since $R_1(P) \neq R_1(P')$.

The counterexample shows that it is not necessary for an agent in an earlier cycle to change her preference to a cycle k or later. She may change her preference to an object in *her own* cycle or later. This is the necessary addition; we present a corrected version.

Lemma 3 (Lemma 1, Corrected)**.** *Let $x = T(N, P)$ and let $x' = T(N, P')$. If there is an $i \in S_k(P)$ such that $x'_i P_i x_i$, then there exists a $j \in S_{k'}(P)$ where $k' < k$ and $h \in \cup_{\ell=k'}^{K} S_\ell$ such that $w_h P'_j x_j$.*

That is: if any agent wants to get a more preferred good, he needs to get an agent in an earlier cycle to change her preference to a good that went in *her own cycle* or a later cycle.

Proof. Assume the contrary. So there exists $i \in S_k(P)$ such that $x'_i P_i x_i$, and for all $j \in S_{k'}(P), k' < k$ and for all $h \in \cup_{\ell=k'}^{K} S_\ell$ we have that $x_j P'_j w_h$ or $x_j = w_h$. That is, all agents in cycles before k prefer their original allocation over any objects in their own cycle or later. Alternatively, only objects in earlier cycles can be ranked above the original allocation. We show by induction on cycles that all agents in cycles $k' < k$ receive their original allocations.

Claim. $S_\ell(P) = S_{k'}(P')$ for all $k' < k$.

Base case. For $j \in S_1(P)$, x_j was top-ranked under P. It must be that x_j is still top-ranked under P'. By the definition of TTC, $S_1(P) = S_1(P')$.

k' case. Suppose the claim is true for $S_1(P)$ through $S_{k'-1}(P)$. Consider step k' of TTC under P'. For each $j \in S_{k'}(P)$, of remaining objects, x_j must be top-ranked under P'_j. This is because the premise requires that any better-ranked objects according to P'_j were in earlier cycles. Then $S_{k'}(P) = S_{k'}(P')$.

Thus $x_j = x'_j$ for all $j \in S_{k'}(P), k' < k$. However, to have $x'_i P_i x_i$, x'_i must have been an object in a cycle $k' < k$, leading to a contradiction. □

15 Corrected proof

We now update the proof of Bird's main theorem using the corrected lemma. The argument proceeds in the same manner as the original.

Theorem (Bird, 1984)**.** *Top trading cycles is group strategy-proof.*

Proof. Suppose there is a subset $Q \subseteq N$ reporting P'. Let i be the first agent in Q to enter a trading cycle under P. We will show that i cannot improve.

Let $i \in S_k(P)$ and $x'_i P_i x_i$. Note that we must have $k \geq 2$, since no one in $S_1(P)$ can improve. By Lemma 3, there exists a $j \in S_{k'}(P)$ where $k' < k$ and $h \in \cup_{\ell=k'}^{K} S_\ell$ such that $w_h P'_j x_j$. Then $P'_j \neq P_j$, so $j \in Q$. However, j entered a cycle before i under P, so i could not have been the first agent in Q to under a trading cycle under P. □

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