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OSCILLATION CENTERS AND MODE COUPLING IN NONUNIFORM VLASOV PLASMA

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ABSTRACT

The general coupling coefficient for three electromagnetic linear modes of an inhomogeneous and relativistic plasma is derived from the oscillation-center viewpoint. A concise and manifestly symmetric formula is obtained; it is cast in terms of Poisson brackets of the single-particle perturbation Hamiltonian and its convective time-integral along unperturbed orbits. The simplicity of the compact expression obtained is shown to lead to a new insight into the essence of three-wave coupling and of the Manley-Rowe relations governing such interactions. Thus, the interaction Hamiltonian of the three waves is identified as simply the trilinear contribution to the single-particle (new) Hamiltonian, summed over all non-resonant particles. The relation between this work and the Lie-transform approach to Hamiltonian perturbation theory is discussed.

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1. Introduction

In a recent paper (Johnston, Kaufman and Johnston 1978), we introduced a rather novel formulation of the theory of nonlinear mode coupling in magnetized Vlasov plasma. Our approach was based upon a canonical transformation to "oscillation-center" coordinates, and might be termed the method of "generalized ponderomotive forces." The oscillation-center representation permits one to think in terms of entities which experience purely nonlinear (beat) forces, and leads to a useful and natural decomposition of the nonlinear currents central to problems of coherent mode coupling. The approach was used to extend the conventional ponderomotive-scalar-potential method (Drake et al. 1974) to the domain of strongly magnetized plasma.

Although our above-mentioned paper treated the case of a hot magnetized plasma, it restricted consideration to the infinite uniform model and to non-relativistic particle velocities. The purpose of this paper is to extend the method of generalized ponderomotive forces to nonuniform and relativistic plasma. Our aim will be to derive a very general expression for the three-wave coupling coefficient, an expression which is compact and which clearly manifests symmetry in the three modes. Such an expression would include all uniform-plasma results as limiting cases. We have succeeded in obtaining this master formula; our final expression for the coupling coefficient is remarkably concise and is cast in terms of Poisson brackets of the single-particle perturbation Hamiltonian and its convective time-integral along unperturbed orbits. A schematic outline of some of the present work was reported in a previous publication (Johnston and Kaufman 1977). The simplicity of the concise form derived here will be shown to lead to a new insight (Johnston and Kaufman 1976) into the essence of the wave coupling and of the Manley-Rowe relations governing such interactions.

Striving for generality, then, we consider a nonuniform and (possibly) relativistic Vlasov plasma which is confined in space by inhomogeneous electric and magnetic fields. The linear normal modes of the configuration are treated as fully electromagnetic, but under the assumption that their eigenfrequencies are nearly real. Confinement of the plasma is interpreted to stem from the invariants associated with single-particle orbits; accordingly, the existence of action-angle variables ($\underline{I}$, $\underline{\theta}$) associated with the unperturbed particle Hamiltonian $H_o(\underline{I})$ is assumed. It follows that all the plasma particles can be separated into two categories: the vast majority which comprise the non-resonant particles, and the small subset of "resonant" particles which satisfy (Kaufman 1971, 1972)

$$\omega_a \approx \underline{\ell}_a \cdot \partial H_o/\partial \underline{I} \equiv \underline{\ell}_a \cdot d\underline{\theta}/dt \quad . \tag{1}$$

In (1), ω_a denotes the real part of the eigenfrequency for normal mode a, and the vector $\underline{\ell}_a$ represents a set of three integers.

Since three-wave coupling in plasma arises from the motion of non-resonant particles, the behaviour of the resonant particles is suppressed in this work. The resonant particles are filtered from the problem by means of a smooth decomposition of the unperturbed distribution function,

$$f_o(\underline{I}) = \tilde{f}_o(\underline{I}) + f_o^{res}(\underline{I}), \tag{2}$$

where $\tilde{f}_o(\underline{I})$ represents the non-resonant distribution. The mathematical foundation for this smooth separation is discussed in the Appendix. The proper treatment of resonant particles in the context of three-wave interaction is presently under study.

The paper is organized as follows. In §2, the formalism of the oscillation-center transformation is reviewed. In §3, the action-transfer and frequency-shift equations for each interacting wave are derived. Then in §4, the three-wave coupling coefficient is evaluated using the method of generalized ponderomotive forces. Our final formula is concise, quite general and manifestly symmetric in the three waves. In §5, our calculations are related to some earlier work (Al'tshul and Karpman 1965, Laval and Pellat 1975). In §6, an alternate derivation of the coupling coefficient leads to a new insight into the nature of three-wave coupling. It is shown there that the interaction Hamiltonian of the three waves is simply the trilinear contribution to the single-particle (new) Hamiltonian K, summed over all the (non-resonant) particles. Finally, in §7, our results and methods are discussed further with reference to the Lie-transform approach to Hamiltonian perturbation theory (Kaufman 1978, Johnston and Kaufman 1978).

2. Oscillation-center Transformation:

Consider a perturbed Hamiltonian system

$$H(\underline{q}, \underline{p}, t) = H_o(\underline{q}, \underline{p}, t) + \delta H(\underline{q}, \underline{p}, t) ,$$

where δH denotes a small perturbation of order ε. A near-identity canonical transformation

$$(\underline{q}, \underline{p}, H) \longrightarrow (\underline{Q}, \underline{P}, K), \quad K = H_o + \delta K,$$

can be characterized by a perturbative generating function $S(\underline{q}, \underline{P}, t)$ which satisfies the equation (Johnston, Kaufman and Johnston 1978)

$$(\partial/\partial t) S(\underline{q}, \underline{p}, t) + H(\underline{q}, \underline{p} + \partial S/\partial \underline{q}, t) = K(\underline{q} + \partial S/\partial \underline{p}, \underline{p}, t). \tag{3}$$

Our approach to solving (3) will be to expand S, H and K in powers of the perturbation parameter ε, and then to satisfy the equation in each order. Let us first define the operators

$$\begin{aligned} \underline{\nabla} Q &\equiv \partial Q/\partial \underline{q}, \quad \underline{\partial} Q \equiv \partial Q/\partial \underline{p}, \\ D_t Q &\equiv \partial Q/\partial t + \{Q, H_o\}, \end{aligned} \tag{4}$$

and the Poisson-bracket operation

$$\{A, B\} \equiv \underline{\nabla} A \cdot \underline{\partial} B - \underline{\partial} A \cdot \underline{\nabla} B ,$$

where Q, A and B are arbitrary functions of $(\underline{q}, \underline{p}, t)$. The Hamilton-Jacobi equation (3) can then be written, after expansion, in the form

$$\begin{aligned} \delta K - \delta H - D_t S &= \underline{\nabla} S \cdot \underline{\partial}\, \delta H - \underline{\partial} S \cdot \underline{\nabla}\, \delta K \\ &+ \frac{1}{2} \underline{\nabla} S\, \underline{\nabla} S : \underline{\partial}\, \underline{\partial} H - \frac{1}{2} \underline{\partial} S\, \underline{\partial} S : \underline{\nabla}\, \underline{\nabla} K \\ &+ \frac{1}{6} \underline{\nabla} S\, \underline{\nabla} S\, \underline{\nabla} S \vdots \underline{\partial}\, \underline{\partial}\, \underline{\partial} H_o \\ &- \frac{1}{6} \underline{\partial} S\, \underline{\partial} S\, \underline{\partial} S \vdots \underline{\nabla}\, \underline{\nabla}\, \underline{\nabla} H_o + O(\varepsilon^4) . \end{aligned} \tag{5}$$

The "oscillation-center" transformation corresponds to a certain prescription for satisfying (5). The requirement is imposed that $K^{(1)}$ must vanish (superscripts index the order in ε), and so the new entity (the "oscillation center") sees only a second order perturbation $K^{(2)}$. The requirement that $K^{(1)} = 0$ does not determine $K^{(2)}$ uniquely, i.e., there is flexibility in the choice of $S^{(2)}$. The following transformation is the simplest which (correct to order ε^3) satisfies (5) with $K^{(1)} = 0$:

$$D_t S^{(1)} = -H^{(1)} \quad , \qquad S^{(2)} = \frac{1}{2} \underline{\nabla} S^{(1)} \cdot \underline{\partial} S^{(1)} \quad , \tag{6}$$

$$K^{(2)} = H^{(2)} + \frac{1}{2} \{S^{(1)}, H^{(1)}\} \; . \tag{7}$$

Notice that this choice for $S^{(2)}$ differs from the convection adopted in our earlier paper, namely $S^{(2)} = 0$. Note also that D_t, the conventive time derivative following the unperturbed phase-space orbit, must be inverted in order to obtain $S^{(1)}$. This procedure breaks down for "resonant" perturbations $H^{(1)}$, a difficulty which motivates our extraction of resonant particles from the problem (see Appendix).

The formulae presented so far in this section are valid for any perturbed Hamiltonian system with conjugate variables ($\underline{q}$, $\underline{p}$). The particular problem at hand is that of a plasma particle viewing three frequency-matched waves; thus, H_o corresponds to the equilibrium fields and δH to the perturbing wave fields. We employ conjugate variables ($\underline{r}$, $\underline{p}$), where $\underline{r}$ denotes the Cartesian position vector in physical space. The corresponding oscillation-center coordinates are then

$$\begin{aligned} \underline{R}(\underline{r}, \underline{p}, t) &= \underline{r} + \underline{\partial} S - \underline{\nabla} S \cdot \underline{\partial}\, \underline{\partial} S + 0\,(\varepsilon^3) \quad , \\ \underline{P}(\underline{r}, \underline{p}, t) &= \underline{p} - \underline{\nabla} S + \underline{\nabla} S \cdot \underline{\partial}\, \underline{\nabla} S + 0\,(\varepsilon^3). \end{aligned} \tag{8}$$

The phase-space distribution function for oscillation centers, $F(\underline{R}, \underline{P}, t)$, satisfies the Vlasov equation

$$\partial F/\partial t + \{F, K\} = 0, \tag{9}$$

and is related to the original distribution function $f(\underline{r}, \underline{p}, t)$ by the condition

$$F(\underline{R}, \underline{P}, t) = f(\underline{r}, \underline{p}, t) . \tag{10}$$

As in our earlier work, we find it convenient to inroduce the "polarization density" $\Delta(\underline{r}, \underline{p}, t)$ defined by

$$\Delta(\underline{r}, \underline{p}, t) \equiv f(\underline{r}, \underline{p}, t) - F(\underline{r}, \underline{p}, t) ;$$

substitution of the expansions (8) into relation (10) then leads to the simple result

$$\Delta = - \{S , F\} + \frac{1}{2} \Big\{S , \{S , F\}\Big\} + \frac{1}{2} \{\underline{\nabla} S \cdot \underline{\partial} S , F\} + O(\varepsilon^3) .$$

Accordingly, for the oscillation-center transformation (6) - (7), we have

$$\Delta^{(1)} = - \{ S^{(1)}, \tilde{f}_o\} , \tag{11}$$

$$\Delta^{(2)} = \frac{1}{2} \Big\{S^{(1)}, \{ S^{(1)}, \tilde{f}_o\}\Big\} , \tag{12}$$

where $\tilde{f}_o$ signifies the non-resonant component of the unperturbed distribution function f_o. The canonical-transformation tools needed for this paper are now at our disposal .

3. Action-Transfer and Frequency-Shift Relations;

Having reviewed the apparatus of the oscillation-center transformation, we devote the remainder of the paper to a consideration of the three-wave process in nonuniform plasma. The purpose of this section is to present the equations which goven the slow evolution of the amplitudes and phases of the interacting normal modes.

First, we must consider the normal modes themselves. The linearized Maxwell curl equations can be written

$$c\underline{\nabla}_x \times \underline{\delta E}(\underline{x}, \omega) - i\omega\, \underline{\delta B}(\underline{x}, \omega) = 0 \quad , \tag{13}$$

$$\begin{aligned} c\underline{\nabla}_x \times \underline{\delta B}(\underline{x}, \omega) + i\omega\, \underline{\delta E}(\underline{x}, \omega) \\ = 4\pi \int d^3x'\, \underset{\sim}{\sigma}(\underline{x},\underline{x}' ; \omega) \cdot \underline{\delta E}(\underline{x}', \omega) + 4\pi\, \delta \underline{j}_s(\underline{x},\omega), \end{aligned} \tag{14}$$

where $\underset{\sim}{\sigma}(\underline{x}, \underline{x}'; \omega)$ is the linear conductivity kernel of the plasma configuration (Kaufman 1971, 1972), and $\delta\underline{j}_s(\underline{x}, \omega)$ represents any small current source at frequency ω. Combining (13) and (14), one obtains

$$\underset{\sim}{D}(\omega) \cdot \underline{\delta E}(\underline{x}, \omega) = (4\pi/i\omega)\, \delta\underline{j}_s(\underline{x}, \omega) \quad , \tag{15}$$

where $\underset{\sim}{D}(\omega)$ denotes the integro-differential operator

$$\begin{aligned} \underset{\sim}{D}(\omega) \cdot \underline{F}(\underline{x}) \equiv\ & \underline{F}(\underline{x}) - (c^2/\omega^2)\underline{\nabla}_x \times [\underline{\nabla}_x \times \underline{F}(\underline{x})] \\ & - (4\pi/i\omega) \int d^3x'\, \underset{\sim}{\sigma}(\underline{x}, \underline{x}'; \omega) \cdot \underline{F}(\underline{x}') \quad . \end{aligned}$$

If it is assumed that $\underset{\sim}{D}(\omega)$ is nearly Hermitian (i.e., that damping of the normal modes by resonant particles is slight) then it follows (Kaufman 1971) that the real ports ω_a of the eigenfrequencies and the zero-order eigenfunctions $\underline{E}_a(\underline{x})$ are the solutions of

$$\underset{\sim}{D}_H(\omega_a) \cdot \underline{E}_a(\underline{x}) = 0 \ , \tag{16}$$

where $\underset{\sim}{D}_H(\omega)$ denotes the Hermitian part of $\underset{\sim}{D}(\omega)$ (for ω real). We assume (without loss of generality) that $\omega_a > 0$, and choose to normalize the eigenfunctions $\underline{E}_a(\underline{x})$ to unit magnitude of wave energy, i.e.,

$$\frac{\omega_a}{4\pi} \int d^3x \, \underline{E}_a^*(\underline{x}) \cdot \left. \frac{\partial \underset{\sim}{D}_H(\omega)}{\partial \omega} \right|_{\omega_a} \cdot \underline{E}_a(\underline{x}) = \sigma_a \equiv \pm 1, \tag{17}$$

where σ_a is the energy sign for normal mode a.

Consider now a coupling of three of the plasma eigenmodes (16) which satisfy a frequency matching condition of the form

$$\omega_b + \omega_c = \omega_a + \Delta\omega \ , \tag{18}$$

where $\Delta\omega << \omega_a$ denotes any small mismatch. The corresponding perturbed electric field can be written

$$\underline{\delta E}(\underline{x}, t) = \sum_{a=1}^{3} \alpha_a(t)\underline{E}_a(\underline{x}) \exp(-i\omega_a t) + c.c. \ , \tag{19}$$

where the complex quantity $\alpha_a(t)$ represents the slowly varying amplitude and phase of mode a. According to (15), we must consider three coupled equations of the form

$$\begin{aligned} &\exp(-i\omega_a t) \ \underset{\sim}{D}_H(\omega_a + i\partial_t) \cdot [\alpha_a(t)\underline{E}_a(\underline{x})] \\ &\quad = (4\pi/i\omega_a) \ \tilde{\underline{j}}_a^{(2)}(\underline{x}, t) \ , \end{aligned} \tag{20}$$

where $\tilde{\underline{j}}_a^{(2)}(\underline{x}, t)$ denotes the nonlinear current source near frequency ω_a due to the beating of the other two modes. The tilde on $\tilde{\underline{j}}_a^{(2)}$ is just a reminder that only the contribution of non-resonant particles to the nonlinear current is to be included; thus, all resonant-particle terms have been omitted from (20).

Since each ω_a is a linear eigenfrequency, (16) implies that

$$\underset{\sim}{D}_H(\omega_a + i\partial_t) \cdot [\alpha_a(t)\underline{E}_a(\underline{x})] \simeq i \left[\frac{\partial \underset{\sim}{D}_H(\omega)}{\partial \omega} \bigg|_{\omega_a} \cdot \underline{E}_a(\underline{x}) \right] \frac{d\alpha_a(t)}{dt} . \tag{21}$$

Then insertion of (21) into (20), an inner product of the resultant equation with $\alpha_a^*(t)\underline{E}_a^*(\underline{x}) \exp(i\omega_a t)$, integration over the plasma volume, and use of the normalization condition (17) lead to the result

$$\sigma_a \alpha_a^*(t)\, d\alpha_a(t)/dt = - \int d^3x\, \tilde{\underline{j}}_a^{(2)}(\underline{x}, t) \cdot \underline{E}_a^*(\underline{x})\alpha_a^*(t) \exp(i\omega_a t). \tag{22}$$

Now, by writing the complex number $\alpha_a(t)$ in polar form,

$$\alpha_a(t) = |\alpha_a(t)| \exp[-i\, \delta\theta_a(t)] ,$$

it is simple to show that

$$\alpha_a^* \frac{d\alpha_a}{dt} = \frac{1}{2} \frac{d\, |\alpha_a|^2}{dt} - i\, |\alpha_a|^2\, \delta\omega_a , \tag{23}$$

where $\delta\omega_a \equiv d(\delta\theta_a)/dt$ represents the real frequency shift of mode a produced by the interaction (Sturrock 1960). Also, by virtue of the normalization condition (17), the total energy of wave a is

$$W_a(t) = \sigma_a |\alpha_a(t)|^2 . \tag{24}$$

Using (23) and (24) in (22), and then taking real and imaginary parts, we obtain, respectively, an action-transfer equation,

$$\omega_a^{-1}\, d\, W_a/dt = 2\, \mathrm{Im}\, V_a , \tag{25}$$

and an equation for the frequency shift,

$$\delta\omega_a/\omega_a = W_a^{-1}\, \mathrm{Re}\, V_a , \tag{26}$$

$$V_a \equiv (i\omega_a)^{-1} \int d^3x \, \tilde{\underline{j}}_a^{(2)}(\underline{x}, t) \cdot \underline{E}_a^*(\underline{x}) \, \alpha_a^*(t) \, \exp(i\omega_a t). \quad (27)$$

Equations (25) and (26) are the desired equations governing the slow evolution of the interacting normal modes. Before proceeding, however, let us rewrite the gauge-invariant formula (27) for V_a by expressing the electric field in terms of potentials,

$$\underline{E}_a(\underline{x}) = - \underline{\nabla}_x \, \phi_a(\underline{x}) + (i\omega_a/c)\underline{A}_a(\underline{x}) \; . \quad (28)$$

Partial integration of the term involving $\phi_a(\underline{x})$ will yield a new term involving the divergence of $\tilde{\underline{j}}_a^{(2)}(\underline{x}, t)$. Now it is simple to use the Vlasov equation for species s,

$$\partial f_s/\partial t + \{f_s, H_s\} = 0,$$

to verify that the charge and current densities in the plasma,

$$\rho(\underline{x}, t) = \sum_s e_s \int d^3r \int d^3p \; \delta(\underline{x} - \underline{r}) \; f_s(\underline{r}, \underline{p}, t), \quad (29)$$

$$\underline{j}(\underline{x}, t) = \sum_s e_s \int d^3r \int d^3p \; \delta(\underline{x} - \underline{r}) \; f_s(\underline{r}, \underline{p}, t) \; \underline{\partial} H_s(\underline{r}, \underline{p}, t), \quad (30)$$

satisfy the continuity equation

$$\partial_t \, \rho(\underline{x}, t) + \underline{\nabla}_x \cdot \underline{j}(\underline{x}, t) = 0 \; .$$

It follows that we can substitute the relation

$$\underline{\nabla}_x \cdot \tilde{\underline{j}}_a^{(2)}(\underline{x}, t) = i\omega_a \, \tilde{\rho}_a^{(2)}(\underline{x}, t)$$

into (27) to obtain the equivalent formula

$$V_a = \int d^3x \left[\tilde{\rho}_a^{(2)}(\underline{x}, t)\phi_a^*(\underline{x}) - c^{-1} \, \tilde{j}_a^{(2)}(\underline{x}, t)\cdot\underline{A}_a^*(\underline{x}) \right] \alpha_a^*(t) \, \exp(i\omega_a t). \quad (31)$$

Although it is no longer manifest, (31) retains the gauge invariance of (27).

Finally, in order to deal with true coupling coefficients, we normalize the quantities V_a by dividing out the amplitude factors $\alpha_a(t)$. More precisely, we define the normalized coefficients

$$U_a \equiv V_a / [\alpha_a^* \alpha_b \alpha_c \exp(i\Delta\omega t)] ,$$

$$U_{b,c} \equiv V_{b,c} / [\alpha_a \alpha_b^* \alpha_c^* \exp(-i\Delta\omega t)] ,$$

where $\Delta\omega$ is the frequency mismatch defined by (18). The next section is devoted to an explicit evaluation of these coupling coefficients U.

4. General Three-wave Coupling Coefficient

The goal of this section is an evaluation of the three-wave coupling coefficient (31) and a verification of its symmetry properties. Our method of evaluation involves an application of the oscillation-center formulae of §2 to the Hamiltonian of a single plasma particle. The unperturbed Hamiltonian of a relativistic plasma particle can be written

$$H_o(\underline{r}, \underline{p}) = e\phi_o(\underline{r}) + \{[\underline{p}c - e\underline{A}_o(\underline{r})]^2 + m^2c^4\}^{1/2} \quad ,$$

where $\phi_o(\underline{x})$ and $\underline{A}_o(\underline{x})$ denote the equilibrium scalar and vector potentials in the plasma. The perturbation (19) under study consists of three linear modes of the plasma configuration, coupled together by the frequency-matching condition (18). The perturbed Hamiltonian $\delta H(\underline{r}, \underline{p}, t)$ then has the form

$$\delta H = e\delta\phi + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{e}{c}\right)^n (\partial_{\gamma 1} \cdots \partial_{\gamma n} H_o)(\delta A_{\gamma 1} \cdots \delta A_{\gamma n}) \quad ,$$

where $\delta\phi(\underline{x}, t)$ and $\underline{\delta A}(\underline{x}, t)$ are the perturbed potentials. Accordingly, representative frequency components of δH are

$$H_a^{(1)}(\underline{r}, \underline{p}) = (e/c)[\phi_a(\underline{r})c - \underline{A}_a(\underline{r}) \cdot \underline{\partial} H_o(\underline{r}, \underline{p})] ,$$

$$H_b^{(2)}(\underline{r}, \underline{p}) = (e^2/c^2)\, \underline{A}_a(\underline{r})\, \underline{A}_c^*(\underline{r}) : \underline{\partial}\, \underline{\partial}\, H_o(\underline{r}, \underline{p}) \quad ,$$

$$H_o^{(3)}(\underline{r}, \underline{p}) = -(e^3/c^3)\, \underline{A}_a^*(\underline{r})\, \underline{A}_b(\underline{r})\, \underline{A}_c(\underline{r}) \,\vdots\, \underline{\partial}\, \underline{\partial}\, \underline{\partial}\, H_o(\underline{r}, \underline{p}) \quad ,$$

where the potentials $\phi_a(\underline{x})$ and $\underline{A}_a(\underline{x})$ are defined by (28). The superscripts index the order in the perturbation, and the subscripts identify the time dependence.

According to (31), the evaluation of the coupling coefficient U_a requires determination of the nonlinear charge and current densities produced by the beating of modes b and c. From (29) and (30), we can write

$$\tilde{\rho}_a^{(2)}(\underline{x}) = e \int d^3r \int d^3p \;\; \delta(\underline{x}-\underline{r})(F_a^{(2)} + \Delta_a^{(2)}) \;\; , \tag{32}$$

$$\begin{aligned} \tilde{\underline{j}}_a^{(2)}(\underline{x}) = e \int d^3r \int d^3p \; \delta(\underline{x}-\underline{r}) \Big[& \tilde{f}_o \, \underline{\partial} H_a^{(2)} + \Delta_b^{(1)} \, \underline{\partial} H_c^{(1)} \\ & + \Delta_c^{(1)} \, \underline{\partial} H_b^{(1)} + (F_a^{(2)} + \Delta_a^{(2)}) \, \underline{\partial} H_o \Big] \;\; , \end{aligned} \tag{33}$$

where we have invoked the decomposition $\tilde{f} = F + \Delta$, and suppressed the sum over species s. Insertion of (32) and (33) into (31), and use of the relation

$$-(e/c) \, \underline{\delta A} \cdot \underline{\partial} H = \partial H / \partial \varepsilon - e \, \delta\phi \qquad ,$$

then lead to the formula

$$U_a = \int d\Gamma \left[\tilde{f}_o H_o^{(3)} + \Delta_b^{(1)} H_b^{(2)*} + \Delta_c^{(1)} H_c^{(2)*} + (F_a^{(2)} + \Delta_a^{(2)}) H_a^{(1)*} \right] \;\; , \tag{34}$$

where $d\Gamma \equiv d^3r \, d^3p$. It remains now to eliminate the perturbed phase-space densities $\Delta^{(1)}$, $\Delta^{(2)}$ and $F^{(2)}$ in favor of the generating functions $S^{(1)}$ for the oscillation-center transformation.

The polarization terms in (34) involving $\Delta^{(1)}$ and $\Delta^{(2)}$ are easily dealt with. From (11) and (12), the required polarization densities are given by the Poisson brackets

$$\Delta_b^{(1)} = - \{ S_b^{(1)}, \tilde{f}_o \} \quad ,$$

$$\Delta_a^{(2)} = \frac{1}{2} \left\{ S_b^{(1)}, \{ S_c^{(1)}, \tilde{f}_o \} \right\} + (b \leftrightarrow c),$$

where, from (6), $S_b^{(1)} = - D_t^{-1} H_b^{(1)}$. Partial integration of the corresponding terms in (34) then yields

$$\int d\Gamma \; \Delta_b^{(1)} H_b^{(2)*} = \int d\Gamma \; \tilde{f}_o \; \{ S_b^{(1)}, H_b^{(2)*} \} \;\; , \tag{35}$$

$$\int d\Gamma \; \Delta_a^{(2)} H_a^{(1)*} = \int d\Gamma \; \tilde{f}_o \left[\frac{1}{2} \left\{ S_b^{(1)}, \{ S_c^{(1)}, H_a^{(1)*} \} \right\} + (b \leftrightarrow c) \right] \; , \tag{36}$$

where we have deliberately sought to remove all derivatives from $\tilde{f}_o$.

To obtain the perturbed oscillation-center distribution $F_a^{(2)}$, it will be necessary to invert the convective time derivative D_t appearing in the Vlasov equation

$$D_t F_a^{(2)} = \left\{ K_a^{(2)}, \tilde{f}_o \right\} ; \tag{37}$$

$K_a^{(2)}$ is given by formula (7). Inversion of D_t can be achieved by exploiting the existence of action-angle variables ($\underline{I}$, $\underline{\theta}$) for the unperturbed particle trajectories. Since the variables $\underline{\theta}$ are cyclic, any function of them δQ can be expanded in a Fourier series of the form

$$\delta Q(\Gamma) = \sum_{\underline{\ell}} \delta Q_{\underline{\ell}}(\underline{I}) \exp(i\underline{\ell}\cdot\underline{\theta}),$$

with the inversion

$$\delta Q_{\underline{\ell}}(\underline{I}) = (2\pi)^{-3} \oint d^3\theta \;\; \delta Q(\underline{I}, \underline{\theta}) \exp(-i\underline{\ell}\cdot\underline{\theta}) \;.$$

It follows that we can write

$$D_t^{-1}[\delta Q(\Gamma) \exp(-i\omega t)]$$
$$= \sum_{\underline{\ell}} i \;\delta Q_{\underline{\ell}}(\underline{I}) \;(\omega - \underline{\ell}\cdot\underline{\omega})^{-1} \exp(i\underline{\ell}\cdot\underline{\theta}-i\omega t) \;, \tag{38}$$

where we have defined the bounce-frequency vector

$$\underline{\omega}(\underline{I}) \equiv \partial H_o(\underline{I})/\partial \underline{I} \;.$$

Now, from (6) and (37), we have

$$\begin{aligned} \int d\Gamma \; F_a^{(2)} H_a^{(1)*} &= \int d\Gamma (-D_t S_a^{(1)*}) D_t^{-1} \{K_a^{(2)}, \tilde{f}_o\} \\ &= \int d\Gamma \; S_a^{(1)*} \; \{K_a^{(2)}, \tilde{f}_o\} \;, \end{aligned} \tag{39}$$

where the last step is easily justified using (38). Insertion of formula (7) for $K_a^{(2)}$ and partial integration to extract $\tilde{f}_o$ then yield

$$\int d\Gamma\, F_a^{(2)} H_a^{(1)*} = \int d\Gamma\, \tilde{f}_o(\Gamma) \left[\left\{ S_a^{(1)*},\, H_a^{(2)} \right\} + \frac{1}{2} \left\{ S_a^{(1)*},\, \{S_b^{(1)},\, H_c^{(1)}\} \right\} + \frac{1}{2} \left\{ S_a^{(1)*},\, \{S_c^{(1)},\, H_b^{(1)}\} \right\} \right] . \tag{40}$$

Inspection of our results (36) and (40) shows that there are <u>four</u> triple-Poisson-bracket contributions of the type $\left\{ S^{(1)},\, \{S^{(1)},\, H^{(1)}\} \right\}$ to formula (34) for U_a. It is simple to symmetrize these terms in the three waves, i.e., to show that

$$(1/2)\ (4 \text{ terms}) = (1/3)\ (6 \text{ terms}) . \tag{41}$$

Relation (41) is easily proved by appealing to equations (4) and (6), the frequency-matching condition (18), the fact that $\{H_o,\, \tilde{f}_o\} = 0$, and the Jacobi identity

$$\left\{A,\, \{B,\, C\}\right\} + \left\{C,\, \{A,\, B\}\right\} + \left\{B,\, \{C,\, A\}\right\} = 0 .$$

Substitution of our results (35), (36) and (40) into (34) thus leads to the following concise and general formula for the mode-coupling coefficient:

$$\begin{aligned} U_a = \int d\Gamma\, \tilde{f}_o(\Gamma) \Bigg[& H_o^{(3)} + \left\{ S_a^{(1)*},\, H_a^{(2)} \right\} + \left\{ S_b^{(1)},\, H_b^{(2)*} \right\} + \left\{ S_c^{(1)},\, H_c^{(2)*} \right\} \\ & + \frac{1}{3} \Bigg(\left\{ S_a^{(1)*},\, \{S_b^{(1)},\, H_c^{(1)}\} \right\} + \left\{ S_b^{(1)},\, \{S_a^{(1)*},\, H_c^{(1)}\} \right\} \\ & + \left\{ S_b^{(1)},\, \{S_c^{(1)},\, H_a^{(1)*}\} \right\} + \left\{ S_c^{(1)},\, \{S_b^{(1)},\, H_a^{(1)*}\} \right\} \\ & + \left\{ S_c^{(1)},\, \{S_a^{(1)*},\, H_b^{(1)}\} \right\} + \left\{ S_a^{(1)*},\, \{S_c^{(1)},\, H_b^{(1)}\} \right\} \Bigg) \Bigg] . \end{aligned} \tag{42}$$

Note that formula (42) exhibts manifest symmetry in the three modes (a^*, b, c). Note also its pleasing Poisson-bracket structure.

It is straightforward to repeat the preceding calculation for modes b and c, and to show explicitly that $U_b = U_c = U_a^* \equiv U$. The action-transfer equation (25) therefore implies relations of the Manley-Rowe type (Sturrock 1960):

$$-\frac{1}{\omega_a}\frac{dW_a}{dt} = \frac{1}{\omega_b}\frac{dW_b}{dt} = \frac{1}{\omega_c}\frac{dW_c}{dt}$$

$$= 2\ \mathrm{Im}\left[U\ \alpha_a \alpha_b{}^* \alpha_c{}^*\ \exp(-i\ \Delta\omega t)\right] . \tag{43}$$

Our derivation of the relations (43) has been purely classical, and so is independent of any heuristic quantum picture of the three-wave interaction. The foundations of the trilinear symmetry of U will be explored further in §6.

5. Relation to Some Earlier Work

The purpose of this section is to establish a relation between our oscillation-center approach and some earlier treatments based upon a direct perturbation expansion of the Vlasov-Maxwell equations (Al'tshul' and Karpman 1965, Laval and Pellat 1975). Similar Poisson-bracket expressions for the coupling coefficients were obtained by these authors, although there was no recourse to canonical transformations. In effect, our use of the oscillation-center representation has simply reorganized the perturbation expansions; our thesis is that this reorganization is helpful. Our formulation of the perturbation theory is compact and systematic, and analyzes the nonlinear currents in terms of the intuitive notion of oscillation centers responding to beat forces. The convective time-integral of the perturbed Hamiltonian which appears in the final formula for the coupling coefficient has an explicit and natural role in our formulation as the generator for the oscillation-center transformation.

In order to relate notations, let us suppose that the perturbing waves were turned on adiabatically at $t = -\infty$, and define D_t^{-1} causally so that there is no initial phase information. Thus, we write

$$D_t^{-1}\ [\delta Q(\Gamma,\ t)] = \int_{-\infty}^{t} d\tau\ \delta Q(\tau)\ ,$$

where δQ is an arbitrary perturbed quantity, and $\delta Q(\tau)$ means $\delta Q[\Gamma_o(\tau),\ \tau]$, where $\Gamma_o(\tau)$ is the unperturbed trajectory in phase space which satisfies $\Gamma_o(t) = \Gamma$. From (6) and (37), it follows that

$$S_b^{(1)}(\Gamma,\ t) = -\int_{-\infty}^{t} d\tau\ H_b^{(1)}(\tau),$$

$$F_a^{(2)}(\Gamma,\ t) = \int_{-\infty}^{t} d\tau\ \left\{K_a^{(2)}(\tau),\ \tilde{f}_o\right\}\ ,$$

where $K_a^{(2)}(\Gamma, t)$ is given by (7). Accordingly, the contributions (36) and (40) to U_a can be rewritten as

$$\int d\Gamma \Delta_a^{(2)} H_a^{(1)*} = \int d\Gamma \tilde{f}_o(\Gamma) \left[\frac{1}{2} \int_{-\infty}^{t} d\tau \int_{-\infty}^{t} d\tau' \left\{ H_b^{(1)}(\tau'), \{H_c^{(1)}(\tau), H_a^{(1)*}(t)\} \right\} + (b \leftrightarrow c) \right], \tag{44}$$

$$\int d\Gamma \, F_a^{(2)} H_a^{(1)*} = - \int d\Gamma \, \tilde{f}_o(\Gamma) \int_{-\infty}^{t} d\tau \left\{ H_a^{(2)}(\tau), H_a^{(1)*}(t) \right\} \tag{45}$$

$$- \frac{1}{2} \int d\Gamma \, \tilde{f}_o(\Gamma) \int_{-\infty}^{t} d\tau \int_{-\infty}^{t} d\tau' \left[\left\{ H_a^{(1)*}(t), \{H_b^{(1)}(\tau'), H_c^{(1)}(\tau)\} \right\} + (b \leftrightarrow c) \right] .$$

The triple-Poisson-bracket terms in (44) and (45) can be combined by changing variables of time integration and using the Jacobi identity. Collecting our results, we obtain from (34) the formula

$$\begin{aligned} U_a = \int d\Gamma \, \tilde{f}_o(\Gamma) \Bigg[H_o^{(3)} &+ \int_{-\infty}^{t} d\tau \left(\left\{ H_a^{(1)*}(t), H_a^{(2)}(\tau) \right\} \right. \\ &+ \left\{ H_b^{(2)*}(t), H_b^{(1)}(\tau) \right\} + \left. \left\{ H_c^{(2)*}(t), H_c^{(1)}(\tau) \right\} \right) \\ &+ \int_{-\infty}^{t} d\tau \int_{-\infty}^{\tau} d\tau' \left(\left\{ H_b^{(1)}(\tau'), \left\{ H_c^{(1)}(\tau), H_a^{(1)*}(t) \right\} \right\} \right. \\ &+ \left. \left\{ H_c^{(1)}(\tau'), \left\{ H_b^{(1)}(\tau), H_a^{(1)*}(t) \right\} \right\} \right) \Bigg] \end{aligned} \tag{46}$$

Formula (46) has been derived independently of any gauge condition. If we specialize to purely electrostatic modes and choose the Coulomb gauge $\underline{\delta A} = 0$, then $H^{(2)}$ and $H^{(3)}$ vanish and (46) reduces to the result of Laval and Pellat (1975). These final terms in expression (46) have a deceptive "causal" structure, seeming to imply that the coupling to wave a at time t comes from particles which have encountered waves b and c at some <u>earlier</u> times in the past. However, such a causal coupling clearly could not be symmetric in the three fields. The real requirement for non-zero three-wave coupling is a region of spatial overlap of the three waves; if the fields were spatially separated, a non-resonant particle would "forget" seeing them, emerging from each unaffected. For this reason, we prefer the overlap-integral form (42) for the coupling coefficient U_a, where the convective tiem integrals have been replaced by the generating functions $S^{(1)}$.

6. Trilinear Interaction Hamiltonian

In this section, we present an alternate derivation of the coupling coefficient U_a which leads to a new insight into the nature of three-wave coupling and the associated Manley-Rowe relations. It has been shown by Sturrock (1970) that relations of the Manley-Rowe type follow immediately when the Hamiltonian of a physical system corresponds to a discrete set of coupled oscillators. It is simple to devise such a model Hamiltonian for the waves in the problem at hand; we write

$$= \sum_{a=1}^{3} \omega_a \beta_a^* \beta_a + (U\beta_a \beta_b^* \beta_c^* + \text{c.c.}), \qquad (47)$$

where the canonically conjugate variables are β_a and $i\beta_a^*$. The identification

$$\beta_a \equiv \alpha_a \exp(-i\omega_a t)$$

then leads to the Hamiltonian equations

$$i\dot{\alpha}_a = U\alpha_b \alpha_c \exp(-i\ \Delta\omega t),$$

$$i\dot{\alpha}_b = U\alpha_a \alpha_c^* \exp(i\ \Delta\omega t)\ ,$$

$$i\dot{\alpha}_c = U\alpha_a \alpha_b^* \exp(i\ \Delta\omega t)\ ,$$

which are consistent with the Manley-Rowe relations (43). The bilinear terms in (47) represent the unperturbed energies of each oscillator (wave) and the trilinear term the interaction energy. We are thereby led to speculate that the mode-coupling coefficient U might represent simply the trilinear interaction energy of a single particle (new Hamiltonian $K^{(3)}$), summed over all the non-resonant particles. This speculation is strongly supported by the work of Burshtein and Solov'ev (1962), who showed that Bogoliubov's "method of averaging" leads to an averaged Hamiltonian with the same Poisson-bracket structure as the coupling coefficient (42). We show now that the conjecture is true.

Returning to the expanded Hamilton-Jacobi equation (5), suppose we construct a canonical transformation which eliminates not only $H^{(1)}$ but also $H^{(2)}$. Such a transformation is characterized by the requirement that the new Hamiltonian K have the form

$$K = H_o + \overline{K}^{(2)} + K^{(3)} + O(\varepsilon^4) , \tag{48}$$

where $\overline{K}^{(2)}$ represents the static component of (7) due to the beating of each wave with itself. $\overline{K}^{(2)}$ can not be transformed away since it is a "resonant" perturbation, corresponding to a null resonance condition satisfied by all particles. It represents the particle component of the wave energy (Dewar 1973); the forces derived from it are the single-wave ponderomotive forces (Cary and Kaufman 1977).

A transformation satisfying requirement (48) can be constructed by matching the terms in (5) at each order in ε. After some algebra, we find that the following generating function S effects such a transformation:

$$S = S^{(1)} + S^{(2)} + S^{(3)} + O(\varepsilon^4),$$

where $S^{(1)}$ and $S^{(2)}$ are found by solving

$$D_t S^{(1)} = - H^{(1)} ,$$

$$D_t (S^{(2)} - \frac{1}{2} \underline{\nabla} S^{(1)} \cdot \underline{\partial} S^{(1)}) = - \tilde{K}^{(2)}$$

$$\tilde{K}^{(2)} \equiv H^{(2)} + \frac{1}{2} \{S^{(1)}, H^{(1)}\} - \overline{K}^{(2)} ,$$

and $S^{(3)}$ is arbitrary. The new Hamiltonian $K^{(3)}$ is then given by the formula

$$K^{(3)} = H^{(3)} + \{S^{(1)}, H^{(2)}\} + \frac{1}{3} \left\{ S^{(1)}, \{S^{(1)}, H^{(1)}\} \right\} + D_t Q^{(3)} , \tag{49}$$

where the quantity $Q^{(3)}$ depends on the choice of $S^{(3)}$. Indeed, we can arrange that $Q^{(3)} = 0$ by choosing

$$S^{(3)} = - \underline{\partial}\, S^{(1)} \cdot \underline{\nabla} S^{(2)} + \frac{1}{3} \underline{\partial}\, S^{(1)}\, \underline{\partial}\, S^{(1)} : \underline{\nabla}\, \underline{\nabla}\, S^{(1)}$$
$$+ \frac{1}{3} \underline{\partial}\, S^{(1)}\, \underline{\nabla} S^{(1)} : \underline{\partial}\, \underline{\nabla}\, S^{(1)} - \frac{1}{6} \underline{\nabla} S^{(1)}\, \underline{\nabla} S^{(1)} : \underline{\partial}\, \underline{\partial}\, S^{(1)} \quad . \tag{50}$$

The static component of formula (49) for $K^{(3)}$, trilinear in the three waves, reproduces the coupling coefficient (42) when summed over all the non-resonant particles. The conjecture is thus proved. The fact that U is symmetric in the three waves can now be viewed as an immediate consequence of the trilinearity of $\overline{K}^{(3)}$.

7. Concluding Remarks

In summary, the method of generalized ponderomotive forces (oscillation-center approach) has been extended to the general case of nonuniform and relativistic plasma. As an illustration, we have derived a "master formula" for three-wave coupling coefficients [formula (42)], cast in terms of Poisson brackets of the single-particle perturbation Hamiltonian and its convective time-integral along unperturbed trajectories. This master formula includes, as limiting cases, all uniform-plasma results; its direct use in determining growth rates for three-wave processes can circumvent a laborious calculation of the nonlinear currents in such problems.

The simplicity of the general form (42) led us in §6 to investigate and prove the conjecture that the three-wave coupling coefficient represents simply the trilinear interaction energy (new Hamiltonian $K^{(3)}$) of a single particle in the fields of three waves, summed over all the non-resonant particles in the plasma. The explicit demonstration of this fact, however, required some rather lengthy algebra [see equation (50)]. The source of the algebraic complexity is our conventional mixed-generating-function approach to performing the canonical transformations; in mixed variables, the desired Poisson-bracket form for $K^{(3)}$ does not arise naturally, since Poisson brackets are defined in terms of unmixed (conjugate) variables. It is clear that a more economical formulation, in which Poisson brackets play on essential role, should be possible.

The tools for such a concise reformulation can be found in the Lie-transform approach to Hamiltonian perturbation theory (Hori 1966, Deprit 1969, Dewar 1976). The hallmark of the Lie method is the Poisson bracket, and its use of unitary operators to effect canonical transformations avoids the usual mixing of old and new variables. The Lie approach indeed permits an elegant and remarkably simple derivation of the relation between $K^{(3)}$ and the three-wave coupling coefficient (Johnston and Kaufman 1978). This relation can, in fact, be viewed as a special case of a more general relation between the field-plasma interaction energy and the transformed single-particle Hamiltonian.

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Appendix: Resonant Particles and the Problem of Small Divisors

The mathematical foundation for a smooth filtration of all "resonant" actions $\underline{I}$ from the unperturbed distribution function $f_o(\underline{I})$ can be found in the work of Arnol'd (1963). To illustrate the essential ideas, we begin with an "integrable" Hamiltonian system with n degrees of freedom, i.e., a system which possesses n first integrals and hence (Arnol'd 1963) action-angle variables $(\underline{I}, \underline{\theta})$. Let us subject this system to a small perturbation of order ε,

$$H(\underline{I}, \underline{\theta}) = H_o(\underline{I}) + \varepsilon H'(\underline{I}, \underline{\theta}),$$

where H' is periodic in $\underline{\theta}$ and so can be expanded in a Fourier series of the form

$$H'(\underline{I}, \underline{\theta}) = \langle H' \rangle_{\underline{\theta}} + \sum_{\underline{\ell} \neq o} H'_{\underline{\ell}}(\underline{I})\, e^{i\underline{\ell}\cdot\underline{\theta}} \quad .$$

If follows that the canonical transformation which eliminates the $\underline{\theta}$-dependent terms of order ε in H' has for its generating function

$$S_{\underline{\ell}}(\underline{I}) = \frac{i\varepsilon H'_{\underline{\ell}}(\underline{I})}{\underline{\ell}\cdot\underline{\omega}(\underline{I})} , \qquad \underline{\omega}(\underline{I}) \equiv \frac{\partial H_o(\underline{I})}{\partial \underline{I}} \quad . \tag{A.1}$$

The denominator $(\underline{\ell}\cdot\underline{\omega})$ might vanish for certain "resonance" values of $\underline{\ell}$, and for any $\underline{\omega}$ is arbitrarily small for suitable $\underline{\ell}$. These small denominators raise serious doubts concerning the validity of the formal perturbation theory.

Nevertheless, since there are more irrational numbers than rational, it follows that the components of a randomly selected vector $(\omega_1, \omega_2, \cdots, \omega_n)$ are "incommensurable" (i.e., not in rational ratio). Therefore, for almost all vectors $\underline{\omega}$ (except for a set of Lebesgue measure zero), one has $\underline{\ell}\cdot\underline{\omega} \neq 0$ for all integers $\underline{\ell} \neq 0$. This idea is expressed more precisely in the following theorem (Arnol'd 1963): As $\kappa \rightarrow 0$, the measure of the set of vectors $\{\underline{\omega}\}$ which violate the inequality

$$|\underline{\ell}\cdot\underline{\omega}| \geq \kappa \; |\ell|^{-(n+1)}, \; \left(|\ell| \equiv |\ell_1| + \cdots + |\ell_n|\right)$$

also tends to zero. Thus, for the majority of vectors $\underline{\omega}$, the denominators $(\underline{\ell}\cdot\underline{\omega})$ not only do not vanish, but can be bounded from below by a power of $|\ell|$. Accordingly, one might hope that the perturbation series $S_{\underline{\ell}}$ in (A.1) is valid for the majority of vectors $\underline{\omega}$ (i.e., for the majority of actions $\underline{I}$).

However, there remains a problem, namely that the resonances are "dense" in action space. In any small neighborhood of a point $\underline{I}$, there is always a point $\underline{I}'$ where the frequencies $\underline{\omega}(\underline{I}')$ are commensurable (excluding the case $\underline{\omega}$ = constant). If one attempted to remove from the unperturbed distribution function $f_o(\underline{I})$ all resonant actions, one would be left with a wildly discontinuous function of $\underline{I}$. The corresponding generating function $S_{\underline{\ell}}(\underline{I})$ could not be differentiated or integrated by parts.

Fortunately, there is a resolution to this dilemma. The need to deal with everywhere-discontinuous functions of $\underline{I}$ can be avoided by truncation of the Fourier expansion for H' after a finite number of terms:

$$\varepsilon \; H'(\underline{I}, \underline{\theta}) \simeq \varepsilon \; \langle H' \rangle_{\underline{\theta}} + \sum_{o<|\ell|<N} \varepsilon \; H'_{\underline{\ell}}(\underline{I}) \; e^{i\underline{\ell}\cdot\underline{\theta}} \quad .$$

In the framework of a first-order perturbation theory, this procedure is acceptable provided that the omitted terms are of order ε^2. Since the Fourier coefficients of an analytic function decrease in geometric progression (Arnol'd 1963), it suffices to choose N to be of order $\ell n(\varepsilon^{-1})$.

To understand how this truncation facilitates the extraction of resonant actions, it is helpful to consider an integer lattice in $\underline{\ell}$-space. For simplicity, we limit our discussion to the case n=2 which is shown in Fig. 1. The integer lattice is of finite extent, corresponding to some given value of N. Now, for each value of $\underline{I}$, we can draw a line in the direction of the two-vector $\underline{\omega}(\underline{I})$. A second line drawn perpendicular to the first then represents the locus of points (ℓ_1, ℓ_2) which satisfy

$$\underline{\ell}\cdot\underline{\omega} = 0 \Longrightarrow \frac{\ell_2}{\ell_1} = -\frac{\omega_1}{\omega_2} \quad .$$

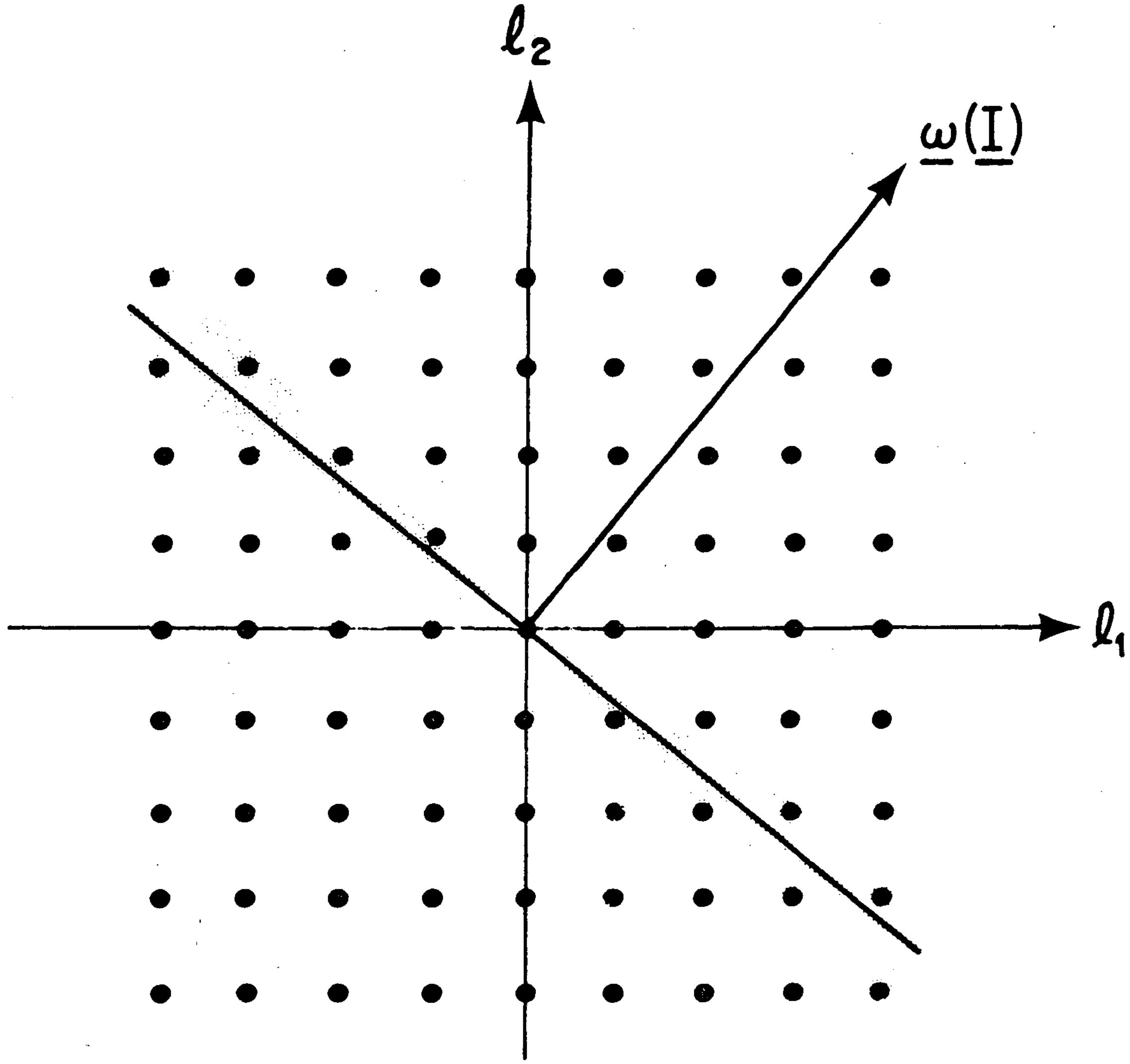
If the components of $\underline{\omega}(\underline{I})$ are incommensurable, then this locus does not pass through any points of the integer lattice; note, however, that as $N \to \infty$, lattice points can be found arbitrarily close to the locus.

From (A.1), it can be seen that the smallness of the generating function is violated when

$$\underline{\ell}\cdot\underline{\omega}(\underline{I}) = 0(\varepsilon) \quad . \tag{A.2}$$

For each $\underline{I}$, condition (A.2) corresponds to a narrow cone about the orthogonal locus whose angular width is small with ε (see Fig. 1). The points $\underline{\ell}$ lying within the cone are the ones which satisfy (A.2); if a lattice point lies within the cone, then the corresponding action $\underline{I}$ must be classified as "resonant". The key point is that non-resonant actions are allowed since the lattice is of finite extent.

Now as $\underline{I}$ varies, $\underline{\omega}(\underline{I})$ also varies and the corresponding orthogonal cone sweeps across the lattice. Suppose we agree to extract all the resonant actions $\underline{I}$ for which points of the integer lattice fall inside the orthogonal cone. Clearly, if ε is sufficiently small, this procedure defines isolated zones in action space whose total area tends to zero with ε. These zones can then be removed smoothly from the distribution function $f_o(\underline{I})$ to form the non-resonant distribution $\tilde{f}_o(\underline{I})$ employed in this paper.



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