

UC Irvine

UC Irvine Previously Published Works

Title

Neoclassical theory of elementary charges with spin of 1/2

Permalink

<https://escholarship.org/uc/item/2dg1d47c>

Journal

Journal of Mathematical Physics, 55(8)

ISSN

0022-2488

Authors

Babin, Anatoli
Figotin, Alexander

Publication Date

2014-08-01

DOI

10.1063/1.4893336

Peer reviewed

Neoclassical Theory of Elementary Charges with Spin of $1/2$

Anatoli Babin and Alexander Figotin
 Department of Mathematics
 University of California at Irvine
 Irvine, CA 92697-3875

August 5, 2014

Abstract

We advance here our neoclassical theory of elementary charges by integrating into it the concept of spin of $1/2$. The developed spinorial version of our theory has many important features identical to those of the Dirac theory such as the gyromagnetic ratio, expressions for currents including the spin current, and antimatter states. In our theory the concepts of charge and anticharge relate naturally to their "spin" in its rest frame in two opposite directions. An important difference with the Dirac theory is that both the charge and anticharge energies are positive whereas their frequencies have opposite signs.

1 Introduction

In a series of papers including [BF6]-[BF9] we have developed a neoclassical theory of electromagnetic (EM) interactions between elementary charges without spin. One of our key motivations for introducing such a theory was a desire to account for particle properties as well as for wave phenomena in a single mathematically sound Lagrangian relativistic field theory. In this theory all particle properties come out naturally from the field equations as approximations. We have shown that the theory implies in the non-relativistic limit: (i) the non-relativistic particle mechanics governed by the Newton equations with the Lorentz forces and (ii) the frequency spectrum for hydrogenic atoms. We have studied also in [BF8], [BF9] relativistic aspects of the theory and have demonstrated that the relativistic point mass equation is an approximation of the field equations when the charge wave function is well localized, and derived the Einstein energy-mass relation $E = Mc^2$ for the accelerated motion.

A primary goal of this paper is to integrate into our neoclassical theory of elementary charges the concept of spin of $1/2$. As in the cited above papers, an elementary charge is not a charged mass point, but it is described by a field distribution, and now we want to add to its properties an intrinsic magnetic moment and a spin. We have accomplished that goal by constructing a spinorial version of the mentioned above neoclassical theory. When developing this theory we kept in mind that it has to incorporate in one form or another some features of

the Dirac theory of spin 1/2 particles that are verified experimentally. To integrate the spin into our Lagrangian relativistic field theory we used methods developed by D. Hestenes and other authors, see [Hes2-03], [HesNF], [DeSDat], [DorLas], [Sny] and references therein. In particular, we used D. Hestenes's "real" form of the Dirac equation based on the spacetime algebra (STA), that is the Clifford algebra of the Minkowski vector space. The geometric transparency of the STA combined with a rich multivector algebraic structure was a decisive incentive for using it instead of the Dirac γ -matrices.

Since the spinorial version of our neoclassical theory is obtained by a modification of its original spinless version it is useful to take a look at its basic features. In our original spinless theory a single elementary charge is described by a pair (ψ, A^μ) , where ψ is its complex valued *wave function* and $A^\mu = (\varphi, \mathbf{A})$ is its 4-vector *elementary potential* with the corresponding *elementary EM field* defined by the familiar formula $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. *An elementary charge does not interact with itself electromagnetically. Its wave function ψ represents its matter properties and the elementary potential A^μ mediates its EM interactions with all other elementary charges. Importantly, (i) all internal forces of an elementary charge are exclusively of non-electromagnetic origin; (ii) every elementary charge is a source of its elementary EM field which represents force exerted by this charge on any other charge but not upon itself.*

A system of any number of elementary charges is furnished with a relativistic Lagrangian that yields EM interactions with the following features: (i) elementary charges interact only through their elementary EM potentials and fields; (ii) the field equations for the elementary EM fields are exactly the Maxwell equations with proper conserved currents; (iii) the wave functions evolution is governed by nonlinear Klein-Gordon equations; (iv) EM force density is described exactly by its well known Lorentz expression; (v) the Newton equations with the Lorentz forces hold approximately when charges are well separated and move with non-relativistic velocities; (vi) a free charge moves uniformly preserving up to the Lorentz contraction its shape. Since an overwhelming number of EM phenomena are explained within the classical EM theory by the Maxwell equations and the Lorentz forces our neoclassical EM theory is equally successful in explaining the same phenomena.

Particle-like states of elementary charges are recovered in our theory from the original field concepts as localized states. A possibility for an elementary charge to localize is facilitated by internal forces of non-electromagnetic origin. These forces are introduced in the Lagrangian in the form of a nonlinear term $G(|\psi|^2)$ defined by the following expression

$$G(s) = G_a(s) = -a^{-2}s [\ln(a^3|s|) + \ln\pi^{3/2} + 2], \quad -\infty < s < \infty, \quad (1)$$

where $a > 0$ is the size parameter. The free charge then has Gaussian shape and is of the size a . As we have shown in [BF6] *the specific expression (1) for the nonlinearity $G(s)$ is derived from a physically sound requirement that the Planck-Einstein relation $E = \hbar\omega$ holds exactly in the non-relativistic approximation to our fully relativistic theory.*

To give a flavor of the proposed here spinorial version of our neoclassical theory, we can state that the Euler-Lagrange field equation governing the motion of a free charge is a spinorial version of the nonlinear Klein-Gordon equation, namely

$$\mathcal{P}^2\psi - \left[\kappa_0^2 + G' \left(\langle \psi \tilde{\psi} \rangle \right) \right] \psi = 0, \quad \kappa_0 = \frac{mc}{\chi}, \quad (2)$$

where \mathcal{P} is a certain spinorial version of the momentum operator similar to the same in the Dirac theory, G is a nonlinearity defined by (1) χ is a constant which is equal approximately

to the Planck constant. We show that *the spinorial version of our theory developed here has many important features identical to those of the Dirac theory such as the gyromagnetic ratio, expressions for currents including the spin current, and antimatter states.* We treat here mostly the case of a free charge, since the difference with the spinless scalar case shows itself already in this case. Detailed analysis of more complex cases including charge in external electromagnetic field or systems of interacting charges is left for future work.

While our theory has many features identical to the Dirac theory as pointed out above, it differs significantly from the Dirac theory. The first significant difference is that, by its very design, our neoclassical theory is a consistent relativistic Lagrangian field theory. This difference is manifest in the treatment of the energy. The Dirac theory following to the quantum mechanics (QM) framework essentially identifies the energy with the frequency through the Planck-Einstein relation $E = \hbar\omega$ considered to be fundamentally exact. In our neoclassical Lagrangian field theory the energy-momentum density is constructed based on the system Lagrangian and the Noether theorem and its relation to frequencies in relevant regimes is non-trivial. In particular, in our theory the Planck-Einstein relation holds only approximately as a non-relativistic approximation for time harmonic states taking the form $E \approx \hbar|\omega|$. Observe that it is the compliance of the Dirac theory with the foundations of QM requires the identity $E = \hbar\omega$ resulting in unbounded negative energy problem. Indeed, the QM evolution equation $\hbar i\partial_t\psi = H\psi$ with H being the energy operator is just an operator form of the Planck-Einstein relation $E = \hbar\omega$. This evolution equation requires naturally negative frequencies which then according to $E = \hbar\omega$ have to be interpreted as negative energies.

The second significant difference is that our Lagrangian theory is free from any "infinities" which constitute a known problem for the QM and the quantum electrodynamics.

The third difference is that in our theory an elementary charge is an extended object which can be localized whereas in the QM it is a point-like object. As R. Feynman put it, [Feynman III, p. 21-6] : "The wave function $\psi(\mathbf{r})$ for an electron in an atom does not, then, describe a smeared-out electron with a smooth charge density. The electron is either here, or there, or somewhere else, but wherever it is, it is a point charge." In our theory the localization property of an elementary charge in relevant situations is provided by a nonlinear non-electromagnetic self-interaction G defined by (1). As a result the free charge spinor wave function in our theory is a plane wave modulated by a Gaussian amplitude factor compared to a plane wave free charge solution in the Dirac theory. In particular, the localization property of the neoclassical free charge solutions allows to evaluate the conserved quantities by integration whereas that is not possible in the case of plane waves. The presence of the nonlinearity in our theory also invalidates to some degree the linear superposition principle, whereas it is of fundamental importance in the QM.

The fourth important difference is that in our theory there is no electromagnetic self-interaction for an elementary charge.

The paper is organized as follows. In Section 2 we provide basic information on the STA needed to carry out computations. In Section 3 we discuss important properties of the STA version of the Dirac theory. In Section 4 we develop the spinorial version of our neoclassical theory and in Section 5 we study properties of a free charge. In Section 6 we consider the interpretation of the neoclassical solutions and compare main features of the developed here theory with those of the Dirac theory.

2 Basics of Spacetime Algebra (STA)

In this section we formulate the basic properties of the *Spacetime Algebra* (STA), for it is perfectly suited for our conceptual purposes as well the computation. The spacetime algebra is a particular case of the *Clifford algebra*. A general *Clifford algebra*, also called *Geometric Algebra* (GA), is an associative algebra generated by an n -dimensional vector space V over the set of real scalars \mathbb{R} furnished with a symmetric quadratic form g . We denote such a Clifford algebra by $\text{Cl}(V, g)$ and call its elements *multivectors* referring to elements of the generating linear space V as vectors. The *Clifford product*, also called *geometric product*, of any two multivectors A and B is denoted by juxtaposition, that is AB . The Clifford product is fundamentally determined by the requirement to satisfy the following identity for any two vectors a and b from the generating vector space V :

$$ab + ba = 2g(a, b)1, \quad (3)$$

where 1 is the multiplicative identity which we often skip in notation. The Clifford algebra is naturally furnished with *inner* "·" (*interior, dot*) *product* and *outer* (*exterior, Grassmann*) *product* " \wedge " so that for any two vectors a and b in V

$$a \cdot b = \frac{ab + ba}{2} = g(a, b), \quad a \wedge b = -b \wedge a = \frac{ab - ba}{2}, \quad (4)$$

implying

$$ab = a \cdot b + a \wedge b. \quad (5)$$

According to (4), the orthogonality of two vectors a and b in the Clifford algebra, that is $a \cdot b = 0$, has an equivalent algebraic representation as the anticommutativity of a and b .

The *Spacetime Algebra* is the Clifford algebra based on the real 4-dimensional Minkowski space \mathbb{M}^4 and it is denoted by $\text{Cl}(1, 3)$, where (1, 3) is the signature of the Minkowski space metric. The 3-dimensional Euclidian space is denoted by \mathbb{R}^3 and the corresponding to it Clifford Algebra is $\text{Cl}(3, 0)$. In setting up the STA we follow to [Hes2-03], [DorLas] and [Sny]. Though many properties of Clifford Algebras hold universally across different dimensions and signatures, we formulate them mostly for the case of our primary interest which is the Spacetime Algebra $\text{Cl}(1, 3)$. We want to stress that the concise review of the STA presented here is not meant to be complete and/or systematic presentation of the Clifford Algebras theory, but rather it is a selection of important for our purposes properties of the STA. We want to acknowledge the work done by D. Hestenes who pioneered and developed many aspects of the STA and its applications to physics. There is number of excellent presentations of Clifford Algebras and their applications to physics written by D. Hestenes and his followers, see, for instance, [HesSob], [HesNF], [Hes2-03], [DorLas], [Sny], [DorFonMan].

The standard model for the spacetime is the real Minkowski vector space \mathbb{M}^4 with the standard metric $g_{\mu\nu}$ defined by

$$\{g_{\mu\nu}\} = \{g^{\mu\nu}\}, \quad g_{00} = 1, \quad g_{jj} = -1, \quad j = 1, 2, 3, \quad g_{\mu\nu} = 0, \quad \mu \neq \nu. \quad (6)$$

A basis for the STA can be generated by a standard frame $\{\gamma_\mu : \mu = 0, 1, 2, 3\}$ of orthonormal vectors, with a *timelike vector* γ_0 in the forward light cone, and γ_μ are assumed to satisfy the following relations:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\gamma_\mu \cdot \gamma_\nu = 2g_{\mu\nu}, \quad (7)$$

$$\gamma_0^2 = 1, \quad \gamma_i^2 = -1, \quad \gamma_0 \cdot \gamma_i = 0, \quad \gamma_i \cdot \gamma_j = -\delta_{ij}, \quad i, j = 1, 2, 3. \quad (8)$$

Notice that (7)-(8) are the defining relations of the *Dirac matrix algebra*. That explains our choice to denote an orthonormal frame by $\{\gamma_\mu\}$, but it must be remembered that the $\{\gamma_\mu\}$ are basis vectors and not a set of matrices in "isospace".

To facilitate algebraic manipulations, it is convenient to introduce the reciprocal frame $\{\gamma^\mu\}$ defined by the equations

$$\gamma^\mu = g^{\mu\nu} \gamma_\nu, \quad \gamma_\mu \cdot \gamma^\nu = \delta_\mu^\nu, \quad (9)$$

with the summation convention understood. Observe that two different vectors γ_μ anticommute. Since we are in a space of mixed signature, we distinguish between a *frame* $\{\gamma_\mu\}$ and its *reciprocal* $\{\gamma^\mu\}$, namely

$$\gamma^0 = \gamma_0, \quad \gamma^i = -\gamma_i, \quad i = 1, 2, 3. \quad (10)$$

Notice that, following to the common practice, we use Greek letters μ, ν, \dots for indices taking values 0, 1, 2, 3 and Latin letter i, j, \dots for indices taking values 1, 2, 3. The γ_μ determine a unique *right-handed unit pseudoscalar*

$$\mathbf{I} = \gamma_0 \gamma_1 \gamma_2 \gamma_3, \quad \mathbf{I}^2 = -1, \quad (11)$$

that anticommutes with vectors γ_μ

$$\mathbf{I} \gamma_\mu = -\gamma_\mu \mathbf{I}, \quad \mu = 0, 1, 2, 3. \quad (12)$$

For any vector a a frame $\{\gamma_\mu\}$ determines a set of *rectangular coordinates*

$$a = a^\mu \gamma_\mu = a^0 \gamma_0 + a^i \gamma_i, \quad \{a^\mu\} = \{a^0, \mathbf{a}\}. \quad (13)$$

In particular, for any spacetime point x

$$x = x^\mu \gamma_\mu = ct \gamma_0 + x^i \gamma_i, \quad \{x^\mu\} = \{x^0, \mathbf{x}\} = \{ct, \mathbf{x}\}. \quad (14)$$

The frame $\{\gamma_\mu\}$ defines also an explicit basis for this algebra as follows:

$$\begin{array}{cccccc} 1 & \{\gamma_\mu\} & \{\gamma_\mu \wedge \gamma_\nu\} & \{\mathbf{I} \gamma_\mu\} & \{\mathbf{I}\} & \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar} & \end{array}, \quad (15)$$

where " \wedge " is the external (Grassman) product. This is the spacetime algebra $\text{Cl}(1, 3)$. The structure of this algebra tells us practically all one needs to know about (flat) spacetime and the Lorentz transformation group. A general element M of the spacetime algebra is called *multivector* and can be written as

$$M = \alpha + a + B + \mathbf{I}b + \mathbf{I}\beta, \quad (16)$$

where α and β are scalars, a and b are vectors, and B is a bivector. The representation (16) is a decomposition of M into its k -vector parts (grades), and that can be expressed more explicitly by putting it in the form

$$\begin{aligned} M &= \sum_{0 \leq k \leq 4} \langle M \rangle_k, \quad \text{where } \langle M \rangle_0 = \langle M \rangle = \alpha, \\ \langle M \rangle_1 &= a, \quad \langle M \rangle_2 = B, \quad \langle M \rangle_3 = \mathbf{I}b, \quad \langle M \rangle_4 = \mathbf{I}\beta, \end{aligned} \quad (17)$$

where the subscript (k) means “ k -vector part”. Notice the special notation $\langle M \rangle = \langle M \rangle_0$ for the scalar part for the multivector M . The space of k -vectors, that is multivectors of the grade k , is denoted by Λ^k .

It is instructive to see the grade decomposition for the geometric product of two multivectors $A_r \in \Lambda^r$ and $B_s \in \Lambda^s$, [HesSob, 1.1], [HesZie, 2.2], [RodOli, 2.4.2]

$$\begin{aligned} A_r B_s &= \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s} = \\ &= \sum_{k=0}^m \langle A_r B_s \rangle_{|r-s|+2k}, \text{ where } m = \frac{1}{2}(r+s-|r-s|), \end{aligned} \quad (18)$$

where it is understood that for any multivector M

$$\langle M \rangle_k \equiv 0 \text{ for any } k > 4. \quad (19)$$

The inner (dot) “ \cdot ” and outer (Grassman) “ \wedge ” products are defined first for homogeneous multivectors $A_r \in \Lambda^r$ and $B_s \in \Lambda^s$ by, [HesSob, 1.1],

$$A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|}, \text{ if } r, s > 0; \quad (20)$$

$$A_r \cdot A_s = 0, \text{ if } r = 0 \text{ or } s = 0;$$

$$A_r \cdot B_s = (-1)^{r(s-r)} B_s \cdot A_r \text{ for } s \geq r; \quad (21)$$

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s} = (-1)^{rs} B_s \wedge A_r; \quad (22)$$

with consequent extension by linearity to arbitrary multivectors A and B . In particular, if a is a vector and A_r is multivector of the grade r , we have, [HesSob, 1-1], [DorLas, 4.1.2]

$$a A_r = a \cdot A_r + a \wedge A_r, \quad A_r a = A_r \cdot a + A_r \wedge a, \quad (23)$$

where

$$\begin{aligned} a \cdot A_r &= (-1)^{r-1} A_r \cdot a = \frac{1}{2}(a A_r - (-1)^r A_r a), \\ a \wedge A_r &= (-1)^r A_r \wedge a = \frac{1}{2}(a A_r + (-1)^r A_r a). \end{aligned} \quad (24)$$

The STA has a much richer structure than the algebra of complex numbers, and it can be furnished with several natural conjugations (involutions) operations, [DeSDat, 5.3], [Bay, 1.4.8], [Perw, 3.1, 3.2]. The most important of those is called *reversion* (principal anti-automorphism), and the *reverse* \tilde{M} of a general multivector M is defined by

$$\tilde{M} = \alpha + a - B - Ib + I\beta, \quad (25)$$

$$\langle \tilde{M} \rangle_k = \widetilde{\langle M \rangle_k} = (-1)^{\frac{k(k-1)}{2}} \langle M \rangle_k, \quad 0 \leq k \leq 4. \quad (26)$$

The reversion operation justifies its name since it reverses the order of the multipliers:

$$(MN)^\sim = \tilde{N} \tilde{M}. \quad (27)$$

Grade involution is a conjugation defined by

$$\langle \hat{M} \rangle_k = (-1)^k \langle M \rangle_k, \quad 0 \leq k \leq 4. \quad (28)$$

There is yet another *Hermitian conjugation* also called *relative reversion* M^\dagger of a multivector M defined by

$$M^\dagger = \gamma_0 \tilde{M} \gamma_0, \quad (29)$$

and it corresponds to the Hermitian conjugation in the Dirac Algebra. Every multivector then can be decomposed into γ_0 -even and γ_0 -odd components

$$M = M_e + M_o, \text{ where } M_e = \frac{1}{2} (M^\dagger + M), \quad M_o = \frac{1}{2} (M^\dagger - M), \quad (30)$$

and evidently

$$M_e^\dagger = M_e, \quad M_o^\dagger = -M_o, \quad \tilde{M}^\dagger = \widetilde{M^\dagger}. \quad (31)$$

The grade structure (16) of STA and the grade involution operator defined by (28) provide for a natural decomposition of any multivector M into the sum of an *even part* M_+ and an *odd part* M_- as follows:

$$M_+ = \alpha + B + I\beta, \quad M_- = a + Ib, \quad M_\pm = \frac{1}{2} (M \pm \hat{M}) = \frac{1}{2} (M \mp IMI). \quad (32)$$

Notice that the *even and odd parts respectively commute and anticommute with I*, that is

$$M_+I = IM_+, \quad M_-I = -M_-I. \quad (33)$$

Importantly, the set of all even elements M_+ of the STA $\text{Cl}(1,3)$ forms a Clifford algebra on its own, we denote it by $\text{Cl}_+(1,3)$. This even subalgebra $\text{Cl}_+(1,3)$ is isomorphic to the geometric algebra (GA) $\text{Cl}(3,0)$ of the three-dimensional Euclidean space with multivectors of the form, [Hes-96, 1], [Hes1-03, VI],

$$N = \alpha + I\beta + a + Ib \in \text{Cl}(3,0), \quad (34)$$

where α and β are scalars, a and b are vectors and I is the unit pseudoscalar in $\text{Cl}(3,0)$. The even subalgebra $\text{Cl}_+(1,3)$ is very important to the STA version of the Dirac electron theory where it is the space of values of the Dirac spinorial wave function.

Notice that the scalar part of $\langle M \rangle$ has the following properties

$$\langle M \rangle = \langle \tilde{M} \rangle, \quad \langle MN \rangle = \langle NM \rangle, \quad \langle \langle M \rangle_k \langle N \rangle_s \rangle = 0, \quad \text{if } k \neq s, \quad (35)$$

where M and N are multivectors. The above equalities imply the following identities involving Hermitian conjugation

$$\langle MN^\dagger \rangle = \langle N^\dagger M \rangle = \langle NM^\dagger \rangle. \quad (36)$$

Based on the above we define first a *scalar-valued *-product* for any two arbitrary multivectors A and B by, [HesSob, 1.1], [DorLas, 4.1.3], [DorstIP], [DorFonMan, 3.1.2], [Perw, 3.2.3]

$$A * B = \langle AB \rangle = \sum_{0 \leq k \leq 4} \langle A_{(k)} B_{(k)} \rangle. \quad (37)$$

The above scalar *-product is symmetrical and reversible

$$A * B = B * A = \tilde{A} * \tilde{B} = \tilde{B} * \tilde{A}. \quad (38)$$

Another (fiducial) *scalar product* $A \bullet B = \langle A, B \rangle$ of two arbitrary multivectors A and B is defined by, [Sny, 3.4], [Moya, 4.2.4]

$$\begin{aligned} A \bullet B &= \langle A, B \rangle = \tilde{A} * B = \langle \tilde{A} B \rangle = \langle A \tilde{B} \rangle \\ &= \sum_{0 \leq k \leq 4} \left\langle \langle \tilde{A} \rangle_k \langle B \rangle_k \right\rangle = \sum_{0 \leq k \leq 4} (-1)^{\frac{k(k-1)}{2}} \langle A_{(k)} B_{(k)} \rangle. \end{aligned} \quad (39)$$

Notice that we use the symbol " \bullet " for the scalar product since the "normal" dot symbol " \cdot " is already taken for the inner product. Unfortunately, the symbols " $*$ " and " \bullet " are used differently in different texts and one has to pay attention when using those symbols. For detailed and insightful analysis of relations between different products and their geometric meaning see [DorstIP]. The relations (35)-(42) readily imply the following useful properties of the scalar products

$$(AB) * C = A * (BC) = \langle ABC \rangle \quad (40)$$

$$(AB) \bullet C = B \bullet (\tilde{A}C) = A \bullet (C\tilde{B}), \quad (41)$$

$$A \bullet (BC) = (\tilde{B}A) \bullet C = (A\tilde{C}) \bullet B.$$

A grade- r multivector A is called *simple* or a *blade* if it is a product of r anticommuting vectors, that is

$$A = a_1 \wedge a_2 \cdots \wedge a_r, \text{ where } a_k a_j = -a_k a_j \text{ for } k \neq j. \quad (42)$$

Blades naturally correspond to subspaces, and they are instrumental to establishing relations between geometric and algebraic properties. An important property of every grade- r blade A_r is that it has the inverse, [HesSob, 1-1], [DorFonMan, 3.5.2]

$$A_r^{-1} = \frac{\tilde{A}_r}{A_r * \tilde{A}_r} = (-1)^{r(r-1)/2} \frac{A_r}{A_r * \tilde{A}_r}. \quad (43)$$

In the case where A_r and B_r are simple r -vectors, the scalar products (38), (39) have the following representations via the determinant, [DorFonMan, 3.1.2]

$$A_r * B_r = \langle A_r B_r \rangle = A_r \cdot B_r = \det \begin{bmatrix} \langle a_1, b_r \rangle & \cdots & \langle a_1, b_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle a_r, b_r \rangle & \cdots & \langle a_r, b_1 \rangle \end{bmatrix}, \quad r > 0, \quad (44)$$

$$A_r \bullet B_r = A_r * \tilde{B}_r = \det \begin{bmatrix} \langle a_1, b_1 \rangle & \cdots & \langle a_1, b_r \rangle \\ \vdots & \ddots & \vdots \\ \langle a_r, b_1 \rangle & \cdots & \langle a_r, b_r \rangle \end{bmatrix}, \quad r > 0, \quad (45)$$

and

$$A_r * B_s = 0, \quad r \neq s; \quad a * b = ab, \text{ if } a \text{ and } b \text{ are scalars.} \quad (46)$$

Observe that in the case of the Clifford Algebra $\text{Cl}(3,0)$ of 3-dimensional Euclidian space, for any multivector $A \in \text{Cl}(3,0)$ the scalar product is positive, $A \bullet A = \langle \tilde{A}A \rangle \geq 0$; and

that is the primary motivation to define the scalar product by the formula (35). The scalar product allows also to define a *positive definite magnitude* $|M|$ for any multivector M by

$$|M|^2 = \left| \langle \tilde{M}M \rangle \right| = \left| \langle M\tilde{M} \rangle \right|. \quad (47)$$

Notice that in the case of vectors we always have

$$A \cdot B = A \bullet B = A * B \text{ if } A, B \in \Lambda^1. \quad (48)$$

Being given a basis $\{\gamma_\mu\}$ for \mathcal{M}^4 , we define a basis $\{\sigma_k\}$ for the 3-dimensional Euclidean space \mathcal{P}^3 by

$$\sigma_k = \gamma_k \wedge \gamma_0 = \gamma_k \gamma_0 = -\tilde{\sigma}_k = -\sigma^k, \quad \sigma_k^2 = 1, \quad k = 1, 2, 3, \quad (49)$$

$$\sigma_i \sigma_j = -\gamma_i \gamma_j = -\gamma_i \wedge \gamma_j = \epsilon_{ijk} \mathbf{I} \sigma_k, \quad i \neq j, \quad \sigma_1 \sigma_2 \sigma_3 = \mathbf{I}, \quad (\mathbf{I} \sigma_k)^2 = -1, \quad (50)$$

where ϵ_{ijk} is the *alternating tensor*, also called *Levi-Civita symbol*, defined by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is a cyclic permutation of } 123, \\ -1 & \text{if } ijk \text{ is an anticyclic permutation of } 123, \\ 0 & \text{if otherwise.} \end{cases} \quad (51)$$

Notice also that the following identities hold

$$\gamma_0 \sigma_k = -\sigma_k \gamma_0, \quad \gamma_0 \mathbf{I} = -\mathbf{I} \gamma_0, \quad \gamma_0 \mathbf{I} \sigma_k = \mathbf{I} \sigma_k \gamma_0, \quad \sigma_k \mathbf{I} = \mathbf{I} \sigma_k, \quad k = 1, 2, 3. \quad (52)$$

$$\sigma_i \cdot \sigma_j = \delta_{ij}, \quad \frac{1}{2} (\sigma_i \sigma_j - \sigma_j \sigma_i) = \epsilon_{ijk} \mathbf{I} \sigma_k, \quad \frac{1}{2} (\mathbf{I} \sigma_i \mathbf{I} \sigma_j - \mathbf{I} \sigma_j \mathbf{I} \sigma_i) = \epsilon_{ijk} \mathbf{I} \sigma_k. \quad (53)$$

The bivectors σ_k are called *relative vectors* and they correspond to timelike planes. The *relative vectors* σ_k generate the even subalgebra $\text{Cl}_+(1, 3)$ which is isomorphic to the geometric algebra (GA) $\text{Cl}(3, 0)$ of the three-dimensional Euclidean space, [Hes-86, 3], [Hes-96, 1]. Relative bivectors $\mathbf{I} \sigma_k$ according to (50) are spacelike bivectors.

Observe that using $\mathbf{I}^2 = -1$, we can recast the relations (50) as

$$\mathbf{I} \sigma_i \mathbf{I} \sigma_j = -\sigma_i \sigma_j = \gamma_i \gamma_j = \gamma_i \wedge \gamma_j = -\epsilon_{ijk} \mathbf{I} \sigma_k, \quad i \neq j, \quad (54)$$

implying that the span $\langle 1, \mathbf{I} \sigma_1, \mathbf{I} \sigma_2, \mathbf{I} \sigma_3 \rangle$ is a subalgebra \mathbf{Q} which is isomorphic to the even subalgebra $\text{Cl}_+(3, 0)$ of the geometric algebra $\text{Cl}(3, 0)$ of the three-dimensional Euclidean space. Since $\text{Cl}_+(3, 0)$ is isomorphic to the *quaternion algebra*, [HesNF, 2.3], [DorLas, 2.4.2], [DeSDat, 6.1], the subalgebra \mathbf{Q} is also isomorphic to the *quaternion algebra* and we refer to it by that name, that is

$$\mathbf{Q} = \langle 1, \mathbf{I} \sigma_1, \mathbf{I} \sigma_2, \mathbf{I} \sigma_3 \rangle \text{ is the quaternion subalgebra.} \quad (55)$$

Notice that the quaternion subalgebra \mathbf{Q} can also be characterized as the one consisting of even multivectors which are also γ_0 -even, that is

$$\mathbf{Q} = \left\{ M \in \text{Cl}_+(3, 0) : M^\dagger = \gamma_0 \tilde{M} \gamma_0 = M \right\}. \quad (56)$$

The quaternion subalgebra \mathbf{Q} is very important to the STA version of the Pauli electron theory where it is the space of values of the Pauli spinorial wave function.

3 The Dirac equation in STA

Since the Dirac theory has been very thoroughly analyzed and tested experimentally, we would like to consider its STA version in sufficient detail and compare it with developed here neoclassical theory. In addition to that, the Dirac equation in the STA and its analysis provides us with a number of valuable tools useful for our own constructions, and we consider its important features in this section.

The STA version of Dirac spinor Ψ is *the wave function ψ taking values in the even subalgebra $\text{Cl}_+(1,3)$ of the Clifford algebra $\text{Cl}(1,3)$* , and we refer to it as *Dirac spinor* or just *spinor*. Notice that for any ψ from $\text{Cl}_+(1,3)$ we have $\psi\tilde{\psi} = \tilde{\psi}\psi$ implying that this product is a linear combination of the scalar and the pseudoscalar I , that is

$$\psi\tilde{\psi} = \tilde{\psi}\psi = \varrho e^{I\beta} = \varrho (\cos \beta + I \sin \beta), \quad \text{where } \varrho \geq 0 \text{ and } \beta \text{ are scalars.} \quad (57)$$

This leads to the following *canonical Lorentz invariant decomposition* which holds for every even multivector ψ , [Hes-75], [Hes2-03, VII.D], [DorLas, 8.2], [DeSDat, 9.3],

$$\psi = \varrho^{\frac{1}{2}} e^{\frac{I\beta}{2}} R = R \varrho^{\frac{1}{2}} e^{\frac{I\beta}{2}}, \quad R\tilde{R} = R\tilde{R} = 1, \quad (58)$$

where $\varrho > 0$ and β are scalars, and R is the Lorentz rotor, that is $x' = Rx\tilde{R}$ is the Lorentz transformation. According to D. Hestenes, the canonical decomposition (57) can be regarded as an invariant decomposition of the Dirac wave function into a 2-parameter *statistical factor* $\varrho^{\frac{1}{2}} e^{\frac{I\beta}{2}}$ and a 6-parameter *kinematical factor* R .

It is worth to point out that the identity (57) clearly shows that though the reversion operation $\tilde{\psi}$ is analogous to the complex conjugation for complex numbers, the even subalgebra $\text{Cl}_+(1,3)$ is a richer entity than the set of complex numbers allowing $\psi\tilde{\psi}$ to be *negative and not scalar valued*.

To introduce an STA form of the Dirac equation, we define first an STA version of the *Dirac operator* denoted sometimes by nabla dagger, [ItzZub, 2-1-2], [GreRQM, 3]. We denote this STA version of the Dirac operator by $\partial = \partial_x$. It is often called *vector derivative* with respect to vector x and defined by, [Hes1-03], [Hes2-03, II], [HesSob], [DorLas],

$$\partial = \partial_x = \gamma^\mu \partial_\mu, \quad \text{where } \partial_\mu = \frac{\partial}{\partial x^\mu}. \quad (59)$$

Notice that since ∂ is a vector, it may not commute with other multivectors.

In the case of the Clifford algebra $\text{Cl}(3,0)$ of the 3-dimensional Euclidian space, *the vector derivative* ∇ is defined by

$$\nabla = \sum_{j=1}^3 \sigma_j \partial_j, \quad \text{where } \partial_j = \frac{\partial}{\partial x^j}, \quad \text{and } \sigma_j \text{ is a basis of } \text{Cl}(3,0). \quad (60)$$

The *covariant Dirac equation in STA*, known also as the real Dirac equation, was obtained by D. Hestenes [Hes2-03, VII], [DorLas, 13.3.3, 13.3.3.4], [RodOli, 6.7, 6.8] and it is

$$\hbar \partial \psi I \sigma_3 - \frac{e}{c} A \psi = mc \psi \gamma_0, \quad \text{where } I \sigma_3 = \gamma_1 \gamma_2. \quad (61)$$

The real Dirac equation (61) is equivalent to the original Dirac equation. The equation (61) can be recast also as

$$(\mathcal{P} - mc \overleftarrow{\gamma}_0) \psi = 0, \quad \text{or } \mathcal{P} \psi = mc \psi \gamma_0, \quad (62)$$

where the momentum operator \mathcal{P} and the operator $\overleftarrow{\gamma}_0$ are defined by

$$\mathcal{P}\psi = \hbar\partial\psi\mathbf{I}\sigma_3 - \frac{e}{c}A\psi, \quad \overleftarrow{\gamma}_0\psi = \psi\gamma_0. \quad (63)$$

The momentum operator \mathcal{P} can be alternatively represented by

$$\mathcal{P}\psi = \gamma^\mu\mathcal{P}_\mu\psi, \quad \text{where } \mathcal{P}_\mu\psi = \hbar\partial_\mu\psi\mathbf{I}\sigma_3 - \frac{e}{c}A_\mu\psi. \quad (64)$$

We refer to the equations (61), (62) as the *Dirac-Hestenes equations*. Observe that \mathcal{P} and $\overleftarrow{\gamma}_0$ commute since the multivectors $\mathbf{I}\sigma_3 = \gamma_1\gamma_2$ and γ_0 commute, that is

$$(\mathbf{I}\sigma_3)\gamma_0 = \gamma_0(\mathbf{I}\sigma_3), \quad \overleftarrow{\gamma}_0\mathcal{P} = \mathcal{P}\overleftarrow{\gamma}_0. \quad (65)$$

The free electron canonical momentum operator $\mathring{\mathcal{P}}$ is obtained as a particular case of \mathcal{P} in (63) when $A = 0$, that is

$$\mathring{\mathcal{P}}\psi = \hbar\partial\psi\mathbf{I}\sigma_3, \quad \text{and } \mathcal{P}\psi = \mathring{\mathcal{P}}\psi - \frac{e}{c}A\psi = \hbar\partial\psi\mathbf{I}\sigma_3 - \frac{e}{c}A\psi. \quad (66)$$

One can also introduce for spinor valued ψ the *covariant derivative operator* \mathcal{D} :

$$\mathcal{D}\psi = \partial\psi + \frac{e}{\hbar c}A\psi\mathbf{I}\sigma_3, \quad \text{implying } \mathcal{P} = \hbar\partial\psi\mathbf{I}\sigma_3 - \frac{e}{c}A\psi\hbar = \mathcal{D}\psi\mathbf{I}\sigma_3. \quad (67)$$

Conserved quantities of interest, including the electric current and the energy-momentum tensor (EnMT), can be obtained from the following *real Dirac-Hestenes Lagrangian* density for electron in external electromagnetic field, [LDG, 4.4], [Hes-96, Ap. B], [Hes-STC, App. B]

$$L = c \left\langle \hbar\partial\psi\mathbf{I}\gamma_3\tilde{\psi} - \frac{e}{c}A\psi\gamma_0\tilde{\psi} - mc\psi\tilde{\psi} \right\rangle. \quad (68)$$

Using expressions (67) for the canonical momentum \mathcal{P} and the covariant derivative \mathcal{D} , we can transform the Dirac-Hestenes Lagrangian into the following form

$$L = c \left\langle [(\mathcal{P} - mc\overleftarrow{\gamma}_0)\psi] \gamma_0\tilde{\psi} \right\rangle = c \left\langle [(\hbar\mathcal{D}\psi\mathbf{I}\sigma_3 - mc\overleftarrow{\gamma}_0)\psi] \gamma_0\tilde{\psi} \right\rangle. \quad (69)$$

The Lagrangian representation (69) implies

$$L = 0 \quad \text{for any } \psi \text{ satisfying the Dirac equation (62),} \quad (70)$$

and that is typical for the first order systems, [DorLas, 13.3]. One can also verify that the corresponding Euler-Lagrange field equation is equivalent to the Dirac-Hestenes equation (61).

The *free electron Dirac-Hestenes Lagrangian* \mathring{L} (when $A = 0$) equals

$$\begin{aligned} \mathring{L} &= c \left\langle \hbar\partial\psi\mathbf{I}\gamma_3\tilde{\psi} - mc\psi\tilde{\psi} \right\rangle = c \left\langle [(\mathring{\mathcal{P}} - mc\overleftarrow{\gamma}_0\psi)] \gamma_0\tilde{\psi} \right\rangle = \\ &= c \left\langle [(\hbar\partial\psi\mathbf{I}\sigma_3 - mc\overleftarrow{\gamma}_0\psi)] \gamma_0\tilde{\psi} \right\rangle, \quad \text{where } \mathring{\mathcal{P}}_\mu\psi = \hbar\partial_\mu\psi\mathbf{I}\sigma_3. \end{aligned} \quad (71)$$

3.1 Conservation laws

Our treatment of the charge and energy-momentum conservation laws is based on the Dirac-Hestenes Lagrangian and the Noether theorem.

3.1.1 Electric charge conservation

We introduce the so-called *global electromagnetic gauge transformation* as follows, [Hes-73, 3], [LDG, 3.2],

$$x' = x, \quad \psi'(x') = \psi(x) e^{\mathbf{I}\sigma_3 \epsilon}, \quad \epsilon \text{ is any real number.} \quad (72)$$

Consequently, the global electromagnetic gauge transformation preserves the vector derivative $\partial\psi$, that is

$$\partial' = \partial, \quad \partial'\psi'(x') = \partial\psi(x) e^{\mathbf{I}\sigma_3 \epsilon}. \quad (73)$$

The infinitesimal form of (72) for small ϵ is

$$\delta x' = 0, \quad \bar{\delta}\psi = \psi \mathbf{I}\sigma_3 \epsilon. \quad (74)$$

The *local electromagnetic gauge transformation* is conceived to keep the covariant derivative $\mathcal{D}\psi$ defined by (67) invariant. It involves both the ψ and A and is of the form

$$x' = x, \quad \psi'(x) = \psi(x) e^{\mathbf{I}\sigma_3 \frac{e}{\hbar c} \epsilon(x)}, \quad A'(x) = A(x) - \partial\epsilon, \quad (75)$$

where $\epsilon(x)$ is real valued function of x is . Then, since $\partial = \gamma^\mu \partial_\mu$, we consequently obtain

$$\partial' = \partial, \quad \partial'\psi'(x') = \left[\partial\psi(x) + \frac{e}{\hbar c} \partial\epsilon\psi(x) \mathbf{I}\sigma_3 \right] e^{\mathbf{I}\sigma_3 \frac{e}{\hbar c} \epsilon(x)}, \quad (76)$$

$$\mathcal{D}'\psi'(x') = \mathcal{D}\psi(x). \quad (77)$$

The infinitesimal form of (75) for small $\epsilon(x)$ is

$$\delta x' = 0, \quad \bar{\delta}\psi = \psi \mathbf{I}\sigma_3 \frac{e}{\hbar c} \epsilon, \quad \bar{\delta}\partial\psi = \partial\psi + \frac{e}{\hbar c} \partial\epsilon\psi \mathbf{I}\sigma_3, \quad \bar{\delta}A = -\partial\epsilon. \quad (78)$$

The last equality in (78) indicates that to have the local gauge invariance, we have to couple the spinor field ψ with a vector field to "compensate" for the term $\frac{e}{\hbar c} \partial\epsilon\psi \mathbf{I}\sigma_3$. And this exactly what the electromagnetic potential A does yielding the well known *minimal coupling*.

One readily verifies that the Dirac-Hestenes Lagrangian (68) is invariant with respect to electromagnetic gauge transformation (72). Then, according to Noether's theorem, there is a conserved electric current J^μ defined by

$$\begin{aligned} \pi^\mu &= \frac{\partial L}{\partial \psi_{,\mu}} = c\hbar \mathbf{I}\gamma_3 \tilde{\psi} \gamma^\mu, \\ J^\mu &= \pi^\mu * \bar{\delta}\psi = c \left\langle \hbar \mathbf{I}\gamma_3 \tilde{\psi} \gamma^\mu \psi \mathbf{I}\sigma_3 \epsilon \right\rangle = c \left\langle \hbar \tilde{\psi} \gamma^\mu \psi \gamma_0 \epsilon \right\rangle. \end{aligned} \quad (79)$$

Multiplying the current expression in (79) by a proper constant and using the STA properties (27), (35)-(38) together with momentum \mathcal{P} representation (67), we obtain the following expression for the electric current

$$J^\mu = ec \left\langle \gamma^\mu \psi \gamma_0 \tilde{\psi} \right\rangle, \quad J = ec\psi \gamma_0 \tilde{\psi}, \quad J^\mu = (c\rho, \mathbf{J}). \quad (80)$$

Assuming that ψ satisfies the Dirac equation (62), (63), we can recast the above expression into

$$J^\mu = \frac{e}{m} \left\langle \gamma^\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle, \quad J = J^\mu \gamma_\mu = \frac{e}{m} (\mathcal{P}\psi) \tilde{\psi} = \frac{e}{m} \left(\chi \partial\psi \mathbf{I}\sigma_3 - \frac{e}{c} A\psi \right) \tilde{\psi}. \quad (81)$$

The expression $c\psi\gamma_0\tilde{\psi}$ is known as the *Dirac probability current* in the QM, [Hes2-03, VII.D], [Hes-STC, 10], whereas the expression $\frac{e}{m}(\mathcal{P}\psi)\tilde{\psi}$ is known as the *Gordon current* [Hes-75, 5], [Hes2-03, VII.H]. We want to stress that the current J expressions (80) and (81) are evidently two very different expressions which are equal only because ψ satisfies the Dirac equation (62), (63). Consequently, one may interpret the Dirac equation as a requirement that two generally different currents defined by (80) and (81) must be the same. In addition to that, notice that the expression $\frac{e}{m}(\chi\partial\psi\mathbf{I}\sigma_3 - \frac{e}{c}A\psi)\tilde{\psi}$ in (81) for a general even ψ , that is ψ not necessarily satisfying the Dirac equation, can take multivector values. So in the case of general even ψ the proper Gordon current expression based on its components J^μ is

$$J = J^\mu\gamma_\mu = \left\langle \frac{e}{m}(\mathcal{P}\psi)\tilde{\psi} \right\rangle_1 = \frac{e}{m} \left\langle \left(\chi\partial\psi\mathbf{I}\sigma_3 - \frac{e}{c}A\psi \right) \tilde{\psi} \right\rangle_1, \quad (82)$$

and in the special case when ψ satisfies the Dirac equation, the projection operation $\langle \rangle_1$ on the vector space can be naturally omitted since $(\chi\partial\psi\mathbf{I}\sigma_3 - \frac{e}{c}A\psi)\tilde{\psi}$ has to be a vector in this case. The current J satisfies the conservation law

$$\partial \cdot J = 0 \text{ or } \partial_\mu J^\mu = 0, \quad J^\mu = (c\rho, \mathbf{J}), \quad (83)$$

where ρ is the charge density and \mathbf{J} is the charge current.

The Gordon current expression (81) satisfies the following *Gordon decomposition* law, [Hes-96, 3]

$$J^\mu = J_c^\mu + J_s^\mu, \quad J_c^\mu = \frac{e}{m} \left\langle (\mathcal{P}^\mu\psi)\tilde{\psi} \right\rangle, \quad J_s^\mu = \frac{\hbar e}{m} \left\langle [\gamma^\mu, \gamma^\nu] \partial_\nu\psi\mathbf{I}\sigma_3\tilde{\psi} \right\rangle, \quad (84)$$

where J_c^μ and J_s^μ are respectively the *convection* and *magnetization (spin) currents*. To justify the use of magnetization and spin terms, let us recall that the *magnetization bivector* M and intimately related to it *spin angular momentum bivector* S are defined in the STA by the following expressions, [Hes-75, 4], [Hes-96, 2, 3], [Hes2-03, VII.C]

$$S = \frac{\hbar}{2}R\mathbf{I}\sigma_3\tilde{R} = \frac{\hbar}{2}R\gamma_2\gamma_1\tilde{R}, \quad \psi = \varrho^{\frac{1}{2}}e^{\frac{I\beta}{2}}R = R\varrho^{\frac{1}{2}}e^{\frac{I\beta}{2}}, \quad (85)$$

$$M = \frac{\hbar e}{2mc}\psi\mathbf{I}\sigma_3\tilde{\psi} = \frac{\hbar e}{2mc}\psi\gamma_2\gamma_1\tilde{\psi} = \frac{e}{mc}\varrho e^{I\beta}S, \quad (86)$$

where ψ satisfies the canonical relations (57), (58). Then the following relations between components J_s^μ (81) and magnetization bivector M hold:

$$\begin{aligned} J_s^\mu &= J_s^\mu\gamma_\mu = c\partial \cdot M, & J_s^\mu &= c\gamma^\mu \cdot (\partial \cdot M) = c\gamma^\mu \cdot (\gamma^\nu \cdot \partial_\nu M) = \\ &= c(\gamma^\mu \wedge \gamma^\nu) \cdot \partial_\nu M = \frac{\hbar e}{m} \left\langle (\gamma^\mu \wedge \gamma^\nu) \partial_\nu\psi\mathbf{I}\sigma_3\tilde{\psi} \right\rangle. \end{aligned} \quad (87)$$

It is instructive to see that the STA Gordon current decomposition representation (84) perfectly matches a similar formula in the conventional Dirac theory, [GreRQM, 8.1], [Sny, 8.1], [Wach, p. 148]:

$$J^\mu = ec\bar{\Psi}\gamma^\mu\Psi = J_c^\mu + J_s^\mu = \frac{e}{2m} \left[\bar{\Psi}\hat{P}^\mu\Psi - \overline{(\hat{P}^\mu\Psi)}\Psi \right] - \frac{ie}{2m}\partial^\nu(\bar{\Psi}\sigma^\mu{}_\nu\Psi), \quad (88)$$

where $\sigma^\mu{}_\nu$ is defined by

$$\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \quad (89)$$

and

$$J_c^\mu = \frac{e}{2m} \left[\bar{\Psi} \hat{P}^\mu \Psi - \overline{(\hat{P}^\mu \Psi)} \Psi \right] \text{ is the convection current density,} \quad (90)$$

$$J_s^\mu = -\frac{ie}{2m} \hat{P}^\nu (\bar{\Psi} \sigma^\mu{}_\nu \Psi) \text{ is the spin current density.}$$

Notice that the spin (magnetization) current J_s in view of the representation $J_s = \partial \cdot M$ in (87) is conserved since

$$\partial \cdot J_s = c \partial \cdot (\partial \cdot M) = c (\partial \wedge \partial) \cdot M = 0. \quad (91)$$

Combining (83) and (84), we obtain also the conservation law for the convection current J_c^μ :

$$\partial_\mu J_c^\mu = 0. \quad (92)$$

Notice also that, assuming that J is the Dirac probability current defined by (80), we can recast the Lagrangian in (69)

$$L = \dot{L} - \frac{1}{c} \langle AJ \rangle = c \left\langle [(\hbar \partial \psi \mathbf{I} \sigma_3 - mc \overleftarrow{\gamma}_0 \psi)] \gamma_0 \tilde{\psi} \right\rangle - \frac{1}{c} \langle AJ \rangle, \quad (93)$$

indicating that the current J definition by (80) is in accord with the classical theory. Indeed, in the classical theory the EM interaction between the EM field four-potential A and the current J is described by the expression $\frac{1}{c} \langle AJ \rangle$.

3.1.2 Energy-momentum conservation

Applying the Noether theorem to the Dirac Lagrangian (68), (69) and using relations (66), (70), we consequently obtain for the canonical EnMT $\dot{T}^{\mu\nu}$

$$\pi^\mu = \frac{\partial L}{\partial \psi_{,\mu}} = c \hbar \mathbf{I} \gamma_3 \tilde{\psi} \gamma^\mu, \quad \dot{T}^{\mu\nu} = \pi^\mu * \partial^\nu \psi = c \left\langle \hbar \mathbf{I} \gamma_3 \tilde{\psi} \gamma^\mu \partial^\nu \psi \right\rangle. \quad (94)$$

The above expression for EnMT $\dot{T}^{\mu\nu}$ after an elementary transformation turns into

$$\dot{T}^{\mu\nu} = c \left\langle \hbar \mathbf{I} \gamma_3 \tilde{\psi} \gamma^\mu \partial^\nu \psi \right\rangle = c \left\langle \gamma_0 \tilde{\psi} \gamma^\mu \hbar \partial^\nu \psi \mathbf{I} \sigma_3 \right\rangle = c \left\langle \gamma^\mu \left(\dot{\mathcal{P}}^\nu \psi \right) \gamma_0 \tilde{\psi} \right\rangle. \quad (95)$$

Then the EnMT conservation law takes the form

$$\partial_\mu \dot{T}^{\mu\nu} = -\partial^\nu L = \frac{1}{c} (\partial^\nu A^\mu) J_\mu. \quad (96)$$

Observe that the canonical EnMT $\dot{T}^{\mu\nu}$ involves $\dot{\mathcal{P}}^\nu$ and evidently is not gauge invariant. To find its gauge invariant modification $T^{\mu\nu}$ we use the charge conservation law $\partial^\mu J_\mu = 0$ to obtain the following identity

$$\begin{aligned} (\partial^\nu A^\mu) J_\mu &= (\partial^\nu A^\mu - \partial^\mu A^\nu) J_\mu + (\partial^\mu A^\nu) J_\mu = \\ &= F^{\nu\mu} J_\mu + \partial^\mu (A^\nu J_\mu) = F^{\nu\mu} J_\mu + \partial_\mu (A^\nu J^\mu), \end{aligned} \quad (97)$$

where $F^{\nu\mu} = \partial^\nu A^\mu - \partial^\mu A^\nu$ are components of the EM field bivector $F = \frac{1}{2}F_{\nu\mu}\gamma^\nu \wedge \gamma^\mu$. The above identity allows to recast the conservation law (96) as

$$\partial_\mu \left(\dot{T}^{\mu\nu} - \frac{1}{c} A^\nu J^\mu \right) = \frac{1}{c} F^{\nu\mu} J_\mu. \quad (98)$$

The equality (98) in turn suggests to introduce the following gauge invariant modification $T^{\mu\nu}$ of the canonical EnMT $\dot{T}^{\mu\nu}$:

$$T^{\mu\nu} = \dot{T}^{\mu\nu} - \frac{1}{c} A^\nu J^\mu = c \left\langle \gamma^\mu (\mathcal{P}^\nu \psi) \gamma_0 \tilde{\psi} \right\rangle. \quad (99)$$

Then (98) can be recast into the conservation law

$$\partial_\mu T^{\mu\nu} = \frac{1}{c} F^{\nu\mu} J_\mu, \quad (100)$$

where $\frac{1}{c} F^{\nu\mu} J_\mu$ are the components of the Lorentz force. Using the identity

$$F^{\nu\mu} J_\mu = (\gamma^\nu \wedge \gamma^\mu) \cdot F J_\mu = \gamma^\nu \cdot (\gamma^\mu \cdot F) J_\mu = \gamma^\nu \cdot (J \cdot F), \quad (101)$$

and introducing the vectors

$$T^\mu = T^{\mu\nu} \gamma_\nu, \quad (102)$$

we can recast the EnMT conservation (99) into a concise vector form

$$\partial_\mu T^\mu = \frac{1}{c} J \cdot F, \text{ where } J \cdot F \text{ is the Lorentz force vector.} \quad (103)$$

The properties of the gauge invariant EnMT $T^{\mu\nu}$ and related to T^μ are thoroughly studied in [Hes-96, 3].

3.2 Free electron solutions to the Dirac equation

This section provides basic information on the plane wave solutions to the Dirac-Hestenes equations following to [Hes-81, 6], [Hes-96, 4], [Hes2-03, VIII.B], [DorLas, 8.3.2]. Free electron satisfies the Dirac equation (61) with $A = 0$, that is

$$\hbar \partial \psi \mathbf{I} \sigma_3 = mc \psi \gamma_0, \text{ where } \mathbf{I} \sigma_3 = \gamma_1 \gamma_2. \quad (104)$$

A *positive energy plane-wave solution* ψ_- to the Dirac equation (104) for electron is defined to be of the form

$$\text{positive energy solution: } \psi_- = \psi_0 e^{-\mathbf{I} \sigma_3 k \cdot x}, \text{ where } \gamma_0 \cdot k > 0, \quad (105)$$

and ψ_0 is a constant spinor. Notice that in ψ_- the subindex "−" signifies the sign of the electron charge. Recall that the *wave vector* k is related to the *momentum vector* p by $p = \hbar k$, and we obtain the following spacetime split representations in terms of relative vectors:

$$k \gamma_0 = \frac{\omega}{c} + \mathbf{k}, \quad p \gamma_0 = \hbar k \gamma_0 = \frac{\hbar \omega}{c} + \hbar \mathbf{k} = \frac{E}{c} + \mathbf{p}. \quad (106)$$

If the charge is at rest in the γ_0 -frame interpreted as $\mathbf{p} = \mathbf{0}$ then according the above formula

$$p = p \cdot \gamma_0 = p_0 = \frac{\hbar\omega_0}{c} = mc. \quad (107)$$

Since $\partial = \gamma^\mu \partial_\mu$ we have

$$\partial\psi = \partial\psi_0 e^{-\mathbf{I}\sigma_3 k \cdot x} = -k\psi_0 e^{-\mathbf{I}\sigma_3 k \cdot x} \mathbf{I}\sigma_3 = -k\psi \mathbf{I}\sigma_3, \quad (108)$$

implying that $\psi = \psi_0 e^{-\mathbf{I}\sigma_3 k \cdot x}$ is a solution to the Dirac equation (104) if and only if ψ_0 satisfies

$$p\psi_0 = mc\psi_0\gamma_0. \quad (109)$$

Multiplying the above equation from the right by $\tilde{\psi}_0$ we obtain

$$p\psi_0\tilde{\psi}_0 = mc\psi_0\gamma_0\tilde{\psi}_0. \quad (110)$$

We assume the constant spinor ψ_0 to be normalized with the following canonical representation (58):

$$\begin{aligned} \psi_0 &= e^{\frac{i\beta_0}{2}} R_0, & \psi_0\tilde{\psi}_0 &= e^{\beta_0 \mathbf{I}}, \text{ where } \beta_0 \text{ is real,} \\ R_0 &\text{ is the Lorentz rotor: } R_0\tilde{R}_0 = \tilde{R}_0 R_0 = 1. \end{aligned} \quad (111)$$

Then it follows from (110) and (111) that

$$pe^{\beta_0 \mathbf{I}} = mcR_0\gamma_0\tilde{R}_0, \quad (112)$$

and since both the p and $R_0\gamma_0\tilde{R}_0$ are vectors, we must have

$$\psi_0\tilde{\psi}_0 = e^{\beta_0 \mathbf{I}} = \pm 1, \text{ that is } \beta_0 = 0, \pi. \quad (113)$$

Since $\gamma_0 \cdot p > 0$ and $\gamma_0 \cdot R_0\gamma_0\tilde{R}_0 > 0$ as it follows from (105), we must have $e^{\beta_0 \mathbf{I}} = 1$ in (112), that is

$$p = mcR_0\gamma_0\tilde{R}_0. \quad (114)$$

The rotor R_0 solving the problem (114) is the product

$$R_0 = LU, \quad (115)$$

where the boost L is defined by

$$L = \frac{1 + v\gamma_0}{[2(1 + v \cdot \gamma_0)]^{1/2}}, \quad v = \frac{p}{mc} = \gamma \left(1 + \frac{\mathbf{v}}{c}\right) \gamma_0 = \frac{1}{mc} \left(\frac{E}{c} + \mathbf{p}\right) \gamma_0, \quad (116)$$

or in view of (105)

$$L = L(\mathbf{p}) = \frac{E_0 + E(\mathbf{p}) + \mathbf{c}\mathbf{p}}{[2E_0(E_0 + E(\mathbf{p}))]^{1/2}}, \quad (117)$$

$$\text{where } E(\mathbf{p}) = E_0 \sqrt{\frac{\mathbf{p}^2}{c^2} + 1}, \quad E_0 = mc^2 = p_0 c = \hbar k_0 c = \hbar\omega, \quad (118)$$

and the rotor U is a pure rotation in γ_0 -frame, that is $U\gamma_0 = \gamma_0 U$.

A negative energy plane-wave solution ψ_+ to the Dirac equation (104) is defined by a formula similar to (104) but with the phase factor $e^{+I\sigma_3 k \cdot x}$, namely

$$\text{negative energy solution: } \psi_+ = \psi_0 e^{I\sigma_3 k \cdot x}, \text{ where } k \cdot \gamma_0 > 0, \quad (119)$$

Notice that in ψ_+ the subindex ”+” signifies that the sign of the positron charge is opposite to the negative sign of the electron charge. For negative energy solutions in place of (112) we have

$$-pe^{\beta_0 I} = mcR_0\gamma_0\tilde{R}_0, \quad (120)$$

and, consequently, $e^{\beta_0 I} = -1$, implying

$$\text{for negative energy: } p = mcR_0\gamma_0\tilde{R}_0, \quad \psi_0\tilde{\psi}_0 = e^{\beta_0 I} = -1. \quad (121)$$

Positive and negative energy plane wave states are commonly interpreted as respectively electron state and positron (antiparticle) state with positive energy, [Wach, 2.1.6]. Their representations can be summarized by

$$\text{positive energy (electron): } \psi_0\tilde{\psi}_0 = e^{\beta_0 I} = 1: \quad \psi_- = L(\mathbf{p})U_r e^{-I\sigma_3 k \cdot x}, \quad (122)$$

$$\text{negative energy (positron): } \psi_0\tilde{\psi}_0 = e^{\beta_0 I} = -1: \quad \psi_+ = L(\mathbf{p})U_r e^{I\sigma_3 k \cdot x}, \quad (123)$$

where $p = \hbar k$ and the subscript r at the spatial rotor U_r labels the spin state with

$$U_0 = 1, \quad U_1 = -I\sigma_2 = \gamma_1\gamma_3, \quad U_1\gamma_3\tilde{U}_1 = -\gamma_3. \quad (124)$$

Electron and positron states in (122)-(123) can be related to each other by the so-called *charge conjugation* transformation, [Hes2-03, VII.C, VIII.B], [Wach, 2.1.6], defined by

$$\psi^C = \psi\sigma_2, \text{ where } \sigma_2 = \gamma_2\gamma_0. \quad (125)$$

Namely, σ_2 anticommutes with γ_0 and $I\sigma_3$, therefore if ψ solves the Dirac equation (61) with charge e its conjugate ψ^C solves the Dirac equation with the charge $-e$, that is

$$\hbar\partial\psi^C I\sigma_3 + \frac{e}{c}A\psi^C = mc\psi^C\gamma_0, \text{ where } I\sigma_3 = \gamma_1\gamma_2. \quad (126)$$

Notice also that the following identity holds for any real α

$$e^{-I\sigma_3\alpha}\sigma_2 = \sigma_2 e^{I\sigma_3\alpha}, \quad (127)$$

implying together with (11), (52) and (122)-(123) that

$$\psi_-^C = L(\mathbf{p})U'_r e^{-I\sigma_3 k \cdot x}, \quad U'_r = U_r(-I\sigma_2). \quad (128)$$

Observe that ψ_-^C in the above equation is a state similar to ψ_+ in (123) indicating that the charge conjugation transforms an electron state into an antiparticle (positron) state with positive energy. Note that in view of the last equality in (124) the factor $-I\sigma_2 = U_1$ represents a spatial rotation that “flips” the direction of the spin vector, [Hes2-03, VIII.B]. In fact, the charge conjugation $\psi \rightarrow \psi^C$ reverses the charge, energy, momentum, and spin of an electron state transferring it into a positron state describing the antiparticle with opposite charge $-e$ in the same potential A^μ , [Wach, 2.1.6].

4 Basics of neoclassical theory of charges with spin of 1/2

We develop in this section a spinorial version of our neoclassical field Lagrangian theory of elementary charges. The initial step in this development is to assume that the wave function ψ of a single charge such as electron takes values in the even algebra $\text{Cl}_+(1, 3)$ just as in the Dirac theory. We focus here on the theory of a single charge in an external electromagnetic field. Extension of this theory to the case of many elementary charges is similar to the same for spinless charges constructed and studied in [BF7]-[BF8].

The Lagrangian of a *single elementary charge in an external electromagnetic field described by the 4-potential \check{A}* is

$$L = \frac{1}{2m} \left\{ \left\langle \mathcal{P}\psi (\mathcal{P}\psi)^{\check{}} \right\rangle - \chi^2 \left[\kappa_0^2 \left\langle \psi \check{\psi} \right\rangle + G \left(\left\langle \psi \check{\psi} \right\rangle \right) \right] \right\}, \quad \kappa_0 = \frac{mc}{\chi}, \quad (129)$$

where (i) m is the electron mass; (ii) χ is a constant approximately equal to the Planck constant \hbar ; (iii) G is a *nonlinear self-interaction term of not electromagnetic origin*, and (iv)

$$\mathcal{P}\psi = \chi \partial \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A}\psi \quad (130)$$

is the *momentum operator* which is identical to the same in the Dirac-Hestenes equation (62). Notice that we have somewhat departed from the common notations of the Dirac theory denoting the external EM 4-potential by \check{A} instead of A . The reason for such an alteration is that *there is no electromagnetic self-interaction for an elementary charge in our theory*, and every charge is associated with its individual wave function ψ and elementary EM four potential A . So, to avoid any confusion and to distinguish the external 4-potential from the elementary 4-potential A , we use \check{A} for the external one.

The nonlinearity $G(s)$ in (129) is defined by the formula (1). We readily obtain from it

$$G'(s) = G'_a(s) = -a^{-2} [\ln(a^3 |s|) + \ln \pi^{3/2} + 3], \quad -\infty < s < \infty. \quad (131)$$

Notice that (1) and (131) imply the following identity

$$sG'_a(s) - G_a(s) = -a^{-2}s. \quad (132)$$

As it is already explained the nonlinear self-interaction term G of non-electromagnetic origin and its role in theory is to provide for the localization property of the elementary charge in relevant situations.

Just as in the Dirac theory, it is useful to single out the "free" part $\mathring{\mathcal{P}}$ of \mathcal{P} , namely

$$\mathcal{P}\psi = \mathring{\mathcal{P}}\psi - \frac{e}{c} \check{A}\psi, \quad \text{where } \mathring{\mathcal{P}}\psi = \chi \partial \psi \mathbf{I}\sigma_3. \quad (133)$$

The coordinate forms \mathcal{P}_μ and $\mathring{\mathcal{P}}_\mu$ of the above momenta operators are

$$\begin{aligned} \mathcal{P}_\mu \psi &= \chi \partial_\mu \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A}_\mu \psi, & \mathcal{P}\psi &= (\gamma^\mu \mathcal{P}_\mu) \psi, \\ \mathring{\mathcal{P}}_\mu \psi &= \chi \partial_\mu \psi \mathbf{I}\sigma_3, & \mathring{\mathcal{P}}\psi &= (\gamma^\mu \mathring{\mathcal{P}}_\mu) \psi. \end{aligned} \quad (134)$$

When transforming expressions involving reversion operation, we often use the following elementary identities:

$$\widetilde{\sigma}_3 = -\sigma_3, \quad \widetilde{\mathbf{I}} = \mathbf{I}, \quad \widetilde{\mathbf{I}\sigma_3} = -\mathbf{I}\sigma_3 = -\sigma_3\mathbf{I}. \quad (135)$$

Lagrangian treatment of the conservation laws based on a multivector Noether's theorem has been developed in [LDG, 4-6], [DorLas, 12.4, 13], and we adopt most of that approach here. For more details of mathematical aspects of the Lagrangian field theory for multivector-valued fields, we refer the reader to [RodOli, 7]. To obtain the Euler-Lagrange equations for the Lagrangian L defined by (129), we find first its derivatives

$$\frac{\partial L}{\partial \psi} = -\frac{1}{m} \left\{ (\mathcal{P}\psi) \check{\frac{e}{c}} \check{A} + \chi^2 \left[\kappa_0^2 + G' \left(\langle \psi \tilde{\psi} \rangle \right) \right] \tilde{\psi} \right\}, \quad (136)$$

$$\pi^\mu = \frac{\partial L}{\partial \psi_{,\mu}} = \frac{\chi}{m} \mathbf{I}\sigma_3 (\mathcal{P}\psi) \check{\gamma}^\mu. \quad (137)$$

Notice that we have dropped the projection operation $\langle * \rangle_X$ in the right-hand sides of (136), (137) since their expressions take values in the even subalgebra. Using expressions (136), (137), we obtain the Euler-Lagrange equation

$$- (\mathcal{P}\psi) \check{\frac{e}{c}} \check{A} - \chi^2 \left[\kappa_0^2 + G' \left(\langle \psi \tilde{\psi} \rangle \right) \right] \tilde{\psi} - \partial_\mu \chi \mathbf{I}\sigma_3 (\mathcal{P}\psi) \check{\gamma}^\mu = 0. \quad (138)$$

Application of the reversion operation to the above equation yields

$$-\frac{e}{c} \check{A} \mathcal{P}\psi - \left[\kappa_0^2 + G' \left(\langle \psi \tilde{\psi} \rangle \right) \right] \psi + \chi \partial (\mathcal{P}\psi) \mathbf{I}\sigma_3 = 0, \quad (139)$$

which, in turn, in view of the expression (130) for \mathcal{P} , can be transformed into a more concise form of the *field equation*

$$\mathcal{P}^2 \psi - \left[\kappa_0^2 + G' \left(\langle \psi \tilde{\psi} \rangle \right) \right] \psi = 0. \quad (140)$$

Hence, the field equation (140) is the master evolution equation for the wave function in our theory based on the Lagrangian (129). The expression $\mathcal{P}^2 \psi$ in equation (140) can be transformed into the following form showing the external EM field

$$\mathcal{P}^2 \psi = \check{\mathcal{P}}^2 \psi - \frac{\chi e}{c} \left[F + 2\check{A} \cdot \partial \right] \psi \mathbf{I}\sigma_3 + \frac{e^2}{c^2} \check{A}^2 \psi, \quad (141)$$

where $F = \partial \wedge \check{A}$ is the bivector of the electromagnetic field.

Using the commutativity (65) of the operator $\check{\gamma}_0$ and the momentum operator \mathcal{P} , one can factorize the expression $\mathcal{P}^2 \psi - \kappa_0^2 \psi$ in the equation (140) yielding

$$(\mathcal{P} + mc \check{\gamma}_0) (\mathcal{P} - mc \check{\gamma}_0) \psi - \chi^2 G' \left(\langle \psi \tilde{\psi} \rangle \right) \psi = 0. \quad (142)$$

It is instructive to compare the above field equation (140) with the Dirac-Hestenes equation (62), (63). Just by looking at the two equations, one can see two significant differences. First of all, the field equation (140) contains a nonlinear self-interaction term $G' \left(\langle \psi \tilde{\psi} \rangle \right)$, that

is a concept not present in the Dirac theory. For comparison purposes it is instructive to eliminate this nonlinear term from the field equation (140) resulting in

$$\text{the truncated field equation: } (\mathcal{P}^2 - m^2 c^2) \psi = 0. \quad (143)$$

Now one can see another significant difference between the truncated field equation (143) and the Dirac-Hestenes equation (62). Indeed, the Dirac-Hestenes equation (62) is linear in \mathcal{P} whereas the truncated field equation (143) is quadratic in \mathcal{P} . In spite of this difference it is possible to establish an intimate relation between the two equations by factorizing the truncated field equation (143). To do that we use the commutativity (65) of the operator $\overleftarrow{\gamma}_0$ and the momentum operator \mathcal{P} and factorize equation (140) into the following form

$$(\mathcal{P} + mc\overleftarrow{\gamma}_0)(\mathcal{P} - mc\overleftarrow{\gamma}_0)\psi = 0, \text{ truncated field equation factorized.} \quad (144)$$

The above factorization of the truncated field equation is not unique. In fact, one can drop the operator $\overleftarrow{\gamma}_0$ from it, and what is left is still a correct representation of the original field equation (140). An important justification for the factorization (144) with the operator $\overleftarrow{\gamma}_0$ is as follows. For even ψ both the vectors $\mathcal{P}\psi$ and $mc\overleftarrow{\gamma}_0\psi$ are odd and hence each of the equation

$$(\mathcal{P} - mc\overleftarrow{\gamma}_0)\psi = 0, \quad (\mathcal{P} + mc\overleftarrow{\gamma}_0)\psi = 0 \quad (145)$$

can have even solutions. On the other hand, the equations

$$(\mathcal{P} - mc)\psi = 0, \quad (\mathcal{P} + mc)\psi = 0 \quad (146)$$

can not have a nontrivial even solution since for even ψ the multivector $\mathcal{P}\psi$ is always odd.

Observe now that any linear combination of solutions to equations (145) is a solution to the truncated field equations (144). Hence any solution to the Dirac-Hestenes equation (61)-(63) solves also the truncated form (144) of the neoclassical field equation. In particular, let us take the external potential \hat{A} to be the Coulomb potential, that is $\hat{A} = \hat{A}_c^\mu = \left(-\frac{Ze^2}{|\mathbf{x}|}, 0\right)$ where Z is the nucleus charge. Then solutions to the Dirac-Hestenes equation for the Coulomb potential

$$\hbar\partial\psi\mathbf{I}\sigma_3 - \frac{e}{c}\hat{A}_c\psi = mc\psi\gamma_0, \quad \hat{A}_c^\mu = \left(-\frac{Ze^2}{|\mathbf{x}|}, 0\right), \quad (147)$$

are solutions to the truncated field equation (144) and, consequently, are approximate solutions to the neoclassical field equation (140) with neglected nonlinearity G . Notice that the typical spatial scale of electron states in the Coulomb potential is the Bohr radius, and if the size parameter a is much larger than the Bohr radius, then the nonlinearity can be neglected, see [BF7]. Since the Dirac-Hestenes equation for the Coulomb potential (147) is exactly equivalent to the original Dirac equation for the same potential, [Hes2-03, VII], we can claim that the frequency spectrum of the neoclassical field equation (140) includes as an approximation the well known frequency spectrum of the Dirac equation, [Schwabl, 8.2].

Interestingly, in [SanMar] the equation (143) (called the "square of the Dirac equation") is derived by conformal differential geometry. The general setup in [SanMar], though very different from our neoclassical approach, has some common features including the underlying continuum and that the QM is not a starting point but rather an approximation.

4.1 Conservation laws

Our treatment of the charge and energy-momentum conservation is based on the Noether theorem and consequently requires the knowledge of relevant groups of transformations which leave the Lagrangian invariant.

4.1.1 Charge and current densities

The neoclassical Lagrangian (136) is invariant with respect to the global charge gauge transformation (74) as in the case of the Dirac theory. Consequently, Noether's current reduces in this case to the following expression for the electric current

$$\begin{aligned} J^\mu &= \pi^\mu * \bar{\delta}\psi = \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} * \bar{\delta}\psi = \frac{\chi}{m} \left\langle \mathbf{I}\sigma_3 (\mathcal{P}\psi)^\sim \gamma^\mu \psi \mathbf{I}\sigma_3 \epsilon \right\rangle = \\ &= -\frac{\chi}{m} \left\langle (\mathcal{P}\psi)^\sim \gamma^\mu \psi \right\rangle \epsilon = -\frac{\chi}{m} \left\langle \gamma^\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle \epsilon, \end{aligned} \quad (148)$$

where we have used expressions (137) and (74) for π^μ and $\bar{\delta}\psi$ respectively. Multiplication of the above expression for J^μ by a suitable constant and consequent transformations yield the following expressions for the current components

$$\begin{aligned} J^\mu &= \frac{e}{m} \left\langle \gamma^\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle = \frac{e}{m} \gamma^\mu \cdot \left\langle (\mathcal{P}\psi) \tilde{\psi} \right\rangle_1 = \\ &= \frac{e}{m} \left\langle \gamma^\mu \left(\chi \partial \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A} \psi \right) \tilde{\psi} \right\rangle = \frac{e}{m} \gamma^\mu \cdot \left\langle \left(\chi \partial \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A} \psi \right) \tilde{\psi} \right\rangle_1, \end{aligned} \quad (149)$$

implying the following concise form for the current vector

$$J = \frac{e}{m} \gamma_\mu J^\mu = \frac{e}{m} \left\langle (\mathcal{P}\psi) \tilde{\psi} \right\rangle_1 = \frac{e}{m} \left\langle \left(\chi \partial \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A} \psi \right) \tilde{\psi} \right\rangle_1. \quad (150)$$

Observe that our expressions (149) and (150) for the current J are exactly the same as the current expressions (81) and (82) in the Dirac theory

Notice that any even ψ according to (57) satisfies $\psi \tilde{\psi} = \tilde{\psi} \psi = \rho e^{\beta I}$. Consequently, for any vector \check{A} we have

$$\begin{aligned} \left\langle \tilde{\psi} \gamma^\mu \check{A} \psi \right\rangle &= \left\langle \gamma^\mu \check{A} \psi \tilde{\psi} \right\rangle = \left\langle \left(\gamma^\mu \cdot \check{A} + \gamma^\mu \wedge \check{A} \right) \rho e^{\beta I} \right\rangle = \\ &= \left\langle \gamma^\mu \cdot \check{A} \rho e^{\beta I} \right\rangle = \check{A}^\mu \left\langle \rho e^{\beta I} \right\rangle = \check{A}^\mu \left\langle \tilde{\psi} \psi \right\rangle. \end{aligned} \quad (151)$$

Using the above identity we can transform the current components J^μ in (149) as follows:

$$\begin{aligned} \left\langle \gamma^\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle &= \left\langle \gamma^\mu \left(\chi \partial \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A} \psi \right) \tilde{\psi} \right\rangle = \chi \left\langle \gamma^\mu \gamma^\nu \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle - \left\langle \frac{e}{c} \check{A}^\mu \psi \tilde{\psi} \right\rangle = \\ &= \frac{\chi}{2} \left\langle (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle + \frac{\chi}{2} \left\langle (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle - \left\langle \tilde{\psi} \frac{e}{c} \check{A}^\mu \psi \right\rangle = \\ &= \frac{\chi}{2} \left\langle 2g^{\mu\nu} \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle + \chi \left\langle (\gamma^\mu \wedge \gamma^\nu) \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle - \left\langle \frac{e}{c} \check{A}^\mu \psi \tilde{\psi} \right\rangle = \\ &= \left\langle (\mathcal{P}^\mu \psi) \tilde{\psi} \right\rangle + \chi \left\langle (\gamma^\mu \wedge \gamma^\nu) \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle, \end{aligned} \quad (152)$$

yielding

$$J^\mu = \frac{e}{m} \left\langle (\mathcal{P}^\mu \psi) \tilde{\psi} \right\rangle + \frac{\chi e}{m} \left\langle [\gamma^\mu, \gamma^\nu] \partial_\nu \psi \mathbf{I}\sigma_3 \tilde{\psi} \right\rangle, \text{ where } \mathcal{P}^\mu \psi = \chi \partial^\mu \psi \mathbf{I}\sigma_3 - \frac{e}{c} \check{A}^\mu \psi, \quad (153)$$

where we used the commutator product $[\gamma^\mu, \gamma^\nu]$ notation. Observe that the current expression (153) is exactly the same as the Gordon current decomposition (84) for the current in the Dirac theory if we substitute \hbar with χ , namely

$$J^\mu = J_c^\mu + J_s^\mu, \quad J_c^\mu = \frac{e}{m} \left\langle (\mathcal{P}^\mu \psi) \tilde{\psi} \right\rangle, \quad J_s^\mu = \frac{\chi e}{m} \left\langle [\gamma^\mu, \gamma^\nu] \partial_\nu \psi \mathbf{I} \sigma_3 \tilde{\psi} \right\rangle, \quad (154)$$

where J_c^μ and J_s^μ are respectively the convection and magnetization (spin) currents. Consequently, just as in the case of the Dirac theory as indicated by conservation laws (91), (92), these currents are conserved individually

$$\partial_\mu J_c^\mu = 0, \quad \partial \cdot J_s = 0. \quad (155)$$

Notice that the representation (85)-(87) for the magnetization/spin current in the Dirac theory holds in the neoclassical case as well, namely

$$\begin{aligned} J_s = J_s^\mu \gamma_\mu &= c \partial \cdot M, & J_s^\mu &= c \gamma^\mu \cdot (\partial \cdot M) = c \gamma^\mu \cdot (\gamma^\nu \cdot \partial_\nu M) = \\ &= c (\gamma^\mu \wedge \gamma^\nu) \cdot \partial_\nu M = \frac{\hbar e}{m} \left\langle (\gamma^\mu \wedge \gamma^\nu) \partial_\nu \psi \mathbf{I} \sigma_3 \tilde{\psi} \right\rangle. \end{aligned} \quad (156)$$

Observe also that the magnetization bivector M defined in (181) can be related to the spin angular momentum bivector S as follows, [Hes-75, 4], [Hes-96, 2, 3], [Hes2-03, VII.C]

$$S = \frac{\hbar}{2} R \mathbf{I} \sigma_3 \tilde{R} = \frac{\hbar}{2} R \gamma_2 \gamma_1 \tilde{R}, \quad \psi = \rho^{\frac{1}{2}} e^{\frac{i\beta}{2}} R = R \rho^{\frac{1}{2}} e^{\frac{i\beta}{2}}, \quad (157)$$

$$M = \frac{\hbar e}{2mc} \psi \mathbf{I} \sigma_3 \tilde{\psi} = \frac{\hbar e}{2mc} \psi \gamma_2 \gamma_1 \tilde{\psi} = \frac{e}{mc} e^{i\beta} \rho S. \quad (158)$$

4.1.2 Gauge invariant energy-momentum tensor

In the case of the neoclassical Lagrangian (136), the general expression for the canonical EnMT $\hat{T}^{\mu\nu}$ with the help of (137) reduces to

$$\hat{T}^{\mu\nu} = \pi^\mu * \partial^\nu \psi - \delta^{\mu\nu} L, \quad \text{where } \pi^\mu = \frac{\partial L}{\partial \psi_{,\mu}} = \frac{\chi}{m} \mathbf{I} \sigma_3 (\mathcal{P} \psi)^\sim \gamma^\mu, \quad (159)$$

implying

$$\hat{T}^{\mu\nu} = \frac{\chi}{m} \left\langle \gamma^\mu \partial^\nu \psi \mathbf{I} \sigma_3 (\mathcal{P} \psi)^\sim \right\rangle - \delta^{\mu\nu} L, \quad \text{where } \mathcal{P} \psi = \chi \partial \psi \mathbf{I} \sigma_3 - \frac{e}{c} \check{A} \psi. \quad (160)$$

The corresponding conservation law takes then the form

$$\partial_\mu \hat{T}^{\mu\nu} = -\partial^\nu L, \quad \text{where } \partial^\nu L = \frac{\partial L}{\partial x_\nu}. \quad (161)$$

Since the explicit dependence on x_ν in L comes only through the EM potential \check{A} , we find that

$$\begin{aligned} \partial^\nu L &= \partial^\nu \frac{1}{2m} \left\langle (\mathcal{P} \psi) (\mathcal{P} \psi)^\sim \right\rangle = \frac{1}{m} \left\langle [\partial^\nu (\mathcal{P} \psi)] (\mathcal{P} \psi)^\sim \right\rangle = \\ &= -\frac{1}{m} \frac{e}{c} \left\langle (\partial^\nu \check{A}) \psi (\mathcal{P} \psi)^\sim \right\rangle = -\frac{1}{m} \frac{e}{c} (\partial^\nu \check{A}^\mu) \left\langle \gamma_\mu \psi (\mathcal{P} \psi)^\sim \right\rangle = \\ &= -\frac{1}{c} \frac{e}{m} (\partial^\nu \check{A}^\mu) \left\langle \gamma_\mu (\mathcal{P} \psi) \tilde{\psi} \right\rangle = -\frac{1}{c} (\partial^\nu \check{A}^\mu) J_\mu. \end{aligned} \quad (162)$$

The canonical EnMT $\mathring{T}^{\mu\nu}$ defined by (160) is evidently not gauge invariant. To modify it into a gauge invariant form, we use the expression (149) for the current components J^μ and transform EnMT $\mathring{T}^{\mu\nu}$ as follows:

$$\begin{aligned}
\mathring{T}^{\mu\nu} + \delta_\nu^\mu L &= \frac{\chi}{m} \left\langle \gamma^\mu \partial^\nu \psi \mathbf{I} \sigma_3 (\mathcal{P}\psi)^\sim \right\rangle = \frac{1}{m} \left\langle \gamma^\mu \chi \partial^\nu \psi \mathbf{I} \sigma_3 (\mathcal{P}\psi)^\sim \right\rangle = \\
&= \frac{1}{m} \left\langle \gamma^\mu (\mathcal{P}^\nu \psi) (\mathcal{P}\psi)^\sim \right\rangle + \frac{1}{m} \left\langle \gamma^\mu \frac{e}{c} \mathring{A}^\nu \psi (\mathcal{P}\psi)^\sim \right\rangle = \\
&= \frac{1}{m} \left\langle \gamma^\mu (\mathcal{P}^\nu \psi) (\mathcal{P}\psi)^\sim \right\rangle + \frac{1}{c} \mathring{A}^\nu \frac{e}{m} \left\langle \gamma^\mu (\mathcal{P}\psi) \tilde{\psi} \right\rangle = \\
&= \frac{1}{m} \left\langle \gamma^\mu (\mathcal{P}^\nu \psi) (\mathcal{P}\psi)^\sim \right\rangle + \frac{1}{c} \mathring{A}^\nu J^\mu.
\end{aligned} \tag{163}$$

The above equality suggests to introduce the following expression for a gauge invariant EnMT $T^{\mu\nu}$:

$$T^{\mu\nu} = \mathring{T}^{\mu\nu} - \frac{1}{c} \mathring{A}^\nu J^\mu = \frac{1}{m} \left\langle \gamma^\mu (\mathcal{P}^\nu \psi) (\mathcal{P}\psi)^\sim \right\rangle - \delta_\nu^\mu L. \tag{164}$$

Indeed, using the conservation law $\partial_\mu J^\mu = 0$ and the canonical EnMT $\mathring{T}^{\mu\nu}$ conservation law (161), we obtain

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu \mathring{T}^{\mu\nu} - \frac{1}{c} \left(\partial_\mu \mathring{A}^\nu \right) J^\mu = -\partial^\nu L - \frac{1}{c} \left(\partial_\mu \mathring{A}^\nu \right) J^\mu = \\
&= \frac{1}{c} \left(\partial^\nu \mathring{A}^\mu \right) J_\mu - \frac{1}{c} \left(\partial^\mu \mathring{A}^\nu \right) J_\mu = \frac{1}{c} F^{\nu\mu} J_\mu,
\end{aligned} \tag{165}$$

where $\frac{1}{c} F^{\nu\mu} J_\mu$ is the Lorentz force.

5 Neoclassical free charge with spin

In this section we carry out a rather detailed analysis of the basic case of the free charge when $\mathring{A} = 0$. The charge Lagrangian (129) in the case of the free charge takes the form

$$L = \frac{\chi^2}{2m} \left\{ \left\langle \gamma^\beta \gamma^\alpha (\partial_\alpha \psi) (\partial_\beta \tilde{\psi}) \right\rangle - \left[\kappa_0^2 \left\langle \psi \tilde{\psi} \right\rangle + G \left(\left\langle \psi \tilde{\psi} \right\rangle \right) \right] \right\}, \quad \kappa_0 = \frac{mc}{\chi}. \tag{166}$$

The field equation (140) when $\mathring{A} = 0$ after straightforward transformations turns into *free charge spinor field equation*

$$-\chi^2 \partial_\mu \partial^\mu \psi - \left[\kappa_0^2 + G' \left(\left\langle \psi \tilde{\psi} \right\rangle \right) \right] \psi = 0. \tag{167}$$

The above equation is similar to a *scalar nonlinear Klein-Gordon* (NKG) equation that arises in our neoclassical scalar theory, [BF8], namely

$$-\frac{1}{c^2} \partial_t^2 \psi + \nabla^2 \psi - \kappa_0^2 \psi - G' (|\psi|^2) \psi = 0, \tag{168}$$

where ψ is complex-valued. The scalar equation (168) and its solutions are relevant to the analysis of the spinor field equation (167) and its basic properties are considered in the following section.

5.1 Scalar equation

Equation (168) is the Euler-Lagrange equation associated with the Lagrangian

$$L = \frac{\chi^2}{2m} \left\{ \partial_\mu \psi^* \partial^\mu \psi - [\kappa_0^2 \psi^* \psi + G(\psi^* \psi)] \right\}, \quad (169)$$

where the nonlinearity $G = G_a$ is defined by equation (1) for $s \geq 0$. The fundamental rest solution to the scalar NKG equation (168) is of the form, [BF8]

$$\psi_\pm(t, \mathbf{x}) = e^{\mp i \omega_0 t} \hat{u}(|\mathbf{x}|), \quad \text{where } \omega_0 = \frac{mc^2}{\chi} = \kappa_0 c, \quad (170)$$

$$\hat{u}(s) = \hat{u}_a(s) = a^{-3/2} \pi^{-3/4} \exp\left(-\frac{s^2}{2a^2}\right), \quad s \geq 0, \quad (171)$$

where $\hat{u}(|\mathbf{x}|)$ satisfies the equation

$$\nabla^2 \hat{u}(|\mathbf{x}|) - G'(\hat{u}^2(|\mathbf{x}|)) \hat{u}(|\mathbf{x}|) = 0. \quad (172)$$

Observe that the solution ψ_\pm is the product of a time-harmonic factor $e^{\mp i \omega_0 t}$ and the Gaussian factor $\hat{u}(|\mathbf{x}|)$ describing the localized shape of the wave. An STA representation of the above solution which is manifestly coordinate free is as follows:

$$\psi(x) = \psi_\mp(v, x) = e^{\mp i \kappa_0 x \cdot v} \hat{u}\left(\sqrt{(x \cdot v)^2 - x^2}\right), \quad \kappa_0 = \frac{\omega_0}{c}, \quad (173)$$

where v is proper velocity of the free electron, and one can think of v as describing the rest frame of the electron.

It is instructive to find a representation of the solution $\psi(v, x)$ in (173) in the frame of an arbitrary inertial observer γ_0 by relating it to the inertial observer $v = \gamma'_0$. Such a representation can be effectively obtained by introducing a subspace of the vector space $\text{Span}\{v, \gamma_0\}$ and the corresponding orthogonal decomposition as in [PauRT, p. 10], [Hes-74, 1]

$$x = x_\parallel + x_\perp, \quad \text{where } x_\parallel \in \text{Span}\{v, \gamma_0\} \text{ and } x_\perp \text{ is orthogonal to } \text{Span}\{v, \gamma_0\}. \quad (174)$$

We will need also the corresponding relative velocity \mathbf{v} which is defined by

$$\frac{\mathbf{v}}{c} = \frac{v \wedge \gamma_0}{v \cdot \gamma_0}, \quad v = \gamma \left(1 + \frac{\mathbf{v}}{c}\right) \gamma_0 = \gamma \gamma_0 \left(1 - \frac{\mathbf{v}}{c}\right), \quad (175)$$

$$\text{where } \gamma = v \cdot \gamma_0 = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} \text{ is the Lorentz factor.}$$

Then we obtain the following identities:

$$x \cdot v = \gamma \left[x_0 - \mathbf{x}_\parallel \cdot \frac{\mathbf{v}}{c} \right] \text{ where } \mathbf{x}_\parallel = x_\parallel \wedge \gamma_0, \quad (176)$$

$$\sqrt{(x \cdot v)^2 - x^2} = \left| \gamma \left(\mathbf{x}_\parallel - x_0 \frac{\mathbf{v}}{c} \right) + \mathbf{x}_\perp \right|, \text{ where } \mathbf{x}_\perp = x_\perp \wedge \gamma_0. \quad (177)$$

Observe that the right-hand sides correspond to standard Lorentz boost transformations for respectively time and space components of the vector x , [PauRT, p. 10].

Consequently, we get the following representation of the scalar solution (173) in the frame of an arbitrary observer γ_0 :

$$\psi(x) = \exp \left\{ \mp i \kappa_0 \gamma \left[x_0 - \mathbf{x}_{\parallel} \cdot \frac{\mathbf{v}}{c} \right] \right\} \hat{u} \left(\left| \gamma \left(\mathbf{x}_{\parallel} - x_0 \frac{\mathbf{v}}{c} \right) + \mathbf{x}_{\perp} \right| \right). \quad (178)$$

The charge and current densities in the scalar case are given by the expressions

$$\rho = -\frac{\chi q}{mc^2} \operatorname{Im} \frac{\partial_t \psi}{\psi} |\psi|^2, \quad \mathbf{J} = \frac{\chi q}{m} \operatorname{Im} \frac{\nabla \psi}{\psi} |\psi|^2, \quad (179)$$

implying for the solutions ψ_{\pm} in (170) the following representation for the total conserved charge

$$q_{\pm} = \int_{\mathbb{R}^3} \rho_{\pm}(t, \mathbf{x}) \, d\mathbf{x} = \pm \frac{\chi q \omega_0}{mc^2} = \pm q. \quad (180)$$

The conserved energy \mathcal{E} and momentum \mathbf{p} densities are

$$\mathcal{E} = \frac{\chi^2}{2m} \left[\frac{1}{c^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* + \tilde{\nabla} \psi \tilde{\nabla}^* \psi^* + G(\psi^* \psi) + \kappa_0^2 \psi \psi^* \right], \quad (181)$$

$$\mathbf{p} = (p^1, p^2, p^3) = -\frac{\chi^2}{2mc^2} \left(\tilde{\partial}_t \psi \tilde{\nabla}^* \psi^* + \tilde{\partial}_t^* \psi^* \tilde{\nabla} \psi \right). \quad (182)$$

In particular, the expression (181) for the energy density \mathcal{E} implies the following representation for the total conserved energy E_{\pm} for the wave function ψ_{\pm} defined by (173)

$$E_{\pm} = \chi \omega_0 \left(1 + \frac{a_C^2}{2a^2} \right) > 0. \quad (183)$$

A very detailed theory of the scalar Klein-Gordon equation including the Lagrangian treatment can be found in [GreRQM, 1.5], [Wach, 1.1, p. 20]. In particular one can find there studies of EnMT $T^{\mu\nu}$ showing *that the energy of solutions for both the positive and negative frequencies is always positive.*

5.2 Solutions to the spinor field equation

We seek a solution to the free charge spinor equation (167) which is expected to incorporate the features of the scalar solution as in (173) and the plane-wave solution (105) to the Dirac equation. We find that such a spinor solution does exist and is of the form $\psi(x) = \psi_{\mp}(v, x)$

$$\psi_{\mp}(v, x) = \psi_0 e^{\mp i \sigma_3 \kappa_0 x \cdot v} \hat{u} \left(\sqrt{(x \cdot v)^2 - x^2} \right), \quad \kappa_0 = \frac{\omega_0}{c}, \quad (184)$$

where the Gaussian factor \hat{u} is defined by (171) and ψ_0 is a normalized constant spinor from the even subalgebra $\text{Cl}_+(1, 3)$ satisfying the canonical representation (111) and the following special conditions

$$\langle \psi_0 \tilde{\psi}_0 \rangle = 1 \text{ or } \langle \psi_0 \tilde{\psi}_0 \rangle = -1, \quad (185)$$

that is $\beta_0 = 0$ or $\beta_0 = \pi$. One can readily see a distinct feature of the spinor solution (184) compared to the plane-wave solution (105) to the Dirac equation. It is the amplitude factor

$\dot{u} \left(\sqrt{(x \cdot v)^2 - x^2} \right)$ which can be attributed to the nonlinearity $G \left(\langle \psi \tilde{\psi} \rangle \right)$ in the spinor equation (167). The origin of the special constraints (185) can be traced to the particular way ψ enters the nonlinearity, namely as $G \left(\langle \psi \tilde{\psi} \rangle \right)$. For ψ of the form (184) to be a solution to the free charge spinor equation (167), there has to be an effective reduction to the scalar equation (168) with the nonlinearity $G'(\dot{u}^2)$. The constraint (185) is essential for such a reduction. Indeed, if the spinor wave function ψ is defined by (184) and satisfies the condition (185) then

$$\langle \psi \tilde{\psi} \rangle = \langle \psi_0 \tilde{\psi}_0 \rangle \dot{u}^2 \text{ implying } G' \left(\langle \psi \tilde{\psi} \rangle \right) = G'(\pm \dot{u}^2) = G'(\dot{u}^2). \quad (186)$$

When establishing identities (186) we used the identity (135) and that $G'(s)$ defined by (131) is an even function. The identities (186) allow to reduce the spinor equation (167) to the scalar equation (168).

5.3 Charge and current densities

Let us consider the solution $\psi_{\mp}(x)$ as in (184) with $v = \gamma_0$, that is

$$\begin{aligned} \psi_{\mp}(x) &= \psi_{\mp}(\gamma_0, x) = \psi_0 e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} u(x), \text{ where} \\ x_0 &= x \cdot \gamma_0, \quad u(x) = \dot{u} \left((x \cdot \gamma_0)^2 - x^2 \right), \end{aligned} \quad (187)$$

where the Gaussian factor \dot{u} is defined by (171) and ψ_0 satisfies the condition (185), that is $\langle \psi_0 \tilde{\psi}_0 \rangle = \pm 1$. Then the current component J^μ defined by (149) for the free charge with $\dot{A} = 0$ takes the following form:

$$J^\mu = \frac{q}{m} \langle \gamma^\mu \chi \partial \psi \mathbf{I} \sigma_3 \tilde{\psi} \rangle = \frac{q}{m} \langle \gamma^\mu \chi \partial \psi \mathbf{I} \sigma_3 \tilde{\psi} \rangle. \quad (188)$$

Since for the rest solution ψ_{\mp} defined by (187) $\partial_0 u = 0$, the following relations hold:

$$\begin{aligned} \partial_0 \psi_{\mp} &= \psi_0 (\mp \mathbf{I} \sigma_3 \kappa_0) e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} u = \psi_0 e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} (\mp \mathbf{I} \sigma_3 \kappa_0) u = \psi_{\mp} (\mp \mathbf{I} \sigma_3 \kappa_0) u, \\ \partial_j \psi_{\mp} &= \psi_0 e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} \partial_j u = \psi_{\mp} \partial_j \ln u, \end{aligned} \quad (189)$$

$$\partial \psi_{\mp} = \gamma^\mu \partial_\mu \psi_0 e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} u = \left[\gamma^0 \psi_0 (\mp \mathbf{I} \sigma_3 \kappa_0) u + \sum_{1 \leq j \leq 3} \gamma^j \psi_0 \partial_j u \right] e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0}. \quad (190)$$

Notice that in view of the identity (135) we have

$$\tilde{\psi}_{\mp} = e^{\pm \mathbf{I} \sigma_3 \kappa_0 x_0} \tilde{\psi}_0 u(x). \quad (191)$$

The above relation combined with the equality (190) yields

$$\begin{aligned} \langle \gamma^0 \partial \psi_{\mp} \mathbf{I} \sigma_3 \tilde{\psi}_{\mp} \rangle &= \langle \psi_0 (\mp \mathbf{I} \sigma_3 \kappa_0) e^{\mp \mathbf{I} \sigma_3 \kappa_0 x_0} u \mathbf{I} \sigma_3 e^{\pm \mathbf{I} \sigma_3 \kappa_0 x_0} \tilde{\psi}_0 u \rangle + \\ &\quad + \sum_{1 \leq j \leq 3} u \partial_j u \langle \gamma^0 \gamma^j \psi_0 \mathbf{I} \sigma_3 \tilde{\psi}_0 \rangle \\ &= \pm u^2 \kappa_0 \langle \psi_0 \tilde{\psi}_0 \rangle + \frac{1}{2} \sum_{1 \leq j \leq 3} (\partial_j u^2) \langle \gamma^0 \gamma^j \psi_0 \mathbf{I} \sigma_3 \tilde{\psi}_0 \rangle. \end{aligned} \quad (192)$$

Hence, based on (190) and the above equality, we obtain

$$\begin{aligned} J_{\mp}^0 &= c\rho_{\mp} = \frac{\chi q}{m} \left\langle \gamma^0 (\partial\psi_{\mp}) \mathbf{I}\sigma_3 \tilde{\psi}_{\mp} \right\rangle = \\ &= \frac{\chi q}{m} \left[\pm u^2 \kappa_0 \left\langle \psi_0 \tilde{\psi}_0 \right\rangle + \sum_{1 \leq j \leq 3} \frac{1}{2} (\partial_j u^2) \left\langle \gamma^0 \gamma^j \psi_0 \mathbf{I}\sigma_3 \tilde{\psi}_0 \right\rangle \right], \end{aligned} \quad (193)$$

implying the following expressions for the total charge

$$\mathbf{q}_{\mp} = \int \rho_{\mp} \, d\mathbf{x} = \pm \frac{\chi \kappa_0}{mc} q \left\langle \psi_0 \tilde{\psi}_0 \right\rangle = \pm q \left\langle \psi_0 \tilde{\psi}_0 \right\rangle. \quad (194)$$

Observe that the different signs \pm of the charge \mathbf{q}_{\mp} above can be traced to the different signs of the frequencies in expressions (187) for $\psi_{\mp}(x)$.

5.4 Energy-momentum density

Let us find the energy-momentum density for the solutions of the spinorial field equations $\psi_{\mp}(x)$ defined by (187). To facilitate efficient computation, we use the following identities. Suppose p_{α} , c_{α} and φ are multivectors satisfying

$$p_{\alpha} = \varphi c_{\alpha}, \quad c_{\alpha} c_{\beta} = c_{\beta} c_{\alpha}, \quad \tilde{c}_{\alpha} = c_{\alpha}. \quad (195)$$

Then

$$p_{\alpha} \tilde{p}_{\beta} = p_{\beta} \tilde{p}_{\alpha} = \varphi c_{\alpha} c_{\beta} \tilde{\varphi}. \quad (196)$$

Observe now that if we consider the derivatives $\partial_{\alpha} \psi_{\mp}$ defined by (189) and set

$$p_{\alpha} = \partial_{\alpha} \psi_{\mp}, \quad c_0 = \mp \mathbf{I}\sigma_3 \kappa_0 u, \quad c_j = \partial_j \ln u, \quad \varphi = \psi_{\mp}, \quad (197)$$

then the relations (196) are satisfied, that is

$$(\partial_{\alpha} \psi_{\mp}) (\partial_{\beta} \tilde{\psi}_{\mp}) = (\partial_{\beta} \psi_{\mp}) (\partial_{\alpha} \tilde{\psi}_{\mp}). \quad (198)$$

Notice that the following relations hold for the solutions ψ_{\mp} defined by (187)

$$\left\langle \psi_{\mp} \tilde{\psi}_{\mp} \right\rangle = \left\langle \psi_0 \tilde{\psi}_0 \right\rangle u^2, \quad (199)$$

$$\partial_0 \psi_{\mp} = \mp \psi_0 \mathbf{I}\sigma_3 \kappa_0 e^{\mp \mathbf{I}\sigma_3 \kappa_0 x_0} u, \quad \partial_0 \tilde{\psi}_{\mp} = \pm u \mathbf{I}\sigma_3 \kappa_0 e^{\pm \mathbf{I}\sigma_3 \kappa_0 x_0} \tilde{\psi}_0, \quad (200)$$

$$(\partial_0 \psi_{\mp}) (\partial_0 \tilde{\psi}_{\mp}) = -u^2 \psi_0 \mathbf{I}\sigma_3 \kappa_0 e^{\mp \mathbf{I}\sigma_3 \kappa_0 x_0} \mathbf{I}\sigma_3 \kappa_0 e^{\pm \mathbf{I}\sigma_3 \kappa_0 x_0} \tilde{\psi}_0 = u^2 \kappa_0^2 \psi_0 \tilde{\psi}_0, \quad (201)$$

$$\partial_j \psi_{\mp} = \psi_0 e^{\mp \mathbf{I}\sigma_3 \kappa_0 x_0} \partial_j u, \quad \partial_j \tilde{\psi}_{\mp} = e^{\pm \mathbf{I}\sigma_3 \kappa_0 x_0} \tilde{\psi}_0 \partial_j u, \quad (202)$$

$$(\partial_j \psi_{\mp}) (\partial_j \tilde{\psi}_{\mp}) = \psi_0 \tilde{\psi}_0 (\partial_j u)^2. \quad (203)$$

Using then the expression (166) for the Lagrangian L and the identities (198), we find its value on the field fields ψ_{\mp} to be

$$\begin{aligned} L &= \frac{\chi^2}{2m} \left\{ \left\langle \gamma^{\beta} \gamma^{\alpha} (\partial_{\alpha} \psi_{\mp}) (\partial_{\beta} \tilde{\psi}_{\mp}) \right\rangle - \left[\kappa_0^2 \left\langle \psi_{\mp} \tilde{\psi}_{\mp} \right\rangle + G \left(\left\langle \psi_{\mp} \tilde{\psi}_{\mp} \right\rangle \right) \right] \right\} = \\ &= \frac{\chi^2}{2m} \left\{ \left\langle (\partial^{\alpha} \psi_{\mp}) (\partial_{\alpha} \tilde{\psi}_{\mp}) \right\rangle - \left[\kappa_0^2 \left\langle \psi_{\mp} \tilde{\psi}_{\mp} \right\rangle + G \left(\left\langle \psi_{\mp} \tilde{\psi}_{\mp} \right\rangle \right) \right] \right\}. \end{aligned} \quad (204)$$

The canonical EnMT $\mathring{T}^{\mu\nu}$ defined by (160) takes the following form for the free charge with $\mathring{A} = 0$

$$\begin{aligned}\mathring{T}^{\mu\nu} &= \frac{\chi}{m} \left\langle \gamma^\mu \partial^\nu \psi \mathbf{I} \sigma_3 (\chi \partial \psi \mathbf{I} \sigma_3)^\sim \right\rangle - \delta_\nu^\mu L = \frac{\chi^2}{m} \left\langle \gamma^\mu \partial^\nu \psi (\partial \psi)^\sim \right\rangle - \delta_\nu^\mu L = \\ &= \frac{\chi^2}{m} \left\langle \gamma^\mu \partial^\nu \psi \partial_\alpha \tilde{\psi} \gamma^\alpha \right\rangle - \delta_\nu^\mu L = \frac{\chi^2}{m} \left\langle \gamma^\alpha \gamma^\mu \partial^\nu \psi \partial_\alpha \tilde{\psi} \right\rangle - \delta_\nu^\mu L,\end{aligned}\quad (205)$$

where we used the identity (135). The above formula yields the following representation for the energy density \mathcal{E}

$$\mathcal{E} = \mathring{T}^{00} = \frac{\chi^2}{m} \left\langle \gamma^\alpha \gamma^0 \partial^0 \psi \partial_\alpha \tilde{\psi} \right\rangle - L. \quad (206)$$

In particular, for $\psi = \psi_\mp$, we use (206) and (204) to obtain

$$\begin{aligned}\mathcal{E}_\mp &= \frac{\chi^2}{m} \left\langle \gamma^\alpha \gamma^0 \partial^0 \psi_\mp \partial_\alpha \tilde{\psi}_\mp \right\rangle - L = \\ &= \frac{\chi^2}{m} \left\langle \gamma^\alpha \gamma^0 \partial^0 \psi_\mp \partial_\alpha \tilde{\psi}_\mp \right\rangle - \frac{\chi^2}{2m} \left\langle (\partial^\alpha \psi_\mp) (\partial_\alpha \tilde{\psi}_\mp) \right\rangle + \frac{\chi^2}{2m} \left[\kappa_0^2 \left\langle \psi_\mp \tilde{\psi}_\mp \right\rangle + G \left(\left\langle \psi_\mp \tilde{\psi}_\mp \right\rangle \right) \right].\end{aligned}\quad (207)$$

The expression above can be transformed into

$$\begin{aligned}\mathcal{E}_\mp &= \frac{\chi^2}{2m} \left\{ \left\langle (\partial_0 \psi_\mp) (\partial_0 \tilde{\psi}_\mp) \right\rangle + \sum_{1 \leq j \leq 3} \left\langle (\partial_j \psi_\mp) (\partial_j \tilde{\psi}_\mp) \right\rangle + \left[\kappa_0^2 \left\langle \psi \tilde{\psi} \right\rangle + G \left(\left\langle \psi \tilde{\psi} \right\rangle \right) \right] \right\} - \\ &\quad - \sum_{1 \leq j \leq 3} \frac{\chi^2}{m} \left\langle \gamma^j \gamma^0 \partial^0 \psi_\mp \partial_j \tilde{\psi}_\mp \right\rangle.\end{aligned}\quad (208)$$

Using the identities (199)-(203) we transform the above representation further into

$$\mathcal{E}_\mp = \frac{\chi^2 \left\langle \psi_0 \tilde{\psi}_0 \right\rangle}{2m} \left[2u^2 \kappa_0^2 + \sum_{1 \leq j \leq 3} (\partial_j u)^2 + G(u^2) \right] \pm \frac{\chi^2}{2m} \kappa_0 \sum_{1 \leq j \leq 3} \left\langle \gamma^j \gamma^0 \psi_0 \mathbf{I} \sigma_3 \tilde{\psi}_0 \right\rangle \partial_j u^2. \quad (209)$$

Then, using the above formula and relations (172), (132), we obtain the following representation for the total energy \mathbf{E}_\mp of the free charge solutions ψ_\mp :

$$\begin{aligned}\mathbf{E}_\mp &= \int \mathcal{E}_\mp \, d\mathbf{x} = \frac{\chi^2 \left\langle \psi_0 \tilde{\psi}_0 \right\rangle}{2m} \int \left[2u^2 \kappa_0^2 - \sum_{1 \leq j \leq 3} (\partial_j^2 u) u + G(u^2) \right] d\mathbf{x} = \\ &= \frac{\chi^2 \left\langle \psi_0 \tilde{\psi}_0 \right\rangle}{2m} \int [2u^2 \kappa_0^2 - G'(u^2) u^2 + G(u^2)] d\mathbf{x} = \\ &= \frac{\chi^2 \left\langle \psi_0 \tilde{\psi}_0 \right\rangle}{2m} \int \left(2\kappa_0^2 + \frac{1}{a^2} \right) u^2 d\mathbf{x} = \frac{\chi^2 \left\langle \psi_0 \tilde{\psi}_0 \right\rangle}{2m} \left(2\kappa_0^2 + \frac{1}{a^2} \right) = \\ &= \left\langle \psi_0 \tilde{\psi}_0 \right\rangle \chi \omega_0 \left(1 + \frac{c^2}{2\omega_0^2 a^2} \right) = \left\langle \psi_0 \tilde{\psi}_0 \right\rangle \chi \omega_0 \left(1 + \frac{a_C^2}{2a^2} \right), \quad a_C = \kappa_0^{-1} = \frac{\chi}{mc}.\end{aligned}\quad (210)$$

Observe now that if we want the energy \mathbf{E}_\pm defined by (210) to be positive, and we do, then according to the canonical spinor representation (111) for ψ_0 and constraints (185) we require

$\langle \psi_0 \tilde{\psi}_0 \rangle = \cos \beta_0 = 1$, that is $\beta_0 = 0$. This requirement in view of (111) is equivalent to the following constraint for free charge solutions

$$\psi_0 \tilde{\psi}_0 = 1, \text{ that is } \psi_0 \text{ is the Lorentz rotor.} \quad (211)$$

Consequently, under the above constraint the formula (210) turns into

$$\mathbf{E}_{\mp} = \chi \omega_0 \left(1 + \frac{a_C^2}{2a^2} \right), \quad a_C = \kappa_0^{-1} = \frac{\chi}{mc}. \quad (212)$$

Let us take a closer look at the origin of the factor $\langle \psi_0 \tilde{\psi}_0 \rangle$ in expressions (210) and (194) for \mathbf{E}_{\mp} and \mathbf{q}_{\mp} . The similar dependence on this factor of evidently spinorial nature occurs in the quadratic part of the charge Lagrangian L (without the nonlinear term G) defined by (129). Observe that multiplication of the Lagrangian by any constant positive or negative does not change the Euler-Lagrange equation but it does alter the energy and the charge densities defined canonically by the Lagrangian. *To summarize, the presence of the factor $\langle \psi_0 \tilde{\psi}_0 \rangle$, which can be positive or negative altering the sign of the Lagrangian, the energy and the charge, is special to the spinorial wave functions since for complex-valued ones the similar factor $\psi_0 \psi_0^*$ is always positive.*

6 Neoclassical solutions interpretation and comparison with the Dirac theory

As to general aspects of the interpretation of the wave function and observables in the STA settings we rely mostly on works of D. Hestenes, see [Hes-75, 4], [Hes-96, 2], [Hes2-03, VII.D] and references therein. The key of those aspects are as follows. First of all, based on the general canonical representation (57) for the wave function $\psi = \psi(x)$, we assign at each spacetime point x the *local rotor* $R = R(x)$. This rotor determines the Lorentz rotation of a given fixed frame of vectors $\{\gamma_{\mu}\}$ into the *local rest frame of vectors* $\{e_{\mu} = e_{\mu}(x)\}$ given by

$$e_{\mu} = e_{\mu}(x) = R \gamma_{\mu} \tilde{R}, \quad R = R(x). \quad (213)$$

Importantly, in view of the canonical representation (57) we have

$$\psi \gamma_{\mu} \tilde{\psi} = \varrho R \gamma_{\mu} \tilde{R} = \varrho e_{\mu}. \quad (214)$$

The interpretation of the above fields in the Dirac theory is as follows. The vector field

$$\psi \gamma_0 \tilde{\psi} = \varrho e_0 = \varrho v \quad (215)$$

is the *Dirac current* (*probability current* in the standard Born interpretation) that determines the local rest frame v . *The local spin vector density* is defined by

$$s = \frac{1}{2} \hbar \psi \gamma_3 \tilde{\psi} = \frac{1}{2} \hbar \varrho e_3. \quad (216)$$

The *spin angular momentum* $S = S(x)$ (*proper spin*) is a bivector field related to the spin vector field $s = s(x)$ by, [Hes-75, 4], [Hes-96, 2]

$$S = \frac{1}{2} \hbar e_2 e_1 = \frac{1}{2} \hbar R I \sigma_3 \tilde{R} = \frac{1}{2} \hbar R \gamma_2 \gamma_1 \tilde{R} = \frac{1}{2} \hbar R I \sigma_3 \tilde{R} = I s e_0 = I (s \wedge e_0). \quad (217)$$

Notice that according to (157), (158) that the *proper spin density* is ϱS and the *magnetization* or *magnetic moment density* M of the charge is defined by the following expression,

$$M = \frac{\hbar e}{2mc} \psi \gamma_2 \gamma_1 \tilde{\psi} = e^{I\beta} \frac{q}{mc} \varrho S. \quad (218)$$

Let us turn now to our neoclassical free charge solutions (187) satisfying energy positivity constraint (211), that is

$$\psi_{\mp}(x) = \psi_0 e^{\mp I \sigma_3 \kappa_0 x_0} u(x), \quad \text{where } u(x) = \hat{u}((x \cdot \gamma_0)^2 - x^2), \quad \psi_0 \tilde{\psi}_0 = 1, \quad (219)$$

$$\hat{u}(s) = a^{-3/2} \pi^{-3/4} \exp\left(-\frac{s^2}{2a^2}\right), \quad s \geq 0, \quad (220)$$

where the time-like vector unit vector γ_0 describes the constant rest frame $v = \gamma_0$ of charge for every x . Then formulas (194) and (212) yield the following ultimate expressions for the total charge \mathbf{q}_{\mp} and the total energy \mathbf{E}_{\mp} of the solutions ψ_{\mp}

$$\mathbf{q}_{\mp} = \int \rho_{\mp} d\mathbf{x} = \pm q, \quad \text{for the total charge,} \quad (221)$$

$$\mathbf{E}_{\mp} = \int \mathcal{E}_{\mp} d\mathbf{x} = \chi \omega_0 \left(1 + \frac{a_C^2}{2a^2}\right), \quad \text{for the total energy,} \quad (222)$$

where

$$\kappa_0 = \frac{mc}{\chi}, \quad \omega_0 = \kappa_0 c = \frac{mc^2}{\chi}, \quad a_C = \kappa_0^{-1} = \frac{\chi}{mc}, \quad (223)$$

and χ is a constant approximately equal to the Planck constant \hbar . Notice that subindices \mp in \mathbf{q}_{\mp} and \mathbf{E}_{\mp} are picked so that if the charge is electron then its index is "−" to match the sign of the charge.

Observe that formula (221) demands the total charges \mathbf{q}_{\mp} associated with the wave functions ψ_- and ψ_+ to have opposite signs. Following to the established tradition we call them *charge and anticharge* (for instance, electron and positron). To see "spinning" of the local rest frame as a defining basis for charge and anticharge as two different states of the same single charge we introduce the following representation of $\psi_{\mp}(x)$ based on (219) and (211)

$$\psi_{\mp}(x) = R_{\mp}(x) u(x), \quad \text{where } R_{\mp} = \psi_0 e^{\mp I \sigma_3 \kappa_0 x_0}, \quad \psi_0 \tilde{\psi}_0 = 1 \quad R_{\mp} \tilde{R}_{\mp} = 1. \quad (224)$$

Then following to D. Hestenes, [Hes-81, 6, 9] and using the above representation we obtain a formula involving the rotational velocities Ω_{\mp} :

$$\frac{d\psi_{\mp}}{dt} = \frac{cd\psi_{\mp}}{dx_0} = c \frac{1}{2} \Omega_{\mp} \psi_{\mp}, \quad \Omega_{\mp} = \mp 2\kappa_0 c R_{\mp} (I \sigma_3) \tilde{R}_{\mp}. \quad (225)$$

In other words, *the local rest frame rotates "about the spin axis" $s = R_{\mp}(\gamma_3) \tilde{R}_{\mp}$ with the angular speed $|\Omega_{\mp}| = 2\kappa_0 c = 2\omega_0$ equal to twice the frequency ω_0 corresponding the rest energy $mc^2 = \chi \omega_0$. Observe that according to relations (225) the *charge and "anticharge"* (for instance, electron and positron) wave functions ψ_- and ψ_+ differ only by ascribing opposite sense to the rotation described by the factor $e^{\mp I \sigma_3 \kappa_0 x_0}$ in (224).*

Let consider now the magnetization M and the proper spin ϱS bivector densities defined by (217),(218) for the neoclassical free charge solutions ψ_{\mp} described by formulas (219) and

(224). Notice that for the free charge at rest we have $\rho = u^2(x)$ and the following relation holds for the magnetization and proper spin densities

$$M_{\mp} = \frac{\hbar e}{2mc} \psi_{\mp} \gamma_2 \gamma_1 \tilde{\psi}_{\mp} = \frac{q}{mc} u^2(x) S_{\mp}, \text{ where } S_{\mp} = \frac{1}{2} \hbar \psi_0 \mathbf{I} \sigma_3 \tilde{\psi}_0, \quad (226)$$

where we took into account that $\beta = 0$ since $\psi_0 \tilde{\psi}_0 = 1$. Observe also that the above formula shows that the both states ψ_{\mp} have the same and constant M_{\mp} and S_{\mp} . Integrating then the magnetization density of the free charge at rest M_{\mp} in (226) we obtain the total generalized magnetic momentum bivector

$$\mathcal{M}_{\mp} = \int_{\mathbb{R}^3} M_{\mp} d\mathbf{x} = \frac{q}{mc} S_{\mp}. \quad (227)$$

For general issues of the STA treatment of localized charge distributions and the proper momentum bivector M see [Hes2-74, 1].

Let us compare now the neoclassical free charge solutions (219) with the Dirac free charge solutions (104). First of all, the spinorial aspect of the proposed here neoclassical theory is identical to that in the Dirac theory since in the both cases the wave function ψ takes values in the even algebra $\text{Cl}_+(1, 3)$. The governing field equation for the neoclassical spinor field is (167) and the Dirac spinor field satisfies the Dirac equation (61). Structurally the neoclassical field equation (167) can be viewed as a spinorial version of the Klein-Gordon equation with added nonlinearity and consequently it is related to the Dirac equation. In particular, *solutions to the Dirac equations are also solutions to our field equations if the nonlinearity there is neglected*. But when it comes to the structure of solutions the first significant difference of the neoclassical free charge solution (219) compare to the Dirac free charge plane wave solution is the Gaussian factor $u(x)$. In other words, *the neoclassical free charge solution is a localized soliton-like wave whereas the Dirac free charge solution is a plane wave*.

In what follows we compare other features of the neoclassical free charge wave function and the Dirac free charge plane wave function.

6.1 The gyromagnetic ratio and currents

Recall that the gyromagnetic ratio g is defined as a coefficient that relates the magnetic dipole moment \mathbf{m} and the angular momentum \mathbf{L} for a system of localized currents, namely

$$\mathbf{m} = \frac{gq}{2mc} \mathbf{L}. \quad (228)$$

Comparing the above relation (228) with (227) we conclude that in our theory the gyromagnetic ratio $g = 2$ just as in the Dirac theory. *The expressions (154)-(158) for the current and its Gordon decomposition which includes the magnetization (spin) current in the neoclassical are identical to the same in the Dirac theory (84)-(87)*. This identity of the above mentioned currents is very important since they were extensively analyzed and thoroughly tested experimentally. The value of the gyromagnetic ratio $g = 2$ in our theory is in fact not so surprising since the minimal coupling as in (130) *implies that $g = 2$* , [ItzZub, 2-2-3]. Interestingly, there is an example of a *classical particle with the gyromagnetic ratio $g = 2$* , [Hes2-03, V].

6.2 The energies and frequencies

The issue of negative energies in the Dirac theory constitutes a well known serious problem discussed extensively in the literature, see for instance [GreRQM, 12], [ItzZub, 2.4.2], [Wach, 2.1.6] and references therein. One of the proposed ways to deal with it is *to reinterpret an electron state of negative energy/frequency as a positron state of positive energy/frequency using the charge conjugation transformation (125)*. This operation in the conventional setting involves complex conjugation of the Dirac wave function and reverses the sign of the charge, its frequency, energy, momentum, and spin, [HalMar, 5.4], [Wach, 2.1.6, p. 109]. Effectively, the charge conjugation operation changes the sign of the frequency of the wave function which satisfies then a complex conjugate version of the original Dirac equation with the opposite sign of the charge there.

In quantum mechanics and in the Dirac theory in particular, the energy is identified with the frequency via the Planck-Einstein relation $\mathcal{E} = \hbar\omega$. In contrast, in our neoclassical theory the energy and the frequency are two distinct though closely related concepts, see [BF7]. The relation between them in the relativistic case by no means is as explicit as the Planck-Einstein relation $\mathcal{E} = \hbar\omega$. Importantly, in our theory the frequencies may be positive or negative when the energy is positive. The positivity of the energies of the free charge in our theory was obtained by simply limiting the values of the spinor constant ψ_0 for the free charge solutions ψ_{\mp} in (219) to be a Lorentz rotor, that is to satisfy the energy positivity constraint (211), i.e. $\psi_0\tilde{\psi}_0 = 1$. Consequently, in our theory a positron state differs from the quantum mechanical positron state and, importantly, it is not obtained by applying the charge conjugation (125) to an electron state. Also there were no changes of frequencies or any transformation of the evolution equation. If the energy positivity constraint (211) is satisfied, then according to (222) the energies E_{\mp} of the free charge at rest satisfy approximately $E_{\mp} \approx \hbar|\omega_0|$, and they stay positive whereas the corresponding frequencies $\mp\kappa_0c = \mp\omega_0$ of the solutions ψ_{\mp} in (219) can be positive or negative. The above analysis indicates a significant difference in the treatment of negative energies in our theory compare to the Dirac theory or the QM. One may notice though that we analyzed this far only energies of free charges. We expect the treatment of a charge in external field to be more complex, but this study is left for the future work.

6.3 Antimatter states

Similarly to the Dirac theory our theory naturally integrates into it the concept of an antiparticle. According to (224) there are two directions of "spinning" in the rest frame and that naturally leads to the concepts of charge and anticharge with the frequencies of the opposite signs. Note that since the value of a charge is preserved even in external EM field the charge can not turn into the anti-charge as a result of electromagnetic interactions. All the properties of the charge and anticharge are exactly the same except for the difference in sign. We would like stress once again a noticeable difference between the antimatter states in our theory and the same in the Dirac theory. In our theory the matter and antimatter states correspond to $\langle\psi_0\tilde{\psi}_0\rangle = 1$, whereas in the Dirac theory the usual way to introduce the antimatter state (positron) is by applying charge conjugation (125) that requires $\langle\psi_0\tilde{\psi}_0\rangle = \langle e^{\beta_0\mathbf{I}}\rangle = \cos\beta_0 = -1$ as in (111), (123). Consequently, in order to introduce the antimatter state in the Dirac theory one has to invoke the parameter β of the canonical

spinor representation (58), but the interpretation of the parameter β has known difficulties, [Hes-96, 3], [Hes2-03, VII.D, G].

Acknowledgment. The research was supported through Dr. A. Nachman of the U.S. Air Force Office of Scientific Research (AFOSR), under grant number FA9550-11-1-0163.

References

- [Adl] Adler S., *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford, 1995.
- [AitHey] Aitchison I. and Hey A., *Gauge Theories in Particle Physics, Vol. 1, From Relativistic Quantum Mechanics to QED*, 3rd ed., IoP 2003.
- [BF6] Babin A. and Figotin A., *Some mathematical problems in a neoclassical theory of electric charges*, *Discrete and Continuous Dynamical Systems A*, **27**(4), 1283-1326 (2010).
- [BF7] Babin A. and Figotin A., *Electrodynamics of balanced charges*, *Found. Phys.*, **41**: 242–260, (2011).
- [BF8] Babin A. and Figotin A., *Relativistic Dynamics of Accelerating Particles Derived from Field Equations*, *Found. Phys.*, **42**: 996–1014, (2012).
- [BF9] Babin A. and Figotin A., *Relativistic Point Dynamics and Einstein Formula as a Property of Localized Solutions of a Nonlinear Klein-Gordon Equation*, *Comm. Math. Phys.* 2013
- [BagGit] Bagrov V. and Gitman D., *Exact solutions of relativistic wave equations*, Kluwer, 1990.
- [Basdevant] Basdevant J. and Dalibard J., *Quantum mechanics*, Springer, 2002.
- [Bar] A. Barut, *Electrodynamics of and Classical Theory of Fields and Particles*, Dover, 1980.
- [Bay] Baylis W., *Electrodynamics. A modern Geometric Approach*, Birkhouser, 1998.
- [BorWol] Born M. and Wolf E., *Principles of optics*, 7th ed., Cambridge, 2005.
- [Bou] Boudet R., *Quantum Mechanics in the Geometry of Space-Time*, Springer, 2011.
- [Bre] Breit G., *Approximately Relativistic Equations*, *Phys. Rev*, **53**, 153-173, (1938).
- [Cat] Catoni F. et. al., *Geometry of Minkowski Space–Time*, Springer, 2011.
- [DeSDat] de Sabbata V. and Datta B., *Geometric Algebra and Applications to Physics*, CRC, 2007.

- [DorLas] Doran C. and Lasenby A., *Geometric Algebra for Physicists*, Cambridge Univeristy Press, 2003.
- [DorstIP] Dorst L., *The inner products of geometric algebra*, In J. Lasenby, L. Dorst, and C. Doran (Eds), *Applications of Geometric Algebra in Computer Science and Engineering*, pp. 35–46, Birkhauser, 2002.
- [DorFonMan] Dorst L., Fontijne D. and Mann S., *Geometric Algebra for Computer Science*, Elsevier, 2007.
- [Esposito] Esposito G., Marmo G. and Sudarshan G., *From classical to quantum mechanics*, Cambridge University Press, 2004.
- [Feynman III] Feynman R., Leighton R. and Sands M., *The Feynman Lectures on Physics*, Addison-Wesley, Reading, Vol. III, 1964.
- [Gold] Goldstein H., Poole C., Safko J., *Classical Mechanics*, 3rd ed., Addison-Wesley, 2000.
- [Gro] Gross F., *Relativistic Quantum Mechanics and Field Theory*, Wiley, 2004.
- [Gul93] Gull S. et. al., *Electron paths, tunnelling, and diffraction in the spacetime algebra*, *Found. Phys.* **23**, 1329-1356, 1993.
- [Hes-71] Hestenes D., *Vectors, Spinors, and Complex Numbers in Classical and Quantum Physics*, *Am. J. Phys.*, **39**, 1013–1027, (1971).
- [Hes-73] Hestenes D., *Local observables in the Dirac theory*, *J. Math. Phys.*, **14** (7), 893-905, (1973).
- [Hes-74] Hestenes D., *Proper particle mechanics*, *J. Math. Phys.*, **15** (10), 1768-1777, (1974).
- [Hes2-74] Hestenes D., *Proper dynamics of a rigid point particle*, **15** (10), 1778-1786, (1974).
- [Hes-75] Hestenes D., *Observables, operators, and complex numbers in the Dirac theory*, *J. Math. Phys.*, **16** (3), 556-572, (1975).
- [Hes-79] Hestenes D., *Spin and uncertainty in the interpretation of quantum mechanics*, *Am. J. Phys.*, **47** (5), 399-415, (1979).
- [Hes-81] Hestenes D., *Geometry of the Dirac theory*, In: A Symp. on the Math. of Physical Space-Time, (1981).
- [Hes-86] Hestenes D., *A Unified Language for Mathematics and Physics*, in J. Chisholm/A.K. Commons (Eds.), *Clifford Algebras and their Applications in Mathematical Physics*, Reidel, 1-23, (1986).
- [Hes-96] Hestenes D., *Real Dirac Theory*, in J. Keller and Z. Oziewicz (Eds.), "The Theory of the Electron", UNAM, Cuautitlan, Mexico, 1-50, (1996).

- [HesZie] Hestenes D. and Ziegler R., *Projective Geometry with Clifford Algebra*, Acta Appl. Math., **23**, 25–63, (1991).
- [Hes1-03] Hestenes D., *Oersted Medal Lecture 2002: Reforming the mathematical language of physics*, Am. J. Phys, **71**, (2), 104–121 (2003).
- [Hes2-03] Hestenes D., *Spacetime physics with geometric algebra*, Am. J. Phys, **71**, (7), 691-714 (2003).
- [HesNF] Hestenes D., *New Foundations for Classical Mechanics*, 2nd ed., Kluwer, 1999.
- [HesSob] Hestenes D. and Sobczyk G., *Clifford Algebra to Geometric Calculus*, D. Reidel Publishing Co., 1987.
- [Hes-STC] Hestenes D., *Space-time Calculus*, draft, 1998, <http://geocalc.clas.asu.edu/pdf/SpaceTimeCalc..>
- [HesGur71] Hestenes D. and Gurtler R., *Local Observables in Quantum Theory*, Am. J. Phys., **39**, 1028–1038 (1971).
- [Hil] Hillion P., *Constitutive relations and Clifford Algebra in Electromagnetism*, in Advances in Applied Clifford Algebras **5**, No. 2, 140-158, (1995).
- [ItzZub] Itzykson C. Zuber J., *Quantum Field Theory*, McGraw-Hill, 1980.
- [GurHes] Gurtler R. and Hestenes D., *Consistency in the formulation of the Dirac, Pauli, and Schrodinger theories*, J. Math. Phys., **16** (3) 573-584 (1975).
- [Jackson] Jackson J., *Classical Electrodynamics*, 3rd ed., Wiley, 1999.
- [Janc] Jancewicz B., *Multivectors and Clifford Algebra in Electrodynamics*, World Scientific, 1989.
- [Gottfried] Gottfried K. and Yan T., *Quantum Mechanics: Fundamentals*, 2nd ed., Springer, 2003.
- [GreRQM] Greiner W., *Relativistic Quantum Mechanics*, 3rd edition, Springer, 2000.
- [GreQM] Greiner W., *Quantum Mechanics - An Introduction*, 4th edition, Springer, 2001.
- [GreRei] Greiner W. and Reinhardt J., *Field Quantization*, Springer, 1996.
- [HalMar] Halzen F. and Martin A., *Quarks and Leptons - An Introductory Course in Modern Particle Physics*, Wiley, 1984.
- [Kon] Konopinski E., *Electromagnetic fields and relativistic particles*, McGraw-Hill, 1981.
- [Kos] Kosyakov B., *Introduction to the Classical Theory of Particles and Fields*, Springer, 2007.

- [LDG] Lasenby A., Doran C. and Gull S., *A Multivector Derivative Approach to Lagrangian Field Theory*, Found. Phys., **23**, 1295-1327 (1993).
- [LDG1] Lasenby A., Doran C. and Gull S., *Gauge theories and gemetric algebra*, Phil. Trans. R. A, **356**, 487–582 (1998), improved and corrected version arXiv:gr-qc/0405033v1, 2004.
- [Loun] Lounesto P., *Clifford Algebras and Spinors*, 2nd ed., Cambridge, 2001.
- [Marx] Marx E., *Klein-Gordon Equation for Spinors*, J. Math. Phys., **8**, 1559-1566 (1967).
- [MilSchw] Milton K. and Schwinger J., *Electromagnetic Radiation: Variational Methods, Waveguides and Accelerators*, Springer, 2006.
- [Mol] Møller C., *The Theory of Relativity*, 2nd edition, Oxford, 1982.
- [MorFeshI] Morse P. and Feshbach H., *Methods of Theoretical Physics*, Vol. I, McGraw-Hill, 1953.
- [Moya] Moya A., Fernandez V. and Rodrigues W., *Advances in Clifford Algebras*, Springer, 2011.
- [Oha] Ohanian H., *Classical Electrodynamics*, Allyn and Bacon, 1988.
- [Lev1] Levy-Leblond J., *Nonrelativistic particles and wave equations*, Comm. Math. Phys, **6**, 286–311 (1967).
- [PanPhi] Panofsky W. and Phillips M., *Classical Electricity and Magnetism*, 2nd ed., Addison-Wesley, 1962.
- [PauRT] Pauli W., *Theory of Relativity*, Pergamon, 1958.
- [Perw] Perwass C., *Geometric Algebra with Applications in Engineering*, Springer, 2009.
- [PesSch] Peskin M. and Schroeder D., *An Introduction To Quantum Field Theory*, Perseus Books, 1995.
- [Ram] Ramond P., *Field Theory - A Modern Primer*, 2nd ed., Westview, 2001.
- [RaaLan] Raab R. and de Lange O., *Multipole Theory In Electromagnetism*, Oxford University Press, 2005.
- [Rich] Richards P., *Manual of mathematical physics*, Pergamon Press, 1959.
- [RodOli] Rodrigues W. and de Oliveira E., *The Many Faces of Maxwell, Dirac and Einstein Equations, A Clifford Bundle Approach*, Springer, 2007.
- [SanMar] Santamato E. and De Martini F., *Derivation of the Dirac Equation by Conformal Differential Geometry*, Found. Phys., **43**, 631–641, (2013)
- [Schwabl] Schwabl F., *Advanced Quantum Mechanics*, 4th ed., Springer, 2008.

- [SchwiE] Schwinger J. et. al., *Classical Electrodynamics*, Perseus Books, 1998.
- [Sexl] Sexl R. and Urbantke H., *Relativity, Groups, Particles, Special Relativity and Relativistic Symmetry in Field and Particle Physics*, Springer, 2000.
- [Sny] Snygg J., *Clifford Algebra. A Computational Tool for Physicists*, Cambridge, 1997.
- [Stern-Gerlach] Gerlach, W. and Stern, O. (1922). *Der experimentelle nachweis der richtungsquantelung im magnetfeld*, Z. Phys. 9, 349–52.
- [Stra] Stratton J., *Electromagnetic Theory*, McGraw-Hill, 1941.
- [Tak] Takabayasi T., *Relativistic Hydrodynamics of the Dirac Matter*, Prog. Theor. Phys. Supplement No.4, 2-80, (1957).
- [vanBS] Van Bladel J., *Singular Electromagnetic Fields and Sources*, Oxford University Press, 1995.
- [VandE] Vanderlinde J., *Classical Electromagnetic Theory*, 2nd ed., Springer, 2004.
- [Wach] Wachter A., *Relativistic Quantum Mechanics*, Springer, 2011
- [WacHoe] Wachter A. and Hoerber H., *Compendium of Theoretical Physics*, Springer, 2009.
- [Yag] Yaglom I., *A Simple Non-Euclidean Geometry and Its Physical Basis*, Springer, 1979.