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The Growth and Decline of the Western Roman Empire: Quantifying the Dynamics of Army Size, Territory, and Coinage

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We model the Western Roman Empire from 500 BCE to 500 CE, aiming to understand the interdependent dynamics of army size, conquered territory and the production and debasement of coins within the empire. The relationships are represented through feedback relationships and modelled mathematically via a dynamical system, specified as a set of ordinary differential equations. We analyze the stability of a subsystem and determine that it is neutrally stable. Based on this, we find that to prevent decline, the optimal policy was to stop debasement and reduce the army size and territory during the rule of Marcus Aurelius. Given the nature of the stability of the system and the kind of policies necessary to prevent decline, we argue that a high degree of centralized control was necessary, in line with basic tenets of structural-demographic theory.

1. Introduction

Mathematical and numerical modelling has increasingly been passing through disciplinary boundaries, with quantitative models in the social sciences becoming more common. Cliodynamics is one such recent development (Turchin 2008, 2011), wherein the integration of quantitative methods and historical knowledge brings new insights into human behavior and social institutions.

The present study lies within this domain. Our research focus is to quantify feedback relationships between a set of variables in the many factors at play in the decline of the Western Roman Empire. There are centuries' worth of explanations for the decline of the Western Roman Empire. An early starting point for the study of the Roman Empire and its decline is the work of Gibbon (1776). Since then, at least 210 reasons and theories have been put forth for the fall of the Roman Empire (Storey and Storey 2017), a list of which was originally compiled by Demandt

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(1984). In Appendix A, we sort these possible reasons into different categories and discuss their viability as explanations of collapse. Inspired by Tainter (1988), we identify the following categories: exogenous causes that are either natural (e.g., calamities), or social (e.g., foreign invaders), endogenous causes that are either always present (e.g., wealth differences) or episodic (e.g., civil wars) and other reasons that have a moral or mystical nature (e.g., egoism, lack of dignity).

Any one given cause is unlikely to explain why a complex phenomenon, such as the decline of a civilization, occurred. As such, we focus on determining the relationships between several interrelated factors, such as the size of the army, that of the territory, and the quality and quantity of coins, and how these dependencies affected them over time. In addition, the model we have built is mathematical and computational in nature, which provides a quantitative understanding of the different factors in play. Thus, we move from the mental (or conceptual) models listed in Appendix A to quantitative, precise ones, where the magnitudes of the variables are known or determined.

The maximum time horizon for our model is from 500 BCE to 500 CE, and we aim to understand the interdependent dynamics of the size of the army, the territorial expanse and the production and debasement of coins within the empire. These are the historical data we used, which pertain to the full lifespan of the empire or to key periods (such as the height of the empire) and are representative of the entire western empire. The relationships between these quantities are represented through feedback mechanisms and modelled mathematically via a dynamical system specified as a set of ordinary differential equations. By solving the system, we can check to what extent the historical record is recovered by the model. While the parameters of the model are determined so that the model predictions best fit the data, they are nonetheless archaeologically meaningful.

Through a stability analysis, we show that a subsystem of the model has a neutrally stable center and periodic orbits. Excluding negative values for the variables, trajectories at any given distance from the center are possible. The trajectory most closely resembling the historical record for the Western Roman Empire is the one of maximum distance from the center, reaching zero as the minimum of the periodic orbit for each variable. Due to this stability of the system, we can recommend fiscal and military policies that could probably have prolonged the existence of the empire by several centuries.

Our results connect to and partially validate structural-demographic theory (SDT) (Goldstone 1991, 2017; Turchin 2003, 2013). Given the nature of the stability of the system and the kind of policies necessary to prevent decline, we argue that a high degree of centralized control was necessary, in line with basic tenets of SDT. In SDT terminology, the “state” had a significant role to play in the dynamics of the empire, due to its military policies (which affect army size) and fiscal measures (which concern the debasement of coins). Thus, of the key aspects of SDT, the most pertinent aspect with regard to our model is the high degree of centralization that directed military and monetary policy in the Roman Empire.

Societal collapse can be defined as "a rapid, significant loss of an established level of sociopolitical complexity" (Tainter 1988). However, there is a wider debate regarding the notion of collapse and how it applies to specific cases, such as in the case of the Maya (Aimers 2007; Storey and Storey 2017). Furthermore, a process, be it a collapse or any other, is rapid only relative to certain timescales. For the Western Roman Empire, views and terminology range from calling its later centuries a decline and fall (Gibbon 1776), a collapse (Tainter 1988) or a slow collapse (Storey and Storey 2017). We do not enter into this debate, but given the data, we built a model that integrates several factors in a consistent, minimal fashion and gives results in line with the archaeological record.

We do not claim to have achieved an all-encompassing theory of the decline of the Western Roman Empire. We neither compare nor do we address any aspect regarding its relationship with the Eastern Roman Empire. The division of the empire at 395 CE is treated as an exogenous factor in the model, a feature we explain in the modelling section of the paper. Furthermore, our modelling effort is not aimed at incorporating all the complexities of Roman society, the multitude of sociopolitical aspects, and relationships with foreign forces and cultures.

In the following section, we review several existing quantitative models that capture different aspects of the Roman Empire (e.g., military strategies, travel). Next, in section 3, we specify the model, explaining the equations and the factors they take into account. In section 4, we show the fit to the historical record, provide an approximate analytic solution and, on the basis of our model, we present policies that would have prevented the decline of the empire. In section 5, we provide a discussion of the limitations of the model, particularly with regard to the role of gold in the empire. We then discuss our findings, specifically, the feedback mechanisms the model suggests existed within Roman society, how this relates to what is historically known, and how different policies could have affected the long-term evolution of the empire. Lastly, we conclude with a summary of the paper's contribution. In Appendix A, we list the 210 possible reasons that Demandt (1984) found for the decline of the Roman Empire; we classify them and discuss their adequacy as explanations for decline or collapse. In Appendix B, we detail the mathematical aspects of the stability analysis for the dynamical system we propose as a model, along with the main results of the sensitivity analysis for the parameters.

2. Mathematical and Computational Models of the Roman Empire

While the decline of the Roman Empire has been studied for centuries, the mathematical modelling of the possible dynamics of and within the empire has only been attempted in recent decades. Several papers have focused on geographical aspects of the Roman Empire; either related to military activity (Stewart 1999; ReVelle and Rosing 2000; Henning 2003), travel (Graham 2006) or commerce (Scheidel 2013, 2014).

Stewart (1999) introduced a graph-theoretic method for approaching the problem of how to secure the different regions of the Roman Empire from possible attacks. ReVelle and Rosing (2000) analyzed this mathematical problem and possible strategies in greater depth. Henning (2003) explored ways of reducing the substantial costs of maintaining legions and developed a new strategy for tackling the problem. While applicable to the Roman Empire, these graph-theoretic considerations are much more general and not specific to the Roman case.

Graham (2006) carried out a study that focuses more on the Roman Empire. Employing network analysis, he used Antonine itineraries in agent-based simulations of information diffusion along the different routes. An estimate was obtained for the time it would take for information to reach different fractions of the population, and the findings were partially validated against the density of inscriptions in the different regions of the empire. Another model that focuses on geographical aspects is ORBIS: The Stanford Geospatial Network Model of the Roman World, which simulates the time and costs associated to travel via land, rivers or sea in the Roman Empire in conditions approximating the state of the empire at 200 CE. By using this network model, Scheidel (2013) determines the correlation between maximum prices of transported goods and sailing time. Similarly, with the same network model, Scheidel (2014) estimates travel times and costs within the empire at courier and military speeds during the summer and winter. While these geographical models are important for quantifying aspects of Roman communication and travel, they are static in nature, temporally constrained to periods after the first century BCE, at the mature stages of the empire.

The model we develop does not have any spatial resolution, but rather focuses on aggregated quantities and their interdependent dynamics over time, aiming to understand periods of growth and decline. Work in a similar spirit has been done by Gündüz (2002), who aims to fit a power law to the total area versus time during the growth stages of the Roman and Ottoman empires. A good fit to the historical record is achieved, with exponents that are related to the golden ratio and other irrational numbers. While numerical results are presented, no model for different dependencies (e.g., feedback relationships) between quantities is proposed.

Sverdrup and colleagues (2013) propose a model for analyzing the scarcity of resources in the modern world. Beyond these considerations, data on the Roman world is presented and a conceptual model is proposed in the form of a causal loop diagram. The diagram specifies feedback relationships between different factors (e.g., population, resource base, military strength) but no equations are given to capture these dependencies quantitatively. Yaroshenko et al. (2015) conducted a wavelet analysis of changes in population and territory for the Roman Empire and the European Union. The analysis takes as input a time series and determines whether at each point in time the system was in a chaotic state or not, depending on the frequency of the wavelet (higher frequency means more chaotic). However, beyond the evaluation of the states at different points in time, no explanation for the results based on causal mechanisms is proposed.

Building upon previous research, we map feedback mechanisms between the army size, land extent and coin production, which we capture in a system of ordinary differential equations. Thus, by solving this system of equations, we can see how these aggregate quantities evolve over time and compare them with the archaeological record. This allows us to validate the model in the sense that it forms a possible set of dynamics for the real system, at least within the range and scope of the data used.

3. Model Specification

In this section, we first outline the modelling methodology we employed and then introduce our model (3.1)–(3.5) for the growth and decline of the Western Roman Empire. Afterwards, we discuss its structure and how the different terms can be interpreted, along with the parameters and their values, which are given in Table 1.

In the model (3.1)–(3.5), we account for the growth and decline of the army size, the land conquered and the production and debasement of silver coins. The evolution of the army size is estimated from multiple sources (MacMullen 1980; Ward 1990; Roth 1999; Campbell 2006). The extent of land conquests over time is from Taagepera (1979), and the data for the production of coins is from Hopkins (1980), while debasement information is from Tainter (1988, 1996). The time span of the data regarding the army and land is 1000 years, from 500 BCE (-500) to 500 CE (+500), which determines the maximum time horizon for the model. We refer to the historical time series, which we aim to reproduce as reference modes (see Figure 1).

Given this data, the task of developing a dynamical systems model can be understood as constructing a set of ordinary differential equations (ODEs), whose solutions reproduce the observed historical trajectories. The system of ODEs gives the rate of change of variables (e.g., army size, territory), which are typically

Table 1. Parameter and initial values for model (3.1)–(3.5)

Notation	Meaning	Value
p_0	Debasement effect on changes of army size	1142 soldiers/year
p_1	Scaling parameter for debasement	0.5
p_2	Army effect on changes in area of territory	326 km^2/year
p_3	Scaling parameter for army size	200 000 soldiers
p_4	Debasement effect on changes of area of territory	14 276 km^2/year
p_5	Territory effect on changes in debasement	0.002856
p_6	Scaling parameter for area of territory	2 500 000 km^2
p_7	Net growth rate of silver reserves	2.5%
Λ	Army and territory decrease after division	55%
t_0	Initial time	-500
t_d	Time of division	395
$x(t_0)$	Initial value of army size	48 850 soldiers
$y(t_0)$	Initial value of territory area	648 521 km^2
$(z_1/z_2)(t_0)$	Initial value of silver concentration	17.3%
$z_1(t_0)$	Initial amount of silver	0.33 million coins
$z_2(t_0)$	Initial number of coins	1.91 million coins

expressed as linear and nonlinear combinations of the variables themselves. Often, these type of models do not have an explicit time dependence and are called autonomous. Exogenous effects, not modelled directly by the system, can be incorporated as time-dependent terms.

Mathematical models of this type abound in physics, mathematical biology and engineering (Strogatz 2015). Within the social sciences, a substantial number of such models have been developed in the field of system dynamics (Sterman 2000). Beyond fitting the observed time series, the model should have a structure and terms that are overall meaningful from a psychological, sociological and historical perspective. When constructing an ODE model of a complex social system, one main guiding principle is the identification of feedback mechanisms between the different variables. This allows us to uncover dependencies between variables and to ensure that the relationships within the mathematical model reflect real qualitative and quantitative features.

However, developing a model that fits the historical record, is faithful in capturing feedback relationships, has terms and parameter values that can be calibrated and interpreted meaningfully and is robust enough to overcome uncertainties in the data, is challenging (Sterman 2000). Theoretical insights, feedback analysis and modelling principles (such as those employed in population dynamics [Turchin 2003]) can help identify meaningful equations for the model, but the task of verifying the adequate combination of factors is laborious. Our initial aim was to use an automated equation discovery framework, such as SINDy (Brunton et al. 2016) that performs linear regression on gradients (i.e., rates of change) on combinations of variables to identify ODEs that generate trajectories consistent with the reference modes.

However, the uncertainty in the data and its granular nature prevents the determination of reliable gradient information, which then makes automated equation discovery unfeasible. A way to resolve the low resolution problem in the data is to use a smooth approximation and, in the case of the reference modes for the army, land and coin debasement, we found that the data is well approximated by sine and cosine curves, such as in the solution (4.1). Then, we could infer the system of ordinary differential equations of the subsystem (3.1)–(3.3), which has an almost linear structure (except for the discontinuities). Thus, we arrived at part of the model in a semi-analytic way and did not have to resort to searching in high-dimensional feature space, which automated equation discovery frameworks have to do. We detail below further aspects of the model.

In total, there are four independent equations in the model (3.1)–(3.5); specifically, only two of the last three equations (3.3)–(3.5) are independent. These latter equations account for the number of minted silver coins z_2, the amount of silver used z_1 (measured in equivalent number of pure silver coins) and the ratio z_1/z_2 that gives the average silver content of a coin in circulation, which is how the debasement of coins is measured and tracked in the historical data (van Heesh 2011). Data referring to coin production and debasement is available over a shorter time span than for the army or land, and we cannot guarantee accurate results outside the time period for each dataset. We do not attempt to capture in the model the variety of different coins used by the Romans throughout time. Instead, we aim to account for an aggregated measure in z_2, which reflects an equivalent total value. Nevertheless, the representative coins incorporated in the model are mostly closely reflected in denarii: see Figure 1 (d).

Let x be the army size, y the land conquered, z_1 the amount of silver used in minting (measured in coins) and z_2 the number of minted coins. We propose the following dynamics for these quantities:

$$\dot{x} = p_0\left(\frac{z_1}{p_1 z_2} - 1\right) - \lambda\delta(t - t_d)x \tag{3.1}$$

$$\dot{y} = p_2\left(\frac{x}{p_3} - 1\right) + p_4\left(\frac{z_1}{p_1 z_2} - 1\right) - \lambda\delta(t - t_d)y \tag{3.2}$$

$$\left(\frac{z_1}{z_2}\right)' = p_5\left(1 - \frac{y}{p_6}\right) \tag{3.3}$$

$$\dot{z}_1 = p_7 z_1\left(1 - \frac{y}{p_6}\right) \tag{3.4}$$

$$\dot{z}_2 = z_2\left(1 - \frac{y}{p_6}\right)\left(p_7 - p_5\frac{z_2}{z_1}\right) \tag{3.5}$$

where the parameters $p_0, \ldots, p_7$ are assumed non-negative and determined such that the trajectories the system generates match as closely as possible to the reference modes (see Figure 1). The year t_d = 395 CE is when the empire divided into two separate parts for political reasons, with the surface area of the western empire decreasing by λ = 55% (Taagepera 1979). This change is modelled as an exogenous shock to the system, where the land extent and army size are reduced by a fraction λ.

The first equation (3.1) of the model dictates the evolution of the army size x. The first contribution to the rate of change of the army size is given by the debasement of the coins in circulation. We can understand this as follows: if a conquest was successful, then the influx of silver and other resources led to economic benefits, e.g., reduced taxation to citizens (Tainter 1988), which also translated into coins of high purity. This then encouraged further conquest, which remained in proportion to the returns, as measured in the quality of the minted coins. The operations on z_1/z_2 within the parenthesis change the value from the interval *[0; 1]* to *[-1; 1]*, so that contractions can be also be experienced as expected if the returns are too low and the quality of coins declines. The parameter p_1 sets the scale difference between *[-1; 1]* and the range of x.

The second contribution to the army is due to the division of the empire at t_d = 395. The drivers of this split were political in nature (Taagepera 1979) and could not be captured in an autonomous way (with no explicit time dependence). Thus, we model the division occurring at t_d = 395 as an exogenous change (or shock) that reduces the army size and land extent by half. The exogenous shocks are modelled with Dirac delta functions $\delta(t - t_d)$, centered at the time of the division.

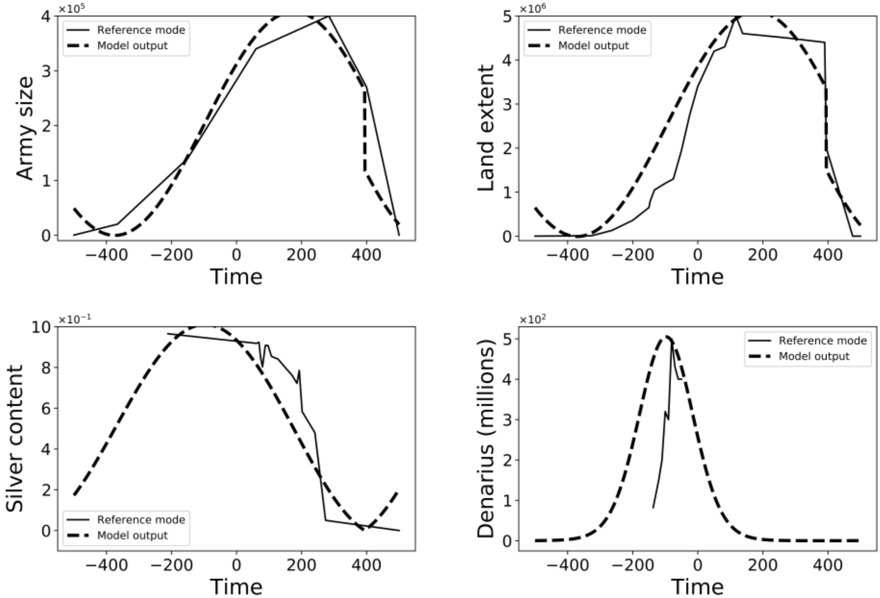

Figure 1. Comparing the historical reference modes (solid lines) with model output (dashed lines). The evolution of (a) army size, (b) land conquered and (c) debasement (measured as percentage silver content of coins) as predicted by the model. In (d), the predicted amount of minted coins follows a similar trend to the historical record and matches well on the downward trajectory.

The second equation (3.2) of the model specifies how the land conquered y changes. There are two contributing terms, distinct from the exogenous shock mentioned. First, the army size contributes to more land being conquered. Second, the conquest of more land takes place if there are sufficiently high returns to this activity, which is measured by the quality of minted coins (given by the silver content), just as for the army size.

Comparing the parameters p_2 and p_4, we can see that the second term, due to the debasement, has a more significant influence on the rate of change of the land conquered y. By looking at the solutions (4.1) of the equations in subsystem (3.1)–(3.3), we can see that the first term, due to the army size, amounts to a delay of δ = 4 years (see Table 2) in the expansion of land compared to the growth of the army. This is consistent with what we would expect, namely that an increase in land extent is subsequent to an increase in the army size. Furthermore, most Roman wars ranged in duration from a year to a decade. The delay δ = 4 years is close to the average time we have estimated for Roman wars prior to the third century. Thus, the value for δ is consistent with what we would expect on average for a delay between an increase in army size (before a war) and an increase in territorial expansion (after a war).

The third equation (3.3) gives the rate of change of the silver content of coins z_1/z_2, whose dynamics depend on the land conquered, y. We expect a law of diminishing returns to hold, meaning that early conquests are highly profitable. When the empire has grown extensively, later conquests are less profitable and affect the quality of coins in a detrimental manner. This assumption is consistent with the historical data shown in Figure 1(b) and (c). Mathematically, this dependence is captured by a setting an upper limit to profitable land conquests, given by p_6. The value of p_6 = B can be determined by fitting the output of the model to the data: see section 4.

The structure of the equation for z_1/z_2 and the parameter values are thus sufficient for reproducing the reference mode, and so are consistent with the historical record. However, we can also interpret p_6 in a different way. According to Marchetti and Ausubel (2012), a state is stable if the distance from its center to the border does not exceed 14 days of travel. From the value of p_6, we can estimate a characteristic speed of $\sqrt{p_6/\pi}/14$ = 63 km/day (assuming a disk shape), which is consistent with the speed of the 67 km/day of the *cursus publicus* (postal service) (Scheidel 2014) of the Roman Empire. A higher value of $\sqrt{p_6}/14$ = 112 km/day can be estimated, which is in line with the highest speeds that the Romans could achieve of 120 km/day, navigating downstream by rivers (Scheidel 2014). Thus, the equation for z_1/z_2 and the value of p_6 are compatible with the theory of Marchetti and Ausubel (2012), in which stability is lost if travel time exceeds 14 days from the center to the border.

The fourth equation (3.4) gives the dynamics of the amount of silver used in minting z_1. We assume that z_1 grows exponentially at a rate p_7 = 2.5% per year, provided that the empire's land extent y is sufficiently small. Thus, the parameter p_7 is the net growth rate of silver, incorporating both discovery and mining, and attrition and wear. If a separate attrition and wear term is introduced, this would amount to a more rapid decline in z_1 and z_2. By making adequate changes in p_7 and in the initial conditions, the higher rate of decline would not substantially alter the observed trajectory in Figure 1 (d).

Once the empire grows beyond a certain point, there are diminishing returns to conquests and the subsequent fiscal policy leads to decreasing amounts of silver being used in coin production. We do not have historical data to validate the equation for z_1 directly. The fifth equation (3.5) gives the dynamics of the number of coins minted z_2 and is deduced from equation (3.3) combined with the dynamics for z_1. The dynamics of z_2 were inferred from the equation for z_1/z_2 and z_1 as follows:

$$\begin{aligned}\dot{z}_2 &= \frac{\dot{z}_1}{z_1} z_2 - \frac{z_2^2}{z_1}\left(\frac{z_1}{z_2}\right)' \\ &= \frac{\dot{z}_1}{z_1} z_2 - p_5 \frac{z_2^2}{z_1}\left(1 - \frac{y}{p_6}\right)\end{aligned} \tag{3.6}$$

The equation that results for z_2 is similar to that for z_1 but with an additional term that captures a counteracting/balancing (negative) feedback loop if too many coins are produced relative to amount of silver z_1 available. Furthermore, we can compare the trajectory for z_2 with the historical record in Figure 1 (d). Thus, we can aim to reproduce the reference mode for silver coin production in Figure 1 (d) without affecting the output of subsystem (3.1)–(3.3) and the fit to the other reference modes in Figure 1 (a)–(c).

4. Results

In Figure 1, the reference modes of the data are compared to the model output. Within the uncertainty and granularity of the data, the army size, land conquered and the debasement are well reproduced by the subsystem (3.1)–(3.3). With regard to the number of coins minted, we see that equation (3.5) recovers the downward trajectory well but seems to overestimate the number of silver coins prior to 137 BCE.

The model has a subsystem (3.1)–(3.3) for which we can write down an approximate closed-form solution. This helps us in fitting the trajectories of the dynamical system to the historical record. For the subsystem of ordinary differential equations (3.1)–(3.3), we find the following approximation for the solution up to $t \leq t_d$:

$$\begin{aligned} x(t) &= A[1 + \sin(w[t + \Delta])] \\ y(t) &= B[1 + \sin(w[t + \Delta - \delta])] \\ \left(\frac{z_1}{z_2}\right)(t) &= C[1 + \cos(\omega[t + \Delta])] \end{aligned} \tag{4.1}$$

The parameters in the subsystem (3.1)–(3.3) and in the solution (4.1) are related as follows: $p_0 = wA$, $p_1 = C$, $p_2 = \omega B \sin(\omega\delta)$, $p_3 = A$, $p_4 = \omega B \cos(\omega\delta)$, $p_5 = \omega C$, $p_6 = B$. Thus, we see that not all the parameters $p_0, \ldots, p_6$ are independent, and it is sufficient to determine A, B, C, ω, Δ and δ. In Table 1, the initial conditions of the subsystem (3.1)–(3.3) have been set using the solution (4.1) at time t_0.

Fitting the output of the model to the data requires us to take into account the following aspects: (1) the overall fit to the historical trajectory, which involves (a) the fit as measured by e.g., an L_1 loss (least absolute deviations) and (b) the match of beginning and end values of the model output and the reference modes; and (2) the requirement of having: (a) positive parameters and (b) non-negative output values throughout the time periods in the data.

A minimum of an L_1 loss does not imply a good match of initial and final values of the model output with the reference modes; nor does it guarantee the non-negativity requirements. While it is possible to perform an optimization taking into account these constraints, in practice we found it simpler to provide an initial estimate of the parameters that give a good visual match to the data and then fine-tune the values by performing a grid search to locate the closest local minimum consistent with all the requirements. For details, see Appendix B. Following this procedure, we determined the parameter values in Table 2. The last two equations (3.4)–(3.5) introduce one additional parameter, p_7, which is the maximum growth rate for z_1 and z_2. The parameter p_7 and the initial values $z_1(t_0), z_2(t_0)$ were determined to achieve the best fit with the reference mode in Figure 1 (d).

Table 2. Alternative model parametrization for subsystem (3.1)–(3.3).

Notation	Meaning	Value
A	Scaling parameter for army size	200 000 soldiers
B	Scaling parameter for territory	2 500 000 km^2
C	Scaling parameter for debasement	0.5
ω	Angular frequency	0.005712/year
T	Period	1100 years
Δ	Overall time shift	100 years
δ	Time shift between army size and territory	4 years

With finer tuning, a better fit of the model to the reference modes could possibly be achieved, but there are inherent uncertainties within the data, which imply that a more precise match is not necessarily more accurate. Similarly, a model with more parameters could provide a better fit and include more dependencies between the variables. However, in the interest of parsimony and in the spirit of Occam's razor, we present in the model (3.1)–(3.5) what we found to be the simplest dynamics that can account for the historical record.

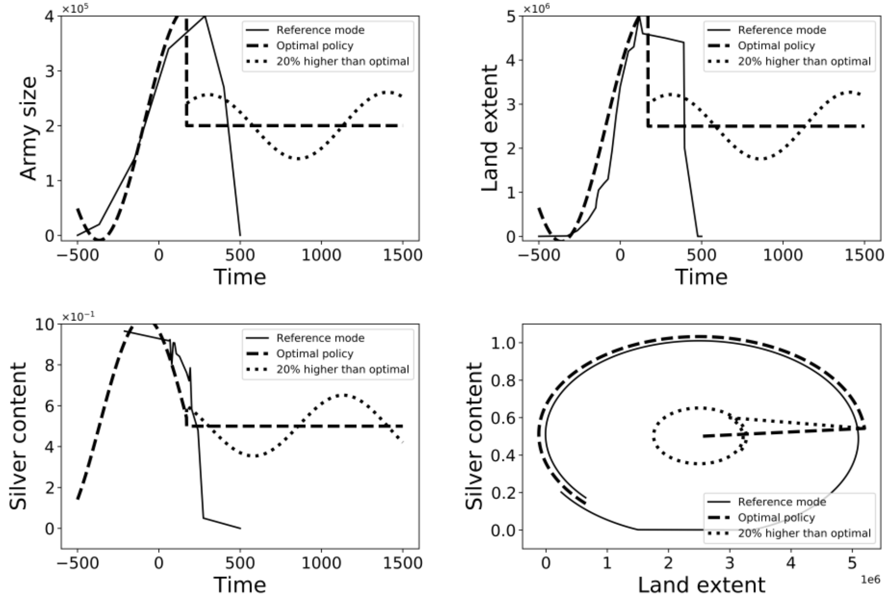

Figure 2. Comparing references modes (solid lines) with policies that change stock values of (a) army size, (b) land extent, (c) debasement (measured as percentage silver content of coins) to prevent the collapse of the Western Roman Empire. Two policies are shown: one that changes the stock values to prevent any oscillation, which we consider optimal (dashed lines) and another policy where stock values are 20% higher than the optimum (dotted lines). In (d), the phase portrait for debasement and land extent is shown for the different policies.

The subsystem (3.1)–(3.3) of the model has one fixed point. In Appendix B, we analyze the stability of this equilibrium and find that it is neutrally stable. The phase portrait of the system is a center, where a neutrally stable trajectory is possible at any given radius from the fixed point (provided stock values remain positive). Thus, if stock values are changed in an exogenous way, then the system can enter a periodic orbit of a different amplitude; see Figure 2 (d). Specifically, we keep the linear part of the subsystem (3.1)–(3.3) and alter the discontinuous part (represented by the delta functions) to implement the exogenous changes.

Given the neutral stability of the subsystem (3.1)–(3.3), we can recommend mitigation policies for the decline of the Western Roman Empire. In Figure 2, we show what happens if all the stock values are changed. We illustrate two scenarios: (1) variables are changed such that the new values are exactly at the fixed point (dashed line), and (2) values are 20% higher than the fixed point (dotted line). If the fixed point is reached then no oscillations occur, which we consider to be the sign of an optimal policy. Otherwise, oscillations do occur, but at a smaller amplitude. In either case, the army size, land extent and silver content of coins do not reach the very low values seen in the archaeological record, and collapse is prevented.

Mathematically, the timing of this change in values is not significant, but in historical terms, this is likely a crucial aspect. For example, once the land extent or silver content of coins drops to smaller values than those of the fixed point, it is much more difficult to increase them afterwards than allowing for a continued decrease. In choosing the time for the exogenous changes in the stock values, we looked at when the silver content of coins reached the fixed point value of 50%. For the optimal policy, at this point the army size and land extent were reduced while keeping the silver content constant. For the sake of comparison, the other exogenous change was also done at the same time.

5. Discussion

In this section, we discuss certain aspects pertaining to the scope of the model. Specifically, we explain what elements we chose or chose not to incorporate; the extent to which the archaeological record is reproduced and how the mismatches can be understood; implications of the model and its relation to broader theories on societal collapse; and the nature of the mitigation policies proposed above and how they relate to other cases of collapse.

The scope of the model is dictated in part by the availability and quality of the data, the possibility of finding system-wide, closed feedback mechanisms between the relevant variables, and the ability of the proposed equations to relate to and reproduce the archaeological record. These criteria are aimed at finding a feasible mathematical description of the system and do not necessarily guarantee that all factors that can be deemed archaeologically important will be incorporated within the model. As such, there are innumerable aspects the model does not address (e.g., trade relations, border attacks, army logistics), which do not lie within the scope of our modelling effort.

A particularly important topic is the multi-metallic monetary system that the Roman Empire used throughout its lifetime. While bronze and silver coins were predominantly used in the early and middle periods of the empire, the later periods enjoyed an increased influx of gold (Bransbourg 2015), to the extent that the high-purity gold coins were commonly issued, most notably the aureus until the third century CE and the solidus thereafter. Given this, our model is not representative of the entire Roman monetary system, as we do not account for gold coinage.

However, this does not undermine the consistency of our modelling effort, nor of our results.

Gold had a different monetary circuit than the other metals, being isolated in its usage and distribution in part due to the inequality (Levitt 2019; Milanovic et al. 2017) and stratification of Roman society (Bransbourg 2015), as well as certain general behavioral tendencies (such as Gresham's law [Sparavigna 2014]). Access to gold was tightly regulated by the Roman bureaucracy and various laws were enacted in the fourth and fifth centuries aimed at restricting the giving of gold solely to the imperial household (Guest 2008). As such, while gold represented a substantial proportion of the monetary supply by value, it had a restricted use to the extent that:

> The contrast between an even more prevalent gold coinage and the increasingly debased base coins implies a sharp decoupling between two very distinct monetary circuits, one involving the upper strata of the Roman society, which used gold, while poorer individuals were left to deal with the inflationary consequences of the bronze currency's ongoing devaluations. (Bransbourg 2015: 271)

In contrast to the aureus, which functioned as a store of wealth for the aristocratic elite (Jongman 2003), it has been argued that there was gradual diffusion of the solidus as a mass currency (Banaji 2002). Thus, the issue of gold circulation in the Roman Empire is a complex one we do not aim to resolve, but what we have shown through our model is that the debasement of the silver currency functions as a good numerical proxy for the factors involved in the general feedback mechanisms of the wider system—giving an emergent, aggregate measure of the numerous, more direct influences on the decline in army size and territory even in the later centuries of the empire.

Nonetheless, gold coinage is an important aspect of empire and other modelling frameworks can accommodate its specific function and distribution. For example, game-theoretic network models have shown how optimal economic behavior can lead to social inequality, where a hub of influential players emerges that controls the most important resources and deprives the majority of the network (or it can also lead to more egalitarian distributions, depending on the game played) (Roman and Brede 2017).

Regarding the fit to the historical data, we have shown that a system of ordinary differential equations (3.1)–(3.5)—which reflect relationships and feedback between variables of interest, specifically the army size, land extent and coin quality and quantity of the Western Roman Empire—can generate trajectories that closely follow the archaeological record for these quantities. In principle, any number of models could achieve a good fit to the historical data; however, this does not guarantee the model is meaningful from a sociopolitical or economic perspective or that it is conservative in the number of parameters and relationships that it hypothesizes (in the spirit of Occam's razor). Given the data we focused on (the reference modes in Figure 1), we have provided what we intended to be a minimal mathematical structure (in the form of a system of ordinary differential equations) sufficient to account for the behavior seen in the time series.

A visible mismatch of the model and data is noticeable in the early periods in Figure 1 (c) and (d). With regard to Figure 1 (c), silver concentration of coins starts at approximately 20%, which is consistent with the silver content of ancient coins (Wickens 1995). Thus, historically and according to the model, coins prior to 137 BCE have a lower silver content. Hence, regarding Figure 1 (d), the model output is an estimate of the total number of coins in circulation, not just of high silver concentration (which is what the data show). Coins with higher silver content are better preserved, so more reliable data is available for these types of coins (van Heesh 2011). This can explain the mismatch between the available data and model output.

Another possible explanation is that, as we can see in Figure 1 (d), at 100 BCE several hundreds of millions of denarii were in circulation, and we assume that the extent of economic activity was proportional to availability of the coins. While coins of high purity could be found before 200 BCE, this does not necessarily mean their number was representative of the extent of economic activity at that time (Bowman and Wilson 2009). The model structure, on the other hand, remains unchanged, and its output shows the equivalent in coin production and purity for the overall economic activity even at early points in the empire's history (just as it also does when the availability and quality of coins in the data is reflective of the economic state of the later empire).

If there is a match of model output and data, this does not mean the model is historically accurate, but only that is consistent with known historical data and cannot be falsified without more data. A model does not have to be completely dismissed if found inconsistent with data, but rather it can be extended to incorporate new ranges and scales, while the previous (valid) models are recovered in certain limits. For example, within the time span in Figure 1 (a), (b) and (c), the subsystem (3.1)–(3.3) matches the historical record well, independent of the fit in Figure 1 (d). The mismatch of model output and data may mean that the model is not valid in a certain domain or that the interpretation is no longer compatible with the data, as may be the case with (3.4)–(3.5) and Figure 1 (d).

Still, there are genuine modelling artefacts inconsistent with the data, within the scope of this model. In particular, in Figure 1 (c), after the year 400 CE the silver content of coins increases, which points to a limitation of the model. This inconsistency arises from the discontinuity at t_d = 395 in the army size and land extent. While this could be corrected by additional terms in the equation for the rate of change of z_1/z_2, we prefer to keep the structure of the model (3.1)–(3.5) simpler and allow for greater transparency of the model's limits.

How does the model structure relate to known historical processes? As Figure 1 shows, the equations are sufficient to reproduce the reference modes well. However, the structure of the model goes beyond this good fit. It implies that the value of the currency was a strong determinant of both army growth and territorial expansion. Historically, it is known that in case of cash shortages (e.g., due to military campaign costs), the central authority of the empire would debase the coins (Garnsey and Saller 2015). Fiscal policy invariably affects the resources of the army and the amount of land that can be conquered, and similarly, the success of campaigns affects revenue of the empire and the fiscal policies in place (including coin debasement). Thus, there were important causal feedback relationships between military costs, territorial expansion and monetary issues.

Furthermore, Tainter (1988) proposes a theory of societal collapse that surpasses the difficulties we outlined in Appendix A, building upon the idea of diminishing returns to investments in problem solving. This is exemplified well by the Roman Empire, whose early conquests were very profitable and allowed for the elimination of taxes for the citizens. With the expanding territory, military and administrative costs grew as well. At a certain point, further conquests and conflicts proved less beneficial and even amounted to a loss of resources, like the wars with the Germanic tribes. Maintenance of the empire ended up having larger costs than revenue, and territory was gradually lost. The Roman currency, the denarius, was being debased to expand the money supply and cover the costs (at least temporarily) (Tainter 2000). Throughout the period 200–500 CE, these negative returns manifested as the decline of the empire. Hence, at least according to Tainter's (1988) theory, the feedback mechanisms that existed in the empire connect directly to its observed long-term development and decline. Similar arguments have been put forward regarding the Chinese dynastic cycle (Lattimore 1940) and the Ottoman Empire (Lewis 1958), as well as the Maya (Culbert 1991).

While such relationships are qualitatively known, the model (3.1)–(3.5) we propose provides in addition a precise, quantitative way to encapsulate these feedback relationships. Thus, it moves from a mental model to a mathematical one (Sterman 2000). Furthermore, except for a discontinuity at a specific time, the subsystem (3.1)–(3.3) has a linear structure. Thus, given the variables that we investigate, it is a particularly simple structure that affords sufficient explanatory power to match the historical record.

Regarding the mitigation policies, we remark that the timing for the optimal policy is 171 CE, at the midpoint of the rule of Marcus Aurelius. What the subsystem

(3.1)–(3.3) suggests is that if at this point the empire had been split into two roughly equal parts and a fiscal policy implemented to keep silver content at approximately 50%, then the Western Empire could have potentially lasted for much longer in this new state. A similar policy was actually implemented in 395 CE, but only with regard to the army size and land extent. Raising the silver content of the more widely and highly circulated coins was not feasible anymore, and gold had different dynamics of circulation, even if it was more abundant than in earlier times. In addition, the real changes were implemented too late to alter the trajectory substantially and prevent decline. Nevertheless, the measures taken in 395 CE are in line with what the phase portrait of the subsystem (3.1)–(3.3) suggests to have been the adequate course of action, namely reduction of stocks to values closer to the fixed point. Of course, the model (3.1)–(3.5) is an idealization of reality and it is debatable whether the policy it suggests could have worked, but it does offer an interesting thought experiment in this regard.

In addition, it is worthwhile to compare the dynamics of the model (3.1)–(3.5), which describes the socioeconomic system of the Roman Empire, with the dynamics for models of socio-ecological systems such as Easter Island (Roman et al. 2017) and the Classic Maya (Roman et al. 2018). In the case of Easter Island and the Classic Maya, the collapse is modelled by a super-critical Hopf bifurcation where, if the parameter representing the harvesting rate of resources per capita exceeds a critical threshold, the system moves from a stable fixed point to an attractive periodic orbit (a limit cycle) of large amplitude. Once the system reached the lower limits of the orbit, the collapse occurred. The necessary change to prevent collapse in this case is a change in the harvesting rate of resources, which translates into a lifestyle change for the entire society and its relationship to its environment.

For the Western Roman Empire, the dynamics for the subsystem (3.1)–(3.3) shows that neutral stability and changes in parameter values are not the course of action best suited to prevent decline—it is the values of the variables (stocks) that require tuning to reduce oscillations. Rather than a lifestyle or cultural change for all inhabitants, the policy in the case of the socioeconomic system of the Roman Empire is more feasibly implemented by centralized state powers and intervention by elites. Thus, to tackle instability (in the sense of territorial loss), the state would need to intervene with a fiscal and military policy supported by elites that affects the wider population but ensures territorial integrity and economic stability (e.g., prevention of inflation via debasement). Thus, the dynamics the model (3.1)–(3.5) uncovers for the Roman Empire represent a specific instance of the feedback and relationships posited by structural-demographic theory (Baker 2011). In this sense, through the example of the socioeconomic system we analyze, our work provides partial validation of the more general framework that structural-demographic theory develops.

6. Conclusion

Hundreds of reasons have been put forward for the fall of the Western Roman Empire; we have classified them into the several categories in Appendix A. Any one reason is not sufficient to explain the decline of such a complex society. We have aimed to develop an understanding that links several dependent factors together. On the other hand, given the complexity of Roman society, it would be unfeasible to attempt a model that covers all possible features.

Thus, we identified certain key feedback relationships in the Western Roman Empire between aggregated variables representative for the whole empire, for which we had data: the army size, the area of the territory, and the debasement and quantity of silver coins. The main focus was on understanding the evolution over time of these variables, and we built a system of ordinary differential equations that captures this feedback quantitatively.

The linear structure of a subsystem of the model allowed us to solve part of the system of equations analytically up to the time of division of the empire at 395 CE. Parameters in the model were optimized to match the historical record as closely as possible. In general, even an optimal choice of parameters does not guarantee a good fit. However, in this case, the structure of a model gives numerical solutions that show a close fit to the historical record. Furthermore, the parameters are archaeologically meaningful, relating to the scale of the variables.

A stability analysis determined that the linear subsystem has a neutral center, with periodic orbits (for details, see Appendix B). This implies that by making adequate exogenous changes to the system at the right times and to the right extent, the life span of the Western Empire could have been increased significantly. We found that the optimal policy is to roughly halve the size of the army and territory and fix the silver content of coins at 50% at 171 CE, during the rule of Marcus Aurelius.

For socio-ecological systems, such as Easter Island (Roman et al. 2017) and the Classic Maya (Roman et al. 2018), the collapse can be modelled as a type of critical transition, in which a stable fixed point changes to an attractive limit cycle, which is an isolated periodic orbit. The critical parameter that determines the sustainability or collapse of the system is the extraction rate of resources per capita, which, if high enough, severely degrades the ecosystem support of the societies. For the Western Roman Empire, the decline is not due to a critical transition but an unsustainable trajectory from the beginning, which could have been changed through a reduction in the values of key variables. Such a change was attempted in 395 CE, when the empire divided, but the debasement of coins was too severe to allow for reversion to higher quality. In this later period, gold coinage became more common but had a different, more restricted circulation (Bransbourgh 2015; Guest 2008) and thus, we argue, does not affect the model results.

We have not solved the deeper problem of why the Western Roman Empire declined, a question that has been posed for centuries. However, we provide insight into the interlocking dynamics of some key aspects of the empire, substantiated by a quantitative model and analysis that offers a precise, mathematically definite view on the problem beyond conceptual models.

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