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**Performance-Driven Control Theory and Applications**

A dissertation submitted in partial satisfaction of the  
requirements for the degree  
Doctor of Philosophy

in

Engineering Sciences (Mechanical Engineering)

by

Daniel J. Riggs

Committee in charge:

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William Michael McEneaney

2012

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The dissertation of Daniel J. Riggs is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

2012

DEDICATION

To my Lizzie. And little Jason Isaiah.

## EPIGRAPH

*By wisdom a house is built, and through understanding it is established; through  
knowledge its rooms are filled with rare and beautiful treasures.*

—Proverbs 24:3-4

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Lastly, I have come to know the true value of this bit of wisdom during my

studies: “As iron sharpens iron, so one man sharpens another” (Proverbs 27:17, NIV). This is also for those who have helped bring meaning to this saying.

Chapter 2, in part, includes reprints of the material as it appears in the following, both of which are copyright IEEE.

**D.J. Riggs and R.R. Bitmead.** Rejection of aliased disturbances in a production pulsed light source. *IEEE Transactions on Control Systems Technology*, to appear.

**D.J. Riggs and R.R. Bitmead.** Rejection of aliased disturbance in a pulsed light source. In *Proceedings of the 48th IEEE Conference on Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference*, pages 8148–8153, Shanghai, P.R. China, December 2009.

Chapter 3, in part, has been submitted for publication of the material as it may appear in the following.

**D.J. Riggs and R.R. Bitmead.** Destruktion and déconstruction of model predictive control. Pre-print submitted to *Automatica*.

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**D.J. Riggs and R.R. Bitmead.** MPC under the hood / sous les capot / unter der haube. In *Proceedings of the IFAC Conference on Nonlinear Model Predictive Control 2012*, pages 363–368, Noordwijkerhout, the Netherlands, August 2012.

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**D.J. Riggs and R.R. Bitmead.** Negotiation of coupled constraints in coordinated vehicles. In *Proceedings of the 49th IEEE Conference on Decision and Control*, pages 479–484, Atlanta, GA, December 2010.

The dissertation author was the primary investigator and author of each of these papers; Professor Bitmead supervised the research.

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D.J. Riggs and R.R. Bitmead. Rejection of aliased disturbances in a production pulsed light source. *IEEE Transactions on Control Systems Technology*, to appear.

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ABSTRACT OF THE DISSERTATION

**Performance-Driven Control Theory and Applications**

by

Daniel J. Riggs

Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

University of California, San Diego, 2012

Robert R. Bitmead, Chair

In this dissertation, we study stochastic disturbance rejection, performance, and optimal control. This study is composed of three distinct investigations: an application, theory, and the development of an algorithm. The studies are linked by optimal control and its associated performance.

In application, we study a disturbance rejection problem in a production pulsed light source to yield quantifiable and guaranteed improved performance over existing control techniques. We apply generalizations of continuous-discrete Kalman filter ideas for actuator and disturbance state estimation and prediction; following Harris, we analyze the variance light source output prediction errors in order to ascertain the theoretical lower bound for closed-loop control performance.

We establish and solve a non-standard regularized minimum variance control problem, and use the derived control law in concert with the continuous-discrete estimator to construct a certainty-equivalence state-feedback controller. We demonstrate on a production light source that the estimator-controller yields closed-loop performance near the derived theoretical lower bound for the hardware.

The theoretical framework is constructed around the application of Nonlinear Model Predictive Control (NMPC) schemes to discrete-time, nonlinear systems which are subject to persistent, stochastic disturbances. We pose a discounted-cost infinite-horizon optimal control problem and use its optimal value function as the performance benchmark to which all subsequent NMPC closed loops are compared. Following Jadbabaie, Hauser, Grüne, and Rantzer, who address performance and stability of NMPC in the undisturbed case, we employ monotonicity of finite-horizon optimal control value functions to establish an upper bound to NMPC loop performance. We highlight assumptions which are required to achieve this upper bound and offer insight as to how one might satisfy these assumptions.

We tackle a third problem which is unrelated to performance of a closed-loop system, but which finds application in real-time MPC calculations. We consider the development of distributed algorithms which cooperating nodes can employ to solve a global optimization problem. The global solution constitutes the performance benchmark of interest and we seek distributed algorithms which nodes can employ in order to achieve this solution. Each node has access to local information which is suitable for solving a local optimization problem subject to local constraints; the nodes are coupled through a coupling constraint and through the structure of the global cost function. Our focus lies in understanding the required information content exchange between nodes to solve the global optimization problem. We show that the amount of information content is related to activity of both local constraints and coupled constraints at the global solution.

# Chapter 1

## Introduction

*Hello, my friend! Stay awhile, and listen.*

— Deckard Cain

This dissertation investigates stochastic disturbance rejection, performance, and optimal control through the lenses of an application, theory, and development of an algorithm. We are motivated by industrial systems, where manufacturing performance is demanded in spite of persistent disturbances which act to adversely impact manufacturing output. Optimal control provides a framework in which we can accomplish control design which at once provides performance guarantees and handles disturbances.

The application considers estimation and control design for a production (laser) light source, which is used in the photolithography stage of semiconductor chipmaking. The theoretical developments address nonlinear model predictive control (NMPC), as applied to discrete-time, stochastic, nonlinear systems. The algorithm is developed to aid the analysis of information exchange required to solve a distributed optimization problem.

While each study – application, theory, and algorithm development – is distinct, it is the necessity of archival publication that dictates this to be so. The overarching presence is optimal control and its associated performance.

The general topics under consideration in this dissertation follow.

**Performance:** The Oxford English Dictionary defines performance as, “the capabilities, productivity, or success of a machine, product, or person when mea-

sured against a standard.” In our context, we consider the performance of a system as can be guaranteed by the insertion of a feedback control system. Prescription of performance bounds is particularly important for industrial systems, e.g., photolithography tools which are responsible for the manufacturing of chips. Performance of these manufacturing tools directly impacts chip yield which in turn impacts chipmaker profitability, device manufacturing, and, eventually, consumer confidence. Theoretical investigations yield performance guarantees, subject to assumptions. A discrepancy between theoretical performance and achieved performance might indicate inappropriate assumptions have been asserted.

**Constrained Optimization:** Optimization brings with it a value function for performance. In constrained optimization, satisfaction of the constraints is a higher-level objective than performance. Indeed, an optimization problem in which there is only one solution that satisfies the prescribed the constraints is, by default, the optimal solution. Optimization problems of present interest are non-dynamic and performance-oriented. We focus on distributed solutions to (not necessarily) static, constrained optimization problems where several nodes cooperate to compute the solution to the optimization problem. We use properties of Lagrange multipliers to develop our distributed algorithms and ascertain when the global solution is achieved.

**Optimal control:** The framework of optimal control is centered on the selection of a control strategy which optimizes a performance function; this performance function can be selected to reflect the behavior of parameters which are pertinent to the successful operation of the system of interest. The connection between the selected control strategy and the performance bounds which can be guaranteed is of particular interest.

Optimal control encompasses the solution of an optimization problem in which a control solution is sought which minimizes a specified performance measure of importance, as evaluated over a specified period of time. The time period can be infinite (infinite-horizon optimal control) or finite (finite-

horizon optimal control). Receding-horizon control can also be considered, where the optimization problem is solved considering a finite horizon yet the control solution is applied (and performance is measured) for infinite time. We address all three time-horizon cases here.

**Stochastic Disturbance Rejection:** Industrial systems are fraught with disturbances which impinge on process and performance variables of interest. It is difficult, in the context of stochastic disturbances, to provide guarantees of system performance; though, optimal control provides a suitable framework and toolset in which this issue can be studied.

We assume the disturbances have a stochastic nature for several reasons. First, the stochastic framework provides a manageable description of uncertain behavior. Second, we can greatly benefit from the multitude of tools available to analyze stochastic disturbances: correlation functions, state estimation and prediction, covariances, distribution functions, etc. This framework and associated tools allow for comprehensive analysis and rigorous proof of results relating to uncertainty.

## 1.1 Motivation

Closed-loop control performance, of both linear and nonlinear systems which are subject to persistent disturbances, is our present interest. Our focus throughout is twofold. First, we aim to minimize the performance function and, following this, pursue understanding of theoretical lower bounds for system performance under closed-loop control; this is the bound to which we compare all subsequently-achieved performance values. Realization of a feedback controller which achieves this theoretical performance level is possible, though unlikely. Rather, our second aim is to derive and understand upper bounds on performance levels under a given feedback controller and contrast these upper bounds with the theoretical lower limit of performance.

In this vein, we tackle two problems. The first problem we consider is an application to photolithography light sources in which novel extensions of existing

control-theoretic ideas are employed to design a feedback control system for the rejection of disturbances, some of which are sinusoids which appear aliased in measurement data. We use ideas introduced by Harris [32] on establishing theoretical performance limits using offline analysis of system operation data. In particular, we construct a disturbance state estimator and create an offline prediction error and evaluate its variance; this represents the best achievable variance level that can be achieved in closed-loop operation. We demonstrate on a production light source that the presented scheme yields a guaranteed level of performance which we show is close to the optimal variance for the light source hardware under consideration.

Second, we develop new theoretical results for establishing upper bounds for performance of Nonlinear Model Predictive Control (NMPC) schemes as applied to discrete-time nonlinear systems which are subject to persistent, stochastic disturbances. Following Grüne and Rantzer [30] and Jadbabaie and Hauser [39], who have recently presented results for the undisturbed case, we employ monotonicity of finite-horizon optimal control value functions to derive an upper bounds for achieved performance of the NMPC controller on the infinite-horizon. We pay particular attention to the assumptions and conditions which are required to obtain these bounds and comment on difficulties which might arise. Of special importance in our setting is the inclusion of constraints on the system state and control input, which are present alongside the persistent disturbances. We show that, using feasibility analysis of this NMPC problem, bounded-input / bounded-state (BIBS) stability can be established.

Our third subject steps back from feedback control as we consider algorithms for the distributed solution of a global optimization problem; the distributed algorithm is one which distributed nodes can employ to cooperatively solve a global problem which is of interest to all nodes. Each node has access to a local cost function and local constraints, but share a common coupling constraint which limits the independent behavior of the nodes. The global cost function is constructed as a sum of nodes' local cost functions, which further couples the behavior of the nodes. We show the distributed algorithm produces iterates which converge to the global solution and hence achieves global performance while minimizing informa-

tion exchange between nodes. Information exchange between nodes is the focus of this subject and we seek to understand exactly what information is required to solve the global problem. While this algorithm is general, we are motivated by its specific application to MPC dynamic optimization and its implementation in real time and in physically distributed systems; such examples are provided.

### 1.1.1 Chapter 2: Production Light Source Application

Light sources for semiconductor photolithography are frequently subject to new performance requirements. As critical dimensions decrease, the performance required of all stages of photolithographic processes must improve. For the light source, these new performance requirements manifest as tighter specifications on variance of energy, bandwidth, and wavelength of light source pulses. Light sources have little margin against light output performance specifications, hence new performance requirements result in light source control systems which no longer operate within specification.

New performance requirements can be met by changing light source hardware or control algorithms; in our application, we pursued changes to both hardware and algorithms. The hardware modifications included a modification to actuator drive electronics and were relatively simple. The software (and firmware) modifications were fairly complex and constituted the introduction of a completely new estimator-based state-feedback control algorithm, along with new control system timing and data synchronization design.

The combined algorithm, software, firmware, and hardware changes yielded a production light source which operated within specification, relative to the new performance requirements. We have shown the production light source operates with a performance level that is provably close to hardware limits.

Very recent semiconductor critical dimension roadmap-induced performance requirement changes have led to fundamental alterations to light source hardware technology and their concomitant algorithm and software challenges. Research, design, implementation, and test of high-performance control algorithms and hardware is ongoing.

### 1.1.2 Chapter 3: Stability and Performance of Nonlinear Model Predictive Control

Model predictive control (MPC) originated as an industrial controller which showed promise for the control of systems which are subject to operating constraints (i.e. on pressures, temperatures, actuator limits, etc) and persistent disturbances. MPC allowed the mixture of optimization and constraints with feedback control, and constitutes a model-based full-state-feedback controller. Practical implementation led theory by a large margin. Though, there were no guarantees of either stability or performance bounds for MPC as applied to these industrial systems. This lack of understanding of MPC as a control law led academics and industrial engineers alike to assess theoretical guarantees of the scheme.

Asymptotic stability of deterministic (or, disturbance-free) systems became a hot topic for MPC researchers. Robust<sup>1</sup> MPC was later introduced to assess stability in the context of worst-case disturbances, and input-to-state stability (ISS) results have been established and existence of performance bounds has been shown. Unfortunately, there has been a dearth of results concerning prescription of performance bounds for MPC when disturbances are present. Grüne and Rantzer [30] have recently addressed the performance question for deterministic, undisturbed systems using infinite-horizon optimal control as the performance benchmark; their approach and results inspire our present developments.

Our aim is to refocus and reignite interest in theoretical developments which concern performance guarantees for systems which are subject to both persistent stochastic disturbances and constraints, as this is the focus of model predictive control as an industrial control application. We have interest in the currently evolved apparent complexity of the MPC recipe and its elimination; we turn recipe-driven approaches into focus on the principle aims of industrial control. Commercially-available MPC algorithms are presently in operation on industrial systems; the lack of guarantee of performance bounds when stochastic disturbances are impinging upon the system is disconcerting and needs to be rectified.

MPC still needs to evolve in the following areas.

---

<sup>1</sup>In as much as a full-state feedback could be robust.

- Robustness to modeling errors.
- Optimal state estimation feedback.
- Tight signal bounds and their dependence on design elements.

### 1.1.3 Chapter 4: Algorithms and Information Exchange in Distributed Optimization

In the area of distributed optimization – where several nodes cooperate to solve a global optimization problem – very little attention has been paid to the analysis of inter-node information exchange which is required to solve the global problem. Of particular importance is the information architecture of these distributed optimization problems, which has also received little attention in the research community.

Our interest is in large-scale optimization problems where limited communication becomes important. As motivating control examples, we consider two ships transferring cargo at sea which are connected by a cable, or, in three dimensions, the mid-air refueling of aircraft. The optimization problems under consideration are nominally static ones, as in, for example finite-horizon optimal control problems or MPC problems.

Parallel processing is also an application of interest, where the focus is on management of communication for the distributed solution of an optimization problem among multiple processors. Parallel processing was of interest in the 1980s in the consideration of systolic arrays for multi-processor computation; it is re-emerging with MPC applications which require fast, reliable implementation. Real-time MPC – meaning, deterministic computation times – is of particular interest in embedded applications. Thus our aim is to understand bounds on required communication to solve optimization problems which might be used in MPC designs.

## 1.2 Contributions

The specific contributions of Chapter 2 follow.

- We analyze light source disturbances and determine the existence of periodic disturbances which appear aliased in the variable-rate pulse data.
- We use continuous-discrete Kalman filtering ideas to design a multi-rate estimator/predictor for the reconstruction of the aliased disturbances. Given the multi-rate nature and time delays associated with the light source data, the estimator we design is generalizable to asynchronous, multi-rate applications.
- We employ the continuous-discrete Kalman filter to perform prediction-error analysis of light source data in an offline setting. We then use the prediction error variance to establish the best-achievable control performance.
- We design and implement a regularized, minimum-variance, state-feedback controller to reject the aliased, periodic disturbances.
- We make modifications to the light source hardware electronics design to increase the bandwidth of the actuators. These design changes are implemented in light source hardware and installed on a production light source.
- We test the new control algorithm and hardware, and show achieved performance which matches the best-achievable for the hardware.
- The control algorithm and hardware modifications have been propagated to several tens of production light sources around the world. Chipmakers in Asia, Europe, and the US have realized the performance improvements which the algorithm provides.
- A patent has been awarded (outside this dissertation work) which addresses the algorithm, hardware, software, signal timing, and firmware design.

The specific contributions of Chapter 3 follow.

- We establish connections between NMPC finite-horizon optimal control design elements (stage cost, terminal cost, horizon length, state constraints, discount factor) and achieved performance.
- We identify and exploit the use of monotonicity of value functions in horizon length in proving performance bound results.
- We analyze feasibility of constrained finite-horizon optimal controllers and recursive feasibility of NMPC controllers from a purely topological viewpoint.
- We establish a link between topological feasibility analysis and bounded-input, bounded-state (BIBS) stability. These state bounds apply without having to require the state to converge to a set in which constraints are inactive.
- We establish quantifiable performance bounds for stochastic, constrained, nonlinear systems under MPC control: the bound relates the finite-horizon value function to the infinite-horizon optimal control value function and the achieved value when the system is under MPC control. The performance bound results we derive are the first in the literature which apply to systems with stochastic disturbances.
- We perform comparisons of our performance bound results for stochastic systems with performance bound results for undisturbed systems. In particular, we compare the assumptions required to achieve our results with those present in current literature.

The specific contributions of Chapter 4 follow.

- We consider the distributed solution of a convex optimization problem, which is subject to constraints. We propose an information architecture which sees local nodes having access to local constraint functions, local cost functions, and elements of a global coupling constraint function.
- We establish that the minimizers which solve optimization problems defined by local constraints and local cost functions solve the global problem if and

only if the combined minimizers are feasible with respect to the global coupling constraint: this is the minimal-information-exchange case.

- We propose a coordinate descent algorithm which sees the nodes exchanging vector information which corresponds to the coupling constraint. The coordinate descent algorithm maintains feasibility and guarantees performance-improving iterates.
- We provide testable conditions under which the coordinate descent algorithm yields the global solution.
- We prove that the coupling constraint is active at the terminating point of the coordinate descent algorithm.
- We propose a constraint negotiation algorithm in which nodes exchange local Lagrange multipliers in an effort to compute a feasible, performance-improving iterate.
- We identify six cases which occur as a result of the constraint negotiation algorithm, which must be evaluated before iterations can continue. Five of these cases maintain bounded communications. Case 6 may require unbounded communication.

# Chapter 2

## Aliased Disturbances

*No matter how many instances of white swans we may have observed, this does not justify the conclusion that all swans are white.*

— Karl Popper

### Abstract

We study a disturbance rejection problem in a production pulsed light source, used in semiconductor photolithography, to yield quantifiable and guaranteed improved performance over existing control techniques. The disturbances of interest include an offset with reset properties and sinusoids which appear aliased in the measured data which is available only at pulse events. The light source is pulsed at varying rates yet actuators move in continuous-time, yielding a system which blends aspects of continuous-time and variable-data-rate discrete-time. We employ novel modifications to standard continuous-discrete Kalman filtering ideas for disturbance state estimation and establish and solve a non-standard regularized minimum variance control problem within a disturbance rejection framework. The controller as discussed is now in production in semiconductor lithography manufacturing lines. We analyze data from these production light sources and show the controller has the capacity to remove aliased sinusoids from the measured output and yields operational performance levels provably close to optimal for the hardware.

## 2.1 Introduction

We study the feedback control of a pulsed light source as forms the backbone of the photolithography stage of the semiconductor industry. Cymer Inc. – the market leader in developing deep-ultraviolet (DUV) and extreme-ultraviolet (EUV) light sources – continues to realize the commercial application of such feedback control research as they develop products for ever-growing lithography performance and throughput demands. The light sources provide a sequence of very narrow-band light pulses which illuminate a mask and expose the photo-resistive material on silicon wafers [55]. Critical performance specifications of the light include stability in energy, wavelength, bandwidth, and divergence which have strong influences on chip Critical Dimension (CD) which is a key process performance metric for lithography [54]. Typical deviations in the light source performance signals (hereafter referred to as outputs) are affected by a range of system disturbances which manifest as offsets, drifts, and periodic signals. These output signals, however, are measurable only when the light source is pulsed, as they are features of the light pulse itself. The pulse-rate is determined by the downstream scanner process, described in detail in the recent tutorial article [16], and is variable between long periodic bursts of pulses, where one burst typically corresponds to one die on the wafer. These bursts are separated by quiescent intervals – corresponding to moving to an adjacent die, a new wafer, or a new set of wafers – where no pulses are fired and, hence, no measurement of light properties is available. As a result of the variable pulse-rate, the following effects are evident.

- The light source control subsystem must operate with measurements of the outputs arriving at the variable pulse-rates, even though the control can be applied at an underlying, faster, fixed rate which is based on the clock of the control computer.
- At restart after a quiescent interval there is an increased uncertainty in the light source state which can lead to errant pulses. These errors must be rapidly accommodated as the exposure requirements dictate a maximal number of errant pulses which can be tolerated per die. Specifically, the controlled

output deviation is averaged over a multi-step window (the window being reset for each burst) and this average deviation must remain below a prescribed threshold.

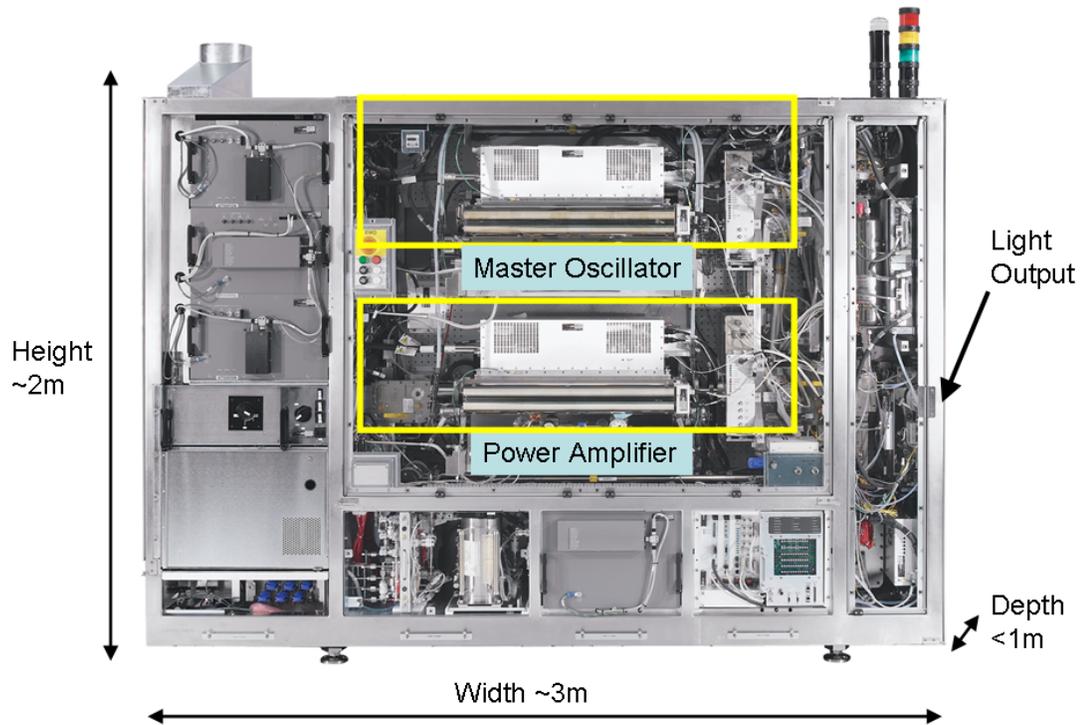
- Because of the variable sample-rate, the apparent discrete frequency of periodic disturbance signals changes. At lower rates, the periodic components of the disturbance can appear aliased in the measured data, although the underlying fundamental frequency is known. Such periodic components increase the variance of the output and hence increase the likelihood of crossing the performance thresholds unless removed by control.

The feedback control approach seeks to use these available measurements of the light source outputs to reject the disturbance effects to improve control performance. One of the main aims and contributions of this chapter is the development and demonstration of a methodology for the reconstruction and subsequent removal of aliased periodic disturbances, though the feedback approach is asynchronous to pulse events and not periodic. The algorithmic techniques we discuss are in use on a large number of production light sources operating at various chip-maker locations around the world and yield significantly higher performance than legacy control designs [90], [89]; the use of the algorithm is protected by pending patent [88]. A further contribution is the analysis of light source performance limitations and calibration of the closed-loop in achieving close to the highest performance possible given available hardware.

The development is supported by the following figures, which will be brought into play subsequently but which we find useful to present now.

- Figure 1 is a photograph of a Cymer deep-ultraviolet (DUV) photolithography light source and is representative of the system which was used for development and validation of the algorithm presented herein. The photograph reveals the various modules that comprise the light source; under normal operation these modules are concealed with large doors. Note the dimensions of this production system.

- Figure 2 is a schematic of the physical elements comprising the photolithography system.
- Figure 3 presents a block diagram of the laser control system with disturbances assumed additive.
- Figure 4 depicts light pulse timing and the corresponding output time series, which illustrates the reset features of the disturbance after a quiescent interval.

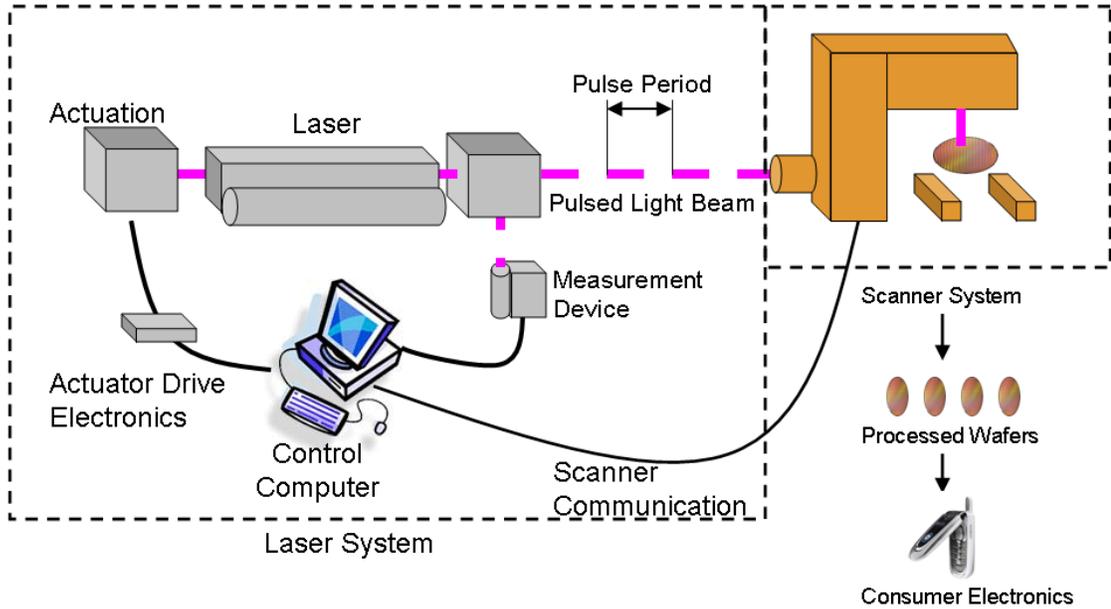


**Figure 2.1:** Deep-ultraviolet laser system and view of system components

### 2.1.1 Description of Physical System

The light source of interest here is one subsystem of the photolithography process. A beam delivery unit directs the light from the source to the scanner,

which is responsible for handling and exposing wafers. The scanner manufacturers are responsible for chip exposure performance and throughput. As industry standards tighten and more transistors are required per square millimeter (smaller critical dimension), exposure performance must also improve [38]. Improving the control of the light output properties is a necessary precursor to improvements of downstream scanner performance.

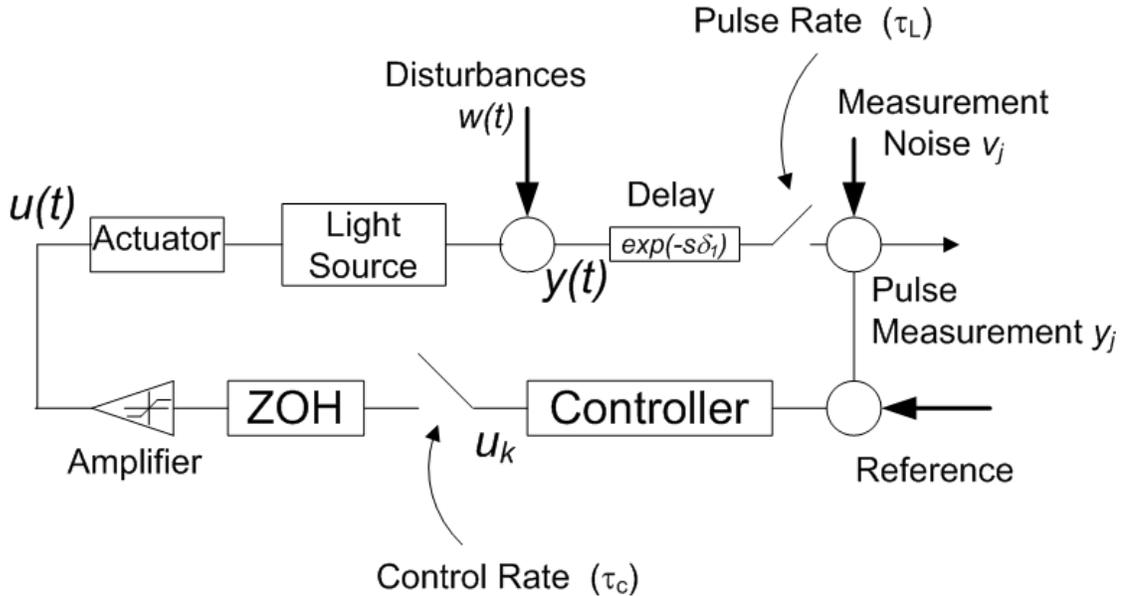


**Figure 2.2:** Photolithography system schematic

The light source provides the scanner with a sequence, or burst, of light pulses at a scanner-specified pulse-rate and starting time, which can vary due to exposure performance needs and throughput demands. The corresponding pulse period can vary from 10 microseconds to 1 millisecond in our application. Light output measurements are provided to the control subsystem after each pulse (with pulse duration on the order of  $10^{-9}$ s) by various measurement devices; for instance, energy is measured using a photodetector module, and wavelength is measured using an etalon, diffraction grating, and a one-dimensional photodiode array. The measurements are provided to the control system computer after a known but device-specific variable delay; as an example, the majority of the delay in the wavelength sensor can be attributed to the readout time of the photodiode array

whereas the delay in the photodetector module for measuring energy is a result of the pulse integration time. Each light property is measured once for each pulse.

The outputs of interest are modified by the actuation of mechanical devices (e.g. solenoids, piezoelectric transducers, stepper motors, etc.) which are affixed to various optical elements, the positions/angles of which directly affect the measured light properties. The very short pulse length relative to the actuator dynamics means that the velocity of the optical element is immaterial to control, which therefore depends solely on the instantaneous position, or angle, of the element at pulse time. We use this feature to reject the aliased disturbances. The electro-mechanical stages are themselves lightly damped resonant systems with resonant frequencies ranging from 1Hz to 15kHz. The actuator drive-electronics may introduce additional low-pass dynamics. Notable features of these actuators include hysteresis, creep, slew-rate limit, and control saturation limits imposed by the drive electronics.



**Figure 2.3:** Light source system diagram depicting disturbances  $w(t)$ , sampling delay  $\delta_1$ , fixed control update rate  $\tau_c$ , and known but variable pulse-rate  $\tau_L$ , which is dictated by the downstream process.

The dynamics of the electro-mechanical elements and the drive electronics are well modeled. We will employ these models as we construct model-based esti-

mators. Throughout the chapter we will refer only to the light source outputs, since the methodology is independently applicable to all performance properties of the light; however, our descriptions and data analysis mainly apply to the wavelength control subsystem.

### 2.1.2 Description of Control Approach

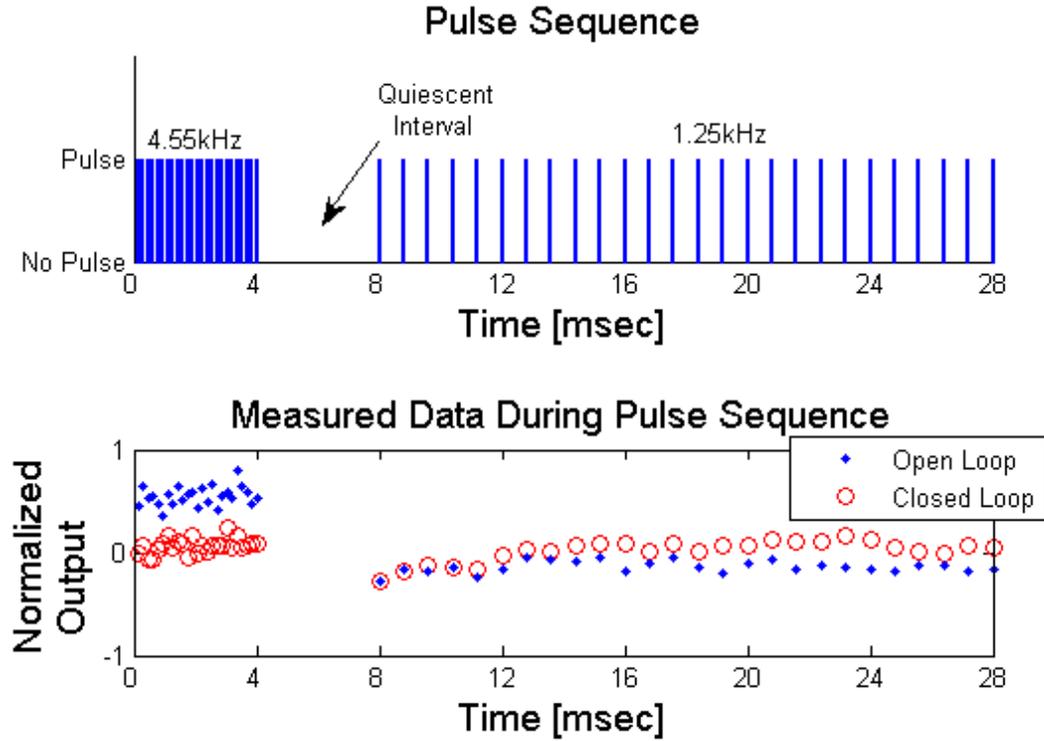
The existing or legacy method of controlling the light source output is achieved via a proportional-plus-integral (PI) feedback control scheme [95]. The legacy PI controller is event-driven, with no control action taken faster than the varying pulse-rate nor during the quiescent interval. Furthermore, it is a fixed-gain controller, with the gain selected to provide sufficient gain and phase margins at all pulse rates.

A final item which deserves special attention is the presence of hardware low-pass filters in the actuator drive electronics. These filters limit achievable control performance by introducing phase lag. Although deemed necessary with the legacy PI controller in the loop, we require that the bandwidth of these low-pass filters be substantially increased so that the full speed of the actuator can be used by the new controller. This requirement represents the single hardware alteration of the legacy control system. This is now implemented in production systems.

Here we present a control design which uses knowledge of the system delays, disturbance dynamics (which we have yet to discuss in detail), and actuator dynamics to achieve improved performance over the legacy controller. We use a generalization of the continuous-discrete estimator [41] to overcome the variable data-rate and, with appropriate modifications to accommodate system delays and disturbance properties, reconstruct the underlying state. This state estimate is then used in a non-standard, time-varying, certainty-equivalence, linear state variable feedback controller that is synchronous to real-time rather than to light source firing events. The period of our controller is equal to the fastest pulse period of the light source under control. This controller approach enables rejection of aliased periodic components as the controller has the capacity to position the actuator

appropriately at the exact moment the light pulse comes into contact with the optical element. The described approach is at first glance similar to the dual-rate control of [93], though we deal with a varying pulse rate and variable measurement delay.

The treatment of aliased disturbances in disturbance rejection control has been considered in Iterative Learning Control and Repetitive Control contexts by Ratcliffe et al. [80], [81]. In these works, aliasing is used as a means to filter and hence remove resonant modes that appear outside of the learning algorithm's bandwidth and therefore prevent their deleterious influence on the convergence. The filtering is achieved by down-sampling in such a way that the resonant mode is placed at DC; hence knowledge of the aliasing process is beneficial to control design. Here we perform an inverse operation, starting with the aliased signal, and reconstructing the unaliased signals to generate a control-sample-rate disturbance rejection feedback.



**Figure 2.4:** (a) Illustration of pulse sequence timing (b) Measured output data during pulse sequence

The contributions contained in this chapter are threefold. First, we demonstrate that aliased signals can be reconstructed and then rejected with a time-varying sampled-data control. We implement and test the algorithmic solutions discussed in the chapter on a production light source and show improved performance over legacy light source algorithms; the performance improvement is close to a 45% reduction in closed-loop variance for the tested control loop. We determine via prediction error analysis that the achieved performance is close to the theoretical upper bound for the hardware and discuss calibration of the closed-loop in achieving this performance level.

The rest of the chapter is organized as follows. In Section 2.2, we discuss the disturbance process and modeling. In Section 2.3, we detail the control system timing and bookkeeping, and estimator design. In Section 2.4, we present the

control problem and discuss its solution. In Section 2.5, we evaluate estimator performance and present performance data taken with an implemented version of the proposed controller on a Cymer light source. The evaluation of and comparison to optimal performance is also presented here. We end in Section 2.6 with closing remarks and suggestions for further control improvement.

## 2.2 Process Modeling

### 2.2.1 Process Modeling

*Actuator Stage.* The dynamic model for the actuator stage is a continuous-time, second-order resonant system with resonant frequency at approximately 1600Hz. The electronics that drive the actuator impose a slew-rate limit and maximum and minimum limits on the control signal, which will be addressed in the control design. Additional features of the actuator stage include hysteresis and creep. The hysteresis proves too small to be of concern whereas the effect of the creep is captured with the disturbance offset models that will be presented.

*Light Source Stage.* The light output is measured only at pulse times. The measurement becomes available to the control computer after an asynchronous processing delay from the measurement device, which we model as a known transport delay and denote by  $\delta_1$ . The data transfer time of the control signal we also model as a known transport delay,  $\delta_2 < \tau_c$  with  $\tau_c$  being the period of the controller. The downstream controlled variation in scanner-determined pulse-rate is modeled as a known variable rate sampling.

### 2.2.2 Disturbance Modeling

The disturbance process is modeled as additive to the light output and dependent on the variable pulse-rate of the light source, as depicted in Figure 2.3. It can be determined from the open-loop signal in Figure 2.4b that the disturbance can be modeled by three principal components.

*Offset.* When the light source commences pulse generation after the quies-

cent interval, an offset is observed in the measured output relative to the desired target. The size of this offset is unknown but within known bounds. We approach the modeling of this component phenomenologically with an offset model.

*Drift.* There is a slowly drifting, or lowpass, component to the disturbance. We capture this by adding driving process noise to our offset model.

*Periodic Signals.* There are multiple periodic components in the open-loop data at known frequencies, some of which appear aliased in the measured data. These components do not manifest themselves until several pulses have been generated. We approximate these periodic components with sinusoidal models at multiple known frequencies.

We incorporate these unstable disturbance models into our control design as we aim to capture these observed signal behaviors and subsequently remove them by feedback control.

### 2.2.3 State Equations

We now present the state equations for the models discussed above. A comprehensive list of symbols and descriptions is included at the end of the chapter.

Let;  $x_p(t)$  be the continuous-time state of the combined actuator stage and plant,  $x_o(t)$  be the disturbance offset state, and  $x_{s,i}(t)$ ,  $i = 1, \dots, n_s$  be the state of the  $i^{th}$  sinusoid, where  $n_s$  is the total number of sinusoids in the disturbance model. Let  $x_D(t) = \left[ x_o^T, x_{s,1}^T, \dots, x_{s,n_s}^T \right]^T$ , be the augmented disturbance state. Let continuous-time  $u(t) = u_k$ ,  $k\tau_c \leq t < (k+1)\tau_c$  be the zero-order hold of the discrete input  $u_k$  which is updated at the control period  $\tau_c$ . Also let  $y(t)$  be the system output deviation from nominal and  $y_j$  be the  $j^{th}$  sample of  $y$  occurring at the  $j^{th}$  light pulse. We assume bandlimited broadband process noise disturbances  $w_p(t)$  and  $w_D(t) = \left[ w_o(t)^T, w_{s,1}^T, \dots, w_{s,n_s}^T \right]^T$  in our model, which will allow us to generate our continuous-discrete estimator in the next section.

With these definitions we have the following augmented state, output, and measurement models,

$$\begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_D(t) \end{bmatrix} = \begin{bmatrix} A_p & 0 \\ 0 & A_D \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_D(t) \end{bmatrix} + \begin{bmatrix} B_p \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} w_p(t) \\ w_D(t) \end{bmatrix}, \quad (2.1)$$

$$y(t) = \begin{bmatrix} C_p & C_D \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_D(t) \end{bmatrix}, \quad (2.2)$$

$$y_j = y(j\tau_L - \delta_1 - \delta_2) + v_j. \quad (2.3)$$

Here we have, for a damped second-order oscillator as our actuator stage,

$$A_p = \begin{bmatrix} -\zeta\omega_n & \omega_n\sqrt{1-\zeta^2} \\ -\omega_n\sqrt{1-\zeta^2} & -\zeta\omega_n \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ \frac{g\omega_n}{\sqrt{1-\zeta^2}} \end{bmatrix},$$

with  $\zeta, \omega_n, g$  the damping coefficient, natural frequency, and DC gain of the model (all known); and our disturbance model,

$$A_D = \text{blkdiag}(1, A_{s,1}, \dots, A_{s,n_s}),$$

with

$$A_{s,i} = \begin{bmatrix} \cos(\omega_i) & \sin(\omega_i) \\ -\sin(\omega_i) & \cos(\omega_i) \end{bmatrix},$$

and  $\{\omega_i\}$  fixed and known.

The output matrices are  $C_p = \begin{bmatrix} 1 & 0 \end{bmatrix}$  and  $C_D = \begin{bmatrix} 1 & C_{s,1} & \dots & C_{s,n_s} \end{bmatrix}$  with  $C_{s,i} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ ; the quantities  $\delta_1$  and  $\delta_2$  are variable, but known, components of the total transport delay, and we have  $v_j$  as a discrete measurement noise sequence. We will write (2.1-2.2) as  $\dot{x} = Ax + Bu + w$  and  $y = Cx$ , respectively.

Note that the laser, per se, is not explicitly identifiable as part of this model. An integral part of the lasing process for DUV lithography is a diffraction grating which acts to filter out undesirable wavelengths that might occur. The actuator stage here controls the light incidence angle on the grating thus acting as a high-resolution wavelength selector. Our aim is to control the actuator in continuous or fast discrete time so that the appropriate incidence angle, hence appropriate wavelength selection, is achieved at the slower discrete pulse time.

## 2.3 Estimator Design

We use the underlying ideas of the continuous-discrete Kalman filter, as developed by Jazwinski [41], as the basis for our estimator, as;

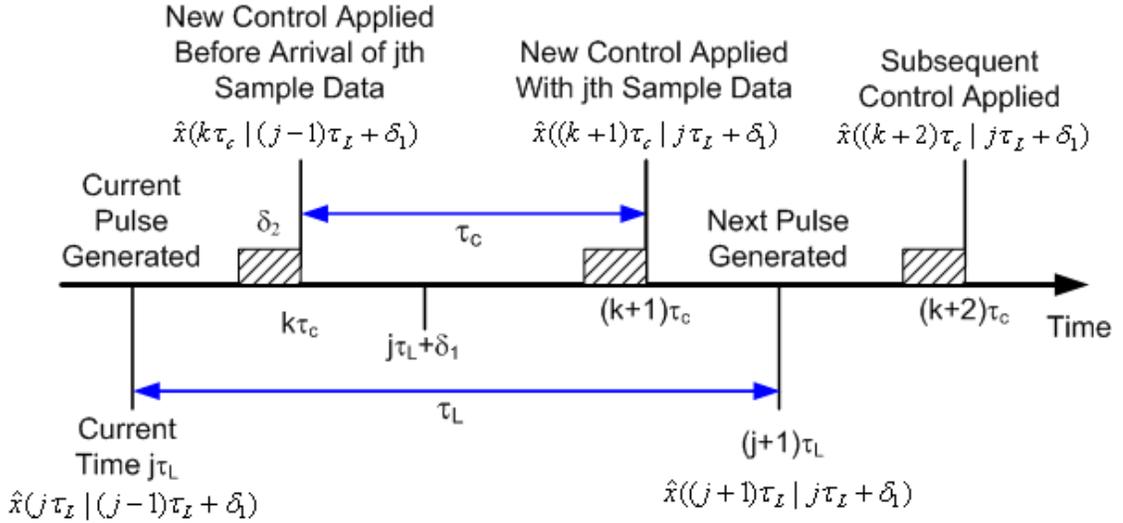
- the structure of the estimator handles directly a variable data rate,
- the estimator provides a continuous-time state prediction, or indeed a fast control-rate prediction, with discrete measurement updates, making it very useful in handling cases where the measurement is taken at a different time from that when the control signal is applied and,
- the estimator can be configured to use measured data to reconstruct the un-aliased, continuous-time version of each aliased periodic signal.

We modify the estimator to accommodate system delays using prediction ideas; the approach is different from the ideas presented in [4], [103], and [51], where the delay states are introduced to accommodate the measurement delay. Furthermore, we perform a reset of the estimator covariance at the beginning of each burst to accommodate the reset in the disturbance value. Filtered state estimates provided by the estimator need further propagation to provide appropriate state feedback at the next pulse time, which is asynchronous with the control update rate but is known. This is accomplished via prediction.

### 2.3.1 Signal Timing

We refer to Figure 2.5 illustrating a representative sequence of signal availabilities and timing requirements based around a current time  $j\tau_L$  of a laser pulse; recall pulse index  $j$  counts pulses into a burst and hence resets to zero at the beginning of each burst. There are two distinct, known but variable, delays:  $\delta_1$ , the delay between the pulse at time  $j\tau_L$  and the arrival of the pulse measurement data;  $\delta_2$ , the computation and communication delay required by the controller and depicted by the cross-hatched region preceding the control signal application at time  $k\tau_c$ .

Because of the non-commensurate timing of the data arrival and the assertion of the control, there is a need for the state estimator to provide predictions at a number of differing horizons. From the measurement availability at time  $j\tau_L + \delta_1$ , state estimates are needed for the next pulse at time  $(j+1)\tau_L$  for the propagation of the filter state, and at times  $(k+1)\tau_c$  and  $(k+2)\tau_c$ , i.e. all controls required before the availability of the measurement due to the pulse at time  $(j+1)\tau_L$ .



**Figure 2.5:** Timeline depicting the arrival of a new measurement (indicated by a small marker below the timeline at time  $j\tau_L + \delta_1$ ) due to a light pulse at the current time  $j\tau_L$ . Above the timeline, the control timing is indicated. The state estimate pertaining at each time and its dependence on data times is displayed by the arguments of  $\hat{x}$ . The relative timing of the control signal and the measurements is explained in the text.

Note that Figure 2.5 is purely illustrative and that control signals might occur several times in sections surrounding the arrival of a measurement. Nevertheless we shall describe the processing for the timing as illustrated in Figure 2.5.

### 2.3.2 Continuous-Discrete Kalman Filter

Denote by  $\hat{x}(t|j\tau_L + \delta_1)$  an estimate of  $x(t)$  based on data available at time  $j\tau_L + \delta_1$ , i.e. data up to that associated with the pulse occurring at time  $j\tau_L$ . From the timing diagram, we observe the measurement at time  $j\tau_L + \delta_1$  contains

information regarding the laser state at time  $j\tau_L$ . Accordingly the state estimate measurement update is,

$$\hat{x}(j\tau_L|j\tau_L + \delta_1) = (I - L_j C) \hat{x}(j\tau_L|(j-1)\tau_L + \delta_1) + L_j y_j. \quad (2.4)$$

$$L_j = \Sigma_{j|j-1} C^T (C \Sigma_{j|j-1} C^T + R)^{-1}, \quad (2.5)$$

$$\Sigma_{j|j} = (I - L_j C) \Sigma_{j|j-1}, \quad \Sigma_{0|-1} = \Sigma_0, \quad (2.6)$$

where the reset to  $\Sigma_0$  occurs at the start of each burst, and the predictive time update state is

$$\begin{aligned} \hat{x}((j+1)\tau_L|j\tau_L + \delta_1) &= e^{A\tau_L} \hat{x}(j\tau_L|j\tau_L + \delta_1) \\ &\quad + \int_{j\tau_L}^{(j+1)\tau_L} e^{A((j+1)\tau_L - \sigma)} B u(\sigma) d\sigma, \end{aligned} \quad (2.7)$$

$$\Sigma_{j+1|j} = e^{A\tau_L} \Sigma_{j|j} (e^{A\tau_L})^T + Q(\tau_L). \quad (2.8)$$

This is a non-standard time-varying discrete-time Kalman filter for the laser state, as the pulse index  $j$ , and hence the covariance, reset each burst. Furthermore, when  $j = 0$ , the sinusoid states are reset to 0 to reflect the understood behavior of the disturbance.

The process noise variance is pulse-period-dependent and is computed as [79],

$$Q(\tau_L) = \int_0^{\tau_L} e^{A\sigma} Q_c (e^{A\sigma})^T d\sigma,$$

which can be pre-computed and implemented in a lookup table, or computed online. The process and measurement noise variances for the filter,  $Q_c$  and  $R$ , are determined and validated empirically using the whiteness of the prediction error [65] and will be tuned in the next section. We will discuss selection of the initial state covariance  $\Sigma_{0|-1}$  later, since it is used to accommodate the uncertainty occurring at the beginning of a burst.

For the control signal occurring at  $k\tau_c$  a predictive state estimate,  $\hat{x}(k\tau_c|(j-1)\tau_L + \delta_1)$ , is required, where  $(j-1)\tau_L + \delta_1$  is the time of the most recently available measurement data. The computation of this state prediction proceeds via continuous-discrete Kalman filter ideas with prediction interval  $\gamma = k\tau_c - (j -$

$1)\tau_L - \delta_1$ . Thus,

$$\hat{x}(k\tau_c|(j-1)\tau_L + \delta_1) = \hat{x}(\gamma + (j-1)\tau_L + \delta_1|(j-1)\tau_L + \delta_1), \quad (2.9)$$

$$\begin{aligned} &= e^{A\gamma} \hat{x}((j-1)\tau_L|(j-1)\tau_L + \delta_1) \\ &+ \int_{j\tau_L}^{j\tau_L + \gamma} e^{A(j\tau_L + \gamma - \sigma)} B u(\sigma) d\sigma. \end{aligned} \quad (2.10)$$

Note the use of prediction ideas to accommodate measurement and control delays. The corresponding prediction covariance is not used in the controller and so is not computed.

### 2.3.3 Estimator Novelties and Digital Implementation

We employ a time-varying Kalman filter gain  $L_j$  to address the time variation of the measurement availability caused by the varying pulse-rate determined by the scanner and the resets that are observed in both the disturbance offset and sinusoid states at the start of a burst. When pulse generation commences after a quiescent interval, we reset the offset and sinusoid state covariances and sinusoid states which allows the estimator to acquire the new state values quickly [45].

The predictors in (2.7), (2.10) are given as the integral representation of (2.1) as it is the form best suited for digital implementation. Specifically, we can write the discrete-time prediction for the laser state as,

$$\hat{x}((j+1)\tau_L|j\tau_L + \delta_1) = e^{A\tau_L} \hat{x}(j\tau_L|j\tau_L + \delta_1) + B(\tau_L)u_k,$$

with  $B(\tau_L) = \int_0^{\tau_L} e^{A\sigma} B d\sigma$  when  $u(t) = u_k$  across the prediction interval  $\tau_L$ . When  $u(t)$  changes, say  $N$  times, across the horizon  $\tau_L$ , the integral (hence  $B(\tau_L)$  matrix) is split into multiple pieces  $[0 \ \gamma_1)$ ,  $[\gamma_1 \ \gamma_2)$ ,  $\dots$ ,  $[\gamma_N \ \tau_L)$  where the control signal changes at instances  $\gamma_1, \gamma_2, \dots, \gamma_N$  seconds into the horizon.

## 2.4 Control Strategy

The control update occurs synchronously to a real-time clock with fixed period  $\tau_c$  which is less than the variable light source pulse generation period,  $\tau_L$ .

The control objective is to minimize the deviation in the output,  $y_j$ , which requires proper positioning of the actuator stage precisely at the light source pulse time. This approach uses an underlying fast ( $\tau_c$ ) timescale model for control with the measurement data being presented only at the slow timescale ( $\tau_L$ ) of light source pulses. The brief light pulse duration,  $\sim 10^{-9}$ s, samples the output of the actuator-plant combination effectively instantaneously and without the introduction of further dynamics other than the measurement and processing delay.

We again refer the reader to Figure 2.5 for the following development. Suppose the next control signal update is to occur at time  $k\tau_c$  and the next pulse generation event is at time  $k\tau_c + \gamma$ . We solve a regularized minimum-variance control problem [21], [43],

$$\min_{u(k\tau_c)} \frac{1}{2} \left( \hat{y}^2(k\tau_c + \gamma | (j-1)\tau_L + \delta_1) + \rho u^2(k\tau_c) \right), \quad (2.11)$$

to yield the regularized minimum-variance control,

$$\begin{aligned} u(k\tau_c) = & \left( \rho + \left( \int_0^\gamma e^{A\sigma} B \, d\sigma \right)^T C^T C \left( \int_0^\gamma e^{A\sigma} B \, d\sigma \right) \right)^{-1} \\ & \times \left( \int_0^\gamma e^{A\sigma} B \, d\sigma \right)^T C^T C e^{A\gamma} \hat{x}(k\tau_c | (j-1)\tau_L + \delta_1). \end{aligned} \quad (2.12)$$

With a single control law we distinguish between control intervals in which the light source will not generate a pulse, and hence no measurement is taken, and those intervals during which a pulse is imminent. If an additional control update is to occur before the next pulse generation event, e.g.  $k\tau_c + \gamma > (k+1)\tau_c$ , then  $\gamma$  is set equal to  $\tau_c$  when computing the control in (2.12); the resulting control value is applied to the actuator drive electronics to appropriately position the optical element at control time  $(k+1)\tau_c$  as if it were to reject the predicted disturbance at that time.

When the light source is due to generate a pulse before the next control update, e.g.  $\gamma < \tau_c$ , then the control law (2.12) is applied without any alteration. In this case, the applied control positions the optical element to reject the disturbance at pulse time. Note the control law is time-varying, as the control update proceeds more frequently than pulse generation; hence throughout a burst the value for  $\gamma$  in (2.12) might vary from being slightly larger than 0 to as large as  $\tau_c$ .

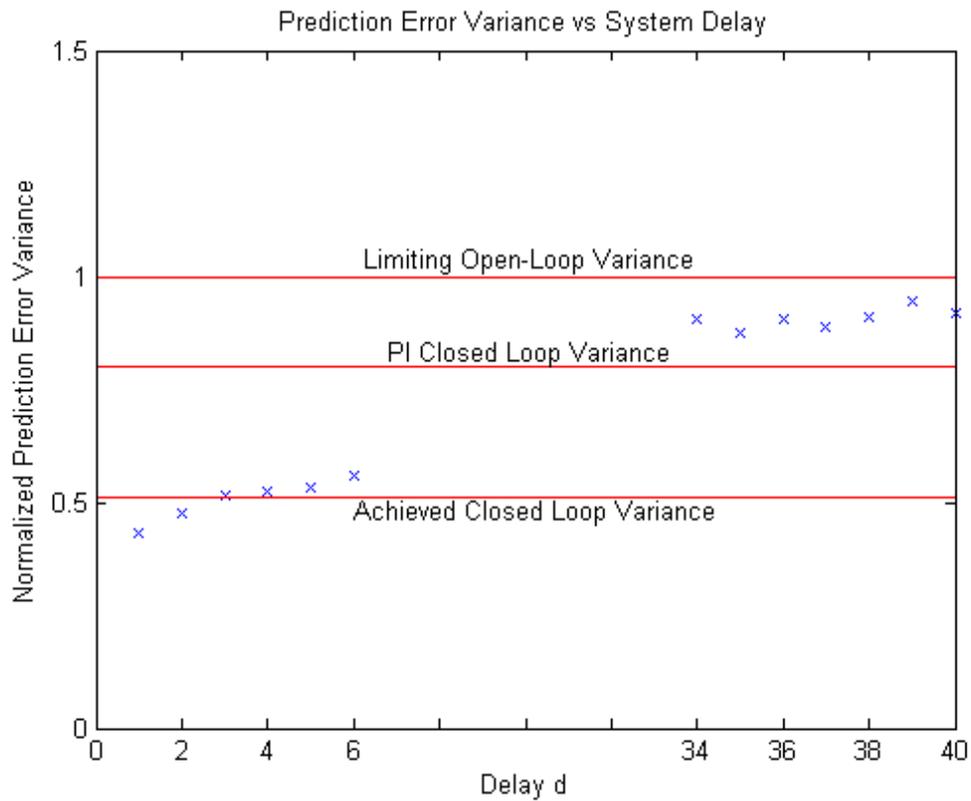
The regularizing control penalty  $\rho$  is imposed to satisfy the slew-rate and saturation constraints imposed by the actuator drive electronics and is determined empirically. The purpose of the control signal,  $u(t)$ , being updated during periods in which the generation of a pulse occurs is to regulate the light output to a suitable value for disturbance rejection; Figure 2.7(bottom) shows the efficacy of the closed-loop control at removing aliased disturbances from the measured output.

During those periods where a pulse is not generated, the control signal is still applied, but now to regulate the predicted state  $\hat{x}((k+1)\tau_c|j\tau_L + \delta_1)$  to a suitably small value in order that the subsequent controls have sufficiently small amplitude and hence the drive electronics constraints are not violated. Were these intermediate values of the control signal not to be applied, the corresponding value required for  $\rho$  would be much higher, resulting in a concomitant reduction of control performance. Measured closed-loop output and commanded control input time series are provided in Figure 2.9. Note that even though the control is rejecting periodic disturbances from the output, the control law is time-varying but not periodic in general.

## 2.5 Evaluating Estimator and Controller Performance, Tuning the Estimator

The  $d$ -step-ahead prediction error variance provides a lower bound to the achievable closed-loop output variance for a plant system with delay- $d$  [78]. This has been exploited by Harris [32] to provide an approach to the monitoring of the control loop to ascertain the potential maximal benefits to be derived from redesign of the feedback controller. In our context, we construct from open-loop (constant input) data a sequence of  $d$ -step-ahead (i.e.  $d \times \tau_L$ -seconds-ahead) predictors based on the disturbance models with differing values of delay  $d$  and plot the empirical prediction error covariance as a function of  $d$ ; details regarding the predictors will be presented later. This prediction error analysis provides an insight into the potentially best achievable closed-loop covariance of the light output as it varies with delay. It also provides a comparative open-loop or uncontrolled output

variance. This is plotted in Figure 2.6, which displays the monotonic increase of the prediction error covariance with plant delay and its relationship to the uncontrolled performance. For the real system, the worst-case (though quite standard) delay is between 2 and 3  $\tau_L$  periods depending on the behavior of the measurement device; that is, the sum of  $\delta_1$  and  $\delta_2$  is between 2 and 3  $\tau_L$ . This would indicate that up to 45% improvement in the output variance is potentially available through control – we achieve close to this figure. We next present the experimental details conducted on a standard calibration cycle of a production light source.



**Figure 2.6:** Graph of normalized prediction error performance versus assumed delay as integer multiples of pulse period  $\tau_L$ . These data were computed using measured operating data and the continuous-discrete estimator with different delay values. The figure also illustrates the closed-loop performance levels and shows that the new controller achieves close to the best possible variance for a delay of three periods  $\tau_L$ .

### 2.5.1 Estimator: Experiment and Results

Open-loop output data were collected from a light source for evaluation of estimator performance. The control signal (e.g. actuator input voltage) was held constant for a long period before and throughout the experiment. In the experiment, 300,000 data were recorded and consisted of a single burst at a nominal sampling rate immediately following a quiescent interval. The test sequence used here is a standard diagnostic check for evaluating open-loop performance on newly-manufactured and fielded light sources. Because the system is operated in open loop, the variation in the output measurement is due solely to the disturbance process with the resonant modes of the actuator stage unexcited. The continuous-discrete filter presented earlier in (2.4-2.8) is used to generate the  $d$ -step-ahead predictions as follows.

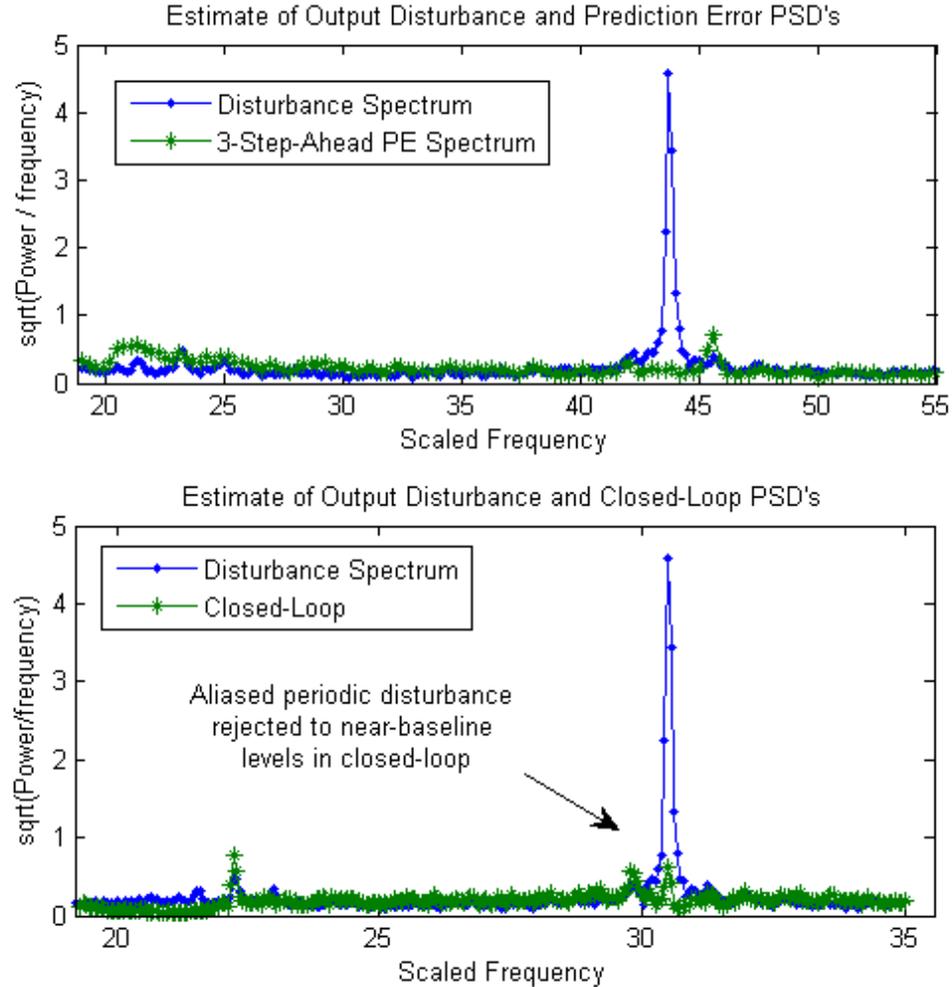
$$\hat{x}((j+d)\tau_L|j\tau_L + \delta_1) = e^{Ad\tau_L}\hat{x}(j\tau_L|j\tau_L + \delta_1),$$

where the integral that appears in (2.7) is omitted as the control signal is constant throughout the experiment.

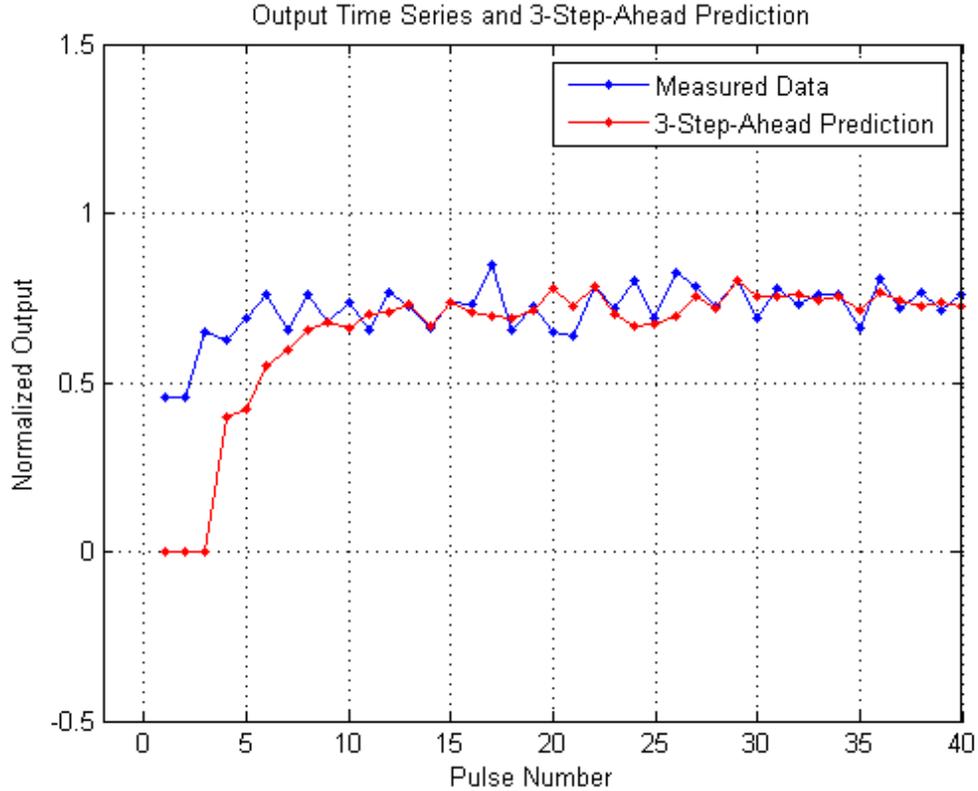
The prediction error data were analyzed in three ways.

- The mean of the prediction error is calculated and shown to be within 0.1% of the output signal indicating that the offset prediction and drift tracking function well in steady-state. This achieved tracking is near the limiting accuracy of the measurement device.
- The de-trended prediction-error data is used to compute an estimate of the power spectral density using Welch's averaged periodogram. This is depicted in the top plot of Figure 2.7 and demonstrates the efficacy of the sinusoidal tracker in the filter in steady-state.
- The transient performance of the filter is examined over the first 15-20 samples of a pulse series in order to tune the initial covariance reset for the rapid acquisition of the offset. Since the operational performance requirement is that the controlled output deviation averaged over a multi-step window fall below a prescribed threshold, this was used to select the appropriate value

for the burst-starting covariance,  $\Sigma_{0|-1}$ . The achieved estimator convergence rate is visible in Figure 2.8 where the 3-step-ahead predicted output is plotted alongside the open loop measured output data.



**Figure 2.7:** (Top) Power spectral density (PSD) estimate of the uncontrolled light property (blue dots and line) versus the estimate for the corresponding prediction error associated with the continuous-discrete filter (4-9). This illustrates the nature of the disturbance signal and the performance of the filter. (Bottom) Power spectral density estimate of the uncontrolled light property (blue dots and line) versus the PSD estimate for the production closed-loop controller. The controller is able to reject the aliased disturbance to near-baseline levels.

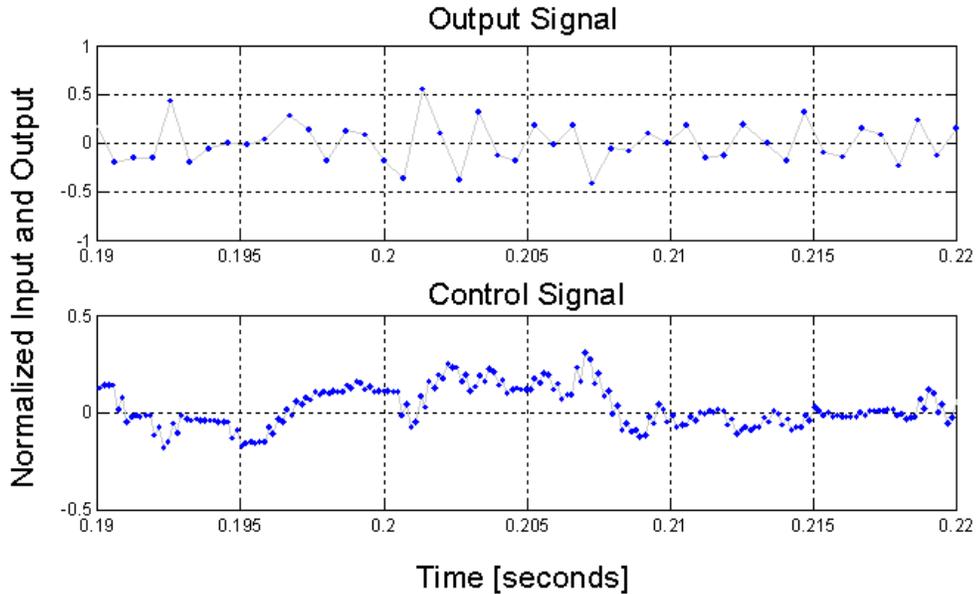


**Figure 2.8:** Measured light output property versus estimator predictions at the commencement of a burst following a quiescent period. The recapturing of the offset disturbance is tied to the initialization of the estimator covariance following a quiescent period.

## 2.5.2 Control: Experiment and Results

Analysis of the prediction error variance in the previous section indicates that up to 45% improvement in output variance might be achieved with control at a single pulse-rate, indicating that our estimator works well. However, the light source is typically operated at multiple pulse-rates within a very short period of time, and we would like to measure control performance in a setting close to the light source's use in the lithography application. There exist standard field diagnostic test patterns that are used to ascertain control performance. Such an experiment typically includes pulse generation at a multitude of pulse-rates in the operational range of the source. We run this experiment using both the

legacy control algorithm and the algorithm proposed in this chapter. The realized improvement over the legacy controller is approximately 35% averaged over the operating space tested using the output variance as the measure of performance. This is illustrated in Figure 2.6. This improvement is very close to the fundamental upper bound of improvement provided via evaluation of the prediction error in the previous section, indicating that the added nonlinearities of the actuator drive electronics and the corresponding control criterion with  $\rho \neq 0$  have only modest effect on the achieved performance. Additional performance data from production light sources can be found in [90], [89].



**Figure 2.9:** Closed loop: measured output and control input time series. Note that the control signal is not periodic even though the dominant disturbance is. This is due to the estimator-controller being asynchronous to pulse events.

## 2.6 Conclusions

In this chapter we present a methodology for designing an observer-based controller that has the capacity to estimate and reject aliased periodic disturbances in a measured output signal. Under this scheme we achieve output variance performance near the theoretical lower bound as given by the variance of the estimator

prediction error.

Key features of the design follow. In estimation we can operate with aliased (or demodulated) data and knowledge of the aliasing (or carrier) to reconstruct the un-aliased signal. After a modicum of thought, this is apparent. What is less obvious is that a related approach to feedback control of aliased disturbances is also possible and hence, because actuation is important only at pulse times, the aliased signal may be removed at the (slow) timescale of the light source pulse events. Under our scheme, control updates are provided even when the light source is not due to generate a pulse and hence the controller regulates the actuator state at control update instances. When a pulse is imminent, the controller pre-positions the actuator in order to reject the predicted disturbance at pulse time. With increased controller update frequency one could equally reject the now un-aliased disturbance, though our analysis suggests a faster control would not yield much performance improvement.

The estimator presented here is based on a generalization of continuous-discrete Kalman filter ideas, with careful timing management. The structure of the filter can be used to handle time variation of the measurement availability as caused by the varying, scanner-determined, pulse-rate. We employ covariance and state resets which we find are key to handling the disturbance resets that are observed at the beginning of a burst. The filtered state estimate is propagated to yield a predicted value correctly timed for the control signal. This requires further detailed timing management to yield a regularized minimum pulse variance control operating at the control update rate, which is asynchronous from the pulse firing rate.

The control has been fielded into a large number of production light sources at semiconductor fabrication facilities around the world.

## Symbol List and Descriptions

$\tau_c$ : Control signal update period, fixed and known (see Figure 3).

$\tau_L$ : Laser pulse generation period, variable and known (see Figure 3).

$\zeta, \omega_n, g$ : Damping, natural frequency, and DC gain of second order actuator model (fixed and known).

$n_s$ : Number of sinusoids in the disturbance model, including possible harmonic frequencies.

$\omega_i$ : Continuous-time frequency of the  $i$ th sinusoid,  $i = 1, \dots, n_s$  (fixed and known).

$\delta_1$ : Measurement delay, which is less than  $\tau_L$  (variable and known).

$\delta_2$ : Digital control signal transport delay from control computer to actuator, which is less than  $\tau_c$ . This delay includes control value computation time (variable and known).

$\gamma$ : Prediction horizon for the estimator-controller.

$t$ : Time from the beginning of a burst.

$k, j$ : Index for number of controller periods and number of pulse generation periods, respectively (reset at the start of a burst).

$x(t)$ : State at time  $t$ .

$\hat{x}(t_2|t_1)$ : Estimate of state at time  $t_2$ , conditioned on data up to and including time  $t_1$ .

$x_p(t)$ : Plant state.

$x_o(t)$ : Disturbance offset state.

$x_{s,i}(t)$  State of the  $i$ th disturbance sinusoid,  $i = 1, \dots, n_s$ .

$x_D(t)$ : Augmented disturbance state which includes offset and sinusoid states.

$u(t), u_k$ : Continuous-time control input at time  $t$  and discrete control input computed at controller update time  $\tau_c k$ , respectively.

$y(t), y_j$ : Continuous-time laser output at time  $t$  and discrete measurement of laser pulse number  $j$ , respectively.

$w_p(t)$ : Broadband, bandlimited, continuous-time (BDBLCT) process noise acting on the model plant state.

$w_o(t)$ : BDBLCT process noise acting on the model offset state.

$w_{s,i}(t)$ : BDBLCT process noise acting on the model  $i$ th sinusoid state.

$w_D(t)$ : Augmented disturbance state which includes offset and sinusoid disturbances.

$v_j$ : Discrete measurement noise acting on measurement  $j$ .

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**D.J. Riggs and R.R. Bitmead.** Rejection of aliased disturbances in a production pulsed light source. *IEEE Transactions on Control Systems Technology*, to appear.

**D.J. Riggs and R.R. Bitmead.** Rejection of aliased disturbance in a pulsed light source. In *Proceedings of the 48th IEEE Conference on Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference*, pages 8148–8153, Shanghai, P.R. China, December 2009.

The dissertation author was the primary investigator and author of these papers; Professor Bitmead supervised the research.

# Chapter 3

## Destruktion and Déconstruction of Model Predictive Control

*Illegitimi non carborundum.*

— Attributed to Second World War British Army Intelligence

### Abstract

The chapter develops achieved performance bounds for Model Predictive Control (MPC) control applied over the infinite horizon to constrained systems with persistent, stochastic disturbances. The analysis is approached from a minimalist perspective of introducing as few as possible assumptions and alterations to the original target infinite-horizon constrained disturbance rejection optimal control problem other than the introduction of finite-horizon MPC. We separate the feasibility analysis, which is shown to be purely topological, from the performance analysis, which is based on discounted value functions and builds on the work of Jadbabaie, Hauser, Grüne and Rantzer. A specific focus is on the requirements for stability, where MPC stability in the sense of bounded-input/bounded-state (BIBS) is addressed. We employ feasibility analysis to derive conditions under which BIBS stability for the MPC controlled system law holds, without resort to a cost function. We further quantify this bound and generalize current results on undisturbed systems to the stochastic disturbance case.

### 3.1 Introduction

Borrowing from Jacques Derrida’s analyses of philosophy and language that seek to examine their internal workings and contradictions [23], our aim is to deconstruct Model Predictive Control (MPC) by examining the relationships and tensions between its inner elements within the broader milieu of control systems design. Our approach is postmodernist, skeptical and questioning of the concept of MPC being formulated to recipe, with the presence of all the requisite components — objective function, horizon, constraints, terminal cost, stage cost, terminal constraint, etc. Rather, proceeding from the explication of the computer-bound finite-horizon MPC *designed* system and of the real-world receding-horizon closed-loop controlled *achieved* or target system, we take a destruktionist approach [33] and investigate what each of the MPC recipe ingredients brings into play, individually or in concert, for the achieved system’s properties. The objective is the reexamination and contextualizing of MPC with a focus on the original target control system behavior and our approach will be to explore, as much as we are able, what we entitle *the most powerful bequeathing tools*. That is, we seek to identify results with the weakest possible assumptions connecting properties of the MPC designed system to inherited or implied features of the achieved system.

MPC is a powerful and widely applicable feedback control approach based on constrained receding-horizon optimal control. A raft of worthy books is available chronicling the formulation, history, computational solution, and practical applications of the method [3, 17, 18, 29, 62, 82, 84, 31, 92, 100, 50]. Behind and in front of these monographs and texts is an even larger body of research literature, including the regular appearance of survey papers and reviews [25, 53, 63, 67, 74]. The core attraction of MPC is its capacity to handle constraints on system inputs, states, and/or outputs. Its principle domain of applications success has been as a disturbance rejection controller in the process industries [85, 22].

The central feature of MPC is its receding-horizon nature, which is responsible for its capacity to handle constraints via the solution of either an open-loop or a closed-loop feedback-policy constrained optimization. The key principle is to state the MPC problem as an explicit finite-dimensional constrained search through the

specification of an objective function, horizon, and a set of constraints. A general form of MPC amenable to such analysis is provided in Section 3.3.

The underlying control application is continuous operation on the infinite horizon. This motivates the *raison d'être* of MPC: as an approximate solution of an infinite-horizon optimal control problem with constraints. This formulation has been driven by the process industry, where control systems must accommodate systems operating on the long term with incessant disturbances and ceaselessly active constraints. The infinite-horizon cost function employed, then, has value to the process engineer as it conveys performance of the control system in terms of measures of interest and importance. We shall therefore concentrate on MPC for long-term disturbance rejection with constraints, since this is where performance can be assessed.

To reflect application of MPC best in the process industry where constraint activity is inevitable, performance functions have meaning, and disturbances are persistent, we impose that the MPC design problem inherits performance functions and constraints from the infinite-horizon optimal control problem. We then investigate the infinite-horizon impact of modifications which the designer might perform to the MPC design problem: terminal cost, terminal constraint, and other constraints; even replacement of the performance function with some other function that might not be related to infinite-horizon performance. We are most interested in the properties bequeathed from the MPC design choices to the achieved infinite-horizon performance and constraint satisfaction.

We focus in particular on feasibility analysis and quantification of performance bounds on the achieved system under MPC control. Feasibility analysis is accomplished via set-theoretic methods of [10, 11]. In particular, we decouple feasibility analysis from analysis of the cost function. Via constraint analysis only, we are able to show existence of a bound on the achieved system state via a bounded-input bounded-state (BIBS) result. The existence of this bound precedes consideration of a cost function and hence can be contrasted with ISS methods for proving existence of bounds, which require a cost function but might possibly yield improved bounds if eventually carefully quantified.

Once we have established simple conditions for the existence of a state bound via the BIBS result, we turn to quantification of a bound on the infinite-horizon performance of the MPC-controlled system. The development here is aided mainly by the results of [40], [39], and [30], who consider stability and performance bounds for undisturbed systems. We generalize performance bound results to stochastic systems with discounted cost functions [48], which are a necessary adjunct for this class of problems in order that value functions remain finite.

The rest of the chapter is organized as follows. In Section 3.2, we pose a stochastic, constrained, infinite-horizon optimal control problem. In Section 3.3, we pose the MPC control problem as an approximation to the infinite-horizon control problem, and define the achieved system and performance under MPC control. In Section 3.4, we present the well-known Stochastic Dynamic Programming Equation (SDPE) and establish its relationship to the MPC control problem. In Section 3.5 we define and analyze feasibility and recursive feasibility of the MPC control problem. Section 3.6 contains analysis of systems subject to stochastic disturbances and establishes quantifiable bounds on achieved performance under MPC control. We conclude in Section 3.7. To streamline the development and discussions, we provide proofs of selected results in the Appendix.

The central contribution of this chapter lies in the refocus of the MPC control problem onto the question of achieved infinite-horizon behavior as a disturbance rejection controller operating with tireless disturbances and active constraints. The deconstructive nature is to examine how the ingredients of MPC contribute to or detract from this underlying objective. This becomes most evident in using the infinite-horizon discounted performance function and optimal value to quantify the MPC achieved infinite-horizon discounted performance. The combination of these discounted performance bounds and a new BIBS stability condition yields sustained properties of the MPC controlled disturbance rejection system on the infinite horizon.

## 3.2 Problem Setup: System, Constraints, Disturbances, Admissible Controls, and Infinite-Horizon Optimal Control

In this section, the problem will be set up mathematically and appropriate definitions will be made and indicated in the text by boldface type. The problem of interest which we aim to solve with MPC is the infinite-horizon optimal control of an input- and state-constrained stochastic nonlinear system.

### 3.2.1 System Dynamics, Available Information, and State / Input Constraint Specification

We commence with the disturbed nonlinear system to be controlled.

$$x_{t+1} = f(x_t, u_t) + w_t, \quad x_0, \quad (3.1)$$

where  $x_t \in \mathbb{R}^n$  is the state and  $u_t \in \mathbb{R}^m$  is the control. The disturbance  $w_t$  is of a stochastic nature which we will describe shortly.

**Assumption 1** (Available Information). *The state  $x_t$  is available at every time  $t$ ; in particular,  $x_0$  is known exactly at time 0. Further, we presume the dynamic model  $f$  is known exactly.*

The state and control are constrained to  $x_t \in \mathcal{X} \subseteq \mathbb{R}^n$  and  $u_t \in \mathcal{U} \subseteq \mathbb{R}^m$ ,  $t \geq 0$ .

### 3.2.2 Stochastic $w_t$

**Assumption 2.** *The stochastic process  $\{w_t, t \geq 0\}$  defined on probability space  $(\Omega, \mathcal{B}, P_w)$  is independent and identically distributed (i.i.d.) and takes values in the compact set  $\mathcal{W} \subset \mathbb{R}^n$ .*

**Remark 1.** *The assumed compactness of the disturbance space  $\mathcal{W}$  allows for analysis of systems which operate with persistently active, hard constraints; that is,*

systems where constraint violation is prohibited but constraint activity has non-zero probability in the future  $\sigma$ -algebra from any state. Further, almost sure constraint satisfaction would not be possible if  $\mathcal{X}$  or  $\mathcal{U}$  were taken to be compact with  $w_t$  with infinite support. Though the specification of stochastic disturbances with infinite support and probabilistic constraint violation can be considered [102, 20, 19], we adopt the formulation of stochastic disturbances with finite support as is discussed in [47]. We also note that the assumption of bounded i.i.d. fundamental stochastic variables  $\{w_t\}$  does not preclude the system being subject to unbounded disturbances, only that their independent increments are required to be bounded.

### 3.2.3 Admissible Control Policies

Our aim is to find feedback policies for controlling the system dynamics (3.1) as a persistent disturbance rejection problem over an infinite horizon and subject to hard state and input constraints.

**Definition 1.** An *admissible, non-anticipative, feedback control policy* is a function  $\pi : \mathcal{X} \rightarrow \mathcal{U}$  such that at time  $t$ ,  $u_t = \pi(x_0, \dots, x_t)$ . We let

$$\Pi = \{\pi : u_t = \pi(x_0, \dots, x_t) \in \mathcal{U}\}, \quad (3.2)$$

denote the set of all admissible, non-anticipative, feedback control policies.

We end this section by presenting the standing assumption that all subsequent optimal control problems considered have an optimal policy.

**Assumption 3.** There exists an optimal policy which achieves the minimum in each optimal control problem which follows.

### 3.2.4 Infinite-Horizon Optimal Control

*Optimal Control Problem of Interest,  $\mathcal{P}_\infty$ .*

The **infinite-horizon value function** is

$$V_\infty(x) = \min_{\pi \in \Pi} \mathbf{E}_w \left[ \sum_{i=0}^{\infty} \alpha^i l(x_i, u_i) \mid x_0 = x \right], \quad (3.3)$$

subject to: (3.1),  $x_t \in \mathcal{X}$  almost surely,  $u_t \in \mathcal{U}$ ,

where: **discount factor**  $\alpha \in [0, 1]$ , the conditional expectation  $E_w[\cdot|\cdot]$  is taken with respect to the measure  $P_w$  on the  $\sigma$ -algebra generated by  $\{w_0, w_1, \dots\}$ , and  $l(\cdot, \cdot)$  is the **stage cost**. Due to time-invariance of the system  $f$  and stationarity of the disturbance  $w_t$ ,  $t \geq 0$ , the **infinite-horizon optimal control law**,  $\pi_\infty^*$ , which minimizes (3.3) is stationary, i.e. at time  $t$ ,  $u_t^* := \pi_\infty^*(x_t) =: \mu_\infty(x_t)$ ,  $t \geq 0$ . The **optimal system**

$$x_{t+1}^o = f(x_t^o, \mu_\infty(x_t^o)) + w_t, \quad x_0, \quad (3.4)$$

is the result of applying the optimal control to system (3.1) and achieves the optimal infinite-horizon discounted value (3.3).

We make the following standing assumptions.

**Assumption 4.** *The set of initial conditions  $x_0$  such that  $\mathcal{P}_\infty$  has a feasible solution is non-empty.*

**Assumption 5** (Realizability of Performance). *The infinite-horizon value function is finite, i.e.  $V_\infty(x) < \infty$ , for all feasible initial conditions  $x_0 = x$ .*

**Assumption 6.** *The stage cost  $l(x, u) \geq W(x)$  for each  $u \in \mathcal{U}$ , where  $W(x)$  is a positive definite function.*

The infinite-horizon value function  $V_\infty(x)$  is a random variable measurable on the  $\sigma$ -algebra generated by  $x_0$ . That is, when  $x_0$  is specified,  $V_\infty(x)$  takes a specific real value.

Persistent disturbances impinging upon the state will prevent the state from converging to an equilibrium point, nominally at the origin. Assumption 6, in turn, will prevent the stage cost from tending to zero in these circumstances. The discount factor,  $\alpha$ , must then be chosen in the interval  $[0, 1)$  in order that Assumption 5 can hold. The inclusion of the discounting factor in the definition of the optimal value function is a central departure from recent results in Nonlinear MPC [29]. Rather than introducing a discount factor, one might consider the infinite-horizon average cost per unit time formulation [91, 101, 7]. See also [1, 34]. This is challenging. We shall return to the consideration of the discounted cost later when achieved performance on the infinite horizon is examined.

### 3.3 Model Predictive Control and Achieved Performance: Problem Formulation

In the Model Predictive Control problem,  $\mathcal{P}_{MPC}$ , a finite-horizon optimal control problem is posed and solved at time  $t$  starting from state  $x_t$ . The stage cost  $l(x, u)$  and control constraint  $\mathcal{U}$  are inherited from  $\mathcal{P}_\infty$ . The preservation of the stage cost allows for evaluation of the sub-optimality of the MPC controller applied over the infinite horizon by comparison to the optimal value and, hence, gauging the performance of the MPC control law with respect to the performance measure of importance. This is an important point, since the stage cost measures the control performance in terms identified as meaningful for the original infinite-horizon problem. Amending this cost to accommodate or induce good behavior of the associated MPC problems introduces a variance from the metric of performance.

*MPC Problem,  $\mathcal{P}_{MPC}$ .*

The **designed value function** is

$$V_N(x) = \min_{\pi_i^N \in \Pi} \mathbb{E}_w \left[ \sum_{i=0}^{N-1} \alpha^i l(x_{t+i}, \pi_i^N(x_{t+i})) + F(x_N) \mid x_t = x \right], \quad (3.5)$$

subject to:

$$(3.1), \quad x_{t+i+1} \in \mathcal{X}_{i+1} \text{ almost surely, } u_{t+i} = \pi_i^{N,*}(x_{t+i}) \in \mathcal{U}, \quad (3.6)$$

where: the **terminal cost**  $F(x)$  is a design choice which will be discussed in detail later, the conditional expectation  $\mathbb{E}_w[\cdot|\cdot]$  is taken with respect to the measure  $P_w$  on the  $\sigma$ -algebra generated by random variables  $\{w_t, \dots, w_{t+N-1}\}$ , and the **designed state constraints** are  $\mathcal{X}_i \subseteq \mathbb{R}^n$ ,  $i = 1, \dots, N$ . The **designed optimal control law** which minimizes (3.5) is non-stationary and is given by the sequence  $\{\pi_0^{N,*}, \dots, \pi_{N-1}^{N,*}\}$ . The **designed system** is

$$x_{t+i+1,t}^d = f \left( x_{t+i,t}^d, \pi_i^{N,*}(x_{t+i,t}^d) \right) + w_{t+i,t}, \quad x_t, \quad (3.7)$$

for:  $i = 0, \dots, N-1$ , initial state  $x_{t,t}^d = x_t$ , and  $\{w_{t,t}, w_{t+1,t}, \dots, w_{t,t+N-1}\}$  a sequence of i.i.d. random variables with probability distribution  $P_w(\cdot)$  and independent from  $x_t$  and  $\{w_0, \dots, w_{t-1}\} \cup \{w_{t+N}, \dots\}$ . The **MPC control law** is  $\mu_N := \pi_0^{N,*}$ . The **MPC algorithm** applies  $u_t = \mu_N(x_t)$  to the system (3.1) at time  $t$  and repeats the solution of problem  $\mathcal{P}_{MPC}$  at time  $t+1$  starting from state  $x_{t+1}$ .

Note the finite-horizon state constraints  $\mathcal{X}_i$ ,  $i = 1, \dots, N$ , might differ from  $\mathcal{X}$  and are left to the designer to select; feasibility analysis and the role of these MPC design constraints will be discussed in a subsequent section.

The present formulation is that of closed-loop MPC, as (3.5) is solved over *policies*  $\pi_i^N \in \Pi$ ,  $i = 0, \dots, N-1$ . Closed-loop controllers are studied and explained in [94] as being functionally block lower triangular in the dependence between  $\pi_i^N$  and  $x_{t+i}$ . Note that closed-loop policies include open-loop policies.

We define the achieved system and achieved expected value, which together convey the performance of the MPC law on the infinite-horizon.

*Achieved System and Value Under MPC Control.*

The **achieved system** under MPC control  $\mu_N$  is

$$x_{t+1}^a = f(x_t^a, \mu_N(x_t^a)) + w_t, \quad x_0. \quad (3.8)$$

Corresponding to the achieved state  $x_t^a$  and MPC control  $u_t^a := \mu_N(x_t^a)$ ,  $t \geq 0$ , the **achieved value function** is

$$V_\infty^{\mu_N}(x) = \mathbb{E}_w \left[ \sum_{i=0}^{\infty} \alpha^i l(x_i^a, u_i^a) \middle| x_0^a = x \right], \quad (3.9)$$

where the conditional expectation  $\mathbb{E}_w[\cdot|\cdot]$  is taken with respect to the measure  $P_w$  on the  $\sigma$ -algebra generated by  $\{w_0, w_1, \dots\}$ .

The achieved value function (3.9) is a random variable and is defined over the same  $\sigma$ -algebra as the original infinite-horizon problem  $\mathcal{P}_\infty$ , though  $\mathcal{P}_{MPC}$  for a single time  $t$  is defined over a sub- $\sigma$ -algebra corresponding to the time interval

$t, \dots, t + N - 1$ . The random variable  $x_{t+1}^a$  is measurable with respect to the  $\sigma$ -algebra generated by  $x_t^a$  and  $w_t$ , the i.i.d. stochastic process entering the actual system, (3.1). Note: the specific value of discount factor,  $\alpha$ , is the same in  $\mathcal{P}_\infty$ ,  $\mathcal{P}_{MPC}$ , and in (3.9), the achieved value calculation.

Following [30], we bound  $V_\infty^{\mu_N}(x)$  by known quantities.

**Remark 2.** *We take the viewpoint that the MPC solution is an approximation of the solution of  $\mathcal{P}_\infty$ , where the approximation is accomplished through the consideration of a finite time, yielding a tractable solution. The MPC approach allows for accommodating constraints which can be readily handled over finite horizons. Others (e.g. [64]) consider the infinite horizon at once but approximate the solution to (3.3) by dividing up the state space.*

**Remark 3.** *The viewpoint of the MPC solution as an approximation of  $\mathcal{P}_\infty$  further emphasizes the importance of using the original stage cost  $l(\cdot, \cdot)$  in the MPC problem, as it is the performance measure of interest over the infinite horizon.*

We conclude this section with the following result, which provides a link between the achieved system and the design system.

**Theorem 1** (Central Observation). *At time  $t$ , the achieved system state and the design system state coincide,*

$$x_{t,t}^d = x_t^a, \quad (3.10)$$

*and at the next time,  $t + 1$ , the achieved state satisfies*

$$x_{t+1}^a = f(x_{t,t}^d, \mu_N(x_{t,t}^d)) + w_t, \quad (3.11)$$

*for bounded random variable  $w_t$  independent from  $x_t$  and  $\{w_0, \dots, w_{t-1}\}$ . In the disturbance-free case where  $w_t \equiv 0$ , (3.10 - 3.11) correspond to  $x_{t,t}^d = x_t^a$  and  $x_{t+1}^a = x_{t+1,t}^d$ .*

This concordance between the states of the achieved and designed systems at times  $t$  and  $t+1$  has been commented upon by [62]. It leads to some very simple but powerful implications between the designed and achieved systems. We remark that the designed state values beyond the first,  $x_{t+1,t}^d$ , i.e.  $x_{t+j+1,t}^d$  for  $j > 0$ , need not be mimicked in the behavior of the achieved state.

**Property 1.** *If  $\mathcal{P}_{MPC}$  possesses a solution at time  $t$ , then the achieved system satisfies  $x_{t+1}^a \in \mathcal{X}_1$  almost surely.*

## 3.4 Stochastic Dynamic Programming

Following [30] and [29], we employ dynamic programming for the analysis of the MPC control law. We specify the stochastic dynamic programming equation (SDPE) and state its relationship to the finite-horizon optimal control problem  $\mathcal{P}_{MPC}$ . We later use the SDPE to derive a relationship between the achieved value function  $V_{\infty}^{\mu_N}(x)$  and the designed value function  $V_N(x)$ .

### 3.4.1 Stochastic Dynamic Programming Equation

As pointed out in [63] for the undisturbed case  $w_t \equiv 0$ , the first control value of the optimal sequence obtained from solving an open-loop model predictive control problem and the first control policy obtained via dynamic programming coincide.

Analogously, closed-loop MPC of (3.1) and the optimal policy supplied by dynamic programming also coincide; hence we base our analysis on the vast literature available on stochastic dynamic programming. The following result from [48] on the SDPE for discounted stochastic optimal control has been adapted to the current setting with state constraint sets.

**Theorem 2** (SDPE, [48]). *Let*

*$\{w_0, \dots, w_{N-1}\}$  be a sequence of i.i.d. random variables with probability measure  $P_w(\cdot)$ . Define the real-valued functions,  $\{V_k(x_{N-k}), k = 0, \dots, N\}$ , by the recur-*

sion,

$$V_k(x_{N-k}) := \min_{\pi_{N-k}^N \in \Pi} \{l(x_{N-k}, \pi_{N-k}^N(x_{N-k})) + \alpha E_w [V_{k-1}(f(x_{N-k}, \pi_{N-k}^N(x_{N-k})) + w_k)]\}, \quad (3.12)$$

subject to:

$$\begin{aligned} f(x_{N-k}, \pi_{N-k}^N(x_{N-k})) + w_k &\in \mathcal{X}_{N-k+1}, \\ u_k = \pi_{N-k}(x_{N-k}) &\in \mathcal{U}, \end{aligned}$$

with initial condition  $V_0(x_N) := F(x_N)$ , for  $x_N \in \mathcal{X}_N$ . Then the designed value function from problem  $\mathcal{P}_{MPC}$  at time  $t$  with horizon  $N$  and corresponding to the designed system state  $x_{t+N-k,t}^d$  satisfies

$$V_k(x_{t+N-k,t}^d) = \min_{\pi_i^k \in \Pi} E_w \left[ \sum_{i=N-k}^{N-1} \alpha^i l(x_{t+i,t}^d, \pi_i^k(x_{t+i,t}^d)) \middle| x_{t+N-k,t}^d \right], \quad (3.13)$$

subject to: (3.1),  $x_{t+i,t}^d \in \mathcal{X}_i$  almost surely,  $u_{t+i} \in \mathcal{U}$ ,

which, when  $k = N$ , is the solution to (3.5).

Equation (3.12) is known as the **stochastic dynamic programming equation** (SDPE). The functions  $V_k(x_{N-k})$  are deterministic and not random. However, the quantity  $V_k(x_{t+N-k,t}^d)$  is a random variable because of this property of its argument.

### 3.5 Feasibility

We proceed with identifying conditions under which problem  $\mathcal{P}_{MPC}$  has a feasible solution; first for a single time  $t$  and later for the MPC algorithm which considers successive solution of  $\mathcal{P}_{MPC}$ . Our results are set-theoretic as we leverage the work of [10], [11], and [76].

The concordance between the designed system state  $x_{t+1,t}^d$  and the achieved system state  $x_{t+1}^a$  as offered by the Central Observation (Theorem 1) allows us to ascertain constraint satisfaction on the infinite-horizon via analysis of the designed

state constraints  $\{\mathcal{X}_k, k = 1, \dots, N\}$  from problem  $\mathcal{P}_{MPC}$ . In particular, we provide a condition on the first constraint set,  $\mathcal{X}_1$ , which yields constraint satisfaction for the achieved system state and control signal for all time.

We conclude the section by establishing conditions under which the achieved system state is bounded under MPC control. This is accomplished via consideration of the designed constraint sets and dynamics  $f$ ; that is, we prove existence of a bound on the achieved system state without specification of stage or terminal cost functions. This is new; existence of a state bound is generally accomplished via input-to-state stability analysis, which requires specification of stage and terminal cost functions (c.f. [75]).

### 3.5.1 Recursive Feasibility

**Definition 2** (*N-feasible set*). *Given the sequence of  $N$  subsets of  $\mathbb{R}^n$ ,  $\{\mathcal{X}_k : k = 1, \dots, N\}$ , from (3.5) in  $\mathcal{P}_{MPC}$ , the **N-feasible set** of states,  $\mathcal{X}_{N\phi}$ , is the subset of  $\mathbb{R}^n$  such that there exists a finite sequence of admissible control policies  $\{\pi_{k-1}^{N,\phi} : k = 1, \dots, N\} \in \Pi$  such that  $x_t \in \mathcal{X}_{N\phi}$  and  $x_{t+k,t}^d = f(x_{t+k-1,t}^d, \pi_{k-1}^{N,\phi}(x_{t+k-1,t}^d)) + w_{t+k-1,t}$  imply  $x_{t+k,t}^d \in \mathcal{X}_k$  almost surely and  $u_{t+k-1}^d = \pi_{k-1}^{N,\phi}(x_{t+k-1,t}^d) \in \mathcal{U}$  for  $k = 1, \dots, N$ .*

**Definition 3** (*Recursive N-feasibility*). *Problem  $\mathcal{P}_{MPC}$  is **recursively N-feasible** if, given that  $x_t \in \mathcal{X}_{N\phi}$  and that  $\pi_0^{N,\phi}$  from Definition 2 is applied, we have  $x_{t+1,t}^d = f(x_t, \pi_0^{N,\phi}(x_t)) + w_{t,t} \in \mathcal{X}_{N\phi}$  almost surely and  $u_{t+k,t}^d = \pi_{t+k-1}^{N,\phi}(x_{t+k-1,t}^d) \in \mathcal{U}$ .*

**Theorem 3.** *Problem  $\mathcal{P}_{MPC}$  is recursively N-feasible if*

$$\mathcal{X}_1 \subseteq \mathcal{X}_{N\phi} \text{ and } \mathcal{X}_1 \neq \emptyset. \quad (3.14)$$

The importance of Theorem 3 is that it demonstrates that recursive  $N$ -feasibility is a topological property of the constraint sets and is a property divorced from the presence of an optimization problem associated with the MPC control. This is particularly evident in the MPC control design from the paper [99], for example, where the optimization objective function is loosely tied to the control and state signals and the MPC design converts to a sequence of feasibility problems. Such an interpretation is not immediately evident from the paper [99].

The condition (3.14) is only sufficient; recursive  $N$ -feasibility as given in Definition 3 implies  $x_{i+1,t}^d \in \mathcal{X}_1 \cap \mathcal{X}_{N\phi}$  almost surely for arbitrary  $\mathcal{X}_1 \subseteq \mathbb{R}^n$ .

We note that Theorem 3 deals with  $\mathcal{P}_{MPC}$  and the possibly different state constraint sets  $\{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_N\}$  in  $\mathbb{R}^n$  and control constraint set  $\mathcal{U}$  in  $\mathbb{R}^m$ .

Denote by  $\mathcal{X}_{\infty\phi}$  the feasible initial state set corresponding to fixed state constraint sets  $\mathcal{X}$  and  $\mathcal{U}$  from (3.3) in problem  $\mathcal{P}_{\infty}$ . By Assumption 4,  $\mathcal{X}_{\infty\phi}$  is non-empty. When the MPC constraints are taken as subsets of  $\mathcal{X}$ , the feasibility of the MPC problem may be simply related to that of the infinite-horizon problem. We have the following direct observations.

**Lemma 1.** *If  $\mathcal{X}_i = \mathcal{X}$ , for  $i = 1, 2, \dots, N$ , then*

$$\mathcal{X}_{1\phi} \supseteq \mathcal{X}_{2\phi} \supseteq \dots \supseteq \mathcal{X}_{N\phi} \supseteq \mathcal{X}_{\infty\phi}.$$

**Lemma 2.** *For state constraint sets  $\{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_N\}$ , if*

(i)  $\mathcal{X}_1 \subseteq \mathcal{X}_{N\phi}$  and  $\mathcal{X}_1 \neq \emptyset$ , so  $\mathcal{P}_{MPC}$  is recursively  $N$ -feasible,

(ii)  $\mathcal{X}_1 \subseteq \mathcal{X}$ ,

then  $\mathcal{X}_1 \subseteq \mathcal{X}_{\infty\phi}$ .

The proof of Lemma 2 follows from recursive feasibility and the evident MPC-control invariance of the set  $\mathcal{X}_1$ . The upshot of Lemmata 1 and 2 is that both the feasible sets and the recursive feasibility of the horizon- $N$   $\mathcal{P}_{MPC}$  and of the infinite-horizon  $\mathcal{P}_{\infty}$  are linked, even though their constraints sets might differ. This becomes apparent in the feature that  $\mathcal{X}_{N\phi}$  need not be a superset of  $\mathcal{X}_{\infty\phi}$ . Accordingly, we identify the set  $\mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$  as important as the set of initial conditions feasible for  $\mathcal{P}_{\infty}$  and recursively feasible for  $\mathcal{P}_{MPC}$ .

**Assumption 7.** *The designed constraint set  $\mathcal{X}_1$  satisfies  $\mathcal{X}_1 \subseteq \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$  and  $\mathcal{X}_1 \neq \emptyset$ .*

The following corollary is immediate and extends the results of Theorem 3 to constraint satisfaction of the achieved system (3.8) on the infinite horizon.

**Corollary 1.** *Suppose Assumption 7 holds and  $x_0 \in \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$ . Then the achieved system state  $x_t^a$  lies in  $\mathcal{X}$  almost surely for all  $t \geq 0$ .*

### 3.5.2 Recursive Feasibility and BIBS Stability

Our aim in this part of the study has been to divorce the optimization objective function from the constraint analysis to determine the role and importance of each in the MPC formulation; noting that such a deconstructionist approach might disrespect the attraction of MPC for practice, where the joint presence of constraints and a performance objective function is key, albeit with a strict precedence of constraints over objective function.

Since the landmark paper of [44] where a terminal state constraint  $\mathcal{X}_N = \{0\}$  was introduced to achieve asymptotic stability of the undisturbed receding-horizon controlled system, the cost-to-go function has been used as a Lyapunov function for the closed-loop. This suggests a close tie between the objective function and closed-loop stability. In the situation with disturbances, this analysis can be generalized by including a compact finite-horizon terminal constraint set  $\mathcal{X}_N$  within which a feasible control is known which renders this set positively invariant. Asymptotic stability is then replaced by demonstrated convergence to this set. However, the Lyapunov analysis using the objective function remains as the core tool for such convergence. These methods then provide, eventually, a Bounded-Input-Bounded-State (BIBS) stability result, since  $\mathcal{X}_N$  is assumed compact.

However, one may establish BIBS stability under MPC control without resort to full specification of the objective function in  $\mathcal{P}_{MPC}$ . This requires an assumption about the system (3.1), that  $f$  be a *proper map* [12].

**Definition 4** (Proper Map). *A function  $g$  is a **proper map** if pre-images in  $g$  of compact sets are compact.*

Effectively, this rules out systems with zero gain over unbounded intervals. In the linear system case, it is a requirement met through having the  $A$ -matrix full rank.

The following theorem expresses the main result of this section. Via specification of a bounded state constraint  $\mathcal{X}_j$  and supposing  $f$  is a proper map, boundedness of the achieved system state  $x_t^a$  under MPC control can be guaranteed.

**Theorem 4.** *Suppose that: problem  $\mathcal{P}_{MPC}$  is recursively  $N$ -feasible for the sequence of constraint sets  $\{\mathcal{X}_k : k = 1, \dots, N\}$ ,  $\mathcal{U}$  is compact, and the set  $\mathcal{X}_k$  is compact for at least one value of  $k \in \{1, \dots, N\}$ . Further, suppose that the function  $f(\cdot, u) : \mathcal{X} \rightarrow \mathcal{X}$  is a proper map for each  $u \in \mathcal{U}$ . Then, provided the initial state is  $N$ -feasible,  $x_0 \in \mathcal{X}_{N\phi}$ , the MPC control law remains feasible and the achieved system state  $x_t^a$  is almost surely bounded for all time  $t \geq 0$ .*

The novelty of this result is that it invokes no property of the optimization objective function, only of the system function  $f$ , of the MPC control constraint set  $\mathcal{U}$  and of the recursive feasibility. The stability established is BIBS stability from the disturbance to the state, which is stability in the sense of Lagrange rather than in the sense of Lyapunov [83], even though we have so far not introduced any assumption about an equilibrium. In incorporating the notion of a compact set and hence of boundedness, we have also moved the analysis to a metric space context from simply topological considerations. Although compactness of sets and thus properness of maps can be defined solely in a topological space.

**Remark 4.** *The upshot of this subsection concerning BIBS stability is that, provided a feasible solution can be found, the stability follows directly from the problem statement in the constraint specification; thus existence of a bound is shown without resort to an objective function. Approaches treating system state convergence to compact sets in which the constraints remain inactive, such as the dual-mode controllers of [66] or [99], cannot accommodate persistently active constraints.*

**Remark 5.** *Competing stability statements using Lyapunov arguments deal with asymptotic properties of the unforced or disturbance-free system [29, 73], or deal directly with disturbances using a min-max objective function [56, 75, 52]. These approaches aim to show input-to-state stability (ISS) of (3.1), only guaranteeing existence of a bound. Quantification of the ISS bound, i.e. via methods as presented in [37], has yet to be applied to the MPC ISS results.*

The BIBS result of Theorem 4 establishes a signal bound. ISS approaches also supply existence of a bound, though with more effort, e.g., the prescription of a cost function. The issue therefore must lie in the quantification of the magnitude

of the bound. We now move towards this goal through performance quantification. But we note that the simple achievement of BIBS is an important adjunct to the description of infinite-horizon performance via discounted value functions to be developed next.

### 3.6 Value Functions: Achieved Performance Bounds

Monotonicity of the designed value function  $V_N(x)$  (3.5) is a property that allows for proving asymptotic properties of the MPC control in the disturbance-free case [63, 40, 39, 27] and in min-max formulations with disturbances [75]. It has also recently been used to derive quantifiable state and performance bounds on disturbance-free systems [30, 29].

In this section, we extend the use of monotonicity of  $V_N(x)$  to prescribing infinite-horizon performance bounds for the stochastic nonlinear system (3.1) with discounted cost. This is accomplished by bounding the achieved value function  $V_\infty^{\mu_N}(x)$  (3.9). The performance-bounding inequalities presented here are generalizations of existing inequalities to the stochastic system (3.1) and discounted cost formulation considered in this chapter. Furthermore, we generalize the quoted results to include consideration of designed state constraints  $\mathcal{X}_k$ .

Before proceeding with performance analysis, consider the following assumption.

**Assumption 8.** *The MPC design problem  $\mathcal{P}_{MPC}$  is recursively feasible with  $\mathcal{X}_j \subseteq \mathcal{X}$ ,  $j = 1, \dots, N$ .*

In the subsequent results, we suppose Assumption 8 holds with  $\mathcal{X}_j \subseteq \mathcal{X}$ ,  $j = 1, \dots, N$  to yield  $\mathcal{P}_{MPC}$  recursively  $N$ -feasible and  $x_t^a \in \mathcal{X}$ ,  $t \geq 0$  if  $x_0 \in \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$ .

### 3.6.1 Value Functions Monotonically Non-Decreasing with Horizon

In this section we consider MPC formulations yielding value functions which are monotonically non-decreasing with horizon. We first present results from [39] for the undisturbed case  $w_t \equiv 0$  which state that, with selection of zero terminal cost and for sufficiently long horizon, the MPC control yields asymptotic stability of the undisturbed system; this is an important result as it does not employ the use of terminal constraints sets [63] to guarantee stability. [30] extend the result of [39] by considering practical stability and deriving performance bounds for a modified stage cost which takes zero value inside a set that is attractive for the MPC control law. We generalize this result to the current stochastic system setting and offer comparison to the new discounted-cost formulation result that is presented at the end of the section.

We begin with an elementary observation concerning the designed value function from  $\mathcal{P}_{MPC}$ , which pertains regardless of discounting in the cost function or the presence of disturbances, assuming the infinite-horizon value function is finite. This is a generalization of observations made in [39] and [30], the generalization being in the consideration of the designed constraint sets involved in problem  $\mathcal{P}_{MPC}$ .

**Theorem 5.** *Consider any  $x \in \mathcal{X}_{\infty\phi}$ . Assume  $V_{\infty}(x) < \infty$ . Let the terminal cost from (3.5)  $F(y) = 0$ ,  $\forall y \in \mathcal{X}$  and let  $\mathcal{X}_j = \mathcal{X}$ ,  $j = 1, 2, \dots$ . Then  $V_N(x)$  from (3.5) has a feasible solution  $\forall N \geq 0$  and*

$$V_i(x) \leq V_{i+1}(x) \leq V_{\infty}(x) < \infty.$$

$$i = 0, 1, \dots$$

The above result employs Lemma 1 and holds for the solution of  $\mathcal{P}_{MPC}$  at a single time  $t$  with infinite-horizon-feasible initial condition  $x \in \mathcal{X}_{\infty\phi}$ , as recursive feasibility is not considered in the theorem statement. As noted by [35], [39], and [30], once the value functions cease to change with horizon, we have infinite-horizon optimality.

**Undisturbed, undiscounted, zero terminal cost case:**  $w_t = 0$ ,  $\alpha = 1$ ,  $F(x) = 0$ .

Consider  $V_\infty(x)$  in (3.3) from problem  $\mathcal{P}_\infty$  with  $w_t = 0$  and  $\alpha = 1$ . Also consider  $V_N(x)$  in (3.5) from  $\mathcal{P}_{MPC}$  with  $w_t = 0$ ,  $F(x) = 0$ ,  $\alpha = 1$ .

Evidently,  $V_j(x) \leq V_\infty(x) < \infty$ , and hence the value function  $V_N(x) \rightarrow V_\infty(x)$  as  $N \rightarrow \infty$  due to the Monotone Convergence Theorem. Further, the problem admits discussion of asymptotic stability, since the state can feasibly tend to an equilibrium. We note that [39, 30] and others develop their results without constraints, in order to demonstrate that asymptotic stability is achievable without resort to constraints to make it so. The following result is a minor extension of the result given in [39] to the case with state constraints to show that this property is not lost when constraints are maintained.

**Theorem 6** ([39]). *For the undisturbed, undiscounted, zero terminal cost problem  $\mathcal{P}_{MPC}$  satisfying Assumption 8 with  $\mathcal{X}_j \subseteq \mathcal{X}$ ,  $j = 1, \dots, N$ , and  $\mathcal{X}$ ,  $\mathcal{U}$  from  $\mathcal{P}_\infty$  compact and convex, there exists  $N^* < \infty$  that yields asymptotic stability of the achieved system (3.8) for all  $N \geq N^*$  under MPC control law  $\mu_N(x)$ . Further,  $\forall x \in \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$ , as  $N \rightarrow \infty$ ,  $V_\infty^{\mu_N}(x) \rightarrow V_\infty(x)$ .*

That is, stability of the MPC achieved system can be obtained without using a terminal cost or a terminal state constraint at the expense of possibly large horizon  $N$  in the design problem  $\mathcal{P}_{MPC}$ . The result is proved using Dini's theorem on the uniform convergence of functions, which can be applied given the monotonic increase of  $V_N$  and compactness of the constraint sets.

Recall that  $V_N(x)$  is related to the infinite-horizon value function (3.3) since it uses the same stage cost,  $l(\cdot, \cdot)$ . Theorem 6 states that MPC with a sufficiently long horizon approaches the optimal infinite-horizon control performance.

[30] extend the results of [39] and establish a relationship between the infinite-horizon achieved value and the finite-horizon designed value functions.

**Theorem 7** ([30]). *For the undisturbed, undiscounted, zero terminal cost, horizon- $N$  problem  $\mathcal{P}_{MPC}$  satisfying Assumption 8 with  $\mathcal{X}_j \subseteq \mathcal{X}$ ,  $j = 1, \dots, N$ , and with corresponding control solution  $\mu_N(x)$ , suppose there exists a scalar function  $\gamma_N \in$*

$[0, 1]$  such that,

$$V_N(f(x, \mu_N(x))) - V_{N-1}(f(x, \mu_N(x))) \leq \gamma_N l(x, \mu_N(x)), \quad (3.15)$$

$\forall x \in \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$ . Then

$$(1 - \gamma_N)V_\infty(x) \leq (1 - \gamma_N)V_\infty^{\mu_N}(x) \leq V_N(x). \quad (3.16)$$

The rate bound on the monotonicity, (3.15), is used to provide a rate of convergence on the achieved value function; this establishes the stated bound on the achieved value with respect to the designed value. We later generalize the inequalities (3.15) and (3.16) to the stochastic system, discounted cost formulation; the above result, then, is included here for ease of comparison.

**Disturbed, undiscounted, zero terminal cost case:**  $w_t \neq 0$ ,  $\alpha = 1$ ,  $F(x) = 0$ .

Here we consider  $V_\infty(x)$  in (3.3) from problem  $\mathcal{P}_\infty$  and  $V_N(x)$  in (3.5) from  $\mathcal{P}_{MPC}$  with: stochastic process  $\{w_t\}$ ,  $F(x) = 0$ , and  $\alpha = 1$ .

[30] further extend the results of [39] and establish a relationship between the infinite-horizon achieved value and the finite-horizon optimal designed value functions for the non-zero disturbance case. Their analysis is couched as ‘‘Practical Optimality’’ and involves an altered stage cost function  $\bar{l}(\cdot, \cdot)$ , which takes the value zero in a certain set,  $\mathcal{L}$ . Here we present a generalization of their result, originally intended to capture the convergence of systems to a neighborhood of the origin, by quoting it with reference to the stochastic system (3.1) outside a limit set and given constraint set  $\mathcal{X}$  from  $\mathcal{P}_\infty$ .

**Theorem 8** ([30]). *Consider the disturbed, undiscounted ( $\alpha = 1$ ), zero terminal cost, horizon- $N$  MPC problem satisfying Assumption 8 with  $\mathcal{X}_j \subseteq \mathcal{X}$ ,  $j = 1, \dots, N$ , and with resultant control law  $\mu_N(x)$ . Define:*

- set  $\mathcal{L} \subset \mathcal{X}$  to be the minimal (almost surely) invariant set under  $\mu_N$ ,
- alternate stage cost

$$\bar{l}(x, u) = \begin{cases} \max\{l(x, u) - \varepsilon, 0\}, & x \notin \mathcal{L}, \\ 0, & x \in \mathcal{L}, \end{cases}$$

and corresponding optimal and achieved infinite-horizon alternate value functions  $\bar{V}_\infty(x)$  and  $\bar{V}_\infty^{\mu_N}(x)$  evaluated using  $\bar{l}$ ,

$$- \text{constant } \sigma = \inf\{E_w[V_N(f(x, \mu_N(x)) + w)] : x \in \mathcal{X} \setminus \mathcal{L}\}$$

Assume that, for some  $\varepsilon > 0$  and for all  $x \in (\mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}) \setminus \mathcal{L}$ , there exists a scalar function  $\gamma_N \in [0, 1]$ , such that

$$\begin{aligned} V_N(x) - E_w[V_N(f(x, \mu_N(x)) + w)] \\ \geq \max[(1 - \gamma_N)l(x, \mu_N(x)) - \varepsilon, 0], \end{aligned} \quad (3.17)$$

Then, for all  $x \in \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$ ,

$$(1 - \gamma_N)\bar{V}_\infty(x) \leq (1 - \gamma_N)\bar{V}_\infty^{\mu_N}(x) \leq V_N(x) - \sigma. \quad (3.18)$$

The implication of condition (3.17) is that  $\mathcal{L}$  is attractive for the MPC control law, though no terminal constraint is explicitly imposed. When the state is inside  $\mathcal{L}$ , the modified stage cost is 0. The cost  $\bar{V}_\infty^{\mu_N}$  then only captures the cost of convergence to  $\mathcal{L}$ . We will see that such a modification to the cost is not required when a discount factor  $\alpha \in [0, 1)$  is introduced, and hence the analysis will not be limited to evaluation of transient performance.

The condition (3.17) can be established using the monotonic increase of the designed value function with horizon and imposing a controllability assumption on the pair  $[f(x, u), l(x, u)]$ . That is, the existence of a control sequence yielding  $l(x_i, u_i) \leq \beta^i$  for some fixed  $\beta > 0$  and  $i = 0, \dots, N$ , is assumed. [27], [30], and [28] discuss such a controllability assumption and encourage modification of the infinite-horizon stage cost (and, hence, modification to the stage cost used in  $\mathcal{P}_{MPC}$ ) to achieve the desired controllability properties which in turn yield convergence to  $\mathcal{L}$ ; then the achieved value function performance bound (3.18) is based on a restricted version of an already-modified stage cost.

Given our aim to evaluate performance against the original infinite-horizon problem of interest  $\mathcal{P}_\infty$  (and associated infinite-horizon value function), we now move toward presenting results which do not require such stage cost modification.

**Disturbed, discounted, zero terminal cost case:**  $w_t \neq 0$ ,  $\alpha \in [0, 1)$ ,  $F(x) = 0$ .

We next consider  $V_\infty(x)$  and  $V_N(x)$  from  $\mathcal{P}_\infty$  and  $\mathcal{P}_{MPC}$  with: stochastic process  $\{w_t\}$ ,  $F(x) = 0$ ,  $\alpha \in [0, 1)$ .

The following result is an extension of the undisturbed, undiscounted performance result, presented in Theorem 7, to the case with stochastic disturbances and discounted stage cost. The result can be contrasted with Theorem 8 as it does not require introduction of a modified stage cost  $\bar{l}$ , nor does it require convergence to a set, provided the conditions of the theorem can be satisfied.

**Theorem 9.** *For the disturbed, discounted, zero terminal cost, horizon- $N$  problem  $\mathcal{P}_{MPC}$  satisfying Assumption 8 with  $\mathcal{X}_j \subseteq \mathcal{X}$ ,  $j = 1, \dots, N$ , and with resultant control law  $\mu_N(x)$ , suppose there exists a  $\gamma_N \in [0, 1]$  such that,*

$$\begin{aligned} E_w[V_N(f(x, \mu_N(x)) + w)] - E_w[V_{N-1}(f(x, \mu_N(x)) + w)] \\ \leq \gamma_N l(x, \mu_N(x)) + \bar{w}, \end{aligned} \quad (3.19)$$

for all  $x \in \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$ . Then,

$$(1 - \alpha\gamma_N)V_\infty(x) \leq (1 - \alpha\gamma_N)V_\infty^{\mu_N}(x) \leq V_N(x) + \frac{\alpha}{1 - \alpha}\bar{w}. \quad (3.20)$$

The inequalities (3.19) and (3.20) are generalizations of the inequalities (3.15) and (3.16), respectively, from Theorem 7 to the stochastic, discounted cost case. However, we note the discount factor  $\alpha$  does not appear in the first of these inequalities.

The scalar  $\bar{w}$  is introduced in the inequality (3.19) to ameliorate difficulties which might arise in achieving an upper bound on the monotonic increase rate due to the nature of the stochastic disturbance  $w$ . The penalty paid for prescription of large  $\bar{w}$  is evident in the performance bound inequality (3.20). When larger horizons  $N$  are specified, the size required of  $\bar{w}$  decreases, as the designed value function approaches the infinite-horizon optimal value function. Then, we have a tradeoff between horizon length and the performance bound we would like to guarantee. The corroboration between horizon and achieved performance bounds

for zero terminal cost in this case is similar to the undisturbed counterpart (3.15), which is also generally easier to satisfy for large horizons  $N$ .

The discount factor can be selected as  $\alpha = 1$  with the incurred penalty of a meaningless performance bound; however, selection of  $\alpha = 1$  and  $\bar{w} = 0$  for the undisturbed case recovers the results in Theorem 7. Specification of a small discount factor might yield tight bounds. Indeed, selection of  $\alpha = 0$  can yield infinite-horizon optimal performance as both  $\mathcal{P}_\infty$  and  $\mathcal{P}_{MPC}$  reduce to the one-step-ahead problem. But useful state bounds on the infinite horizon might be compromised and the state bound could be relegated to that established by the constraint-based BIBS result in Theorem 4.

We have not only established existence of a bound on the achieved performance under MPC control while considering stochastic disturbances, but have quantified this bound. Standard approaches in the literature addressing MPC with disturbances use ISS ideas to establish existence of a bound for the MPC control by either designing a control for the undisturbed system and establishing conditions under which the infinite-horizon system is ISS [73] or considering disturbances directly in a min–max formulation [75]. Others [99, 19] consider stochastic MPC but focus on convergence of the state to unconstrained sets and are not concerned with performance. With the present formulation, existence and quantification of this performance bound occur at once.

Furthermore, Theorem 9 does not require modification of the stage cost. This allows for comparison of the achieved value function under MPC control directly with the original infinite-horizon problem of interest.

The approach using value functions which are monotonically non-decreasing with horizon,  $N$ , yields a sequence of results “for sufficiently large  $N$ .” The monotonicity of the value functions follows in Theorem 5 directly from the zero terminal cost,  $F(\cdot) = 0$ , regardless of discounting or disturbances and assuming that  $V_\infty(x)$  is finite. To study the properties of value functions monotonically non-increasing with horizon, then, is limited to the consideration of the terminal cost, which cannot be taken as zero.

### 3.6.2 Value Functions Monotonically Non-Increasing with Horizon

In this section we consider design of positive definite terminal cost  $F(x)$  to yield designed value functions which are monotonically non-increasing with horizon. The main idea employed here is based on assuming  $F(x)$  is a special type of Control Lyapunov Function (CLF) [2]; then, asymptotic stability and performance bounds can be established. In the undisturbed case, the attractiveness of selecting the terminal cost as a CLF is that asymptotic stability and bounded achieved performance can be established for any horizon.

The papers [40], [57], [30], and [29] discuss selection of the terminal cost as a CLF for the undisturbed, undiscounted case. We cite the corresponding undisturbed performance result here as given in [30] for comparative purposes before generalizing the result to the stochastic system and discounted cost of the present formulation.

**Undisturbed, undiscounted case:**  $w_t = 0$ ,  $\alpha = 1$ .

Here, consider  $\mathcal{P}_\infty$  and  $\mathcal{P}_{MPC}$  with:  $w_t = 0$ ,  $F(x)$  positive definite, and  $\alpha = 1$ .

The following result presents an inequality condition on the terminal cost  $F(x)$  which is sufficient to yield performance bounds for the undisturbed, undiscounted, achieved value function. This inequality is reminiscent of specifying  $F(x)$  as a CLF.

**Theorem 10** ([30]). *For the undisturbed, undiscounted ( $\alpha = 1$ ), horizon- $N$  MPC problem  $\mathcal{P}_{MPC}$  satisfying Assumption 8 with  $\mathcal{X}_j \subseteq \mathcal{X}$ ,  $j = 1, \dots, N$ , suppose*

$$F(f(x, u)) - F(x) \leq -l(x, u), \quad (3.21)$$

$\forall x \in \mathcal{X}$  and all admissible controls. Then,

$$V_\infty(x) \leq V_\infty^{\mu N}(x) \leq V_N(x),$$

$$\forall x \in \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}.$$

Achieving (3.21) can be difficult over the entire space  $\mathcal{X}$  and over all admissible controls. Furthermore, one could use  $F(x)$  as a CLF for the infinite-horizon system if this condition were satisfied over the whole space  $\mathcal{X}$ . The condition might be easier to satisfy for some  $\mathcal{X}_f \subset \mathcal{X}$ .

There are several approaches used in the literature to overcome the need to find  $F(x)$  satisfying (3.21)  $\forall x \in \mathcal{X}$ .

1. Jadbabaie et al. [40] determine a sufficiently large  $N^*$  such that for all  $N \geq N^*$  the optimal designed state trajectory  $x_{t,t}^d$  enters the set  $\mathcal{X}_f$  in  $N$  steps.
2. Grüne and Rantzer [30] assume knowledge of the set of initial conditions  $\mathcal{X}_0$  such that the designed state enters  $\mathcal{X}_f$  in  $N$  steps and impose that the initial condition for problem  $\mathcal{P}_{MPC}$ ,  $x_t$ , start in  $\mathcal{X}_0$ .
3. Mayne et al. [63] and references cited therein suggest specifying an explicit terminal constraint  $x_{t+N,t}^d \in \mathcal{X}_f$  in problem  $\mathcal{P}_{MPC}$ .

The specification of a terminal cost function satisfying (3.21) yields a designed value function which is monotonically non-increasing with horizon. This monotonicity then admits the stated performance bound.

**Remark 6.** *Considering that in problem  $\mathcal{P}_{MPC}$  only the first control,  $\mu_N(x) := \pi_0^{N,*}(x)$ , is applied and, given the Central Observation, the main effect of specifying a cost at the end of the horizon  $N$  is to encourage desirable state behavior at the beginning of the horizon. That is, the aim of (3.21) is to select  $\mu_N(x)$  so that the first state,  $x_{t+1,t}^d$ , enters some particular subset of  $\mathcal{X}_1$  that has desirable properties. This suggests the existence of a terminal cost  $F(x)$  satisfying (3.21) for a single-stage,  $N = 1$ , problem. Indeed, the infinite-horizon value function  $V_\infty(x)$ , if selected as the terminal cost  $F(x)$ , satisfies (3.21) and, as a stationary point of the dynamic programming equation [7], yields a one-step-ahead MPC design problem that achieves the optimal infinite-horizon control  $\mu_\infty$  and optimal infinite-horizon performance. This concordance between the terminal cost and the infinite-horizon value function has been commented on by [35, 30].*

**Disturbed, discounted case:**  $w_t \neq 0$ ,  $\alpha \in [0, 1)$ .

In this section we establish conditions on the terminal cost under which the designed value function, (3.5), satisfies an inequality reminiscent of monotonic decrease with horizon when disturbances and a discount factor  $\alpha \in [0, 1)$  are present. Then, bounds on the achieved performance can be established in a fashion that is similar to the undisturbed, undiscounted case.

In the following lemma, an inequality condition on the terminal cost is established that mirrors the inequality (3.21) from Theorem 10; recall the inequality (3.21) yields monotonically non-increasing designed value functions in the undisturbed, undiscounted case. This new terminal cost inequality suffices to guarantee bounds on the achieved value function for the stochastic system with discounted cost, for all  $N \geq 1$ .

The new terminal cost inequality introduced here possesses an explicit dependence between the discount factor  $\alpha$  and the terminal cost  $F(x)$ . The result, like the monotonic non-decreasing result from Theorem 5, holds for solution of  $\mathcal{P}_{MPC}$  at a single time  $t$  given horizon  $N$  and terminal cost  $F(x)$ .

**Lemma 3.** *Consider the disturbed, discounted, non-zero terminal cost, horizon- $N$  MPC problem  $\mathcal{P}_{MPC}$  satisfying Assumption 8 with  $\mathcal{X}_j = \mathcal{X}$ ,  $j = 1, \dots, N$ . Suppose that for all  $x \in \mathcal{X}$  and all admissible controls, the terminal cost  $F(\cdot)$  satisfies,*

$$\alpha E_w[F(f(x, u) + w)] - F(x) \leq -l(x, u) + \bar{w}, \quad (3.22)$$

for some  $\bar{w} \in [0, \infty)$ . Then the designed value function satisfies

$$V_k(x) \leq V_{k-1}(x) + \alpha^{k-1}\bar{w}, \quad (3.23)$$

for all  $k \geq 1$  and for  $x \in \mathcal{X}_{\infty\phi}$ .

The inequality condition on the terminal cost (3.22) yields the inequality on the designed value function (3.23), which in turn yields bounds on the achieved value function under MPC control, which we will show shortly.

Paralleling the inequality (3.19) in Theorem 9, the positive scalar  $\bar{w}$  in (3.22) relaxes the inequality. In fact, it can be shown that, even for scalar linear systems

with stochastic disturbances and quadratic stage cost, the inequality cannot be satisfied with positive definite  $F(x)$  and  $\bar{w} = 0$ .

We would of course prefer small  $\bar{w}$  in order to guarantee the designed value function is monotonically non-increasing with horizon to reflect the monotonicity achieved in the undisturbed case of Theorem 10; the guaranteed bound on the achieved value function would also be tighter, as we will show shortly. But this might not be possible.

Here, specification of a small discount factor  $\alpha$  might also ease difficulties in establishing inequality (3.22). Though, as discussed in the previous section, small  $\alpha$  might also relegate bounds on the achieved system state,  $x_t^a$ , to those established by the BIBS stability result in Theorem 4.

As in the undisturbed, undiscounted case of Theorem 10, the terminal cost function requirement (3.22) must hold on the entire space  $\mathcal{X}$ . This might appear unfortunate, but the relaxation of the inequality provided by  $\bar{w}$  allows for flexibility in selection of  $F(x)$ . The penalty paid for poor terminal cost selection given large  $\bar{w}$  is a conservative performance bound, which we now show.

The inequality (3.23) implies a bound on the achieved value function and convergence rate of the designed value function. This bound on the achieved value function is contained in the following result, which imitates the performance bound given in Theorem 10.

**Theorem 11.** *For the disturbed, discounted, non-zero terminal cost, horizon- $N$  MPC problem  $\mathcal{P}_{MPC}$  satisfying Assumption 8 with  $\mathcal{X}_j \subseteq \mathcal{X}$ ,  $j = 1, \dots, N$ , suppose*

$$V_N(x) - V_{N-1}(x) \leq \bar{w}, \quad (3.24)$$

for all  $x \in \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$  and some  $\bar{w} \geq 0$ . Then,

$$V_\infty(x) \leq V_\infty^{\mu N}(x) \leq V_N(x) + \frac{\alpha}{1-\alpha}\bar{w}. \quad (3.25)$$

Here we have made it apparent that performance bound (3.25) follows if the designed value function satisfies the inequality (3.24), which in turn can be made possible via judicious selection of the terminal cost.

The achieved performance bound (3.25), like its counterpart with zero terminal cost in Theorem 9, depends on discount factor  $\alpha$  and positive scalar  $\bar{w}$ . The

comments made following Theorem 9 also apply in this case. That is, selection of small discount factor  $\alpha$  might improve the discrepancy between the infinite-horizon value function and the designed value function, but the achieved state bound on the infinite horizon might be compromised. However, contrary to the performance result from Theorem 9, in which long horizons are required to achieve a sensible bound, the bound here can be satisfied for all  $N \geq 1$ , as long as the selected terminal cost satisfies (3.22).

The following corollary extends Theorem 11 to yield a result on the rate of convergence of the designed value function. With this result in place it becomes apparent why selection of small  $\alpha$  in Theorems 9 and 11 might not be appropriate.

**Corollary 2.** *Suppose that the terminal cost  $F(x)$  satisfies (3.22) from Lemma 3,  $\forall x \in \mathcal{X}$  and consider the problem statement and assumptions given in Theorem 11. Then,*

$$E_w[V_N(f(x, \mu_N(x)) + w)] \leq \frac{1}{\alpha} \left( 1 - \frac{l(x, \mu_N(x))}{F(x) + \frac{1}{1-\alpha}\bar{w}} \right) V_N(x) + \bar{w}.$$

For boundedness of the value function and hence boundedness of the state, the term  $\frac{1}{\alpha} \left( 1 - \frac{l(x, \mu_N(x))}{F(x) + \frac{1}{1-\alpha}\bar{w}} \right)$  must be strictly bounded by unity. This encourages selection of larger discount factor  $\alpha$ .

### 3.6.3 Discussion

Our approach has been to extend the analysis of achieved performance to include disturbance rejection and the infinite-horizon control performance with persistently active constraints achieved by horizon- $N$  MPC. In order to do this, we have introduced a value function based on discounted cost value functions. Techniques from the undisturbed analysis due to [40], [39], and [30] have been adapted to accommodate the stochastic system (3.1) and discounted cost functions. The centerpiece of the analysis has been to explore the application of monotonicity properties of the value function with horizon and both its dependence on the terminal cost function and its consequences for achieved performance bounds. The

flavor of the results for value functions monotonically increasing with horizon is that there exists a horizon  $N_0$  large enough so that for any  $N \geq N_0$  the MPC achieved performance is able to be characterized as arbitrarily close to optimal. For the monotonically decreasing value functions, the tenor is towards ensuring performance with small horizons  $N$  and this, in turn, is related to the selection of a sufficiently large terminal cost  $F(x)$  in the MPC problem. These results generalize the unconstrained linear results from [9].

### 3.7 Conclusions

The chapter has considered the recursive feasibility and achieved performance of a disturbed MPC problem. In doing so, we have built on the undisturbed value function approaches due to Jadbabaie, Hauser, Grüne, and Rantzer; among others. But we have included modifications to handle the persistent presence of disturbances to yield performance bounds of more direct application to the usage of MPC as a disturbance rejection feedback controller. We have extended the focus on monotonicity of the value functions with horizon length to gain a handle on modifications to the MPC design problem, chiefly through the terminal cost function.

The benefit of a discounted cost function is that the infinite-horizon optimal value function is finite. A consequence is that normal questions of closed-loop stability become moot. Indeed, because of the presence of the disturbances, asymptotic stability is not achievable and a different measure of satisfactory behavior is needed. We note that BIBS stability arises as a side-effect of the MPC problem formulation via the (almost) topological analysis of recursive feasibility. It is our view that BIBS is the most appropriate form of stability for MPC in application. Our aim in future works is to tie together the rough state bounds accorded by the BIBS result to a better quantification via the value function bounds.

To return to the chapter's opening and title, the thrust of the work here has been to harken back to the genesis and dominant application of MPC as a disturbance rejection controller applied typically in the processing industries

to minimize the deleterious effects of persistent disturbances over an unbounded time interval while observing operational constraints which are disturbance induced and therefore also persistent. The destruktiv and déconstructif analysis has attempted to appreciate the set of design properties which are key to achieving quantitative bounds on achieved performance as a function of designed and optimal performance. Necessarily, this has sidestepped important questions of: solubility and solution calculation, robustness beyond bounded disturbances, evaluation and achievement of feasible problem statements. But we have endeavored to redirect focus onto the performance questions and away from efforts to simplify the MPC problem divorced from the intention to use it in applications.

## Appendix

*Proof of Theorem 4.* Since problem  $\mathcal{P}_{MPC}$  is recursively  $N$ -feasible, the  $N$ -feasibility of  $x_0$  suffices, via Theorem 3, for feasibility for all time and all disturbances provided a feasibility preserving control law  $\pi_0^{N,\phi}$  is applied. The boundedness of the achieved state satisfying the closed-loop dynamics  $x_{t+1}^a = f(x_t^a, \pi_0^{N,\phi}(x_t)) + w_t$ , follows because the finite-horizon designed state sequence  $\{x_{t+j+1,t}^d\}$  from (3.7) contains one element,  $x_{t+k,t}^d$ , which is bounded. Since  $x_t^a$  is in the pre-image set of this point in  $f$ , it too must be bounded because  $f$  is a proper map.  $\square$

*Proof of Theorem 9.* Consider the disturbed, discounted, zero terminal cost problem  $\mathcal{P}_{MPC}$  with horizon  $N$ , with state constraints from the theorem statement, and with  $x_t = x \in \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$  at time  $t$ . Corresponding to the designed value

function (3.5), the stochastic dynamic programming equation (3.12) satisfies,

$$\begin{aligned}
V_N(x) &= l(x, \mu_N(x)) + \alpha \mathbf{E}_w[V_{N-1}(f(x, \mu_N(x)) + w)], \\
&= l(x, \mu_N(x)) + \alpha \mathbf{E}_w[V_{N-1}(f(x, \mu_N(x)) + w)] \\
&\quad - \alpha \mathbf{E}_w[V_N(f(x, \mu_N(x)) + w)] \\
&\quad + \alpha \mathbf{E}_w[V_N(f(x, \mu_N(x)) + w)], \\
&\geq l(x, \mu_N(x)) + \alpha \mathbf{E}_w[V_N(f(x, \mu_N(x)) + w)] \\
&\quad - \alpha \gamma_N l(x, \mu_N(x)) - \alpha \bar{w}.
\end{aligned}$$

We rewrite this as

$$\beta_N l(x, \mu_N(x)) \leq V_N(x) - \alpha \mathbf{E}_w[V_N(f(x, \mu_N(x)) + w)] + \alpha \bar{w}$$

with  $\beta_N := (1 - \alpha \gamma_N)$ . Next, for some horizon  $M$ , write the achieved value for  $\mu_N$  starting at time  $t = 0$  with  $x_0 = x$ .

$$\begin{aligned}
\beta_N V_M^{\mu_N}(x) &= \\
&\beta_N \mathbf{E}_w \left[ \sum_{i=0}^{M-1} \alpha^i l(x_i, \mu_N(x_i)) \middle| x_0 = x \right] \\
&\leq \mathbf{E}_w \{ V_N(x_0) - \alpha \mathbf{E}_w[V_N(x_1)] + \alpha \bar{w} \\
&\quad + \alpha V_N(x_1) - \alpha^2 \mathbf{E}_w[V_N(x_2)] + \alpha^2 \bar{w} \\
&\quad \cdots - \alpha^M \mathbf{E}_w[V_N(x_M)] + \alpha^M \bar{w} \mid x_0 = x \} \\
&= V_N(x) - \alpha^M \mathbf{E}_w[V_N(x_M)] \\
&\quad + (\alpha + \alpha^2 + \cdots + \alpha^M) \bar{w}.
\end{aligned}$$

Taking the limit as  $M \rightarrow \infty$  yields the right-hand inequality, while the left-hand inequality follows from optimality.  $\square$

*Proof of Lemma 3.* Consider the disturbed, discounted, non-zero terminal cost problem  $\mathcal{P}_{MPC}$  at time  $t$  with horizon  $N$ , with state constraints from the lemma statement, and with initial state  $x_t = x$ . We use induction to show (3.23).

Using the SDPE (3.12) we have, for  $x \in \mathcal{X}_{\infty\phi} \subseteq \mathcal{X}_{N\phi}$ ,

$$\begin{aligned}
V_1(x) &= V_0(x) = l(x, \pi_{N-1}^{N,*}(x)) \\
&\quad + \alpha \mathbb{E}_w [F(f(x, \pi_{N-1}^{N,*}(x)) + w)] - F(x), \\
&\leq l(x, \bar{u}) + \alpha \mathbb{E}_w [F(f(x, \bar{u}) + w)] - F(x), \\
&\leq \bar{w}.
\end{aligned}$$

where the last inequality is due to (3.22). So we have monotonicity for  $k = 0$ . Now, assume the inductive hypothesis  $V_k(x) \leq V_{k-1}(x) + \alpha^{k-1}\bar{w}$  and consider i.i.d. random variables  $w_1, w_2$  with probability distribution  $P_w(\cdot)$ . Then,

$$\begin{aligned}
V_{k+1}(x) &= l(x, \pi_n^{N,*}(x)) + \alpha \mathbb{E}_{w_2} [V_k(f(x, \pi_n^{N,*}(x)) + w_2)], \\
&\leq l(x, \pi_{n+1}^{N,*}(x)) + \alpha \mathbb{E}_{w_2} [V_k(f(x, \pi_{n+1}^{N,*}(x)) + w_2)], \\
&\leq l(x, \pi_{n+1}^{N,*}(x)) + \alpha \mathbb{E}_{w_2} [V_{k-1}(f(x, \pi_{n+1}^{N,*}(x)) + w_2)] \\
&\quad + \alpha^k \bar{w}, \\
&= l(x, \pi_{n+1}^{N,*}(x)) + \alpha \mathbb{E}_{w_1} [V_{k-1}(f(x, \pi_{n+1}^{N,*}(x)) + w_1)] \\
&\quad + \alpha^k \bar{w}, \\
&= V_k(x) + \alpha^k \bar{w},
\end{aligned}$$

where the first inequality is due to optimality, the second inequality is the inductive hypothesis, and the second-to-last equality follows from the i.i.d. property of  $w_1, w_2$ .  $\square$

*Proof of Theorem 11.* We start by showing (3.24) implies a bound on the stage cost. Consider the disturbed, discounted, non-zero terminal cost problem  $\mathcal{P}_{MPC}$  at time  $t$  with horizon  $N$ , with state constraints as given in the theorem statement, and with initial state  $x_t = x$ . We have, for  $x \in \mathcal{X}_{\infty\phi} \cap \mathcal{X}_{N\phi}$ ,

$$\begin{aligned}
V_N(x) &= l(x, \mu_N(x)) + \alpha \mathbb{E}_w [V_{N-1}(f(x, \mu_N(x)) + w)], \\
&= l(x, \mu_N(x)) + \alpha \mathbb{E}_w [V_{N-1}(f(x, \mu_N(x)) + w)] \\
&\quad - \alpha \mathbb{E}_w [V_N(f(x, \mu_N(x)) + w)] \\
&\quad + \alpha \mathbb{E}_w [V_N(f(x, \mu_N(x)) + w)], \\
&\geq l(x, \mu_N(x)) + \alpha \mathbb{E}_w [V_N(f(x, \mu_N(x)) + w)] - \alpha \bar{w}.
\end{aligned}$$

Rearranging terms yields,

$$l(x, \mu_N(x)) \leq V_N(x) - \alpha E_w[V_N(f(x, \mu_N(x)) + w)] + \alpha \bar{w}. \quad (3.26)$$

One can use the bound (3.26) in a similar manner to that which was presented in the proof of Theorem 9 to verify the stated result.  $\square$

*Proof of Corollary 2.* Using (3.23) from Lemma 3 and (3.26) from the proof of Theorem 11, we have

$$\begin{aligned} V_N(x) - \alpha E_w[V_N(f(x, \mu_N(x)) + w)] &\geq l(x, \mu_N(x)) - \alpha \bar{w}, \\ &\geq l(x, \mu_N(x)) \frac{V_N(x)}{F(x) + \frac{1}{1-\alpha} \bar{w}} - \alpha \bar{w}. \end{aligned}$$

Rearranging terms yields the stated result.  $\square$

This chapter, in part, has been submitted for publication of the material as it may appear in the following.

**D.J. Riggs and R.R. Bitmead.** Destruktion and déconstruction of model predictive control. Pre-print submitted to *Automatica*.

This chapter, in part, is a reprint of the material as it appears in the following.

**D.J. Riggs and R.R. Bitmead.** MPC under the hood / sous les capot / unter der haube. In *Proceedings of the IFAC Conference on Nonlinear Model Predictive Control 2012*, pages 363–368, Noordwijkerhout, the Netherlands, August 2012.

The dissertation author was the primary investigator and author of these papers; Professor Bitmead supervised the research.

# Chapter 4

## Distributed Optimization with Coupling Constraints: Algorithms and Information Exchange Requirements

*We cannot solve our problems with the same thinking we used when we created them.*

— Albert Einstein

### Abstract

We derive and establish properties for algorithms for the distributed optimization of a global constrained optimization problem through the limited communication of data between cooperating nodes; each node solves a constrained local optimization problem. The structure of the optimization problem is such that it is performed by the distinct nodes, which have their own objective functions and local constraints, but share a common coupling constraint which limits the behavior of the independent nodes. Our interest is in performing a distributed computation of the global solution without the participating nodes needing explicit knowledge of other members' cost functions or constraints. Feasibility of local and coupling con-

straints is the main technical challenge; we propose a set of distributed algorithms which are motivated by maintaining feasibility and solve the global problem given limited communication and identify cases in which the nodes must communicate more than a minimal set of information.

## 4.1 Introduction

We consider the distributed solution of a global optimization problem which involves the cooperation of  $n$  local nodes,  $S_1, \dots, S_n$ . Each node  $i$  is equipped with a local cost function and local constraints, both of which are functions of local decision variables  $u_i \in \mathbb{R}^{p_i}$ ,  $i = 1, \dots, n$ . The global cost function is composed of a sum of nodes' local cost functions and there is a joint constraint which is dependent on all decision variables and thereby coupling the behavior of the nodes. Each node has access to its own objective function and local constraints, and elements of the coupling constraint. Each node does not know the cost functions or constraints of other nodes. This structure prevents the computation of a global constrained solution, and we seek an approach which is capable of yielding solutions convergent to the global optimum via distributed local optimization and *limited communications*. By limited communications, we mean to impose limitations on the information content of any data exchanged between nodes.

### 4.1.1 Global Optimization Problem: Setup and Structure

Consider the following inequality constrained, global optimization problem,  $\mathcal{P}_g$ , over  $n$  vector decision variables  $u_1 \in \mathbb{R}^{p_1}, \dots, u_n \in \mathbb{R}^{p_n}$ .

$$\min_{u_1, \dots, u_n} J_g(u_1, \dots, u_n), \quad (4.1)$$

subject to local vector constraints

$$g_i(u_i) \leq c_i, \quad i = 1, \dots, n, \quad (4.2)$$

and a global coupling vector constraint

$$\sum_{i=1}^n f_i(u_i) \leq b. \quad (4.3)$$

The separability of the coupling constraint (4.3) into a sum of local terms is central to our formulation, as will become clear later. Also, the scalar global cost function is constructed as a sum of local cost functions,

$$J_g(u_1, \dots, u_n) = \sum_{i=1}^n J_i(u_i), \quad (4.4)$$

with  $J_i$ ,  $i = 1, \dots, n$ , and, hence  $J_g$ , strictly convex, nonnegative, and continuously differentiable. The constraint vector  $b$  is an element of  $\mathbb{R}^{m_f}$  while the constraint vectors  $c_i$  are elements of  $\mathbb{R}^{m_{g_i}}$ ,  $i = 1, \dots, n$ . The constraint functions  $f_i(u_i)$  and  $g_i(u_i)$  satisfy

$$f_i(u_i) = [f_i^1(u_i), f_i^2(u_i), \dots, f_i^{m_f}(u_i)]^T, \quad i = 1, \dots, n,$$

and

$$g_i(u_i) = [g_i^1(u_i), g_i^2(u_i), \dots, g_i^{m_{g_i}}(u_i)]^T, \quad i = 1, \dots, n,$$

with  $f_i^j(u_i) : \mathbb{R}^{n_{u_i}} \rightarrow \mathbb{R}$ ,  $j = 1, \dots, m_f$ , and  $g_i^k(u_i) : \mathbb{R}^{n_{u_i}} \rightarrow \mathbb{R}$ ,  $k = 1, \dots, m_{g_i}$ , each strictly convex functions. With this structure, the inequalities (4.2) and (4.3) are to be interpreted component-wise. We suppose that the  $\{J_i\}$ ,  $\{f_i\}$ ,  $\{g_i\}$  are not necessarily identical over  $i$ .

Our investigation here is to consider formulations of local subproblems to be solved by each  $S_i$ , together with inter-node communication, which yield a *distributed* solution of the global constrained problem  $\mathcal{P}_g$ . Of central importance to our study is the content of exchanged information and limited communications. We are interested in iterative algorithms which guarantee iterates provide a non-increasing global cost function and maintain feasibility at every step.

Note the use of “distributed” here differs from that in [8] through the nodes not having access to the global cost function nor to the complete constraint set and needing to use communication to achieve global optimality. The problem addressed in [8] deals with distributed computation with full information, while in this study distributed information is the focus.

The coupling constraint of our present formulation is similar to constraints associated with a shared resource amongst the nodes. This interpretation is employed in mechanism theory as applied in microeconomics and networks [97]. How-

ever, unlike mechanism-theoretic approaches where nodes need to be incentivized to cooperate, the nodes here cooperate willingly to meet the shared objective.

### 4.1.2 Information Localization and Information Exchange

Suppose that local node  $S_i$  has access to  $u_i, J_i, f_i, g_i, b$ , and  $c_i, i = 1, \dots, n$ . Local nodes  $S_1, \dots, S_n$  will be equipped with limited communication and jointly will seek to find  $u_1, \dots, u_n$  which solve the global optimization  $\mathcal{P}_g$  through local computation and negotiation.

We refer to the constraint (4.3) as the *coupling* constraint as each node has access to some part of this constraint set and, as such, nodes must cooperate in order to ensure the constraint is satisfied. The constraint (4.2) is termed the *local* constraint as each node  $S_i$  has full access to all information required to satisfy  $g_i(u_i) \leq c_i$ .

The optimization problem will split into cycles in which the entire set of nodes computes and exchanges information and our focus is on limiting the quantity of information needing exchange within a cycle. For this exchange, the nodes need a network topology, which we assume for simplicity is a ring in which sequential processing occurs. That is, node  $S_j$  receives information from node  $S_{j-1}$  and sends information to node  $S_{j+1}$ ; nodes  $S_n$  and  $S_1$  exchange information to close the ring. We are interested in analyzing the minimal set of communicated information required to solve the global problem  $\mathcal{P}_g$ , and the conditions which necessitate the communication or sharing of additional information. We show that in some cases, which are fully characterized, sharing of local constraint functions  $g_i$  is unavoidable if the global solution is to be achieved.

The nodes *share* information when exchanged information content is to include functional forms of cost functions or constraints. Alternatively, we say the nodes *share* information, when a possibly unlimited number of function evaluations is required for communication between nodes. The nodes *communicate* when sending particular values of cost functions or constraints, i.e., for decision variable value  $\bar{u}_i$ , node  $i$  might communicate the vector  $J_i(\bar{u}_i)$ .

Our aim is to find and understand the minimal set of information exchanges

between nodes  $S_i$  required to solve the global problem  $\mathcal{P}_g$ ; we find that the required information exchange depends on constraint activity at the global solution. This minimal-information approach is relevant in cases where: (1) communication or sharing of local cost functions, local constraints, and coupling constraint components, which are local to node  $S_i$ , might be highly undesirable from, say, a security or stealth perspective or (2) communications required to exchange information are expensive and need to be kept to a minimum or (3) where the capability to achieve performance speedup via, say, parallelization is limited by interprocess communication.

### 4.1.3 Motivating Examples: Distributed Control and Network Optimization

We consider here three examples to set ideas and motivate the problem formulation under consideration. The first example is distributed control in the context of multi-vehicle control and collision-avoidance constraints; the second is network optimization, where  $n$  nodes cooperate to maximize network utility subject to local buffer size constraints and shared resource constraints.

#### Distributed Control

Consider two ships which are transporting cargo at sea. Each ship is a dynamic system with local state, local controls, local dynamic models, and local state and actuator constraints. Each ship is subject to local ocean wave disturbances. Each ship seeks a model-predictive-control solution to a local constrained infinite-horizon optimal control problem formulated to minimize fuel consumption while maintaining a reference course. Each ship's dynamics are unknown to the other, though it is assumed that the local cost functions and coupling constraints are designed in comparable units.

Consider the case in which the ships become connected by cables with length restrictions, i.e. for mid-sea refueling or other cargo exchange. These cables introduce a coupling constraint as the ships must not drift too far apart lest

the cables break, or come too close together lest they collide. The local control problems then become coupled due to this new constraint as both ships' control decisions affect the separation distance. The two ships must then cooperate to maintain this constraint through limited communication of local information and negotiation of control decisions; local performance and actuator constraints must also be maintained. It would be prudent to require a more nimble ship to carry out maneuvers to maintain the prescribed separation distance in response to ocean wave disturbances, hence a global cost function is constructed as a sum of local cost functions and the global objective becomes the minimization of total fuel consumption while maintaining course. The functional form of others' local models, cost functions, and constraints are not locally used and communication bandwidth is limited.

### **Network Optimization**

In network resource assignment, we consider a network of  $n$  nodes, each node having access to a local utility function. Each node has a local packet buffer size (i.e. queue length in a router) which should not be exceeded – this is the local constraint. The network has a total resource capacity which is shared by all nodes – this is the coupling constraint. The goal of the  $n$  nodes is to maximize total network utility while minimizing data traffic due to network node management packets, while maintaining the local and coupling constraints – this is the global optimization problem. The exchange of local information between nodes is limited due to the network management communication bandwidth limit. Hence information exchange is to be kept to a minimum as the nodes cooperate to solve the global optimization problem.

### **Multiprocessor Computation**

Multiple processors can be used to speed up certain parallelizable computations. The interprocess communication and the topology of connection introduce limits to the achievable performance benefit over centralized, single-processor approaches [49, 14]. Deterministic a priori bounds on the communication complexity

are required for the success of these designs. While the early works on systolic arrays concentrated on computer clusters, more recent interest in parallelization is driven by embedded systems, notably in Model Predictive Control [59], where constrained optimization is the core operation.

#### 4.1.4 Review of the Literature

Distributed optimization – in the context of both distributed control and distributed network optimization – has recently received the attention of several researchers. In the context of distributed networks, Nedić and others address the design and analysis of algorithms for the solution of distributed optimization problems which are subject to convex constraints and convex cost functions that have the structure (4.4) [96], [69], [77], [104], and [68]. Gossip algorithms [58], [77], broadcast algorithms [68], and consensus algorithms [71], [69], [42], [36], and [5] are among the algorithmic approaches taken to solve these distributed problems, though the main thrust of these approaches is distributed computation, à la the text [8]. The connection between the above algorithms and distributed computation is particularly evident in the paper [61], in which a gossip algorithm is employed to find the optimal solution to an unconstrained, global optimization problem; though the algorithm requires exchange of local cost functions to find the optimal solution. Synchronicity of the information exchange between nodes and delayed arrival of the information (or intermittent communication) [26], [77], [68], [72] are also addressed by these algorithms, but these issues are not of present concern.

In the vein of distributed computation, the above papers which consider coupling constraints like (4.3), assume availability of these constraints to all participants involved in solving the optimization problem. A distributed optimization formulation which is less concerned with distributed computation and instead considers limited information exchange and coupled constraints can be found in [46]. This recent paper addresses solution of a more general distributed optimization problem than considered here; the coupling constraint is not separable, unlike (4.3). Though, the algorithms discussed do not guarantee feasibility during itera-

tions and, in some cases, feasibility is not achieved at convergence. Our primary goal is to ensure feasibility is maintained throughout iterations, hence our approach has the added benefit to allow for early termination of iterations if necessary. Low and Lapsley [60] also address distributed solution of a global optimization problem with a coupling constraint but the formulation does not contain local constraints. Consideration of local and coupling constraints at once is difficult; we later show activity of local constraints at the global solution might necessitate excessive communication in order for the nodes to cooperatively achieve the global solution.

Our focus, as previously stated, is the analysis of distributed solutions to global constrained optimization problems in which we limit information exchange between nodes. The recent paper [24] provides a step towards limited communications, as we define here, in the context of distributed control. The subsystems considered (which are dynamically coupled and have coupled constraints like (4.3)) are restricted to communicate only reference trajectories and communication of models, states, and cost functions is prohibited. The approach in [24] keeps information content exchange to a minimum but requires tightening of local constraints and a presumption that the subsystems have access to an auxiliary constraint set, the satisfaction of which ensures the coupled constraint is met amongst all subsystems. These restrictions result in reduction of the size of the feasible region and hence the solution to the global optimization problem might not be achieved.

#### 4.1.5 Commentary and Contributions

The information exchange which occurs between cooperating nodes is of central importance to the distributed solution of global optimization problems. In the non-distributed case, a number of numerically-focused algorithms are available for computing the solution to the optimization problem (4.1)-(4.3) given complete access to  $J_g$ ,  $f_i$ ,  $g_i$ ,  $b$ , and  $c_i$  [70]. The present chapter, like the paper [98], applies numerical distributed optimization techniques; though we consider here coupling constraints, which are absent from [98].

One possible solution to this problem involves nodes  $S_2, \dots, S_n$  sharing cost functions, local constraints, and coupling constraint functions with a single

node,  $S_1$ , and letting  $S_1$  solve the full-information problem; however, as we are more interested in a limited-information-exchange setup, we refrain from sharing local constraints and local cost functions, unless exchange of such information is absolutely necessary to solve the global problem. We suggest that in some cases the algorithms developed herein may require less data transfer to compute the global solution than the full-information-exchange approach.

We present iterative algorithms which converge to the global solution; ensuring feasibility of local and coupling constraints is the main challenge and hence feasibility is the focus of all presented algorithms. We first evaluate global feasibility of solutions to locally-defined optimization problems which are based entirely on information local to each node. Global feasibility of the resulting local solutions implies a single-step computation of the global solution using only local information; otherwise, the nodes proceed by executing a series of iterative algorithms, the first of which comprises a distributed coordinate descent; the second algorithm (which we term *constraint negotiation* for reasons which will become clear later) executes steps similar to those of Binding Direction Method algorithms [70] and contains a number of distributed line search subroutines for determining suitable step size. We are motivated by the application of current state-of-the-art in numerical optimization routines to our distributed information problem and the majority of our results are extensions of these accepted approaches.

We pay particular attention to the information exchange required in each of the algorithms. We would like avoid sharing of cost functions and local constraints, while limiting communication of cost function and constraint values. The algorithms are developed in such a way as to limit communications and restrict sharing unless absolutely required to solve the global problem. In some cases, local constraint functions need to be shared in order to obtain the global solution; we detail the conditions under which such information exchange is required.

The rest of the chapter is organized as follows. In Section 4.2, we provide the full-information solution to the global solution and discuss properties of this solution; these properties will be used to ascertain convergence of the distributed algorithms. In Section 4.3 we present a series of algorithms which employ inter-

node communication and local computation to enable local nodes to compute the solution to the global problem, and present various results related to algorithm properties. We conclude in Section 4.4.

## 4.2 Global Problem Solution Structure and Properties

In this section we present the solution to the global constrained optimization problem  $\mathcal{P}_g$ . Our aim is to construct distributed algorithms which nodes  $S_1, \dots, S_n$  can employ to cooperatively achieve the global solution. We use properties of the global solution to ascertain convergence of distributed algorithms to the global solution.

As is standard in convex optimization, we assume the Slater condition holds [6], [13].

**Assumption 9** (Slater Condition). *There exists a point  $(\bar{u}_1, \dots, \bar{u}_n)$  such that*

$$\sum_{i=1}^n f_i(\bar{u}_i) < b, \quad (4.5)$$

$$g_i(\bar{u}_i) < c_i, \quad i = 1, \dots, n. \quad (4.6)$$

*That is, there exists a point which lies in the interior of the constraint set.*

With strict convexity of  $J_g, f_i$ , and  $g_i$ , and given Assumption 9, there is a unique  $(u_1^*, \dots, u_n^*)$  which solves the global problem  $\mathcal{P}_g$  if and only if it is a Karush-Kuhn-Tucker (KKT) point [6], [13], [70], [15]. That is, the following conditions hold.

1. *Feasibility.* The point  $(u_1^*, \dots, u_n^*)$  satisfies

$$\sum_{i=1}^n f_i(u_i^*) \leq b,$$

$$g_i(u_i^*) \leq c_i, \quad i = 1, \dots, n.$$

2. *Stationarity.* There exist Lagrange multipliers  $\lambda_0 \in \mathbb{R}^{m_f}$ ,  $\lambda_i \in \mathbb{R}^{m_{g_i}}$ ,  $i = 1, \dots, n$ , satisfying

$$\begin{aligned} \nabla J_i(u_i)|_{u_i^*} &= -(\nabla f_i(u_i)|_{u_i^*})^T \lambda_0 \\ &\quad - (\nabla g_i(u_i)|_{u_i^*})^T \lambda_i, \quad i = 1, \dots, n. \end{aligned} \quad (4.7)$$

3. *Non-negativity.* The multipliers  $\lambda_0 \geq 0, \dots, \lambda_n \geq 0$ .

Furthermore, for a KKT point  $(u_1^*, \dots, u_n^*)$ , the Slater condition of Assumption 9 implies *strict complementarity* of the multipliers [6], [13], [70], i.e.,  $\lambda_i^T (g_i(u_i^*) - c_i) = 0$  with  $\lambda_i = 0$  when  $g_i(u_i^*) < c_i$  and  $\lambda_i \geq 0$  when  $g_i(u_i^*) = c_i$ .

We use these properties of the global solution to construct distributed algorithms for establishing global optimality.

## 4.3 Distributed Optimization

In this section we pose and solve a series of local constrained optimization subproblems in which  $S_1, \dots, S_n$  use local information and limited communication to compute the global solution; the intent is to compute the solution to the global problem  $\mathcal{P}_g$  (4.1) cooperatively.

The algorithms herein detail the local optimization problems and communication between local subsystems  $S_i$ ,  $i = 1, \dots, n$ . We establish connections between the type of communicated information needed to solve the global problem and properties of the global constraint sets.

### 4.3.1 When Local Optima Solve the Global Problem

Let  $u_i^\dagger$ ,  $i = 1, \dots, n$  be the optimum for isolated subproblem  $\mathcal{P}_{iso}^i$ , repeated from Section I-C, that is local to  $S_i$  and recall this subproblem excludes the coupling constraint set (4.3).

$$\begin{aligned} \min_{u_i} \quad & J_i(u_i), \\ \text{subject to} \quad & g_i(u_i) \leq c_i. \end{aligned}$$

Suppose  $S_1$  communicates vector  $f_1(u_1^\dagger)$  to  $S_2$ . Let  $S_2$  compute vector  $f_1(u_1^\dagger) + f_2(u_2^\dagger)$  and communicate this sum to  $S_3$ .  $S_3$  appropriately incorporates local vector  $f_3(u_3^\dagger)$  and communicates the resulting vector to  $S_4$ . This compute-and-communicate scheme continues until  $S_n$  has access to vector  $\sum_{i=1}^n f_i(u_i^\dagger)$ .

The following result establishes a simple test that allows  $S_n$  to evaluate global optimality of the point  $(u_1^\dagger, \dots, u_n^\dagger)$ .

**Theorem 12.** *Let Assumption 9 hold. The point  $(u_1^\dagger, \dots, u_n^\dagger)$  is the unique solution to the global problem  $\mathcal{P}_g$  if and only if it is feasible with respect to the coupling constraint set. That is,*

$$\sum_{i=1}^n f_i(u_i^\dagger) \leq b.$$

Furthermore, at this point there exist unique non-negative Lagrange multipliers  $\lambda_{g_1}, \dots, \lambda_{g_n}$ , satisfying

$$\nabla J_i(u_i)|_{u_i^\dagger} = -(\nabla g_i(u_i)|_{u_i^\dagger})^T \lambda_{g_i}, \quad i = 1, \dots, n.$$

This result captures the case under which the global solution is obtained by combining the  $n$  local solutions. Note the Lagrange multipliers from (4.7) for the global problem  $\mathcal{P}_g$  satisfy  $\lambda_0 = 0$  and  $\lambda_i = \lambda_{g_i}$ ,  $i = 1, \dots, n$ . That is, the components of the coupling constraint set are either inactive or active with equality at the locally-constrained minimum of  $J_g$ , which is evident from the coupling constraint set Lagrange multiplier  $\lambda_0 = 0$ .

### 4.3.2 When Local Optima Are Infeasible: Coordinate Descent and Constraint Negotiation

Consider the case where  $\sum_{i=1}^n f_i(u_i^\dagger) \not\leq b$  and hence the point  $(u_1^\dagger, \dots, u_n^\dagger)$  does not solve the global problem, and make the following assumption.

**Assumption 10** (Globally Feasible Point Availability). *An initial globally feasible solution  $(u_1^f, \dots, u_n^f)$  is available.*

The approach to determining such a solution is left for later study. Let  $S_1$  have access to  $u_1^f$ ,  $S_2$  have access to  $u_2^f$ , and so forth for nodes  $S_3$  through  $S_n$ .

With these decision variable values, we have

$$\begin{aligned} \sum_{i=1}^n f_i(u_i^f) &\leq b, \\ g_i(u_i^f) &\leq c_i, \quad i = 1, \dots, n, \end{aligned}$$

by feasibility and we identify the active rows of the vector coupling constraint set.

$$f_1^a(u_1^f) + f_2^a(u_2^f) + \dots + f_n^a(u_n^f) = b^a,$$

for some (possibly empty) vectors  $f_i^a(u_i^f)$  and vector  $b^a$ . Make the following assumption.

**Assumption 11.** *For all sets of feasibly active coupling constraints, the gradients  $\nabla f_i^a(u_i)$  are linearly independent.*

**Remark 7.** *This is a structural property of the constraint set, which eliminates the possibility of linearly dependent constraints involving  $u_1, \dots, u_n$ .*

Suppose  $S_1$  communicates vector  $f_1(u_1^f)$  to  $S_2$  and  $S_2$  computes  $f_1(u_1^f) + f_2(u_2^f)$  and communicates this vector to  $S_3$ . Continue this computation and communication until  $S_n$  receives vectors  $\sum_{i=1}^{n-1} f_i(u_i^f)$  and computes vector  $\sum_{i=1}^n f_i(u_i^f)$  and communicates this vector to  $S_1$ .

Consider one iteration of the following distributed coordinate descent algorithm which starts from the feasible point  $(u_1^f, \dots, u_n^f)$ .

**Algorithm 1.** *Distributed Coordinate Descent.*

1.  $S_1$  computes  $u_1^{cd}$  which satisfies<sup>1</sup>

$$\begin{aligned} \min_{u_1} \quad & J_1(u_1), \\ \text{s.t.} \quad & f_1(u_1) \leq - \sum_{i=2}^n f_i(u_i^f) + b, \\ & g_n(u_1) \leq c_1, \end{aligned}$$

and communicates vector  $\sum_{i=2}^n f_i(u_i^f) + f_1(u_1^{cd})$  to  $S_2$ .

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<sup>1</sup>The notation  $u^{cd}$  is chosen to reflect the value  $u$  which is computed using the coordinate descent (CD) algorithm

2. FOR  $i = 2$  TO  $n-1$

$S_i$  computes  $u_i^{cd}$  which satisfies

$$\begin{aligned} & \min_{u_i} J_i(u_i), \\ & \text{s.t.} \\ & f_i(u_i) \leq - \sum_{j=i+1}^n f_j(u_j^f) - \sum_{j=1}^{i-1} f_j(u_j^{cd}) + b, \\ & g_i(u_i) \leq c_i, \end{aligned}$$

and communicates vector  $\sum_{j=i+1}^n f_j(u_j^f) + \sum_{j=1}^i f_j(u_j^{cd})$  to  $S_{i+1}$ .

3.  $S_n$  computes  $u_n^{cd}$  which satisfies

$$\begin{aligned} & \min_{u_n} J_n(u_n), \\ & \text{s.t. } f_n(u_n) \leq - \sum_{j=1}^{n-1} f_j(u_j^{cd}) + b, \\ & g_n(u_n) \leq c_n, \end{aligned}$$

and communicates vector  $\sum_{j=1}^n f_j(u_j^{cd})$  to  $S_1$ .

At the end of a single iteration of Algorithm 1, node  $S_1$  has access to vectors  $\sum_{j=1}^n f_j(u_j^{cd})$  and  $\sum_{j=1}^n b_j$ . We have the following result.

**Lemma 4.** *The point  $(u_1^{cd}, \dots, u_n^{cd})$  is globally feasible, satisfying (4.2) and (4.3). Furthermore,  $J_g(u_1^{cd}, \dots, u_n^{cd}) \leq J_g(u_1^f, \dots, u_n^f)$  with equality holding if and only if  $u_1^{cd} = u_1^f, \dots, u_n^{cd} = u_n^f$ .*

Algorithm 1 describes a single iteration of a distributed coordinate descent that starts at a globally feasible point and either produces another globally feasible point that reduces the global cost or returns the starting point and the global cost remains unchanged, at which point iterations can terminate. Nodes  $S_1, \dots, S_n$  can notify each other if their respective local costs are reduced in order to determine whether to terminate or continue. The next result establishes convergence and hence termination of Algorithm 1.

**Theorem 13.** *Subject to Assumptions 9 and 11, multiple iterations of Algorithm 1 yield a sequence of globally feasible points which converge.*

Each iteration of Algorithm 1 requires each node  $S_i$  to communicate a single vector and solve an optimization problem. The algorithm does not require explicit communication of the cost functions  $J_i$ , constraint functions  $f_i, g_i$ , or local constraint vectors  $c_i, i = 1, \dots, n$ .

Coordinate descent can be terminated after any number of iterations; convergence is not required for our next results. Denote the terminating point of coordinate descent by  $(u_1^{cd,*}, \dots, u_n^{cd,*})$ . The following result establishes properties of this point.

**Theorem 14.** *At  $(u_1^{cd,*}, \dots, u_n^{cd,*})$  there exist nonempty, linearly independent active constraint gradients  $\nabla f_i^a(u_i), i = 1, \dots, n$  and vectors  $\sum_{i=1}^n f_i^a(u_i^{cd,*})$  and  $b^a$  satisfying*

$$\sum_{i=1}^n f_i^a(u_i^{cd,*}) = b^a,$$

and Lagrange multipliers  $\lambda_{u_i} \geq 0, i = 1, \dots, n$ , and  $\lambda_{g_{u_i}}, i = 1, \dots, n$ , satisfying

$$\begin{aligned} \nabla J_i(u_i)|_{u_i^{cd,*}} &= -(\nabla f_i(u_i)|_{u_i^{cd,*}})^T \lambda_{u_i} \\ &\quad - (\nabla g_i(u_i)|_{u_i^{cd,*}})^T \lambda_{g_{u_i}}, \quad i = 1, \dots, n. \end{aligned}$$

Furthermore,  $(u_1^{cd,*}, \dots, u_n^{cd,*})$  solves the global problem  $\mathcal{P}_g$  if and only if  $\lambda_{u_1} = \lambda_{u_2} = \dots = \lambda_{u_n}$ .

*Proof.* Unique non-negative Lagrange multipliers exist as  $u_1^{cd,*}, \dots, u_n^{cd,*}$  are each solutions to strictly convex inequality constrained optimization problems with cost and constraint functions  $J_i, f_i, g_i, i = 1, \dots, n$ .

Now suppose the coupling constraint set  $\sum_{i=1}^n f_i(u_i^{cd,*}) < b$ . Then by strict complementarity as implied by Assumption 9, the Lagrange multipliers  $\lambda_{u_1} = \dots = \lambda_{u_n} = 0$ . Thus we can write,

$$\nabla J_i(u_i)|_{u_i^{cd,*}} = -(\nabla g_i(u)|_{u_i^{cd,*}})^T \lambda_{g_{u_i}}, \quad i = 1, \dots, n.$$

But this implies  $u_1^{cd,*} = u_1^\dagger, \dots, u_n^{cd,*} = u_n^\dagger$  which we already know are infeasible. Thus the active set is nonempty.

We have yet to prove how to test optimality of the point  $(u_1^{cd,*}, \dots, u_n^{cd,*})$ . Certainly if  $\lambda_{u_1} = \dots = \lambda_{u_n} = \lambda^* \geq 0$  then

$$\begin{aligned} \nabla J_g(u_1, \dots, u_n)|_{(u_1^{cd,*}, \dots, u_n^{cd,*})} &= \begin{bmatrix} \nabla J_1(u_1)|_{u_1^{cd,*}} \\ \vdots \\ \nabla J_n(u_n)|_{u_n^{cd,*}} \end{bmatrix}, \\ &= \begin{bmatrix} -(\nabla f_1(u_1)|_{u_1^{cd,*}})^T \lambda^* - (\nabla g_1(u_1)|_{u_1^{cd,*}})^T \lambda_{g_{u_1}} \\ \vdots \\ -(\nabla f_n(u_n)|_{u_n^{cd,*}})^T \lambda^* - (\nabla g_n(u_n)|_{u_n^{cd,*}})^T \lambda_{g_{u_n}} \end{bmatrix} \end{aligned}$$

so that  $(u_1^{cd,*}, \dots, u_n^{cd,*})$  is a first-order KKT point of the global problem and hence its unique solution. This gives us necessity.

Now let  $(u_1^{cd,*}, \dots, u_n^{cd,*})$  solve the global problem and suppose  $\lambda_u \neq \lambda_v$ . Write

$$\begin{aligned} \nabla J_i(u_i)|_{u_i^{cd,*}} &= -(\nabla f_i(u_i)|_{u_i^{cd,*}})^T \lambda_{u_i} \\ &\quad - (\nabla g_i(u_i)|_{u_i^{cd,*}})^T \lambda_{g_{u_i}}, \quad i = 1, \dots, n. \end{aligned}$$

But the global solution satisfies

$$\begin{aligned} \nabla J_i(u_i)|_{u_i^{cd,*}} &= -(\nabla f_i(u_i)|_{u_i^{cd,*}})^T \lambda_0 \\ &\quad - (\nabla g_i(u_i)|_{u_i^{cd,*}})^T \lambda_i, \quad i = 1, \dots, n. \end{aligned}$$

with *unique*  $\lambda_j$ ,  $j = 0, \dots, n$  which requires  $\lambda_0 = \lambda_{u_1} = \dots = \lambda_{u_n}$  and  $\lambda_i = \lambda_{g_{u_i}}$ ,  $i = 1, \dots, n$ . This completes the proof.  $\square$

The distributed coordinate descent algorithm terminates with a subset of the coupling constraint set being active. Let  $G_i(x) = \nabla J_i(u_i)|_x$  and  $H_i(x) = \nabla^2 J_i(u_i)|_x$ ,  $i = 1, \dots, n$ , and consider subproblems  $\mathcal{P}_{S_i}$ ,  $\mathcal{P}'_{S_i}$ , and  $\mathcal{P}''_{S_i}$ . These

subproblems will be employed by subsystems  $S_i$  in Algorithm 2 which follows.

$$\begin{aligned} \mathcal{P}_{S_i} : \min_{p_i} & \left[ \left( \frac{1}{n-1} \sum_{j \neq i} \lambda_{u_j}^T \right) \nabla f_i(u_i)|_{u_i^{cd,*}} \right. \\ & \left. + G_i(u_i^{cd,*})^T \right] p_i + \frac{1}{2} p_i^T H_i(u_i^{cd,*}) p_i, \\ \text{s.t.} & \quad g_i(u_i^{cd,*} + p_i) \leq c_i, \end{aligned}$$

where  $\lambda_{u_i}$ ,  $i = 1, \dots, n$  are vectors of Lagrange multipliers for  $S_i$  corresponding to the active coupling constraint set.

$$\begin{aligned} \mathcal{P}'_{S_i} : \min_{p_1, \dots, p_n} & \left[ \left( \frac{1}{n-1} \sum_{j \neq i} \lambda_{u_j}^T \right) \nabla f_i(u_i)|_{u_i^{cd,*}} \right. \\ & \left. + G_i(u_i^{cd,*})^T \right] p_i + \frac{1}{2} p_i^T H_i(u_i^{cd,*}) p_i, \\ \text{s.t.} & \quad g_j(u_j^{cd,*} + p_j) \leq c_j, \quad j = 1, \dots, n. \end{aligned}$$

$$\begin{aligned} \mathcal{P}''_{S_i} : & \quad \min_{p_i^j} J_i(u_i^{cd,*} + p_i^j), \\ \text{s.t.} & \quad g_i(u_i^{cd,*} + p_i^j) \leq c_i, \\ & \quad f_i(u_i^{cd,*} + p_i^j) = x_j, \end{aligned}$$

where  $x_j$  will be appropriately defined in the algorithm.

In Algorithm 2 below, the  $S_i$  negotiate a step within the active coupling constraint set described by  $f_i^a$  and  $b^a$ . We view the algorithm as a distributed version of well-known Binding Direction Methods [70] which compute points along the active constraint subspace that improve upon the starting point. The algorithm contains six case statements which correspond to various outcomes of the initial steps. In all but the first and fifth case,  $S_1, \dots, S_n$  execute some form of a distributed line search to find a step size  $\alpha$  which yields a feasible and global-cost-reducing point. Excluding the sixth case, the  $S_i$  need not explicitly communicate constraint matrices or cost functions. This makes the algorithm attractive for finding a distributed

solution to a global optimization problem with a coupling constraint set. Before we present the algorithm, define as functions of  $\alpha$ ,  $u_i^{cn}(\alpha) = u_{cd}^* + \alpha p_i^2$ .

**Algorithm 2.** *Constraint Negotiation.*

1. (*Lagrange Multiplier Exchange*).  $S_1$  communicates vector  $\lambda_{u_1}$ , corresponding to the active constraints in the coupling constraint set, to  $S_2$  and  $S_2$  computes  $\lambda_{u_1} + \lambda_{u_2}$  and passes the sum to  $S_3$ . This sum-and-communicate scheme continues until  $S_n$  computes  $\sum_{i=1}^{n-1} \lambda_{u_i} + \lambda_{u_n}$ . The resulting sum is then passed around all the  $S_i$ .
2. ( *$S_j$  Move Proposal*). Pick some  $j \in \{1, \dots, n\}$ . Subsystem  $S_j$  computes  $p_j$ , the solution to  $\mathcal{P}_{S_j}$ .  $S_j$  then computes  $\Delta J_j^j = J_j(u_j^{cd,*} + \alpha_0^j p_j) - J_j(u_j^{cd,*})$  for some  $0 < \alpha_0^j \leq 1$  and communicates vector  $f_j^a(u_j^{cd,*} + \alpha_0^j p_j)$  and scalars  $\alpha_0^j$  and  $\Delta J_j^j$  to  $S_{j+1}$  (or  $S_1$  if  $j = n$ ). Call this subsystem  $S_i$ .
3. ( *$p_j$  Proposal Evaluation*).  $S_i$  solves  $\mathcal{P}_{S_i}''$  with  $x_j := -f_j^a(u_j^{cd,*} + \alpha_0^j p_j)$  for  $p_i^j$ , computes  $\Delta J_i^j = J_i(u_i^{cd,*} + p_i^j) - J_i(u_i^{cd,*})$  (set  $\Delta J_i^j = \infty$  if infeasible), and communicates vector  $f_j^a(u_j^{cd,*} + \alpha_0^j p_j) + f_i^a(u_i^{cd,*} + p_i^j)$  and scalar  $\Delta J_i^j$  to subsystem  $S_{i+1}$ . Subsystem  $S_{i+1}$  then solves  $\mathcal{P}_{S_{i+1}}''$  with  $x_j := f_j^a(u_j^{cd,*} + \alpha_0^j p_j) + f_i^a(u_i^{cd,*} + p_i^j)$  and communicates appropriate vectors to  $S_{i+2}$ . This procedure repeats until subsystem  $S_j$  receives information from  $S_{j-1}$ .
4. Steps 2 to 3 repeat until each subsystem  $S_j$ , computes  $p_j$  from Step 2 and all other subsystems have responded by accomplishing Step 3 for each  $p_j$ .
5. ( *$p_j$  Direction Selection*). Pick  $j^*$  such that  $\sum_{i=1}^n \Delta J_i^j$  is smallest. Then  $p_{j^*}$  is chosen as the direction to move, and all other subsystems move  $p_i^{j^*}$  as computed from Step 3.

### Case 1.

*Conditions:*

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<sup>2</sup>The notation  $u_{cn}$  is chosen to reflect the value  $u$  which is computed using the constraint negotiation (CN) algorithm

- **Local feasibility:** vectors  $p_i^{j^*}$ ,  $i = 1, \dots, n$ , satisfying the problem in Step 3 are found.
- **Global feasibility:**  $\sum_{i \neq j^*} f_i(u_i^{cd,*} + p_i^{j^*}) + f_{j^*}(u_{j^*}^{cd,*} + \alpha_0^{j^*} p_{j^*}) \leq b$ .
- **Decreasing:**  $\sum_{i=1}^n \Delta J_i^{j^*} < 0$ .

Run coordinate descent Algorithm 1 to ensure positivity of Lagrange multipliers and then proceed to subsequent iteration of Algorithm 2.

### Case 2.

Conditions:

- Local feasibility.
- Global feasibility.
- **Non-decreasing:**  $\sum_{i=1}^n \Delta J_i^{j^*} \geq 0$  with  $p_{j^*} \neq 0$ .

$\alpha^*$ -Step: Distributed line search to compute  $0 < \alpha^* < \alpha_0^{j^*}$  such that

1.  $(u_1^{cn}, \dots, u_n^{cn})$  is feasible,
2.  $\sum_{i=1}^n \Delta J_i^{j^*} < 0$ , and
3. the Lagrange multipliers for the coupling constraint at this new point  $\lambda_{u_i} \geq 0$ ,  $i = 1, \dots, n$ . Then proceed with subsequent iteration.

### Case 3.

Conditions:

- Local feasibility.
- **Global infeasibility:**  $\sum_{i \neq j^*} f_i(u_i^{cd,*} + p_i^{j^*}) + f_{j^*}(u_{j^*}^{cd,*} + \alpha_0^{j^*} p_{j^*}) \not\leq b$ .
- Decreasing.

$\alpha_{max}^{j^*}$ -Step: Distributed computation of  $0 < \alpha_{max}^{j^*} < \alpha_0^{j^*}$  such that

1.  $\sum_{i \neq j^*} f_i(u_i^{cd,*} + p_i^{j^*}) + f_{j^*}(u_{j^*}^{cd,*} + \alpha_{max}^{j^*} p_{j^*}) \leq b$ ,
2. the active set vectors  $f_i^a$  and  $b^a$  increase by a single row, and
3. the Lagrange multipliers at this new point  $\lambda_{u_i} \geq 0$ ,  $i = 1, \dots, n$ .

**Case 4.**

Conditions:

- *Local feasibility.*
- *Global infeasibility.*
- *Non-decreasing.*

**$\hat{\alpha}$ -Step:** Execute  **$\alpha^*$ -Step** and  **$\alpha_{max}$ -Step**. Compute  $\hat{\alpha}^{j*} = \min\{\alpha_{max}^{j*}, \alpha^*\}$  that is globally feasible and decreasing. If  $\hat{\alpha} = \alpha_{max}$  then the active constraint set increases by a single constraint.

**Case 5.**

Conditions:

- **Stationarity:** The solution  $p_j = 0$  to  $\mathcal{P}_{S_j}$ ,  $\forall j = 1, \dots, n$ , yielding  $\sum_{i=1}^n \Delta J_i^j = 0$ .

Iterations terminate; global solution achieved.

**Case 6.**

Conditions:

- **Local infeasibility:** The subproblem in Step 3 is infeasible for all  $p_i$  from Step 2, for given  $\alpha_0^i$ .

**Case-6 Steps:** Distributed line search to compute  $0 \leq \alpha^+ < \alpha_0$  such that the subproblem in Step 3 is feasible. If  $\alpha^+ \neq 0$  then proceed to one of Cases 1-5 with  $\alpha_0^{j*} = \alpha^+$ . If  $\alpha^+ = 0$  then execute the following steps.

1.  $S_i$ ,  $i = 2, \dots, n$  share local constraint functions  $g_i$  and communicate vectors  $u_i^{cd,*}$  and  $c_i$  to  $S_1$
2.  $S_1$  computes the solution  $p_1$  to  $\mathcal{P}'_{S_1}$
3.  $S_1$  computes  $\Delta J_1 = J_1(u_1^{cd,*} + \alpha_0^1 p_1) - J_1(u_1^{cd,*})$  for some  $0 < \alpha_0^1 \leq 1$ .

*This solution  $p_1$  guarantees feasibility of the Step 3 subproblem, at which point all other  $S_i$  can solve the Step 3 subproblem for given  $p_1$  and, subsequently, Cases 1-5 can be evaluated.*

Denote the resulting feasible, decreasing point of Algorithm 3 by  $(u_1^{cn,*}, \dots, u_n^{cn,*})$ . Start a subsequent iteration of Algorithm 2 with this point as the initial condition. We are now ready to state our main result.

**Theorem 15.** *Subject to Assumptions 9 and 11, successive iterations of Algorithm 2 result in a sequence of globally feasible points which converge to the global solution.* ▽▽▽

In solving subproblem  $\mathcal{P}_{S_i}$  in Step 2 of Algorithm 2, node  $S_i$  aims to find a direction  $p_i$  that improves its local cost while noting the effect of  $p_i$  on the other nodes. That is, we interpret the quantity  $\frac{1}{n-1} \sum_{j \neq i} \lambda_{u_j}$  as the average, first-order effect of a move  $p_i$  affecting the other subsystems through the coupled constraint. This allows  $S_i$  to select a global-cost-reducing move  $p_i$ , to first order. Furthermore, when the solution to  $\mathcal{P}_{S_i}$ ,  $p_i = 0$ ,  $\forall i$ , we have  $\frac{1}{n-1} \sum_{j \neq i} \lambda_{u_j}$  yields the same value for all  $i$ .

### 4.3.3 Sharing and Communication

Algorithms 1-2 require inter-node communication to compute a solution to the global problem  $\mathcal{P}_g$ . Algorithm 1 and Cases 1-5 of Algorithm 2 do not require explicit sharing of local cost functions or local constraint matrices. These cases correspond to situations in which nodes' local constraints are inactive at the global solution. In particular, these cases capture problems in which the nodes do not have local constraints. Information exchange for these cases may amount to less information than would be exchanged were explicit sharing of constraint and cost functions done up front. Thus we have identified distributed algorithms which may reduce the necessary quantity of information exchange.

The issue identified with Case 6 of Algorithm 2 is that there is no prescribed upper limit on the communication required to reach a feasible update within one

iterative cycle of the complete set of nodes. This could be manifested as a requirement for the sharing of local constraint functions or through repeated calls for local function evaluation and communication, without a guarantee of completion in bounded time. This occurs when local constraints and the coupling constraint both are active at the global optimum. In light of one of the motivating distributed control problem, Case 6 captures situations in which local actuators saturate or maximum slew-rate limits are engaged at the optimum; these local constraints then become bound to the coupling constraints thus necessitating additional information exchange. We would of course prefer to achieve the global solution without such sharing of constraint functions. But this may not be possible. Case 6 can occur due solely to the coupling constraint and without the local constraints being active at the global optimum if Assumption 11 is relaxed, i.e. when a coupling constraint set arises where any of  $f_i^a$ ,  $i = 1, \dots, n$ , is not linearly independent.

## 4.4 Conclusions and Future Work

In this chapter we present distributed algorithms which enable  $n$  nodes to solve a constrained global optimization problem cooperatively. The  $n$  nodes have independent cost functions and independent local constraints but are coupled in the global optimization via a coupling constraint set. We impose that each node only has access to its own cost function, local constraints, a portion of the coupling constraint, and structure of the global cost function; limited communications are allowed.

The case of most interest occurs when solutions to local optimization problems, based only on local cost functions and constraints, do not satisfy the coupling constraint set. This corresponds to situations in which the coupling constraint set is active at the global solution. These situations are important to understand, as distributed control problems which admit disturbances might see persistently active coupling constraints and hence necessitate cooperation and negotiation. We present two algorithms; the first algorithm is a distributed version of familiar coordinate descent algorithms. The point at which the coordinate descent algorithm

terminates initializes the second algorithm which resembles a distributed Binding Direction Method and employs distributed line search methods throughout. We have identified conditions under which the nodes do need to communicate local constraint functions but explicit communication of cost functions is not necessary for the considered class of problems. Rather, communication is limited to components of the coupling constraint, as well as Lagrange multipliers which are employed for coordinating moves along coupling constraint set subspaces.

The algorithms have been derived with limited communication in mind. Our interest is in performing a distributed computation of the global solution without the participating subsystems needing explicit knowledge of other members' models, constraints, cost functions, and state. Referring back to our motivating example of two ships, we would like to handle cases where the ships are of different type and hence have different dynamics and different local constraints, and communication of such local information becomes a burden.

Throughout our developments and results, we have supposed availability of an initial globally feasible point. Future work may address distributed computation of such a point, under the imposition of limited communication considered here. We also intend to extend these results to problems in which the communication channel is noisy.

This chapter, in part, has been submitted for publication of the material as it may appear in the following.

**D.J. Riggs and R.R. Bitmead.** Distributed optimization with coupling constraints: algorithms and information exchange requirements. Pre-print submitted to *IEEE Transactions on Automatic Control*.

This chapter, in part, is a reprint of the material as it appears in the following, which is copyright IEEE.

**D.J. Riggs and R.R. Bitmead.** Negotiation of coupled constraints in coordinated vehicles. In *Proceedings of the 49th IEEE Conference on Decision and Control*, pages 479–484, Atlanta, GA, December 2010.

The dissertation author was the primary investigator and author of these papers; Professor Bitmead supervised the research.

# Chapter 5

## Conclusions and Future Works

*Hasta la vista, baby.*

— The Terminator

In this dissertation, we consider a control application to a production pulsed light source, a theoretical study of performance bounds for nonlinear model predictive control (NMPC), and the development of an algorithm which several nodes can employ to compute cooperatively the distributed solution of a global optimization problem. The studies are distinct, and the developments in each chapter are mutually independent, though the framework of optimal control is common to the three problems we address.

The optimal control framework provides a plethora of tools, which we have employed to establish new results: aliased periodic disturbance rejection in the light source application using estimation and minimum-variance control concepts, monotonicity of optimal value functions in horizon length for establishing quantifiable performance bounds in NMPC, and Lagrange multipliers for the negotiation and determination of the optimal solution to a global optimization problem, as computed in a distributed manner.

## 5.1 Chapter Summaries and Future Works

### 5.1.1 Chapter 2: Light Source Application

In Chapter 2, we develop and implement an estimator/controller for a production pulsed light source in order to reject aliased periodic disturbances. The new control design is necessitated by light source non-compliance to new performance requirements. The new requirements are mandated by semiconductor chipmakers, in order that light source pulse properties are provided which match decrease in critical dimensions size, which is driven by the semiconductor technology roadmap. The designed estimator/controller generalizes continuous-discrete Kalman filtering ideas to a multi-rate setting to match the variable data rate of the light source. A regularized-minimum-variance, full-state feedback controller is designed which rejects the aliased disturbances. Design changes are also made to light source hardware electronics in order to speed up actuator response. The resultant controller provides performance levels which meet chipmaker requirements, and we show the achieved performance is close to optimal for the hardware.

As the semiconductor technology roadmap dictates further decrease in critical dimension, light source hardware will need to undergo significant technology advancements. New light source hardware is currently under development which generates soft x-rays, with a wavelength of approximately 13.5nm. This new light source technology is expected to be used in chipmaking for the next several critical dimension nodes. Concurrent with hardware development, new software and algorithm development needs to occur. Application of high performance control algorithms as addressed in this dissertation will need to be researched, developed, and implemented on the next generation of production light sources.

### 5.1.2 Chapter 3: Nonlinear Model Predictive Control

In Chapter 3, we investigate the relationship between NMPC design elements which appear in a stochastic, constrained, finite-horizon optimal control problem (horizon, state constraints, stage cost, terminal cost, discount factor) and feasibility, stability, and performance bounds. Theoreticians rarely consider the

performance question for nonlinear systems. This motivates our developments. To our knowledge, we are the first to consider the performance question for systems with stochastic disturbances and persistently-active constraints. We employ topological analysis of feasibility and recursive feasibility of NMPC to establish existence of bounds on the state of the stochastic system. Generally, existence of state bounds is accomplished via the prescription of a stage cost and the employment of input-to-state stability (ISS) analysis; our results for existence of state bounds do not require specification of a cost function. We further the state bound results by providing a quantification for performance bounds. We use monotonicity of the finite-horizon optimal value function in horizon to establish these performance bounds. We investigate both monotonically-increasing and monotonically-decreasing value functions in horizon and analyze the role of the discount factor, horizon length, and disturbance size on the achieved performance bounds.

Our study offers preliminary insight into the difficulties which might arise in establishing performance bounds for NMPC controllers. Further work is required in understanding the assumptions which are required to yield our present results. We postulate that we have provided a framework which is sufficient for the analysis and relaxing of key assumptions. In particular, further work is needed in three areas. The first is robustness of the NMPC controller to modeling errors, which is important to consider in concert with the analysis and development system identification tools and modeling error bounds. The second area which requires attention is the incorporation of optimal state estimates for NMPC full-state feedback, and the corresponding analysis of performance bounds when state estimation errors are present. Lastly, the performance bounds we present are not proven to be tight. Tight performance bounds and the corresponding analysis and insight into connections between these bounds and the NMPC design elements would prove useful to industrial control engineers who implement and operate NMPC controllers in practice.

### 5.1.3 Chapter 4: Distributed Optimization

In Chapter 4, we develop an algorithm for computing a solution to a global optimization problem in a distributed manner. We assert an information architecture which sees local nodes having access to local cost functions and local constraints. This allows each node to solve its own local optimization problem for its own local minimizers. The nodes become coupled through the prescription of a global cost function which is composed as a sum of local cost functions and a coupling constraint which is a function of all nodes' minimizers. The nodes must cooperate and communicate to solve the resulting global optimization problem. We are motivated by limited communications, which are important in large-scale optimization problems, i.e. those which occur in parallel processing or coordinated vehicles. We identify several algorithms and their corresponding information exchange requirements and establish cases in which unbounded communication might be required.

The next generation of MPC challenges arise in implementing MPC controllers in embedded systems. Fast and real-time, i.e. deterministic, computational performance is required in the embedded applications of interest. In order to achieve the required computation speed, parallel processing will likely be employed. This requires algorithms which can solve NMPC optimization problems in a distributed manner with deterministic computation times. This requires a known (and preferably small) bound on the inter-processor communication required to solve the optimization problem of interest. Our results are preliminary in the sense that we establish information content required to solve these distributed problems; we do not provide explicit bounds for communications. Further work needs to occur to establish these bounds.

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