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DERIVATIONS ON REAL AND COMPLEX JB*-TRIPLES

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1. *Introduction*

At the regional conference held at the University of California, Irvine, in 1985 [**24**], Harald Upmeier posed three basic questions regarding derivations on JB*-triples:

(1) Are derivations automatically bounded?

(2) When are all bounded derivations inner?

(3) Can bounded derivations be approximated by inner derivations?

These three questions had all been answered in the binary cases. Question 1 was answered affirmatively by Sakai [**17**] for C*-algebras and by Upmeier [**23**] for JB-algebras. Question 2 was answered by Sakai [**18**] and Kadison [**12**] for von Neumann algebras and by Upmeier [**23**] for JW-algebras. Question 3 was answered by Upmeier [**23**] for JB-algebras, and it follows trivially from the Kadison–Sakai answer to question 2 in the case of C*-algebras.

In the ternary case, both question 1 and question 3 were answered by Barton and Friedman in [**3**] for complex JB*-triples. In this paper, we consider question 2 for real and complex JBW*-triples and question 1 and question 3 for real JB*-triples. A real or complex JB*-triple is said to have the *inner derivation property* if every derivation on it is inner. By pure algebra, every finite-dimensional JB*-triple has the inner derivation property. Our main results, Theorems 2, 3 and 4 and Corollaries 2 and 3 determine which of the infinite-dimensional real or complex Cartan factors have the inner derivation property.

2. *Background*

We recall that a JB*-*algebra* is a complete normed Jordan complex algebra (say $\mathscr{A}$) endowed with a conjugate-linear algebra involution * satisfying $\|U_x(x^*)\| = \|x\|^3$ for every $x \in \mathscr{A}$. Here, for every Jordan algebra $\mathscr{A}$, and every $x \in \mathscr{A}$, U_x denotes the operator on $\mathscr{A}$ defined by $U_x(y) := 2x \circ (x \circ y) - x^2 \circ y$, for all $y \in \mathscr{A}$.

A JB-*algebra* is a complete normed Jordan real algebra (say A) satisfying the following two additional conditions for $a, b \in A$:

(i) $\|a^2\| = \|a\|^2$.

(ii) $\|a^2\| \leqslant \|a^2 + b^2\|$.

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It is due to Wright (see [**25**]) that the complexification of a JB-algebra is a JB*-algebra under a unique norm extending the given norm on the JB-algebra. Conversely, the self-adjoint part of a JB*-algebra is a JB-algebra under the restricted norm.

If H is a complex Hilbert space, then the real Banach space $\mathscr{H}(H)$ of all bounded hermitian operators on H is a JB-algebra with respect to the Jordan product

$$x \circ y := \tfrac{1}{2}(xy + yx).$$

A uniformly (respectively weakly) closed unital real subalgebra of $\mathscr{H}(H)$ is called a JC-*algebra* (respectively JW-*algebra*) on H. A norm (respectively weakly) closed (complex) Jordan*-subalgebra of a C*-algebra (respectively von Neumann algebra) is called a JC*-*algebra* (respectively JW*-*algebra*). For more details on JB-algebras and JB*-algebras we refer the reader to [**9**].

We recall that a (complex) JB*-*triple* is a complex Banach space $\mathscr{J}$ with a continuous triple product $\{\cdot,\cdot,\cdot\} : \mathscr{J} \times \mathscr{J} \times \mathscr{J} \longrightarrow \mathscr{J}$ that is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and that satisfies the following conditions.

(i) (Jordan identity) $L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\} - \{x,L(b,a)y,z\} + \{x,y,L(a,b)z\}$ for all a,b,x,y,z in $\mathscr{J}$, where $L(a,b)x := \{a,b,x\}$.

(ii) For all $a \in \mathscr{J}$, the map $L(a,a)$ from $\mathscr{J}$ to $\mathscr{J}$ is a hermitian operator with non-negative spectrum.

(iii) $\|\{a,a,a\}\| = \|a\|^3$ for all a in $\mathscr{J}$.

It is worth mentioning that every C*-algebra is a (complex) JB*-triple with respect to $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$. Also, every JB*-algebra is a JB*-triple with respect to $\{a,b,c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$. Conversely, every JB*-triple with a unitary element u (that is, $\{u,u,z\} = z$ for every z) is a unital JB*-algebra with product $a \circ b = \{a,u,b\}$, involution $a^* = \{u,a,u\}$, and unit u. We refer to [**5**, **15**, **16**] for recent surveys on the theory of JB*-triples.

Following [**11**], we recall that a *real* JB*-*triple* is a norm-closed real subtriple of a complex JB*-triple. Given a real JB*-triple J, there exists a unique complex JB*-triple structure on the complexification $\hat{J} = J \oplus iJ$, and a unique conjugation (that is, conjugate-linear isometry of period 2) τ on $\hat{J}$ such that $J = \hat{J}^{\tau} := \{x \in \hat{J} : \tau(x) = x\}$. From this point of view, the real JB*-triples are real forms of complex JB*-triples.

The class of real JB*-triples includes all JB-algebras [**9**], all real C^*-algebras [**8**], and all J*B-algebras [**2**].

A *triple derivation* or simply a *derivation* δ on a real or complex JB*-triple U is a linear operator satisfying

$$\delta\{a,b,c\} = \{\delta a,b,c\} + \{a,\delta b,c\} + \{a,b,\delta c\}$$

for all $a,b,c \in U$.

If U is a real or complex JB*-triple, we can conclude from the Jordan identity that $\delta(a,b) := L(a,b) - L(b,a)$ is a derivation, for every $a,b \in U$. An *inner triple derivation* δ on U is a finite sum of derivations of the form $\delta(a,b)$ $(a,b \in U)$, that is,

$$\delta = \sum_{i=1}^{n} \delta(a_j, b_j). \tag{2.1}$$

The *degree* of an inner derivation is the least number of terms in a representation of the form (2.1). Any derivation that is not inner is called *outer*.

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REMARK 1. Let E be a real JB*-triple and δ a derivation on E. Then δ can be extended to a derivation $\hat{\delta}$, on the complexification of E, defined by $\hat{\delta}(x+iy) := \delta(x)+i\delta(y)$.

It is due to Barton and Friedman [**3**] that every derivation on a complex JB*-triple is automatically continuous, and so, by the previous comment, every derivation on a real JB*-triple is also continuous.

3. *Inner derivation property*

We say that a real or complex JB*-triple U has the *inner derivation property* if every derivation on U is inner.

By [**14**, Chapter 8], every finite-dimensional real or complex JB*-triple has the inner derivation property. The next proposition shows that a real JB*-triple has the inner derivation property whenever its complexification satisfies this property.

PROPOSITION 1. *Let E be a real* JB**-triple. Suppose that the complexification $\hat{E}$ of E has the inner derivation property. Then E has the inner derivation property. Moreover, if M is a bound of the degree of all inner derivations of $\hat{E}$, then $2M$ is a bound of the degree of all inner derivations of E.*

Proof. Suppose that E is a real JB*-triple such that $\hat{E}$ has the inner derivation property. Let δ be a derivation of E. We denote by $\hat{\delta}$ the derivation on $\hat{E}$, extending δ to $\hat{E}$. Since $\hat{E}$ has the inner derivation property, then $\hat{\delta}$ is an inner derivation of degree n, that is,

$$\hat{\delta} = \sum_{k=1}^{n} \delta(a_k, b_k),$$

where $a_k, b_k \in \hat{E}$. Since $\hat{E} = E \oplus iE$, it follows that $a_k = a_{k,1}+ia_{k,2}$ and $b_k = b_{k,1}+ib_{k,2}$ for suitable $a_{k,l}, b_{k,l} \in E$, $l = 1,2$ and $k = 1,\ldots,n$.

Consider now $x \in E$. We can compute

$$\begin{aligned}
\delta(a_k, b_k)x &= \delta(a_{k,1}+ia_{k,2}, b_{k,1}+ib_{k,2})x \\
&= \{a_{k,1}+ia_{k,2}, b_{k,1}+ib_{k,2}, x\} - \{b_{k,1}+ib_{k,2}, a_{k,1}+ia_{k,2}, x\} \\
&= \{a_{k,1}, b_{k,1}, x\} + \{a_{k,2}, b_{k,2}, x\} + i(\{a_{k,2}, b_{k,1}, x\} - \{a_{k,1}, b_{k,2}, x\}) \\
&\quad - \{b_{k,1}, a_{k,1}, x\} - \{b_{k,2}, a_{k,2}, x\} - i(\{b_{k,2}, a_{k,1}, x\} - \{b_{k,1}, a_{k,2}, x\}) \\
&= \delta(a_{k,1}, b_{k,1})(x) + \delta(a_{k,2}, b_{k,2})(x) \\
&\quad + i(\{a_{k,2}, b_{k,1}, x\} - \{a_{k,1}, b_{k,2}, x\} - \{b_{k,2}, a_{k,1} x\} + \{b_{k,1}, a_{k,2}, x\}).
\end{aligned}$$

Therefore

$$\begin{aligned}
E \ni \delta(x) = \hat{\delta}(x) &= \sum_{k=1}^{n} \delta(a_{k,1}+ia_{k,2}, b_{k,1}+ib_{k,2})x \\
&= \left(\sum_{k=1}^{n} (\delta(a_{k,1}, b_{k,1}) + \delta(a_{k,2}, b_{k,2}) \right) x \\
&\quad + i \sum_{k=1}^{n} (L(a_{k,2}, b_{k,1}) - L(a_{k,1}, b_{k,2}) - L(b_{k,2}, a_{k,1}) + L(b_{k,1}, a_{k,2}))(x)
\end{aligned}$$

Since the elements $a_{k,l}, b_{k,l} \in E$, we have

$$\left(\sum_{k=1}^{n}(\delta(a_{k,1}, b_{k,1}) + \delta(a_{k,2}, b_{k,2}))\right)(E) \subset E$$

and

$$\left(i\sum_{k=1}^{n}(L(a_{k,2}, b_{k,1}) - L(a_{k,1}, b_{k,2}) - L(b_{k,2}, a_{k,1}) + L(b_{k,1}a_{k,1}))\right)(E) \subset iE.$$

Therefore

$$\left(\sum_{k=1}^{n}(L(a_{k,2}, b_{k,1}) - L(a_{k,1}, b_{k,2}) - L(b_{k,2}, a_{k,1}) + L(b_{k,1}, a_{k,1}))\right)(x) = 0$$

for all $x \in E$. Thus

$$\delta(x) = \hat{\delta}(x) = \sum_{k=1}^{n}(\delta(a_{k,1}, b_{k,1}) + \delta(a_{k,2}, b_{k,2}))(x)$$

for all $x \in E$, which proves that δ is an inner derivation with degree at most $2n$. □

From Proposition 1, it is easy to see that if E is a real JB*-triple which does not satisfy the inner derivation property, then its complexification also does not satisfy the inner derivation property.

3.1. *Reversible unital* JB*-*algebras*

We recall that the (complex) *type* 1 *Cartan factor* can be defined as the complex Banach space $\mathrm{BL}(H,K)$ of all bounded linear operators between two complex Hilbert spaces H and K, with triple product given by

$$\{a, b, c\} = \tfrac{1}{2}(ab^*c + cb^*a).$$

Next we give a brief description of the (complex) Cartan factors of type 2 and 3. Let H be a complex Hilbert space equipped with a conjugation (conjugate-linear isometry of period 2) $j : H \longrightarrow H$; then for any $z \in \mathrm{B}(H)$ we can define its transpose as $z^{\mathrm{t}} := jz^*j$. The *type* 2 *Cartan factor* coincides with the Banach space of all t-skew symmetric elements in $\mathrm{B}(H)$ ($z^{\mathrm{t}} = -z$), and the *type* 3 *Cartan factor* is defined as the Banach space of all t-symmetric elements of $\mathrm{B}(H)$ ($z^{\mathrm{t}} = z$). The triple product of these Cartan factors is the restriction of the triple product in $\mathrm{B}(H)$.

We recall that a JC-algebra (or a JC*-algebra) A is said to be *reversible* if $x_1x_2 \ldots x_n + x_nx_{n-1} \ldots x_1 \in A$, for all $n \in \mathrm{N}$ and $x_1, \ldots, x_n \in A$.

PROPOSITION 2. *Cartan factors of type* 1 *with* $\dim H = \dim K$, *Cartan factors of type* 2 *with* $\dim H$ *even, or infinite, and all Cartan factors of type* 3 *are reversible* JW*-*algebras.*

Proof. Let C^3 be a type 3 Cartan factor. Since $x^{\mathrm{t}} = x$ for all $x \in C^3$, we have $(x_1 \ldots x_n + x_n \ldots x_1)^{\mathrm{t}} = x_n \ldots x_1 + x_1 \ldots x_n \in C^3$.

Let C^2 be a type 2 Cartan factor with $\dim H$ even or infinite. Then C^2 contains a distinguished unitary

$$u = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdot & \cdot \\ -1 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 & \cdot & \cdot \\ 0 & 0 & -1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

In this case we can provide a new C*-algebra structure for B(H) with product $a \cdot b = au^*b$ and involution $a^\| = ua^*u$ in which C^2 becomes a JC*-subalgebra under $a \circ b = (a \cdot b + b \cdot a)/2$. With this Jordan product, C^2 is reversible since

$$\begin{aligned}(x_1u^*x_2 \dots u^*x_n + x_nu^* \dots x_2u^*x_1)^{\mathrm{t}} \\ &= (-1)^{n+(n-1)}(x_nu^*x_{n-1} \dots x_2u^*x_1 + x_1u^*x_2 \dots x_{n-1}u^*x_n) \\ &= -(x_1u^*x_2 \dots u^*x_n + x_nu^* \dots x_2u^*x_1).\end{aligned}$$ □

We recall that if $\mathscr{A}$ is an algebra, then a *derivation* D of $\mathscr{A}$ is a linear mapping $D : \mathscr{A} \longrightarrow \mathscr{A}$ satisfying $D(ab) = D(a)b + aD(b)$, for all $a, b \in \mathscr{A}$. If $\mathscr{A}$ is a Jordan algebra, an *inner algebra derivation* of $\mathscr{A}$ is a finite sum of commutators of the form $[L_a, L_b]$ for some $a, b \in \mathscr{A}$, where $L_a x := a \circ x$. For an inner algebra derivation D, the *degree* of D is the least natural number n satisfying $D = \sum_{i=1}^n [L_{a_i}, L_{b_i}]$.

LEMMA 1. *Let Z be a* JB*-*algebra, with unit u, regarded as a complex* JB*-*triple. If δ is a triple derivation of Z, then $L_{\delta(u)}$ is an inner triple derivation of Z of degree* 1.

Proof. Simply note that for every triple derivation δ of Z, we have

$$\begin{aligned}\delta u = \delta\{u,u,u\} &= \{\delta u, u, u\} + \{u, \delta u, u\} + \{u, u, \delta u\} \\ &= 2\{\delta u, u, u\} + \{u, \delta u, u\} = 2\delta u \circ u + (\delta u)^*\end{aligned}$$

and hence

$$(\delta u)^* = -\delta u.$$

Now consider

$$\begin{aligned}L_{\delta u} z = \delta u \circ z &= \tfrac{1}{2}(\delta u \circ z - (-\delta u) \circ z) \\ &= \tfrac{1}{2}(\delta u \circ z - (\delta u)^* \circ z) = \tfrac{1}{2}(\{\delta u, u, z\} - \{u, \delta u, z\});\end{aligned}$$

it follows that $L_{\delta u}$ is an inner triple derivation of degree 1. □

LEMMA 2 [**3**, p. 263]. *Let Z be a unital* JB*-*algebra and D be an algebra derivation of Z that commutes with the involution of Z. Then D is a triple derivation of Z.*

Conversely, if Z is a JB*-*triple with a unitary element u and δ is a triple derivation of Z, then $\delta - L_{\delta u}$ is an algebra derivation of Z that commutes with the involution on Z. In particular, if δ is an inner derivation of degree* 1, *that is, $\delta = \delta(x, y)$, then*

$$\delta - L_{\delta(u)} = \tfrac{1}{2}([L_{x+x^*}, L_{y+y^*}] + [L_{-i(x-x^*)}, L_{-i(y-y^*)}]).$$

Proof. The first statement is clear. To prove the second one, let δ be a triple

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derivation of Z. It is easy to check that

$$\begin{aligned}(\delta - L_{\delta u})(x \circ y) &= \delta\{x, u, y\} - \{\delta u, u, \{x, u, y\}\} \\ &= \{\delta x, u, y\} + \{x, \delta u, y\} + \{x, u, \delta y\} - \{\delta u, u, \{x, u, y\}\} \\ &= \{\delta x, u, y\} + \{x, \delta u, y\} + \{x, u, \delta y\} \\ &\quad - \{\{\delta u, u, x\}, u, y\} + \{x, \{u, \delta u, u\}, y\} - \{x, u, \{\delta u, u, y\}\} \\ &= \delta x \circ y + \{x, \delta u, y\} + x \circ \delta y \\ &\quad - (\delta u \circ x) \circ y + \{x, (\delta u)^*, y\} - x \circ (\delta u \circ y)\end{aligned}$$

(applying $(\delta u)^* = -\delta u$)

$$\begin{aligned}&= (\delta - L_{\delta u})(x) \circ y + \{x, \delta u, y\} + x \circ (\delta - L_{\delta u})(y) - \{x, \delta u, y\} \\ &= (\delta - L_{\delta u})(x) \circ y + x \circ (\delta - L_{\delta u})(y).\end{aligned}$$

Thus $\delta - L_{\delta u}$ is an algebra derivation.

The verification of the last formula is left to the reader. □

By [**22**, Theorem 13] (see also [**1**, p. 255], each JW-algebra A admits a decomposition into weakly closed ideals of the form

$$A = \mathrm{I}_{\mathrm{fin}} \oplus \mathrm{I}_{\infty} \oplus \mathrm{II}_1 \oplus \mathrm{II}_{\infty} \oplus \mathrm{III}.$$

See [**22**] and [**1**] for the meaning of these symbols. A JW-algebra A is called *properly non-modular* if its modular part $\mathrm{I}_{\mathrm{fin}} \oplus \mathrm{II}_1$ vanishes.

In 1980, Upmeier showed that each algebra derivation on a properly non-modular JW-algebra is the sum of six commutators of the form $[L_a, L_b]$ [**23**, Theorem 3.8], and each algebra derivation on a reversible JW-algebra of type $\mathrm{I}_{\mathrm{fin}}$ is the sum of five commutators [**23**, Theorem 3.9].

The proof of the following theorem is implicitly contained in [**23**], and we include it here for completeness.

THEOREM 1. *Let A be a reversible* JW*-algebra of type* II_1. *Then each derivation of A is a sum of at most* 140 *commutators of the form* $[L_a, L_b]$.

Proof. Let A be a reversible JW-algebra of type II_1. We denote by $\mathcal{U}(\mathcal{A})$ its complex enveloping von Neumann algebra (the smallest von Neumann algebra containing A). By [**1**, Theorem 8], $\mathcal{U}(\mathcal{A})^+$ (that is, $\mathcal{U}(\mathcal{A})$ with the Jordan product $w_1 \circ w_2 = (w_1 w_2 + w_2 w_1)/2$) is also of type II_1. Thus if we follow the proof of [**23**, Theorem 3.10], it follows that each derivation of A has the form $D(x) = \mathrm{ad}(w)(x) := [w, x]$ $(x \in A)$, where $w = -w^* \in \mathcal{U}(\mathcal{A})$. Moreover, since $\mathcal{U}(\mathcal{A})^+$ is of type II_1, w is the sum of ten commutators in $\mathcal{U}(\mathcal{A})$ (see [**7**, Theorem 2.3]), so that each derivation of A has the form

$$D = \sum_{j=1}^{10} \mathrm{ad}([w_{1,j}, w_{2,j}]).$$

Since A is the self-adjoint part of $\mathcal{R}(\mathcal{A})$ [**19**], where $\mathcal{R}(\mathcal{A})$ is the real enveloping algebra of A, we have, by [**20**, Lemma 6.1; **21**, Lemma 2.3, Theorem 2.4], $\mathcal{U}(\mathcal{A}) = \mathcal{R}(\mathcal{A}) + i\mathcal{R}(\mathcal{A})$.

Hence every element $w_{l,j}$ is the sum $w_{l,j} = u_{l,j} + iv_{l,j}$, where $u_{l,j}, v_{l,j} \in \mathcal{R}(\mathcal{A})$.

Since, for every $u_l, v_l \in \mathscr{R}(\mathscr{A})$, the equalities

$$[u_1 + iv_1, u_2 + iv_2] = [u_1, u_2] - [v_1, v_2] + i([u_1, v_2] + [v_1, u_2])$$
$$[u_1 + iu_2, x] = [u_1, x] + i[u_2, x]$$

hold for all $x \in A$, and since D maps A in A, we have

$$\sum_{j=1}^{10} [[u_{1,j}, v_{2,j}] + [v_{1,j}, u_{2,j}], x] = 0$$

for all $x \in A$. Thus

$$D = \mathrm{ad}(w) = \sum_{j=1}^{10} \mathrm{ad}([u_{1,j}, u_{2,j}] - [v_{1,j}, v_{2,j}]) = \sum_{j=1}^{20} \mathrm{ad}([z_{1,j}, z_{2,j}]),$$

where $z_{i,j} \in \mathscr{R}(\mathscr{A})$ and $w = \sum_{j=1}^{20} [z_{1,j}, z_{2,j}]$.

Our next goal is to prove that every element $[z_{1,j}, z_{2,j}]$ is a finite sum of commutators of elements in A.

Let $z_{1,j}, z_{2,j} \in \mathscr{R}(\mathscr{A})$, and $l \in \{1, 2\}$. We denote by $z^s_{l,j}$ (respectively $z^a_{l,j}$) the symmetric part (respectively the skew-symmetric part) of $z_{l,j}$. Since, for every j, $[z^a_{1,j}, z^s_{2,j}]$ and $[z^s_{1,j}, z^a_{2,j}]$ are symmetric elements and $w^* = -w$, we deduce that

$$w = \sum_{j=1}^{20} [z^s_{1,j}, z^s_{2,j}] + [z^a_{1,j}, z^a_{2,j}].$$

Again, since A is the self-adjoint part of $\mathscr{R}(\mathscr{A})$, we have $z^s_{1,j}, z^s_{2,j} \in \mathscr{A}$. Therefore it is enough to show that every commutator $[z^a_{1,j}, z^a_{2,j}]$ is a finite sum of commutators of elements in A.

By [**4**, p. 121], $\mathscr{R}(\mathscr{A})$ is isomorphic to the matrix algebra $M_2(B)$, where B is a suitable real associative $*$-algebra.

If we follow the proof of [**23**, Lemma 3.11], it follows that each commutator of skew-symmetric elements in $M_2(B)$ has the form

$$\begin{pmatrix} a & -c^* \\ c & b \end{pmatrix},$$

with

$$a + b = [a_1, a_2] + [b_1, b_2] + [c_1, c_2] + [d_1, d_2],$$

where a_j, b_j and c_j are skew-symmetric elements in B, while d_1 and d_2 are symmetric elements in B.

On the other hand, since for a, b, c, α_j and $\beta_j \in B$, with $a^* = -a$, $b^* = -b$, $\alpha_j^* = \alpha_j$ and $\beta_j^* = -\beta_j$, the following identities hold,

$$\begin{pmatrix} 0 & -c^* \\ c & 0 \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix} \right],$$

$$2 \begin{pmatrix} a - b & 0 \\ 0 & b - a \end{pmatrix} = \left[\begin{pmatrix} 0 & a - b \\ b - a & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right],$$

$$\begin{pmatrix} [\alpha_1, \alpha_2] & 0 \\ 0 & [\alpha_1, \alpha_2] \end{pmatrix} = \left[\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \alpha_2 \end{pmatrix} \right],$$

$$\begin{pmatrix} [\beta_1, \beta_2] & 0 \\ 0 & [\beta_1, \beta_2] \end{pmatrix} = \left[\begin{pmatrix} 0 & -\beta_2 \\ \beta_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\beta_1 \\ \beta_1 & 0 \end{pmatrix} \right],$$

$$\begin{pmatrix} a & -c^* \\ c & b \end{pmatrix} = \begin{pmatrix} 0 & -c^* \\ c & 0 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} a - b & 0 \\ 0 & b - a \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} a + b & 0 \\ 0 & a + b \end{pmatrix},$$

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it can be concluded that each commutator $[z^a_{1,j}, z^a_{2,j}]$ is the sum of six commutators of elements in A. Therefore we have proved that

$$w = \sum_{j=1}^{140} [x_{1,j}, x_{2,j}],$$

where $x_{l,j} \in A$, for all l, j, which proves that

$$D = \sum_{j=1}^{140} \operatorname{ad}([x_{1,j}, x_{2,j}]) = \sum_{j=1}^{140} [L_{x_{1,j}}, L_{x_{2,j}}]. \qquad \square$$

Recall that a derivation on a JB-algebra is automatically continuous and that a JB-algebra has an approximate unit [**9**, 3.5.4]. Thus a derivation leaves each closed ideal invariant. By combining Theorem 1 with the comments preceding it, we have the following corollary.

COROLLARY 1. *Each derivation on a reversible* JW*-algebra is a sum of at most* 151 *commutators of the form* $[L_a, L_b]$.

The next theorem is the main result of this section.

THEOREM 2. *Cartan factors of type* 1 *with* $\dim H = \dim K$, *Cartan factors of type* 2 *with* $\dim H$ *even or infinite, and all Cartan factors of type* 3 *have the inner derivation property. Moreover, every derivation of the above Cartan factors has degree at most* 153.

Proof. By Proposition 2, such factors are unital reversible JW*-algebras. Thus it is enough to prove the statement for a unital reversible JW*-algebra Z.

It is well known that Z decomposes in the form $Z = X + iX$, where X is the symmetric part of Z, and hence X is a reversible JW-algebra.

If δ is a triple derivation of Z, then, by Lemma 2, $\delta - L_{\delta u}$ is a derivation of the JB*-algebra Z that commutes with the involution, and hence its restriction to X is a derivation of X. From the identity

$$(\delta - L_{\delta u})(z) = (\delta - L_{\delta u})(x + iy) = (\delta - L_{\delta u})|_X(x) + i(\delta - L_{\delta u})|_X(y),$$

it follows that $(\delta - L_{\delta u})|_X$ determines $(\delta - L_{\delta u})$. Now, Corollary 1 gives (except for summing the 0 commutator)

$$\begin{aligned}(\delta - L_{\delta u})(z) &= \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](x) + i\sum_{j=1}^{152} [L_{a_j}, L_{b_j}](y)\\ &= \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](x + iy) = \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](z).\end{aligned}$$

Now, applying the identity

$$[L_a, L_b] + [L_c, L_d] = 2(\tilde{\delta} - L_{\tilde{\delta} u}),$$

for all a, b, c and d in X, where

$$\tilde{\delta} = \delta\left(\frac{a + ic}{2}, \frac{b + id}{2}\right),$$

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we obtain

$$(\delta - L_{\delta u})(z) = \sum_{j=1}^{152} [L_{a_j}, L_{b_j}](z) = 2\sum_{j=1}^{76} (\delta(c_j, d_j) - L_{\delta(c_j,d_j)u})(z).$$

Finally, if we apply Lemma 1, it follows that

$$\delta = 2\sum_{j=1}^{76} (\delta(c_j, d_j) - L_{\delta(c_j,d_j)u}) + L_{\delta u}$$

is an inner derivation with degree at most 153. □

Following [**13**], we define a *real Cartan factor* to be a real form of a complex Cartan factor. Combining Theorem 2 and Proposition 1, we obtain the following result for real Cartan factors.

COROLLARY 2. *If E is a real form of a type* 1 *Cartan factor with* $\dim H = \dim K$*, or a real form of a Cartan factor of type* 2 *with* $\dim H$ *even or infinite, or a real form of a Cartan factor of type* 3*, then every derivation on E is inner with degree at most* 306.

3.2. *Real or complex spin factors*

In this subsection, we prove that no infinite-dimensional real spin factor satisfies the inner derivation property. Thus, by Proposition 1, it can be concluded that no complex spin factor satisfies the inner derivation property.

We recall that a *complex spin Cartan factor* is a JB*-triple that can be equipped with a complete inner produce $(.|.)$ and a conjugation $*$ such that the triple product satisfies

$$\{x, y, z\} = (x|y)z + (z|y)x - (x|z^*)y^*,$$

and the norm is given by

$$\|x\|^2 := (x|x) + ((x|x)^2 - |(x|x^*)|^2)^{1/2}.$$

By a *real spin factor*, we mean any real form of a complex spin factor. By [**13**, Theorem 4.1], we know that every real spin factor E is an l_1-sum

$$E = X_1 \oplus^{\ell_1} X_2,$$

where X_1 and X_2 are closed subspaces of a real Hilbert space X satisfying $X_2 = X_1^{\perp}$, and the triple product on E is given by

$$\{x, y, z\} = (x|y)z + (z|y)x - (x|\bar{z})\bar{y},$$

where $(.|.)$ is the inner product of X and the map $x \longmapsto \bar{x}$ is given by $\bar{x} = (x_1, -x_2)$ for all $x = (x_1, x_2) \in E$.

Our goal is to build a derivation that is not inner in the case of an infinite-dimensional real spin factor $E = X_1 \oplus^{\ell_1} X_2$. Without loss of generality, we can assume that X_1 is also infinite-dimensional.

First we suppose that E is separable. Let $\{e_n : n \in \mathrm{N}\}$ be a countable orthonormal basis of X_1. Since $\overline{e_n} = e_n$, it is easy to check that $\{e_n, e_n, e_n\} = e_n$ and

$\|\delta(e_{2k-1}, e_{2k})\| \leqslant 2$, and hence the operator

$$\delta_0 := \sum_{k=1}^{\infty} \frac{1}{2^k} \delta(e_{2k-1}, e_{2k})$$

is a well defined derivation on E. Our goal is to show that δ_0 is not inner. Suppose that δ_0 is inner; then

$$\delta_0 = \sum_{j=1}^{P} \delta(a_j, b_j)$$

for suitable $a_j, b_j \in E$, with $a_j = a_{j,1} + a_{j,2}$ and $b_j = b_{j,1} + b_{j,2}$, where $a_{j,i}$ and $b_{j,i}$ are in X_i ($j = 1, \ldots, P$, $i = 1, 2$). Hence

$$\begin{aligned}\delta_0 &= \sum_{j=1}^{P} \delta(a_j, b_j) \\ &= \sum_{j=1}^{P} \delta(a_{j,1}, b_{j,1}) + \delta(a_{j,1}, b_{j,2}) + \delta(a_{j,2}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}).\end{aligned}$$

It is easy to check that, for all $x_1 \in X_1$,

$$\delta(a_{j,2}, b_{j,2})(x_1) = \delta(a_{j,1}, b_{j,2})(x_1) = \delta(a_{j,2}, b_{j,1})(x_1) = 0$$

and $\delta_0(X_2) = 0$. Therefore

$$\delta_0(x_1) = \sum_{j=1}^{P} \delta(a_{j,1}, b_{j,1})(x_1)$$

for all $x_1 \in X_1$.

Now we define K as the linear span of $\{a_{j,1}, b_{j,1} : j = 1, \ldots, P\}$. Let $x_1 \in K^{\perp} \cap X_1$; then

$$\begin{aligned}0 = \sum_{j=1}^{P} \delta(a_{j,1}, b_{j,1})(x_1) = \delta_0(x_1) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \delta(e_{2k-1}, e_{2k})(x_1) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^k} (\{e_{2k-1}, e_{2k}, x_1\} - \{e_{2k}, e_{2k-1}, x_1\}) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^k} ((e_{2k-1}|e_{2k})x_1 + (x_1|e_{2k})e_{2k-1} - (e_{2k-1}|x_1)e_{2k} \\ &\quad -(e_{2k}|e_{2k-1})x_1 - (x_1|e_{2k-1})e_{2k} + (e_{2k}|x_1)e_{2k-1}) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} ((x_1|e_{2k})e_{2k-1} - (e_{2k-1}|x_1)e_{2k}).\end{aligned}$$

Thus $(x_1|e_{2k}) = (e_{2k-1}|x_1) = 0$ for all $k \in \mathrm{N}$, and so $x_1 = 0$ since $\{e_n\}$ is a basis of X_1. Therefore $K^{\perp} \cap X_1 = 0$, and hence $X_1 = K$ is finite-dimensional, which is impossible.

This proves that δ_0 is not an inner derivation. Suppose now that $\dim X_1 > \aleph_0$, and let $\{e_n\}_{\mathrm{N}}$ be a countable set of orthonormal vectors in X_1. Let us denote by H the real separable Hilbert space generated by $\{e_n\}_{\mathrm{N}}$, and by δ_0 the derivation on E

given by

$$\delta_0 := \sum_{k=1}^{\infty} \frac{1}{2^k} \delta(e_{2k-1}, e_{2k}).$$

Since $\delta_0(H) \subseteq H$, it follows that $\delta_0|_H$ is a derivation of the real spin factor H, which is not inner by the previous case. Actually, we claim that δ_0 is not an inner derivation on E. Suppose, contrary to our claim, that δ_0 is inner on E; then

$$\delta_0 = \sum_{j=1}^{P} \delta(a_j, b_j)$$

with $a_j, b_j \in E$. Since

$$E = (H \oplus^{\ell_2} H^{\perp}) \oplus^{\ell_1} X_2,$$

the elements a_j and b_j can be expressed as $a_j = h_j + x_{j,3}$ and $b_j = k_j + y_{j,3}$, where h_j and k_j are in H and $x_{j,3}, y_{j,3} \in H^{\perp} \oplus^{\ell_1} X_2$ $(j = 1, \ldots, P)$. Thus

$$\begin{aligned}\delta_0 &= \sum_{j=1}^{P} \delta(a_j, b_j)\\ &= \sum_{j=1}^{P} \delta(h_j, k_j) + \delta(h_j, y_{j,3}) + \delta(x_{j,3}, k_j) + \delta(x_{j,3}, y_{j,3}).\end{aligned}$$

It is easy to check that

$$\delta(h_j, y_{j,3})h = -(h_j|h)\overline{y_{j,3}} - (h|h_j)y_{j,3} \in H^{\perp} \oplus^{\ell_1} X_2$$

$$\delta(x_{j,3}, k_j)h = (h|k_j)x_{j,3} + (k_j|h)\overline{x_{j,3}} \in H^{\perp} \oplus^{\ell_1} X_2$$

and

$$\delta(x_{j,3}, y_{j,3})(h) = 0$$

for all $h \in H$. From the last identity, we have

$$\delta_0(h) = \sum_{j=1}^{P} \delta(h_j, k_j)(h) + \sum_{j=1}^{P}(\delta(h_j, y_{j,3}) + \delta(x_{j,3}, k_j))(h)$$

for all $h \in H$. Since $\delta_0(H) \subseteq H$ and $\sum_{j=1}^{P}(\delta(h_j, y_{j,3}) + \delta(x_{j,3}, k_j))(H) \subseteq H^{\perp} \oplus^{\ell_1} X_2$, we have

$$\delta_0(h) = \sum_{j=1}^{P} \delta(h_j, k_j)(h)$$

for all $h \in H$. Therefore $\delta_0|_H$ is an inner derivation on H, which is impossible, and hence δ_0 is not an inner derivation on E.

We have thus proved the following theorem.

THEOREM 3. *Every infinite-dimensional real or complex spin factor has a derivation that is not inner, that is, none of the infinite-dimensional real or complex spin factors has the inner derivation property.*

3.3. *Non-square type* 1

As in the case of a real or complex spin factor, we are going to build an outer derivation in every real form of an infinite-dimensional and non-square ($\dim H \neq \dim K$) type 1 Cartan factor. Again, using Proposition 1, we will conclude that no complex infinite-dimensional non-square type 1 Cartan factor satisfies the inner derivation property.

By [**13**, Theorem 4.1], we know that the real forms of a complex type 1 Cartan factor are precisely the real Banach spaces $\mathrm{BL}(X, Y)$ of all bounded linear operators between two real Hilbert spaces X and Y or the real Banach spaces $\mathrm{BL}(P, Q)$ of all bounded linear operators between two Hilbert spaces P, Q over the quaternion field. Thus it is enough to prove that $\mathrm{BL}(X, Y)$, with $+\infty = \dim(X) > \dim(Y)$, possesses an outer derivation. We will divide the proof into several steps. In a first step, we suppose that $Y = \mathbf{R}$. In this case, $\mathrm{BL}(X, \mathbf{R})$ is isometrically isomorphic, as a real JB*-triple, to X equipped with the triple product

$$\{x, y, z\} = \tfrac{1}{2}((x|y)z + (z|y)x)$$

for all $x, y, z \in X$.

Let δ be a derivation on X; then

$$\delta\{x, y, z\} = \{\delta x, y, z\} + \{x, \delta y, z\} + \{x, y, \delta z\} \tag{$*$}$$

for all $x, y, z \in X$. Now from the expression of the triple product, we have

$$\begin{aligned}
\delta\{x, y, z\} &= \tfrac{1}{2}((x|y)\delta z + (z|y)\delta x),\\
\{\delta x, y, z\} &= \tfrac{1}{2}((\delta x|y)z + (z|y)\delta x),\\
\{x, \delta y, z\} &= \tfrac{1}{2}((x|\delta y)z + (z|\delta y)x),\\
\{x, y, \delta z\} &= \tfrac{1}{2}((x|y)\delta z + (\delta z|y)x).
\end{aligned}$$

Thus it follows from ($*$) that

$$\tfrac{1}{2}(((\delta x|y) + (x|\delta y))z + ((z|\delta y) + (\delta z|y))x) = 0$$

for all $x, y, z \in X$. In particular, we have

$$(x|\delta y) = -(\delta x|y)$$

for all $x, y \in X$, that is, $\delta^* = -\delta$. Therefore every derivation on X, regarded as the real type 1 Cartan factor $\mathrm{BL}(X, \mathbf{R})$, is a skew-symmetric operator on X. Conversely, the following holds.

LEMMA 3. *If X is a real Hilbert space, regarded as the real Cartan factor* $\mathrm{BL}(X, \mathbf{R})$, *then the derivations on X coincide with the skew-symmetric operators on X.*

Proof. Suppose that T is a skew-symmetric operator on X. The identities

$$\begin{aligned}
T\{x, y, z\} &= \tfrac{1}{2}((x|y)Tz + (z|y)Tx),\\
\{Tx, y, z\} &= \tfrac{1}{2}((Tx|y)z + (z|y)Tx)),\\
\{x, Ty, z\} &= -\tfrac{1}{2}((Tx|y)z + (Tz|y)x),\\
\{x, y, Tz\} &= \tfrac{1}{2}((x|y)Tz + (Tz|y)x),
\end{aligned}$$

show that T is a derivation on X. $\square$

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The next proposition characterizes the inner derivations on X.

PROPOSITION 3. *The inner derivations on X, regarded as the real Cartan factor* $\mathrm{BL}(X,\mathbf{R})$*, coincide with the finite rank operators on X that are skew-symmetric.*

Proof. Let

$$\delta = \sum_{j=1}^{P} \delta(a_j, b_j)$$

be an inner derivation on X. Since

$$\delta(a_j, b_j)(x) = \tfrac{1}{2}((x|b_j)a_j - (x|a_j)b_j),$$

it follows that δ is a finite rank operator. The other implication follows from Lemma 3. □

REMARK 2. Since for every infinite-dimensional Hilbert space X there exists a skew-symmetric operator T on X satisfying $T^2 = -\mathrm{Id}$, we conclude from Lemma 3 and Proposition 3 that T is an outer derivation on X. It follows that $\mathrm{BL}(X,\mathbf{R})$ does not satisfy the inner derivation property.

Our next goal is to build derivations on $\mathrm{BL}(X,Y)$ from derivations on $X = \mathrm{BL}(X,\mathbf{R})$.

LEMMA 4. *Let δ be a derivation on a real Hilbert space X (regarded as the real Carton factor* $\mathrm{BL}(X,\mathbf{R})$*), and let Y be another real Hilbert space. Then the operator*

$$\tilde{\delta} : \mathrm{BL}(X,Y) \longrightarrow \mathrm{BL}(X,Y)$$
$$\tilde{\delta}a = a\delta$$

is a derivation on $\mathrm{BL}(X,Y)$.

Proof. Since δ is a derivation on X, $\delta^* = -\delta$ (see Lemma 3). Given $a,b,c \in \mathrm{BL}(X,Y)$, we have

$$\begin{aligned}
\{\tilde{\delta}a, b, c\} &+ \{a, \tilde{\delta}b, c\} + \{a, b, \tilde{\delta}c\} \\
&= \tfrac{1}{2}(a\delta b^*c + cb^*a\delta + a\delta^*b^*c + c\delta^*b^*a + ab^*c\delta + c\delta b^*a) \\
&= \tfrac{1}{2}(a\delta b^*c + cb^*a\delta - a\delta b^*c - c\delta b^*a + ab^*c\delta + c\delta b^*a) \\
&= \tfrac{1}{2}(cb^*a\delta + ab^*c\delta) = \{a,b,c\}\delta = \tilde{\delta}\{a,b,c\},
\end{aligned}$$

which proves that $\tilde{\delta}$ is a derivation. □

At this moment, we need the following identification. Let us fix a norm one element $y_0 \in Y$. In the sequel, we will identify each $h \in X$, with the operator

$$f_h : \quad X \longrightarrow Y$$
$$f_h(x) := (x|h)y_0 \qquad x \in X.$$

In this way, X can be regarded as the subspace of $\mathrm{BL}(X,Y)$ formed by all operators of the form f_h with $h \in X$. Using this identification, it is easy to check that if δ and $\tilde{\delta}$ are as in Lemma 4, then $\tilde{\delta}(X) \subseteq X$. In fact,

$$\begin{aligned}
\tilde{\delta}(f_h)(x) &= f_h(\delta x) = (\delta x|h)y_0 \\
&= (x|\delta^*h)y_0 = (x|-\delta h)y_0 = f_{-\delta h}(x).
\end{aligned}$$

The next lemma is the key tool of the main result of this subsection.

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LEMMA 5. *Let δ and $\tilde{\delta}$ be as in Lemma 4, and suppose that $\tilde{\delta}$ is an inner derivation. Then δ has rank less than or equal to the hilbertian dimension of Y.*

Proof. Since $\tilde{\delta}$ is an inner derivation on $\mathrm{BL}(X,Y)$, $\tilde{\delta}$ is the sum

$$\tilde{\delta} = \sum_{j=1}^{P} \delta(a_j, b_j)$$

for suitable $a_j, b_j \in \mathrm{BL}(X,Y)$. As we have seen previously for each $h \in X$, $\tilde{\delta} f_h = f_{-\delta h} \in X$. On the other hand,

$$\begin{aligned} f_{-\delta h} = \tilde{\delta}(f_h) &= \sum_{j=1}^{P} \delta(a_j, b_j)(f_h) \\ &= \sum_{j=1}^{P} \tfrac{1}{2}(a_j b_j^* f_h + f_h b_j^* a_j - b_j a_j^* f_h - f_h a_j^* b_j) \\ &= \left(\sum_{j=1}^{P} \tfrac{1}{2}(a_j b_j^* - b_j a_j^*) \right) f_h + f_h \left(\sum_{j=1}^{P} \tfrac{1}{2}(b_j^* a_j - a_j^* b_j) \right) \\ &= R f_h + f_h T, \end{aligned}$$

where

$$R = \sum_{j=1}^{P} \tfrac{1}{2}(a_j b_j^* - b_j a_j^*) : Y \longrightarrow Y$$

$$T = \sum_{j=1}^{P} \tfrac{1}{2}(b_j^* a_j - a_j^* b_j) : X \longrightarrow X$$

are two skew-symmetric operators. Moreover,

$$f_h T(x) = (Tx|h) y_0 = (x| - Th) y_0 = f_{-Th}(x)$$

for all $x \in X$, so that $f_h T = f_{-Th}$, and

$$R f_h = \tilde{\delta} f_h - f_h T = f_{-\delta h - Th} = f_{h'} \in X.$$

Therefore, for all $x, h \in X$, the equality

$$R f_h(x) = (x|h) R(y_0) = (x|h') y_0$$

holds. Thus we have $R(y_0) = \lambda y_0$ for a suitable $\lambda \in \mathbf{R}$. Since R is a skew-symmetric operator and λ is a real eigenvalue of R, $\lambda = 0$.

In this way, since $R f_h = 0$, we have

$$f_{-\delta h} = \tilde{\delta}(f_h) = f_h T = f_{-Th}$$

for all $h \in X$, and hence $T = \delta$.

Since each $b_j^* a_j$ and each $a_j^* b_j$ are operators that factorize through Y, they have rank at most the hilbertian dimension of Y. Therefore so does

$$\delta = T = \sum_{j=1}^{P} \tfrac{1}{2}(b_j^* a_j - a_j^* b_j). \qquad \square$$

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THEOREM 4. *Let X be an infinite-dimensional real Hilbert space, and Y be a real Hilbert space with hillbertian dimension less than the hilbertian dimension of X. Then* $\mathrm{BL}(X, Y)$ *does not satisfy the inner derivation property.*

Proof. We recall that, since X is infinite-dimensional, there exists a bounded linear operator T on X such that $T^2 = -\mathrm{Id}_X$ and $T^* = -T$. Hence T has rank equal to the hilbertian dimension of X. Since $T^* = -T$, Lemma 3 states that T is a derivation on X. Moreover, by Lemma 4, the operator $\tilde{T}$ given by $\tilde{T}a = aT$ ($a \in \mathrm{BL}(X, Y)$) is a derivation on $\mathrm{BL}(X, Y)$. If $\tilde{T}$ is an inner derivation, then Lemma 5 states that T has rank at most the hilbertian dimension of Y, which is impossible, since $\dim(X) > \dim(Y)$. □

Again combining Theorem 3 and Proposition 1, we obtain the following corollary.

COROLLARY 3. *The complex infinite-dimensional non-square type* 1 *Cartan factors and their real forms do not satisfy the inner derivation property.*

By virtue of the previous results, we know that there exist real and complex JB*-triples having outer derivations. Therefore it is natural to ask if any derivation can be approximated (in a convenient topology) by inner derivations. Upmeier [**23**] proved that there exists a unital JB-algebra X and a derivation D on X that cannot be approximated in norm by inner algebra derivations. Let $\hat{X}$ denote the complexification of X, and $\hat{D}$ the complex linear extension of D to $\hat{X}$. Then $\hat{X}$ is a unital JB*-algebra with unit u, and hence a JB*-triple, and $\hat{D}$ is a triple derivation, since $\hat{D}$ is an algebra derivation that commutes with the involution (see Lemma 2). We claim that $\hat{D}$ cannot be approximated in norm by inner triple derivations. Otherwise, for $\varepsilon > 0$, there would exist an inner triple derivation

$$\delta = \sum_j^P \delta(e_j, f_j)$$

such that

$$\|\hat{D} - \delta\| < \varepsilon.$$

Now, by Lemma 2,

$$\begin{aligned} \delta - L_{\delta(u)} &= \sum_j^P \delta(e_j, f_j) - L_{\delta(e_j, f_j)(u)} \\ &= \tfrac{1}{2} \sum_j^P [L_{aj}, L_{cj}] + [L_{bj}, L_{dj}], \end{aligned}$$

where $e_j = \frac{1}{2}(a_j + ib_j)$, $f_j = \frac{1}{2}(c_j + id_j)$ with a_j, b_j, c_j, d_j in X. Therefore $\delta - L_{\delta(u)}$ is an inner derivation on X such that

$$\begin{aligned} \|D - (\delta - L_{\delta(u)})\| &= \|D - L_{D(u)} - (\delta - L_{\delta(u)})\| \\ &\leqslant \|\hat{D} - \delta\| + \|L_{D(u)} - L_{\delta(u)}\| \\ &\leqslant \|\hat{D} - \delta\| + \|L_{D(u) - \delta(u)}\| \\ &\leqslant \|\hat{D} - \delta\| + \|(\hat{D} - \delta)(u))\| \leqslant 2\varepsilon, \end{aligned}$$

which is impossible, since D cannot be approximated in norm by an inner derivation.

On the other hand, D is also a derivation on the real JB*-triple X. If D could be approximated in norm by inner triple derivations on X, then, for every $\varepsilon > 0$, there exists

$$\delta = \sum_{j}^{P} \delta(e_j, f_j)$$

with e_j, $f_j \in X$ such that $\|D - \delta\| \leqslant \varepsilon$. In this case, $\delta = \sum_j^P \delta(e_j, f_j)$ is an inner derivation on $\hat{X}$ and

$$\|(\hat{D} - \delta)\| \leqslant 2\varepsilon.$$

This is impossible.

Upmeier [**23**], also proved that every algebra derivation on a JB-algebra can be approximated in the strong operator topology by inner derivations. In [**3**, Theorem 4.6], Barton and Friedman proved that the set of all inner derivations on a JB*-triple is dense in the set of all derivations with respect to the strong operator topology. This result can be extended to real JB*-triples.

THEOREM 5. *The set of all inner derivations on a real* JB**-triple is dense in the set of all derivations with respect to the strong operator topology.*

Proof. Let E be a real JB*-triple and δ a derivation on E. We consider

$$\hat{\delta} : \hat{E} \longrightarrow \hat{E}$$
$$\hat{\delta}(x + iy) := \delta(x) + i\delta(y)$$

the natural extension of δ to $\hat{E}$. Since $\hat{E}$ is a complex JB*-triple, by [**3**, Theorem 4.6], it follows that for every $x_1, \dots, x_n \in E \subset \hat{E}$ and every $\varepsilon > 0$ that there exists an inner derivation

$$\delta_1 = \sum_{j=1}^{P} \delta(a_j, b_j)$$

on $\hat{E}$ such that $\|\hat{\delta}(x_1) - \delta_1(x_1)\| \leqslant \varepsilon$ for all $l = 1, \dots, n$.

Since $a_j = a_{j,1} + ia_{j2}$ and $b_j = b_{j,1} + ib_{j,2}$, where $a_{j,k}$ and $b_{j,k}$ are in E, it is easy to check that

$$\delta_1(x_l) = \sum_{j=1}^{P} (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}) \\ + i(L(a_{j,2}, b_{j,1}) + L(b_{j,1}, a_{j,2}) - L(a_{j,1}, b_{j,2}) - L(b_{j,2}, a_{j,1})))x_l.$$

Since $a_{j,k}, b_{j,k}$ and x_l are elements in E, it follows that

$$\sum_{j=1}^{P} i(L(a_{j,2}, b_{j,1}) + L(b_{j,1}, a_{j,2}) - L(a_{j,1}, b_{j,2}) - L(b_{j,2}, a_{j,1}))x_l \in iE.$$

Thus

$$\|\delta(x_l) - \sum_{j=1}^{P} (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}))(x_l)\|$$
$$\leqslant \|\delta(x_l) - \sum_{j=1}^{P} (\delta(a_{j,1}, b_{j,1}) + \delta(a_{j,2}, b_{j,2}))(x_l))$$

$$-i\sum_{j=1}^{P}(L(a_{j,2},b_{j,1})+L(b_{j,1},a_{j,2})-L(a_{j,1},b_{j,2})-L(b_{j,2},a_{j,1}))(x_l))\|$$
$$=\|\hat{\delta}(x_l)-\sum_{j=1}^{P}(\delta(a_{j,1},b_{j,1})+\delta(a_{j,2},b_{j,2})$$
$$+i(L(a_{j,2},b_{j,1})+L(b_{j,1},a_{j,2})-L(a_{j,1},b_{j,2})-L(b_{j,2},a_{j,1})))(x_l)\|$$
$$=\|\hat{\delta}(x_l)-\delta_1(x_l)\|\leqslant\varepsilon$$

for all $l=1,\ldots,n$. □

PROBLEM 1. If we could obtain a universal bound for the degree of all derivations in a type 2 Cartan factor with $\dim H$ odd, we could try to determine all JBW*-triples of type I satisfying the inner derivation property following the techniques contained in Ho's dissertation [**10**].

References

1. S. A. AJUPOV, 'Extension of traces and type criterions for Jordan algebras of self-adjoint operators', *Math. Z.* 181 (1982) 253–268.
2. K. ALVERMANN, 'Real normed Jordan algebras with involution', *Arch. Math.* 47 (1986) 135–150.
3. T. J. BARTON and Y. FRIEDMAN, 'Bounded derivations of JB*-triples', *Quart. J. Math. Oxford Ser.* (2) 41 (1990) 255–268.
4. S. K. BERBERIAN, *Baer *-rings*, Grundlehren der Mathematischen Wissenschaften 195 (Springer, 1972).
5. C-H CHU and P. MELLON, 'Jordan structures in Banach spaces and symmetric manifolds', *Expo. Math.* 16 (1998) 157–180.
6. S. DINEEN, 'The second dual of a JB*-triple system', *Complex analysis, functional analysis and approximation theory* (North-Holland, 1986).
7. T. FACK and P. DE LA HARPE, 'Sommes de commutateurs dans les algèbres de von Neumann finies continues', *Ann. Inst. Fourier* (*Grenoble*) 30 (1980) 49–73.
8. K. R. GOODEARL, *Notes on real and complex* C*-*algebras* (Shiva, 1982).
9. H. HANCHE-OLSEN and E. STØRMER, *Jordan operator algebras*, Monographs and Studies in Mathematics 21 (Pitman, 1984).
10. T. HO, 'Derivations of Jordan Banach triples', Dissertation, University of California, Irvine, 1992.
11. J. M. ISIDRO, W. KAUP and A. RODRÍGUEZ, 'On real forms of JB*-triples', *Manuscripta Math.* 86 (1995) 311–335.
12. R. KADISON, 'Derivations of operator algebras', *Ann. of Math.* 83 (1966) 280–293.
13. W. KAUP, 'On real Cartan factors', *Manuscripta Math.* 92 (1997) 191–222.
14. O. LOOS, 'Bounded symmetric domains and Jordan pairs', Mathematics Lectures, University of California, Irvine, 1977.
15. A. RODRÍGUEZ, 'Jordan structures in analysis', *Jordan algebras. Proceedings of Oberwolfach Conference, August* 9–15, 1992 (ed. W. Kaup, K. McCrimmon and H. Petersson, de Gruyter, Berlin, 1994) 97–186.
16. B. RUSSO, 'Structure of JB*-triples', *Jordan algebras. Proceedings of Oberwolfach Conference, August* 9–15, 1992 (ed. W. Kaup, K. McCrimmon and H. Petersson, de Gruyter, Berlin, 1994) 209–280.
17. S. SAKAI, 'On a conjecture of Kaplansky', *Tôhoku Math. J.* 12 (1960) 31–33.
18. S. SAKAI, 'Derivations of W*-algebras', *Ann. of Math.* 83 (1966) 273–279.
19. E. STORMER, 'On the Jordan structure of C*-algebras', *Trans. Amer. Math. Soc.* 120 (1965) 438–447.
20. E. STORMER, 'Jordan algebras of type I', *Acta Math.* 115 (1966) 165–184.
21. E. STORMER, 'Irreducible Jordan algebras of self-adjoint operators', *Trans. Amer. Math. Soc.* 130 (1968) 153–166.
22. D. TOPPING, 'Jordan algebras of self-adjoint operators', *Mem. Amer. Math. Soc.* 53 (1965).
23. H. UPMEIER, 'Derivations of Jordan C*-algebras', *Math. Scand.* 46 (1980) 251–264.

24. H. Upmeier, 'Jordan algebras in analysis, operator theory, and quantum mechanics', CBMS Regional Conference Series in Mathematics 67 (Conference Board for Mathematical Sciences, Washington DC, 1987).

25. J. M. Wright, 'Jordan C*-algebras', *Michigan Math. J.* (1977) 291–302.

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