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Singular Isoperimetric Regions and Twisted Jacobi Fields on Locally Stable CMC Hypersurfaces with Isolated Singularities

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Singular Isoperimetric Regions and Twisted Jacobi Fields on Locally Stable CMC Hypersurfaces with Isolated Singularities

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Mathematics

by

Gongping Niu

Committee in charge:

Professor Luca Spolaor, Chair
Professor Bennett Chow, Co-Chair
Professor Ken Intriligator
Professor Lei Ni

2024

The Dissertation of Gongping Niu is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2024

DEDICATION

To my parents and grandparents.

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VITA

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ABSTRACT OF THE DISSERTATION

Singular Isoperimetric Regions and Twisted Jacobi Fields on Locally Stable CMC Hypersurfaces with Isolated Singularities

by

Gongping Niu

Doctor of Philosophy in Mathematics

University of California San Diego, 2024

Professor Luca Spolaor, Chair
Professor Bennett Chow, Co-Chair

In this dissertation, we study singular isoperimetric hypersurfaces and singular constant mean curvature hypersurfaces in closed Riemannian manifolds. It is well known that isoperimetric regions in a smooth, compact $(n+1)$-manifold have smooth boundaries, except possibly on a closed set of codimension at most 8. For $n \geq 7$, we construct an $(n+1)$-dimensional compact, smooth manifold whose unique isoperimetric region, containing half the volume of the manifold, exhibits isolated singularities. These are the first known examples of singular isoperimetric regions.

We then explore the twisted Jacobi field of singular constant mean curvature hy-

persurfaces under certain regularity assumptions. This exploration provides a direction for studying the generic regularity of isoperimetric and constant mean curvature hypersurfaces in eight dimensions.

Chapter 1

Introduction

In this chapter, we will discuss two main topics covered in this dissertation. For each, we will provide an overview of my work and outline my research objectives. Then I will talk about my future research plan for each topic.

1.1 Singular Isoperimetric Hypersurfaces

Given an $(n+1)$-dimensional smooth closed Riemannian manifold (M, g), an isoperimetric region is a subset $\Omega \subset M$ that minimizes perimeter for a given volume. We refer to the boundary of Ω as the isoperimetric hypersurface, denoted by $\partial\Omega$. More precisely, for a positive number $0 < t < |M|_g$, the volume of M, we seek to find a solution to the following constrained variational problem:

$$\mathcal{I}_g(M,t) := \inf\{\mathbf{P}_g(\Omega) :\, \Omega \in \mathcal{C}(M,t)\} \tag{1.1.1}$$

Here, $\mathbf{P}_g(\Omega)$ denotes the perimeter of the boundary of Ω with respect to the Riemannian metric g, and $\mathcal{C}_g(M,t)$ represents the class of sets with finite perimeters and enclosed volume t.

We are interested in investigating the existence and regularity of isoperimetric hypersurfaces. The existence can be established using the direct method when we formulate the variational problem with sets Ω of finite perimeter (Caccioppoli sets), (e.g., see Maggi

[25] or Chapter 2). Therefore, our interest lies in the regularity aspects of $\partial\Omega$.

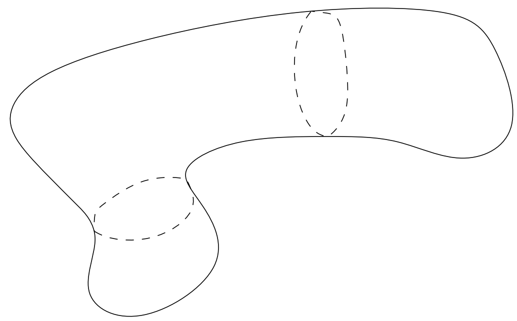

Figure 1.1. Isoperimetric regions in different volumes

A point on the boundary is regular if it is locally a smooth hypersurface. Ideally, we hope that the boundaries of isoperimetric regions are smooth, but the best result that has been achieved is that they are regular outside of a closed set of codimension 8, which becomes discrete when $n+1=8$ (see Gonzalez, Massari, and Tamanini [19]).

A natural question arises: Is the regularity result from [19] optimal? Note that for any point on the isoperimetric hypersurface, the tangent cone (a limit current which is a blow up at the point p) is an area minimizing integral current. So we may want to see that whether there is an equivalent relation of the regularity theorem of isoperimetric hypersurfaces and Plateau's problem (i.e. area minimizing surfaces). In the case of area-minimizing integral currents, this question has been answered affirmatively through examples such as the Simons cone in $\mathbb{R}^8$: $\mathbf{C} := \{(x,y) : |x| = |y| \text{ for } x, y \in \mathbb{R}^4\}$. However, isoperimetric regions in $\mathbb{R}^n$ are Euclidean balls and are therefore smooth for every $n \in \mathbb{N}$. To construct a singular isoperimetric hypersurface in dimension 8, we need to create a manifold that is not a space form.

In Chapter 3, we solve the optimal regularity problem by constructing an 8-dimensional compact smooth manifold whose unique isoperimetric region, with half the volume of the manifold, exhibits two isolated singularities on the boundary. These are the first examples of isoperimetric regions with singularities.

Theorem 1.1.1. *There exists a smooth closed Riemannian 8-manifold (M, g) whose unique isoperimetric region with volume $|M|_g/2$ has two isolated singularities. The unique tangent cone at each singular point is a Simons cone.*

The construction of the Riemannian manifold in Theorem 3.1.16 Similar ideas can be used to generalize our result to $n > 7$. By suitably modifying the proof in Theorem 3.1.16, we can construct singular isoperimetric hypersurfaces in higher dimensions.

Theorem 1.1.2. *For any integer $n \geq 7$, there exists a closed smooth $(n+1)$-dimensional Riemannian manifold (M, g), such that its unique isoperimetric region Ω with half the volume is singular.*

Therefore, Theorem 1.1.1 and Theorem 1.1.2 show that the regularity theorem of isoperimetric hypersurfaces in [19, 28] is as sharp. This is analogous to regularity results for area-minimizing current problems (e.g. [38]). In fact, we prove further results regarding the singular part. For any integers $n \geq 7$ and $p \in [3, \frac{n-1}{2}]$, we demonstrate the existence of a closed smooth $(n+1)$-dimensional Riemannian manifold (M, g) whose unique isoperimetric region Ω with half the volume has a singular part that is a closed submanifold diffeomorphic to $\mathbb{S}^{n-2p-1}$ (denoting $\mathbb{S}^0$ as a single point rather than two points).

1.2 Twisted Jacobi Field on Locally Stable CMC Hypersurfaces

From Theorem 3.1.16 and Theorem 3.1.18 above, we know that singularities are inevitable in higher dimensions. The emergence of singularities is the major stumbling block in the study of isoperimetric hypersurfaces. So, we may ask whether arbitrary small perturbations of the Riemannian metric would prevent isoperimetric hypersurfaces from having singularities. This is called the generic regularity property. In the case of homological area-minimizing currents, generic regularity has been proved in dimensions 8, 9, and 10 (see [39] for dimension 8 and [12] for dimensions 9 and 10). For isoperimetric regions, we notice that the local surgery method of constructing a generic Riemannian metric (similar to the ideas in [39]) is quite challenging to apply to isoperimetric regions since the volume changes upon perturbing the Riemannian metrics. Therefore, we investigated another approach by studying the (twisted) Jacobi operator on the isoperimetric hypersurfaces. This operator has been studied in smooth immersed CMC hypersurfaces ([4]), and it also has applications for isoperimetric problems. In [11, Corollary 5.3], it is shown that in lower dimensions (where isoperimetric hypersurfaces are all smooth), under generic metrics, all isoperimetric regions (for a fixed volume) are (weakly) strictly stable. Roughly speaking, they define a projection map from Banach manifolds (similar to the argument in [44]) and show that the kernel of the derivative of the projection map has the same dimension as the twisted Jacobi fields (see definition below for smooth case). Using this, they obtain the generic result from the Sard-Smale theorem. This observation suggests that for higher dimensions (where isoperimetric hypersurfaces may have singularities), it is necessary to define a twisted Jacobi operator, which ideally should have finite dimensional kernel under some regularity assumptions.

In Chapter 2, we see that isoperimetric hypersurface is an embedded constant mean curvature (CMC) hypersurface. Therefore, we will generalize our object to study the twisted Jacobi field on the CMC hypersurfaces rather than the isoperimetric hypersurfaces. We will study the CMC hypersurfaces with isolated singularities (see definition in Chapter 4), which is the easiest scenario of singular structure.

To explain our result, for simplicity we first consider the case that Σ^n is a compact manifold and $i : \Sigma^n \to M^{n+1}$ is an immersion of an oriented constant mean curvature (CMC) hypersurface (possible with boundary). In contrast to the minimal surface case, the Euler-Lagrange equation we study has the volume-preserving requirement. Then the area varation $\frac{d}{dt}\big|_{t=0} \mathbf{M}(\Sigma_t) = 0$ for all volume-preserving variations $\{\Sigma_t\}$ with $\Sigma_0 = \Sigma$. In addition, if $\Sigma = \partial\Omega$ is a boundary of an isoperimetric region Ω, or it is a minimizer among all volume-preserving variations, then the second derivative for for all volume-preserving variations $\{\Sigma_t\}$ is $\frac{d^2}{dt}\Big|_{t=0} \mathbf{M}(\Sigma_t) \geq 0$. This implies that for any $\psi \in C^\infty(\Sigma)$ with $\int_\Sigma \psi = 0$, we have that

$$Q_\Sigma(\psi, \psi) := \int_\Sigma -\psi\Delta\psi - (|A_\Sigma^g|^2 + Ric_g(\nu,\nu))\psi^2 \geq 0,$$

where ν denotes the unit normal vector field on Σ. For our analysis, define

$$\mathscr{D}_T(\Sigma) := \left\{\psi \in C_c^\infty(\Sigma) : \int_\Sigma \psi = 0\right\}, \quad L_T^2(\Sigma) := \left\{\psi \in L^2(\Sigma) : \int_\Sigma \psi = 0\right\}.$$

We refer to $\psi \in \mathscr{D}_T(\Sigma)$ as a twisted Jacobi field if $\psi \in \operatorname{Ker} Q_\Sigma$, i.e.,

$$Q_\Sigma(\psi, \phi) = 0 \qquad \text{for } \forall \phi \in \mathscr{D}_T(\Sigma).$$

By [3], we see that $\psi \in \operatorname{Ker} Q_\Sigma$ if and only if ψ is a solution of the following equation:

$$L^g\psi - \frac{1}{|\Sigma|_g}\int_\Sigma L^g\psi = 0 \tag{1.2.1}$$

where $L^g := \Delta\psi + \left(|A_\Sigma^g|_g^2 + Ric_g(\nu)\right)\psi$. Therefore, we call the twisted Jacobi operator by $\widetilde{L}^g := L^g - \Psi^g$, where

$$\Psi^g(\psi) := \frac{1}{|\Sigma|_g}\int_\Sigma L^g\psi.$$

Note that for $\psi, \phi \in \mathscr{D}_T(\Sigma)$, Q_Σ is a symmetric quadratic form:

$$\langle\psi, \widetilde{L}\phi\rangle_{L^2} = \langle\phi, \widetilde{L}\psi\rangle_{L^2} = Q_\Sigma(\psi, \phi).$$

Therefore, it is clear that we want to study the linear operator $\Delta + b$, where b is a continuous function on Σ. By not considering Σ extrinsically as an immersion of CMC hypersurface, [4] provides a stronger result for this operator.

Proposition 1.2.1. *[3, Proposition 2.2] Suppose (Σ^n, g) is compact Riemannian manifold* *with boundary. Denote* $H^1_{0,T}(\Sigma) := H^1_0(\Sigma) \cap L^2_T(\Sigma)$. *Suppose* $b \in C^0(\Sigma)$, *the quadratic form*

$$q_T := (H^1_{0,T}(\Sigma), Q|H^1_{0,T}(\Sigma))$$

is closed and is associated with the self-adjoint operator:

$$(H^2(\Sigma) \cap H^1_{0,T}(\Sigma), \Delta_g + b - \Psi_g)$$

on $L^2_T(\Sigma)$. *The corresponding eigenvalue problem on* $L^2_T(\Sigma)$ *is given by:*

$$\begin{cases} (\Delta_g + b)u - \Psi_g(u) = -\lambda u, \\ u|_{\partial\Sigma} = 0, \\ \Phi_g(u) = 0. \end{cases} \tag{1.2.2}$$

In addition, the spectrum $\delta_T(\Sigma; g, b)$ *of the Dirichlet problem consists of eigenvalues with finite multiplicities:*

$$\lambda_1^T(\Sigma) < \lambda_2^T(\Sigma) \leq \lambda_3^T(\Sigma) \leq \ldots$$

For immersed CMC hypersurface, the function b is a bounded function and thus we would have a coercivity result, i.e., for any $\phi \in C^1_c(\Sigma)$, there exists a constant $C > 0$ such that

$$\int_\Sigma |\nabla\psi|^2 \leq Q_\Sigma(\psi, \psi) + \|\psi\|_{L^2}.$$

Then by the compactly embeddedness of H^1 in L^2, we will have the Proposition above by argument of standard elliptic PDEs.

Unfortunately, similar result is not clear for CMC hypersurface with isolated singularities. Note that as $x \in \Sigma$ approaches to the singularities, the second fundamental form $|A_\Sigma|^2$ blows up to infinity. So the function b is unbounded, so we do not know whether we have the coercivity of the quadratic form. Similar problems arise in minimal surfaces with singularities. In [42], we see that we could define a spectral theorem for minimal hypersurfaces with isolated singularities for Jacobi field. Similarly, in Chapter 4, we particularly study the twisted Jacobi field for CMC hypersurfaces which have isolated singularities (see definition in Chapter 4). We establish the Hilbert space $\mathscr{B}_{0,T}$ to substitute the Sobolev space $H^1_{0,T}$ and construct a spectral theorem for the Dirichlet problem concerning singular locally stable CMC hypersurfaces with isolated singularities. Consequently, we generalize the following spectral theorem, Fredholm alternative, and the generic property of nondegeneracy of the twisted Jacobi operator for the singular locally stable CMC hypersurface Σ. We also prove the equivalence of finite index and local stability (see (4) in the Theorem below).

Theorem 1.2.2. *Suppose $\Sigma \subset (M^{n+1}, g)$ is a LSCMC hypersurface with isolated singularities. Let $U \subset M$ be an open subset such that ∂U is smooth and intersects Σ transversely. Denote $\mathcal{U} := U \cap \Sigma$. Then*

(1) For any $f \in L^\infty(\mathcal{U})$, there exists a strictly increasing sequence $\sigma_p(\Sigma) = \{\lambda_j\}_{j=1} \nearrow \infty$ and finite-dimensional pairwise L^2-orthogonal linear subspaces, $\{E_j\}$, of $\mathscr{B}_{0,T}(\Sigma) \cap C^\infty(\mathcal{U})$, such that

$$-\widetilde{L^f}\psi = \lambda_j \psi$$

for all $\psi \in E_j$. Furthermore, $\{E_j\}$ forms the orthonormal basis of the following spaces

$$L^2_T(\mathcal{U}) = span_{L^2}\{E_j\}_j, \qquad \mathscr{B}_{0,T}(\mathcal{U}) = span_{\mathscr{B}}\{E_j\}_j.$$

(2) For any $f \in L^\infty(\mathcal{U})$, if $\widetilde{L}$ is nondegenerate, i.e. $0 \notin \sigma_p(\Sigma)$, Then for each $g \in L^2_T(\Sigma)$

there exists a unique $\psi \in \mathscr{B}_{0,T}(\mathcal{U})$ *such that*

$$-\widetilde{L^f}\psi = g$$

on $\mathcal{U}$.

(3) The subset

$$G = \{f \in C_c^\infty(\mathcal{U}) : (-\widetilde{L^f}) \text{ is nondegenerate}\}$$

is open and dense in $C_c^\infty(\mathcal{U})$.

(4) Σ *has finite index in* U.

Not only are twisted Jacobi fields of interest in their own right for CMC hypersurfaces, but they are also crucial for isoperimetric problems. In dimension 8, the isoperimetric regions have locally stable CMC hypersurfaces with isolated singularities. Thus, the operator in $\mathscr{B}_{0,T}(\mathcal{U})$ is precisely a generalization in dimension 8 of the twisted Jacobi field studied in [11] for lower-dimensional cases.

Open problems

We list several open questions about isoperimetric regions and CMC hypersurfaces which related to our research. The first part is about more general types of singular examples of isoperimetric regions. The second is about the full regularity Theorem of isoperimetric regions under generic metrics and isoperimetric inequality in Riemannian manifolds.

1. About the construction of singular examples with prescribed conditions

There are two natural questions we may proceed with:

1. **Prescribed regular tangent cones:** Our construction in Theorem 3.3.1 and Theorem 3.1.18 require the tangent cones to be Simons' cones. Then we can

guarantee that Γ_+ and Γ_- are diffeomorphic to each other, and we can explicitly compute the topology of the manifolds M. If we generally choose the tangent cones as regular, strictly stable, strictly minimizing hypercones, Γ_+ may not be diffeomorphic to Γ_-, and we cannot study the homology groups of M. So we may ask whether there exists examples of singular isoperimetric regions with prescribed regular minimizing tangent cones.

2. **Prescribed mean curvature:** Note that $\partial\Omega$ is particularly composed of area-minimizing hypersurfaces. So this singular example comes from the existence of singular area minimizers. So now we come up with a new question: whether there exists a singular isoperimetric region with mean curvature constantly non-zero? In Morgan and Johnson [29, Theorem 2.2], we see that regardless of dimensions, if an isoperimetric region encloses a sufficiently small region, it will not have singularities (it will be a nearly round sphere.) However, for an isoperimetric with mean curvature small (close to zero but not zero), a singular example is still unknown.

2. About isoperimetric regions under generic metrics

For $4 \leq k \leq \infty, \alpha \in (0,1)$, consider the collection of normalized Riemannian metrics on M, denoted by $\mathcal{G}^{k,\alpha}(M) := \{g \in \mathrm{Met}^{k,\alpha}(M) : \mathrm{Vol}_g(M) = 1\}$. Our goal is to study the following conjecture:

Conjecture 1.2.3. *Let M be a closed Riemannian manifold of dimension 8, and fix $t \in (0,1)$. Then for a $C^{k,\alpha}$ generic metric $g \in \mathcal{G}^{k,\alpha}(M)$, for every isoperimetric region $\Omega \in \mathcal{C}_g(M,t)$, the boundary is regular and non-degenerate (i.e., they are smooth and no non-trivial twisted Jacobi fields).*

It is worth noting that if we only consider smooth minimal surfaces (or constant mean curvature surfaces), Brian White in [44, 2.2] shows that the "bumpy" metrics theorem holds, i.e., C^k-generically, all smooth minimal surfaces are non-degenerate. In

[24], Li and Wang extend this result to dimension 8, also including locally stable singular minimal surfaces with optimal regularity. Similarly, Conjecture 1.2.3 can be regarded as a full "bumpy" metrics theorem and a generic regularity result for isoperimetric regions in dimension 8.

In the case of lower dimensions (i.e., $\Sigma \subset (M^{n+1}, g)$, an isoperimetric hypersurface with $1 \leq n \leq 6$), there exists a generic metric such that all isoperimetric regions with fixed volume have weakly strictly stable hypersurfaces (see [11]). In Chapter 4, we study the twisted Jacobi field on CMC hypersurfaces with isolated singularities, which provides a method to define non-degeneracy for isoperimetric hypersurfaces in dimension $n = 7$ because the boundary exhibits only isolated singularities (as shown by Theorem 3.1.16). Therefore, we may question whether we can generically "smooth" the singular isoperimetric regions and ensure that the boundaries are weakly strictly stable.

One direct application for the Conjecture 1.2.3 is to generalize Bonnesen-type inequalities (also called quantitative isoperimetric inequalities) in dimension 8. In [18, Theorem 1], Fusco, Maggi, and Pratelli show that for $n \geq 2$, there is a constant $C = C(n)$ such that if $\Omega \in \mathcal{C}_{g_{eul}}(\mathbb{R}^n, 1)$, i.e., Ω is a Cacciopolli set in $\mathbb{R}^n$ under Euclidean metric with volume 1, then

$$\left(\inf_{B = B_1(x) \subset \mathbb{R}^n} |\Omega \Delta B| \right)^2 \leq C(n)\big(\mathbf{P}(\Omega) - \mathbf{P}(B)\big).$$

In [11], Chodosh, Engelstein, and Spolaor prove a sharp quantitative isoperimetric inequality for Riemannian manifolds. However, due to the appearance of singular isoperimetric hypersurfaces, the following result holds in restricted dimensions.

Theorem 1.2.4. *For $1 \leq n \leq 6$, assume that M^{n+1} is a closed Riemannian manifold and that there exists an open and dense subset $\mathcal{G}_0 \subset \mathcal{G}^3(M)$ with the following property. If $g \in \mathcal{G}_0$, then there exists an open dense subset $\mathcal{V} \subset (0, |M|_g)$ so that for $V_0 \in \mathcal{V}$, there* *exists* $C = C(g, V_0) > 0$ *so that*

$$\mathbf{P}_g(E) - \mathcal{I}_g(M, V_0) \geq C\alpha_g(E)^2 \tag{1.2.3}$$

for any $E \in \mathcal{C}_g(M, V_0)$*. Here,* $\alpha_g(E)$ *denotes the "manifold Fraenkel asymmetry":*

$$\alpha_g(E) := \inf\left\{|E\Delta\Omega|_g \,:\, \Omega \in \mathcal{C}_g(M, V_0) \text{ is an isoperimetric region}\right\}.$$

Conjecture 1.2.3 says that generically the isoperimetric hypersurfaces are smooth for a fixed volume. Consequently, the arguments presented in [11] will continue to hold in generic cases, allowing us to extend Theorem 1.2.4 to dimension 8.

Conjecture 1.2.5. *For* M^8 *a closed Riemannian manifold and* $k \leq 4$*, there exists an open and dense subset* $\mathcal{G}_0 \subset \mathcal{G}^k(M)$ *so that* (1.2.3) *holds.*

Chapter 2

Preliminaries

In this chapter, we will cover basic notations and theorems essential to geometric measure theory. And in specific, we will discuss Isoperimetric problems, including their relationships with Constant Mean Curvature (CMC) and minimal surfaces problems.

2.1 Notations

Let $n \geq 3$ and (M, g) be an $n+1$ dimensional closed oriented Riemannian manifold. Extrinsically, we will consider that (M, g) is isometrically embedded in some $\mathbb{R}^L$. And we will use the following standard intrinsic geometric notations:

$B_r^g(p)$ or $B_r(p)$	the open geodesic ball in M of radius r centered at p
$\mathbf{B}_r(p)$	for Σ a submanifold of M, $\mathbf{B}_r(p) := B_r^g(p) \cap \Sigma$
$A_{s,r}(p)$ or $A(p; s, r)$	$:= B_r(p) \setminus \overline{B_s(p)}$ the open geodesic annuli in M
$\mathcal{H}^k$	k-dimensional Hausdorff measure
$\mathcal{C}(M)$	The collection of sets with finite perimeter
$\mathbf{P}_g(E; U)$	The perimeter of E in U over the metric g
$\nu(f)$	For the vector or vector field ν and $f \in C^1(M)$, $\nu(f) := \langle \nabla f, \nu \rangle$
$\mathcal{U} \subset \Sigma$	for Σ a submanifold of M, and U an open set of M, $\mathcal{U} := U \cap \Sigma$

In an Euclidean space $\mathbb{R}^{n+1}$, we use the following notations:

$$\begin{aligned} \mathbb{B}^{n+1}_r(p) \text{ or } \mathbb{B}_r(p) &\quad \text{the open ball of radius } r \text{ centered at } p \\ \mathbb{B}^{n+1}_r \text{ or } \mathbb{B}_r &:= \mathbb{B}^{n+1}_r(0^{n+1}) \\ \mathbb{S}^n_r(p) &\quad \text{the sphere of radius } r \text{ centered at } p \\ \mathbb{A}^{n+1}(p;s,r) \text{ or } \mathbb{A}_{s,r}(p) &\quad \text{the open annuli } \mathbb{B}_r(p) \setminus \overline{\mathbb{B}_s(p)} \text{ centered at } p \end{aligned}$$

Denote $\mathbf{C} := \mathbf{C}^n \subset \mathbb{R}^{n+1}$ a minimal hypercone with isolated singularity at 0.

$$\begin{aligned} \mathbf{B}_r &:= \mathbf{C} \cap \mathbb{B}_r, \\ \mathbf{A}(s,r) &:= \mathbf{B}_r \setminus \overline{\mathbf{B}_s}. \end{aligned}$$

2.2 Sets of Finite Perimeters

Let $n \geq 1$ and (M,g) be an $(n+1)$-dimensional closed oriented Riemannian manifold. In order to have an explicit definition of our variational problem, we first review the sets with well-defined measure theoretical perimeters.

Definition 2.2.1 (Caccioppoli sets/sets of finite perimeter; see e.g. [17])**.** Suppose E is a Lebesgue measurable subset of M, we define the perimeter of E by

$$\mathbf{P}_g(E) = \sup\left\{ \int_E \operatorname{div}_g X \; d\mathcal{H}^{n+1}(x) \;:\; X \in \Gamma^1(M),\; \|X\|_g \leq 1 \right\}.$$

We define the collection of sets of finite perimeters in (M, g) by

$$\mathcal{C}_g(M) := \{\Omega \subset M \,:\, \mathbf{P}_g(\Omega) < \infty\}.$$

In addition, for the subset of $\mathcal{C}(M)$ with a fixed volume, we denote

$$\mathcal{C}_g(M, t) := \{\Omega \subset M \,:\, \mathbf{P}_g(\Omega) < \infty\,,\ |\Omega|_g = t\}.$$

We usually omit the subscript g above if the defining metric is understood.

By the Riesz Representation Theorem (see e.g. [37]), there is a TM-valued Radon measure μ_Ω such that for any $X \in \Gamma^1(M)$, we have

$$\int_\Omega \operatorname{div}_g X = \int_M X \cdot_g d\mu_\Omega.$$

The total variation is denoted by $\|\mu_\Omega\|$. For an open set U, we denote

$$\mathbf{P}(\Omega; U) = \|\mu_\Omega\|(U),$$

the relative perimeter in U. Note that if E and F are of locally finite perimeter and $|\,(E\Delta F) \cap U|_g = 0$, then $\mathbf{P}_g(E; U) = \mathbf{P}_g(F; U)$.

We can equip the set $\mathcal{C}(M)$ with the weak-$*$ topology such that

- sequences of sets with uniformly bounded perimeters have convergent subsequences;
- the perimeter is lower semi-continuous.

For general Radon measures on $\mathbb{R}^n$, we have the following "slicing" property.

Property 2.2.2. *[Countable leafs have positive measures] If $\{E_t\}_{t\in I}$ is a disjoint family of Borel sets in $\mathbb{R}^n$, and μ is a Radon measure on $\mathbb{R}^n$, then $\mu(E_t) > 0$ for at most countably many $t \in I$.*

Proof. For $k \in \mathbb{N}$, denote

$$I_k := \{t \in I : \mu(E_t \cap B_k) > k^{-1} \text{ for some ball } B_k\}.$$

Then $\bigcup_{k\in\mathbb{N}} I_k = \{t \in I : \mu(E_t) > 0\}$. Observe that fixing I_k, denote J an arbitrary finite subindex in I_k, then we have

$$\mu(B_k) \geq \mu\left(\bigcup_{t\in I} E_t \cap B_k\right) \geq \mu\left(\bigcup_{t\in J} E_t \cap B_k\right) = \sum_{t\in J} \mu(E_t \cap B_k) \geq \frac{\#(J)}{k}.$$

Therefore, I_k is finite (otherwise, $\mu(B_k) = \infty$, a contradiction), and $\#(I_k) \leq k\mu(B_k)$ and so $\{t \in I : \mu(E_t) > 0\}$ is countable. $\square$

Definition 2.2.3. Suppose $E \subset \mathbb{R}^n$ a set of locally finite perimeter. We denote $\partial^* E$ the **reduced boundary** the set of points $x \in \operatorname{spt} \mu_E$ such that

$$\lim_{r\to 0^+} \frac{\mu_E(B_r(x))}{|\mu_E|(B_r(x))}$$

exists and the limit belongs to $\mathbb{S}^{n-1}$. We define a the limits by ν_E. So $\nu_E : \partial^* E \to \mathbb{S}^{n-1}$ a Borel function.

Intuitively, ν_E is the outer unit normal of E in the measure-theoretic sense. By the Lebesgue-Besicovitch differentiation theorem, we have

$$\mu_E = \nu_E |\mu_E| \llcorner_{\partial^* E}.$$

To study the locally regularity on the boundary, we define the density of sets of locally finite perimeter.

Definition 2.2.4 (Density of volume)**.** Given $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define

$$\theta_n(E)(x) := \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{\omega_n r^n}$$

if the limit exists. Given $t \in [0, 1]$, the **set of points of density** t of E is defined as

$$E^{(t)} = \{x \in \mathbb{R}^n : \theta_n(E)(x) = t\},$$

and it turns out to be a Borel set.

Remark 2.2.5. The above definition should be distinguished from that of the density of hypersurfaces or the boundary of a set of finite perimeter.

The following Theorem gives us the regularity of sets of locally finite perimeter.

Theorem 2.2.6. *(De Giorgi's structure theorem) If E is a set of locally finite perimeter in $\mathbb{R}^n$, then there exists countably many C^1-hypersurfaces Σ_h in $\mathbb{R}^n$ and compact sets $K_h \subset \Sigma_h$ such that*

$$\partial^* E = \bigcap_{h \in \mathbb{N}} K_h \cup F, \qquad \mathcal{H}^{n-1}(F) = 0,$$

where F is some a Borel set. In addition, the Gauss–Green measure μ_E satisfies

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner_{\partial^* E}, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner_{\partial^* E}.$$

and the generalized Gauss-Green formula:

$$\int_E \nabla \phi = \int_{\partial^* E} \phi \nu_E \, d\mathcal{H}^{n-1}, \qquad \forall \phi \in C_c^1(\mathbb{R}^n).$$

For every $x \in K_h$, $\nu_E(x)^\perp = T_x \Sigma_h$, the tangent space to Σ_h at x.

For simplicity, we will denote $\partial \Omega = \operatorname{spt} \mu_\Omega$. By de Giorgi's structure theorem, $\partial^* \Omega$ is a n-rectifiable set and $\mathcal{H}^{n-1}(\partial E \setminus \partial^* E) = 0$.

Note that if E, F are sets of locally finite perimeter, then so are $E \cap F$, $E \setminus F$, and $E \cup F$. We will frequently use the following theorem, which computes the perimeter of sets under set operations.

Theorem 2.2.7. *[Set operations] Suppose E and F are sets of locally finite perimeter, and denote*

$$\{\nu_E = \nu_F\} := \{x \in \partial^* E \cap \partial^* F : \nu_E(x) = \nu_F(x)\},$$

$$\{\nu_E = -\nu_F\} := \{x \in \partial^* E \cap \partial^* F : \nu_E(x) = -\nu_F(x)\},$$

and we write $E \approx F$ if $\mathcal{H}^{n-1}(E_1 \Delta E_2) = 0$.

Then $E \cap F$, $E \backslash F$ and $E \cup F$ are sets of locally finite perimeter, with

$$\begin{aligned} \mu_{E\cap F} &= \mu_{E \llcorner F^{(1)}} + \mu_{F \llcorner E^{(1)}} + \nu_E \mathcal{H}^{n-1} \llcorner_{\{\nu_E = \nu_F\}}, \\ \mu_{E\backslash F} &= \mu_{E \llcorner F^{(0)}} - \mu_{F \llcorner E^{(1)}} + \nu_E \mathcal{H}^{n-1} \llcorner_{\{\nu_E = -\nu_F\}}, \\ \mu_{E\cup F} &= \mu_{E \llcorner F^{(0)}} + \mu_{F \llcorner E^{(0)}} + \nu_E \mathcal{H}^{n-1} \llcorner_{\{\nu_E = \nu_F\}}, \end{aligned}$$

and

$$\begin{aligned} \partial^*(E \cap F) &\approx (F^{(1)} \cap \partial^* E) \cup (E^{(1)} \cap \partial^* F) \cup \{\nu_E = \nu_F\}, \\ \partial^*(E \backslash F) &\approx (E^{(0)} \cap \partial^* E) \cup (E^{(1)} \cap \partial^* F) \cup \{\nu_E = -\nu_F\}, \\ \partial^*(E \cup F) &\approx (E^{(0)} \cap \partial^* E) \cup (E^{(0)} \cap \partial^* F) \cup \{\nu_E = \nu_F\}. \end{aligned}$$

Moreover, for every Borel set $G \subseteq \mathbb{R}^n$,

$$\begin{aligned} \mathbf{P}(E \cap F; G) &= \mathbf{P}(E; F^{(1)} \cap G) + \mathbf{P}(F; E^{(1)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\} \cap G), \\ \mathbf{P}(E \backslash F; G) &= \mathbf{P}(E; F^{(0)} \cap G) + \mathbf{P}(F; E^{(1)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = -\nu_F\} \cap G), \\ \mathbf{P}(E \cup F; G) &= \mathbf{P}(E; F^{(0)} \cap G) + \mathbf{P}(F; E^{(0)} \cap G) + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\} \cap G). \end{aligned}$$

For more discussions about sets of finite perimeters and the proofs of Theorem 2.2.6 and Theorem 2.2.7, see [25].

2.3 (Λ, r_0)-Perimeter Minimizers

In this section, we work entirely within the ambient space $\mathbb{R}^n$ with $n > 1$ and the Euclidean metric. The applications to Riemannian manifolds can be extended and generalized using standard methods. We present some basic properties and some results on regularity and singularity ([2, 20, 19, 15] and also [28], which involve less regularity assumptions).

Definition 2.3.1. Let $\Sigma \subset M$ be a smooth open hypersurface. We denote

$$\begin{aligned} Reg(\Sigma) &:= \{x \in \overline{\Sigma} \cap U : \overline{\Sigma} \text{ is smooth, embedded hypersurface near } x\}; \\ Sing(\Sigma) &:= \overline{\Sigma} \setminus Reg(\Sigma). \end{aligned}$$

Therefore Σ is open but could be not complete, we can modify it on a set of measure zero, so that we can assume without loss of generality $\Sigma = \mathrm{Reg}(\Sigma)$. Consequently, $\mathrm{Sing}(\Sigma) = \overline{\Sigma} \setminus \Sigma$. In particular, Σ is said to be *regular* if $\mathrm{Sing}(\Sigma) = \emptyset$. We say Σ is **regular**, if $Sing(\Sigma) = \emptyset$. In this dissertation, we will always assume Σ has **optimal regularity**: $\mathcal{H}^{n-7}(Sing(\Sigma)) = 0$ and $\mathcal{H}^n \llcorner_\Sigma$ is locally finite (see more in Definition 4.1.4).

Plateau's problem: Given an open set $A \subset M^{n+1}$, find a surface with least area among all the surfaces with a prescribed boundary data $E_0 \in \mathcal{C}(M)$:

$$\gamma(A, E_0) := \inf \big\{ \mathbf{P}(E) : \ E \setminus A = E_0 \setminus A \big\}.$$

In the 1960s, by the fundamental works of Fleming, De Giorgi, Almgren, Federer, Bombieri–De Giorgi–Giusti, etal., we have the following dimension estimates for singularities:

- if $1 \le n \le 6$, then $Sing(E; A)$ is empty;
- if $n = 7$, then $Sing(E; A)$ has isolated points in A;
- if $n \ge 8$, then $\mathcal{H}^s(Sing(E; A)) = 0$ for every $s > n - 7$, i.e., $\dim(Sing(E; A) \le n - 7$.

Definition 2.3.2. For $0 \le \Lambda < \infty, r_0 > 0$, we call $\Omega \in (M, g)$ is a (Λ, r)-**perimeter minimizer** in an open set U if

$$\mathbf{P}(\Omega; B_r(x)) \le \mathbf{P}(F; B_r(x)) + \Lambda|\Omega \Delta F|,$$

whenever $\Omega \Delta F \Subset B_r(x) \cap A$ and $r < r_0$.

We can consider the $(\Lambda, r_0)-$perimeter minimality is a generalization of perimeter minimality (i.e. choosing $\Lambda = 0$). We allow the "error part" $\Lambda|\Omega \Delta F|$ having perturbation in the order of r^n (because $|\Omega \Delta F| \le \omega(n) r^n$.) In order to study the blow up of a $(\Lambda, r_0)-$perimeter minimizers, we observe that if E a set of locally finite perimeter is a $(\Lambda, r_0)-$perimeter minimizer in an open set A, then for any $x \in \mathbb{R}^n$ and any $r > 0$, $\eta_{x,r\#}E$ is a (Λ', r_0')-perimeter minimizer in $\eta_{x,r\#}A$, with

$$\Lambda' = \Lambda r, \qquad r_0' = \frac{r_0}{r}.$$

As $r \searrow 0^+$, we have $\Lambda' \searrow 0^+$, $r_0' \nearrow +\infty$, and $\eta_{x,r\#}E$ approaches a perimeter minimizer. Note that $\Lambda r_0 = \Lambda' r_0'$, i.e., the product is invariant under blowing up.

Theorem 2.3.3 (Regularity Theorem of (Λ, r_0)-minimizer)**.** *Suppose A an open set in $\mathbb{R}^n$ with $n > 1$, and E is a (Λ, r_0)-perimeter minimizer in A with $\Lambda r_0 \le 1$, then $A \cap \partial^* E$ is a $C^{1,\gamma}$-hypersurface for every $\gamma \in (0, 1/2)$. In addition, $A \cap \partial^* E$ is relatively open in $A \cap \partial E$, and it is $\mathcal{H}^{n-1}$-equivalent to $A \cap \partial E$.*

Theorem 2.3.4 (Singularity Theorem of (Λ, r_0)-minimizer)**.** *Suppose E is a $(\Lambda, r_0)-$ perimeter minimizer in an open set $A \subset \mathbb{R}^n$ with $n > 1$ and $\Lambda r_0 \geq 1$. Denote*

$$Sing(E; A) := A \cap (\partial E \setminus \partial^* E).$$

Then we have the following dimension estimates for singularities:

- if $1 \leq n \leq 6$, then $Sing(E; A)$ is empty;
- if $n = 7$, then $Sing(E; A)$ has isolated points in A;
- if $n \geq 8$, then $\mathcal{H}^s(Sing(E; A)) = 0$ for every $s > n - 7$.

Chapter 3

Existence of Singular Isoperimetric Regions

3.1 Regularity Theorems for Isoperimetric Problems

In this section, we will introduce the following minimization problem, which is the main topic of this dissertation:

$$\mathcal{I}_g(M,t) := \inf\{\mathbf{P}_g(\Omega) \,:\, \Omega \in \mathcal{C}_g(M,t)\},$$

where (M^8, g) is a closed Riemannian manifold and $t \in (0, |\Omega|_g)$. We will call a Cacciopoppli set Ω a **t-isoperimetric region** in (M,g) if

$$\mathbf{P}_g(\Omega) = \mathcal{I}_g(M,t) \quad \text{and} \quad \Omega \in \mathcal{C}_g(M,t).$$

Intuitively, the boundary of Ω is a n-dimensional hypersurface.

We can consider the isoperimetric regions locally in an open set U.

Definition 3.1.1 (volume-constrained perimeter minimizer)**.** We say that $E \in \mathcal{C}(M)$ of

M is a **volume-constrained perimeter minimizer** in an open set U if

$$\mathbf{P}(E;U) \leq \mathbf{P}(F;U)$$

whenever $|E \cap U| = |F \cap U|$ such that $E\Delta F \Subset U$.

Note that the difference between the above definition and the relative perimeter minimizer is that the competitors are required to be volume-preserving. The following property implies that if we drop the volume-preserving condition, the volume-constrained perimeter minimizers are still minimizers, but with an "error term".

Property 3.1.2. *If $E \in \mathcal{C}(M)$ is a volume-constrained perimeter minimizer in an open set U, then there exist constants $0 < \Lambda < \infty$ and $r_0 > 0$ depending on E, U such that*

$$\mathbf{P}(E; B_r(x)) \leq \mathbf{P}(F; B_r(x)) + \Lambda \big||E| - |F|\big|. \tag{3.1.1}$$

whenever $E\Delta F \Subset B_r(x) \cap U$ and $r < r_0$.

We will prove Property 3.1.2 in $\mathbb{R}^n$ later. Now we claim the following lemma which will be frequently used in this section.

Lemma 3.1.3 (Perimeter control for variations)**.** *If E is a set of finite perimeter and A is an open set such that $\mathcal{H}^{n-1}(A \cap \partial^* E) > 0$, then there exist $\sigma_0 > 0, C < \infty$ both depending on E and A such that for every $\sigma \in (-\sigma_0, \sigma_0)$ we can find a set of finite perimeter F with $F\Delta E \subset A$ and*

$$|F| = |E| + \sigma, \quad |\mathbf{P}(F; A) - \mathbf{P}(E; A)| \leq C|\sigma|.$$

Proof. By $\mathcal{H}^{n-1}(A \cap \partial^* E) > 0$ and De Giorgi's structure theorem (Theorem 2.2.6), there exists a vector field $T \in C_c^\infty(A; \mathbb{R}^n)$ such that

$$\gamma := \int_{\partial^* E} T \cdot \nu_E \, d\mathcal{H}^{n-1} > 0.$$

Suppose $\{\Phi_t\}_{|t|<\epsilon}$ represents the local variation associated with T. We can consider the first variations of volume and perimeter associated with the deformation along T.

$$\mathbf{P}(\Phi_t(E);A) = \mathbf{P}(E;A) + t\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} + O(t^2), \tag{3.1.2}$$

$$|\Phi_t(E)| = |E| + t\int_{\partial^* E} T\cdot\nu_E \, d\mathcal{H}^{n-1} + O(t^2). \tag{3.1.3}$$

Since $\gamma > 0$, by the expansions above, $|\Phi_t(E)|$ is locally increasing for t near 0, i.e., there is an $\epsilon_0 > 0$ such that for any $t \in (-\epsilon_0, \epsilon_0)$, we have

$$|\Phi_t(E)| - |E| \geq \frac{\gamma}{2}|t|, \tag{3.1.4}$$

and for perimeters, we have

$$|\mathbf{P}(\Phi_t(E);A) - \mathbf{P}(E;A)| \leq 2\left|\int_{\partial^* E} \mathrm{div} T \, d\mathcal{H}^{n-1}\right| \cdot |t|.$$

We choose $\sigma_0 > 0$ such that the interval $(|E|-\sigma_0, |E|+\sigma_0) \subset (|\Phi_{-\sigma_0}(E)|, |\Phi_{\sigma_0}(E)|)$, and denote

$$C := \frac{4}{\gamma}\left|\int_{\partial^* E} \mathrm{div} T \, d\mathcal{H}^{n-1}\right|.$$

So for any σ with $|\sigma| < \sigma_0$, there exists $|t| < t_0$ such that for $F := \Phi_t(E)$, we have

$$||\Phi_t(E)| - |E|| = \sigma,$$

and for perimeters,

$$\begin{aligned}|\mathbf{P}(\Phi_t(E);A)-\mathbf{P}(E;A)| &\le 2\left|\int_{\partial^* E}\operatorname{div}T\,d\mathcal{H}^{n-1}\right|\cdot|t|\\ &\le 2\frac{C\gamma}{4}|t|\le\frac{C}{2}\gamma|t|\\ &\le C\,||\Phi_t(E)|-|E||\\ &\le C\sigma.\end{aligned}$$

So the $\Phi_t(E)$ satisfies all the requirements. □

Proof of Property 3.1.2. At first, we pick two points $x_1, x_2 \in A\cap\partial E$ and $t_0>0$ such that, for $B_1 := B_{t_0}(x_1)$ and $B_2 := B_{t_0}(x_2)$, we have $B_1\cap B_2=\emptyset$ and $B_1\cup B_2 \Subset A$.

By Lemma 3.1.3, we can find positive constants σ_0 and C_0 (depending on E and A) such that for $|\sigma|<\sigma_0$, there exist two sets of finite perimeter F_1 and F_2 with

$$E\Delta F_k \Subset B_k,\quad |F_k|=|E|+\sigma,\quad |\mathbf{P}(E;B_k)-\mathbf{P}(F_k;B_k)|\le C_0|\sigma|,\qquad \text{where } k=1,2.$$

Denote $t_1=(|x_1-x_2|-2t_0)/2$, geometrically, it means that if a ball of radius t_1 intersects B_1 (respectively, B_2), then it is disjoint from B_2 (respectively, from B_1).

Now we choose

$$\Lambda = C_0,\qquad r_0=\min\left\{\frac{t_0}{2},\frac{\sigma_0^{\frac{1}{n}}}{\omega_n},t_1\right\},$$

and claim that E satisfies the inequality (3.1.1) with constants Λ, r_0 defined above.

Fixing $r<r_0$, suppose that F a set of finite perimeter with $E\Delta F\Subset B_r(x)\cap A$. By the definition of t_1, WLOG, we can assume that $B_r(x)$ does not intersect with B_1. Set $\sigma=||E|-|F||$, obviously,

$$\sigma\le|E\Delta F|\le|B_r(x)|<\omega_n r_0^n\le\sigma_0.$$

Therefore, consider F_1 a modification of F as above by Lemma 3.1.3 in B_1, so we have

$$E\Delta F_1 \subset B_1, \qquad E\Delta F \Subset B_r(x) \subset \mathbb{R}^n \setminus \overline{B_1}, \qquad |F_1| - |E| = \sigma.$$

Now we consider the set

$$G = (F \cap B_r(x)) \cup (F_1 \cap B_1) \cup (E \setminus (B_r(x) \cup B_1)),$$

clearly by Theorem 2.2.7, G is also a set of finite perimeter and

$$|G| = |E|, \qquad E\Delta G \subset A.$$

Because E is a volume-constrained perimeter minimizer in A, we have

$$\mathbf{P}(E; A) \leq \mathbf{P}(G; A).$$

And by our construction above, we have

$$\begin{aligned}
\mathbf{P}(G; A) &= \mathbf{P}(G; A \setminus \overline{B_1}) + \mathbf{P}(G; B_1) + \mathbf{P}(G; \partial B_1) \\
&= \mathbf{P}(F; A \setminus \overline{B_1}) + \mathbf{P}(F_1; B_1) + \mathbf{P}(F; \partial B_1) \\
&\leq \mathbf{P}(F; A \setminus B_1) + \mathbf{P}(E; B_1) + C_0|\sigma| \\
&= \mathbf{P}(F; A) + C_0||E| - |F||.
\end{aligned}$$

So the property is proved. □

Corollary 3.1.4. *Suppose* $\{g_j\}_{j=1}^{\infty}$ *a class of Riemannian manifold on* M *such that* $g_j \to g$ *in* C^3*. Fixing* $t > 0$*, and any sequences* $\Omega_j \in \mathcal{A}_{g_j}(M, t)$*, there exists* $\Omega \in \mathcal{A}_g(M, t)$ *and* $\lim_{j\to\infty} \mathbf{P}_{g_j}(\Omega_j) = \mathbf{P}_g(\Omega)$*. Moreover,* $\lim_{j\to\infty} \mathcal{I}_{g_j}(M, t) = \mathcal{I}_g(M, t)$*.*

Proof. Clearly, we have $\sup_j \mathbf{P}_g(\Omega_j) < \infty$. So by the compactness of sets of finite preimiters.

There exits an $\Omega \in \mathcal{C}(M)$ such that $d_g(\Omega_j, \Omega) \to 0$. Because $g_j \to g$ in C^3 and $|\Omega_j|_{g_j} = t$ for all j, we have $|\Omega|_g = t$. By the lower semicontinuity of perimeters, we have

$$\liminf_{j\to\infty} \mathbf{P}_{g_j}(\Omega_j) = \liminf_{j\to\infty} \mathbf{P}_g(\Omega_j) \geq \mathbf{P}_g(\Omega).$$

Next we need to show that $\limsup_{j\to\infty} \mathbf{P}_{g_j}(\Omega_j) \leq \mathbf{P}_g(\Omega)$. Assume not, then we have $\limsup_{j\to\infty} \mathbf{P}_{g_j}(\Omega_j) > \mathbf{P}_g(\Omega)$.

Define $\epsilon_0 := \limsup_{j\to\infty} \mathbf{P}_{g_j}(\Omega_j) - \mathbf{P}_g(\Omega) > 0$. By Lemma 3.1.3, there exist $\widetilde{\Omega}_j \in \mathcal{C}_{g_j}(M,t)$ such that $|\mathbf{P}_{g_j}(\widetilde{\Omega}_j) - \mathbf{P}_{g_j}(\Omega)| \leq C|\epsilon_j|$, with $\epsilon_j = |\Omega|_{g_j} - t$. So $\epsilon_j \to 0$.

Then we have

$$\mathbf{P}_{g_j}(\Omega_j) = I_{g_j}(M,t) \leq \mathbf{P}_{g_j}(\widetilde{\Omega}_j) \leq \mathbf{P}_{g_j}(\Omega) + C|\epsilon_j| \leq \mathbf{P}_g(\Omega) + C|\epsilon_j| + 1/10\epsilon_0,$$

Therefore, we have $\limsup_{j\to\infty} I_{g_j}(M,t) \leq \mathbf{P}_g(\Omega) + 1/2\epsilon_0$, which leads a contradiction. Therefore, we have $\lim_{j\to\infty} \mathbf{P}_{g_j}(\Omega_j) = \mathbf{P}_g(\Omega)$.

Next we show that $\mathbf{P}_g(\Omega) = \mathcal{I}_g(M,t)$. By the previous part, we have $\lim_{j\to\infty} \mathcal{I}_{g_j}(M,t)$ exists and $\mathcal{I}_g(M,t) \leq \lim_{j\to\infty} \mathcal{I}_{g_j}(M,t)$. Suppose there exists $\widetilde{\Omega} \in \mathcal{C}_g(M,t)$ such that

$$\mathbf{P}_g(\widetilde{\Omega}) = \mathcal{I}_g(M,t) < \mathbf{P}_g(\Omega).$$

Similar as previous part, denote $\epsilon_0 := \mathbf{P}_g(\Omega) - \mathbf{P}_g(\widetilde{\Omega}) = \lim_{j\to\infty} \mathbf{P}_{g_j}(\Omega_j) - \mathbf{P}_g(\widetilde{\Omega})$. By Lemma 3.1.3 again, under the Riemannian metric g_j, there exists $\widetilde{\Omega}_j \in \mathcal{C}_{g_j}(M,t)$ such that

$$|\mathbf{P}_{g_j}(\widetilde{\Omega}) - \mathbf{P}_{g_j}(\widetilde{\Omega}_j)| < C|\epsilon_j|,$$

where $\epsilon_j := |\widetilde{\Omega}|_{g_j} - t$. So we have

$$\begin{aligned}
\mathbf{P}_{g_j}(\widetilde{\Omega}_j) &\leq \mathbf{P}_{g_j}(\widetilde{\Omega}) + C|\epsilon_j| \\
&\leq \mathbf{P}_g(\widetilde{\Omega}) + C|\epsilon_j| + 1/10\epsilon_0 \\
&< \mathbf{P}_g(\Omega) - \epsilon_0 + C|\epsilon_j| + 1/10\epsilon_0 \\
&< \mathbf{P}_{g_j}(\Omega_j)
\end{aligned}$$

for j large enough. The last inequality arises from $\mathbf{P}_{g_j}(\Omega_j) \to \mathbf{P}_g(\Omega)$. So we get a contradiction. □

By the Property 3.1.2, we see that if $E \in \mathcal{C}(M)$ is a volume-constrained perimeter minimizer in an open set U, then

$$\begin{aligned}
\mathbf{P}(E; B_r(x)) &\leq \mathbf{P}(F; B_r(x)) + \Lambda \big||E| - |F|\big| \\
&\leq \mathbf{P}(F; B_r(x)) + \Lambda \big|E\Delta F\big|,
\end{aligned}$$

whenever $E\Delta F \Subset B_r(x) \cap U$ and $r < r_0$. So E is a $(\Lambda, r_0)-$perimeter minimizer in U. Thus, by Theorem 2.3.4, isoperimetric hypersurfaces share the same regularity results as perimeter minimizers.

Theorem 3.1.5 (Existence and Regularity theorems for Isoperimetric regions [2, 19, 20])**.** *If M^{n+1} is compact, then for any $t \in (0, Vol(M))$, there exists an $\Omega \in \mathcal{C}(M)$ such that $Vol(\Omega) = t$ and Ω minimizes perimeter among regions of volume t. Moreover, except for a closed singular set of Hausdorff dimension at most $n-7$, the boundary $\operatorname{spt}\partial\Omega$ of any minimizing region is a smooth embedded hypersurface with constant mean curvature.*

Similar to the (local) area-minimizing hypersurfaces, for $2 \leq n \leq 7$, it is a well-known result that $Sing(\Omega) = \emptyset$ if Ω is an isoperimetric region [2, 19, 20]. So higher dimension is the only possible case that singularities may appear. On the other hand, Simons' cone [7] manifests an example of an area-minimizing hypersurface in Euclidean space, but isoperimetric regions are smooth regardless of dimensions.

Another property of isoperimetric hypersurfaces is their mean curvature. As mentioned in Theorem 3.1.5, the regular part of the isoperimetric hypersurfaces has constant mean curvature.

Example 3.1.6 ([3])**.** Let $M^{n+1}(c)$ denote the simply connected complete Riemannian manifold with constant sectional curvature c. Let $X : \Sigma^n \to M^{n+1}$ be an immersion of a differentiable manifold Σ^n. Suppose $X(\Sigma)$ has constant mean curvature. Then the immersion X is volume-preserving stable (see Definition 3.1.12) if and only if $X(\Sigma) \subset M(c)$ is a geodesic sphere.

The above examples are isoperimetric regions in space forms (see also [34, 23]). Similar to the area functional for the minimal surfaces, we define the following functional for isoperimetric regions. Fixing a metric g, we can consider the functional $\mathcal{F} : \mathbb{R} \times \mathcal{C}(M) \to \mathbb{R}$ by

$$\mathcal{F}(\lambda, \Omega) = \mathbf{P}(\Omega) + \lambda|\Omega|.$$

Suppose X the smooth compactly supported vector field, $\{\phi_t\}$ the corresponding diffeomorphism, H the generalized mean curvature vector, and υ the outer normal vector for $\partial^*\Omega$. By the area formula and first variation for potential energy (e.g., [25] Chapter 17), we can get the first variation with respect to Ω:

$$\delta_2\mathcal{F}(\lambda, \Omega) = \frac{d}{dt}\mathcal{F}(\lambda, \phi_t(\Omega))\Big|_{t=0} \tag{3.1.5}$$

$$= \int_{\partial^*\Omega} \operatorname{div} X \, d\mathcal{H}^{n-1} + \lambda \int_{\partial^*\Omega} X \cdot \upsilon \, d\mathcal{H}^{n+1}. \tag{3.1.6}$$

Remark 3.1.7. In addition, if Ω is the isoperimetric region (i.e. the minimizer of (1.1.1)), we have $\delta_2\mathcal{F}(\lambda, \Omega) = 0$. As remarked in [22, 35], by using the estimates of Hausdorff dimension for the singular set, $\mathcal{H}^{n-2}(\partial\Omega) = 0$, then using a cutoff function argument for the test functions, we can see that $\lambda = -H \cdot \upsilon = -h$ for any points in $Reg(\partial\Omega)$, which provides $Reg(\partial\Omega)$ a constant mean curvature hypersurface in M. We call Σ a H-hypersurface if $H_\Sigma \equiv H$.

Theorem 3.1.8 (Constant mean curvature)**.** *If E is a volume-constrained perimeter minimizer in the open set A, then there exists $\lambda \in \mathbb{R}$ such that*

$$\int_{\partial^* E} div\, X d\mathcal{H}^{n-1} = \lambda \int_{\partial^* E} (X \cdot \nu_E) d\mathcal{H}^{n-1}, \qquad \forall X \in C_c^\infty(A; \mathbb{R}^n).$$

We call E has constant distributional mean curvature in A and it equal to λ.

Proof. Initially, we prove a special case where $\int_{\partial^* E}(X \cdot \nu_E) d\mathcal{H}^{n-1} = 0$ for a certain variation X, which involves deformation within small balls.

Claim: There exists a constant $r_0 > 0$ such that if $X \in C_c^\infty(A; \mathbb{R}^n)$ with spt $X \subset B_{r_0}(x)$ for some $x \in A$, and

$$\int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^{n-1} = 0,$$

then we have

$$\int_{\partial^* E} \text{div } X \, d\mathcal{H}^{n-1} = 0.$$

Proof of the claim: Choosing $r_0 > 0$ (sufficiently small) such that

$$(A \cap \partial^* E) \backslash B_{r_0}(z) \neq \emptyset, \qquad \text{for all } z \in A.$$

Given any $X \in C_c^1(A; \mathbb{R}^n)$ with spt $X \subset B_{r_0}(x)$ such that $\int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^{n-1} = 0$. By Property 2.2.2, we can find a smaller $r > 0$ such that

$$\text{spt } X \subset B_r(x), \qquad \mathcal{H}^{n-1}(\partial^* E \cap \partial B_r(x)) = 0.$$

On the other hand, we can pick $y \in A \cap \partial^* E$ and $s > 0$ such that $B_s(y) \cap B_r(x) = \emptyset$

and (by Property 2.2.2 again)

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial B_s(y)) = 0.$$

Now denote $\{\Phi_t\}_{|t|<\epsilon}$ the local variation in $B_r(x)$ associated with X. By the first variation of volume and perimeter, we have

$$\begin{aligned} |\Phi_t(E)| &= |E| + O(t^2), \\ \mathbf{P}(\Phi_t(E); A) &= \mathbf{P}(E; A) + t \int_{\partial^* E} \operatorname{div} X \, d\mathcal{H}^{n-1} + O(t^2). \end{aligned}$$

Now we use Lemma 3.1.3 with E in the open set $B_s(y)$. Denote σ_0 and C the constants from Lemma 3.1.3. Denote $\sigma(t) = |E| - |\Phi_t(E)|$, then for each small t, we have $|\sigma(t)| < \sigma_0$. And for each small $|t|$, there exists F_t with $E \Delta F_t \Subset B_s(y)$ and

$$|F_t| - |E| = \sigma(t)(\text{ so also} = |E| - |\Phi_t(E)|),$$

$$|\mathbf{P}(F_t; B_s(y)) - \mathbf{P}(E; B_s(y))| \le C|\sigma(t)| = O(t^2).$$

Now we denote

$$E_t = (\Phi_t(E) \cap B_r(x)) \cup (F_t \cap B_s(y)) \cup (E \backslash (B_r(x) \cup B_s(y))),$$

defined for each small t. Note that $|E_t| = |E|$.

Because E is a volume-constrained minimizer, so we have

$$\begin{aligned} 0 &\le \mathbf{P}(E_t; A) - \mathbf{P}(E; A) \\ &\le \mathbf{P}(\Phi_t(E); B_r(x)) + \mathbf{P}(F_t; B_s(y)) - \mathbf{P}(E; B_r(x)) - \mathbf{P}(E; B_s(y)) \\ &= t \int_{\partial^* E} \operatorname{div} X \, d\mathcal{H}^{n-1} + O(t^2). \end{aligned}$$

Because the inequality above holds for all small t around 0, we have

$$\int_{\partial^* E} \text{div } X \, d\mathcal{H}^{n-1} = 0.$$

So the Claim is proved.

Possibly choosing a smaller r_0, we may assume that

$$(A \cap \partial^* E) \setminus (B_{r_0}(x_1) \cup B_{r_0}(x_2)) \neq \emptyset, \qquad \text{for any } x_1, x_2 \in A.$$

Now choosing two vector fields $X_1, X_2 \in C_c^0(A; \mathbb{R}^n)$ such that, for $j = 1, 2$,

$$\text{spt } X_j \subset B_{r_0}(x_j), \qquad \int_{\partial^* E} (X_j \cdot \nu_E) d\mathcal{H}^{n-1} \neq 0.$$

Similar as in the proof of the Claim above, we may find a smaller $r < r_0$ such that, for $j = 1, 2$,

$$\text{spt } X_j \subset B_r(x_j), \quad \mathcal{H}^{n-1}(\partial^* E \cap B_r(x_1) \cup B_r(x_2)) = 0.$$

Now we define $X \in C_c^0(A; \mathbb{R}^n)$ by

$$X := X_1 - \frac{\int_{\partial^* E} (X_1 \cdot \nu_E) d\mathcal{H}^{n-1}}{\int_{\partial^* E} (X_2 \cdot \nu_E) d\mathcal{H}^{n-1}} X_2.$$

Clearly, we have $\int_{\partial^* E} (X \cdot \nu_E) d\mathcal{H}^{n-1} = 0$.

By a slightly revision of Claim 1's proof, we can show that $\int_{\partial^* E} (X \cdot \nu_E) d\mathcal{H}^{n-1} = 0$ implies $\int_{\partial^* E} \text{div } X \, d\mathcal{H}^{n-1} = 0$, that is

$$\frac{\int_{\partial^* E} \text{div}_E X_1 d\mathcal{H}^{n-1}}{\int_{\partial^* E} (X_1 \cdot \nu_E) d\mathcal{H}^{n-1}} = \frac{\int_{\partial^* E} \text{div}_E X_2 d\mathcal{H}^{n-1}}{\int_{\partial^* E} (X_2 \cdot \nu_E) d\mathcal{H}^{n-1}} (=: \lambda).$$

That is to say, for any vector field $X \in C_c^\infty(A; \mathbb{R}^n)$ such that $\text{spt} X \Subset B(x, r_0)$ for some $x \in A$, we have

$$\int_{\partial^* E} \operatorname{div} X d\mathcal{H}^{n-1} = \lambda \int_{\partial^* E} (X \cdot \nu_E) d\mathcal{H}^{n-1}.$$

Now consider a general vector field $X \in C_c^\infty(A; \mathbb{R}^n)$, and let $\{B(z_k, r_0)\}_{k=1}^N$ be a finite cover of spt$X$ by open balls centered in A. Using a partition of unity argument, we can show that

$$\int_{\partial^* E} \operatorname{div}_E X d\mathcal{H}^{n-1} = \lambda \int_{\partial^* E} (X \cdot \nu_E) d\mathcal{H}^{n-1}.$$

□

Example 3.1.9. [Schmidt 1940 Isoperimetric regions in space forms] Let $n \geq 2$, the isoperimetric regions in the simply connected constant curvature spaces $\mathbb{R}^n, \mathbb{S}^n$, and $\mathbb{H}^n$ are exactly the geodesic balls.

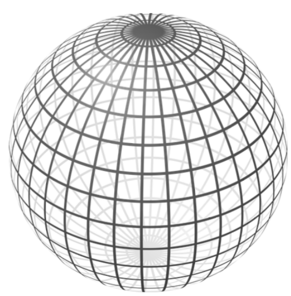

Figure 3.1. Isoperimetric regions in $\mathbb{S}^2$ with different volume constraints.

Example 3.1.10. [33, Theorem. 4.3] Consider the Riemannian manifold $\mathbb{S}^1(r) \times \mathbb{S}^2$ with standard metric, the isoperimetric regions are one of the following cases:

1. balls or their complements, or

2. tubular neighborhoods of the closed geodesics $\mathbb{S}^1(r) \times \{\text{point}\}$ (which are actually diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$), or

3. sections bounded by two totally geodesic $\{\text{point}\} \times \mathbb{S}^2$ (which are diffeomorphic to $[a, b] \times \mathbb{S}^2$).

In addition, case 1 (balls) are solutions for small/large values of the volume. If $r > 1$, then the torus are not solutions. If r small, then the case 3 are not solutions.

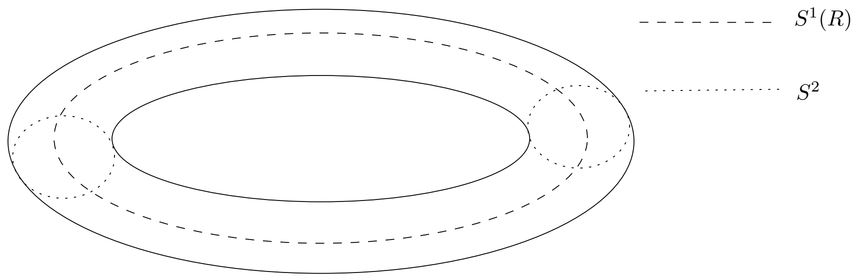

Figure 3.2. Torus $\mathbb{S}^2 \times \mathbb{S}^1(R)$

A analogous result for isoperimetric regions in higher dimension torus still holds if we require the radius R large enough:

Example 3.1.11. [11, Lemma 4.1] There is a dimentional constant $R_0 = R_0(n)$ so that for $R \geq R_0$, if we consider the product metric g_R on $S^1(R) \times S^{n-1}(1)$, then every isoperimetric region $\Omega \subset M$ with half volume $|\Omega| = \frac{1}{2}|S^1(R) \times S^{n-1}(1)|$ is of the form

$$\Omega = (t_0, t_0 + \pi R) \times S^{n-1}$$

for $t_0 \in \mathbb{R}$.

Similar to the area-minimizing hypersurfaces, isoperimetric regions have the following sense of "stability".

Definition 3.1.12. For $\Omega \in \mathcal{C}(M)$, we say Ω is volume-preserving stable if $\partial^*\Omega$ has constant mean curvature and for any diffeomorphism ϕ with $|\phi(\Omega)|_g = |\Omega|_g$, we have $\delta^2\mathbf{P}(\Omega) \geq 0$.

Moreover, suppose $\partial\Omega$ is smooth. By [3, Proposition 2.5], suppose

$$f(x) = \langle \frac{\partial \phi}{\partial t}(x), \nu(x) \rangle,$$

we have

$$\begin{aligned} \delta^2 \mathbf{P}(\Omega) &= \delta_2^2 \mathcal{F}(-H, \Omega) \\ &= \int_{\partial\Omega} -f\Delta f - (|A_\Sigma|^2 + Ric_M(\nu,\nu)) f^2 \, d\mathcal{H}^n(x) \geq 0, \end{aligned}$$

where $\int_{\partial\Omega} f = 0$.

Remark 3.1.13. For the sake of disambiguation, we will say "stable" if Σ is a minimal surface such that $\delta^2(\Sigma) > 0$ for any diffeomorphisms.

Using the definition in [40], we define "strictly stability" as follows, which generalized the definition over isolated cones (see [8]).

Definition 3.1.14. We say Σ is **strictly stable** if Σ is a minimal surface, with singularities on a closed set of codimension 7, there is a positive constant C such that

$$\int_\Sigma |\nabla u|^2 - (|A|^2 + Ric(v))u^2 \geq C \int_\Sigma u^2 \rho^{-2},$$

for all $u \in W_0^{1,2,-1}(\Sigma)$. Here we define $\rho(x) = dist(x, Sing(\Sigma))$ and $W_0^{1,2,-2}(\Sigma) = \overline{C_0^1(\Sigma \setminus Sing(\Sigma))}$ with norm

$$\|u\|_{1,2}^2 := \int_\Sigma |\nabla u|^2 + u^2 \rho^{-2}.$$

Remark 3.1.15. Smale in [40] shows that after a conformal change of metric, the hypersurface constructed in $\mathbb{S}^8$ is strictly stable in the sense of Definition 3.1.14, which implies the existence of a neighborhood such that it is homological minimizing in it (see Theorem 3.2.1

or [40, Lemma 4]).

In Chapter 2, we see that isoperimetric regions in a smooth compact $(n+1)$-manifold are smooth, up to a closed set of codimension at most 8. A natural question is whether the bound is optimal or not. In the case of area-minimizing integral currents, the question is answered in the positive by the Simons' cone in $\mathbb{R}^8$: $\mathbf{C} := \{(x,y) : |x| = |y| \text{ for } x, y \in \mathbb{R}^4\}$. However, minimizers of the problem above in $\mathbb{R}^n$ are euclidean balls and hence smooth for every $n \in \mathbb{N}$. Therefore, to construct a singular minimizer in dimension 8, we need to construct a manifold that is not a space form (see more explanation in chapter 2.) In this chapter, we first construct an 8-dimensional compact smooth manifold whose unique isoperimetric region with half volume that of the manifold exhibits two isolated singularities. And then, for $n \geq 7$, using Smale's construction of singular homological area minimizers for higher dimensions, we construct a Riemannian manifold such that the unique isoperimetric region of half volume, with singular set the submanifold $\mathbb{S}^{n-7}$.

For dimension $n+1 = 8$, we first prove the following theorem for singular isoperimetric region:

Theorem 3.1.16 (Singular isoperimetric region in 8-manifold)**.** *[32, Theorem 1.1] There exists a smooth closed Riemannian* 8*-manifold* (M, g) *whose unique isoperimetric region with volume* $|M|_g/2$ *has two isolated singularities. The unique tangent cone at each singular point is a Simons' cone.*

Remark 3.1.17.

- In [40], we can prescribe the singularity for the homological area minimizers to be any strictly stable, strictly minimizing (tangent) cone with an isolated singularity (see chapter 2 for the definitions), but in our construction, for the technical reason of the setting, we need in addition to assume that the unique (up to scaling) smooth area minimizing hypersurface on one side of the cone (see [21, Theorem 2.1]) is diffeomorphic to the one on the other side (e.g., Simons' cone). However, it is promising that we can eliminate this requirement by modifying the construction. As far as the author's knowledge, these are the first examples of isoperimetric regions with singularities.

- The metric in the above theorem is only C^∞. It is an open question whether the same result would hold for an analytic metric.

By slightly modifying the proof in Theorem 3.1.16, we can generalize it to higher dimensions.

Theorem 3.1.18 (Singular isoperimetric regions in higher dimensional manifolds)**.** *[32, Theorem 1.1] For any integers $n \geq 7, p \in [3, \frac{n-1}{2}]$, there exists a closed smooth $(n+1)$-dimensional Riemannian manifold (M, g) such that whose unique isoperimetric region Ω with volume half has the singular part a closed submanifold diffeomorphic to $\mathbb{S}^{n-2p-1}$ (denote $\mathbb{S}^0$ as a point). Denote the Simons' cone in $\mathbb{R}^{2p+2}$,*

$$\mathbf{C}^{p,p} := \{(x, y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{p+1} : \ |x| = |y|\}.$$

Near $Sing(\partial\Omega)$, $\partial\Omega$ looks like $\mathbb{S}^{n-2p-1} \times \mathbf{C}^{p,p}$, i.e., there exists $\sigma > 0$, and an isometric map Φ from the tubular neighborhood of the singular set $\mathcal{N}(\sigma) := \{x \in M : d_M(x, Sing(\Sigma)) < \sigma\}$ to $(\mathbb{B}^{2p+2}(\sigma) \times \mathbb{S}^{n-2p-1}, g_{eucl} + g_S)$, here g_{eucl}, g_S are the standard metrics in $\mathbb{R}^{2p+2}$ and $\mathbb{S}^{n-2p-1}$ respectively. Moreover,

$$\Phi(\partial\Omega \cap \mathcal{N}(\sigma)) = \mathbf{C}^{p,p}(\sigma) \times \mathbb{S}^{n-2p-1}.$$

We could replace $\mathbb{S}^{n-2p-1}$ to quite numerous varieties as the singular part, with only some topological restrictions (see details in section 3.4). The construction comes from the fruitful examples of singular homological area minimizing codimension 1 currents (see also [41, Theorem A]). The proof of Theorem 3.1.18 strongly uses the result from [41] and a slight modification of the proof in Theorem 3.1.16. In section 3.2 to 3.3, we will focus on constructing examples with isolated singularities. In section 3.4, we will prove Theorem 3.1.18.

Idea of the Construction for Isolated Singularities

As in [40], the starting point is constructing a singular minimal surface in $\mathbb{S}^8$. We denote $\mathbf{C}$ as a 7-dimensional Simons' cone. Then the product space $\mathbf{C} \times \mathbb{R}$ will be an area-minimizing cone in $\mathbb{R}^9$. Now consider $\Sigma = \mathbf{C} \times \mathbb{R} \cap \mathbb{S}^8$ in $\mathbb{S}^8$. Clearly, Σ is a minimal hypersurface in $(\mathbb{S}^8, g_S)$ with two isolated singularities, where g_S is the round metric.

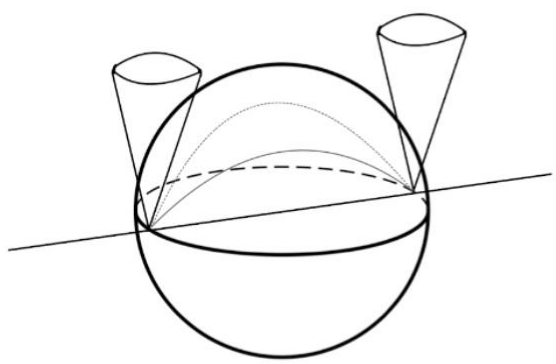

Figure 3.3. The singular hypersurface $\Sigma := (\mathbf{C} \times \mathbb{R}) \cap \mathbb{S}^8$.

The important part of Smale's work is to prove Theorem 3.2.1 [cf. [40, Lemma 4]]: under a conformal change of the standard metric of $\mathbb{S}^8$, there exists a smooth neighborhood V of Σ such that Σ is the unique homological area-minimizing current in V. Moreover, Σ splits $\overline{V}$ into two parts, V_+ and V_-, and ∂V has exactly two components (denoted by Γ_+, Γ_-) such that they lie in $\overline{V}_+, \overline{V}_-$ respectively. Each part of $\{\Gamma_\pm\}$ is homologous to Σ and Γ_+ is diffeomorphic to Γ_- (Theorem 3.2.2 below).

Next, we will construct a manifold with a singular isoperimetric region. Consider (Γ, g_l) the 7-manifold which is diffeomorphic to Γ_+ and Γ_-, endowed with a "larger" metric (see Theorem 3.2.3 for details); denote $T_R := \Gamma \times [0, R]$ a tube with length $R > 0$, with the product metric $g = g_l + dr^2$. We glue the Γ_+, Γ_- with the boundary of the tube, $\widetilde{\Gamma}_- := \Gamma \times \{0\}, \widetilde{\Gamma}_+ := \Gamma \times \{R\}$ respectively to form a torus, calling it M_R. Finally, we prove that (in section 3.3) for sufficiently large R, the unique isoperimetric region with half the volume of M_R has boundary $(\Gamma \times \{t_0\}) \cup \Sigma$ for some $t_0 \in (0, R)$, where Σ is the one described in the previous paragraph, with two isolated singularities.

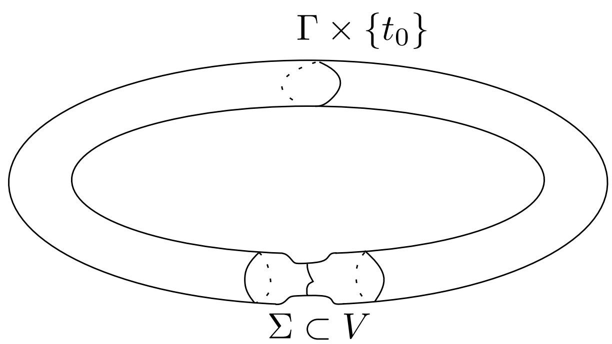

Figure 3.4. $M(R) := \Gamma \times [0, R] \cup V/\Gamma_\pm \sim \tilde{\Gamma}_\pm$.

3.2 Constructing the Manifolds with Smale's Ideas

This section is dedicated to constructing a family of manifolds, a suitable choice of which will later give us the main Theorem 3.1.16. We divide it into two parts: first, using a result of Smale [40], we obtain the first piece of our manifold, then we suitably modify it to glue it to a cylinder to obtain the desired construction. The second part is more similar to an 8-dimensional torus example in [11, Lemma 4.1] by Chodosh, Engelstein, and Spolaor.

3.2.1 Smale's Main Result

We recall here the main result from [40], which will be the starting point of our construction.

Theorem 3.2.1 ([40, Lemma 4])**.** *Let* $\mathbf{C}$ *be any strictly stable and strictly minimizing cone, e.g., Simons' cone. Let* $\Sigma := (\mathbf{C} \times \mathbb{R}) \cap \mathbb{S}^8$*. There exists a* C^∞ *metric* g *on* $\mathbb{S}^8$ *and* $\delta > 0$ *such that* Σ *is uniquely homologically area-minimizing in the tubular neighborhood*

$$U_\delta := \{x \in \mathbb{S}^8 : d_{\mathbb{S}^8}(x, \Sigma) \leq \delta\},$$

with respect to the metric g*.*

Proof. The proof of this result can be found in [40, Lemma 4]. □

In order to glue the U_δ along the boundaries with a manifold (with boundary), we need the boundary of U_δ to be smooth. Because of the singularities of Σ, we cannot expect the smoothness of the boundaries of U_δ for any small $\delta > 0$. Fortunately, as remarked in the proof of [40, Lemma 4], we can find a smaller neighborhood $V \subset U_\delta$ such that Σ is still homological area-minimizing in V, and the boundary ∂V is smooth. The basic idea is to glue in pieces of foliations.

Theorem 3.2.2. *Denote* $\mathbf{C}$ *the Simons cone. Let* $\Sigma := (\mathbf{C} \times \mathbb{R}) \cap \mathbb{S}^8$ *and let* U_δ *and* g *be as in the previous theorem. There exists an open subset* V *such that* $\overline{V} \Subset U_\delta$ *and*

- $\Sigma \subset V$ *and it is homologically minimizing in* V *with respect to the metric* g*;*
- $V \setminus \Sigma$ *consist of* 2 *connected components,* $V_\pm$ *and* $\partial V_\pm = \Sigma \cup \Gamma_\pm$*, disjoint union, and* $\Gamma_\pm$ *are smooth and diffeomorphic to each other;*
- Σ *is a deformation retract of* V*.*

Proof. The construction of V is almost the same as the argument in [40].

For δ small, Σ splits U_δ into two parts, call them U_+, U_-. Note that $Sing(\Sigma) = \{p_+, p_-\}$. For $\sigma > 0$ smaller than $\delta/8$, we have $B_\sigma(p_\pm) \subset U_\delta$. Consider the Fermi's coordinate

$$\{(x, t) : x \in \Sigma, \quad \rho(x) > \sigma, \quad |t| < \delta\},$$

around Σ, here we denote $\rho(x) := d_{\mathbb{S}^8}(x, p_+ \cup p_-)$. We denote $\Sigma_\sigma = \{q \in \Sigma : \rho(q) < \sigma\}$, then consider the constant graph on $\Sigma \setminus \Sigma_\sigma$:

$$\Gamma_t = graph_{\Sigma \setminus \Sigma_\sigma} t,$$

with $|t| < \delta$.

Let $p := p_+$ or p_-, and denote $S_t = \{(x,t) : \rho(x,0) = 4\sigma\}$. By [21, Theorem 5.6], S_t bounds a 7-dimensional smooth submanifold R_t which is area-minimizing in $B_{5\sigma}(p)$, and as $t \to 0$, $R_t \to \Sigma_{4\sigma}$ in both the current and Hausdorff senses. Denote $\lambda_t := d_{\mathbb{S}^8}(p, R_t)$; as argued in [21, Theorem 2.1], we have $\eta_{p,\lambda_t\#} R_t \to S$, where S is the unique minimal hypersurface in E with $d(S,p) = 1$, where E is one side of $\mathbf{C}_p$ such that $S \subset E$. Because S is smooth, $\eta_{p,\lambda_t\#} R_t \to S$ in C^2_{loc}. By [21, Theorem 2.1], S has the following properties:

1. for any vector $\xi \in E$, the ray $\{\lambda\xi : \lambda > 0\}$ intersects S at a single point, and the intersection is transverse;

2. there exists a constant $C := C(\mathbf{C}_p)$ such that $S_{\llcorner \mathbb{R}^8 \setminus B_C(p)}$ is a graph of a function on $\mathbf{C}_p$.

On the other hand, for any small t, there exists a C^2 function u_t on $\Sigma_{4\sigma} \setminus \Sigma_{C\lambda_t}$ such that $R_t \setminus B_{C\lambda_t}(p)$ can be represented as the graph of u_t. We denote $\epsilon_t := C\lambda_t$.

We can find a smooth cutoff function χ on Σ such that

$$\chi(x) = \begin{cases} 1 & \text{for } \rho(x) \geq 3\sigma, \\ 0 & \text{for } \rho(x) \leq 2\sigma. \end{cases}$$

We can get a smooth hypersurface by gluing the smooth submanifold Γ_t with R_t

through a function w on $\Sigma \setminus \Sigma_\sigma$ by

$$w_t = \chi u_t + (1-\chi)t. \tag{3.2.1}$$

Therefore,according to our construction above, we can denote V_{t_0} as a foliation in the following sense: there exists $t_0 > 0$ such that

$$V_{t_0} := \bigcup_{t\in[-t_0,t_0]} W_t,$$

where $W_0 := \Sigma$ and for each $t \neq 0$, W_t is the smooth hypersurface formed by gluing the smooth submanifold Γ_t with R_t through the function w_t defined in (3.2.1).

Then, we explore the topology of V_{t_0}. As mentioned above, for small enough t, for any $q \in \operatorname{spt} \eta_{p,\lambda_t\#}R_t \cap B_C(p)$, there is a unique ray from p connecting p and q such that the ray intersects $\eta_{p,\lambda_t\#}R_t \cap B_C(p)$ only at q. So, after rescaling back, there exists $\bar{t} > 0$ (smaller than t_0) such that

$$\left(\bigcup_{t\in[-\bar{t},\bar{t}]} W_t\right) \cap B_{\varepsilon_{\bar{t}}}(p)$$

forms a cone centered at p, which could collapse to the point p. Denote

$$V := \left(\bigcup_{t\in[-\bar{t},\bar{t}]} W_t\right)$$

After this deformation retraction, the new space is homotopy equivalent to V. Denote by Σ' the Σ under this deformation retract. Because $W_{\bar{t}}$ and $W_{-\bar{t}}$ can be represented by graphs on $\Sigma \setminus B_{\epsilon_{\bar{t}}}$, there consequently exists $u \in C(\overline{\Sigma'})$ with $u \geq 0$ on $\operatorname{Reg}(\Sigma')$ and $u = 0$ at $p := p_\pm$, such that V is a deformation retract of a space X, where

$$X = \{(x,t) : x \in \Sigma' \text{ and } -u(x) \leq t \leq u(x)\}.$$

Therefore, Σ is a deformation retract of X. And thus, Σ is a deformation retract of V.

Denote by $\Gamma_+ := W_{\bar{t}}$ (resp. $\Gamma_- := W_{-\bar{t}}$), the smooth hypersurface in U_+ (resp. U_-). Moreover, note that $\mathbf{C}$ splits $\mathbb{R}^8$ into two parts, by the symmetry of the Simons cone, R_t is diffeomorphic to R_{-t} for all $|t| \leq \bar{t}$. Therefore, Γ_+ and Γ_- are diffeomorphic to each other.

In summary, Σ splits V into two parts: V_+, V_-. And ∂V_+ has two components: the smooth part Γ_+, and Σ (similarly for ∂V_-). Then we have $\partial V = \Gamma_+ \cup \Gamma_-$, where Γ_+, Γ_- are both smooth and homologous to Σ. Moreover, Γ_+ is diffeomorphic to Γ_-. We will use the set V in the next subsection to construct the manifolds with singular isoperimetric regions.

□

3.2.2 Construction of the Toric manifolds

Next, we will construct a collection of 8-dimensional closed Riemannian manifolds, which we will use in the next section. Again we denote V, the smooth neighborhood of Σ from the last subsection.

Theorem 3.2.3. *Denote Γ the 7-manifold which is diffeomorphic to Γ_+ and Γ_- by $F_\pm : \Gamma_\pm \to \Gamma$. Consider the smooth manifold defined by gluing the boundaries of V and $\Gamma \times [0, R]$:*

$$M(R) := \Gamma \times [0, R] \cup V/\Gamma_\pm \times \{0, R\} \sim_{F_\pm} \tilde{\Gamma}_\pm, \tag{3.2.2}$$

where

$$\tilde{\Gamma}_+ := \Gamma \times \{0\}, \qquad \tilde{\Gamma}_- := \Gamma \times \{R\}.$$

There exist $R_0 > 0$ and a one parameter family of C^∞-metrics $(g_R)_{R>0}$ on $M(R)$ such that, if V is as in Theorem 3.2.2, then for every $R > R_0$, the followings hold:

1. *V is isometrically embedded into $(M(R), g_R)$;*

2. *$H_7(M(R); \mathbb{Z}_2) \cong \mathbb{Z}_2 \cong \langle[\Gamma]\rangle \cong \langle[\Sigma]\rangle$, where $[\Gamma], [\Sigma]$ denote the homology classes represented by Γ and Σ respectively;*

3. *Σ is the unique homological area minimizer in $(M(R), g_R)$.*

The idea is the following: suppose h is some Riemannian metric of Γ to be determined. At first, we conformally change (enlarge) the metric g of V in U_δ near Γ_+, Γ_- such that the outside of a tubular neighborhood of Γ_+ (and Γ_-) is isometric to the cylindrical metric $(\Gamma \times [0, \epsilon], h + dr^2)$ for sufficiently small ϵ, . Additionally, we require that Σ is still homological area-minimizing in $(V, \widetilde{g})$, where $\widetilde{g}$ is the metric after the conformal change.

Finally, we glue $(V, \widetilde{g})$ with $(\Gamma \times [0, R], h + dr^2)$ along the boundaries respectively. There is a Riemannian metric on $M(R)$ depending on the length R. And we denote

$$(M(R), g_R) := \Gamma \times [0, R] \cup V/\Gamma_\pm \sim \tilde{\Gamma}_\pm.$$

Proof of Theorem 3.2.3. For each $i = \pm$, consider $U_i(\supset \Gamma_i)$ the Fermi's coordinate (x, t) for $x \in \Gamma_i$, here choosing t with the positive direction towards outside of Σ, then there exists an $\epsilon > 0$ such that

(1) for $|t| \leq \epsilon$, we have $(x, t) \in U_i$ respectively;

(2) there exists an $\epsilon_1 > 0$ such that

$$\inf_{|t| \leq \epsilon} \inf_{x \in \Gamma_\pm} dist((x, t), \Sigma) > \epsilon_1.$$

For each small t, we define

$$\Gamma_i(t) := \{(x, t) : \, x \in \Gamma_i\},$$

again here we define the positive signs for each Γ_i the side opposite from Σ, so each $\Gamma_i(t)$ forms a layer in U_i.

Note that for any $|t| \leq \epsilon$, $\Gamma_+(t), \Gamma_-(t)$ bound an open neighborhood of Σ as well. For $t = -\epsilon$, we denote V_0 the neighborhood of Σ with boundary $\Gamma_+(-\epsilon) \cup \Gamma_-(-\epsilon)$. In another word, $\Gamma_+(-\epsilon), \Gamma_-(-\epsilon)$ split the neighborhood U_ϵ into three parts, and Σ lies in the middle part.

For $-\epsilon < \sigma \leq \epsilon$, denote

$$V_+(\sigma) := \{(x,t) : \ x \in \Gamma_+, \ -\epsilon \leq t < \sigma\};$$
$$V_-(\sigma) := \{(x,t) : \ x \in \Gamma_-, \ -\epsilon \leq t < \sigma\}.$$

We can denote the neighborhood (depending on σ) by

$$V(\sigma) := V_0 \cup V_+(\sigma) \cup V_-(\sigma). \tag{3.2.3}$$

Note that for each small σ, $V(\sigma)$ is a neighborhood of Σ with two smooth boundaries $\Gamma_i(t)$ such that each $\Gamma(t)$ is homologous to Σ. In addition, by Theorem 3.2.2, Σ is uniquely homological area-minimizing in $V(\sigma)$ for any $-\epsilon \leq \sigma \leq \epsilon$.

In addition, we observe that $V := V(0)$. Following the ideas at the beginning of this section, we will glue the smooth neighborhood $V(\epsilon)$ along the boundary with a manifold (with boundary) endowed with a cylindrical metric.

Now, we will keep the metric on V_0 and deform the metrics on $V_\pm(\epsilon)$ to the cylindrical metric (near $\Gamma_\pm(\epsilon)$). Note that for each $i = \pm$, under the Fermi coordinates in $V_i(\epsilon)$, the metric has the form:

$$g(x,t) = g_i(x,t) + \eta_i(x,t)dt^2,$$

where for each small t, $g_i(\cdot, t)$ is a metric on Γ_i respectively, and η_i is a smooth positive function.

In order to construct a deformation of the metric on $V(\epsilon)$, we first define new metrics on $\Gamma_+(\epsilon), \Gamma_-(\epsilon)$. Note that $\Gamma_+(\epsilon), \Gamma_-(\epsilon)$ are both diffeomorphic to Γ. Consider h the Riemannian metric on Γ and the diffeomorphisms $F_\pm$:

$$F_+ : \Gamma_+(\epsilon) \longrightarrow \Gamma(= (\Gamma, h)), \qquad F_- : \Gamma_-(\epsilon) \longrightarrow \Gamma,$$

such that for any $x \in \Gamma_\pm(\epsilon)$ and any $v \in T_x\Gamma_\pm(\epsilon)$, we have

$$F_\pm^* h(v, v) \geq 2g(v, v),$$

where g denotes the original metric in $V(\epsilon)$.

To construct a new metric on $V(\epsilon)$, for $i = \pm$, we define the metric $\overline{g}_i$ on Γ_i and the number $\overline{\eta}_i$ by

$$\overline{g}_i(x) := F_i^*(h)(x), \tag{3.2.4}$$

$$\overline{\eta}_i := \max_{\overline{V}_i(\epsilon)} \eta_i(x, t). \tag{3.2.5}$$

Denote the cutoff function $\phi : \mathbb{R} \to \mathbb{R}$ a smooth non-negative function such that

$$\phi(x) = \begin{cases} 0 & t \leq 0; \\ 1 & t \geq \epsilon/2, \end{cases}$$

Then we can define new metrics $\widetilde{g}_i$ on $V_i(\epsilon)$, where $i = +$ or $-$, by the following:

$$\begin{aligned} \widetilde{g}_i(x, t) =& (1 - \phi(t))g_i(x, t) + \phi(t)\overline{g}_i(x) \\ &+ [(1 - \phi(t))\eta_i(x, t) + \phi(t)\overline{\eta}_i]dt^2. \end{aligned} \tag{3.2.6}$$

So clearly, each of $\widetilde{g}_+$ and $\widetilde{g}_-$ is simply an interpolation such that, as t increases from 0 to $\epsilon/2$, the original metric deforms to the cylindrical metric. Moreover, we leave the metric unchanged when t is non-positive.

Now we can define a new metric $\widetilde{g}$ on the whole neighborhood $V(\epsilon)$ by the following:

$$\widetilde{g}(p) = \begin{cases} g(p), & p \in V_0, \\ \widetilde{g}_i(p), & p \in V_i(\epsilon),\ i = + \text{ or } - . \end{cases}$$

Therefore, we can construct the collection of manifolds $\{(M(R), g_R)\}_R$. Note that $V(\epsilon)$ has cylindrical metric on $V(\epsilon) \setminus V(\epsilon/2)$, and $\Gamma_\pm(\epsilon)$ are both isometric to (Γ, h). Then consider the maps $F_\pm : \Gamma_\pm \to \Gamma$ defined above. Denote $W(R) = [0, R] \times \Gamma$ for $R > 0$ a Riemannian manifold with the boundary equipped with the product metric. Along with the diffeomorphisms $F_\pm$, we glue $\Gamma_+(\epsilon)$ with $\{0\} \times \Gamma$ and glue $\Gamma_-(\epsilon)$ with $\{R\} \times \Gamma$. Then we get a connected smooth Riemannian manifold depending on the positive number R, denote it by $(M(R), g_R)$.

Note that under the Riemannian metric $\widetilde{g}$, (3.2.4)-(3.2.6) show that we leave the metric in V unchanged and enlarge the metric on $V_+(\epsilon), V_-(\epsilon)$. So Σ is still the unique homological area minimizer in $V(\epsilon)$. Moreover, by [40, Lemma 5], we see that for sufficiently large $R > 0$, under the metric defined above, Σ is the unique homological area minimizer in $M(R)$.

Finally, we study the homology group $H_7(M(R), \mathbb{Z}_2)$. It is evident that $\Gamma \times [0, R]$ is homotopy equivalent to Γ. Note that, by our construction, topologically, Σ is a suspension of $\mathbb{S}^3 \times \mathbb{S}^3$, and the smooth hypersurface Γ is $\mathbb{S}^3 \times \mathbb{S}^3 \times [-1, 1]$, with two boundaries filled in by two copies of $\mathbb{S}^3 \times \mathbb{B}^4$. Thus, Γ is homotopy equivalent to the product of $\mathbb{S}^3$ with the suspension of $\mathbb{S}^3$. Therefore, $H_6(\Gamma, \mathbb{Z}_2) = 0$. On the other hand, by Theorem 3.2.2, Σ is a deformation retract of V. Then, by computing the Mayer–Vietoris sequence for M,

$$\cdots \to H_7(\Gamma \sqcup \Gamma) \to H_7(V, \mathbb{Z}_2) \oplus H_7(\Gamma \times [0, R], \mathbb{Z}_2) \to H_7(M(R), \mathbb{Z}_2) \to H_6(\Gamma \sqcup \Gamma) \to \dots$$

We have

$$\begin{aligned} \cdots \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to H_7(M, \mathbb{Z}_2) \to 0 \to \dots \\ (1,0) \mapsto (1,1) \\ (0,1) \mapsto (1,1) \end{aligned}$$

we see that $H_7(M, \mathbb{Z}_2) \cong \mathbb{Z}_2 \cong \langle [\Gamma] \rangle$.

□

3.3 Proof of the Main Theorem 3.1.16

In section 3.2.2, we have constructed a collection of closed manifolds that we need later as ambient spaces. Then, in section 3.3.1, we will construct the singular isoperimetric region in the corresponding manifold. In this section, we will redefine V for convenience by

$$V := V(\epsilon) \tag{3.3.1}$$

where $V(\epsilon)$ is defined in (3.2.3).

3.3.1 Construction of Singular Isoperimetric Region

Theorem 3.3.1. *Let $M(R)$ be the family of manifolds constructed in Theorem 3.2.3. There exists $R_1 > 0$ such that for any $R > R_1$, there is $t_0 \in (0, R)$ such that the boundary of the* *two unique isoperimetric regions (the one and its complement) with volume $|M(R)|_g/2$ is of the form*

$$\Sigma \cup [\Gamma \times \{t_0\}] .$$

This theorem directly implies Theorem 3.1.16.

Example 3.3.2. Consider the torus $\mathbb{S}^7 \times \mathbb{S}^1(R)$ with product metric for large length R. [11] shows that for large R, the boundary of an isoperimetric region with half volume is of the form

$$\left[\mathbb{S}^7 \times \{0\}\right] \cup \left[\mathbb{S}^7 \times \{\pi R\}\right] .$$

To prove Theorem 3.3.1, we want to have a uniform bound on the mean curvature of isoperimetric regions with volume bounded away from zero and that of the manifold. For a closed manifold with dimension $2 \leq n \leq 7$, this lemma is proved in [11]. In the higher dimensional cases, the boundary can have singularities; we want to have the mean curvature bounded in $Reg(\partial\Omega)$. Adapting the argument of [11, Lemma C.1] shows the following lemma:

Lemma 3.3.3 (**cf.** [11, Lemma C.1] and [30, 10])**.** *For $n \geq 2$, fix $\delta > 0$, and (M^n, g) a closed Riemannian manifold with C^3-metric, there is $C = C(M, g, \delta) < \infty$ so that if $\Omega \in \mathcal{C}(M, t)$ is an isoperimetric region with volume $|\Omega|_g \in (\delta, |M|_g - \delta)$, then the mean curvature of $Reg(\partial\Omega)$ satisfies $|H| \leq C$.*

Proof. [11, Lemma C.1] proves the case $2 \leq n \leq 7$. Next, we assume $n \geq 8$, the proof is similar to [11, Lemma C.1].

Assuming not, we would have a sequence of isoperimetric regions $\Omega_j \subset (M, g)$ with $|\Omega_j|_g \in (\delta, |M|_g - \delta)$ with divergent constant mean curvature H_j such that $\lambda_j := |H_j| \to \infty$.

Choosing any $x_j \in \partial\Omega_j$, we can rescale the metric in λ_j at the point x_j, denote $\widetilde{g}_j = \lambda_j^2 g$. So we have $(M, \widetilde{g}_j, x_j)$ converges in C^3_{loc} to the flat metric on $(\mathbb{R}^n, 0)$. Also, for each j, we have $\widetilde{\Omega}_j$ an isoperimetric region in $(M, \widetilde{g}_j)$. Passing to a subsequence, there is a locally isoperimetric region[1] $\widetilde{\Omega}$ in $\mathbb{R}^n$ such that $\widetilde{\Omega}_j$ converges to $\widetilde{\Omega}$ in the local Hausdorff sense, and $|\partial^*\widetilde{\Omega}_j| \to |\partial^*\widetilde{\Omega}|$ locally in the varifold sense. Therefore, for any $p \in Reg(\partial\widetilde{\Omega})$, there exists a neighborhood $B_r(p)$ around p for some $r > 0$ such that $\partial\widetilde{\Omega}_j \cap B_r(p) \subset Reg(\partial\widetilde{\Omega}_j)$, and therefore $Reg(\partial\widetilde{\Omega}_j) \cap B_r(p)$ converges in $C^{2,\alpha}$ to $Reg(\partial\widetilde{\Omega}) \cap B_r(p)$. Therefore, the mean curvature of $\widetilde{\Omega}$ is ± 1. Moreover, because $\widetilde{\Omega}$ is a locally isoperimetric region, $\partial\widetilde{\Omega}$ is thus volume-preserving stable.

Next, we claim that there exists a sequence of choices of $x_j \in \partial\Omega_j$ such that the boundary of the limit space $\partial\widetilde{\Omega}$ is non-compact. Suppose not, and assume that $\partial\widetilde{\Omega}$ is compact for any choices of $x_j \in \partial\Omega_j$. Then, either $\widetilde{\Omega}$ or its complement is compact and must be an isoperimetric region in $\mathbb{R}^n$. By [25, Theorem 14.1] (or Example 3.1.9), it must be a ball. Given the mean curvature $|\widetilde{H}| = 1$, $\widetilde{\Omega}$ or its complement $\widetilde{\Omega}^c$ is necessarily a unit ball.

Therefore, $\partial\Omega_j$ will also be regular. Because the only possible $\widetilde{\Omega}$ is the unit ball or its complement, $(\partial\widetilde{\Omega}_j, \widetilde{g}_j)$ would be composed of a union of connected components, each approximating a unit geodesic sphere.

For λ_j sufficiently large, $\frac{1}{2\lambda_j}$ will be the lower bound of the diameters of each connected component (almost a geodesic sphere) of $\partial\Omega_j$. Without loss of generality, we assume M is connected. Therefore, either $\widetilde{\Omega}_j$ consists of a union of balls, or $\widetilde{\Omega}_j^c$ consists of a union of balls. Consider the first case and denote by $V_1 > 0$ the upper bound of the volumes of $(\Omega_j)_j$. Thus, for any sufficiently large j, $\widetilde{\Omega}_j$ contains at most $V_1\lambda_j^n$ balls. Consequently, we have

$$\mathbf{P}(\Omega_j) \geq C(n)V_1\lambda_j^n\mathbf{P}(B_{\lambda_j}) \geq C(n)\frac{V_1}{\lambda_j^{n-1}}\lambda_j^n.$$

Therefore, as $\lambda_j \to \infty$, $\mathbf{P}(\Omega_j) \to \infty$, which is a contradiction. In the second case, where

[1]We say Ω is a locally isoperimetric region if for any $R > 0$ and $\widetilde{\Omega}$ with $\Omega\Delta\widetilde{\Omega} \Subset B_R$ and $|\Omega \cap B_R| = |\widetilde{\Omega} \cap B_R|$, we have $\mathbf{P}(\Omega, B_R) \leq \mathbf{P}(\widetilde{\Omega}, B_R)$.

$\widetilde{\Omega}_j^c$ consists of a union of balls, we may consider $V_2 > 0$ as the uniform lower bound of the volumes of $(\Omega_j)_j$ and similarly obtain a contradiction. Therefore, we can assume that $\partial\widetilde{\Omega}$ is non-compact.

Denote $\widetilde{H} :=$ the mean curvature of $Reg(\partial\widetilde{\Omega})$, and $\widetilde{A} :=$ the second fundamental form of $Reg(\partial\widetilde{\Omega})$. Because $|\widetilde{H}| = 1$, we have $|\widetilde{A}|^2 \geq \frac{1}{n}$. Because $\widetilde{\Omega}$ is a locally isoperimetric region, $\partial\widetilde{\Omega}$ is volume preserving stable (see Definition 3.1.12). Therefore, for any $\phi \in C_c^1(Reg(\partial\widetilde{\Omega}))$ with $\int_{Reg(\partial\widetilde{\Omega})} \phi \, d\mathcal{H}^{n-1} = 0$, we have

$$\int_{Reg(\partial\widetilde{\Omega})} \phi^2 \, d\mathcal{H}^{n-1} \leq \int_{Reg(\partial\widetilde{\Omega})} n|\widetilde{A}|^2 \phi^2 \, d\mathcal{H}^{n-1} \leq n \int_{Reg(\partial\widetilde{\Omega})} |\nabla\phi|^2 \, d\mathcal{H}^{n-1}. \tag{3.3.2}$$

Following the remark from [16] and [4, Proposition 2.2], we claim that $\partial\widetilde{\Omega}$ is **strongly** stable outside of a compact set, i.e., there exists a large enough $R > 0$ such that for any $\phi \in C_c^1(Reg(\partial\widetilde{\Omega}) \setminus B_R)$, we have

$$\int_{Reg(\partial\widetilde{\Omega})} \phi^2 \, d\mathcal{H}^{n-1} \leq n \int_{Reg(\partial\widetilde{\Omega})} |\nabla\phi|^2 \, d\mathcal{H}^{n-1}. \tag{3.3.3}$$

Assume not, there is $R_1 > 0$ and some $\phi_1 \in C_c^1(Reg(\partial\widetilde{\Omega}) \setminus B_{R_1})$ such that (3.3.3) fails for $\phi := \phi_1$. Denote $R_2 > 0$ such that $\operatorname{spt}\phi_1 \Subset B_{R_2}$, then there exists $\phi_2 \in C_c^1(Reg(\partial\widetilde{\Omega}) \setminus B_{R_2})$ such that for $\phi := \phi_2$. Denote $\phi_3 := c_1\phi_1 + c_2\phi_2$ for some $c_1, c_2 \in \mathbb{R}$ such that $\int_{Reg(\partial\widetilde{\Omega})} \phi_3^2 \, d\mathcal{H}^{n-1} = 0$. So we also have

$$\int_{Reg(\partial\widetilde{\Omega})} \phi_3^2 \, d\mathcal{H}^{n-1} > n \int_{Reg(\partial\widetilde{\Omega})} |\nabla\phi_3|^2 \, d\mathcal{H}^{n-1},$$

which contradicts (3.3.2). Thus we prove the claim.

Consider the C^1 radial cutoff function ϕ in $\mathbb{R}^n$ such that: fixing any $\rho > R+1$,

$$\phi(x) = \begin{cases} 1 & |x| \in [R+1, \rho], \\ 0 & |x| \in [0, R] \cup [2\rho, \infty]. \end{cases}$$

with $|D\phi(x)| \leq C(n)\rho^{-1}$ for $|x| > \rho$ and $|D\phi(x)| \leq C(R)$ where $C(R)$ depending on R, and D is the Euclidean connection on $\mathbb{R}^n$. Because $\widetilde{\Omega}$ is a locally isoperimetric region, we have $\mathcal{H}^{n-3}(Sing(\partial\widetilde{\Omega})) = 0$. Given any $\epsilon > 0$, consider $\{B_{r_j}(p_j)\}_j$ a collection of geodesic balls which cover $Sing(\partial\widetilde{\Omega})$, such that

$$\sum_j r_j^{n-3} < \epsilon.$$

We define ψ_j a smooth cutoff function by

$$\psi_j(x) = \begin{cases} 1 & \text{if } x \notin B_{2r_j}(p_j), \\ 0 & \text{if } x \in B_{r_j}(p_j). \end{cases}$$

with $|D\psi_j| \leq C(n)r_j^{-1}$. Now we define

$$\psi_\epsilon := \inf_j \psi_j$$

$$\Phi_\epsilon := (\phi)^{\frac{n-1}{2}} \cdot \psi_\epsilon.$$

So we note that Φ_ϵ is a Lipschitz compactly supported function on $Reg(\partial\widetilde{\Omega})$. So by (3.3.3), we have

$$\int_{Reg(\partial\widetilde{\Omega})} \Phi_\epsilon^2 \, d\mathcal{H}^{n-1} \le n \int_{Reg(\partial\widetilde{\Omega})} |\nabla \Phi_\epsilon|^2 \, d\mathcal{H}^{n-1}$$
$$\le 2n \int_{Reg(\partial\widetilde{\Omega})} |D\phi^{\frac{n-1}{2}}|^2 \cdot \psi_\epsilon^2 + \phi^{n-1} \cdot |D\psi_\epsilon|^2 \, d\mathcal{H}^{n-1}$$
$$= 2n \int_{Reg(\partial\widetilde{\Omega})} \left(\frac{n-1}{2}\right)^2 \phi^{n-3}|D\phi|^2 \cdot \psi_\epsilon^2 + \phi^{n-1} \cdot |D\psi_\epsilon|^2 \, d\mathcal{H}^{n-1}$$
$$= \int_{Reg(\partial\widetilde{\Omega})} \left(\frac{n-3}{n-1} \cdot \phi^{n-1} + C(n)|D\phi|^{n-1}\right) \cdot \psi_\epsilon^2 \, d\mathcal{H}^{n-1} + 2n \int_{Reg(\partial\widetilde{\Omega})} \phi^{n-1} \cdot |D\psi_\epsilon|^2 \, d\mathcal{H}^{n-1}.$$

Therefore, abusing the notations of constants $C(n)$, we have

$$\begin{aligned} \int_{Reg(\partial\widetilde{\Omega})} \Phi_\epsilon^2 \, d\mathcal{H}^{n-1} \le & C(n) \int_{Reg(\partial\widetilde{\Omega})} |D\phi|^{n-1} \cdot \psi_\epsilon^2 \, d\mathcal{H}^{n-1} + \\ & C(n) \int_{Reg(\partial\widetilde{\Omega})} \phi^{n-1} \cdot |D\psi_\epsilon|^2 \, d\mathcal{H}^{n-1}. \end{aligned} \tag{3.3.4}$$

For the first part of the right hand side of (3.3.4), by the definition of ϕ, ψ_ϵ, we have

$$\int_{Reg(\partial\widetilde{\Omega})} |D\phi|^{n-1} \cdot \psi_\epsilon^2 \, d\mathcal{H}^{n-1} \le \rho^{1-n} \mathcal{H}^{n-1}(\partial\widetilde{\Omega} \cap B_{2\rho}) + C(R)\mathcal{H}^{n-1}(\partial\widetilde{\Omega} \cap A_{R,R+1}),$$

where $A_{R,R+1} := B_{R+1} \setminus \overline{B}_R$. For the second part of (3.3.4), we have

$$\begin{aligned} \int_{Reg(\partial\widetilde{\Omega})} \phi^{n-1} \cdot |D\psi_\epsilon|^2 \, d\mathcal{H}^{n-1} &\le \sum_j \int_{Reg(\partial\widetilde{\Omega}) \cap B_{2r_j}(p_j)} r_j^{-2} \, d\mathcal{H}^{n-1} \\ &\le \sum_j r_j^{-2} \cdot C r_j^{n-1} \\ &\le C\epsilon. \end{aligned}$$

Here the volume bound comes from the monotonicity formula [37, 17.6], i.e., we have

$$\mathcal{H}^{n-1}(Reg(\partial\widetilde{\Omega}) \cap B_{2r_j}(p_j)) \le C r_j^{n-1}$$

for the constant C depending on $|\widetilde{H}|$, which is constantly 1 in our case.

Therefore, as $\epsilon \to 0$, the dominated convergence theorem implies that

$$\mathcal{H}^{n-1}(\partial\widetilde{\Omega} \cap (B_\rho \setminus B_{R+1})) \le C(1 + \rho^{1-n}\mathcal{H}^{n-1}(\partial\widetilde{\Omega} \cap B_{2\rho})), \tag{3.3.5}$$

where the constant C is independent with ρ. Note that as $\rho \to \infty$, and $\partial\widetilde{\Omega}$ is not compact, we can cover $\partial\widetilde{\Omega}$ by a countable collection of geodesic balls $(B_j)_j$ such that the concentric balls in $(B_j)_j$ with the half radius are pairwise disjoint. Then because $|\widetilde{H}| = 1$, by the monotonicity formula (from below), we have that

$$\mathcal{H}^{n-1}(\partial\widetilde{\Omega} \cap (B_\rho \setminus B_R)) \to \infty.$$

On the other hand, note that $\widetilde{\Omega}$ is a locally isoperimetric region. For any ρ large, consider $0 < r(\rho) \le \rho$ with the property that

$$|\widetilde{\Omega} \cap B_\rho| = |(\widetilde{\Omega} \setminus B_\rho) \cup B_{r(\rho)}|,$$

i.e., $[\widetilde{\Omega} \cap B_\rho]\Delta[(\widetilde{\Omega} \setminus B_\rho) \cup B_{r(\rho)}] \Subset B_{\rho+\epsilon}$ and have the same volume in $B_{\rho+\epsilon}$ for any $\epsilon > 0$. Because $\widetilde{\Omega}$ is a local isoperimetric region, so for any large ρ, we have

$$\mathbf{P}(\widetilde{\Omega}; B_{\rho+\epsilon}) \le \mathbf{P}((\widetilde{\Omega} \setminus B_\rho) \cup B_{r(\rho)}; B_{\rho+\epsilon}).$$

Denote $E := \widetilde{\Omega} \setminus B_\rho, F := B_{r(\rho)}$, $G := B_{\rho+\epsilon}$, and ν_E, ν_F the outer normal vector fields on the reduced boundaries. By the set operations on Gauss–Green measures (Theorem 2.2.7), we have

$$\begin{aligned}\mathbf{P}(E\cup F;G)=&\mathbf{P}((\widetilde{\Omega}\setminus B_\rho)\cup B_{r(\rho)};B_{\rho+\epsilon})\\=&\mathbf{P}(\widetilde{\Omega}\setminus B_\rho;B^{(0)}_{r(\rho)}\cap B_{\rho+\epsilon})+\mathbf{P}(B_{r(\rho)};(\widetilde{\Omega}\setminus B_\rho)^{(0)}\cap B_{\rho+\epsilon})\\&+\mathcal{H}^{n-1}(\{\nu_E=\nu_F\}\cap B_{\rho+\epsilon})\\\leq&\mathbf{P}(\widetilde{\Omega}\setminus B_\rho;B_{\rho+\epsilon}\setminus\overline{B_{r(\rho)}})+C\rho^{n-1}\end{aligned}$$

As $\epsilon\to 0^+$, we get

$$\mathbf{P}(\widetilde{\Omega};B_\rho)\leq C\rho^{n-1}.$$

So we have a contradiction with (3.3.5). □

Lemma 3.3.4. *Let $\Omega_R\in(M(R),g_R)$ be the isoperimetric region with volume $|M(R)|/2$, and let $\partial\Omega_R=\bigcup_{i=1}^L I_i$, where $\{I_i\}$ are the connected components of $\partial\Omega_R$. Then there exists a nonnegative constant $R_0=R_0(V)$ (V is defined in (3.3.1)), such that for any $R>R_0$, the following hold:*

1. $\mathbf{P}(\Omega_R)\leq\mathbf{M}(\Gamma)+\mathbf{M}(\Sigma)$;
2. *each I_i has uniformly bounded diameter;*
3. *there exists an integer $L_0=L_0(V,R_0)>1$ such that the number of connected components satisfies $1<L<L_0$;*
4. *I_1 and I_2 are homologous to $[1]:=[\Gamma]\in H_7(M(R),\mathbb{Z}_2)$.*

Proof. (1) Note that

$$\mathbf{P}(\Omega_R)\leq\mathbf{M}(\Gamma)+\mathbf{M}(\Sigma),\tag{3.3.6}$$

because there is clearly a $U_R\in C(M(R),g_R,|M(R)|_{g_R}/2)$ such that $U_R=V_+\cup(\Gamma\times[0,t_0])$

for some t_0, which has half volume and with the boundary

$$\big[\{t_0\}\times\Gamma\big]\cup\Sigma.$$

(2) Next suppose that there is a sequence $R_k\to\infty$, denote with $\Omega_k\subset(M(R_k),g_k)$ the corresponding isoperimetric regions with $|\Omega_k|_{g_k}=\frac{1}{2}|M(R_k)|_{g_k}$. Here we denote $g_k:=g_{R_k}$. Suppose $I_k\in\partial\Omega_k$ is a component such that $\operatorname{diam}(I_k)\to\infty$. We observe that all the metrics g_R are locally isometric, so by Lemma 3.3.3, we have a uniform bound of mean curvature for $Reg(\partial\Omega_k)$ and all $k\in\mathbb{N}$. Then monotonicity formula [1, 10] shows that there exists $r_0>0$, for any $x\in\partial\Omega_k$, we have

$$f(r):=e^{7(\|H_k\|_\infty+C)r}\frac{\mathbf{P}(\Omega_k;B_r(x))}{\omega_7 r^7}, \tag{3.3.7}$$

is non-decreasing for $r\in(0,r_0]$. Where C depends on the upper bound of sectional curvatures of $M(R_k)$ and r_0 depends on both the sectional curvatures bound and the injectivity radius (so they are independent with R_k), H_k is the mean curvature of $Reg(\partial\Omega_k)$, and ω_7 the volume of a unit ball of dimension 7.

If $\operatorname{diam}(I_k)\to\infty$, there must exist $t_k\geq 0$ and $T_k\to\infty$ such that I_k intersects with $\{t\}\times\Gamma$ for all $t\in[t_k,t_k+T_k]$. Thus we can cover $I_k\cap([t_k,t_k+T_k]\times\Gamma)$ by balls with radius r_0 and half radius are pairwise disjoint, the cover contains at least $\frac{T_k}{2r_0}$ many balls. Then by the monotonicity formula (3.3.7), we have $\mathbf{M}(I_k)\to\infty$. Hence, we obtain a contradiction with (1).

(3) Given any sequence $R_k\to\infty$, first, suppose that the isoperimetric regions are connected. Note that by (1) and (2):

$$\mathbf{P}(\Omega_k)\leq\mathbf{M}(\Gamma)+\mathbf{P}(\Sigma), \tag{3.3.8}$$

and they have a uniform bound of diameters. So for each k large, we have

$$\partial\Omega_k \subset M(R_k) \setminus \left([\frac{1}{8}R_k, \frac{7}{8}R_k] \times \Gamma \right),$$

therefore, $|\Omega_k|$ cannot be equal to $\frac{1}{2}|M(R_k)|$. This leads to a contradiction.

In addition, by the uniform bound of the mean curvatures for Ω_k with all large R, the monotonicity formula (3.3.7) implies there exists a $c = c(V)$ such that the perimeter of each component of $\partial\Omega_R$ is bounded below by c. By (3.3.6), we get a uniform bound L about the number of components of $\partial\Omega$ for all large R.

(4) Denote $\partial\Omega_R = \bigcup_i I_i$. Suppose (4) fails. Note that by Theorem 3.2.3 (2), $H_7(M;\mathbb{Z}_2) \cong \langle[\Gamma]\rangle$, then by the boundedness of diameter, all $\{I_i\}$ are boundaries. Reasoning as in the first part of (3), there exists $(t_i)_i$ and $(T_i)_i$ such that

$$I_i \setminus V \subset \mathcal{T}_i := [t_i, t_i + T_i] \times \Gamma, \qquad \text{and} \qquad T_i < T_0 < \infty,$$

for every $i = 1, \dots, L$, where T_0 is independent of i and exists by (2). In particular, since each I_i is a boundary, we have

$$\Omega_R \subset V \cup \bigcup_{i=1}^{L} \mathcal{T}_i.$$

Therefore we conclude

$$\mathrm{Vol}(\Omega_R) \leq \mathrm{Vol}\left(V \cup \bigcup_{i=1}^{L} \mathcal{T}_i \right) \leq \mathrm{Vol}(V) + \sum_{i=1}^{L} T_0 \cdot \mathbf{M}(\Gamma).$$

which for sufficiently large R cannot be half the volume of the whole manifold. □

Proof of Theorem 3.3.1. We still denote Ω_R the isoperimetric region in $M(R)$ with half volume. In order to prove the isoperimetric region is of the form we want, we will consider the connected components of $\partial\Omega_R$ into two types:

$$\{\Lambda_i\} := \{\Lambda_i \subset \partial\Omega_R : \Lambda_i \cap \overline{V} = \emptyset\};$$
$$\{\Delta_i\} := \{\Delta_i \subset \partial\Omega_R : \Delta_i \cap \overline{V} \neq \emptyset\}.$$

Combining the results in Lemma 3.3.4, we claim the following result for isoperimetric regions $\partial\Omega_R$ with large R.

Claim: With R_0 from Lemma 3.3.4, there exists $R_1 > R_0$, such that for any $R > R_1$, $\partial\Omega_R$ has exactly 2 components Λ, Δ, where $\Lambda \in \{\Lambda_i\}$ and $\Delta \in \{\Delta_i\}$. Specifically, $[\Lambda] = [\Delta] = [1]$ where $[1] := [\Sigma] \in H_7(M(R), \mathbb{Z}_2)$.

Proof of the claim:

Step 1. $\{\Delta_i\} \neq \emptyset$ and at least one Δ_i is homologous to Σ.

Suppose not, then $I_1, I_2 \subset [0,T] \times \Gamma$, where I_1, I_2 are as in Lemma 3.3.4(4). Since $[I_1] = [I_2] = [1] \in H_7(M(R), \mathbb{Z}_2)$ and $I_1, I_2 \in \{\Lambda_i\}$, this implies that

$$\mathbf{P}(\Omega_R) \geq \mathbf{M}(I_1) + \mathbf{M}(I_2) \geq 2\mathbf{M}(\Gamma) > \mathbf{M}(\Sigma) + \mathbf{M}(\Gamma) \overset{(1)}{\geq} \mathbf{P}(\Omega_R)\,.$$

A contradiction.

Step 2. $\{\Lambda_i\} \neq \emptyset$ and at least one Λ_i is homologous to Σ.

Suppose not, then we have that $\partial\Omega_R = \bigcup_{i=1}^{L_1} \Delta_i \cup \bigcup_{i=1}^{L_2} \Lambda_i$, with $L_2 = 0$ if $\{\Lambda\} = \emptyset$; and in the other case, each $\Lambda_i = \partial U_i$, for some open connected $U_i \subset \Omega_R$. Moreover, by Lemma 3.3.4(3), $\text{diam}(\Delta_i) < d_0, \text{diam}(\Lambda_i) < d_0$ and $L_1 + L_2 < L_0$, with L_0, d_0 independent of R. Now notice that since $\Delta_i \cap V \neq \emptyset$, there exist $T_0 > 0$, depending only on d_0, such that

$$\Omega_R \subset \Big(V \cup [(0, T_0) \cup (R - T_0, R)] \times \Gamma\Big) \cup \bigcup_{i=1}^{L_2} U_i\,,$$

with $\text{Vol}(U_i) \leq T_0 \cdot \Gamma$.This yields

$$\text{Vol}(\Omega_R) \leq (L_2 + 2) \cdot T_0 \cdot \mathbf{M}(\Gamma)\,,$$

which leads a contradiction for R sufficiently large.

Step 3. $\partial\Omega_R$ has no other components.

We have concluded that there is at least one $\Delta \in \{\Delta_i\}$ and at least one $\Lambda \in \{\Lambda_i\}$ such that

$$[\Delta] = [1] \in H_7(M(R), \mathbb{Z}_2),$$
$$[\Lambda] = [1] \in H_7(M(R), \mathbb{Z}_2).$$

Clearly $\Lambda \subset [0, R] \times \Gamma$ where each slice $\{t\} \times \Gamma$ is a homological area-minimizing in the tube $Tube(R)$. So we directly have

$$\mathbf{M}(\Lambda) \geq \mathbf{M}(\Gamma).$$

On the other hand, we know that Σ is the unique homological area minimizer in $M(R)$. So clearly

$$\mathbf{M}(\Delta) \geq \mathbf{M}(\Sigma).$$

So overall, by the perimeter upper bound (3.3.6), the only case that would happen is

$$\partial\Omega_R = [\{t_0\} \times \Gamma] \cup \Sigma.$$

We have two isolated singularities on Σ, so we prove Theorem 3.1.16. □

3.4 Proof of Theorem 3.1.18

The proof of Theorem 3.1.18 strongly relies on the construction of singular homological area minimizers in higher dimensions, i.e., the following lemma from [41]:

Lemma 3.4.1 (cf. [41, Lemma 1]**).** *Suppose (M, g_0) a smooth,closed Riemannian manifold of dimension $n+1$, with $n \geq 7$, $\Sigma \subset M$ an orientable hypersurface with $Sing(\Sigma)$ of Hausdorff dimension less or equal to $n-7$. In addition, there exists $\sigma > 0$ such that $\mathcal{N}(\sigma)$, the tubular neighborhood of $Sing(\Sigma)$ is the finite disjoint union $\mathcal{N}(\sigma) = \bigcup_{i=1}^{k} \mathcal{N}_i(\sigma)$, and assume that there are isometrics:*

$$\Phi : \mathcal{N}(\sigma) \to (\mathbb{B}^{n_i+1}(\sigma) \times \Lambda_i, g_{eucl} + h_i).$$

where (Λ_i, h_i) is a compact Riemannian manifold of dimension k_i, and $n_i + k_i = n$, $n_i \geq 7, k_i \geq 0$. Furthermore, assume that

$$\Phi_i(\Sigma \cap \mathcal{N}(\sigma)) = \mathbf{C_i}(\sigma) \times \Lambda_i,$$

where $\mathbf{C_i}$ is any strictly stable, strictly minimizing, regular hypercone in $\mathbb{R}^{n_i+1}$. Then, there exists a metric g on M, with $g \equiv g_0$ on $\mathcal{N}(\sigma_1)$ for some $\sigma_1 < \sigma$, and $\delta > 0$, such that Σ is the unique, homologically area minimizing current in $U(\delta)$ relative to the metric g.

Proof of Theorem 3.1.18. At first, we construct a singular homological area minimizer Σ as described in [41]. Fix $n \geq 7$ and $p \geq 3$. We arbitrarily choose M a smooth closed $(n+1)-$manifold with S a smooth connected, oriented, embedded hypersurface, representing a nontrivial element of $H_n(M, \mathbb{Z})$. Denote $\mathbf{C} := \mathbf{C}^{p,p}$ the Simons' cone in $\mathbb{R}^{2p+2}$. We first construct a Riemannian metric g on M and a singular hypersurface Σ in (M, g) such that Σ is homologous to S. Denote $\mathcal{B}$ an open set of M such that $p \in \mathcal{B}$ for some $p \in S$, and S divides $\mathcal{B}$ into two parts $\mathcal{B}_{\pm}$.

Next, we will put a "cap" on the Simons' cone to make it compact. As constructed in [41, Proposition], note that $\partial \mathbf{C}(1) := \partial \mathbf{C} \cap \mathbb{B}^{2p+2}(1)$ is a compact embedded $2p$-manifold, and so it bounds a smooth compact $(2p+1)$-manifold Y. So $C(1) \cup Y$ is piecewisely smooth (away from $\{0\}$). We can approximate $\mathbf{C}(1) \cup Y$ to a compact hypersurface without boundary, denoted by $\hat{\mathbf{C}} \subset \mathbb{R}^{2p+2}$, such that $\hat{\mathbf{C}}$ is smoothly embedded in $\mathbb{R}^{2p+2}$ except at the origin, and a $\sigma > 0$, such that $\hat{\mathbf{C}} \cap \mathbb{B}^{2p+2}(\sigma) = \mathbf{C} \cap \mathbb{B}^{2p+2}(\sigma)$. Furthermore, $\hat{\mathbf{C}}$ is contained in the unit ball (by scaling if needed).

On the other hand, $\mathbb{S}^{n-2p-1}$ is embedded into $\mathbb{R}^{n+1}$ with a trivial $(2p+2)$-normal bundle. So we have an embedding map[2]

$$\overline{\mathbb{B}}^{2p+2} \times \mathbb{S}^{n-2p-1} \to \mathbb{R}^{n+1}. \tag{3.4.1}$$

Theorefore, there is an embedding $\Psi : \mathbb{B}^{2p+2} \times \mathbb{S}^{n-2p-1} \to \mathcal{B}_+$. Denote $\hat{\Sigma} := \Psi(\hat{\mathbf{C}} \times \mathbb{S}^{n-2p-1})$. Let D be a n-disc in $\hat{\Sigma}$ which is in the image of the annulus $\mathbb{A}^{2p+2}(0, \frac{1}{2}, 1) \times \mathbb{S}^{n-2p-1}$, and let D' be a n-disc in $S \cap \mathcal{B}$. Delete D and D' and smoothly connect S and $\hat{\Sigma}$ by a handle (i.e., a hypersurface in N diffeomorphic to an $n-1$ sphere times an interval) in $\mathcal{B}_+$, and gluing the boundaries of the handle with $D \cup D'$. Denote Σ be the resulting hypersurface in M. Note that Σ is homologous to S. Finally, we will define a metric g_0 in M. For points in $\Psi(\mathbb{B}^{2p+2} \times \mathbb{S}^{n-2p-1})$, we require g_0 the the product metric by the pullback metric with Ψ^{-1}. Therefore, we can assign a metric g_0 on M such that $g_0 = (\Psi^{-1})^*(g_{eucl} + g_S)$ on $\Psi(\mathbb{B}^{2p+2}(\frac{1}{2}) \times \mathbb{S}^{n-2p-1})$. Therefore, we get a (M, g_0) and Σ that satisfy the hypotheses of Lemma 3.4.1. So there exists a Riemannian metric g on M and a $\delta > 0$ such that Σ is homological area minimizing in (U_δ, g).

Then we can do the same argument as in Theorem 3.2.2 to get V a smooth neighborhood of Σ. Moreover, we need $V \subset U_\delta$, and $\partial V := \Gamma_+ \cup \Gamma_-$ for two smooth

[2]As remarked in [41], we can replace sphere to any smooth, connected, compact, orientable Λ such that there is an embedding $\overline{\mathbb{B}}^{2p+2} \times \Lambda \to \mathbb{R}^{n+1}$ with $\dim \Lambda + 2p + 2 = n + 1$.

hypersurfaces Γ_+, Γ_- which are homologous to Σ. Since Σ is orientable, for δ small, Σ splits U_δ into two parts, denoted by U_+ and U_-. Consider $\mathcal{N}(\sigma/8)$ the tubular neighborhood of $Sing(\Sigma)$. Now consider (x, t) the Fermi coordinate on M for $x \in \Sigma \setminus \mathcal{N}(\sigma/8)$ and $|t| < \delta$. We denote $\rho(x) := d_M(x, Sing(\Sigma))$, $\Sigma_\sigma = \{q \in \Sigma : \rho(q) < \sigma\}$, and the constant graph on $\Sigma \setminus \Sigma_{\sigma/8}$:

$$\Gamma_t = graph_{\Sigma \setminus \Sigma_{\sigma/8}} t,$$

with $|t| < \delta$.

We need to construct smooth barriers near $Sing(\Sigma)$. Denote

$$\Lambda_t := \{(x, t) : \rho(x, 0) = \sigma/4\}.$$

Λ_t bounds an area minimizing n-current R_t lies in $\mathcal{N}(\sigma/4)$. Consider the isometry

$$\Phi : (\mathcal{N}(\sigma/4), g) \to (\mathbb{B}^{2p+2}(\sigma/4) \times \mathbb{S}^{n-2p-1}, g_{eucl} + g_S).$$

So $\Phi(R_t)$ is area minimizing in $\mathbb{B}^{2p+2}(\sigma/2) \times \mathbb{S}^{n-2p-1}$. Furthermore, by [21, Theorem 2.1], we have

$$\partial\Phi(R_t) = graph_{\partial C(\sigma/4)} t \times \mathbb{S}^{n-2p-1}, \quad \Phi(R_t) = S_t \times \mathbb{S}^{n-2p-1},$$

where we denote $S_t \subset \mathbb{B}^{2p+2}(\sigma/2)$ the area minimizing n-current with the boundary

$$graph_{\partial C(\sigma/4)} t.$$

By [21, Theorem 2.1, 5.6], as $t \to 0$, $R_t \to \Sigma_{\sigma/4}$ in the current and Hausdorff sense, and there exists $\epsilon_t > 0$, a C^2 function u_t on $\Sigma_{\sigma/4} \setminus \Sigma_{\epsilon_t}$ such that $R_t \setminus \mathcal{N}(\epsilon_t)$ can be described as a graph of u_t.

On the other hand, we can find a smooth cutoff function χ supported on the annulus $\Sigma_{\sigma/4} \setminus \Sigma_{\epsilon_t}$ such that

$$\chi(x) = \begin{cases} 1 & \text{for } \rho(x) \geq \sigma/5, \\ 0 & \text{for } \rho(x) \leq \sigma/7. \end{cases}$$

We can get a smooth hypersurface by gluing the smooth submanifold Γ_t with R_t through a function w on $\Sigma \setminus \Sigma_{\sigma/8}$ by

$$w = \chi u_t + (1-\chi)t.$$

Therefore, similar to Theorem 3.2.2, for each side of Σ, we can employ any small positive (resp. negative) t to find a smooth hypersurface, denoted by Γ_+ (resp. Γ_-) in U_+ (resp. U_-), which is homologous to Σ. Denote V the neighborhood of Σ bounded by Γ_+, Γ_-, V clearly has a smooth boundary. Moreover, by the symmetry of $\mathbf{C}^{p,p}$, Γ_+, Γ_- are diffeomorphic to each other. Denote Γ the $(2p+1)$-manifold which is diffeomorphic to Γ_+ and Γ_- by $F_\pm : \Gamma_\pm \to \Gamma$. Consider the smooth manifold defined by gluing the boundaries of V and $\Gamma \times [0, R]$:

$$M(R) := \Gamma \times [0, R] \cup V / \Gamma_\pm \times \{0, R\} \sim_{F_\pm} \tilde{\Gamma}_\pm.$$

Then using the same argument as Theorem 3.2.3, there exists $R_1 > 0$ such that for any $R > R_1$, we get Riemannian metrics g_R on $M(R)$ such that Σ is the unique homological area minimizer in $(M(R), g_R)$. Then Theorem 3.3.1, Lemma 3.3.3, and Lemma 3.3.4 implies that for sufficiently large $R > 0$, the boundary of the two unique isoperimetric regions (the one and its complement) with volume $|M(R)|_{g_R}/2$ is of the form

$$\Sigma \cup [\Gamma \times \{t_0\}].$$

So we get a singular isoperimetric region in any higher dimension. □

Chapter 4

Locally Stable CMC Hypersurfaces

4.1 Properties of LSCMC Hypersurfaces

In this section, we assume that (M^{n+1}, g) is a closed (i.e. compact with empty boundary) Riemannian manifold; and denote Σ by C^2 hypersurface in M (may not be complete). At first, we will define the locally stable constant mean curvature hypersurfaces, which are the main objects of our study in this chapter.

Definition 4.1.1. Suppose Σ is a 2-sided hypersurface with the unit normal bundle ν. We say that Σ is **stable** in an open set $U \subset M$ if

$$\int_\Sigma |\nabla\phi|^2 \geq \int_\Sigma (|A_\Sigma|^2 + Ric(\nu))\phi^2.$$

for any $\phi \in C_c^1(\Sigma \cap U)$.

We say that Σ is **weakly stable** in an open set $U \subset M$ if

$$\int_\Sigma |\nabla\phi|^2 \geq \int_\Sigma (|A_\Sigma|^2 + Ric(\nu))\phi^2.$$

for any $\phi \in C_c^1(\Sigma \cap U)$ with $\int_\Sigma \phi = 0$.

Remark 4.1.2. Note that we are associating the stability inequality (or more precisely, its associated bilinear form) for the second variation of area with an integral zero space in order to capture a suitable notion of stability for constant mean curvature hypersurfaces (see, for example, [3] and [6]). An obvious example is that Euclidean spheres are weakly stable but not stable.

Unlike the immersion of CMC hypersurfaces discussed in Chapter 1, here we allow the hypersurfaces to have singularities. We will introduce the concept of singularity with the simplest structure.

Definition 4.1.3 (Strongly isolated singularities)**.** We say that $V \in \mathcal{IV}_n(M)$, i.e., an n-dimensional integral varifold in M, has strongly isolated singularity at p if the tangent cone of V at p is a multiplicity one regular minimal hypercone. By regular hypercone, we mean a hypercone, $\mathbf{C}$, with $\mathrm{Sing}(\mathbf{C}) \subset \{0\}$.

Next we will define the locally stable constant mean curvature hypersurfaces.

Definition 4.1.4. We call $\Sigma \subset (M^{n+1}, g)$ a locally stable constant mean curvature (or LSCMC) hypersurface with isolated singularities if Σ satisfies the following:

(1) Σ is a 2-sided hypersurface with the unit normal bundle ν.

(2) Σ is an embedded hypersurface with only isolated singularities with multiplicity 1 tangent cones.

(3) the mean curvature of Σ is constant.

(4) For any $p \in M$, there exists an open set U containing p such that for any $\phi \in C^1_c(\Sigma \cap U)$, we have

$$\int_\Sigma |\nabla \phi|^2 \geq \int_\Sigma (|A_\Sigma|^2 + Ric(\nu))\phi^2.$$

On the other hand, we say that $\Sigma \subset (M^8, g)$ is a locally weakly stable CMC hypersurface if Σ satisfies conditions (1)–(3) above, and for inequality (4), we additionally require that $\int_\Sigma \phi = 0$.

Lemma 4.1.5. *Suppose $\Sigma \subset (M, g)$ is weakly stable. Then for any two disjoint open set $\mathcal{U}, \mathcal{V} \subset \Sigma$, Σ is stable in at least one of $\mathcal{U}, \mathcal{V}$.*

Proof. Suppose not, so there are two non-zero functions $\phi_1, \phi_2 \in C^1_c(\Sigma)$ with $\operatorname{spt}\phi_1 \Subset \mathcal{U}$ and $\operatorname{spt}\phi_2 \Subset \mathcal{V}$ such that for $i = 1, 2$,

$$\int_\Sigma |\nabla \phi_i|^2 < \int_\Sigma (|A_\Sigma|^2 + Ric(\nu))\phi_i^2.$$

We can choose a constant c such that $\int_\Sigma \phi_1 = c \int_\Sigma \phi_2$. Now consider the function $\psi := \phi_1 - c\phi_2$. So we have $int_\Sigma \psi = 0$ and

$$\begin{aligned}\int_\Sigma |\nabla \psi|^2 &= \int_\Sigma |\nabla \phi_1|^2 + c^2 \int_\Sigma |\nabla \phi_2|^2 \\ &< \int_\Sigma (|A_\Sigma|^2 + Ric(\nu))\phi_1^2 + \int_\Sigma (|A_\Sigma|^2 + Ric(\nu))c^2\phi_2^2 \\ &= \int_\Sigma (|A_\Sigma|^2 + Ric(\nu))\psi^2,\end{aligned}$$

where the last equality arises from the fact that $\operatorname{spt}\phi_1$ and $\operatorname{spt}\phi_2$ are disjoint. Thus we obtain a contradiction with the assumption that Σ is weakly stable. □

Proposition 4.1.6. *Suppose $\Sigma \subset (M^8, g)$ is a locally weakly stable CMC hypersurface, then it is locally stable.*

Proof. We prove by contradiction. Without loss of generality, suppose that there exists $p \in \overline{\Sigma}$ and $R > 0$ such that Σ is weakly stable in $B_R(p)$ but not stable in $B_R(p)$. For any $r \in (0, R)$, then Σ is stable either in $B_r(p)$ or $A_{r,R}(p)$. Therefore, Σ is stable in some $B_r(p)$ or Σ is stable in $A_{0,R}(p)$. We are done for the first case or in the second case with $p \in Sing(\Sigma)$. Therefore, we can assume the case that $p \in \Sigma$ and Σ is stable in $A_{0,R}(p)$. Suppose there exists $\phi \in C^1_c(\Sigma \cap B_R(p))$ with $p \in \operatorname{spt}\phi$ and

$$\int_\Sigma |\nabla \phi|^2 < \int_\Sigma (|A_\Sigma|^2 + Ric(\nu))\phi^2.$$

By a capacity argument, there exists a $\widetilde{\phi} \in C_c^1(\Sigma \cap A_{0,R}(p))$ which violates the stability inequality, which leads a contradiction. $\square$

Next we will introduce the coordinate of Σ that is defined near singularities. As in [24, Definition 2.2.] and [9], for every $x \in \Sigma$, we define the regularity scale $r_S(x) := r_S(x; M, g, \Sigma)$ of Σ at x with respect to the metric g to be the supremum among all $r \in (0, \mathrm{injrad}(x; M, g)/2)$ such that,

1. $r^2 \|\mathrm{Rm}_g\|_{C^0, B_r^g(x)} + r^3 \|\nabla \mathrm{Rm}_g\|_{C^0, B_r^g(x)} \le 1/10$;
2. After pulling back by $\exp_x^g$ into $T_x M$,

$$\frac{1}{r}(\exp_x^g)^{-1}(\Sigma) \cap \mathbb{B}_1 = \mathrm{graph}_L\, u \cap \mathbb{B}_1,$$

for some linear hyperplane $L \subset T_x M$ and $u \in C^3(L)$ with $\|u\|_{C^3} \le 1/10$.

Definition 4.1.7. For $\phi \in C^k_{\mathrm{loc}}(\Sigma)$, we define the scale (w.r.t. $\lambda^2 g$) invariant C^k norm of a measurable subset $E \subset \Sigma$ to be

$$\|\phi\|_{C^k_*, E} := \sup_{x \in E} \sum_{j=0}^{k} r_S(x)^{j-1} \left|\nabla^j \phi\right|(x).$$

For $f \in C^k(M)$, we define the pointwise norm

$$[f]_{x, C^k_*} := \sum_{j=0}^{k} r_S(x)^j \sup_{B_{r_S}(x)} \left|\nabla^j f\right|(x).$$

Remark 4.1.8. If Σ is a LSCMC hypersurface, the by [36], the tangent cone of Σ are unique. And there exists small ball centered at each singular point such that Σ is locally a C^1 graphical perturbation on the cone. $\phi \in C^k_{\mathrm{loc}}(\Sigma)$ is scaling invariant in the sense that for any $\lambda > 0$, we have

- $graph_{\Sigma, \lambda^2 g}(\lambda \phi) = graph_{\Sigma, g}(\phi)$.

- $\|\lambda\phi\|_{C^k_*,\mathcal{U},\lambda^2 g} = \|\phi\|_{C^k_*,\mathcal{U},g}$ for any $\mathcal{U} \subset \Sigma$.

Remark 4.1.9. Suppose $\Sigma \subset (M, g)$ is a LSCMC hypersurface. For $p \in Sing(\Sigma)$, we denote by $\mathbf{C}_p$ the unique tangent cone at p. By [36], we have that there is a $\delta_0 = \delta_0(M, g, \Sigma) > 0$ such that for the pullback of $\Sigma \cap B_\delta(0)$ to T_pM, and for any $0 < \delta < \delta_0$ (which for convenience we still denote as $\Sigma \cap B_\delta(0)$), we can represent $\Sigma \cap B_\delta(p) = graph_{\mathbf{C}_p}(f)$ for some $f \in C^2(\mathbf{C}_p)$, and $\epsilon : [0, \delta] \to \mathbb{R}^+$ a decreasing function with $\epsilon(0) = 0$ with

$$\|f\|_{C^2_*(\mathbf{B}_r)} < \epsilon(r).$$

We will frequently refer to the above representation of $\Sigma \cap B_\delta(p)$ simply as **conical coordinates**.

4.2 Twisted Jacobi fields

In this subsection we introduce a suitable notion of twisted Jacobi field for a constant mean curvature hypersurface with isolated singularities and establish a spectral theorem.

4.2.1 Definitions and Properties

Associated to the second variation of the area functional we have the following quadratic form, defined on functions $\phi \in C^1_c(\Sigma)$ by

$$Q_\Sigma(\phi, \phi) = \int_\Sigma |\nabla\phi|^2 - (|A_\Sigma|^2 + \mathrm{Ric}(\nu, \nu))\phi^2. \tag{4.2.1}$$

To define it locally on a part of Σ, we adopt the notations from [42]: for an open set, $U \subset M$, with smooth boundary such that ∂U transversely intersects Σ and $\partial U \cap \mathrm{Sing}(\Sigma) = \emptyset$, we denote $\mathcal{U} = U \cap \Sigma$ and set $\overline{\mathcal{U}} := \overline{U} \cap \Sigma$. Note that the topological closure of $\mathcal{U}$ contains the singularity part $Sing(\Sigma) \cap U$, but $\overline{\mathcal{U}}$ consists of only the smooth part of $\Sigma \cap U$ and $\partial U \cap \Sigma$.

Similar to case in minimal surface (see [42]), Q_Σ is semi-bounded in the following sense.

Lemma 4.2.1. *Suppose $\Sigma \subset (M, g)$ is a locally stable CMC hypersurface, there exists a $C_0 = C_0(\Sigma, g, \mathcal{U})$ such that*

$$Q_\Sigma(\psi, \psi) + C_0 \|\psi\|^2_{L^2(\Sigma)} \geq 0$$

for all $\psi \in C^1_c(\mathcal{U})$.

Proof. This is a direct adaptation of the proof of [42, Lemma 3.1] to the case where Σ is locally stable with constant mean curvature. In particular, we compute the inequality

$$Q_\Sigma(\psi, \psi) \geq \int_\Sigma \psi^2 \sum_j \eta_j \Delta_g \eta_j,$$

where $\{\eta_j\}$ is a partition of unity subordinate to a finite open cover (in each set of which Σ is stable) of $\overline{\mathcal{U}}$, just as in [42, Lemma 3.1]. Note now that for a general hypersurface $\Sigma \subset M$, we have

$$\Delta_g f = \operatorname{Hess} f(\nu, \nu) + H_\Sigma \nu(f) + \Delta_\Sigma f.$$

In particular, because Σ has constant mean curvature, we compute that

$$|\Delta_\Sigma \eta_j| = |\Delta_g \eta_j - \operatorname{Hess} \eta_j(\nu, \nu) - H_\Sigma \nu(\eta_j)| \leq n |\operatorname{Hess} \eta_j(\nu, \nu)| + |H_\Sigma||\nabla \eta_j|.$$

Thus we conclude that $|\sum_j \eta_j \cdot \Delta \eta_j| \geq -C_0$ for some $C_0 = C_0(\Sigma, \mathcal{U}, g)$. □

We now fix $C_0 = C_0(\Sigma, g, U)$ as in Lemma 4.2.1 and proceed to make the following definitions.

Definition 4.2.2. For $\psi \in C^1_c(\mathcal{U})$, we define the norm

$$\|\psi\|_{\mathscr{B}(\mathcal{U})} = Q_\Sigma(\psi,\psi) + (C_0+1)\|\psi\|^2_{L^2(\mathcal{U})}, \tag{4.2.2}$$

where $C_0 = C_0(\Sigma, g, U)$ is as in in Lemma 4.2.1. We then define the following two Hilbert spaces,

$$\mathscr{B}(\mathcal{U}) = \overline{C^1_c(\overline{\mathcal{U}})}^{\mathscr{B}} \text{ and } \mathscr{B}_0(\mathcal{U}) = \overline{C^1_c(\mathcal{U})}^{\mathscr{B}},$$

equipped with the L^2 inner product.

We note that by Lemma 4.2.1, Q_Σ extends to a well defined quadratic form on $\mathscr{B}(\mathcal{U})$ and $\mathscr{B}_0(\mathcal{U})$ and both continuously embed into L^2. In Subsection 4.2.3 we will in fact show that this embedding is compact provided our hypersurface satisfies an appropriate L^2-nonconcentration property.

Remark 4.2.3. The above Hilbert spaces will serve as a suitable replacement for $W^{1,2}(\mathcal{U})$ for inverting the twisted Jacobi operator in the presence of isolated singularities. In general, as remarked in [42], $W^{1,2}_0$ is only subset of $\mathscr{B}_0$ but, when every singularity is strongly isolated with strictly stable tangent cone, we have $W^{1,2}_0 = \mathscr{B}_0$. We refer the reader to [42, Example 3.3] for an example where this equality fails (which is always the case if some isolated singularities have non strictly stable tangent cone).

As in [3] we define the L^2 function with integral zero by

$$L^2_T(\mathcal{U}) = \left\{\psi \in L^2(\mathcal{U}) : \int_{\mathcal{U}} \psi \, d\mathcal{H}^g = 0\right\},$$

where, for convenience, we may omit the superscript g if the metric is clear from context. With the above in place we set $\mathscr{B}_T(\mathcal{U}) = \mathscr{B}(\mathcal{U}) \cap L^2_T(\mathcal{U})$, $\mathscr{B}_{0,T}(\mathcal{U}) = \mathscr{B}_0(\mathcal{U}) \cap L^2_T(\mathcal{U})$ and conclude this subsection with the following definition.

Definition 4.2.4. We say that a function $u \in \mathscr{B}_{0,T}(\Sigma)$ is a **(weak) twisted Jacobi field** on Σ if $Q_\Sigma(u,\psi) = 0$ for all $\psi \in \mathscr{B}_{0,T}(\Sigma)$, i.e., $u \in \operatorname{Ker} Q_\Sigma$.

Remark 4.2.5. Note that this definition generalizes to the case that Σ a locally stable CMC hypersurface with singularities in 1.2.1. In the case that $\Sigma \subset (M,g)$ is a smooth immersion of CMC hypersurface, consider the set

$$\mathscr{F} := \{u \in C^2(\Sigma) : \int_\Sigma u = 0 \text{ and } u|_{\partial\Sigma} = 0\}.$$

As the computation in [3, Proposition 2.9], $Q_\Sigma(u,\psi) = 0$ for all $\psi \in \mathscr{F}$ if and only if u satisfies

$$\widetilde{L}u := Lu - \Psi(u) = 0.$$

4.2.2 Nonconcentration for Isolated Singularities

We define a suitable notion of L^2 norm nonconcentration in a similar manner to [42]. This property will be crucially utilised in the proof of the spectral theorem in Subsection 4.2.3.

Definition 4.2.6. We say that Σ has the L^2**-nonconcentration property** in U if, for any $\varepsilon > 0$, there exists an open neighborhood $V_\epsilon \supset Sing(\Sigma) \cap U$ such that

$$\int_{\Sigma \cap V_\epsilon} \phi^2 \leq \epsilon \cdot \|\phi\|^2_{\mathscr{B}(\mathcal{U})}$$

for all $\phi \in C^1_c(\mathcal{U})$.

In [42], it is shown that we have the L^2-nonconcentration property for the minimal surface case, with an a priori assumption for the singularities.

Proposition 4.2.7. *[42, Lemma 3.9] Suppose that Σ a locally stable minimal hpyersurface with isolated singularities and $U \subset M$ is open with $\partial U \cap \operatorname{Sing}(\Sigma) = \emptyset$. Then Σ satisfies* *the L^2-nonconcentration property in U. I.e., if, for any $\varepsilon > 0$, there exists an open neighborhood $V_\epsilon \supset Sing(\Sigma) \cap U$ such that*

$$\int_{\Sigma \cap V_\epsilon} \phi^2 \le \epsilon \cdot \|\phi\|^2_{\mathscr{B}(\mathcal{U})}$$

for all $\phi \in C^1_c(\mathcal{U})$.

In this section, we will prove a similar L^2-nonconcentration property for locally stable CMC hypersurfaces with isolated singularities. Due to the fact that the space of test functions used is the same as in [42, 43], which also heavily relies on the Michael–Simon–Sobolev inequality (see [27]) through an embedding into Euclidean space, the same proof essentially carries through (c.f. [42, Lemma 3.9]). So here we will describe the adaptations needed for the constant mean curvature case and provide more detailed explanations.

Similar to the minimal surface case, we define $Q_\Sigma : C^\infty_c(\Sigma) \times C^\infty_c(\Sigma) \to \mathbb{R}$ by

$$Q_\Sigma(\psi, \phi) := \int_\Sigma \langle \psi, \phi \rangle - \psi(|A_\Sigma|^2_g + \mathrm{Ric}_g(\nu))\phi,$$

for $\psi, \phi \in C^\infty_c(\Sigma)$. Note that here we do not impose zero integral restrictions.

Then the associated Jacobi operator is

$$L_\Sigma := \Delta_\Sigma + |A_\Sigma|^2_g + \mathrm{Ric}_g(\nu).$$

In order to have a coercivity of the quadratic form Q_Σ, we proceed to perturb the Jacobi operator and, for $s \in (0, 1)$, consider

$$L^s = L_\Sigma - s|A_\Sigma|^2$$

with associated quadratic form

$$Q^s_\Sigma(\phi,\phi) = Q_\Sigma(\phi,\phi) + s\int_\Sigma |A_\Sigma|^2\phi^2.$$

(notice here again we use the Jacobi operator and not the twisted version since we consider $\phi \in C^1_c(\mathcal{U})$ and not just those that are average free).

Proposition 4.2.8. *Suppose that Σ is a locally stable CMC hypersurface has only strongly isolated singularities and $U \subset M$ is open with $\partial U \cap \mathrm{Sing}(\Sigma) = \emptyset$. Then Σ satisfies the L^2-nonconcentration property in U.*

Lemma 4.2.9. *For any $\phi \in C^1_c(\Sigma)$, and $\mathcal{U} \subset \Sigma$, there exists $C = C(\Sigma, g, n)$ such that*

$$\int_{\mathcal{U}} \phi^2 \leq C \cdot \|\Sigma\|(\mathcal{U})^{2/n} \cdot \left(\int_\Sigma |\nabla\phi|^2 + \phi^2\, d\|\Sigma\|\right). \tag{4.2.3}$$

Proof. Note that by Hölder inequality, we get

$$\int_{\mathcal{U}} \phi^2 \leq \|\Sigma\|(\mathcal{U})^{2/n} \cdot \left(\int_\Sigma |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

As usual, we embedded the manifold into the Euclidean space, and denote H the mean curvature of Σ, then by the Simon's Sobolev inequality, there exists a dimentional constant c_n such that

$$\begin{aligned} c_n\left(\int_\Sigma |\phi|^{\frac{2(n-1)}{n-2}\cdot\frac{n}{n-1}}\right)^{\frac{n-1}{n}} &\leq \int_\Sigma |\phi|^{\frac{2(n-1)}{n-2}}|H| + \frac{2(n-1)}{n-2}\phi^{\frac{2(n-1)}{n-2}-1}|\nabla\phi| \\ &= \int_\Sigma \frac{2(n-1)}{n-2}|\phi|^{\frac{n}{n-2}}|\nabla\phi| + |\phi|^{\frac{2(n-1)}{n-2}}|H|. \end{aligned}$$

And by Hölder's inequality, we have

$$\begin{aligned}\int_\Sigma |\phi|^{\frac{2(n-1)}{n-2}}|H| &\leq \left(\int_\Sigma \phi^2|H|^2\right)^{\frac{1}{2}} \cdot \left(\int_\Sigma |\phi|^{(\frac{2(n-1)}{n-2}-1)2}\right)^{\frac{1}{2}} \\ &\leq \left(\int_\Sigma \phi^2|H|^2\right)^{\frac{1}{2}} \cdot \left(\int_\Sigma |\phi|^{\frac{2n}{n-2}}\right)^{\frac{1}{2}}\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left(\int_\Sigma |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-1}{n}} &\leq C_n \int_\Sigma |\phi|^{\frac{n}{n-2}}|\nabla\phi| + C_n\left(\int_\Sigma \phi^2|H|^2\right)^{\frac{1}{2}} \cdot \left(\int_\Sigma |\phi|^{\frac{2n}{n-2}}\right)^{\frac{1}{2}} \\ &\leq C_n\left(\int_\Sigma |\phi|^{\frac{2n}{n-2}}\right)^{\frac{1}{2}} \cdot \left(\int_\Sigma |\nabla\phi|^2\right)^{\frac{1}{2}} + C_n\left(\int_\Sigma \phi^2|H|^2\right)^{\frac{1}{2}} \cdot \left(\int_\Sigma |\phi|^{\frac{2n}{n-2}}\right)^{\frac{1}{2}}.\end{aligned}$$

Dividing $\left(\int_\Sigma |\phi|^{\frac{2n}{n-2}}\right)^{\frac{1}{2}}$ on both side, we have

$$\left(\int_\Sigma |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}} \leq C_n\left(\int_\Sigma |\nabla\phi|^2\right)^{\frac{1}{2}} + C_n\left(\int_\Sigma \phi^2|H|^2\right)^{\frac{1}{2}}.$$

So under squaring the both side, we have

$$\left(\int_\Sigma |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_n \int_\Sigma |\nabla\phi|^2 + \phi^2|H|^2.$$

By the bounded of the mean curvature, we have

$$c_n\left(\int_\Sigma |\phi|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C \int_\Sigma |\nabla\phi|^2 + \phi^2.$$

So the lemma is proved. □

Therefore, for fixed ϵ and s, we can choose the region $\mathcal{U}$ with a small $||\Sigma||$ in Lemma 4.2.9. However, we still require an extra L^2-bound for the energy of ϕ. Based on the observations in [42, Page 21] and [43, Page 146], if we additionally assume that $\mathbf{C}_p$ is strictly stable, we can obtain Proposition 4.2.8 through the following lemma.

Lemma 4.2.10. *Suppose $\mathbf{C}_p$ is strictly stable, there exists some small radius, $r > 0$ and $\delta_{\mathbf{C}_p} > 0$ such that*

$$\delta_{\mathbf{C}_p} \int_\Sigma |\nabla \phi|^2 \leq Q_\Sigma(\phi, \phi) \quad \forall \phi \in C_c^\infty(B_r(p)),$$

Proof. Under the conical coordinate, for $x \in \mathbf{C} := \mathbf{C}_p(\Sigma)$ with $\mathrm{dist}(p, x) < r_1$, suppose

$$\mathrm{graph}_{\mathbf{C}}(u) \llcorner_{\mathbb{B}_{r_1}} = \Sigma \llcorner_{B_{r_1}(p)}.$$

denote $\phi(x) := \widetilde{\phi}(x, u(x))$ with $\widetilde{\phi} \in C_c^1(B_{r_1}(p) \cap \Sigma)$. We suffice to prove that there exists $\delta_1 \in (0, 1)$ and $r_2 \in (0, r_1)$ such that for any $\widetilde{\phi} \in C_c^1(B_{r_2}(p) \cap \Sigma)$,

$$\delta_1 \int_\Sigma |\nabla \widetilde{\phi}|^2 \geq \int_\Sigma (|A_\Sigma|^2 + \mathrm{Ric}(\nu)) \widetilde{\phi}^2.$$

By the strictly stability of the tangent cone $\mathbf{C} := \mathbf{C}_p(\Sigma)$, we have that for any $\phi \in C_c^1(\mathbf{C})$,

$$\int_{\mathbf{C}} |\nabla_{\mathbf{C}} \phi|^2 \geq \int_{\mathbf{C}} (|A_{\mathbf{C}}|^2 + \frac{a}{r^2}) \phi^2,$$

where $a > 0$ depending on the first eigenvalue of the Jacobi operator on $\mathbf{C}$ (see Lemma B.2.2 and Theorem B.2.3).

By Chapter A and Remark 4.1.9, there exists $\tau_1 \in (0, r_1)$ such that for any $\phi \in C_c^1(\mathbf{B}_{r_1}(p))$,

$$\int_{\mathbf{C}} (|A_{\mathbf{C}}|^2 + \frac{a}{r^2}) \phi^2 \geq (1 + \delta_2) \int_\Sigma (|A_\Sigma|^2 \circ (x, u(x))) + \mathrm{Ric}(\nu) \circ (x, u(x))) \widetilde{\phi}^2,$$

for some $\delta_2 > 0$ small.

By Theorem A.0.2 again, we pick $r_2 \in (0, \tau_1)$ such that for $\phi \in C_c^1(\mathbf{B}_{r_2}(p))$,

$$\int_\Sigma |\nabla \widetilde{\phi}|^2 \geq (1-\delta_3) \int_{\mathbf{C}} |\nabla \phi|^2,$$

for some $\delta_3 > 0$ and

$$(1-\delta_3) \cdot (1+\delta_2) > 1.$$

Therefore we have

$$\begin{aligned}
\int_\Sigma |\nabla \widetilde{\phi}|^2 &\geq (1-\delta_3) \int_{\mathbf{C}} |\nabla \phi|^2 \\
&\geq (1-\delta_3) \int_{\mathbf{C}} (|A_{\mathbf{C}}|^2 + \frac{a}{r^2}) \phi^2 \\
&\geq (1-\delta_3) \cdot (1+\delta_2) \int_\Sigma (|A_\Sigma|^2 + \mathrm{Ric}(\nu)) \widetilde{\phi}^2 \\
&\geq \frac{1}{\delta_1} \int_{\mathbf{C}} (|A_\Sigma|^2 + \mathrm{Ric}(\nu)) \widetilde{\phi}^2.
\end{aligned}$$

where we define $\delta_1 := \frac{1}{(1-\delta_3)\cdot(1+\delta_2)}$. □

Therefore, it suffices to prove the result when C_p is not strictly stable, i.e., $\mu_1(C_p) = -\left(\frac{n-2}{2}\right)^2$. We deal the non-strictly stable case by perturbing the Jacobi operator.

Suppose α is a continuous function on Σ such that $1 \leq \alpha(x) \leq 1 + \log(\frac{r_1}{r})$ for $x \in \mathbf{B}_{r_1}(p) := B_{r_1}(p) \cap \Sigma$ (for $r_1 > 0$ sufficiently small enough such that Σ is stable in $B_{2r_1}(p)$) and we have conical coordinate in $B_{2r_1}(p)$. Note that $\alpha(x) \to \infty$ as $x \to p \in Sing(\Sigma)$.

Corollary 4.2.11. *For every $\varepsilon > 0$ and $s \in (0,1)$ there exists some open neighbourhood, $V_{\varepsilon,s}$, of p such that for each $\phi \in C_c^1(\mathbf{B}_{r_1}(p))$ we have*

$$\int_{V_{\varepsilon,s}} \phi^2 \alpha \leq \varepsilon \left(Q_\Sigma^s(\phi,\phi) + \int_\Sigma \phi^2 \right). \tag{4.2.4}$$

Proof. Fixing ϵ, s, by Lemma 4.2.9, for a undetermined open set $V_{\epsilon,s} \ni p$, we have

$$\int_{V_{\epsilon,s}} \phi^2 \leq C \cdot \|\Sigma\|(V_{\epsilon,s})^{2/n} \cdot \left(\int_\Sigma |\nabla \phi|^2 + \phi^2 \, d\|\Sigma\| \right) \tag{4.2.5}$$

$$\leq C \cdot \|\Sigma\|(V_{\epsilon,s})^{2/n} \cdot \left(\frac{1}{s} Q^s_\Sigma(\phi, \phi) + C \int_\Sigma \phi^2 \, d\|\Sigma\| \right). \tag{4.2.6}$$

where the last inequality comes from the coercivity of Q^s_Σ:

$$Q^s_\Sigma(\phi, \phi) \geq s \int_\Sigma |\nabla \phi|^2 - s || \operatorname{Ric}_M ||_\infty \int_\Sigma \phi^2.$$

Next we choose the neighborhood $V_{\epsilon,s}$. By the Poincaré inequality and the definition of the function α, we have

$$\int_{V_{\varepsilon,s}} \phi^2 \alpha \leq \int_{V_{\varepsilon,s}} \phi^2 \left(1 + \log \left(\frac{r_1}{r} \right) \right)$$

For the first term on the right hand side, by (4.2.6), we could choose $V_{\epsilon,s}$ with

$$C \|\Sigma\|(V_{\epsilon,s})^{2/n}/s < \epsilon/2.$$

For the second term, we need to again utilise Lemma 4.2.9 and the Simon's Sobolev inequality to choose the size of the neighbourhood. Similar to the bound for the first term, we get

$$\int_{V_{\varepsilon,s}} \phi^2 \log \left(\frac{r_1}{r} \right) \leq C(\|\Sigma\|(V_{\epsilon,s})^{2/n}, s) \int_\Sigma (|\nabla \phi|^2 + \phi^2)$$

where we could choose the neighbourhood $V_{\varepsilon,s}$ sufficiently small in measure so that $C = C(\|\Sigma\|(V_{\epsilon,s})^{2/n}, s) \leq \varepsilon/2$. Choose the $V_{\epsilon,s}$ satisfies the bounds of the two terms will then prove (4.2.4). $\square$

Lemma 4.2.12. *Fixing $s > 0$, parameterize $\mathbf{B}_{2r_1}(p)$ with the conical coordinate by the cross section, there exists a unique $\phi \in C^0_{loc}(\mathbf{B}_{2r_1}(p)) \cap W^{1,2}_0(\mathbf{B}_{2r_1}(p))$, such that the following properties hold*

1. $\phi(r, \cdot) = 0$ *if* $r \geq r_1$,

2. $\phi(r, \cdot) > 0$ *if* $0 < r < r_1$,

3. $\int_\Sigma \phi^2 \alpha = 1$,

4. $Q^s_\Sigma(\phi, \phi) = \inf \left\{ Q^s_\Sigma(\psi, \psi) \mid \psi \in C^1_c(\mathbf{B}_{r_1}(p), \int_\Sigma \psi^2 \alpha = 1 \right\}$.

In addition, $\inf \left\{ Q^s_\Sigma(\psi, \psi) \mid \psi \in C^1_c(\mathbf{B}_{r_1}(p), \int_\Sigma \psi^2 \alpha = 1 \right\} > 0$.

Proof. Note that for any $\psi \in C^1_c(\mathbf{B}_{r_1}(p)$, by the stability of Σ in $B_{2r_1}(p)$, we have

$$Q^s_\Sigma(\psi, \psi) = Q_\Sigma(\psi, \psi) + s \int_\Sigma |A_\Sigma|^2 \psi^2 \geq 0,$$

So $\inf \left\{ Q^s_\Sigma(\psi, \psi) \mid \psi \in C^1_c(\mathbf{B}_{r_1}(p), \int_\Sigma \psi^2 \alpha = 1 \right\} > 0$. Denote the infimum by λ_s.

Next we show that the existence of the minimizer. Suppose there exists a sequence $\phi_j \in C^1_c$ such that

$$Q^s_\Sigma(\phi_j, \phi_j) \searrow \inf \left\{ Q^s_\Sigma(\phi, \phi) \mid \phi \in C^1_c(\mathbf{B}_{r_1}(p), \int_\Sigma \phi^2 \alpha = 1 \right\}.$$

By the coercivity of Q^s_Σ and the L^2 α-normalization of ϕ_j, we have

$$\begin{cases} \int_\Sigma |\nabla \phi_j|^2 < \infty, \\ \int_\Sigma \phi_j^2 < \infty. \end{cases}$$

there exists $\phi \in W^{1,2}_0(\mathbf{B}_{r_1}(p))$ such that $\phi_j \rightharpoonup \phi$ weakly in $W^{1,2}$ ($\phi_j \to \phi$ strongly in L^2).

On the other hand, fixing $\epsilon > 0$, by Corollary 4.2.11, for any $j \geq 1$, we have

$$\int_{V_{\varepsilon,s}} \phi_j^2 \alpha \leq \varepsilon \left(Q_\Sigma^s(\phi_j, \phi_j) + \int_\Sigma \phi_j^2 \right).$$

Note that by $\int_{V_{\varepsilon,s}^c} \phi_j^2 \alpha \to \int_{V_{\varepsilon,s}^c} \phi^2 \alpha \leq 1$, $Q_\Sigma^s(\phi_j, \phi_j) \to \lambda_s$ and $\phi_j \to \phi$ strongly in L^2, we have

$$1 - \epsilon(\lambda_s + \int_\Sigma \phi^2) \leq \int_{V_{\varepsilon,s}^c} \phi^2 \alpha \leq 1.$$

As $\epsilon \searrow 0^+$, we have $\int_\Sigma \phi^2 \alpha = 1$.

Therefore, we have $Q_\Sigma^s(\phi, \phi) \geq \lambda_s$. Thus

$$\int_\Sigma [(1-s)|A_\Sigma|^2 + \mathrm{Ric}(\nu)]\phi^2 \leq -\lambda_s + \int_\Sigma |\nabla \phi|^2,$$

So $[(1-s)|A_\Sigma|^2 + \mathrm{Ric}(\nu)]\phi^2 \in L^1(B_{r_1}(p))$. By the convexity of the Q_Σ^s, we have

$$Q_\Sigma^s(\phi_j, \phi_j) \geq Q_\Sigma^s(\phi, \phi) + 2\int_\Sigma \langle \nabla(\phi_j - \phi), \nabla \phi \rangle + \int_\Sigma [(1-s)|A_\Sigma|^2 + \mathrm{Ric}(\nu)](\phi^2 - \phi_j^2). \tag{4.2.7}$$

By the weak convergence of ϕ_j in $W^{1,2}$ and dominated convergence theorem, we have

$$\liminf_{j \to \infty} Q_\Sigma^s(\phi_j, \phi_j) \geq Q_\Sigma^s(\phi, \phi).$$

Therefore, $\phi \in W_0^{1,2}(\mathbf{B}_{r_1}(p))$ is a minimizer of (4).

On the other hand, note that ϕ is also a critical point of the functional in (4), i.e., for any $\psi \in C_c^1(\mathbf{B}_{r_1}(p))$ and $e > 0$, denote

$$f(\epsilon) = \frac{Q_\Sigma^s(\phi + \epsilon\psi, \phi + \epsilon\psi)}{\int_\Sigma (\phi + \epsilon\psi)^2 \alpha}.$$

We have $f'(0) = 0$. From which we can check it follows that ϕ is a weak solution of the eigenvalue equation

$$-L_{\Sigma}^{s}\phi = \lambda_s \alpha \phi, \tag{4.2.8}$$

on $\mathbf{B}_{r_1}(p)$. Therefore, by the standard regularity theorems of elliptic PDEs, ϕ is smooth in $\mathbf{B}_{r_1}(p)$ and continuous on Σ.

Finally, by the same argument as [14, 6.5, Theorem 2] for Rayleigh's quotient, we can show that the weak solution of the eigenvalue equation above is unique, and positive in $\mathbf{B}_{r_1}(p)$. □

Remark 4.2.13. From the local stability of Σ and the definition of the perturbed Jacobi operator, L^s, we ensure that if $0 < s < \tilde{s} < 1$ then for any $\phi \in C_c^1(\mathbf{B}_{r_1}(p))$ both $0 \leq Q_{\Sigma}^{s}(\phi, \phi) \leq Q_{\Sigma}^{\tilde{s}}(\phi, \phi)$; using this with the definitions above we then have $0 \leq \lambda_s \leq \lambda_{\tilde{s}}$.

By Lemma 4.2.12, for each fixed $s \in (0, 1)$, there exists a unique eigenfunction, denoted by u_s. We want the u_s is not collapsing as s small. By the observation in [43, 42], we have the following lower bound of the eigenfunction u_s (with weight function α). The proof is identical for CMC hypersurfaces and minimal hypersurfaces. Note that the lower bound relies on the choice of the function α near singularities.

Lemma 4.2.14. *[42, Lemma 3.10] there is some $s_0 \in (0, 1)$ and $\delta > 0$ such that for all $0 < s < s_0$ we have the uniform lower bound*

$$\sup \left\{ (u_s(r, w))^2 \,\middle|\, \frac{r_1}{2} < r < r_1 \right\} \geq \delta. \tag{4.2.9}$$

Using Lemma 4.2.14 we are now able to send $s \to 0$ to guarantee, by standard elliptic estimates, there exists some function u_0 and λ_0 such that, up to a sub-sequence we have

$$\begin{cases} u_s \to u_0 \text{ in } C^2_{loc}(\mathbf{B}_{r_1}(p)) \cap W^{1,2}_{loc}(\mathbf{B}_{r_1}(p)) \\ \lambda_s \to \lambda_0 \end{cases},$$

where

$$\lambda_0 \geq \inf\left\{ Q_\Sigma(\phi,\phi) \mid \phi \in C^1_c(\mathbf{B}_{r_1}(p), \int_\Sigma \phi^2\alpha = 1 \right\} \geq 0,$$

and so that in $\mathbf{B}_{r_1}(p)$ we have

$$-Lu_0 = \lambda_0 u_0 \alpha.$$

Moreover, Lemma 4.2.14 provides a uniformly positive lower bound on u_s, which guarantees that $u_0 \neq 0$. The remainder of the argument for the proof of Proposition 4.2.8 is identical to that of the minimal surface case ([42]). We sketch the proof for completeness.

Proof of Proposition 4.2.8. One now proves that $\lambda_0 \neq 0$ by a contradiction argument, based on the fact that $u_0 \neq 0$ (from the uniform positive lower bounds on the u_s) along with the strong maximum principle on the equation $Lu_0 = 0$ on $B_{2r_1}(p)$ (which would imply the contradiction that $u_0 = 0$).

To conclude we define, for a given $\varepsilon > 0$, the open neighbourhood, V_ε, of $p \in \operatorname{Sing}(\Sigma)$ by

$$V_\varepsilon = \left\{ x \in B_{r_1}(p) \mid \alpha(x) > \frac{1}{\lambda_0 \varepsilon} \right\},$$

which is non-empty by virtue of the fact that $\alpha(x) \to \infty$ as $x \to p$. Then, for a fixed $\phi \in C^1_c(\mathbf{B}_{r_1}(p))$ we have that

$$Q_\Sigma(\phi,\phi) \geq \lim_{s\to 0} \lambda_s \cdot \int_\Sigma \phi^2\alpha = \lambda_0 \int_\Sigma \phi^2\alpha \geq \frac{\lambda_0}{\lambda_0\varepsilon}\int_{V_\varepsilon} \phi^2 = \frac{1}{\varepsilon}\int_{V_\varepsilon}\phi^2.$$

One then establishes the desired L^2-nonconcentration for the neighbourhood V_ε for each $\varphi \in C^1_c(\mathcal{U})$ through multiplication with a cutoff function supported in $B_{r_1}(p)$. □

4.2.3 Spectral Theorem for LSCMC Hypersurfaces

In this subsection, we will use ideas in [42, 14] to deduce a spectral theorem for LSCMC (with singular boundaries). In particular, we will show that the Hilbert space $\mathscr{B}_0$

compactly embeds into L^2, from which we are able to establish a spectral theorem for the twisted Jacobi operator, by standard arguments.

As in [3] we define the L^2 function with integral zero by

$$L^2_T(\Sigma) = \left\{ \psi \in L^2(\Sigma) : \int_\Sigma \psi \, d\mathcal{H}^g = 0 \right\},$$

where, for convenience, we may omit the superscript g if the metric is clear from context.

Definition 4.2.15. We say $u \in \mathscr{B}_{0,T}(\Sigma)$ is a (weak) solution of $-\widetilde{L}u = g$ if

$$Q_\Sigma(u, \psi) = \langle g, \psi \rangle_{L^2}$$

for any $\psi \in \mathscr{B}_{0,T}(\Sigma)$.

In addition, for $h \in L^\infty(\Sigma)$, we say $u \in \mathscr{B}_{0,T}(\Sigma)$ is a (weak) solution of $-\widetilde{L^h}u = g$ if

$$Q_\Sigma(u, \psi) + \langle hu, \psi \rangle_{L^2} = \langle g, \psi \rangle_{L^2}$$

for any $\psi \in \mathscr{B}_{0,T}(\Sigma)$.

Theorem 4.2.16. *Suppose Σ is an embedded two-sided locally stable constant mean curvature hypersurface with only strongly isolated singularities in an open set U. Then $\mathscr{B}_{0,T}(\mathcal{U}) = \mathscr{B}_0 \cap L^2_T(\mathcal{U})$ is compactly embedded in $L^2_T(\mathcal{U})$.*

Proof. We suppose that $\{\psi_j\}_j \subset \mathscr{B}_{0,T}(\mathcal{U})$ is a sequence with $\|\psi_j\|_{\mathscr{B}(\mathcal{U})} \leq 1$. By the Banach–Saks theorem for bounded sequences in Hilbert spaces, there exists $\psi \in \mathscr{B}_{0,T}(\mathcal{U})$ and sub-sequence $\{\psi_{j_k}\}$ such that $\frac{1}{m}\sum_{k=1}^{m} \psi_{j_k} \to \psi$ as $m \to \infty$ strongly in $\mathscr{B}_{0,T}$. We now show that $\psi_j \to \psi$ strongly in $L^2(\mathcal{U})$.

Claim: Given any $\Omega \subset\subset \mathcal{U}$ and any $\psi \in C^1_c(\mathcal{U})$, we have

$$\|\psi\|_{W^{1,2}(\Omega)} \leq C\|\psi\|_{\mathscr{B}(\mathcal{U})},$$

for some constant $C = C(\Omega, \mathcal{U}, \Sigma, g)$.

Proof of the claim: Choose a finite open cover $\{\Omega_j\}_{j\geq 0}$ of $Clos(\mathcal{U}) := \overline{\mathcal{U}} \cap \Sigma$ in M satisfies that:

1. $\overline{\Omega} \subset \Omega_0 \subset\subset M \setminus Sing(\Sigma)$.

2. For $j > 0$, $\Omega_j \cap \Omega = \emptyset$ and Σ is stable in Ω_j.

Let $\{\eta_j^2\}_j$ be a smooth partition of unity subordinate to the cover $\{\Omega_j\}_j$, with $\eta_0 = 1$ on Ω. Note that

$$\begin{aligned}
Q_\Sigma(\psi,\psi) &= \sum_{j\geq 0} \int_\Sigma |\nabla\psi|^2\eta_j^2 - \left(|A_\Sigma|^2 + Ric(\nu,\nu)\right)\psi^2\eta_j^2 \\
&= \sum_{j\geq 0} Q_\Sigma(\eta_j\psi, \eta_j\psi) - \sum_{j\geq 0}\int_\Sigma 2\langle \eta_j\nabla\psi, \psi\nabla\eta_j\rangle + \psi^2|\nabla\eta_j|^2 \\
&= \sum_{j\geq 0} Q_\Sigma(\eta_j\psi, \eta_j\psi) + \sum_{j\geq 0}\int_\Sigma \psi^2\left(\mathrm{div}_\Sigma(\eta_j\nabla\eta_j) - |\nabla\eta_j|^2\right) \\
&= Q_\Sigma(\eta_0\psi,\eta_0\psi) + \sum_{j\geq 1} Q_\Sigma(\eta_j\psi,\eta_j\psi) + \int_\Sigma \psi^2\left(\sum_{j\geq 0}\eta_j\cdot\Delta\eta_j\right)
\end{aligned}$$

Denote $\vec{H}_\Sigma = H_\Sigma\nu$ the mean curvature of Σ. Then for each $\eta = \eta_j$, we have

$$|\Delta_\Sigma\eta| = |\Delta_g\eta + H_\Sigma\nu(\eta) - \mathrm{Hess}\,\eta(\nu,\nu)| \leq n|\mathrm{Hess}\,\eta| + H_\Sigma|\nabla\eta|.$$

Therefore, as in Lemma 4.2.1, we have $\sum_{j\geq 0}\eta_j\cdot\Delta\eta_j \geq -C_0$ for some $C_0 = C_0(\Sigma,\Omega,\mathcal{U},g)$, and so

$$\begin{aligned}
Q_\Sigma(\psi,\psi) &\geq Q_\Sigma(\eta_0\psi,\eta_0\psi) + \sum_{j\geq 1} Q_\Sigma(\eta_j\psi,\eta_j\psi) - C_0\|\psi\|^2_{L^2(\mathcal{U})} \\
&\geq \int_\Omega |\nabla\psi|^2 - C(\Omega, C_0)\|\psi\|^2_{L^2(\mathcal{U})}.
\end{aligned}$$

Where the last inequality comes from the stability of Σ in each Ω_j for $j > 0$. In addition, by definition we have $\|\psi\|^2_{L^2(\Omega)} \le \|\psi\|^2_{\mathscr{B}(\mathcal{U})}$, the claim is therefore proved. $\square$

Returning to proof, suppose that $\{V_k\}_k$ is a decreasing sequence of relatively closed sets, $V_k \subset \Sigma$, with smooth boundaries such that $\bigcap_k V_k = Sing(\Sigma) \cap \mathcal{U}$. Since $\|\psi_j\|_{\mathscr{B}} \le 1$, for each fixed k, if we let $\Omega_k = \mathcal{U} \setminus V_k \Subset \Sigma$, then by the claim above we have that $\|\psi_j\|_{W^{1,2}(\Omega_k)} \le C(\Sigma, \Omega_k, \mathcal{U}, g)$. By Rellich–Kondrachov compactness Theorem, there exists $\phi_k \in L^2(\mathcal{U} \setminus V_k)$ such that, up to a sub-sequence, ψ_j converges to ϕ_k in $L^2(\mathcal{U} \setminus V_k)$. Because $\bigcap_k V_k = Sing(\Sigma) \cap \mathcal{U}$, there exists a subsequence such that $\psi_j \to \phi$ almost everywhere and $\sup_j \|\psi_j\|_{\mathscr{B}(\mathcal{U})} < \infty$.

On the other hand, from the above we have that $\frac{1}{m}\sum_{k=1}^{m} \psi_{j_k} \to \psi$ in $\mathscr{B}(\mathcal{U})$, $\frac{1}{m}\sum_{k=1}^{m} \psi_{j_k} \to \psi$ in $L^2(\mathcal{U})$, which implies that $\psi = \phi$ almost everywhere and thus $\phi \in \mathscr{B}_{0,T}(\mathcal{U})$. We now apply Proposition 4.2.8 and, for $\epsilon > 0$, let V_ϵ be the set in Definition 4.2.6 and consider k sufficiently large so that $V_k \subset\subset V_\varepsilon$. Noting that $\|\psi_j - \psi\|_{L^2(\mathcal{U} \setminus V_k)} \to 0$, we then have

$$\begin{aligned}
\limsup_{j\to\infty} \int_{\mathcal{U}} |\psi_j - \psi|^2 &\le \limsup_{j\to\infty} \int_{V_\epsilon} |\psi_j - \psi|^2 + \limsup_{j\to\infty} \int_{\mathcal{U}\setminus V_\epsilon} |\psi_j - \psi|^2 \\
&\le \epsilon \cdot \limsup_{j\to\infty} \|\psi_j - \psi\|^2_{\mathscr{B}} \\
&\le 4\epsilon
\end{aligned}$$

So as $\epsilon \to 0$, we have $\psi_k \to \psi$ in $L^2(\mathcal{U})$, establishing the embedding. $\square$

Using the embedding theorem in Theorem 4.2.16, we can show the following spectral theorem for the twisted Jacobi operator. This can be considered as a generalization of the smooth immersed CMC hypersurfaces ([4, Proposition 2.2]). The argument in parts (1) and (2) is similar to the standard argument for elliptic PDEs (see [14, Section 6.2]).

Theorem 4.2.17. *Suppose $\Sigma \subset (M^{n+1}, g)$ is a LSCMC hypersurface with isolated singularities. Let $U \subset M$ be an open subset such that ∂U is smooth and intersects Σ transversely. Denote $\mathcal{U} := U \cap \Sigma$. Then*

(1) For any $f \in L^\infty(\mathcal{U})$, there exists a strictly increasing sequence $\sigma_p(\Sigma) = \{\lambda_j\}_{j=1} \nearrow \infty$ and finite-dimensional pairwise L^2-orthogonal linear subspaces, $\{E_j\}$, of $\mathscr{B}_{0,T}(\Sigma) \cap C^\infty(\mathcal{U})$, such that

$$-\widetilde{L^f}\psi = \lambda_j \psi$$

for all $\psi \in E_j$. Furthermore, $\{E_j\}$ forms the orthonormal basis of the following spaces

$$L^2_T(\mathcal{U}) = span_{L^2}\{E_j\}_j, \qquad \mathscr{B}_{0,T}(\mathcal{U}) = span_{\mathscr{B}}\{E_j\}_j.$$

(2) For any $f \in L^\infty(\mathcal{U})$, if $\widetilde{L^f}$ is nondegenerate, i.e. $0 \notin \sigma_p(\Sigma)$, then for each $g \in L^2_T(\Sigma)$ there exists a unique $\psi \in \mathscr{B}_{0,T}(\mathcal{U})$ such that

$$-\widetilde{L^f}\psi = g$$

on $\mathcal{U}$.

(3) The subset

$$G = \{f \in C^\infty_c(\mathcal{U}) : (-\widetilde{L^f}) \text{ is nondegenerate}\}$$

is open and dense in $C^\infty_c(\mathcal{U})$.

Proof. For (1) and (2), we note that the definition in (4.2.2) provides a natural coercivity bound for quadratic form Q_Σ, i.e., for any $\psi \in \mathscr{B}_{0,T}(\mathcal{U})$, we have

$$\|\psi\|^2_{\mathscr{B}} = Q_\Sigma(\psi, \psi) + (C_0 + 1)\|\psi\|^2_{L^2(\Sigma)}, \tag{4.2.10}$$

where C_0 is constructed from Lemma 4.2.1.

On the other hand, by Theorem 4.2.16 and Lemma 4.2.1, $(\mathscr{B}_{0,T}(\mathcal{U}), I)$ forms a Hilbert space with

$$I(\phi,\psi) := Q_\Sigma(\phi,\psi) + (C_0+1)\langle\phi,\psi\rangle_{L^2(\Sigma)}$$

for any $\phi, \psi \in \mathscr{B}_{0,T}(\mathcal{U})$. Therefore, by the definition above and Cauchy inequality, we have

$$\begin{cases} \|\phi\|_{\mathscr{B}}^2 = I(\phi,\phi), \\ |I(\phi,\psi)| \leq \|\phi\|_{\mathscr{B}}\|\psi\|_{\mathscr{B}}. \end{cases}$$

Thus, by Theorem 4.2.16, (4.2.10), and (4.2.3), we employ the Riesz Representation Theorem. Therefore, for any $h \in L^2_T(\mathcal{U})$, there exists a unique $u \in \mathscr{B}_{0,T}(\mathcal{U})$ such that

$$I(u,\psi) = \langle h,\psi\rangle \qquad \text{for all } \psi \in \mathscr{B}_{0,T}(\mathcal{U}). \tag{4.2.11}$$

For simplicity, denote $\gamma := C_0 + 1$ and write $u = J_\gamma^{-1}h$.

Note that $-\widetilde{L}u = g$ is a weak solution if and only if

$$I(u,\psi) = \langle \gamma u + g,\psi\rangle \qquad \text{for all } \psi \in \mathscr{B}_{0,T}(\mathcal{U}).$$

By the notation above, it is equivalent to $u = J_\gamma^{-1}(\gamma u+g)$. Denote the operator $S := \gamma J_\gamma^{-1}u$. So we get

$$u - Su = J_\gamma^{-1}g.$$

We claim that $S : L^2_T(\mathcal{U}) \to L^2_T(\mathcal{U})$ is a compact, bounded, symmetric linear operator. It is trivial to see that S is a symmetric linear operator.

Next we show that S is compact. By (4.2.11), we have

$$\begin{aligned}\|u\|_{\mathscr{B}}^2 = I(u,u) = \langle h, u\rangle &\le \|h\|_{L^2}^2 \cdot \|u\|_{L^2}^2 \\ &\le \|h\|_{L^2}^2 \cdot \|u\|_{\mathscr{B}}^2.\end{aligned}$$

So we get $\|u\|_{\mathscr{B}} \le \|h\|_{L^2}^2$. Then Theorem 4.2.16 implies that S is a compact operator. Therefore, we get the Fredholm alternative for the operator S, i.e., one of the following hold:

$$\begin{cases}\text{For any } \phi \in L_T^2(\mathcal{U}) \\ u - Su = \phi \\ \text{ has a unique solution } u \in L_T^2(\mathcal{U})\end{cases} \qquad \begin{cases} u - Su = 0 \\ \text{ has non-trivial solution } u \in L_T^2(\mathcal{U}).\end{cases}$$

For the first case, we choose $\phi \in L_T^2(\mathcal{U})$ such that $\phi = J_\gamma^{-1} g$. So we have a unique solution $u \in \mathscr{B}_{0,T}(\mathcal{U})$ such that $-\widetilde{L}u = g$. Similar for the second case.

Finally, because S is a compact symmetric linear operator, we have the spectral theorem in part (1). The regularity of the eigenfunctions are from Corollary 4.2.22. Thus, we are done with parts (1) and (2).

We now establish (3). For openness we argue by contradiction and suppose there exists $f \in G$, and a sequence $f_j \to f$ smoothly, but with $-\widetilde{L^{f_j}}$ degenerate. Suppose $\psi_j \in \operatorname{Ker}(-\widetilde{L^{f_j}})$ with $\|\psi_j\|_{L^2(\mathcal{U})} = 1$, so that $Q_\Sigma(\psi_j, \psi_j) = \langle -(f_j)\psi_j, \psi_j\rangle$ and thus $\limsup_j \|\psi_j\|_{\mathscr{B}} < +\infty$. By (1), there is thus a sub-sequence (not relabelled) with $\psi_j \to \psi$ weakly in $\mathscr{B}(\mathcal{U})$ and strongly in $L^2(\mathcal{U})$ for some $\psi \in \mathscr{B}_{0,T}(\mathcal{U})$. Thus $\psi \neq 0$ and $\psi \in \operatorname{Ker}(-\widetilde{L^f})$ (by the weakly convergence in $\mathscr{B}$), contradicting the assumption that $f \in G$.

Next we show denseness. The argument is similar to the Jacobi operator studied for minimal surfaces (see [42]). We suppose we have $f \in C_c^\infty(\mathcal{U})$ such that $-\widetilde{L^f}$ is degenerate. We show that there exists some $g \in C_c^\infty(\mathcal{U})$ such that $-\widetilde{L^{f+tg}}$ is nondegenerate for all small non-zero t whenever $\mathrm{Ker}(-\widetilde{L^f}) \neq 0$.

We construct $g \in C_c^\infty(\mathcal{U})$ as follows. Denote $\mathcal{U}_j \subset\subset \mathcal{U}$ an increasing exhaustion of $\mathcal{U}$, and g_j a corresponding cut-off function with $g_j \in C_c^\infty(\mathcal{U}; [0,1])$ and $g_j|_{\mathcal{U}_j} \equiv 1$ (so $g_j \to 1$ weakly in $L^\infty(\mathcal{U})$). For each j we then consider the bilinear forms

$$I_j : Ker(-\widetilde{L^f}) \times Ker(-\widetilde{L^f}) \to \mathbb{R}, \quad (\phi_1, \phi_2) \to \int_{\mathcal{U}} g_j \phi_1 \phi_2.$$

$\mathrm{Ker}(-\widetilde{L^f})$ has finite dimension by (2). We ensure that I_j is a nondegenerate bilinear form. Indeed, if not, then there exists a sequence $\phi_j \in \mathrm{Ker}(-\widetilde{L^f})$ with $\|\phi_j\|_{L^2} = 1$ and $I_j(\phi_j, \psi) = 0$ for all $\psi \in \mathrm{Ker}(-\widetilde{L^f})$. Because $\phi_j \in \mathrm{Ker}(-\widetilde{L^f})$ for all j and Theorem 4.2.16 above, there exists $\phi \in \mathrm{Ker}(-\widetilde{L^f})$ such that $\phi_j \to \phi$ weakly in $\mathscr{B}$, so we have $\int_{\mathcal{U}} \phi\psi = 0$ for all $\phi \in \mathrm{Ker}(-\widetilde{L^f})$, which leads a contradiction. Next we will choose sufficiently large j so that I_j is nondegenerate; and denote $g = g_j, I = I_j$.

We now show the nondegeneracy of $-\widetilde{L^{f+tg}}$ for all small non-zero t. Assume there is a sequence $t_j \to 0$, such that for each j the operator $-\widetilde{L^{f+t_j g}}$ is degenerate. Choose $u_j \in \mathrm{Ker}(-\widetilde{L^{f+t_j g}})$ with $\|u_j\|_{L^2} = 1$ and denote by w_j the L^2-projection of u_j onto $\mathrm{Ker}(-\widetilde{L^f})$. Thus we have

$$\widetilde{L^f}(u_j - w_j) = t_j g u_j.$$

From (2), we denote by $\{E_j\}_j$ the L^2-orthogonal eigenspaces of $-\widetilde{L^f}$ with corresponding eigenvalues, $\{\lambda_j\}_j$. We can represent $u_j - w_j = \sum_{s=0}^{\infty} T_s \phi_s$ for some coefficients, T_s, with ϕ_s the eigenfunction with nonzero eigenvalue λ_s (with multiplicity). We have by the above that for any fixed s,

$$\langle u_j - w_j, \lambda_s \phi_s \rangle = Q(u_j - w_j, \phi_s) + \langle f\phi_s, u_j - w_j \rangle = \langle t_j g u_j, \phi_s \rangle.$$

We thus compute that

$$\begin{aligned} \|u_j - w_j\|_{L^2}^2 &= \sum_s |\langle u_j - w_j, \phi_s \rangle|^2 \\ &\leq \sum_s \frac{1}{\lambda^2} |\langle t_j g u_j, \phi_s \rangle|^2 \\ &\leq \frac{1}{\lambda^2} |t_j|^2 |g|_\infty \sum_s |\langle u_j, \phi_s \rangle|^2 \\ &= \frac{1}{\lambda^2} |t_j|^2 |g|_\infty^2 \cdot \|u_j\|_{L^2}^2 \\ &= \frac{1}{\lambda^2} |t_j|^2 |g|_\infty^2. \end{aligned}$$

where $\lambda = \inf_s \{|\lambda_s| \mid \lambda_s \neq 0\} > 0$ (as the eigenvalues are discrete and diverge). So we have

$$\|u_j - w_j\|_{L^2} \leq \frac{1}{\lambda} |t_j|.$$

Thus in particular we have that $\|u_j - w_j\|_{L^2} \to 0$ as $j \to \infty$.

For each j, we denote $\widetilde{u}_j = (u_j - w_j)/t_j$ so that $\|\widetilde{u}_j\|_{L^2} \leq \frac{1}{\lambda}$ and

$$Q_\Sigma(\widetilde{u}_j, \widetilde{u}_j) = t_j \langle g \widetilde{u}_j, \widetilde{u}_j \rangle + \langle g w_j, \widetilde{u}_j \rangle + \langle f \widetilde{u}_j, \widetilde{u}_j \rangle.$$

Therefore, $\limsup_j \|\widetilde{u}_j\|_{\mathscr{B}} < +\infty$. Similarly, since $w_j \in Ker(\widetilde{L^f})$ and $\|w_j\|_{L^2} \leq \|u_j\|_{L^2} = 1$, we have $\limsup_j \|w_j\|_{\mathscr{B}} < +\infty$. By (1), there are sub-sequences (not relabelled) and $\widetilde{u}, w \in L^2(\Sigma)$ such that both $\widetilde{u}_j \to \widetilde{u}$ and $w_j \to w$ strongly in $L^2(\Sigma)$. such that $\widetilde{L}^f \widetilde{u} = gw$ with $\|w\|_{L^2} = 1$ (as $\|u_j - w_j\|_{L^2} \to 0$ as $j \to \infty$). Note that for any $\psi \in \mathrm{Ker}(\widetilde{L^f})$, we then have

$$\langle gw, \psi \rangle = \langle \widetilde{u}, \widetilde{L^f}\psi \rangle = 0,$$

which contradicts with the nondegeneracy of $I = I_j := \langle g\cdot, \cdot \rangle_{L^2}$. □

An application of Theorem 4.2.17 is its relation to the index theorem. For constant mean curvature hypersurfaces, we can define the index in two different ways. One is the same as for minimal surfaces (similar to [13]), which, roughly speaking, is the number of directions in which the second variation of the area functional is decreasing. The other definition restricts the variations to those that are volume-preserving; we call this the weak index (or ind_0 below). For instance, an isoperimetric hypersurface may have a positive index, but its ind_0 is zero.

Definition 4.2.18. Suppose $\Sigma \subset (M^{n+1}, g)$ is a LSCMC hypersurface. $U \subset M$ is an open subset. As in [5], we define

$$\begin{aligned} ind(\Sigma; U) = \sup\{\dim E : E \subset C_c^\infty(\Sigma \cap U) \text{ linear subspace such that} \\ \int_\Sigma |\nabla\phi|^2 < \int_\Sigma (|A_\Sigma|^2 + Ric(\nu))\phi^2 \text{ for any } \phi \in E\}, \\ ind_0(\Sigma; U) = \sup\{\dim E : E \subset \mathscr{D}_T^1(\Sigma \cap U) \text{ linear subspace such that} \\ \int_\Sigma |\nabla\phi|^2 < \int_\Sigma (|A_\Sigma|^2 + Ric(\nu))\phi^2 \text{ for any } \phi \in E\}. \end{aligned}$$

By Theorem 4.2.17, we directly obtain the following equivalence of definitions of the index.

Corollary 4.2.19. *Suppose $\Sigma \subset (M^{n+1}, g)$ is a LSCMC hypersurface. Let $U \subset M$ be an open subset such that ∂U is smooth and intersects Σ transversely. Then we have*

- $ind_0(\Sigma; U) \le ind(\Sigma; U) \le ind_0(\Sigma; U) + 1,$

- *for W_j is the eigenspace corresponding to the eigenvalue λ_j defined in Theorem 4.2.17, we have*

$$ind_0(\Sigma; U) = \sum_{\lambda_j < 0} \dim W_j.$$

- *$ind(\Sigma; U)$ and $ind_0(\Sigma; U)$ are both finite.*

As mentioned in Chapter 1, this theorem generalizes the Jacobi operator studied in isoperimetric hypersurfaces in low dimensions ([11]). For any isoperimetric hypersurface $\Sigma \subset (M^8, g)$, Σ is a locally stable CMC hypersurface with isolated singularities. Currently, we are working on isoperimetric hypersurfaces under generic metrics in dimension 8. Note that, unlike in lower dimensions, we can use methods from functional analysis to study the generic regularity problem, i.e., the method in [11, Proposition 5.2/Corollary 5.3] does not include all isoperimetric regions (e.g., the singular isoperimetric region constructed in Theorem 3.1.16). In dimension 8, suppose there is a sequence of isoperimetric regions $\Omega_j \in \mathcal{A}_{g_j}(M, t)$ and $\Omega \in \mathcal{A}_g(M, t)$ such that $\Omega_j \to \Omega$ in L^1 and the Riemannian metrics $g_j \to g$ in C^4. In this scenario, we have $|\partial^* \Omega_j| \to |\partial^* \Omega|$ in the varifold sense with multiplicity 1. By Allard's regularity theorem, $\partial^* \Omega_j$ is a graph of $\partial^* \Omega$ for the domain on $\partial^* \Omega$ which is away from singularities. So, the twisted Jacobi field generated by Ω_j may only be in $C^2_{loc}(\partial^* \Omega)$. Thus, we may want to study whether a strong solution of the Jacobi field will belong to $\mathcal{B}_T(\Sigma)$. In this case, we may need to have extra assumptions on the twisted Jacobi field near the singularities on $\partial \Omega$. We provide a necessary condition where we have a weak solution. Next, we denote Σ a LSCMC hypersurface with isolated singularities; $U \subset M$ an open subset such that ∂U is smooth and intersects Σ transversely. Denote $\overline{\mathcal{U}} := \Sigma \cap \mathrm{Clos}(U)$ and $\partial \mathcal{U} := \partial U \cap \Sigma$. Consider the asymptotic rate (definition also used in the minimal surface case [42, 24]):

$$\mathcal{AR}_p(u) := \sup \left\{ \gamma : \limsup_{s \to 0} \int_{A_{s,2s}(p)} u^2 \cdot \rho^{-n-2\gamma} = 0 \right\},$$

where $p \in Sing(\Sigma)$ and $\rho(x) = \text{dist}(x, Sing(\Sigma))$. Then, we have the following proposition.

Proposition 4.2.20. *Suppose $u \in C^2_{loc}(\mathcal{U})$ satisfies the following:*

(1) $Lu = C$ for some constant C,

(2) $u|_{\partial\mathcal{U}} = 0$,

(3) $\int_{\mathcal{U}} u = 0$,

(4) $\mathcal{AR}_p(u) > -(n-2)/2$ for all $p \in Sing(\mathcal{U})$,

Then $u \in \mathscr{B}_{0,T}(\mathcal{U})$ is a weak solution of $\widetilde{L}u = 0$.

Proof. At first, we show that $u \in \mathscr{B}_{0,T}(\mathcal{U})$. By (4) and the same argument as the proof of [24, Lemma 2.4 (iii)], we have

$$\int_{\Sigma} |\nabla u|^2 + \rho^{-2}u^2 < +\infty.$$

Therefore, by (2) and (3), we have $u \in L^2_T(\mathcal{U})$ and $\|u\|_{\mathscr{B}} < +\infty$. So we have $u \in \mathscr{B}_{0,T}(\mathcal{U})$.

Fix any $v \in \mathscr{B}_{0,T}(\mathcal{U})$, suppose $\{\phi_j\}$ is a sequence of $C^\infty_c(\mathcal{U}) \cap L^2_T(\mathcal{U})$ such that $\phi_j \to v$ in $\mathscr{B}$. Then by (1), we have

$$0 = \int_{\mathcal{U}} -Lu\phi_j = Q_\Sigma(u, \phi_j),$$

for all $j \geq 1$. Therefore, we have $Q_\Sigma(u, v) = 0$. So $\widetilde{L}u = 0$ weakly. □

On the other hand, we are also interested in the regularity of weak solutions. Next, we present the interior regularity result for the twisted weak solutions. Because we are only interested in the local interior regularity, we denote the open subsets $\mathcal{V}, \mathcal{N}, \mathcal{O} \Subset \Sigma$, where $\Sigma \subset (M, g)$ is still a locally stable CMC hypersurface.

Theorem 4.2.21. *(Interior H^2-regularity). Suppose furthermore that $u \in H^1_{loc}(\mathcal{O}) \cap L^2(\mathcal{O})$ is a weak solution of*

$$\widetilde{L}u = f \tag{4.2.12}$$

for some $f \in L^2(\mathcal{O})$, i.e., for any $\phi \in \mathscr{B}_{0,T}(\mathcal{O})$, we have

$$Q_\Sigma(u, \phi) = \langle f, \phi \rangle_{L^2(\mathcal{O})}.$$

Then

$$u \in H^2_{loc}(\mathcal{O}). \tag{4.2.13}$$

In addition, fixing any open subset $\mathcal{V}, \mathcal{N} \Subset \mathcal{O}$ such that $\overline{\mathcal{V}}$ is disjoint with $\overline{\mathcal{N}}$, we have

$$\|u\|_{H^2(\mathcal{V})} \leq C_1 \left(\|f\|_{L^2(\mathcal{O})} + \|u\|_{L^2(\mathcal{O})} \right) + C_2 \|u\|_{H^1(\mathcal{N})}, \tag{4.2.14}$$

for some constant $C_1 := C(\mathcal{V}, \mathcal{O}, g, \Sigma)$ and $C_2 := C(\mathcal{V}, \mathcal{O}, \mathcal{N}, g, \Sigma)$.

Proof. Consider an open set $\mathcal{N} \Subset \mathcal{O}$ and $\mathcal{N}, \mathcal{V}$ are disjoint. Note that fixing any $\phi \in C^1_c(\mathcal{V})$, and any function $\psi \in C^1_c(\mathcal{N})$ such that $\int_\Sigma \phi - \psi = 0$, we have

$$Q_\Sigma(u, \phi - \psi) = \langle f, \phi - \psi \rangle. \tag{4.2.15}$$

We have the equality with a large choice of ϕ, and only an integral constraint with ψ, i.e., it holds for any $\phi \in C^1_c(\mathcal{V})$ as long as $\int_\Sigma \phi - \psi = 0$. We will empoly it to get a similar regularity result as standard elliptic PDEs.

For simplicity, we denote $b(x) := |A_\Sigma|^2 + \mathrm{Ric}(\nu)$. We rewrite (4.2.15) to get

$$\int_\Sigma \langle \nabla u, \nabla \phi \rangle = \langle f, \phi - \psi \rangle + Q_\Sigma(u, \psi) + \int_\Sigma b u \phi.$$

We will first estimate the LHS. Choose open sets $\mathcal{W}, \mathcal{U}$ such that $\mathcal{V} \Subset \mathcal{W} \Subset \mathcal{U}$ and $\mathcal{U}$ is also disjoint with $\mathcal{N}$. WLOG, we first assume that there is a coordinate $\Phi : \mathbb{B}_1 \to \mathcal{U}$, and denote W, V, U by the open sets under the coordinate. Then select a smooth function ζ satisfying

$$\begin{cases} \zeta \equiv 1 \text{ on } V \\ 0 \leq \zeta \leq 1, \\ \operatorname{spt} \zeta \subset W. \end{cases}$$

Now for small $|h| > 0$ and integer $k \in \{1, \ldots, n\}$, we consider the following test function

$$\phi := -D_k^{-h}(\zeta^2 D_k^h u)$$

where $D_k^h u$ denotes the difference quotient (see also [14, Chapter 5])

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h} \quad (h \in \mathbb{R}, h \neq 0).$$

Therefore, we have

$$\begin{aligned}
LHS := & \int_\Sigma \langle \nabla u, \nabla \phi \rangle = \int_U a^{ij} u_{x_i} \phi_{x_j} \, dx \\
= & - \int_U a^{ij} u_{x_i} \left[D_k^{-h} \left(\zeta^2 D_k^h u \right) \right]_{x_j} dx \\
= & \int_U D_k^h (a^{ij} u_{x_i}) \left(\zeta^2 D_k^h u \right)_{x_j} dx \\
= & \int_U a^{ij,h} \left(D_k^h u_{x_i} \right) \left(\zeta^2 D_k^h u \right)_{x_j} + \left(D_k^h a^{ij} \right) u_{x_i} \left(\zeta^2 D_k^h u \right)_{x_j} dx \\
= & \int_U a^{ij,h} D_k^h u_{x_i} D_k^h u_{x_j} \zeta^2 \, dx + \int_U a^{ij,h} D_k^h u_{x_i} D_k^h u 2 \zeta \zeta_{x_j} \, dx \\
& + \int_U \left(D_k^h a^{ij} \right) u_{x_i} D_k^h u_{x_j} \zeta^2 \, dx + \left(D_k^h a^{ij} \right) u_{x_i} D_k^h u 2 \zeta \zeta_{x_j} \, dx,
\end{aligned}$$

where a^{ij} depends on the Riemannian metric and $a^{ij,h}(x) := a^{ij}(x + he_k)$.

The coefficients of Riemannian metric implies that there is a constant $\theta := \theta(g, U) >$

0 such that

$$\int_U a^{ij,h} D_k^h u_{x_i} D_k^h u_{x_j} \zeta^2 \, dx \geq \theta \int_U \zeta^2 |D_k^h Du|^2 \, dx.$$

And the rest part implies that

$$\begin{aligned} |L| :=& \left| \int_U a^{ij,h} D_k^h u_{x_i} D_k^h u 2\zeta \zeta_{x_j} \, dx + \int_U \left(D_k^h a^{ij}\right) u_{x_i} D_k^h u_{x_j} \zeta^2 \, dx + \left(D_k^h a^{ij}\right) u_{x_i} D_k^h u 2\zeta\zeta_{x_j} \, dx \right| \\ &\leq \theta/2 \int_U \zeta^2 |D_k^h Du|^2 \, dx + C/\theta \int_W \left(|D_k^h u|^2 + |Du|^2\right) \, dx \\ &\leq \theta/2 \int_U \zeta^2 |D_k^h Du|^2 \, dx + C(\theta) \int_U |Du|^2 \, dx, \end{aligned}$$

Therefore, we have

$$LHS \geq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 \, dx - C \int_U |Du|^2 \, dx.$$

Next we study the RHS, i.e.,

$$RHS = \langle f + bu, \phi \rangle_{L^2(\mathcal{O})} + Q_\Sigma(u, \psi) - \langle f, \psi \rangle_{L^2(\mathcal{O})}.$$

So

$$|\langle f + bu, \phi \rangle_{L^2(\mathcal{O})}| \leq C \int_U (|f| + |u|) \, |\phi| \, dx.$$

We construct the cutoff function $\psi \in C_c^1(\mathcal{N})$ by the following. At first, we consider the smooth function η by

$$\begin{cases} \operatorname{spt} \eta \subset \mathcal{N} \\ \int_{\mathcal{N}} \eta = 1 \end{cases}$$

Then we define $\psi(x) := A \cdot \eta$ where A is the constant $A := \int_{\mathcal{N}} \phi$. Therefore, a direct computation implies that

$$|RHS| \leq C \int_U (|f| + |u|)\,|\phi|\,dx + |A| \left(\int_{\mathcal{N}} |\nabla u| \cdot |\nabla \eta| + (|f| + |bu|) \cdot |\eta| \right) \tag{4.2.16}$$

$$\leq C \int_U (|f| + |u|)\,|\phi|\,dx + |A| C(\mathcal{N}) \left(\int_{\mathcal{N}} |\nabla u| + (|f| + |bu|) \right). \tag{4.2.17}$$

On the other hand, by the properties of quotient difference, we have

$$\begin{aligned}\int_U |\phi|^2\,dx &\leq C \int_U |D(\zeta^2 D_k^h u)|^2\,dx \\ &\leq C \int_W |D_k^h u|^2 + \zeta^2 |D_k^h Du|^2\,dx \\ &\leq C \int_U |Du|^2 + \zeta^2 |D_k^h Du|^2\,dx.\end{aligned}$$

Then the interpolation inequality implies that

$$|RHS| \leq \frac{\theta}{4} \int_U \zeta^2 |D_k^h Du|^2\,dx + C \int_U f^2 + u^2 + |Du|^2\,dx + C(\|f\|_{L^2(\mathcal{N})} + \|u\|_{H^1(\mathcal{N})}).$$

Note that the constants are depending on $C := C(\Sigma, U, \mathcal{N}, g)$. Therefore, $LHS = RHS$ implies that

$$\int_V |D_k^h Du|^2\,dx \leq \int_U \zeta^2 |D_k^h Du|^2\,dx \leq C \int_U f^2 + u^2 + |Du|^2\,dx$$

for $k = 1, \ldots, n$ and all sufficiently small $|h| \neq 0$. Therefore, we have $Du \in H^1_{\text{loc}}(U; \mathbb{R}^n)$, and thus $u \in H^2_{\text{loc}}(U)$. In addition, we have the estimate

$$\|u\|_{H^2(\mathcal{V})} \leq C \left(\|f\|_{L^2(\mathcal{U})} + \|u\|_{H^1(\mathcal{U})} \right) + C(\|f\|_{L^2(\mathcal{N})} + \|u\|_{H^1(\mathcal{N})}).$$

Note that because $\mathcal{V} \Subset \mathcal{W} \Subset \mathcal{U}$, by a suitable change of the constant (depending on $\mathcal{V}$ and $\mathcal{W}$), we can let $\mathcal{W}$ play the role of $\mathcal{U}$ and get

$$\|u\|_{H^2(\mathcal{V})} \leq C\left(\|f\|_{L^2(\mathcal{W})} + \|u\|_{H^1(\mathcal{W})}\right) + C(\|f\|_{L^2(\mathcal{N})} + \|u\|_{H^1(\mathcal{N})}). \tag{4.2.18}$$

Finally, choose a new cutoff function ζ satisfying

$$\begin{cases} \zeta \equiv 1 \text{ on } W, \operatorname{spt} \zeta \subset U, \\ 0 \leq \zeta \leq 1. \end{cases}$$

And let $\phi = \zeta^2 u$, ψ same as above, i.e., we have $\psi = \eta A$ with $\int_\Sigma \phi - \psi = 0$), where

$$\begin{cases} \operatorname{spt} \eta \subset \mathcal{N}, \\ \int_{\mathcal{N}} \eta = 1. \end{cases}$$

Then by the assumption that u a weak solution, and similar interpolation inequality argument, we have

$$\int_U \zeta^2 |Du|^2 \, dx \leq C \int_U f^2 + u^2 \, dx + +C(\|f\|_{L^2(\mathcal{N})} + \|u\|_{H^1(\mathcal{N})}).$$

Thus

$$\|u\|_{H^1(\mathcal{W})} \leq C\left(\|f\|_{L^2(\mathcal{O})} + \|u\|_{L^2(\mathcal{O})}\right) + +C\|u\|_{H^1(\mathcal{N})}.$$

Then combined with (4.2.18), we get (4.2.14).

□

Inductively, we can obtain a higher-order regularity result for the weak solutions. The method is similar to that used for weak solutions of a standard elliptic PDE, with the only differences being that we employ the integral zero property for test functions and additionally estimate the extra terms (similar to Theorem 4.2.21).

Corollary 4.2.22. *Suppose $u \in H^1_{loc}(\mathcal{O}) \cap L^2(\mathcal{O})$ is a weak solution of $\widetilde{L}u = f$ for some $f \in C^\infty(\mathcal{O}) \cap L^2(\mathcal{O})$, then $u \in C^\infty(\mathcal{O})$.*

Proof. Using the same notation as above, we denote the open sets $\mathcal{V} \Subset \mathcal{W} \Subset \mathcal{U} \Subset \mathcal{O}$ and $\overline{\mathcal{U}}$ is disjoint with $\overline{\mathcal{N}}$. Suppose we have a coordinate patch in $\mathcal{W}$ and denote W under the coordinate. Next we prove the regularity by induction. The base case is proved above. Now suppose $u \in H^{m+2}_{\text{loc}}(\mathcal{O})$ and $f \in H^{m+1}(\mathcal{O})$, and the inequality

$$\|u\|_{H^{m+2}(\mathcal{W})} \leq C(\|f\|_{H^m(\mathcal{O})} + \|u\|_{L^2(\mathcal{O})}) + C\|u\|_{H^1(\mathcal{N})}.$$

Let α is a index with $|\alpha| = m+1$. Denote $\phi \in C^\infty_c(\mathcal{W})$ and $\psi \in C^\infty_c(\mathcal{N})$ such that $\psi := A \cdot \eta$ with $\operatorname{spt} \eta \subset \mathcal{N}$, $\int_{\mathcal{N}} \eta = 1$ and $A := \int_{\mathcal{W}} \phi$.

Because u is a weak solution, we have

$$Q_\Sigma(u, \phi - \psi) = \langle f, \phi - \psi \rangle,$$

i.e.,

$$Q_\Sigma(u, \phi) = \langle f, \phi \rangle + Q_\Sigma(u, \psi) - \langle f, \psi \rangle.$$

So we get

$$\int_{\mathcal{W}} \langle \nabla u, \nabla \phi \rangle = \langle f + bu, \phi \rangle + Q_\Sigma(u, \psi) - \langle f, \psi \rangle.$$

For any $\tilde{\phi} \in C^\infty_c(\mathcal{W})$, we use the test function

$$\phi := (-1)^{|\alpha|} D^\alpha \tilde{\phi}.$$

Therefore, we have

$$\int_W a_{ij}u_{x_i}[(-1)^{|\alpha|}D^\alpha\tilde{\phi}]_{x_j}\,dx = \int_W (f+bu)\phi\,dx + Q_\Sigma(u,\psi) - \langle f,\psi\rangle.$$
$$\Rightarrow \int_W D^\alpha(a_{ij}u_{x_i})\tilde{\phi}_{x_j}\,dx = \int_W D^\alpha(f+bu)\tilde{\phi}\,dx + Q_\Sigma(u,\psi) - \langle f,\psi\rangle.$$

By the Leibniz Rule, we obtain

$$\int_W \sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}a_{ij}D^\beta u_{x_j}\tilde{\phi}_{x_j}\,dx = \int_W D^\alpha f\tilde{\phi}\,dx + \int_W \sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}bD^\beta u\tilde{\phi}\,dx$$
$$+ Q_\Sigma(u,\psi) - \langle f,\psi\rangle.$$

Therefore,

$$\int_W a_{ij}D^\alpha u_{x_i}\tilde{\phi}_{x_j}\,dx - \int_W bD^\alpha u\tilde{\phi}\,dx = Q_\Sigma(D^\alpha u,\tilde{\phi})$$
$$= \left\langle D^\alpha f + \sum_{\beta\le(\neq)\alpha}\binom{\alpha}{\beta}\left((D^{\alpha-\beta}a_{ij}D^\beta u_{x_i})_{x_j} + D^{\alpha-\beta}bD^\beta u\right) + d(x), \tilde{\phi}\right\rangle_{L^2(\mathcal{W})},$$

where d is a function such that

$$\int_{\mathcal{W}} d(x)\cdot\tilde{\phi} = Q_\Sigma(u,\psi) - \langle f,\psi\rangle.$$

We define the function $d(x)$ by the following, note that

$$Q_\Sigma(u,\psi) - \langle f,\psi\rangle = \int_{\mathcal{N}} \langle\nabla u,\nabla\psi\rangle - (bu+f)\psi \tag{4.2.19}$$
$$= \int_{\mathcal{N}} \langle\nabla u,\nabla(A\cdot\eta)\rangle - (bu+f)(A\cdot\eta) \tag{4.2.20}$$
$$= A\cdot\int_{\mathcal{N}} \langle\nabla u,\nabla\eta\rangle - (bu+f)\eta \tag{4.2.21}$$
$$= A\cdot C(\mathcal{N},\|f\|_{L^2(\mathcal{N})},\|u\|_{H^1(\mathcal{N})}), \tag{4.2.22}$$

where

$$
\begin{aligned}
A :&= \int_W (-1)^{|\alpha|} D^\alpha \tilde{\phi} \sqrt{g}\, dx \\
&= \int_W \tilde{\phi} \frac{D^\alpha(\sqrt{g})}{\sqrt{g}} \sqrt{g}\, dx,
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
Q_\Sigma(u,\psi) - \langle f, \psi \rangle &= A \cdot C(\mathcal{N}, \|f\|_{L^2(\mathcal{N})}, \|u\|_{H^1(\mathcal{N})}), \\
&= C(\mathcal{N}, \|f\|_{L^2(\mathcal{N})}, \|u\|_{H^1(\mathcal{N})}) \int_W \tilde{\phi} \frac{D^\alpha(\sqrt{g})}{\sqrt{g}} \sqrt{g}\, dx \\
&= \int_{\mathcal{W}} d(x) \cdot \tilde{\phi}.
\end{aligned}
$$

We denote

$$
\tilde{f} := D^\alpha f + \sum_{\beta \leq (\neq) \alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta} a_{ij} D^\beta u_{x_i})_{x_j} + D^{\alpha-\beta} b D^\beta u) + d(x)
$$

Therefore, we have $D^\alpha u$ is a weak solution in $\mathcal{W}$, i.e.,

$$
Q_\Sigma(D^\alpha u, \tilde{\phi}) = \langle \tilde{f}, \tilde{\phi} \rangle_{L^2(\mathcal{W})},
$$

for any $\phi \in C_c^\infty(\mathcal{W})$. And

$$
\begin{aligned}
\left\| \tilde{f} \right\|_{L^2(\mathcal{W})} &\leq C(\|f\|_{H^{m+1}(\mathcal{U})} + \|u\|_{H^{m+2}(\mathcal{W})}) + C(\|f\|_{L^2(\mathcal{N})} + \|u\|_{H^1(\mathcal{N})}) \\
&\leq C\|f\|_{H^{m+1}(\mathcal{U})} + C(\|f\|_{H^m(\mathcal{O})} + \|u\|_{L^2(\mathcal{O})}) + C\|u\|_{H^1(\mathcal{N})} \\
&\leq C\|f\|_{H^{m+1}(\mathcal{O})} + C\|u\|_{L^2(\mathcal{O})}) + C\|u\|_{H^1(\mathcal{N})}.
\end{aligned}
$$

Let denote $\tilde{u} = D^\alpha u$, so we have $\tilde{u} \in H^2_{loc}(\mathcal{O})$ and

$$\|\tilde{u}\|_{H^2(\mathcal{V})} \leq C(\|\tilde{f}\|_{L^2(\mathcal{W})} + \|\tilde{u}\|_{L^2(\mathcal{W})} + C\|\tilde{u}\|_{H^1(\mathcal{N})}).$$

$$\Rightarrow \|u\|_{H^{m+3}(\mathcal{V})} \leq C(\|f\|_{H^{m+1}(\mathcal{O})} + \|u\|_{L^2(\mathcal{O})}) + C\|u\|_{H^{m+2}(\mathcal{N})}.$$

Therefore, $u \in H^m_{loc}(\mathcal{O})$ for all $m \in \mathbb{N}$ and thus $u \in C^\infty(\mathcal{O})$. □

This chapter will be partially included in some unpublished collaborative work with Kobe Marshall-Stevens and Davide Parise [26]. The dissertation author was the primary author of this part.

Appendix A

Graphical Functions with Singular Hypersurfaces

Suppose $\Sigma \subset (M^{n+1}, g)$ an embedded two-sided C^3 hypersurface (with singularities). Denote ν the unit normal vector field on Σ. Suppose $u \in C^2(\Sigma)$, we can define

$$\mathrm{graph}_{\Sigma,g}(u) := \{\exp_x^g(u(x)\nu(x)) : x \in \Sigma\}.$$

Clearly, $\mathrm{graph}_{\Sigma,g}(u)$ is a C^2 hypersurface in M. It is natural to inquire about the embededness and geometry of $\mathrm{graph}_{\Sigma,g}(u)$ when u is sufficiently small in the C^2 sense. Because Σ has singularities, we want to define a C^k norm with invariance under scaling. Here we introduce the regularity scale, which is used in multiple papers (e.g. [31, 24, 42]).

Definition A.0.1. For every $x \in \Sigma$, we define the **regularity scale** $r_S = r_S(x; M, g, \Sigma)$ of Σ at x to be the supremum among all $r \in (0, \mathrm{injrad}(x; M, g)/2)$ such that,

- $r^2\|\mathrm{Rm}_g\|_{C^0, B_r^g(x)} + r^3\|\nabla \mathrm{Rm}_g\|_{C^0, B_r^g(x)} \leq 1/10$;
- In T_xM,
$$\frac{1}{r}(\exp_x^{-1}(\Sigma) \cap \mathbb{B}_1) = \mathrm{graph}_L u \cap \mathbb{B}_1,$$
for some linear hyperplane $L \subseteq T_xM$ and $u \in C^3(L)$ with $\|u\|_{C^3} \leq 1/10$.

Using the regularity scale, we define the point-wise C^k-norm for $f \in C^k(M)$ and

$\beta \in Sym(T^*M \otimes T^*M)$: for every $x \in \Sigma$,

$$[f]_{x,g,C^k_*} := \sum_{j=0}^{k} r_{\mathcal{S}}(x)^j \sup_{B^g_{r_{\mathcal{S}}(x)}} |\nabla^j_g f|; \qquad [\beta]_{x,g,C^k_*} := \sum_{j=0}^{k} r_{\mathcal{S}}(x)^j \sup_{B^g_{r_{\mathcal{S}}(x)}} |\nabla^j_g \beta|$$

These norms are also invariant under scaling: for any $\lambda > 0$, we have

$$[f]_{x,\lambda^2 g,C^k_*} = [f]_{x,g,C^k_*}, \qquad [\lambda^2\beta]_{x,\lambda^2 g,C^k_*} = [\beta]_{x,g,C^k_*}.$$

We list the following theorem we will be frequently used in Chapter 4. A full proof can be found in [24, Theorem B.1].

Theorem A.0.2. *There exists $\delta = \delta(n) \in (0,1)$ and $C = C(n) >> 1$ with the following properties.*

(i) If $u \in C^1(\Sigma)$ with $\|u\|_{C^1_*} \le \delta$, then

$$\Phi^u : \Sigma \to M, \quad x \mapsto \exp^g_x(u(x)\cdot\nu(x)),$$

is a C^1 embedding;

(ii) There exists a C^1 area density function $F^f = F^f(x,z,\xi)$, where $x \in \Sigma$, $z \in \mathbb{R}$ with $r_{\mathcal{S}}^{-1}|z| < 1$, $\xi \in T^*_x\Sigma$ with $|\xi|_g < 1$, with pointwise estimate

$$|F^f(x,z,\xi) - 1| \le C(n)(r_{\mathcal{S}}(x)^{-1}|z| + |\xi| + [f]_{x,C^2_*});$$

And such that for every $u \in C^2(\Sigma)$ with $\|u\|_{C^2_*} \le \delta$, every $f \in C^2(M)$ with $[f]_{x,C^2_*} \le \delta$, $\forall x \in \Sigma$, and every $\varphi \in C^0_c(M \setminus \mathrm{Sing}(\Sigma))$, we have

$$\int_M \varphi(x)\ d\|\operatorname{graph}_\Sigma(u)\|_{(1+f)g}(x) = \int_M \varphi\circ\Phi^u(x)\cdot F^f(x,u(x),du(x))\ d\|\Sigma\|_g.$$

(iii) Let F^f be in (ii); let $\mathscr{M}^f : C^2_*(\Sigma) \to C^0_{loc}(\Sigma)$ be the minimal surface operator ($\mathscr{M}^f$ is only defined in the δ-neighborhood of $\mathbf{0}$), in other words, for every $\varphi \in C^1_c(\Sigma)$,

$$\int_\Sigma \mathscr{M}^f(u) \cdot \varphi \, d\|\Sigma\|_g := \frac{d}{dt}\Big|_{t=0} \int_\Sigma F^f\left(x, u + t\varphi, d(u + t\varphi)\right) \, d\|\Sigma\|_g .$$

Then for every pair $f^\pm \in C^2(M)$ with $[f^\pm]_{x,C^2_*} \le \delta$ along $x \in \Sigma$ and every pair $\|u^\pm\|_{C^2_*} \le \delta$, we have

$$\begin{aligned} \mathscr{M}^{f^+}(u^+) - \mathscr{M}^{f^-}(u^-) = & - L_{\Sigma,g}(u^+ - u^-) + \frac{n}{2}\nu(f^+ - f^-) \\ & + div_{\Sigma,g}(\mathcal{E}_1) + r_{\mathcal{S}}^{-1}\mathcal{E}_2, \end{aligned}$$

where $\mathcal{E}_1, \mathcal{E}_2$ are functions on Σ satisfying the pointwise bound along $x \in \Sigma$,

$$\begin{aligned} |\mathcal{E}_1(x)| + |\mathcal{E}_2(x)| \le C(n) & \left(\sum_{i=\pm} [f^i]_{x,C^2_*} + r_{\mathcal{S}}(x)^{-1}|u^i|(x) + |du^i(x)| \right) \\ & \cdot \left([f^+ - f^-]_{x,C^2_*} + r_{\mathcal{S}}(x)^{-1}|u^+ - u^-|(x) + |d(u^+ - u^-)|(x) \right) . \end{aligned}$$

(iv) Let $u \in C^2(\Sigma)$ such that $\|u\|_{C^2_*} \le \delta$, $\tilde{g}$ be a metric on M such that $[\tilde{g} - g]_{x,C^3_*} \le \delta$ for every $x \in \Sigma$. Denote for simplicity $\Sigma_u = \mathrm{graph}_{\Sigma,g}(u)$. Then we have pointwise estimate,

$$\left| |A_{\Sigma_u,\tilde{g}}|^2_{\tilde{g}} \circ \Phi^u - |A_\Sigma|^2_g \right| \le C(n) \left([\tilde{g} - g]_{x,C^3_*} + \sum_{j=0}^{2} r_{\mathcal{S}}^{j-1} |\nabla^j_{\Sigma,g} u| \right) \cdot r_{\mathcal{S}}^{-2} ;$$

And for every $\psi \in C^2_{loc}(\Sigma_u)$, we have

$$(\Delta_{\Sigma_u}\psi) \circ \Phi^u - \Delta_\Sigma(\psi \circ \Phi^u) = \mathrm{div}_\Sigma(\vec{B}_0) + r_{\mathcal{S}}^{-1} \cdot B_1 ,$$

with pointwise estimate on error terms,

$$|\vec{B}_0| + |B_1| \leq C(n) \left([\tilde{g} - g]_{x, C^3_*} + \sum_{j=0}^{2} r_{\mathcal{S}}^{j-1} |\nabla^j_{\Sigma, g} u| \right) \cdot |d\psi|_g \, .$$

Appendix B

Minimal Cones with Isolated Singularities

B.1 Construction and Basic Properties

In this section, we will introduce some properties of the minimal hypersurfaces with isolated singularity. We will mostly introduce the results in [8, 21].

Example B.1.1 (Examples of minimal cones)**.** Given $\Sigma^{n-2} \subset \mathbb{S}^{n-1}$ a smooth minimal hypersurface, the **cone** based on Σ is defined as

$$C(\Sigma) := \{\lambda x :\ x \in \Sigma,\ \lambda > 0\}.$$

We also denote $C(\Sigma) := 0\#\Sigma$ and $\mathbf{C} := C(\Sigma)$ if there is no ambiguity of the cross section Σ; for a $R > 0$, we denote $\mathbf{C}_R := \mathbb{B}_R(0) \cap \mathbf{C}$.

Example B.1.2 (Examples of minimal cones)**. Clifford Hypersurfaces**: Given spheres $\mathbb{S}^p(r_1) \subset \mathbb{R}^{p+1}$ and $\mathbb{S}^q(r_2) \subset \mathbb{R}^{q+1}$, where $r_1^2 + r_2^2 = 1$, note that

$$\Sigma := \mathbb{S}^p(r_1) \times \mathbb{S}^q(r_2) \subset \mathbb{S}^{p+q+1} \subset \mathbb{R}^{p+q+2}.$$

Then $\Sigma \subset \mathbb{S}^{p+q+1}$ is minimal if and only if

$$\frac{p}{r_1^2} = \frac{q}{r_2^2} = p + q.$$

- For $p = 3, q = 3$: We get the Simons cone $C\big(\mathbb{S}^3(\frac{1}{\sqrt{2}}) \times \mathbb{S}^3(\frac{1}{\sqrt{2}})\big)$, which is **stable** and **area minimizing** [Bombieri–De Giorgi–Giusti '69].
- For $p = 1, q = 5$: We get the Simons cone $C\big(\mathbb{S}^1(\frac{1}{\sqrt{6}}) \times \mathbb{S}^5(\sqrt{\frac{5}{6}})\big)$, which is **stable**, but not area minimizing.

Remark B.1.3*.* Because the tangent cones of isoperimetric regions are area-minimizing cones, we will exhibit some properties about cones with isolated singularities, i.e., singular minimizing cones $K \subset \mathbb{R}^{n+1}$ with $Sing(K) = \{0\}$ and $n \geq 7$.

Remark B.1.4*.* For $\mathbf{C}$ a stable minimal hypercones in $\mathbb{R}^8$, that are smooth, closed, embedded hypersurfaces away from the origin. Denote the cross section $\Sigma := \mathbf{C} \cap \mathbb{S}^7(1)$. So Σ is a smooth orientable closed, embedded, codimension one hypersurface of $\mathbb{S}^7(1)$; by the maximum principle or Frankel's theorem on manifolds with positive Ricci curvature (i.e., minimal submanifolds in ambient manifolds of positive Ricci curvature must intersect), Σ is connected.

Definition B.1.5 ([21, section 3])**.** Let $\mathbf{C}$ be a regular hypercone (i.e. $Sing(\mathbf{C}) \subset \{0\}$), we say $\mathbf{C}$ is strictly minimizing if there is an $\theta > 0$ such that

$$\mathbf{M}(\mathbf{C}_1) \leq \mathbf{M}(S) - \theta\epsilon^n,$$

whenever $\epsilon > 0$ and S is an integer multiplicity current with $\operatorname{spt} S \Subset \mathbb{R}^{n+1} \setminus B_\epsilon$ and $\partial S = \partial \mathbf{C}_1$.

Hardt and Simon in [21, Theorem 3.2] exhibit several equivalent definitions of strictly minimizing. The critical property of strictly minimizing hypercone is the following property of graphical local minimizing.

Theorem B.1.6 ([21, Theorem 4.4])**.** *Suppose* $\mathbf{C}$ *be a regular strictly minimizing (multiplicity one) hypercone in* $\mathbb{R}^{n+1}$*. Let* M *be a smooth oriented embedded hypersurface in* $\mathbb{R}^{n+1}$ *with the representation in the form*

$$M = graph_{\boldsymbol{C}_1} h = \{x + h(x)\nu(x) : \ x \in \mathbf{C} \setminus \{0\}\},$$

where ν *is the unit normal vector and* h *is some function in* $C^2(\boldsymbol{C}_1)$ *such that*

$$\left|\frac{h(r\omega)}{r}\right| + |Dh(r\omega)| \leq Cr^q, \quad r\omega \in \boldsymbol{C}_1,$$

for some $q > 0$*.*

Assume that $\mathbb{R}^{n+1}$ *equipped a* C^3 *Riemannian metric* $g = \sum_{i,j=1}^{n+1} g_{ij} dx^i dx^j$*, which satisfies*

$$g_{ij}(0) = \delta_{ij}, \frac{\partial g_{ij}}{\partial x_k}(0) = 0, \quad i, j, k = 1, \dots, n+1.$$

Suppose M *is a minimal surface (mean curvature zero) in* $(\mathbb{R}^{n+1}, g)$*, then there is a* $\rho > 0$ *such that* M *is area minimizing in* $B_\rho(0)$ *with respect to the metric* g*.*

Remark B.1.7.

- In [40], Smale generalized the above local minimizing property in the manifold setting (Theorem 3.2.1 below), with additionally assuming the cone $\mathbf{C}$ is strictly stable. Moreover, Zhihan Wang in [42, Theorem 5.1] also presents a proof of local minimizing property by constructing a foliation over Σ.

- Simons' cone $\mathbf{C}$ is regular, strictly stable, and strictly minimizing. And it splits $\mathbb{R}^8$ into two parts E_+ and E_-. By [21, Theorem 2.1], each E_+, E_- contains one smooth area minimizing hypersurface up to scaling, call them R_+, R_- respectively. By the symmetry of $\mathbf{C}$, R_+, R_- are diffeomorphic to each other. So Simons' cone satisfies the requirement in Remark 3.1.17.

For minimal hypercones with isolated singularities, we have a discrete set of densities. Denote $\mathcal{C}$ the collection of stable minimal hypercones in $\mathbb{R}^8$ with the only singularity at 0. And for a $\Lambda > 0$, $\mathcal{C}_\Lambda := \{\mathbf{C} \in \mathcal{C} : \ \theta_{\mathbf{C}}(0) \le \Lambda\}$.

Proposition B.1.8. *The set of densities* $\{\theta_{\mathbf{C}}(0) : \ \mathbf{C} \in \mathcal{C}\}$ *is a discrete set, i.e.* $1 \le \theta_0 < \theta_1 < \theta_2 < \cdots \nearrow \infty$.

Sketch of proof: Let $\Sigma \subset \mathbb{S}^7$ be a smooth, closed, embedded, minimal hypersurface. Denote $\mu \in (0,1)$ an arbitrary constant. Then, by [36, Theorem 3], we have the existence of constants $\gamma(\Sigma, \mu) \in (0, 1/2)$, $\sigma(\Sigma, \mu) > 0$, such that for any function $u \in C^{2,\mu}(\Sigma, N(\Sigma))$, i.e., taking values in the normal bundle of Σ with $\|u\|_{C^{2,\mu}(\Sigma)} \le \sigma$, we have the Lojasiewicz-Simon inequality

$$|\mathcal{H}^6(G_\Sigma(u)) - \mathcal{H}^6(G_\Sigma(0))|^{1-\gamma} \le \|\mathcal{M}(u)\|_{L^2(\Sigma)},$$

where $\mathcal{M}(\cdot)$ is the mean curvature operator, i.e. the negative L^2-gradient of the area functional $u \mapsto \mathcal{H}^6(G_\Sigma(u))$. Here $G_\Sigma(u)$ denotes the graph of u over Σ, i.e.

$$G_\Sigma(u) := \left\{ \frac{\theta + u(\theta)}{\sqrt{1+|u|^2}}; \ \theta \in \Sigma \right\}.$$

Note that by the Lojasiewicz-Simon inequality, it follows that $G_\Sigma(u)$ is minimal precisely when $\mathcal{M}(u) = 0$, in which case we obtain that $\mathcal{H}^6(G_\Sigma(u)) = \mathcal{H}^6(G_\Sigma(0)) = \mathcal{H}^6(\Sigma)$. By Example B.1.2, we see that the minimality of the link is equivalent to minimality of the cone.

Consider now a sequence of cones $C_i \in \mathcal{C}$ with bounded density for some $\Lambda > 0$. Then, by Allard's compactness Theorem, we can find a limiting cone $\mathbf{C} \in \mathcal{C}$, a natural number $m \in \mathbb{N}$, such that $\mathbf{C}_i \to m\mathbf{C}$. So it converges smoothly with multiplicity m

away from the origin. Indeed, consider the sequence of stable cones $\mathbf{C}_i$ in $\mathbb{B}_1(0)$ with an isolated singularity at the origin. Then, we have uniform mass bounds, and the regular sets $\operatorname{Reg}(\mathbf{C}_i) \cap \mathbb{B}_1(0)$ are orientable being set-theoretically closed hypersurfaces in a simply connected ambient region $\mathbb{B}_1(0) \setminus \{0\}$ (recall that 0 is the singular set of the stable cones). In particular, by [35, Theorem 2] we can pass to a subsequence and obtain a limiting varifold V such that

$$|\mathbf{C}_i \cap \mathbb{B}_{1/2}(0)| \to V,$$

as well as

$$\operatorname{spt} \|V\| \cap \mathbb{B}_{1/2}(0) = \overline{M} \cap \mathbb{B}_{1/2}(0),$$

where M is an orientable hypersurface with optimal regularity, i.e. with a codimension 7 singular set. Note that the smallness of the singular set hypothesis of Theorem 2 of [35] is trivially satisfied in our setting. Furthermore, the varifold V is a cone, hence denote it by $\mathbf{C}$, and we smooth convergence of the supports away from the singular set of $\mathbf{C}$. We can then pass to a further subsequence to fix the multiplicities. Connectedness of the $\mathbf{C}_i \cap \mathbb{S}^7(1)$, and of $\mathbf{C} \cap \mathbb{S}^7(1)$, forces $m = 1$, i.e. we have multiplicity one convergence. In particular, consider a sequence of cones $\{\mathbf{C}_i\} \subset \mathcal{C}_\Lambda$. Then, there exists a subsequence and a cone $\mathbf{C}$ such that $\theta_{\mathbf{C}_i}(0) = \theta_{\mathbf{C}}(0)$. To obtain this last result we used the following

$$\theta_{\mathbf{C}}(0) = \mathcal{H}^7(\mathbf{C} \cap \mathbb{B}_1(0)) = \frac{1}{7}\mathcal{H}^6(\Sigma),$$

where Σ is the link of the cone $\mathbf{C}$.

We can now conclude the proof of the proposition. Indeed, we claim that for a given Λ, the set $\{\theta_{\mathbf{C}}(0);\ \mathbf{C} \in \mathcal{C}_\Lambda\}$ is finite. The proposition will then follow. Arguing by contradiction, if we have a sequence $\{\mathbf{C}_i\} \subset \mathcal{C}_\Lambda$ such that $\{\theta_{\mathbf{C}_j}\}_j$ are pairwise distinct, then we have a subsequence such that the density are equal, contradiction. $\square$

B.2 Asymptotic Analysis of Jacobi Fields over Minimal Cones

In this section, we exhibit some properties of Jacobi fields over stable minimal cones. Some detailed proofs can be found in [38, 8, 42]. Consider $\mathbf{C} \in \mathcal{C}$ to be a hypercone in $\mathbb{R}^{n+1}$, with the cross section $\Sigma \subset \mathbb{S}^n(1)$, a closed, smooth hypersurface within $\mathbb{S}^n$, such that $\Sigma = \mathbf{C} \cap \mathbb{S}^n$ and the singularity set of $\mathbf{C}$ is $\{0\}$. Denote $\mathbf{B}_R := \mathbf{C} \cap \mathbb{B}_R(0)$. Under the polar coordinate parametrization, the Jacobi operator is:

$$L_{\mathbf{C}} = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2} L_\Sigma,$$

where $L_\Sigma := \Delta_\Sigma + |A_\Sigma|^2$ and A_Σ the second fundamental form of Σ in $\mathbb{S}^n(1)$. Denote by $\mu_1 < \mu_2 \leq \cdots \nearrow +\infty$ the eigenvalues of $-L_\Sigma$, and denote $w_1, w_2, \ldots$ the corresponding smooth eigenfunctions on Σ which are L^2-orthonormal. By the standard elliptic PDEs, the first eigenfunction w_1 is positive. Due to the stability condition of $\mathbf{C}$, it holds that

$$\mu_1 \geq -\left(\frac{n-2}{2}\right)^2.$$

Proposition B.2.1. *[8, section 1] Fixing any $0 \leq R_1 < R_2 \leq +\infty$, the general solution of $L_C u = f$ within $\mathbf{C} \cap \mathbb{A}_{R_1,R_2}(0)$ can be represented by*

$$u(r,\omega) = \sum_{k\geq 1} (u_k(r) + v_k(r)) w_k(\omega), \tag{B.2.1}$$

where

$$u_k(r,f) := \begin{cases} \frac{-1}{2b_k}\left(r^{\gamma_k^+}\int_r^1 s^{-\gamma_k^+ + 1} f_k(s)\,ds - r^{\gamma_k^-}\int_r^1 s^{-\gamma_k^- + 1} f_k(s)\,ds\right), & \text{if } b_k \neq 0; \\ r^{-\frac{n-2}{2}}\left(\log r \int_0^r s^{\frac{n}{2}} f_k(s)\,ds - \int_0^r s^{\frac{n}{2}} \log s f_k(s)\,ds\right), & \text{if } b_k = 0. \end{cases} \tag{B.2.2}$$

with

$$f_k(r) := \int_S f(r,\omega) w_k(\omega)\, d\omega,$$

i.e., the Fourier coefficients of f, and

$$v_k(r) = v_k(r; c_k^+, c_k^-) := \begin{cases} c_k^+ r^{\gamma_k^+} + c_k^- r^{\gamma_k^-}, & \text{if } b_k \neq 0; \\ r^{-\frac{n-2}{2}}(c_k^+ + c_k^- \log r), & \text{if } b_k = 0. \end{cases} \tag{B.2.3}$$

Here $c_k^\pm \in \mathbb{R}$ are constants; $\gamma_k^\pm$ are solutions to the characteristic equations

$$\gamma^2 + (n-2)\gamma - \mu_k = 0, \tag{B.2.4}$$

and $b_k = \sqrt{\left(\frac{n-2}{2}\right)^2 + \mu_k}$.

Lemma B.2.2. *[8, Lemma 4.3] Suppose $\mathbf{C} \subset \mathbb{R}^{n+1}$ is a minimal hypercone with isolated singularity.*

- *For $n = 2$, $\mathbf{C}$ is simply a union of planes, so for $\phi \in C_c^2(CC_1) \setminus \{0\}$,*

$$-\int_{C_1} L_{\mathbf{C}}\xi = \int_{\mathbf{C}_1} |\nabla\phi|^2 > 0.$$

- *For $n \geq 3$*

$$\int_{\mathbf{C}_1} -L_{\mathbf{C}}\phi \geq \frac{(n-2)^2}{4} \cdot \mu_C \int_{C_1} r^{-2}\phi^2,$$

  where $\mu_{\mathbf{C}} := 1 - 4(n-2)^{-2} \cdot \mu_1$ and $\mu_1^- = \max\{-\mu_1, 0\}$.

Theorem B.2.3. *[8, Theorem 4.5] For $n \geq 3$, suppose $\mathbf{C} \subset \mathbb{R}^{n+1}$ is a minimal hypercone with isolated singularity.*

- *if $\mathbf{C}$ is stable, then*

$$\inf_{\phi \in C_c^1(\mathbf{C}_1)} \int_{\mathbf{C}_1} -L_{\mathbf{C}}\phi \geq 0 \Leftrightarrow \mu_{\mathbf{C}} \geq 0. \tag{B.2.5}$$

- *if* $\mathbf{C}$ *is strictly stable, then*

$$\inf_{\phi \in C^1_c(\mathbf{C}_1)} \left(r^{-2}\phi^2\right)^{-1} \int_{\mathbf{C}_1} -L_{\mathbf{C}}\phi > 0 \Leftrightarrow \mu_{\mathbf{C}} > 0. \tag{B.2.6}$$

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