UNIFORMITY OF HARMONIC MAP HEAT FLOW AT INFINITE TIME

LONGZHI LIN

ABSTRACT. We show an energy convexity along any harmonic map heat flow with small initial energy and fixed boundary data on the unit 2-disk. In particular, this gives an affirmative answer to a question raised by W. Minicozzi asking whether such harmonic map heat flow converges uniformly in time strongly in the $W^{1,2}$-topology, as time goes to infinity, to the unique limiting harmonic map.

1. INTRODUCTION

Given a compact Riemannian manifold $\mathcal{M}$ and a closed (i.e., compact and without boundary) Riemannian manifold $\mathcal{N}$ which is an isometrically embedded submanifold of $\mathbb{R}^n$, we can define the *Dirichlet* energy of a map $u \in W^{1,2}(\mathcal{M},\mathcal{N})$:

$$(1.1) \qquad \text{Energy}(u) \, = \, E(u) \, = \, \frac{1}{2} \int_{\mathcal{M}} |\nabla u|^2 dv_{\mathcal{M}} \,,$$

where $W^{1,2}(\mathcal{M},\mathcal{N})$ is the class of maps

$$\left\{ u \in L^1_{loc}(\mathcal{M}, \mathbb{R}^n) : \int_{\mathcal{M}} |\nabla u|^2 dv_{\mathcal{M}} < +\infty, \ u(x) \in \mathcal{N} \ \text{a.e.} \ x \in \mathcal{M} \right\}.$$

The tension field $\tau(u) \in \Gamma(u^*(T\mathcal{N}))$ is the vector field along u representing the negative L^2-gradient of $E(u)$. A weakly harmonic map u from $\mathcal{M}$ to $\mathcal{N}$ is a critical point of the energy functional $E(u)$ in the distribution sense, i.e., the tension field $\tau(u)$ vanishes, and it solves the Euler-Lagrange equation

$$(1.2) \qquad\qquad -\Delta_{\mathcal{M}} u \, = \, \Pi(u)(\nabla u, \nabla u) \,,$$

where $u = (u^1, ..., u^n)$ and $\Pi(u)$ denotes the second fundamental form of $\mathcal{N} \hookrightarrow \mathbb{R}^n$ at the point u. We refer to this system of elliptic equations as the harmonic map equation.

A natural way to control the tension field for an energy minimizing sequence of maps and to get the existence of harmonic maps from $\mathcal{M}$ to $\mathcal{N}$ is to consider the initial(-boundary) value problem:

$$(1.3) \qquad \begin{cases} u_t - \Delta_{\mathcal{M}} u = \Pi(u)(\nabla u, \nabla u), & \text{on } \mathcal{M} \times (0, T) \\ u(x,0) = u_0(x), & \text{for } x \in \mathcal{M} \\ u(x,t) = \chi(x) = u_0|_{\partial \mathcal{M}}, & \text{for all } t \geq 0, \ x \in \partial \mathcal{M} \ \text{if} \ \partial \mathcal{M} \neq \emptyset, \end{cases}$$

where $u = (u^1, ..., u^n)$ and $T > 0$. We refer to this system of parabolic equations as the harmonic map heat flow, to the map u_0 as the initial data, and to the map χ

as the boundary data. Given $u_0 \in W^{1,2}(\mathcal{M}, \mathcal{N})$ and $\chi = u_0|_{\partial\mathcal{M}} \in W^{\frac{1}{2},2}(\partial\mathcal{M}, \mathcal{N})$, we define $u \in W^{1,2}(\mathcal{M} \times [0,T], \mathcal{N})$ to be the weak solution of (1.3) if

$$(1.4) \qquad \int_0^T \int_{\mathcal{M}} \langle u_t, \xi \rangle + \langle \nabla u, \nabla \xi \rangle - \langle \Pi(u)(\nabla u, \nabla u), \xi \rangle dx dt = 0$$

for any $\xi \in C_c^\infty(\mathcal{M} \times (0,T), \mathbb{R}^n)$.

In the fundamental paper [ES64] where the harmonic map heat flow was first introduced, Eells and Sampson proved that the harmonic map heat flow exists for all time in the case that the source domain $\mathcal{M}$ (of arbitrary dimensions) is without boundary and the target manifold $\mathcal{N}$ has non-positive sectional curvature. They also proved that there exists some sequence of times $t_i \nearrow +\infty$ such that

$$u_\infty = \lim_{i \to \infty} u(\cdot, t_i)$$

is a harmonic map from $\mathcal{M}$ to $\mathcal{N}$. The case in which the source domain $\mathcal{M}$ has boundary was dealt with by Hamilton in [Ha75] under the same curvature assumption on $\mathcal{N}$. The question of uniformity of the convergence in time of the flow considered by Eells and Sampson in [ES64] was left open at that stage, but it was settled later by Hartman in [Har67]. We shall state their results in the following theorem.

Theorem 1.1 ([ES64, Har67]). *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are two closed Riemannian manifolds and that $\mathcal{N}$ has non-positive sectional curvature. Then given any $u_0 \in C^1(\mathcal{M}, \mathcal{N})$, the harmonic map heat flow has a unique solution $u \in C^1(\mathcal{M} \times [0,\infty), \mathcal{N}) \cap C^\infty(\mathcal{M} \times (0,\infty), \mathcal{N})$. Moreover,*

$$(1.5) \qquad\qquad\qquad u_\infty = \lim_{t \to \infty} u(\cdot, t)$$

exists uniformly in C^k-topology for all $k \geq 0$ and u_∞ is a harmonic map homotopic to u_0.

Other similar uniformity results were obtainable under various of assumptions on the target manifold $\mathcal{N}$, such as $\mathcal{N}$ is real analytic (see Simon [Si83]) or $\mathcal{N}$ admits a strictly convex function, see also Topping's interesting work [To97] in this direction for harmonic map heat flow in a special case in which both the source and target manifolds are 2-spheres $\mathbb{S}^2$.

When the dimension of the source domain $\mathcal{M}$ is two it is particularly interesting because the energy functional $E(u)$ and the harmonic map equation (1.2) are conformally invariant in this critical dimension. Regarding the harmonic map heat flow (1.3) from surfaces to a general closed target manifold $\mathcal{N}$, the first fundamental work was for the case $\partial\mathcal{M} = \emptyset$ which was due to Struwe [Str85], where "bubbles" may occur and have been analyzed in detail. This result was then extended to the case $\partial\mathcal{M} \neq \emptyset$ with Dirichlet boundary condition by Chang in [Ch89]. If the initial energy $E(u_0)$ is sufficiently small, it is well-known by now that the weak solution of (1.3) is smooth (in the interior) by the results of Freire [Fr95, Fr96] using the so called moving frame technique introduced by Hélein (see e.g. [He02]). We state their ε-regularity theorem that we shall require in this paper in the following. For the self-containedness of the paper, we will include an alternative proof of this ε-regularity theorem using the main tool of our current work that we call Rivière's gauge decomposition, see Theorem 3.7.

Theorem 1.2 ([Fr95, Fr96], cf. [Str85, Ch89, Wa11]). *Let $\mathcal{M}$ be a simply-connected compact Riemannian surface and $\mathcal{N}$ be a closed Riemannian manifold. There exists $\varepsilon_0 > 0$ depending only on $\mathcal{M}$ and $\mathcal{N}$ such that the following is true. For each initial data $u_0 \in W^{1,2}(\mathcal{M}, \mathcal{N})$ with $E(u_0) < \varepsilon_0$ and the boundary data $\chi = u_0|_{\partial\mathcal{M}}$ in the case that $\partial\mathcal{M} \neq \emptyset$, there exists a unique global weak solution $u \in W^{1,2}(\mathcal{M} \times [0, \infty), \mathcal{N})$ for which $E(u(\cdot, t))$ is non-increasing in t. Also, u is smooth in $\mathcal{M} \times [1, \infty)$ and for any $t_2 > t_1 \geq 1$ we have*

$$(1.6) \qquad 2 \int_{t_1}^{t_2} \int_{\mathcal{M}} |u_t|^2 = \int_{B_1} |\nabla u(\cdot, t_1)|^2 - \int_{B_1} |\nabla u(\cdot, t_2)|^2 \, .$$

Moreover, there exists some sequence of times $t_i \nearrow +\infty$ such that

$$(1.7) \qquad u_\infty = \lim_{i \to \infty} u(\cdot, t_i)$$

exists in the C^k-topology for any $k \geq 0$ and u_∞ is a harmonic map from $\mathcal{M}$ to $\mathcal{N}$.

Remark 1.3. In particular, in order to avoid the "bubble" (singularity) along the harmonic map heat flow, *a priori* we may choose $\varepsilon_0 < K_1 + K_2$ where

$$K_1 = \inf \left\{ E(v) \,\big|\, v \in W^{1,2}(\mathcal{M}, \mathcal{N}) \text{ and } v|_{\partial\mathcal{M}} = \chi \right\}$$

and

$$K_2 = \inf \left\{ E(v) \,\big|\, v : \mathbb{S}^2 \to \mathcal{N} \text{ is non-constant and harmonic} \right\} > 0 \, .$$

Remark 1.4. Freire's regularity results for harmonic map heat flow in [Fr95, Fr96] is a parabolic version of the regularity theorem of Hélein which states that weakly harmonic maps from surfaces are regular (see e.g. [He02]).

A tempting question to ask is that, for a general closed target manifold $\mathcal{N}$ (without imposing additional geometric assumptions on $\mathcal{N}$), whether one could establish uniformity results for the harmonic map heat flow similar to Theorem 1.1. In particular, is the convergence (1.7) in Theorem 1.2 uniform for all time in the natural $W^{1,2}$-topology, say? In view of the conformal-invariance of the energy functional $E(u)$ in dimension two, the condition of small energy seems to be a natural candidate to work with in order to get such uniformity of the convergence in time for the flow. We will show in the following that this is indeed the case. In what follows we will concentrate on the case that source domain $\mathcal{M}$ is a simply-connected compact Riemannian surface with boundary. More precisely we focus on domains which are conformally equivalent to the unit 2-disk $B_1 \subset \mathbb{R}^2$. From now on we will only work on B_1:

$$(1.8) \qquad \begin{cases} u_t - \Delta u = \Pi(u)(\nabla u, \nabla u), & \text{on } B_1 \times (0, T) \\ u(x, 0) = u_0(x), & \text{for } x \in B_1 \\ u(x, t) = \chi(x) = u_0|_{\partial B_1}, & \text{for all } t \geq 0 \text{ and } x \in \partial B_1 \, , \end{cases}$$

where Δ is the usual Laplacian $\Delta = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2}$ in $\mathbb{R}^2$. All the arguments could be easily modified to apply to the general case.

Notations 1.5. In what follows, $\nabla = (\partial_x, \partial_y)$ is the gradient operator in $\mathbb{R}^2$ and $\nabla^\perp = (-\partial_y, \partial_x)$ denotes the orthogonal gradient (i.e., $\nabla^\perp$ is the ∇-operator rotated by $\pi/2$).

Now we state the main theorem of this paper.

Theorem 1.6. *Let $\mathcal{N}$ be a closed Riemannian manifold. There exist $\varepsilon_0, T_0 > 0$ depending only on $\mathcal{N}$ such that if $u \in W^{1,2}(B_1 \times [0, \infty), \mathcal{N})$ is a global weak solution to the harmonic map heat flow* (1.8) *with $E(u_0) < \varepsilon_0$, $E(u(\cdot, t))$ is non-increasing in t and $u(\cdot, t)|_{\partial B_1} = \chi$ for all $t \geq 0$, then for all $t_2 > t_1 \geq T_0$ we have the energy convexity*

$$(1.9) \qquad \frac{1}{4} \int_{B_1} |\nabla u(\cdot, t_1) - \nabla u(\cdot, t_2)|^2 \leq \int_{B_1} |\nabla u(\cdot, t_1)|^2 - \int_{B_1} |\nabla u(\cdot, t_2)|^2 \,.$$

Remark 1.7. We do not know if the energy convexity (1.9) holds for all $t_2 > t_1 \geq 0$. In the following arguments we agree to let ε_0 be sufficiently small and T_0 be sufficiently large as needed.

Our approach to the proof of Theorem 1.6 is based on the technique that we call Rivière's gauge decomposition introduced by Rivière in [Ri07], see also Section 3. Immediate applications of Theorem 1.6 are the following corollaries.

Corollary 1.8. *Let $\mathcal{N}$ be a closed Riemannian manifold. There exists $\varepsilon_0 > 0$ depending only on $\mathcal{N}$ such that if $u \in W^{1,2}(B_1 \times [0, \infty), \mathcal{N})$ is a global weak solution to the harmonic map heat flow* (1.8) *with $E(u_0) < \varepsilon_0$, $E(u(\cdot, t))$ is non-increasing in t and $u(\cdot, t)|_{\partial B_1} = \chi$ for all $t \geq 0$, then*

$$(1.10) \qquad u(\cdot, t) \to u_\infty \text{ uniformly as } t \to +\infty \text{ strongly in } W^{1,2}(B_1, \mathbb{R}^n) \,,$$

where u_∞ is the unique harmonic map with $E(u_\infty) < \varepsilon_0$ and boundary data χ.

Corollary 1.9. *Let $\mathcal{M}$ be a two dimensional domain that is conformally equivalent to B_1 and has smooth boundary, and let $\mathcal{N}$ be a closed Riemannian manifold. Suppose the initial energy $E(u_0) < \varepsilon_0$, then the harmonic map heat flow* (1.3) *with initial data $u_0 \in C^{2,\alpha}(\overline{\mathcal{M}}, \mathcal{N})$ and boundary data $\chi \in C^{2,\alpha}(\partial \mathcal{M}, \mathcal{N})$, considered by Chang in [Ch89], converges uniformly in time strongly in $W^{1,2}(\mathcal{M}, \mathcal{N})$ to the unique harmonic map $u_\infty \in C_\chi^{2,\alpha}(\overline{\mathcal{M}}, \mathcal{N})$.*

Remark 1.10. We do not know if a harmonic map heat flow can be non-uniform without the small energy assumption. In view of the non-uniqueness results of Brezis and Coron [BC83] and Jost [Jo84] for harmonic maps (with large energy) sharing the same boundary data on ∂B_1, it is quite possible that the small energy assumption is necessary for the energy convexity and uniform convergence of the flow in Theorem 1.6 and Corollary 1.8 to hold.

Remark 1.11. In [CM08-1] Colding and Minicozzi showed an energy convexity for weakly harmonic maps with small energy on B_1: there exists $\varepsilon_0 > 0$ such that if $u, v \in W^{1,2}(B_1, \mathcal{N})$ with $u|_{\partial B_1} = v|_{\partial B_1}, E(u) < \varepsilon_0$, and u is weakly harmonic, then we have the energy convexity

$$(1.11) \qquad \frac{1}{2} \int_{B_1} |\nabla v - \nabla u|^2 \leq \int_{B_1} |\nabla v|^2 - \int_{B_1} |\nabla u|^2 \,.$$

See our recent work [LL12] for an alternative proof of this energy convexity using the same techniques used in the present paper. A direct consequence of (1.11) is that u_∞ in Corollary 1.8 is unique in the class

$$\{v \in W^{1,2}(B_1, \mathbb{R}^n) : E(v) < \varepsilon_0 \text{ and } v|_{\partial B_1} = \chi\} \,,$$

see [CM08-1, Corollary 3.3].

The paper is organized as follows. In Section 2 we present some heuristic arguments and elaborate on the idea of the proof of the main Theorem 1.6. In Section 3 we review the main tool of our proof, namely, Rivière's gauge decomposition technique adapted to the case of harmonic map heat flow. In Section 4 we show improved estimates for Rivière's matrices B and P, which are the two key ingredients of our proof. We finish the proof of our main theorem in Section 5.

Acknowledgement. The author would like to thank Bill Minicozzi for his continued guidance and support. The author would also like to thank his collaborator Tobias Lamm for all the valuable discussions.

2. HEURISTIC ARGUMENTS AND THE IDEA OF THE PROOF

In this section we will present some heuristic arguments and sketch the basic idea of the proof of Theorem 1.6. We will abbreviate $u(\cdot, t)$ to $u(t)$. In order to prove the energy convexity (1.9) along the harmonic map heat flow, i.e., there exists some $T_0 > 0$ such that for all $t_2 > t_1 \geq T_0$ we have

$$(2.1) \qquad \frac{1}{4} \int_{B_1} |\nabla u(\cdot, t_1) - \nabla u(\cdot, t_2)|^2 \leq \int_{B_1} |\nabla u(\cdot, t_1)|^2 - \int_{B_1} |\nabla u(\cdot, t_2)|^2,$$

it suffices to show

$$(2.2) \qquad \Psi \geq -\left(\int_{B_1} |\nabla u(t_1)|^2 - \int_{B_1} |\nabla u(t_2)|^2 \right) - \frac{1}{2} \int_{B_1} |\nabla u(t_1) - \nabla u(t_2)|^2,$$

where (using that $u(\cdot, t)|_{\partial B_1} = \chi$ for all $t \geq 0$ and the flow equation (1.8))

$$\Psi := \int_{B_1} |\nabla u(t_1)|^2 - \int_{B_1} |\nabla u(t_2)|^2 - \int_{B_1} |\nabla u(t_1) - \nabla u(t_2)|^2$$

$$= 2 \int_{B_1} \langle \nabla u(t_1) - \nabla u(t_2), \nabla u(t_2) \rangle$$

$$(2.3) \qquad = -2 \int_{B_1} \langle u(t_1) - u(t_2), u_t(t_2) - \Pi(u)(\nabla u, \nabla u)(t_2) \rangle.$$

Now note that for any $p, q \in \mathcal{N}$, there exists some constant $C > 0$ depending only on $\mathcal{N}$ such that $|(p - q)^\perp| \leq C|p - q|^2$, where the superscript $\perp$ denotes the normal component of a vector (see e.g. [CM08-2, Lemma A.1]). Therefore, using the fact that $\Pi(u)(\nabla u, \nabla u) \perp T_u \mathcal{N}$ and Cauchy-Schwarz inequality, (2.3) yields

$$\Psi \geq -2 \left(\int_{B_1} |u(t_1) - u(t_2)|^2 \right)^{\frac{1}{2}} \left(\int_{B_1} |u_t(t_2)|^2 \right)^{\frac{1}{2}} - C \int_{B_1} |(u(t_1) - u(t_2))^\perp| |\nabla u(t_2)|^2$$

$$\geq -2\sqrt{t_2 - t_1} \left(\int_{t_1}^{t_2} \int_{B_1} |u_t|^2 \right)^{\frac{1}{2}} \left(\int_{B_1} |u_t(t_2)|^2 \right)^{\frac{1}{2}} - C \int_{B_1} |u(t_1) - u(t_2)|^2 |\nabla u(t_2)|^2,$$

where we also used the smoothness and compactness of the target manifold $\mathcal{N}$. Here and throughout the rest of the paper, $C > 0$ will denote a universal constant depending only on $\mathcal{N}$ unless otherwise stated.

Since we have (1.6) and ε_0 can always be chosen sufficiently small, we know that (2.2) will be achieved if we can show the following two key propositions.

Proposition 2.1. *Let $u(x,t)$ be as in Theorem 1.6, then there exists $T_0 > 0$ such that for all $t_2 > t_1 \geq T_0$ we have*

$$(2.4) \qquad \int_{B_1} |u_t(t_2)|^2 \leq \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \int_{B_1} |u_t|^2 \,.$$

Remark 2.2. The key point of Proposition 2.1 is that (2.4) is valid for all $t_2 > t_1 \geq T_0$. We will see that, in fact, $\int_{B_1} |u_t(t)|^2$ is non-increasing along the flow after T_0, which yields (2.4), cf. Lemma 2.5 and (5.16) below. A similar but weaker estimate was shown in [Str85] when the source domain of the heat flow is boundaryless (see equation (3.5) of [Str85]), which turned out to be the key estimate needed in Struwe's proof.

Proposition 2.3. *Let $u(x,t)$ be as in Theorem 1.6, then there exists $T_0 > 0$ such that for all $t_2 > t_1 \geq T_0$ we have*

$$(2.5) \qquad \int_{B_1} |u(t_1) - u(t_2)|^2 |\nabla u(t_2)|^2 \leq C\varepsilon_0 \int_{B_1} |\nabla u(t_1) - \nabla u(t_2)|^2 \,.$$

If one was able to get

$$(2.6) \qquad \|\nabla u(t_2)\|_{L^\infty(B_1)} \leq C\sqrt{\varepsilon_0}\,,$$

then (2.5) would have been automatically true by Poincaré's inequality. However, without imposing any regularity information on the boundary data χ, it will be hopeless to get such strong global pointwise gradient estimate. In fact, even if we look at the stationary case, i.e., $W^{1,2}$-weakly harmonic maps on B_1, it is easy to convince oneself that it is unreasonable to expect regularity with global estimates on the whole B_1 better than $W^{2,2}$ in general.

Nevertheless, not all hope is lost to show estimates (2.4) and (2.5). Indeed, the following lemma is true which validates Proposition 2.3 under some extra assumptions.

Lemma 2.4. *Let $u(x,t)$ be as in Theorem 1.6 and suppose that for all $t_2 > t_1 \geq T_0 \geq 1$ we can solve the following Dirichlet problem for $\psi \in W_0^{1,2} \cap L^\infty(B_1)$:*

$$(2.7) \qquad \begin{cases} \Delta\psi = |\nabla u(t_2)|^2 & \text{in } B_1\,, \\ \psi = 0 & \text{on } \partial B_1\,, \end{cases}$$

with estimate

$$(2.8) \qquad \|\psi\|_{L^\infty(B_1)} + \|\nabla\psi\|_{L^2(B_1)} \leq C\varepsilon_0\,.$$

Then Proposition 2.3 holds.

Proof. The proof is essentially taken from [CM08-1]. Substituting (2.7) into the left-hand side of (2.5) yields (using also that $u(t_1) = u(t_2) = \chi$ on ∂B_1)

$$\int_{B_1} |u(t_1) - u(t_2)|^2 |\nabla u(t_2)|^2 = \int_{B_1} |u(t_1) - u(t_2)|^2 \Delta\psi \leq \int_{B_1} |\nabla|u(t_1) - u(t_2)|^2||\nabla\psi|$$

$$(2.9) \qquad \leq 2 \left(\int_{B_1} |\nabla u(t_1) - \nabla u(t_2)|^2 \right)^{1/2} \left(\int_{B_1} |u(t_1) - u(t_2)|^2 |\nabla\psi|^2 \right)^{1/2},$$

where we have applied Stokes' theorem to $\text{div}(|u(t_1)-u(t_2)|^2\nabla\psi)$ and used Cauchy-Schwarz inequality. Now applying Stokes' theorem to $\text{div}(|u(t_1)-u(t_2)|^2\psi\nabla\psi)$ and using that $\Delta\psi \geq 0$ and (2.9), we have

$$\int_{B_1}|u(t_1)-u(t_2)|^2|\nabla\psi|^2 \leq \int_{B_1}|\psi|(|u(t_1)-u(t_2)|^2\Delta\psi + |\nabla|u(t_1)-u(t_2)|^2||\nabla\psi|)$$

$$(2.10) \qquad \leq 4\|\psi\|_{L^\infty}\left(\int_{B_1}|\nabla u(t_1)-\nabla u(t_2)|^2\right)^{1/2}\left(\int_{B_1}|u(t_1)-u(t_2)|^2|\nabla\psi|^2\right)^{1/2},$$

and so that

$$(2.11) \quad \left(\int_{B_1}|u(t_1)-u(t_2)|^2|\nabla\psi|^2\right)^{1/2} \leq 4\|\psi\|_{L^\infty}\left(\int_{B_1}|\nabla u(t_1)-\nabla u(t_2)|^2\right)^{1/2}.$$

Finally, substituting (2.11) back into (2.9) and combining with (2.8) (and choosing ε_0 sufficiently small) yield

$$\int_{B_1}|u(t_1)-u(t_2)|^2|\nabla u(t_2)|^2 \leq C\|\psi\|_{L^\infty}\int_{B_1}|\nabla u(t_1)-\nabla u(t_2)|^2$$

$$\leq C\varepsilon_0\int_{B_1}|\nabla u(t_1)-\nabla u(t_2)|^2,$$

which is just (2.5). $\qquad\qquad\square$

Similarly we can show the following lemma which states, under some extra condition, $\int_{B_1}|u_t(t)|^2$ is non-increasing along the harmonic map heat flow after some $T_0 > 0$ and Proposition 2.1 could be validated in this case.

Lemma 2.5. *Let $u(x,t)$ be as in Theorem 1.6. For any $t_2 > t_1 \geq T_0 \geq 1$, suppose that for any $t_0 \in [t_1, t_2]$ we can solve the following Dirichlet problem for $\psi \in W_0^{1,2}\cap L^\infty(B_1)$:*

$$(2.12) \qquad \begin{cases} \Delta\psi = |\nabla u(t_0)|^2 & in \ B_1\,, \\ \psi = 0 & on \ \partial B_1\,, \end{cases}$$

with estimate

$$(2.13) \qquad \|\psi\|_{L^\infty(B_1)} + \|\nabla\psi\|_{L^2(B_1)} \leq C\varepsilon_0\,.$$

Then we have

$$(2.14) \qquad \int_{B_1}|u_t(t_2)|^2 \leq \int_{B_1}|u_t(t_1)|^2\,.$$

In particular, Proposition 2.1 holds if (2.12) and (2.13) are valid for any $t_0 \in [t_1, t_2]$ and any $t_2 > t_1 \geq T_0 \geq 1$.

Proof. Differentiate the flow equation (1.8) with respect to t, multiply with u_t, and integrate over $B_1 \times [t_1, t_2]$, we have

$$\frac{1}{2}\int_{t_1}^{t_2}\int_{B_1}\partial_t|u_t|^2 + \int_{t_1}^{t_2}\int_{B_1}|\nabla u_t|^2 \leq C\int_{t_1}^{t_2}\int_{B_1}|u_t|^2|\nabla u|^2 + |u_t||\nabla u||\nabla u_t|$$

$$(2.15) \qquad\qquad \leq \frac{1}{2}\int_{t_1}^{t_2}\int_{B_1}|\nabla u_t|^2 + C\int_{t_1}^{t_2}\int_{B_1}|u_t|^2|\nabla u|^2\,.$$

Since (2.12) and (2.13) are valid for any $t_0 \in [t_1, t_2]$, we can use the same arguments as in the proof of Lemma 2.4 to get estimate for

$$\int_{B_1} |u_t|^2 |\nabla u|^2$$

at the time t_0 slice. Indeed, similar to (2.5) (i.e., replacing $u(t_1) - u(t_2)$ by $u_t(t_0)$), for any $t_0 \in [t_1, t_2]$ we have

$$(2.16) \qquad \int_{B_1} |u_t|^2 |\nabla u|^2 (t_0) \leq C \|\psi\|_{L^\infty} \int_{B_1} |\nabla u_t(t_0)|^2 \leq C\varepsilon_0 \int_{B_1} |\nabla u_t(t_0)|^2 \,.$$

Inserting (2.16) back into (2.15) (for any $t_0 \in [t_1, t_2]$) we see that the right-hand side of (2.15) can be absorbed into the the left-hand side if we choose ε_0 sufficiently small. This implies that we have (2.14) for any such $t_2 > t_1 \geq T_0$. In the above calculations, we should treat u_t as a difference quotient: $u_t(\cdot, t) = \lim_{h \to 0^+} (u(\cdot, t + h) - u(\cdot, t))/h$ which is zero on ∂B_1 for all $t \geq 1$; moreover, we have denoted $\nabla u_t(\cdot, t) = \lim_{h \to 0^+} (\nabla (u(\cdot, t + h) - u(\cdot, t)))/h$ and all the calculations are valid for any fixed $h > 0$ and then we take $h \to 0^+$ to conclude (2.14).

If (2.12) and (2.13) are valid for any $t_0 \in [t_1, t_2]$ and any $t_2 > t_1 \geq T_0 \geq 1$, then, in view of (2.14), estimating by the mean value of $|u_t|^2$ over $B_1 \times [t_1, t_2]$ gives Proposition 2.1. $\qquad \square$

Therefore, everything boils down to validating the assumptions in Lemmas 2.4 and 2.5, i.e., the existence of such functions ψ's satisfying (2.7), (2.8) and (2.12), (2.13), respectively, for any $t_0 \geq T_0$ for some $T_0 \geq 1$. We point out that, a prior we only know that the energy density $|\nabla u(t)|^2$ lies in $L^1(B_1)$ with global estimate $\||\nabla u(t)|^2\|_{L^1(B_1)} \leq \varepsilon_0$ for any fixed t. But L^1 is the borderline case in which the standard L^p-theory for the Dirichlet problem (2.7) with estimate (2.8) fails!

However, the following regularity theorem for boundary value problems in the local Hardy space $h^1(B_1)$ sheds a new light to validate the assumptions in Lemmas 2.4 and 2.5. Here the local Hardy space $h^1(B_1)$ is a strict subspace of $L^1(B_1)$ and we will recall its definition in Definition 2.8 below.

Theorem 2.6. *(cf. [Se94, Theorem 1.100] and [CKS93, Theorem 5.1]) Let $f \in h^1(B_1)$ such that $f \geq 0$ a.e. in B_1. Then there exists a function $\psi \in L^\infty \cap W_0^{1,2}(B_1)$ solving the Dirichlet problem*

$$(2.17) \qquad \begin{cases} \Delta \psi = f & in \ B_1 \,, \\ \quad \psi = 0 & on \ \partial B_1 \,. \end{cases}$$

Moreover, there exists a constant $C > 0$ such that

$$(2.18) \qquad \|\psi\|_{L^\infty(B_1)} + \|\nabla \psi\|_{L^2(B_1)} \leq C \|f\|_{h^1(B_1)} \,.$$

Proof. For the self-containedness of the paper, we include an elementary proof of this theorem in Appendix A. $\qquad \square$

Remark 2.7. This theorem can be thought of as a generalization of a result of Müller [Mu90], cf. Wente's lemma 3.6 in Section 3. For a more general version of this theorem we refer to Chang, Krantz and Stein's work in [CKS93].

Definition 2.8. ([Mi90]) Choose a Schwartz function $\phi \in C_0^\infty(B_1)$ such that $\int_{B_1} \phi \, dx = 1$ and let $\phi_t(x) = t^{-2}\phi\left(\frac{x}{t}\right)$. For a measurable function f defined in B_1 we say that f lies in the local Hardy space $h^1(B_1)$ if the radial maximal function of f

$$(2.19) \quad f^*(x) = \sup_{0 < t < 1-|x|} \left| \int_{B_t(x)} \frac{1}{t^2}\phi\left(\frac{x-y}{t}\right) f(y)dy \right|(x) = \sup_{0 < t < 1-|x|} |\phi_t * f|(x)$$

belongs to $L^1(B_1)$ and we define

$$(2.20) \qquad\qquad \|f\|_{h^1(B_1)} = \|f^*(x)\|_{L^1(B_1)}.$$

It follows immediately that $h^1(B_1)$ is a strict subspace of $L^1(B_1)$ and

$$\|f\|_{L^1(B_1)} \leq \|f\|_{h^1(B_1)}.$$

It is also clear that if $f \in L^p(B_1)$ for some $p > 1$ then $\|f\|_{h^1(B_1)} \leq C\|f\|_{L^p(B_1)}$.

We shall remark that the local Hardy spaces h^1 (or the global version $\mathcal{H}^1$) act as replacements to L^1 in Calderon-Zygmund estimates. Therefore by Theorem 2.6, if we can somehow manage to obtain a "slightly" improved global estimate for $|\nabla u|^2$ from $L^1(B_1)$ to $h^1(B_1)$ for all $t_0 \geq T_0$, it will be sufficient to validate the assumptions in Lemmas 2.4 and 2.5. As mentioned above, the subtlety is that, without imposing any regularity information on the boundary data χ, global estimates are very difficult to obtain.

The rest of the paper is devoted to validating the assumptions in Lemmas 2.4 and 2.5. Namely, in view of Theorem 2.6, it suffices to show there exists $T_0 > 0$ such that

$$(2.21) \qquad\qquad \||\nabla u(t_0)|^2\|_{h^1(B_1)} \leq C\varepsilon_0 \quad \text{for any } t_0 \geq T_0.$$

The point here is that no pointwise estimate on ∇u such as (2.6) is needed, and instead, a (weaker) improved global integral estimate (2.21) will be sufficient and turns out to be the key to the proof of the main Theorem 1.6.

3. Analysis of harmonic map heat flow using Rivière's gauge

Regarding the regularity of weakly harmonic maps from surfaces, Hélein (see e.g. [He02]) proved the interior regularity with the help of the so called Coulomb or moving frame, and Qing [Qi95] showed the continuity up to the boundary in the case of continuous boundary data based on Hélein's technique. More recently, Rivière [Ri07] succeeded in writing the 2-dimensional conformally invariant non-linear system of elliptic PDE's (which includes the weakly harmonic map equation (1.2)) in the following form:

$$(3.1) \qquad -\Delta u^i = \Omega_j^i \cdot \nabla u^j \quad i = 1, 2, ..., n \quad \text{or} \quad -\Delta u = \Omega \cdot \nabla u,$$

with $\Omega = (\Omega_j^i)_{1 \leq i,j \leq n} \in L^2(B_1, so(n) \otimes \wedge^1 \mathbb{R}^2)$ and $\Omega_j^i = -\Omega_i^j$ (antisymmetry). Here and throughout the paper, the Einstein summation convention is used. We refer to the system of equations (3.1) as Rivière's equation. In particular, this special form of the nonlinearity enabled Rivière to obtain a conservation law for this system of PDE's (see (3.8) below), which is accomplished via a technique that we call Rivière's gauge decomposition. More precisely, following the strategy of Uhlenbeck in [Uh82], Rivière [Ri07] used the algebraic feature of Ω, namely Ω being

antisymmetric, to construct $\xi \in W_0^{1,2}(B_1, so(n))$ and a gauge transformation matrix $P \in W^{1,2} \cap L^\infty(B_1, SO(n))$ (which pointwise almost everywhere is an orthogonal matrix in $\mathbb{R}^{n \times n}$) satisfying some good properties.

Theorem 3.1. *([Ri07, Lemma A.3]) There exist $\varepsilon > 0$ and $C > 0$ such that for every Ω in $L^2(B_1, so(n) \otimes \wedge^1 \mathbb{R}^2)$ satisfying*

$$\int_{B_1} |\Omega|^2 \leq \varepsilon \,,$$

there exist $\xi \in W_0^{1,2}(B_1, so(n))$ and $P \in W^{1,2}(B_1, SO(n))$ such that

(3.2) $$\nabla^\perp \xi \,=\, P^T \nabla P + P^T \Omega P \text{ in } B_1 \quad with \quad \xi \,=\, 0 \text{ on } \partial B_1,$$

and

(3.3) $$\|\nabla \xi\|_{L^2(B_1)} + \|\nabla P\|_{L^2(B_1)} \,\leq\, C\|\Omega\|_{L^2(B_1)} \,.$$

Here the superscript T denotes the transpose of a matrix.

Remark 3.2. Multiplying both sides of equation (3.2) by P from the left gives that (with indices and $1 \leq m, z \leq n$)

(3.4) $$\nabla P_j^i \,=\, P_m^i \nabla^\perp \xi_j^m - \Omega_z^i P_j^z, \quad 1 \leq i, j \leq n \,.$$

Remark 3.3. Besides Uhlenbeck's method there is another way to construct the gauge tranformation matrix P, namely one can minimize the energy functional

(3.5) $$E(R) \,=\, \int_{B_1} \left| R^T \nabla R + R^T \Omega R \right|^2$$

among all $R \in W^{1,2}(B_1, SO(n))$, see e.g. [Cho95] and [Sc09].

Another key result from Rivière's work is the following theorem, which was proved based on Theorem 3.1.

Theorem 3.4. *([Ri07, Theorem I.4]) There exist $\varepsilon > 0$ and $C > 0$ such that for every Ω in $L^2(B_1, so(n) \otimes \wedge^1 \mathbb{R}^2)$ satisfying*

$$\int_{B_1} |\Omega|^2 \,\leq\, \varepsilon \,,$$

there exist $\widehat{A} \in W^{1,2} \cap C^0(B_1, Gl_n(\mathbb{R})), A = (\widehat{A} + Id) \, P^T \in L^\infty \cap W^{1,2}(B_1, Gl_n(\mathbb{R}))$ and $B \in W_0^{1,2}(B_1, M_n(\mathbb{R}))$ such that

(3.6) $$\nabla A - A\Omega \,=\, \nabla^\perp B$$

and

(3.7) $$\|\widehat{A}\|_{W^{1,2}(B_1)} + \|\widehat{A}\|_{L^\infty(B_1)} + \|B\|_{W^{1,2}(B_1)} \,\leq\, C\|\Omega\|_{L^2(B_1)} \,.$$

Remark 3.5. Combining (3.6) with (3.1) one obtains the conservation law (in the distribution sense) for Rivière's equation (3.1):

(3.8) $$\text{div} \left(A\nabla u + B\nabla^\perp u \right) \,=\, 0 \,.$$

Equation (3.1), first considered in such generality by Rivière in [Ri07], generalizes a number of interesting equations appearing naturally in geometry, including the harmonic map equation (1.2)), the H-surface equation and, more generally, the Euler-Lagrange equation of any conformally invariant elliptic Lagrangian which is

quadratic in the gradient. We remark that the harmonic map equation (1.2) can be written in the form of (3.1) if we set

$$(3.9) \qquad \Omega := (\Omega^i_j)_{1 \le i,j \le n} \quad \text{where } \Omega^i_j := [\Pi^i(u)_{j,l} - \Pi^j(u)_{i,l}]\nabla u^l \,.$$

A central issue is the regularity of the weak solution u to this system of equations (3.1). Based on the conservation law (3.8), Rivière proved the (interior) continuity of any $W^{1,2}$ weak solution u to (3.1). This also resolved two conjectures by Heinz and Hildebrandt respectively, see [Ri07]. We point out that the harmonic map heat flow (1.8) on B_1 can be written in the form

$$(3.10) \qquad u_t - \Delta u = \Omega \cdot \nabla u \quad \text{on } B_1 \times (0, T) \,,$$

where Ω is as in (3.9).

The deep reason for Rivière's argument to work is that once the conservation law (3.8) is established, then equation (3.1) can be rewritten in the form

$$\operatorname{div}(A\nabla u) \,=\, \nabla^\perp B \cdot \nabla u \,,$$

The right hand side of this new equation lies in the Hardy space $\mathcal{H}^1$ by a result of Coifman, Lions, Meyer and Semmes [CLMS93]. Moreover, using a Hodge decomposition argument, one can show that u lies locally in $W^{2,1}$ which embeds into C^0 in two dimensions, cf. the proof of Theorem 3.7 below. The key to this fact is a special "compensation phenomena" for Jacobian determinants, which was first observed by Wente in [We69]. We will refer it to the following Wente's lemma, for which an elementary proof can be found in [BC83] and [He02, Theorem 3.1.2]. It will also be the key ingredient of our proof in this paper.

Lemma 3.6 (Wente's lemma [We69])**.** *If $a, b \in W^{1,2}(B_1, \mathbb{R})$ and w be the solution of*

$$(3.11) \qquad \begin{cases} \Delta w \,=\, \dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x} - \dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y} \,=\, \nabla a \cdot \nabla^\perp b & in \ B_1 \,, \\[2mm] w \,=\, 0 \quad or \quad \dfrac{\partial w}{\partial \nu} \,=\, 0 & on \ \partial B_1 \,. \end{cases}$$

Then $w \in C^0 \cap W^{1,2}(B_1, \mathbb{R})$ and the following estimate holds

$$(3.12) \qquad \|w\|_{L^\infty(B_1)} + \|\nabla w\|_{L^2(B_1)} \,\le\, C\|\nabla a\|_{L^2(B_1)}\|\nabla b\|_{L^2(B_1)} \,,$$

where we choose $\int_{B_1} w = 0$ for the Neumann boundary data.

Now let $u(x,t) \in W^{1,2}(B_1 \times [0,\infty), \mathcal{N})$ be a global weak solution to the harmonic map heat flow (1.8) with $E(u_0) < \varepsilon_0$, $E(u(\cdot,t))$ is non-increasing in t and $u(\cdot,t)|_{\partial B_1} = \chi$ for all $t \ge 0$ as in Theorem 1.6. First note that for a.e. $t_0 \in (0,\infty)$ we have $u_t(t_0) \in L^2(B_1)$. Then for any fixed t_0 such that $u_t(t_0) \in L^2(B_1)$, as in (3.9), we have

$$\Omega(t_0) = (\Omega^i_j(t_0))_{1 \le i,j \le n} \quad \text{where} \quad \Omega^i_j(t_0) = [\Pi^i(u(t_0))_{j,l} - \Pi^j(u(t_0))_{i,l}]\nabla u^l(t_0).$$

We will abbreviate $\Omega(t_0)$ to $\Pi(u(t_0))\nabla u(t_0)$. Moreover,

$$(3.13) \qquad \int_{B_1} |\Omega(t_0)|^2 \,\le\, CE(u(t_0)) \,\le\, C\varepsilon_0 \,.$$

Therefore Rivière's Theorems 3.1 and 3.4 on the existence of gauge apply to this time t_0 slice, and we find the existence of matrices $P(t_0) \in W^{1,2}(B_1, SO(n))$,

$$A(t_0) = (\widehat{A}(t_0) + Id) P^T(t_0) \ \in \ L^\infty \cap W^{1,2}(B_1, Gl_n(\mathbb{R}))$$

and

$$B(t_0) \ \in \ W_0^{1,2}(B_1, M_n(\mathbb{R}))$$

such that

$$(3.14) \qquad\qquad \nabla A(t_0) - A(t_0)\Omega(t_0) \ = \ \nabla^\perp B(t_0)$$

with the corresponding estimates (3.3) and (3.7).

Combining (3.14) with the harmonic map heat flow equation (3.10) yields (omitting the index t_0)

$$
\begin{aligned}
\text{div } (A\nabla u + B\nabla^\perp u) \ &= \ \nabla A \cdot \nabla u + A\Delta u + \nabla B \cdot \nabla^\perp u \\
&= \nabla A \cdot \nabla u + A(-\Omega \cdot \nabla u + u_t) + \nabla B \cdot \nabla^\perp u \\
(3.15) \qquad &= \nabla A \cdot \nabla u + (\nabla^\perp B - \nabla A) \cdot \nabla u + A u_t + \nabla B \cdot \nabla^\perp u \\
&= A u_t \,.
\end{aligned}
$$

We refer to (3.15) as an "almost" conservation law. By the results of [CLMS93] and the standard L^p theory, (3.15) readily implies that $u(t_0) \in C^0(B_1, \mathbb{R}^n)$. In fact, we have the following ε-regularity theorem.

Theorem 3.7. *There exist $\varepsilon_0 > 0$ depending only on $\mathcal{N}$ such that if $u \in W^{1,2}(B_1 \times [0,\infty), \mathcal{N})$ is a global weak solution to the harmonic map heat flow (1.8) with $E(u_0) < \varepsilon_0$, $E(u(\cdot, t))$ is non-increasing in t and $u(\cdot, t)|_{\partial B_1} = \chi$ for all $t \geq 0$, then $u \in C^\infty(B_1 \times [1, \infty), \mathcal{N})$.*

Proof. For any fixed t_0 such that $u_t(t_0) \in L^2(B_1)$, by Hodge decomposition (see e.g. [IM01, Corollary 10.5.1]), there exist $D(t_0), E(t_0) \in W^{1,2}(B_1, \mathbb{R}^n)$ such that (omitting the index t_0)

$$(3.16) \qquad\qquad A\nabla u = \nabla D + \nabla^\perp E \,.$$

Note that (3.15) implies

$$(3.17) \qquad \begin{cases} \text{div } (A\nabla u) = -\nabla B \cdot \nabla^\perp u + A u_t \\ \text{curl } (A\nabla u) = \nabla^\perp A \cdot \nabla u \,. \end{cases}$$

Combining (3.16) and (3.17) we have

$$(3.18) \qquad \begin{cases} \Delta D = -\nabla B \cdot \nabla^\perp u + A u_t \\ \Delta E = \nabla^\perp A \cdot \nabla u \,. \end{cases}$$

Then by the results of [CLMS93] and via an extension argument, using the fact that $Au_t(t_0) \in L^2(B_1)$, we get $A\nabla u(t_0) \in W_{loc}^{1,1}(B_1)$. Therefore $u(t_0) \in W_{loc}^{2,1}(B_1)$ which embeds into $C^0(B_1)$.

Indeed, $A\nabla u(t_0) \in W_{loc}^{1,1}(B_1)$ implies immediately that

$$(3.19) \qquad\qquad \Omega(t_0) = \Pi(u(t_0))\nabla u(t_0) \ \in \ W_{loc}^{1,1}(B_1) \,.$$

Then by [Ri12, Theorem IV.4], the flow equation (3.10), (3.19) and the fact that $u_t(t_0) \in L^2(B_1)$ yield that $\nabla u(t_0) \in L_{loc}^p(B_1)$ for some $p > 2$. Note that this is valid for a.e. $t_0 \in (0, \infty)$. Then via a standard bootstrapping argument we have

$\nabla u \in L^q_{loc}(B_1 \times [1, T])$ for all $q > 1$ and any $T > 1$, see e.g. [Li96], and all higher order interior regularity follows.   □

Again, we see that the "compensation phenomenon" enjoyed by the special Jacobian structure (see Wente's lemma 3.6) has played an important role here, and these Wente type estimates have many interesting applications as observed already in e.g. [We69, BC83, BC84, Ta85, CLMS93, He02, Ri07, Ri08-1, Ri08-2] and our recent work [LL12].

4. Improved estimates on the matrices B and P

Our main observation in this section are two hidden Jacobian structures for ΔB and ΔP, which are valid only for Ω being in some special form, i.e., in our case we have $\Omega = \Pi(u)\nabla u$. This observation allows us to gain improved global estimate on the matrix B and improved local estimate on the matrix P.

4.1. Improved global estimate on the matrix B.
We will first show an improved global estimate on the matrix B.

Proposition 4.1. *Let $u(x,t)$ be as in Theorem 1.6, then for any $t_0 \in [1, \infty)$ we have*

$$(4.1) \qquad \|B(t_0)\|_{L^\infty(B_1)} \leq C \int_{B_1} |\nabla u(t_0)|^2 \leq C\varepsilon_0 \,.$$

Proof. We recall that Ω is given by $\Pi(u)\nabla u$ as in (3.9) and therefore $\|\Omega(t_0)\|^2_{L^2(B_1)} \leq C\varepsilon_0$ for all $t_0 \geq 1$. Now let ε_0 be so small that Theorems 3.1 and 3.4 apply. Taking the curl on both sides of equation (3.14) yields

$$(4.2) \quad \Delta B(t_0) = -\mathrm{curl}(A(t_0)\Pi(u(t_0))\nabla u(t_0)) = -\nabla u(t_0) \cdot \nabla^\perp(A(t_0)\Pi(u(t_0)))\,.$$

Combining the Jacobian structure of the right-hand side of (4.2) with the zero boundary condition of B and estimates (3.3) and (3.7), the Wente's lemma 3.6 gives (4.1). Here we have also used $E(u(t)) < \varepsilon_0$ for all $t \geq 0$, the smoothness and compactness of the target manifold $\mathcal{N}$.   □

4.2. Improved local estimate on the matrix P.
We next show that ΔP also has a special Jacobian structure.

Lemma 4.2. *Let $u(x,t)$ be as in Theorem 1.6, then for any $t_0 \in [1, \infty)$ such that $u_t(\cdot, t_0) \in L^2(B_1)$, there exist*

$$\xi(t_0) \in W_0^{1,2}(B_1, so(n)), \quad \eta(t_0) \in W^{1,2}(B_1, \mathbb{R}^n), \quad \zeta(t_0) \in W_0^{2,2}(B_1, \mathbb{R}^n)$$

and

$$Q_k(t_0), R_k(t_0) \in W^{1,2} \cap L^\infty(B_1, Gl_n(\mathbb{R})), \quad k = 1, ..., n$$

with

$$\|\nabla \xi(t_0)\|_{L^2(B_1)} + \|\nabla \eta(t_0)\|_{L^2(B_1)} + \|\nabla \zeta(t_0)\|_{L^2(B_1)}$$
$$+ \sum_k \left(\|\nabla Q_k(t_0)\|_{L^2(B_1)} + \|\nabla R_k(t_0)\|_{L^2(B_1)}\right) \leq C\sqrt{\varepsilon_0}$$

and

$$(4.3) \qquad \|\zeta(t_0)\|_{W^{2,2}(B_1)} \leq C\|u_t(t_0)\|_{L^2(B_1)}$$

such that

$$\Delta P(t_0) = \nabla P(t_0) \cdot \nabla^\perp \xi(t_0) + \nabla Q_k(t_0) \cdot \nabla^\perp \eta^k(t_0)$$
(4.4)
$$+ \nabla R_k(t_0) \cdot \nabla^\perp u^k(t_0) + div \left(Q_k(t_0) \nabla \zeta^k(t_0) \right).$$

Proof. We omit the index t_0 in the proof. Hodge decomposition and the estimates for the L^∞-norms of A and B imply the existence of $\eta \in W^{1,2}(B_1, \mathbb{R}^n)$ and $\zeta \in W_0^{1,2}(B_1, \mathbb{R}^n)$ such that

(4.5)
$$\nabla^\perp \eta + \nabla \zeta \ = \ A\nabla u + B\nabla^\perp u$$

with

(4.6)
$$\|\nabla \eta\|_{L^2(B_1)} + \|\nabla \zeta\|_{L^2(B_1)} \le C\|\nabla u\|_{L^2(B_1)} \le C\sqrt{\varepsilon_0}\,.$$

Moreover, by the "almost" conservation law (3.15) we have

$$\Delta \zeta \ = \ Au_t \ \in \ L^2(B_1) \quad \text{and} \quad \zeta|_{\partial B_1} = 0\,,$$

which gives (4.3) by the standard L^p-theory. Multiplying both sides of equation (4.5) by A^{-1} from the left gives (with indices)

(4.7)
$$\nabla u^l \ = \ (A^{-1})_k^l \nabla^\perp \eta^k - (A^{-1}B)_k^l \nabla^\perp u^k + (A^{-1})_k^l \nabla \zeta^k\,, \quad l = 1, 2, ..., n\,.$$

Taking divergence on both sides of equation (3.4) yields

(4.8)
$$\Delta P_j^i \ = \ \nabla P_m^i \cdot \nabla^\perp \xi_j^m - div\left(\Omega_z^i P_j^z \right), \quad 1 \le i, j \le n\,.$$

Since $\Omega_z^i = [\Pi^i(u)_{z,l} - \Pi^z(u)_{i,l}]\nabla u^l$, combining (4.7) and (4.8) gives

$$\Delta P_j^i = \nabla P_m^i \cdot \nabla^\perp \xi_j^m - div\left[(\Pi^i(u)_{z,l} - \Pi^z(u)_{i,l})\left((A^{-1})_k^l \nabla^\perp \eta^k \right.\right.$$
$$\left.\left. - (A^{-1}B)_k^l \nabla^\perp u^k + (A^{-1})_k^l \nabla \zeta^k \right) P_j^z \right]$$
(4.9)
$$= \nabla P_m^i \cdot \nabla^\perp \xi_j^m - \nabla\left[(\Pi^i(u)_{z,l} - \Pi^z(u)_{i,l})P_j^z (A^{-1})_k^l \right] \cdot \nabla^\perp \eta^k$$
$$+ \nabla\left[(\Pi^i(u)_{z,l} - \Pi^z(u)_{i,l})P_j^z (A^{-1}B)_k^l \right] \cdot \nabla^\perp u^k$$
$$- div\left[\left((\Pi^i(u)_{z,l} - \Pi^z(u)_{i,l})P_j^z (A^{-1})_k^l \right) \nabla \zeta^k \right].$$

Defining

$$(Q_k)_j^i = -(\Pi^i(u)_{z,l} - \Pi^z(u)_{i,l})P_j^z (A^{-1})_k^l$$

and

$$(R_k)_j^i = (\Pi^i(u)_{z,l} - \Pi^z(u)_{i,l})P_j^z (A^{-1}B)_k^l\,,$$

where $1 \le k, i, j \le n$, completes the proof. $\square$

Next we prove a local estimate on the oscillation of the matrix P based on Lemma 4.2. A key observation here is that whether a function is in the local Hardy space $h^1(B_1)$ essentially depends on its local behavior (see Definition 2.8). This local oscillation estimate on P provides important information that we need to control the local behavior of $|\nabla u|^2$. This point will become apparent in Section 5. As we shall see, the Jacobian structure of ΔP enters in a crucial way.

Lemma 4.3. *Let $u(x, t)$ be as in Theorem 1.6, then for any $t_0 \in [1, \infty)$ such that $u_t(\cdot, t_0) \in L^2(B_1)$, any $x \in B_1$, any $r > 0$ such that $B_{2r}(x) \subset B_1$ and any $y \in B_r(x)$ we have*

(4.10)
$$|P(y, t_0) - P(x, t_0)| \ \le \ C\left(\sqrt{\varepsilon_0} + \|u_t(t_0)\|_{L^2(B_1)} \right).$$

Proof. We will omit the index t_0 in the proof. Let $\widetilde{P} \in W^{1,2}(B_1, M_n(\mathbb{R}))$ be the weak solution of

$$\begin{cases} \Delta\widetilde{P} = \nabla P \cdot \nabla^\perp \xi + \nabla Q_k \cdot \nabla^\perp \eta^k + \nabla R_k \cdot \nabla^\perp u^k + \text{div}\,(Q_k \nabla \zeta^k) & \text{in } B_1\,, \\ \widetilde{P} = 0 & \text{on } \partial B_1\,, \end{cases}$$

where Q_k, R_k, η^k and ζ^k's are from Lemma 4.2.

Then by Wente's lemma 3.6 and the standard L^p-theory (and $W^{2,2}(B_1) \hookrightarrow C^0(B_1)$), we have $\widetilde{P} \in C^0(B_1, M_n(\mathbb{R}))$ and

$$(4.11) \qquad \|\widetilde{P}\|_{L^\infty(B_1)} + \|\nabla\widetilde{P}\|_{L^2(B_1)} \leq C\left(\varepsilon_0 + \|u_t(t_0)\|_{L^2(B_1)}\right)\,.$$

Since

$$\Delta(P - \widetilde{P}) = 0 \quad \text{in} B_1\,,$$

we know that $V = P - \widetilde{P} \in C^\infty(B_1, M_n(\mathbb{R}))$ is harmonic. Now for any $x \in B_1$, any $r > 0$ such that $B_{2r}(x) \subset B_1$ and any $y \in B_r(x)$ we have

$$|V(y) - V(x)| \leq Cr\|\nabla V\|_{L^\infty(B_r(x))} \leq Cr^{-1}\|\nabla V\|_{L^1(B_{2r}(x))}$$

$$(4.12) \qquad \leq C\|\nabla V\|_{L^2(B_{2r}(x))} \leq C\left(\|\nabla P\|_{L^2(B_{2r}(x))} + \|\nabla\widetilde{P}\|_{L^2(B_{2r}(x))}\right)$$

$$\leq C\left(\sqrt{\varepsilon_0} + \|u_t(t_0)\|_{L^2(B_1)}\right)\,,$$

where we have used the mean value property of V and (4.11), (3.3). Combining (4.11) and (4.12) yields that for any $x \in B_1$, any $r > 0$ such that $B_{2r}(x) \subset B_1$ and any $y \in B_r(x)$ we have

$$(4.13) \qquad |P(y, t_0) - P(x, t_0)| \leq C\left(\sqrt{\varepsilon_0} + \|u_t(t_0)\|_{L^2(B_1)}\right)\,,$$

which gives the desired estimate (4.10). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Validation of (2.21) and completion of the proof of Theorem 1.6

With the results so far at our disposal, we are now in a position to validate (2.21). As mention above, the local estimate on the oscillation of the transformation matrix P in Lemma 4.3 will be the key ingredient.

Lemma 5.1. *Let $u(x, t)$ be as in Theorem 1.6, then for any $t_0 \in [1, \infty)$ such that $\|u_t(t_0)\|_{L^2(B_1)} < \sqrt{\varepsilon_0}$ we have*

$$(5.1) \qquad\qquad\qquad |\nabla u(t_0)|^2 \in h^1(B_1)$$

with estimate

$$(5.2) \qquad\qquad\qquad \||\nabla u(t_0)|^2\|_{h^1(B_1)} \leq C\varepsilon_0\,.$$

Remark 5.2. Lemma 5.1 continues to hold for the flow (3.10) with a more general Ω in the form $\Omega_j^i = \sum_{l=1}^n f_{jl}^i \nabla u^l + g_{jl}^i \nabla^\perp u^l$ (which includes $\Omega = \Omega_j^i = [\Pi^i(u)_{j,l} - \Pi^j(u)_{i,l}]\nabla u^l$ for the harmonic map heat flow as a special case), see [LL12]. Moreover, the condition $\|u_t(t_0)\|_{L^2(B_1)} < \varepsilon_0$ can be replaced by that $\|u_t(t_0)\|_{L^p(B_1)}$ is sufficiently small for some $p > 1$.

Proof. (of Lemma 5.1) By the assumption $\|u_t(t_0)\|_{L^2(B_1)} < \sqrt{\varepsilon_0}$ and Lemma 4.3, for any $x \in B_1$, any $r > 0$ such that $B_{2r}(x) \subset B_1$ and any $y \in B_r(x)$ we have

$$(5.3) \qquad\qquad |P(y, t_0) - P(x, t_0)| \leq C\sqrt{\varepsilon_0}\,.$$

We will omit the index t_0 from now on. By Proposition 4.1, Theorem 3.1 and Theorem 3.4, for any $x \in B_1$, any $r > 0$ such that $B_{2r}(x) \subset B_1$ and any $y \in B_r(x)$ we have (choosing ε_0 sufficiently small)

$$0 \leq \frac{1}{2}|\nabla u|^2(y) \leq \left(A\nabla u + B\nabla^\perp u\right) \cdot (P^T \nabla u)(y)$$

$$(5.4) \qquad = \left(A\nabla u + B\nabla^\perp u\right) \cdot \left[\left(P^T(x) + \left(P^T - P^T(x)\right)\right)\nabla u\right](y)\,,$$

and therefore by (4.5) and (5.3)

$$\left(\nabla^\perp \eta + \nabla \zeta\right) \cdot \left(P^T(x)\nabla u\right)(y) = \left(A\nabla u + B\nabla^\perp u\right) \cdot \left(P^T(x)\nabla u\right)(y)$$

$$(5.5) \quad \geq \frac{1}{2}|\nabla u|^2(y) - \left(A\nabla u + B\nabla^\perp u\right) \cdot \left[\left(P^T - P^T(x)\right)\nabla u\right](y) \geq \frac{1}{4}|\nabla u|^2(y)\,.$$

Now we choose a function

$$(5.6) \quad \phi \in C_0^\infty(B_1) \text{ with } \phi \geq 0,\, \mathrm{spt}(\phi) \subseteq B_{\frac{1}{2}},\, \phi = 2 \text{ on } B_{\frac{3}{8}},\, \text{and } \int_{B_1} \phi\, dx = 1\,.$$

Moreover, we additionally assume that $\|\nabla\phi\|_{L^\infty(B_1)} \leq 100$. Using (4.3) and (5.5), one verifies directly that (by Definition 2.8)

$$\||\nabla u|^2\|_{h^1(B_1)} = \int_{B_1} \sup_{0 < t < 1 - |x|} \phi_t * |\nabla u|^2 dx$$

$$\leq 4 \int_{B_1} \sup_{0 < t < 1 - |x|} \phi_t * \left(\left(\nabla^\perp \eta + \nabla\zeta\right) \cdot \left(P^T(x)\nabla u\right)\right) dx$$

$$= 4 \int_{B_1} \sup_{0 < t < 1 - |x|} \phi_t * \left[(P^T(x))_{ij}\left(\nabla^\perp \eta^i \cdot \nabla u^j + \nabla\zeta^i \cdot \nabla u^j\right)\right] dx$$

$$\leq C \sum_{i,j=1}^n \left(\|\nabla^\perp \eta^i \cdot \nabla u^j\|_{h^1(B_1)} + \|\nabla\zeta^i\|_{W^{1,2}(B_1)}\|\nabla u^j\|_{L^2(B_1)}\right)$$

$$\leq C \|\nabla^\perp \eta\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)} + C\sqrt{\varepsilon_0}\|u_t\|_{L^2(B_1)} \leq C\varepsilon_0\,,$$

where we have used the facts

(1) $\nabla^\perp \eta^i \cdot \nabla u^j \in h^1(B_1)$ and $\|\nabla^\perp \eta^i \cdot \nabla u^j\|_{h^1(B_1)} \leq C\|\nabla\eta\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}$ for all $i, j = 1, 2, ..., n$;

(2) $\|\nabla\zeta^i \cdot \nabla u^j\|_{L^p(B_1)} \leq C\|\nabla\zeta^i\|_{W^{1,2}(B_1)}\|\nabla u^j\|_{L^2(B_1)}$ for any $1 < p < 2$ and $\|f\|_{h^1(B_1)} \leq C\|f\|_{L^p(B_1)}$ for any $p > 1$.

To see (1), we first extend

$$\eta^i - \frac{1}{|B_1|}\int_{B_1}\eta^i \quad \text{and} \quad u^j - \frac{1}{|B_1|}\int_{B_1}u^j$$

from B_1 to $\mathbb{R}^2$ which yields the existence of $\tilde{\eta}^i, \tilde{u}^j \in W_c^{1,2}(\mathbb{R}^2)$ such that

$$(5.7) \qquad \int_{\mathbb{R}^2}|\nabla\tilde{\eta}^i|^2 \leq C\int_{B_1}|\nabla\eta^i|^2 \quad \text{and} \quad \int_{\mathbb{R}^2}|\nabla\tilde{u}^j|^2 \leq C\int_{B_1}|\nabla u^j|^2$$

and

$$(5.8) \qquad\qquad \nabla\tilde{\eta}^i = \nabla\eta^i \quad \text{and} \quad \nabla\tilde{u}^j = \nabla u^j \quad \text{a.e. in } B_1\,.$$

Then by the results of [CLMS93] we know that

$$\|\nabla^\perp \tilde{\eta}^i \cdot \nabla \tilde{u}^j\|_{\mathcal{H}^1(\mathbb{R}^2)} := \int_{\mathbb{R}^2} \sup_{\phi \in \mathcal{T}} \sup_{t>0} \left| \int_{B_t(x)} \frac{1}{t^2} \phi\left(\frac{x-y}{t}\right) \left(\nabla^\perp \tilde{\eta}^i \cdot \nabla \tilde{u}^j\right)(y) dy \right| dx$$

$$(5.9) \qquad \leq C\|\nabla\tilde{\eta}^i\|_{L^2(\mathbb{R}^2)}\|\nabla\tilde{u}^j\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla\eta\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)},$$

where $\mathcal{T} = \{\phi \in C^\infty(\mathbb{R}^2) : \mathrm{spt}(\phi) \subset B_1$ and $\|\nabla\phi\|_{L^\infty} \leq 100\}$. By (5.8), (5.9) and Definition 2.8, it is clear that

$$\|\nabla^\perp \eta^i \cdot \nabla u^j\|_{h^1(B_1)} = \|\nabla^\perp \tilde{\eta}^i \cdot \nabla \tilde{u}^j\|_{h^1(B_1)}$$

$$(5.10) \qquad \leq \|\nabla^\perp \tilde{\eta}^i \cdot \nabla \tilde{u}^j\|_{\mathcal{H}^1(\mathbb{R}^2)} \leq C\|\nabla\eta\|_{L^2(B_1)}\|\nabla u\|_{L^2(B_1)}.$$

This completes the proof of the lemma. $\qquad\square$

Now since $u(x,t) \in W^{1,2} \cap C^\infty(B_1 \times [1,\infty), \mathcal{N})$ and the energy $E(u(\cdot,t))$ is non-increasing along the flow as shown in (1.6), there exists $T_0 \geq 1$ such that

$$(5.11) \qquad \|u_t(T_0)\|_{L^2(B_1)} < \sqrt{\varepsilon_0}.$$

Then by Lemma 5.1 we know that $|\nabla u(T_0)|^2 \in h^1(B_1)$ with estimate

$$(5.12) \qquad \||\nabla u(T_0)|^2\|_{h^1(B_1)} \leq C\varepsilon_0.$$

Therefore, in view of Lemma 5.1, in order to validate the global estimate (2.21) we are left to show

$$(5.13) \qquad \|u_t(t_0)\|_{L^2(B_1)} < \sqrt{\varepsilon_0} \quad \text{for all } t_0 \geq T_0.$$

We will next show this is indeed the case.

Lemma 5.3. *Let $u(x,t)$ be as in Theorem 1.6, then there exists $T_0 > 0$ such that*

$$(5.14) \qquad \|u_t(t_0)\|_{L^2(B_1)} < \sqrt{\varepsilon_0} \quad \text{for all } t_0 \geq T_0.$$

Proof. Let $T_0 \geq 1$ be as in (5.11) so that we have $\|u_t(T_0)\|_{L^2(B_1)} < \sqrt{\varepsilon_0}$. Since $u(x,t) \in W^{1,2} \cap C^\infty(B_1 \times [1,\infty), \mathcal{N})$ and by the continuity of $\int_{B_1} |u_t(t)|^2$ in t, there exists $\delta = \delta(T_0, \varepsilon_0) > 0$ such that for any $t_0 \in [T_0, T_0 + \delta]$ we have

$$(5.15) \qquad \|u_t(t_0)\|_{L^2(B_1)} < 2\sqrt{\varepsilon_0}.$$

Therefore by our previous arguments (especially Theorem 2.6, Lemma 5.1 and (5.12) with T_0 replaced by t_0), Lemma 2.5 applies to any subinterval of $[T_0, T_0 + \delta]$ and yields

$$(5.16) \qquad \int_{B_1} |u_t(t_2)|^2 \leq \int_{B_1} |u_t(t_1)|^2 \quad \text{for any } t_1, t_2 \text{ s.t. } T_0 \leq t_1 < t_2 \leq T_0 + \delta.$$

This shows, instead of (5.15), for any $t_0 \in [T_0, T_0 + \delta]$ we have

$$(5.17) \qquad \|u_t(t_0)\|_{L^2(B_1)} \leq \|u_t(T_0)\|_{L^2(B_1)} < \sqrt{\varepsilon_0}.$$

We can then continue and iterate this process beyond $T_0 + \delta$ and we see that $\int_{B_1} |u_t(t)|^2$ is indeed non-increasing along the flow after T_0. $\qquad\square$

This completes the validation of (2.21) and therefore the assumptions in Lemmas 2.4 and 2.5 in view of Theorem 2.6, finishing the proof of our main Theorem 1.6 as shown in Section 2.

Appendix A. A proof of Theorem 2.6

Proof. The idea of the proof follows [Se94, Proposition 1.68]. Since the Green's function of Δ on B_1 is given by $\frac{1}{2\pi} \ln|x|$, for $x \in B_1$, we can write

$$(A.1) \qquad \psi(x) = \frac{1}{2\pi} \int_{B_1} f(y) \left(\ln|x-y| - \ln\left(\left| \frac{x}{|x|} - |x|y \right| \right) \right) dy.$$

Let $\theta \in C_0^\infty(B_1)$ be a smooth bump function such that $0 \le \theta \le 1$, $\theta = 1$ in $B_{\frac{1}{16}}$ and $\mathrm{spt}(\theta) \subset B_{\frac{1}{8}}$. For $x \in B_1$ we define

$$(A.2) \qquad l_x(y) := \sum_{j=0}^\infty \theta\left(2^j (1-|x|)^{-1}(x-y) \right) \quad \text{for } y \in B_1.$$

We claim that for any $x, y \in B_1$

$$(A.3) \qquad -20\ln 2 \le \ln|x-y| - \ln\left(\left| \frac{x}{|x|} - |x|y \right| \right) + l_x(y)\ln 2 \le 20\ln 2,$$

To see this, it is clear that for $x, y \in B_1$ such that

$$(A.4) \qquad 2^{-k} \le |x-y| \le 2^{-k+1}, \quad k \in \mathbb{N}_0$$

we have

$$(A.5) \qquad -k\ln 2 \le \ln|x-y| \le (-k+1)\ln 2.$$

Now note that

$$1 - |x| - |x-y| \le 1 - |x| + |x| - |y| = 1 - |y| \le 1 - |x| + |x-y|,$$

and therefore for $x \in B_{1-2^{-i-1}} \setminus B_{1-2^{-i}}$, i.e., $1-|x| \in [2^{-i-1}, 2^{-i}], i \in \mathbb{N}_0$ (with $\bar{B}_0 = \emptyset$) and any $y \in B_1$ satisfying (A.4), we have

$$1 - |y| \in \begin{cases} \left[2^{-i-1} - 2^{-k+1}, 2^{-i} + 2^{-k+1} \right] & \text{if } k \ge i+4; \\ \left[0, 2^{-i} + 2^{-k+1} \right] & \text{if } k \le i+3. \end{cases}$$

We also have

$$0 \le (1-|x|)(1-|y|) \le (1-|x|^2)(1-|y|^2)$$

$$(A.6) \qquad = \left| \frac{x}{|x|} - |x|y \right|^2 - |x-y|^2 \le 2^2(1-|x|)(1-|y|),$$

and thus

$$\left| \frac{x}{|x|} - |x|y \right|^2 - |x-y|^2 \in \begin{cases} \left[2^{-2i-2} - 2^{-i-k}, 2^{-2i+2} + 2^{-i-k+3} \right] & \text{if } k \ge i+4; \\ \left[0, 2^{-2i+2} + 2^{-i-k+3} \right] & \text{if } k \le i+3. \end{cases}$$

Combining this with (A.4) we get

$$\left| \frac{x}{|x|} - |x|y \right|^2 \in \begin{cases} \left[2^{-2i-2} - 2^{-i-k} + 2^{-2k}, 2^{-2i+2} + 2^{-i-k+3} + 2^{-2k+2} \right] & \text{if } k \ge i+4; \\ \left[2^{-2k}, 2^{-2i+2} + 2^{-i-k+3} + 2^{-2k+2} \right] & \text{if } k \le i+3. \end{cases}$$

Now using the facts that for $k \ge i+4$ we have

$$2^{-2i-2} - 2^{-i-k} + 2^{-2k} \ge 2^{-2i-4} \quad \text{and} \quad 2^{-2i+2} + 2^{-i-k+3} + 2^{-2k+2} \le 2^{-2i+4}$$

and for $k \le i+3$ we have

$$2^{-2i+2} + 2^{-i-k+3} + 2^{-2k+2} \le 2^{-2k+10},$$

we arrive at

$$\left|\frac{x}{|x|} - |x|y\right|^2 \in \begin{cases} \left[2^{-2i-4}, 2^{-2i+4}\right] & \text{if } k \geq i+4; \\ \left[2^{-2k}, 2^{-2k+10}\right] & \text{if } k \leq i+3, \end{cases}$$

and hence

$$(A.7) \qquad -\ln\left|\frac{x}{|x|} - |x|y\right| \in \begin{cases} \left[(i-2)\ln 2, (i+2)\ln 2\right] & \text{if } k \geq i+4; \\ \left[(k-5)\ln 2, k\ln 2\right] & \text{if } k \leq i+3. \end{cases}$$

Combining (A.5) and (A.7) we get
(A.8)
$$\ln|x-y| - \ln\left(\left|\frac{x}{|x|} - |x|y\right|\right) \in \begin{cases} \left[(-k+i-2)\ln 2, (-k+i+3)\ln 2\right] & \text{if } k \geq i+4; \\ \left[-5\ln 2, \ln 2\right] \quad (\text{in fact, } [-5\ln 2, 0]) & \text{if } k \leq i+3, \end{cases}$$

for any $x \in B_{1-2^{-i-1}} \setminus B_{1-2^{-i}}, i \geq 0$, and any $y \in B_1$ satisfying (A.4) for some $k \geq 0$.

Now for any $x \in B_{1-2^{-i-1}} \setminus B_{1-2^{-i}}, i \geq 0$, and any $y \in B_1$ satisfying (A.4), since $0 \leq \theta \leq 1$, $\theta = 1$ in $B_{\frac{1}{16}}$ and $\text{spt}(\theta) \subset B_{\frac{1}{8}}$, we get that for any $j \geq 0$

$$\theta\left(2^j(1-|x|)^{-1}(x-y)\right) = 0 \quad \text{for} \quad |x-y| \geq 2^{-j-3}(1-|x|) \in [2^{-j-i-4}, 2^{-j-i-3}]$$

and

$$\theta\left(2^j(1-|x|)^{-1}(x-y)\right) = 1 \quad \text{for} \quad |x-y| \leq 2^{-j-4}(1-|x|) \in [2^{-j-i-5}, 2^{-j-i-4}].$$

Therefore (combining with (A.4)),

$$(A.9) \qquad \theta\left(2^j(1-|x|)^{-1}(x-y)\right) = 0 \quad \text{for} \quad j \geq k-i-3$$

and

$$(A.10) \quad \theta\left(2^j(1-|x|)^{-1}(x-y)\right) = 1 \quad \text{if} \quad k-1 \geq j+i+5 \quad (\text{i.e. } j \leq k-i-6).$$

Hence for any $x \in B_{1-2^{-i-1}} \setminus B_{1-2^{-i}}, i \geq 0$ and any $y \in B_1$ such that $2^{-k} \leq |x-y| \leq 2^{-k+1}$ for some $k = 0, 1, 2, ...,$ (A.2), (A.9) and (A.10) imply

$$(A.11) \qquad \begin{cases} k-i-10 \leq l_x(y) \leq k-i+10 & \text{if } k \geq i+4; \\ l_x(y) = 0 & \text{if } k \leq i+3. \end{cases}$$

Combining (A.8) and (A.11) gives (A.3).

Therefore, in order to obtain the L^∞-bound of ψ on B_1 as in (2.18), it suffices to bound $\int_{B_1} f(y)l_x(y)dy$ since we have (A.1), (A.3) and $\|f\|_{L^1(B_1)} \leq \|f\|_{h^1(B_1)}$.

In order to bound $\int_{B_1} f(y)l_x(y)dy$, we next claim that for any $x \in B_1$, $j \geq 0$ and $z \in B_{2^{-j-4}(1-|x|)}(x)$ we have
(A.12)
$$\int_{B_1} f(y)2^{2j+2}(1-|x|)^{-2}\theta\left(2^j(1-|x|)^{-1}(x-y)\right) dy \leq \int_{B_t(z)} \frac{1}{t^2}\phi\left(\frac{z-y}{t}\right) f(y)dy,$$

where

$$t = 2^{-j-1}(1-|x|)$$

and ϕ is a nonnegative Schwartz function as in (5.6). To see (A.12), we first note that since $\mathrm{spt}(\theta) \subset B_{\frac{1}{8}}$, we have for any $x \in B_1$ and $j \geq 0$

$$\int_{B_1} f(y) 2^{2j+2}(1-|x|)^{-2}\theta\left(2^j(1-|x|)^{-1}(x-y)\right) dy$$

(A.13)
$$= \int_{B_{2^{-j-3}(1-|x|)}(x)} f(y) 2^{2j+2}(1-|x|)^{-2}\theta\left(2^j(1-|x|)^{-1}(x-y)\right) dy.$$

Now since
$$\frac{3}{8}(2^{-j-1}) = 2^{-j-4} + 2^{-j-3}$$

and
$$2^{-j-4} + 2^{-j-1} = \frac{9}{16}2^{-j} < 1$$

for any $j \geq 0$, we see that for any $z \in B_{2^{-j-4}(1-|x|)}(x)$

(A.14)
$$B_{2^{-j-3}(1-|x|)}(x) \subseteq B_{\frac{3t}{8}}(z) \subset B_t(z) = B_{2^{-j-1}(1-|x|)}(z) \subseteq B_1.$$

Using the facts that $f \geq 0$, $0 \leq \theta \leq 1$, $\phi \geq 0$ and $\phi = 2$ on $B_{\frac{3}{8}}$ we conclude

$$\int_{B_{2^{-j-3}(1-|x|)}(x)} f(y) 2^{2j+2}(1-|x|)^{-2}\theta(2^j(1-|x|)^{-1}(x-y))dy$$

$$\leq \int_{B_{2^{-j-3}(1-|x|)}(x)} f(y) 2^{2j+2}(1-|x|)^{-2}dy \leq \int_{B_{\frac{3t}{8}}(z)} f(y) 2^{2j+2}(1-|x|)^{-2}dy$$

$$\leq \int_{B_t(z)} f(y) 2^{2j+2}(1-|x|)^{-2}\phi\left(\frac{z-y}{t}\right) dy = \int_{B_t(z)} \frac{1}{t^2}\phi\left(\frac{z-y}{t}\right) f(y)dy.$$

Combining this with (A.13) gives (A.12). Therefore by (A.12) and the definition (2.19) of the radial maximal function f^*, for any $x \in B_1$ and $j \geq 0$ we have

$$\left|\int_{B_1} f(y)\theta(2^j(1-|x|)^{-1}(x-y))dy\right| \leq 2^{-2j-2}(1-|x|)^2 \inf_{z \in B_{2^{-j-4}(1-|x|)}(x)} f^*(z).$$

Therefore, by (A.2), for any $x \in B_1$ we have

$$\left|\int_{B_1} f(y)l_x(y)dy\right| \leq \sum_{j=0}^{\infty}\left|\int_{B_1} f(y)\theta(2^j(1-|x|)^{-1}(x-y))dy\right|$$

$$\leq \sum_{j=0}^{\infty} 2^{-2j-2}(1-|x|)^2 \inf_{z \in B_{2^{-j-4}(1-|x|)}(x)} f^*(z)$$

(A.15)
$$\leq \frac{2^8}{3\pi}\sum_{j=0}^{\infty}\int_{B_{2^{-j-4}(1-|x|)}(x)\setminus B_{2^{-j-5}(1-|x|)}(x)} f^*(z)dz$$

$$\leq \frac{2^8}{3\pi}\int_{B_1} f^*(z)dz \leq \frac{2^8}{3\pi}\|f\|_{h^1(B_1)}.$$

Combining (A.6), (A.3) and (A.15) yields (using $\|f\|_{L^1(B_1)} \leq \|f\|_{h^1(B_1)}$)

$$|\psi(x)| = -\frac{1}{2\pi}\int_{B_1} f(y)\left(\ln|x-y| - \ln\left(\left|\frac{x}{|x|} - |x|y\right|\right)\right) dy \leq C\|f\|_{h^1(B_1)}.$$

This gives the desired L^∞-bound of ψ on B_1. The L^2-estimate for $\nabla\psi$ simply follows from an integration by parts argument. □

References

[BC83] H. Brezis and J. Coron, Large solutions for harmonic maps in two dimensions, Comm. Math. Phys. 92 (1983), no. 2, 203–215.

[BC84] H. Brezis and J. Coron, Multiple solutions of H-systems and Rellich's conjecture, Commun. Pure Appl. Math. 37(2) (1984), 149–187.

[Ch89] K.C. Chang, Heat flow and boundary value problem for harmonic maps, Annales de l'institut Henri Poincaré (C) Analyse non linéaire, 6 no. 5 (1989), p. 363–395.

[Cho95] P. Chone, A regularity result for critical points of conformally invariant functionals, Potential Anal., 4 (1995), 269–296.

[CKS93] D. Chang, S. G. Krantz and E.M. Stein, H^p theory on a smooth domain in R^N and elliptic boundary value problems, J. Funct. Anal. 114 (1993), no. 2, 286–347.

[CLMS93] R. R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes, Compensated compactness and Hardy spaces, J. Math. Pures Appl. (9) 72 (1993), no. 3, 247–286.

[CM08-1] T.H. Colding and W.P. Minicozzi II, Width and finite extinction time of Ricci flow, Geometry and Topology, 12 (2008), no. 5, 2537–2586.

[CM08-2] T.H. Colding and W.P. Minicozzi II, Width and mean curvature flow, Geometry and Topology, 12 (2008), no. 5, 2517–2535.

[ES64] J. Eells and J.H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86(1964) 109-164.

[Fr95] A. Freire, Uniqueness for the harmonic map flow from surfaces to general targets, Commentarii Mathematici Helvetici, 70(2), 310–338, 1995.

[Fr96] A. Freire, Correction to "Uniqueness for the harmonic map flow from surfaces to general targets", Commentarii Mathematici Helvetici, 71(2), 330–337, 1996.

[Ha75] R. Hamilton, Harmonic maps of manifolds with boundary, volume 471 of Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1975.

[Har67] P. Hartman, On homotopic harmonic maps, Canad. J. Math. 19 (1967) 673–687.

[He02] F. Hélein, *Harmonic maps, conservation laws, and moving frames, Cambridge Tracts in Mathematics*, 150., Cambridge University Press, Cambridge, 2002.

[IM01] T. Iwaniec and G. Martin, *Geometric Function Theory and Non-linear Analysis*, Oxford University Press, Clarendon, 2001.

[Jo84] J. Jost, The Dirichlet problem for harmonic maps from a surface with boundary onto a 2-sphere with nonconstant boundary values, J. Differential Geom., Volume 19, Number 2 (1984), 393–401.

[LL12] T. Lamm and L. Lin, Estimates on the energy density for critical points of conformally invariant variational problems, preprint 2012.

[Li96] G.M.Lieberman, *Second order parabolic differential equations*, World Sci. Publishing Company, 1996.

[Mi90] A. Miyachi, H^p spaces over open subsets of $\mathbb{R}^n$, Studia Math. 95 (1990), no. 3, 205–228.

[Mu90] S. Müller, Higher integrability of determinants and weak convergence in L^1, J. Reine Angew. Math. 412 (1990), 20–34.

[Qi95] J. Qing, On singularities of the heat flow for harmonic maps from surfaces into spheres, Comm. Anal. Geom. 3 (1995), no. 1-2, 297–315.

[Ri07] T. Rivière, Conservation laws for conformally invariant variational problems, Inventiones Mathematicae, v.168 (2007), no.1, 1–22.

[Ri08-1] T. Rivière, The role of Integrability by Compensation in Conformal Geometric Analysis, to appear in Analytic aspects of problems from Riemannian Geometry S.M.F. (2008).

[Ri08-2] T. Rivière, Analysis aspects of Willmore surfaces, Invent. Math. 174 (2008), no. 1, 1–45.

[Ri12] T. Rivière, Conformally Invariant Variational Problems, arXiv:1206.2116.

[Se94] S. Semmes. A primer on Hardy spaces, and some remarks on a theorem of Evans and Müller, Commun. Partial Differ. Equations, 19(1–2), 277–319, 1994.

[Sc09] A. Schikorra, A remark on Gauge transformations and the moving frame method, Annales de l'Institut Henri Poincaré - Analyse non linéaire (2009), doi:10.1016/j.anihpc.2009.09.004.

[Si83] L. Simom, Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems, Ann. Math. 118 (1983) 525–571.

[St69] E. Stein, Note on the class $L \log L$, Studia Math., 32(1969), 305–310.

[Str85] M.Struwe, On the evolution of harmonic mappings of Riemannian surfaces, Commentarii Mathematici Helvetici, v.60, No.1, 1985.

[Ta85] L. Tartar, Remarks on oscillations and Stokes' equation. In: Frisch, U., et al. (eds.) Macroscopic Modelling of Turbulent Flows, vol. 230, pp. 24–31. Nice, 1984. Lecture Notes Phys., Springer, Berlin (1985).

[To97] P. Topping, Rigidity in the harmonic map heat flow, J. Differential Geom. 45 (1997), no. 3, 593–610.

[Uh82] K. Uhlenbeck, Connections with L^p bounds on curvature, Comm. Math. Phys. v. 83 (1982), no. 1, 31–42.

[Wa11] L. Wang, Harmonic map heat flow with rough boundary data, preprint 2011, to appear in Transactions of the AMS.

[We69] H. Wente, An existence theorem for surfaces of constant mean curvature, J. Math. Anal. Appl., 26 (1969), 318-344.

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019,

  E-mail address: `lzlin@math.rutgers.edu`