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Effective Equidistribution in Homogeneous Dynamics with Applications in Number Theory

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Effective Equidistribution in Homogeneous Dynamics with Applications in Number Theory

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Taylor Jane McAdam

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- Professor Alina Bucur
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2019

The dissertation of Taylor Jane McAdam is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California San Diego

2019

DEDICATION

To my family and friends, whose unwavering love, guidance, trust, and support have filled my life with joy and given me the strength to endure and succeed. This accomplishment is also yours.

EPIGRAPH

The Red Queen shook her head. “You may call it ‘nonsense’ if you like,” she said, “but I’ve *heard nonsense, compared with which that would be as sensible as a dictionary.”*

—from *Through the Looking Glass* by Lewis Carroll

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Chapter 6, in part, is currently being prepared for submission for publication of the material. Luethi, Manuel; McAdam, Taylor. The dissertation author is one of the primary investigators and authors of this material.

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ABSTRACT OF THE DISSERTATION

Effective Equidistribution in Homogeneous Dynamics with Applications in Number Theory

by

Taylor Jane McAdam

Doctor of Philosophy in Mathematics

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Professor Amir Mohammadi, Chair

We study the asymptotic distribution of almost-prime entries in horospherical flows on the quotient of $\mathrm{SL}(n,\mathbb{R})$ by a lattice, where the lattice is either cocompact or $\mathrm{SL}(n,\mathbb{Z})$. In the cocompact case, we obtain a result that implies density for almost-primes in horospherical flows where the number of prime factors is independent of the basepoint, and in the space of lattices we show the density of almost-primes in abelian horospherical orbits of points satisfying a certain Diophantine condition. Along the way we give an effective equidistribution result for arbitrary horospherical flows on the space of lattices, as well as an effective rate for the equidistribution of arithmetic progressions in abelian horospherical flows.

Chapter 1

Introduction

In the study of dynamical systems, one seeks to answer questions about the trajectory of a point in some mathematical space that evolves according to a prescribed set of rules. For example, consider a ball rolling without slipping on a billiard table in the shape of a polygon. The ball will continue rolling in a straight line at a constant speed until it hits one of the sides of the table, at which point it will bounce off in such a way that the angle of incidence equals the angle of reflection. If we assume that no energy is lost, then the ball will continue bouncing around this table forever, tracing out some potentially complicated path, or *orbit*. Some interesting questions we might ask about this system include: Are there any orbits that are dense in the whole space? Are there any that are periodic? Are there any that have fractal closure? What does a typical orbit look like? Although the "rules" for this dynamical system seem quite simple, they lead to surprisingly complex behavior, and many basic facts remain unknown. For example, it is still not known whether every triangular billiard table admits a periodic orbit.

Classically, dynamical systems have been used to model natural phenomena, such as the motion of planets or the growth and dispersal of populations. These models generally take the form of either a discrete-time dynamical system, characterized by a $\mathbb{Z}$-group (or $\mathbb{Z}^+$-semigroup) action on the underlying space, or a continuous dynamical system, characterized by an $\mathbb{R}$-group (or $\mathbb{R}^+$-semigroup) action. It turns out that many of the methods used to study such systems can be generalized to study the actions of other groups with a sufficient amount of structure, and that doing so can yield valuable insight into more theoretical areas of mathematics, such as number theory and geometry. This point of view has been particularly fruitful in the field of homogeneous dynamics, which studies the actions of subgroups of a Lie group G on the quotient of G by a discrete subgroup. Examples of dynamical systems of this form include linear flows on tori and geodesic and horocycle flows on the modular surface.

Many recent breakthroughs in a variety of areas are the result of reformulating old problems into the language of homogeneous dynamics. Foundational works such as Margulis's proof of the Oppenheim conjecture [Mar87] and Ratner's classification of unipotent orbits and unipotent-invariant measures [Rtn91; Rtn94] have led the way for others to seek rigidity phenomena in different contexts with an abundance of useful applications. Such results include Lindenstrauss's proof of arithmetic quantum unique ergodicity [Lin06], Einsiedler-Katok-Lindenstrauss's partial result toward Littlewood's conjecture [EKL06], Benoist-Quint's rigidity theorems [BQ11], Eskin-Mozes-Shah and Gorodnik-Oh's counting results for integer and rational points on homogeneous varieties [EMS96; GO11], and Venkatesh's work on the subconvexity problem for L-functions [Ven10].

Adapting the techniques of homogeneous dynamics to analogous settings has also proven quite powerful. For example, the groundbreaking work of McMullen [McM07], Eskin-Mirzakhani [EM13], and Eskin-Mirzakhani-Mohammadi [EMM15] classifying the invariant measures and orbit closures for the $\mathrm{SL}_2(\mathbb{R})$-action on moduli spaces of abelian differentials was inspired largely by results and techniques from homogeneous dynamics, such as Ratner's classification theorems for unipotent flows. This work has numerous applications in geometry and physics, including in the study of billiard trajectories described above.

1.1 Motivation and Related Work

Equidistribution results play an important role in dynamical systems and their applications. A subset of an orbit is said to equidistribute with respect to a given probability measure if it spends the expected amount of time in different regions of the space, i.e., if the proportion of the subset landing within any measurable set converges to the measure of that set. Often in applications to number theory it is important that an equidistribution result be effective—that is, that the rate of convergence is known. Effective results can be used to derive explicit bounds for number theoretic questions, such as quantitative solutions to the Oppenheim conjecture [GM10; LM14], or to obtain information about the distribution of certain "sparse" subsets of the orbit, such as prime times or polynomial sequences (e.g. [SU15; Ven10; GT12]).

Despite their usefulness for applications in number theory and beyond, effective results for many homogeneous systems remain elusive. For example, effective versions of Ratner's theorems would have far-reaching consequences and are highly sought after; however the problem in its complete generality appears to be very challenging (although some progress has been made, see e.g. [Str15; LM14; EMV09; EMMV]). Nonetheless, there are two contexts for which we have strong effective results—that of nilflows, established by Green-Tao in [GT12], and that of horospherical flows, which have a long and rich history.

Geodesic and horocycle flows on $\mathrm{T}^1(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ (or quotients of these) are classical objects. They are defined, respectively, by actions of the subgroups

$$\left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \right\}_{t\in\mathbb{R}} \qquad \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}_{t\in\mathbb{R}}$$

via multiplication. Observe that $\lim_{t\to\infty} a_t^{-1} u_s a_t = e$ for any $s \in \mathbb{R}$, and moreover that the only elements for which this holds are of the form u_s for some such s. We may expand this notion in the following way: A subgroup of a Lie group G is said to be horospherical if it is contracted (conversely, expanded) under iteration of the adjoint action of some element $g \in G$, i.e.

$$U = \{u \in G \mid g^{-n} u g^n \text{ as } n \to \infty\}.$$

(see Section 2.2 for a more thorough discussion of horospherical subgroups). It can be shown that any horospherical subgroup is unipotent, although not every unipotent subgroup can be realized as the horospherical subgroup corresponding to an element of G. In general, horospherical flows are easier to study than arbitrary unipotent flows, as the mixing action of the expanding element can be used to provide dynamical information about a given horospherical flow.

Actions by horospherical and unipotent subgroups have been studied extensively. Hedlund proved in [Hed36] that the horocycle flow on $\Gamma \backslash \mathrm{SL}_2(\mathbb{R})$ for Γ cocompact is minimal, and Furstenberg later showed it to be uniquely ergodic [Fur73]. These results were extended in [Vee77] and [EP78] to more general horospherical flows on compact quotients of suitable Lie groups. For Γ non-uniform, Margulis proved that orbits of unipotent (hence horospherical) flows cannot diverge to infinity [Mar71], which was later refined in Dani's nondivergence theorem [Dan84; Dan86a]. Dani also showed in [Dan78] (for the case of $\Gamma \backslash \mathrm{SL}_2(\mathbb{R})$ noncompact) and [Dan81; Dan86b] (for more general noncompact homogeneous spaces) that horocyclic/horospherical flows have nice (finite volume, homogeneous submanifold) orbit closures and that every ergodic probability measure invariant under such a flow is the natural Lebesgue measure on some such orbit closure. In a series of breakthrough papers culminating in [Rtn91] and summarized in [Rtn94], Ratner resolved conjectures of Raghunathan and Dani by giving a complete description of unipotent orbit closures and unipotent-invariant measures on homogeneous spaces. Although similar in form to Dani's theorems for horospherical flows, Ratner's theorems require a very different method of proof that cannot be easily modified to provide a rate of convergence.

Many of the above results for horospherical flows have since been effectivized, in particular with polynomial rates (see [DM91; KM98; SU15; Bur90; FF03; Str13; Ven10]). Many authors have also considered the effective equidistribution of periods (that it, closed horospherical orbits) in a variety of settings (e.g. [Sar81; Str04; KM96; LO12; DKL16], although this list is by no means complete). In studying both periods and long pieces of generic horospherical orbits, one can make use of the "thickening" argument developed by Margulis in his thesis [Mar04], which uses a known rate of mixing for the semisimple flow with respect to which the given subgroup is horospherical along with the expansion property to get a rate for the horospherical flow.

One topic that has long interested both number theorists and dynamicists is the distribution of sparse sequences of times, especially primes. In [Bou88] Bourgain proved the remarkable result that ergodic averages over primes converge almost-everywhere, however this is often not sufficient for applications in number theory which require information about a specific orbit or about every orbit for a given system. For example, Sarnak's Möbius disjointness conjecture seeks to formalize the heuristic that primes are essentially randomly distributed by positing that the Möbius function

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not squarefree} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \end{cases}$$

is asymptotically uncorrelated with *any* sequence of the form $f(T^n x)$ for $x \in X$, a compact metric space, $f \in C(X)$, and $T : X \to X$ a continuous map of zero topological entropy. The conjecture has been established in a variety of settings [Bou13; Dav80; LS15; GT12; CE19], including for unipotent flows [BSZ13; Pec18].

In [GT12], Green and Tao prove effective Möbius disjointness for nilflows with strong rates of convergence, which allows them to further prove an equidistribution result for prime times. Similarly, it is conjectured that prime times in horospherical (and more generally, unipotent) flows are equidistributed in the orbit closures containing them (see the conjecture of Margulis in [Gor07]). However, the proofs of Möbius disjointness for unipotent flows by Bourgain-Sarnak-

Ziegler [BSZ13] and Peckner [Pec18] rely on Ratner's joining classification theorem, which is not effective. As a result, these theorems cannot be used directly to prove the equidistribution of prime times. Nonetheless, Sarnak and Ubis have shown that primes are dense in a set of positive measure for equidistributing horocycle orbits on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ and that almost-primes (i.e. integers having fewer than a fixed number of prime factors) are dense in the whole space [SU15].

Another example in the area of sparse equidistribution is Venkatesh's use of effective equidistribution of the horocycle flow to provide a partial solution to a conjecture of Shah [Sha94] by showing that orbits along sequences of times of the form $\{n^{1+\gamma}\}_{n\in\mathbb{N}}$ equidistribute in compact quotients of $\mathrm{SL}_2(\mathbb{R})$ for small γ [Ven10]. He accomplishes this by first proving effective equidistribution for arithmetic progressions of times in the horocycle flow. In this document, we extend the work of Venkatesh and Sarnak-Ubis to study the distribution of sparse subsets of horospherical orbits consisting of almost-prime entries via effective equidistribution of arithmetic progressions. These results are significant in that they lend further support to the conjecture that prime times in horospherical flows are generically dense and possibly equidistributed.

1.2 Main Results

We are interested in the asymptotic distribution of almost-primes in horospherical flows on the space of lattices and on compact quotients of $\mathrm{SL}_n(\mathbb{R})$. Our main results are summarized in the following two theorems.

Theorem 1. *Let $\Gamma < \mathrm{SL}_n(\mathbb{R})$ be a cocompact lattice and $u(\mathbf{t})$ be a horospherical flow on $\Gamma\backslash\mathrm{SL}_n(\mathbb{R})$ of dimension d.[1] Then there exists a constant M (depending only on n, d, and Γ) such that for any $x \in \Gamma\backslash\mathrm{SL}_n(\mathbb{R})$, the set*

$$\{xu(k_1,k_2,\cdots,k_d) \mid k_i \in \mathbb{Z} \text{ has fewer than } M \text{ prime factors}\}$$

[1] We have reduced the problem to a class of horospherical flows with a particular parameterization. See (4.1) and (5.9).

is dense in $\Gamma\backslash\mathrm{SL}_n(\mathbb{R})$.

In fact, the method of proof yields an effective density statement which describes how long it takes for almost-prime times in an orbit to hit any ball of fixed radius (see the remark at the end of Section 5.1). The constant M depends polynomially on n and d (see equation (5.8) for an explicit formula). The dependence of M on Γ arises from the spectral gap of the action of $\mathrm{SL}_n(\mathbb{R})$ on $\Gamma\backslash\mathrm{SL}_n(\mathbb{R})$. Since $\mathrm{SL}_n(\mathbb{R})$ has Kazhdan's property (T) for $n \geq 3$, the dependence on Γ can be removed in that setting, or in any setting where Γ varies over a family of lattices with bounded spectral gap, such as when Γ is a congruence lattice in $n = 2$.

In the noncompact case, the number of prime factors allowed in our almost-primes will depend on the rate of equidistribution for the continuous flow, which in turn depends on the basepoint due to the existence of proper orbit closures. The following definition provides a condition under which the basepoint has a persistently good rate of equidistribution.

Definition 2. For a horospherical flow $u(\mathbf{t})$ on $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$, we say that $x = \mathrm{SL}_n(\mathbb{Z})g$ is *strongly polynomially* δ*-Diophantine* if there exists some sequence $T_i \to \infty$ as $i \to \infty$ such that

$$\inf_{\substack{w\in\Lambda^j(\mathbb{Z}^n)\backslash\{0\}\\ j=1,\cdots,n-1}} \sup_{\mathbf{t}\in[0,T_i]^d} \|wgu(\mathbf{t})\| > T_i^{\delta}$$

for all $i \in \mathbb{N}$.

The definition of $\Lambda^j(\mathbb{Z}^n)$ and the norm are given in Section 2.8, and further motivation for this definition is provided in Section 5.2. See also Definition 3.1 of [LMMS] and the related discussion of its algebraic and dynamical implications.

With this definition, we have the following theorem.

Theorem 3. *Let* $u(\mathbf{t})$ *be an abelian*[2] *horospherical flow on* $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$ *of dimension d, and let* $x \in \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$ *be strongly polynomially* δ*-Diophantine for some* $\delta > 0$*. Then there exists*

[2]For Theorem 1, we first prove the result for abelian horospherical flows and then extend the result to arbitrary horospherical flows. However, the method we use to do this cannot be applied in the noncompact setting (see the remark at the end of Section 5.2).

a constant M_δ (depending on δ, n, and d) such that

$$\{xu(k_1,k_2,\cdots,k_d)\mid k_i\in\mathbb{Z} \textit{ has fewer than } M_\delta \textit{ prime factors}\}$$

is dense in $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$.

The constant M_δ depends polynomially on n and d and inversely on δ (see equation (5.14) for an explicit formula).

A brief outline of this dissertation is as follows: In Chapter 2, we establish the basic notation that will be used throughout this document and introduce the key facts and theorems that we use in our analysis. In Chapter 3, we prove an effective equidistribution result for long orbits of arbitrary horospherical flows using the "thickening" argument of Margulis, which leverages the exponential mixing properties of the subgroup with respect to which the flow of interest is horospherical. In Chapter 4, we use the theorem from the previous chapter to derive an effective bound for equidistribution along multivariate arithmetic sequences in abelian horospherical flows following the method used in Section 3 of [Ven10]. In Chapter 5, we use this bound along arithmetic sequences as well as a combinatorial sieve theorem to obtain an upper and lower bound on averages over almost-prime entries in horospherical flows, from which Theorems 1 and 3 follow as immideiate corollaries. Finally, in Chapter 6, we make some closing remarks and indicate possible extensions and areas for future research.

Chapter 1, in part, has been submitted for publication of the material as it may appear in the Journal of Modern Dynamics, 2019, McAdam, Taylor, American Institute of Mathematical Sciences (AIMS), 2018. The dissertation author was the primary investigator and author of this material.

Chapter 2

Notation and Preliminaries

2.1 Some Basic Notation

Let $G = \mathrm{SL}_n(\mathbb{R})$ for $n \geq 2$. Throughout most of this document, Γ will denote $\mathrm{SL}_n(\mathbb{Z})$, but we will also discuss the case where $\Gamma \leq G$ is a cocompact lattice. We are interested in the right multiplication actions of certain subgroups of G on the space of cosets $X = \Gamma \backslash G$.

The Haar measure m_G on G projects to a G-invariant measure m_X on X. In this document, we will always take m_X and m_G to be normalized so that m_X is a probability measure and so that the measure of a small set in G equals the measure of its projection in X. We will use $|\cdot|$ to denote the standard Lebesgue measure on $\mathbb{R}^d$ and $d\mathbf{t}$ to denote the differential with respect to Lebesgue measure for $\mathbf{t} \in \mathbb{R}^d$.

We will use gothic letters to represent the Lie algebra of a Lie group (e.g. $\mathfrak{g}$ is the Lie algebra of G). Fix an inner product on $\mathfrak{g}$. This extends to a Riemannian metric on G via left translation, which defines a left-invariant metric d_G and a left-invariant volume form, which (by uniqueness) coincides with the Haar measure on G up to scaling. This then induces a metric d_X

on X of the form

$$d_X(\Gamma g_1, \Gamma g_2) = \inf_{\gamma_1,\gamma_2 \in \Gamma} d_G(\gamma_1 g_1, \gamma_2 g_2) = \inf_{\gamma \in \Gamma} d_G(g_1, \gamma g_2).$$

The same construction can be used to define a left-invariant metric d_H for any subgroup $H \leq G$ by restricting the inner product to $\mathfrak{h} \subseteq \mathfrak{g}$. Note, however, that in general $d_H \neq d_G|_H$. Instead, we have that $d_G(h_1,h_2) \leq d_H(h_1,h_2)$ for $h_1,h_2 \in H$, since the infimum used to define the distance d_G is taken over a larger set than in d_H. We will use the notation $B_r^H(h)$ to denote a ball of radius r with respect to the metric d_H around a point $h \in H$ (this is to distinguish these balls from the sets B_T that we will define in Section 2.2). Also observe that every point has a neighborhood in which the left-invariant metric is Lipschitz equivalent to the metric derived from any matrix norm on $\mathrm{Mat}_{n\times n}(\mathbb{R})$ (see Lemma 9.12 in [EW11] for details).

Define the adjoint representation of $g \in G$ as the map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ given by $Y \mapsto gYg^{-1}$ for $Y \in \mathfrak{g}$.

In considering equidistribution questions, our space of test functions will be $C_c^\infty(X)$, the set of smooth, compactly supported (real- or complex-valued) functions on X. Define the action of G on this space by $[g \cdot f](x) = f(xg)$ for $g \in G$ and $f \in C_c^\infty(X)$.

Finally, we will use the notation $a \ll b$ to indicate that a is less than a fixed constant times b and $a \asymp b$ to indicate that $a \ll b$ and $b \ll a$. In general, the implied constants may depend on n and on the data of the dynamical system (more specifically, on d, the dimension of the horospherical subgroup). Any additional dependence of the constants will be indicated by a subscript (e.g. $\ll_f$ indicates that the implicit constant may depend on n, d, and f). In principle, the constants may also depend on the lattice Γ, although since we are primarily considering $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, we will not indicate this dependence with a subscript when Γ is understood to be fixed in this way. We will also use the standard notation $O(f(x))$ to indicate a function whose absolute value is bounded by a constant times $|f(x)|$ as $x \to \infty$, where as before the constant may depend on n and d, and any additional dependence will be indicated with a subscript.

2.2 Horospherical Subgroups

A subgroup U of G is (expanding) horospherical with respect to an element $g \in G$ if $U = \{u \in G \mid g^{-j} u g^{j} \to e \text{ as } j \to \infty\}$, where e is the identity. In other words, elements of U are contracted under conjugation by g^{-1} and expanded under conjugation by g.

Define the one-parameter subgroup $\{a_t\}_{t \in \mathbb{R}} \in G$ by

$$a_t = \exp(t \operatorname{diag}(\underbrace{\lambda_1, \cdots, \lambda_1}_{m_1}, \underbrace{\lambda_2, \cdots, \lambda_2}_{m_2}, \cdots, \underbrace{\lambda_N, \cdots, \lambda_N}_{m_N})) \tag{2.1}$$

where $m_1 + \cdots + m_N = n$ and $m_1\lambda_1 + \cdots + m_N\lambda_N = 0$. Without loss of generality we may also assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.

Let U denote the block-upper-triangular unipotent subgroup given by

$$U = \left\{ \begin{pmatrix} I_{m_1} & & & & * \\ & I_{m_2} & & & \\ & & \ddots & & \\ & 0 & & I_{m_{N-1}} & \\ & & & & I_{m_N} \end{pmatrix} \right\} \tag{2.2}$$

where I_m is the $m \times m$ identity matrix. Notice that U is the horospherical subgroup corresponding to a_t for $t > 0$. Similarly, define the contracting subgroup U^- by

$$U^- = \left\{ \begin{pmatrix} \begin{array}{c|c} I_{m_1} & \\ \hline & I_{m_2} \end{array} & & 0 \\ & \ddots & \\ * & & \begin{array}{c|c} I_{m_{N-1}} & \\ \hline & I_{m_N} \end{array} \end{pmatrix} \right\}$$

which is horospherical with respect to a_t for $t < 0$, and define U^0 to be the centralizer of a_t $(t \neq 0)$, given by

$$U^0 = \left\{ \begin{pmatrix} \begin{array}{c|c} B_1 & \\ \hline & B_2 \end{array} & & 0 \\ & \ddots & \\ 0 & & \begin{array}{c|c} B_{m_{N-1}} & \\ \hline & B_{m_N} \end{array} \end{pmatrix} \middle| \begin{array}{l} B_i \in \mathrm{GL}_{m_i}(\mathbb{R}) \\ \det B_1 \cdots \det B_N = 1 \end{array} \right\}.$$

Let $d_0 = \sum_{i=1}^N m_i^2$ and note that $d := \dim U = \dim U^- = \frac{1}{2}\left(n^2 - d_0\right)$ and $\dim U^0 = d_0 - 1$. All horospherical subgroups of $G = \mathrm{SL}_n(\mathbb{R})$ are conjugate to a subgroup of the form given in (2.2), so we restrict our attention to U of this form.

Observe that U is diffeomorphic to $\mathbb{R}^d$ through identification $\mathbf{t} \leftrightarrow u(\mathbf{t})$ of the coordinates of $\mathbb{R}^d$ with the matrix entries in the upper-right corner of (2.2).[1] Note, however, that U and $\mathbb{R}^d$ are only isomorphic as groups in the case that U is abelian, which occurs when a_t has precisely two eigenvalues.

[1]One could also use the more standard map $u(\mathbf{t}) = \exp(\iota(\mathbf{t}))$, where $\iota : \mathbb{R}^d \mapsto \mathfrak{u}$ is any identification of $\mathbb{R}^d$ with the Lie algebra $\mathfrak{u}$ of U. We have chosen to use the former embedding for the ease of certain computations and because we will later restrict our attention to abelian horosphericals, for which the two maps coincide (up to scaling and permutations of the coordinates). However, whichever map is used does not substantively change the results presented here.

The bi-invariant Haar measure m_U on U is the pushforward of Lebesgue measure on $\mathbb{R}^d$ under this identification, and we may normalize it so that $u([0,1]^d)$ has unit measure. Define an expanding family of Følner sets in U by

$$B_T = a_{\log T} u([0,1]^d) a_{-\log T}$$

for $T > 0$. One may verify that the preimage of B_T in $\mathbb{R}^d$ is given by a box where $x_k \in [0, T^{\lambda_i - \lambda_j}]$ for $i < j$ if the coordinate x_k is mapped to the (i, j)-block of (2.2) under our identification. Hence, $m_U(B_T) = T^p$, where $p = \sum_{i<j} m_i m_j (\lambda_i - \lambda_j)$.

2.3 Measure Decomposition

The product map $U \times U^0 \times U^- \to G$ given by $(u, u^0, u^-) \mapsto uu^0u^-$ is a biregular map onto a Zariski open dense subset of G (see Proposition 2.7 in [MT94]). In particular, if we let $H = U^0U^-$, this means that $m_G(G \setminus UH) = 0$ and that the product map $(u, h) \mapsto uh$ is open and continuous. Additionally, it is not difficult to see that $U \cap H = \{e\}$. Then by virtue of the fact that G is unimodular, we have that m_G restricted to UH is proportional to the pushforward of $m_U \times m_H^r$ by the product map, where m_H^r is the right Haar measure on H (see, e.g., Lemma 11.31 in [EW11] or Theorem 8.32 in [Kna96]). Note that we could equivalently use the left Haar measure on H and multiply by the modular function $\triangle_H$, but for convenience of notation we will use the right Haar measure.

2.4 Sobolev Norms

Fix a basis $\mathcal{B}$ for the Lie algebra $\mathfrak{g}$ of G. Define the (right) differentiation action of $\mathfrak{g}$ on $C_c^\infty(X)$ by

$$Yf(x) = \frac{d}{dt} f(x \exp(tY))|_{t=0}$$

for $Y \in \mathcal{B}$ and $f \in C_c^\infty(X)$. Higher order derivatives of f can then be expressed as monomials in the basis $\mathcal{B}$.

For $p \in [1,\infty]$ and $\ell \in \mathbb{N}$, the (p,ℓ)-Sobolev norm of $f \in C_c^\infty(X)$ simultaneously controls the L^p-norm of all derivatives of f up to order ℓ. More precisely, let

$$\mathcal{S}_{p,\ell}(f) = \sum_{\deg(\mathcal{D}) \le \ell} \|\mathcal{D}f\|_{L^p(X)}$$

where $\mathcal{D}$ ranges over all monomials in $\mathcal{B}$ of degree $\le \ell$. Observe that the Sobolev norm can be defined similarly for $C_c^\infty(G)$ and $C_c^\infty(H)$ where $H \le G$, given a choice of basis for $\mathfrak{h} \subseteq \mathfrak{g}$.[2]

We will only require the $(2,\ell)$- and (∞,ℓ)-Sobolev norms. When $p = 2$, we will drop the notation, letting $\mathcal{S}_\ell(f) = \mathcal{S}_{2,\ell}(f)$. When needed, we will use a superscript $\mathcal{S}^X$ to indicate a Sobolev norm for functions defined on X.

Some useful properties of these norms are as follows (see [Ven10] or [KM96]):

(i) For X a probability space, $f \in C_c^\infty(X)$, $p \in [1,\infty]$, and $k \le \ell$, $\mathcal{S}_{p,k}(f) \le \mathcal{S}_{\infty,\ell}(f)$.

(ii) For $f_1, f_2 \in C_c^\infty(X)$, $\mathcal{S}_{\infty,\ell}(f_1 f_2) \ll_\ell \mathcal{S}_{\infty,\ell}(f_1)\mathcal{S}_{\infty,\ell}(f_2)$.

(iii) For $f \in C_c^\infty(X)$ and $g \in G$, $\mathcal{S}_{\infty,\ell}(g \cdot f) \ll_\ell \left\|\mathrm{Ad}_{g^{-1}}\right\|^\ell \mathcal{S}_{\infty,\ell}(f)$, where $\|\cdot\|$ is the operator norm on linear functions $\mathfrak{g} \to \mathfrak{g}$.

(iv) Let $L \subset G$ be compact. For $f \in C_c^\infty(X)$, $x \in X$,

$$|f(xg) - f(x)| \ll_L \mathcal{S}_{\infty,1}(f) d_G(g,e)$$

for all $g \in L$.

[2] The choice of the basis $\mathcal{B}$ is unimportant in the sense that choosing a different basis will lead to an equivalent norm. Likewise, we could use any norm on the components $\|\mathcal{D}f\|_{L^p(X)}$ (here we have used the l^1-norm), but as all such norms are equivalent, the choice is unimportant.

(v) Let X and Y be Riemannian manifolds. For $f_1 \in C_c^\infty(X)$ and $f_2 \in C_c^\infty(Y)$,

$$\mathcal{S}_\ell^{X\times Y}(f_1 \cdot f_2) \ll_{X,Y} \mathcal{S}_\ell^X(f_1)\mathcal{S}_\ell^Y(f_2).$$

2.5 Approximation to the Identity

At times we will want to use smooth bump functions with small support as approximations to the identity, but we will need to know that the Sobolev norm of such functions can be controlled. For this we have the following lemma, which can be found in [KM96].

Lemma 4 ([KM96, Lemma 2.4.7 (b)]). *Let Y be a Riemannian manifold of dimension k. Then for any $0 < r < 1$ and $y \in Y$, there exists a function $\theta \in C_c^\infty(Y)$ such that:*

(i) $\theta \geq 0$

(ii) *supp* $(\theta) \subseteq B_r^Y(y)$

(iii) $\int_Y \theta = 1$

(iv) $\mathcal{S}_\ell^Y(\theta) \ll_{Y,y} r^{-(\ell+k/2)}$.

2.6 The Space of Unimodular Lattices

For $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, X is noncompact and can be understood as the space of unimodular lattices (that is, lattices of covolume one) in $\mathbb{R}^n$ under the identification $\Gamma g \leftrightarrow \mathbb{Z}^n g$.

For $0 < \varepsilon \leq 1$, define L_ε to be the set of lattices in $X = \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$ with no nonzero vectors shorter than ε. That is, let

$$L_\varepsilon = \{\Gamma g \in X \mid \|vg\| \geq \varepsilon \text{ for all } v \in \mathbb{Z}^n \setminus \{0\}\}$$

where the norm above can be taken to be any norm on $\mathbb{R}^n$, but for convenience we will use the max norm. By Mahler's Compactness Criterion, L_ε is a compact set (for details and a proof, see [Rag72] Corollary 10.9, [BM00] Theorem 5.3.2, or [EW11] Theorem 11.33).

2.7 Radius of Injection

Given small $\varepsilon > 0$, we want to find a radius $r > 0$ (depending on ε) such that projection at x, given by

$$\pi_x : B_r^G(e) \to B_r^X(x)$$
$$g \mapsto xg$$

is injective for all $x \in L_\varepsilon$ (in fact, it is not difficult to see from the definition of the metric on X that this will be an isometry). For this, we have the following lemma, which is proved in a much more general setting in [BO12] (see the proof of Lemma 11.2). A proof of the lemma as it is stated here can be found in Appendix A.

Lemma 5. *There exist constants $c_1, c_2 > 0$ (depending only on n) such that for any $0 < \varepsilon < c_1$, the projection map $\pi_x : B_r^G(e) \to B_r^X(x)$ is injective for all $x \in L_\varepsilon$, where $r = c_2\varepsilon^n$.*

2.8 Quantitative Nondivergence

Let $\{e_1, \cdots, e_n\}$ be the standard basis on $\mathbb{R}^n$. Let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_j}$ for a multi-index $I = (i_1, \cdots, i_j)$, where $1 \leq i_1 < \cdots < i_j \leq n$. Then $\{e_I\}$ is a basis for $\Lambda^j(\mathbb{R}^n)$, the j^{th} exterior power of $\mathbb{R}^n$. Define the norm of $w = \sum_I w_I e_I \in \Lambda^j(\mathbb{R}^n)$ to be $\|w\| = \max_I |w_I|$. Denote by $\Lambda^j(\mathbb{Z}^n)$ the discrete subset of $\Lambda^j(\mathbb{R}^n)$ composed of linear combinations of basis vectors with integer coefficients. Notice that $g \in \mathrm{GL}_n(\mathbb{R})$ acts on $\Lambda^j(\mathbb{R}^n)$ on the right by

$$(e_{i_1} \wedge \cdots \wedge e_{i_j})g = (e_{i_1}g) \wedge \cdots \wedge (e_{i_j}g)$$

where the action extends to all of $\Lambda^j(\mathbb{R}^n)$ via linearity. A vector in $\Lambda^j(\mathbb{Z}^n)$ is called *primitive* if it is not a multiple of any other vector in $\Lambda^j(\mathbb{Z}^n)$.

The following theorem quantitatively describes how often certain polynomial maps from $\mathbb{R}^d$ to X land inside a compact set L_ε. This is a special case of Theorem 5.2 in [KM98], which itself extends results of [Dan86a] and [Mar75]. The original theorem is stated for much more general (C,α)-good functions, but we will only need the version below, which can be found as Theorem 3.1 in [KM12].

In fact, this can be improved due to an old result called the Remez inequality (see [Rem36; BG73]), so that the bound becomes $(\varepsilon/\rho)^{1/(n-1)}|B|$. We have decided to use the following less sharp bound in our treatment because we feel it may be more familiar to the reader and because it would take some time reformulate the Remez inequality for application to the space of lattices, which is not a primary concern of this document.

Theorem 6 ([KM98, Theorem 5.2])**.** *Let $d,n \in \mathbb{N}$ and $0 < \rho \leq 1$. Let $B \subset \mathbb{R}^d$ be a ball and suppose that $\xi : B \to \mathrm{GL}_n(\mathbb{R})$ satisfies:*

(i) *all matrix entries of $\xi(\mathbf{t})$ are degree 1 polynomials in the coordinates of $\mathbf{t}$,*

(ii) $\sup_{\mathbf{t}\in B} \|w\xi(\mathbf{t})\| \geq \rho$ *for all primitive $w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\}$ and $j \in \{1, \cdots, n-1\}$.*

Then for any $0 < \varepsilon \leq \rho$,

$$|\{\mathbf{t} \in B \mid \Gamma\xi(\mathbf{t}) \notin L_\varepsilon\}| \ll_{d,n} (\varepsilon/\rho)^{1/d(n-1)} |B|.$$

From this theorem we may derive the following corollary, which we will use in the proof of Theorem 11 to say that the orbit of a point satisfying a certain Diophantine condition spends a relatively large proportion of time in L_ε when pushed by the flow a_t.

Corollary 7. *Let $T,R>1$ and $x_0=\Gamma g_0\in X$. Then suppose $R_0>0$ is such that*

$$\sup_{\mathbf{t}\in[0,1]^d}\left\|wg_0a_{\log T}u(\mathbf{t})a_{-\log T}\right\|\geq R_0$$

for all primitive $w\in\Lambda^j(\mathbb{Z}^d)\backslash\{0\}$ and $j\in\{1,\cdots,n-1\}$ and define $\rho=\min(1,R_0/R^q)$ where $q=\sum_{\lambda_i<0}-m_i\lambda_i$ from (2.1). Then for any $0<\varepsilon<\rho$,

$$\left|\{\mathbf{t}\in[0,1]^d\mid x_0a_{\log T}u(\mathbf{t})a_{-\log T}a_{\log R}\notin L_\varepsilon\}\right|\ll(\varepsilon/\rho)^{1/d(n-1)}.$$

Proof. Let $\xi(\mathbf{t})=g_0a_{\log T}u(\mathbf{t})a_{-\log T}a_{\log R}$. We want to demonstrate that conditions (i) and (ii) hold in Theorem 6 for $\rho=\min(1,R_0/R^q)$ and $B=[0,1]^d$.

Recall that our identification $u(\mathbf{t})$ places one coordinate of $\mathbf{t}$ in each matrix entry in the upper-right corner of (2.2). Then since multiplication by a_t on either the left or the right only changes matrix entries by scaling, each entry in the upper-right corner of $a_{\log T}u(\mathbf{t})a_{-\log T}a_{\log R}$ only depends linearly on a single coordinate of $\mathbf{t}$. This means that for any matrix g_0, all entries of $\xi(\mathbf{t})=g_0a_{\log T}u(\mathbf{t})a_{-\log T}a_{\log R}$ will be affine, satisfying condition (i).

Notice that $e_ka_{\log R}=R^{\lambda_i}e_k$ if λ_i is k^{th} eigenvalue in the definition of a_t in (2.1). Then the right action of $a_{\log R}$ scales $e_{i_1}\wedge\cdots\wedge e_{i_j}\in\Lambda^j(\mathbb{R}^d)$ by the product of all such corresponding factors. Since $R>1$, the most $a_{\log R}$ can therefore contract any basis element is by the product of all scaling factors corresponding to negative eigenvalues of (2.1), that is, by R^{-q}, where $q=\sum_{\lambda_i<0}-m_i\lambda_i$. It then follows from the definition of the norm that

$$\left\|wa_{\log R}\right\|\geq R^{-q}\left\|w\right\| \tag{2.3}$$

for any $w \in \Lambda^j(\mathbb{R}^d) \setminus \{0\}$ and $j \in \{1, \cdots, n-1\}$. Then for $\rho = \min(1, R_0/R^q)$, we have $0 < \rho \leq 1$ and also

$$\begin{aligned}
\sup_{\mathbf{t} \in [0,1]^d} \|w\xi(\mathbf{t})\| &= \sup_{\mathbf{t} \in [0,1]^d} \left\| w g_0 a_{\log T} u(\mathbf{t}) a_{-\log T} a_{\log R} \right\| \\
&\geq R^{-q} \sup_{\mathbf{t} \in [0,1]^d} \left\| w g_0 a_{\log T} u(\mathbf{t}) a_{-\log T} \right\| \\
&\geq R_0/R^q \\
&\geq \rho
\end{aligned}$$

for $j \in \{1, \cdots, n-1\}$ and primitive $w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\}$, satisfying condition (ii).

Hence, by Theorem 6, we have

$$\left| \{\mathbf{t} \in [0,1]^d \mid \Gamma\xi(\mathbf{t}) \notin L_\varepsilon\} \right| \ll (\varepsilon/\rho)^{1/d(n-1)}.$$

$\square$

2.9 Decay of Matrix Coefficients

In order to obtain effective rates of equidistribution in Chapters 3 and 4, we will need to use results on the effective decay of matrix coefficients.

Estimates of this type have a long and rich history, including Selbrerg's celebrated $3/16$ theorem for congruence quotients of $\mathrm{SL}_2(\mathbb{Z})$, Kazhdan's property (T), and works of Harish-Chandra, Cowling, Haagerup, Howe, and Oh. Far reaching extensions of Selberg's work are also in place thanks to works of Jacques-Langlands, Burger-Sarnak, and Clozel. Our formulation here is taken from [KM96] (see [KM96; KS94; GMO08; EMMV] for a more comprehensive history and discussion). In [Oh02], Oh gives optimal bounds for $\mathrm{SL}_n(\mathbb{R})$, $n \geq 3$.

Theorem 8 ([KM96, Corollary 2.4.4]). *Let $G = \mathrm{SL}_n(\mathbb{R})$ and $X = \Gamma/G$ for a lattice Γ. There* *exists a constant* $0 < \tilde{\beta} < 1$ *such that for* $f_1, f_2 \in C_c^\infty(X)$ *and* $g \in G$*,*

$$\left| \langle g \cdot f_1, f_2 \rangle_{L^2(X)} - \int_X f_1 dm_X \int_X \overline{f}_2 dm_X \right| \ll e^{-\tilde{\beta} d_G(e,g)} \mathcal{S}_\ell(f_1) \mathcal{S}_\ell(f_2)$$

where ℓ *is the dimension of maximal compact subgroup of* G*. When* $n \geq 3$*, the constant* $\tilde{\beta}$ *is independent of the lattice* Γ*, and when* $n = 2$ *it is independent of the lattice if* Γ *is a congruence lattice.*

For our specific applications, we have the following corollaries.

Corollary 9. *Let the setting be as above. There exist* $\beta_1, \beta_2 > 0$ *such that*

(i) *For* $f_1, f_2 \in C_c^\infty(X)$ *and* $t \geq 0$*, we have*

$$\left| \int_X f_1(xa_t) f_2(x) dm_X(x) - \int_X f_1 dm_X \int_X \overline{f}_2 dm_X \right| \ll e^{-\beta_1 t} \mathcal{S}_\ell(f_1) \mathcal{S}_\ell(f_2)$$

(ii) *For* $f \in C_c^\infty(X)$ *and* $\mathbf{t} \in \mathbb{R}^d$*,*

$$\left| \langle u(\mathbf{t}) f, f \rangle_{L^2(X)} - \left| \int_X f dm_X \right|^2 \right| \ll \max(1, |\mathbf{t}|)^{-\beta_2} \mathcal{S}_\ell(f)^2$$

Chapter 2, in part, has been submitted for publication of the material as it may appear in the Journal of Modern Dynamics, 2019, McAdam, Taylor, American Institute of Mathematical Sciences (AIMS), 2018. The dissertation author was the primary investigator and author of this material.

Chapter 3

Effective Equidistribution of Horospherical Flows

3.1 The Space of Lattices

Let $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, $X = \Gamma\backslash G$, and $U < G$ a horospherical subgroup of dimension d.

The following qualitative equidistribution statement is well known and follows from Ratner's Theorems, although in this case simpler proofs may be used. As mentioned in the introduction, Dani proved a density result in this setting, and equidistribution results were proved in some special cases prior to Ratner's work [Dan78; FF03; Str13]. The method of proof for this general qualitative result is the same as for these special cases, however we were unable to locate an explicit reference for a result of this form which avoids the use of Ratner's theorem.

Theorem 10. *For every $x_0 = \Gamma g_0 \in X$, either*

$$\frac{1}{m_U(B_T)}\int_{B_T} f(x_0u)\,dm_U(u) \xrightarrow[T\to\infty]{} \int_X f\,dm_X \quad \forall f \in C_c^\infty(X) \tag{3.1a}$$

or

$$\exists j \in \{1, \cdots, n-1\} \text{ and primitive } w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\}$$
$$\text{such that } wg_0u = wg_0 \ \ \forall u \in U. \tag{3.1b}$$

The subgroup $L = \mathrm{Stab}(wg_0)$ is the well-known intermediate subgroup $U \leq L \leq G$ such that $\overline{x_0U} \subset x_0L$ with x_0L supporting an L-invariant probability measure. We will call a point $x_0 \in X$ *generic* if it satisfies (3.1a). By ergodicity of the horospherical flow, the set of generic points has full measure in X.

Our main objective in this chapter is to prove the following quantitative refinement of this theorem.

Theorem 11. *There exist constants $\gamma, C > 0$ (depending[1] only on n, d, and β_1) such that for every $x_0 = \Gamma g_0 \in X$ and $T > R > C$, either*

$$\left| \frac{1}{m_U(B_T)} \int_{B_T} f(x_0u)dm_U(u) - \int_X f dm_X \right| \ll R^{-\gamma} \mathcal{S}_{\infty,\ell}(f) \quad \forall f \in C_c^\infty(X) \tag{3.2a}$$

or

$$\exists j \in \{1, \cdots, n-1\} \text{ and primitive } w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\}$$
$$\text{such that } \|wg_0u\| < R^q \ \ \forall u \in B_T \tag{3.2b}$$

where $q = \sum_{\lambda_i < 0} -m_i\lambda_i$ and $\ell = n(n-1)/2$ is the dimension of maximal compact subgroup of G.

Intuitively, this theorem says that either the U-orbit of x_0 equidistributes in X with a fast rate, or x_0 is close to a proper subspace of X that is fixed by the action of U, where our notions of "fast" and "close" are quantitatively related.

[1] Specifically, $\gamma = 2\beta_1/(nd(3n+1)(n-1)^2+2)$, where β_1 is the constant from Corollary 9. If the Remez inequality is used instead of Theorem 6, then γ depends only on n, with $\gamma = 2\beta_1/(n(3n+1)(n-1)^2+2)$.

Remark. Observe that for fixed R, any generic point x_0 will fail (3.2b) for large enough T. This is because there are only finitely many $w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\}$ with $j \in \{1, \cdots, n-1\}$ such that $\|wg_0\| < R^q$ to begin with, so for (3.2b) to hold for all large times T it must be the case that there is a single w satisfying $\|wg_0 u\| < R^q$ for all $u \in U$. Moreover, $\|wg_0 u\|$ is the maximum of several polynomials in the coordinates of U. Hence, in order to be bounded for all time, it must in fact be constant. This implies wg_0 is fixed by U, which recovers the qualitative result of Theorem 10.

Remark. It is worth noting that this theorem says something even in the case that the basepoint is not generic. Consider the following finitary version of genericity: $x_0 \in X$ is said to be R-*generic* if it satisfies (3.2a) for all sufficiently large T. If x_0 is R-generic for all $R > 0$, then it is generic. On the other hand, suppose that $x_0 = \Gamma g_0$ is not generic, so that there exists $w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\}$ with $j \in \{1, \cdots, n-1\}$ such that wg_0 is fixed by U. If additionally $\|wg_0\| > R^q$ for large enough R, then the theorem tells us that (3.2a) holds for all $T > R$ and thus x_0 is R-generic. In fact, if R is quite large, then the orbit of x_0 will be very nearly equidistributed, even though it is not generic. This relates to the equidistribution of proper orbit closures as their volume goes toward infinity (see, e.g. [Sar81; Str04; KM96]).

Remark. The condition that w in (3.2b) be primitive is conceptually useful but technically unnecessary, in that if there exists any $w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\}$ satisfying (3.2b), then there will also exist a primitive vector that does so. Moreover, although the theorem is stated for Følner sets of the form $B_T = a_{\log T} u([0,1]^d) a_{-\log T}$, it should hold equally well for sets of the form $B_T = a_{\log T} u(B) a_{-\log T}$ for any ball $B \subset \mathbb{R}^d$.

Remark. The "either/or" in the theorem statement is not meant to imply an exclusive or. The structure of the proof will be to show that for x_0, T, and R as in the theorem, not (3.2b) implies (3.2a). This leads us to define the following (T,R)-Diophantine basepoint condition for $x_0 =$

$\Gamma g_0 \in X$, which is simply the negation of condition (3.2b):

$$\forall j \in \{1, \cdots, n-1\} \text{ and } w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\},\ \exists\, u \in B_T \text{ s.t. } \|wg_0u\| \geq R^q. \tag{3.2c}$$

Proof. Let $x_0 = \Gamma g_0 \in X$ satisfy the basepoint condition in (3.2c) for some $T > R$. Then consider $f \in C_c^\infty(X)$ and write, via a change of variables,

$$\begin{aligned} I_0 &:= \frac{1}{m_U(B_T)} \int_{B_T} f(x_0u)dm_U(u) \\ &= \frac{1}{m_U(B_{T/R})} \int_{B_{T/R}} f(x_0 a_{\log R} u a_{-\log R}) dm_U(u) \\ &= \frac{1}{m_U(B_{T/R})} \int_U \mathbf{1}_{B_{T/R}}(u) f(x_0 a_{\log R} u a_{-\log R}) dm_U(u). \end{aligned} \tag{3.3}$$

We want to show that this quantity is close to $\int f dm_X$, and from (3.3) it almost looks as if we could apply the exponential mixing result of Corollary 9 (i) to achieve this, however there are several significant barriers to doing so. Most obviously, the integral in (3.3) is over U instead of X. Furthermore, the "basepoint" $x_0 a_{\log R} u$ varies with u, and will eventually spend time outside of any fixed compact subset of X for u coming from a large enough set. Finally, the function $\mathbf{1}_{B_{T/R}}$ is not smooth.

We will first address the issue of smoothness by convolving the indicator function with a smooth approximation to the identity (**Step 1**). We will then apply the "thickening" argument of Margulis to obtain an integral over X from our integral over U (**Step 2**). Finally, we will deal with the moving basepoint by demonstrating that for most $u \in B_{T/R}$ we have a uniformly good rate of equidistribution and that the size of the set on which this does not occur can be quantitatively controlled (**Step 3**). This last step is where we will use the nondivergence result of Section 2.8.

Step 1. Let r be a small, positive number (to be determined) and let $\theta \in C_c^\infty(U)$ be a nonnegative bump function supported on $B_r^U(e)$ satisfying the approximate identity properties of Lemma 4.

Then the convolution $\int_U \theta(u')\mathbf{1}_{B_{T/R}}(u(u')^{-1})dm_U(u')$ is a smooth function approximating our original indicator function. If we substitute this function for $\mathbf{1}_{B_{T/R}}$ in (3.3) and use the invariance property of the Haar measure, we get the integral

$$\begin{aligned} I_{\text{smth}} &:= \frac{1}{m_U(B_{T/R})}\int_U\int_U \theta(u')\mathbf{1}_{B_{T/R}}(u(u')^{-1})dm_U(u')f(x_0a_{\log R}ua_{-\log R})dm_U(u) \\ &= \frac{1}{m_U(B_{T/R})}\int_U\int_U \theta(u')\mathbf{1}_{B_{T/R}}(u)f(x_0a_{\log R}uu'a_{-\log R})dm_U(u)dm_U(u'). \end{aligned} \tag{3.4}$$

Now observe that since $\int\theta = 1$, we may again use the invariance of the Haar measure to rewrite (3.3) as

$$\begin{aligned} I_0 &= \frac{1}{m_U(B_{T/R})}\int_U \mathbf{1}_{B_{T/R}}(uu')f(x_0a_{\log R}uu'a_{-\log R})dm_U(u)\int_U \theta(u')dm_U(u') \\ &= \frac{1}{m_U(B_{T/R})}\int_U\int_U \theta(u')\mathbf{1}_{B_{T/R}}(uu')f(x_0a_{\log R}uu'a_{-\log R})dm_U(u)dm_U(u'). \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we can see that

$$\begin{aligned} &|I_0 - I_{\text{smth}}| \\ &\qquad \leq \frac{1}{m_U(B_{T/R})}\int_U \theta(u')\mathcal{S}_{\infty,0}(f)\left(\int_U \left|\mathbf{1}_{B_{T/R}}(uu') - \mathbf{1}_{B_{T/R}}(u)\right| dm_U(u)\right)dm_U(u') \\ &\qquad = \frac{\mathcal{S}_{\infty,0}(f)}{m_U(B_{T/R})}\int \theta(u')m_U(B_{T/R}\triangle B_{T/R}(u')^{-1})dm_U(u'). \end{aligned} \tag{3.6}$$

But notice that since $\operatorname{supp}\theta \subseteq B_r^U(e)$, we know that u' is close to the identity, so u in this region can only shift $B_{T/R}$ by a small amount. In fact, by pulling the measure back to $\mathbb{R}^n$, one may compute directly that the size of the symmetric difference is bounded by

$$m_U(B_{T/R}\triangle B_{T/R}(u')^{-1}) \ll (T/R)^{p-p_0}r \tag{3.7}$$

for any $u' \in B_r^U(e)$, where $p_0 = \min_{i<j}(\lambda_i - \lambda_j)$. Combining this with (3.6) above and again

using the fact that θ integrates to 1, we see that

$$|I_0 - I_{\text{smth}}| \ll \frac{(T/R)^{p-p_0}}{m_U(B_{T/R})} r\mathcal{S}_{\infty,0}(f) = (R/T)^{p_0} r\mathcal{S}_{\infty,0}(f) \leq r\mathcal{S}_{\infty,0}(f) \tag{3.8}$$

since $m_U(B_{T/R}) = (T/R)^p$ and $T \geq R$.

Now that we know I_0 and I_{smth} can be made close, we want to know that I_{smth} is not too far from $\int f dm_X$. Using Fubini's Theorem, we can say

$$I_{\text{smth}} = \frac{1}{m_U(B_{T/R})} \int_{B_{T/R}} \int_U \theta(u') f(x_0 a_{\log R} u u' a_{-\log R}) dm_U(u') dm_U(u)$$

and we may also write

$$\int_X f dm_X = \frac{1}{m_U(B_{T/R})} \int_{B_{T/R}} \left(\int_X f dm_X \right) dm_U(u).$$

Hence,

$$\begin{aligned} &\left| I_{\text{smth}} - \int_X f dm_X \right| \\ &\qquad \leq \frac{1}{m_U(B_{T/R})} \int_{B_{T/R}} \left| \int_U \theta(u') f(x_0 a_{\log R} u u' a_{-\log R}) dm_U(u') - \int_X f dm_X \right| dm_U(u). \end{aligned} \tag{3.9}$$

Step 2. Now the expression inside the absolute value looks more similar to the mixing result of Corollary 9 (i), but we are still integrating with respect to the wrong measure. We want an integral with respect to m_X, and although functions on X integrate locally like their pullback by projection over G, the integral with which we are concerned is over the lower-dimensional ("thin") subspace U.

Define

$$I_U(u) := \int_U \theta(u') f(x_0 a_{\log R} u u' a_{-\log R}) dm_U(u'). \tag{3.10}$$

to be the integral from inside (3.9) above. In order to apply Corollary 9 (i), we will need to "thicken" this integral over U to an integral over a neighborhood of the orbit in G and then project to X.

Recall from Section 2.3 that $m_G = m_U \times m_H^r$, where m_H^r is the right Haar measure on $H = U^0 U^-$. Then let $\psi \in C_c^\infty(H)$ be an approximate identity supported on $B_r^H(e)$ as described in Lemma 4. Since $\int \psi = 1$, we may rewrite (3.10) as

$$I_U(u) = \int_H \int_U \theta(u')\psi(h) f(x_0 a_{\log R} u u' a_{-\log R}) dm_U(u') dm_H^r(h). \tag{3.11}$$

Now define

$$I_X(u) := \int_H \int_U \theta(u')\psi(h) f(x_0 a_{\log R} u u' h a_{-\log R}) dm_U(u') dm_H^r(h) \tag{3.12}$$

which differs from $I_U(u)$ only by the presence of the variable h inside f. To see that $I_U(u)$ and $I_X(u)$ are close, observe that

$$|I_U(u) - I_X(u)| \leq \int_H \int_U \theta(u')\psi(h) \left| f(\tilde{x}) - f(\tilde{x} a_{\log R} h a_{-\log R}) \right| dm_U(u') dm_H^r(h) \tag{3.13}$$

where $\tilde{x} = x_0 a_{\log R} u u' a_{-\log R}$. But since f has bounded derivative,

$$\left| f(\tilde{x}) - f(\tilde{x} a_{\log R} h a_{-\log R}) \right| \ll \mathcal{S}_{\infty,1}(f) d_G(e, a_{\log R} h a_{-\log R}) \tag{3.14}$$

by Sobolev property (iv). Furthermore, since conjugation by a_t is non-expanding on the subgroup

H (recall that it fixes U^0 and contracts U^-), we may see that[2]

$$d_G(e, a_{\log R} h a_{-\log R}) \le d_G(e,h) \ll d_H(e,h) \le r \tag{3.15}$$

for $h \in \text{supp}\, \psi \subseteq B_r^H(e)$.

Then from (3.13), (3.14), and (3.15) and the fact that both θ and ψ integrate to 1, we have

$$|I_U(u) - I_X(u)| \ll \mathcal{S}_{\infty,1}(f) r. \tag{3.16}$$

Now we want to verify that $I_X(u)$ is not far from $\int f dm_X$. By our measure decomposition, we can see (3.12) as an integral over G:

$$I_X(u) = \int_G \phi(g) f(x_0 a_{\log R} u g a_{-\log R}) dm_G(g) \tag{3.17}$$

where the function $\phi(uh) = \theta(u)\psi(h)$ is defined for all $g \in UH$, hence it is defined almost-everywhere. In order to apply mixing, we want to further interpret $I_X(u)$ as an integral over X. To do this, let $y = x_0 a_{\log R} u$, keeping in mind that y depends on u. Then define $\phi_y \in C_c^\infty(X)$ by $\phi_y = \phi \circ \pi_y^{-1}$ where $\pi_y : G \to X$ is natural projection at y. Note, however, that ϕ_y is only well-defined if π_y is injective on $\text{supp}\, \phi = \text{supp}\, \theta\ \text{supp}\, \psi \subseteq B_r^U(e) B_r^H(e)$. In a neighborhood of the identity, $B_r^U(e) B_r^H(e) \subseteq B_{cr}^G(e)$ for a positive constant c, since

$$d_G(uh,e) \le d_G(uh,u) + d_G(u,e) = d_G(h,e) + d_G(u,e) \ll d_H(h,e) + d_U(u,e) \le 2r.$$

Therefore, if π_y is injective on $B_{cr}^G(e)$ for $y = x_0 a_{\log R} u$ (an assumption we will reutrn to later) we

[2]There is a slight subtlety here because we used the right Haar measure on H, so the corresponding metric d_H is right-invariant, while d_G is left-invariant. In general, d_G restricted to H will be less than or equal to the corresponding left-invariant metric on H. However, any left-invariant metric is Lipschitz equivalent to any right-invariant metric in a suitable neighborhood of the identity, so the above series of inequalities goes through for r small enough.

can say from (3.17) that

$$\begin{aligned} I_X(u) &= \int_G \phi(g) f(yga_{-\log R}) dm_G(g) \\ &= \int_G \phi_y(yg) f(yga_{-\log R}) dm_G(g) \\ &= \int_X \phi_y(x) f(xa_{-\log R}) dm_X(x). \end{aligned}$$

Since $\int \phi_y dm_X = \int \phi dm_G = \int \theta dm_U \int \psi dm_H^r = 1$, we can now apply the effective mixing result from Section 2.9 to obtain

$$\begin{aligned} \left| I_X(u) - \int_X f dm_X \right| &= \left| \int_X \phi_y(x) f(xa_{-\log R}) dm_X(x) - \int_X \phi_y dm_X \int_X f dm_X \right| \\ &\ll R^{-\beta_1} \mathcal{S}_\ell(\phi_y) \mathcal{S}_\ell(f). \end{aligned}$$

Then from property (v) in Section 2.4 and our bound on the Sobolev norm of an approximate identity (property (iv) in Section 2.5), we can say

$$\mathcal{S}_\ell^X(\phi_y) = \mathcal{S}_\ell^G(\phi) \ll \mathcal{S}_\ell^U(\theta) \mathcal{S}_\ell^H(\psi) \ll r^{-(\ell+d/2)} r^{-(\ell+\tilde{d}/2)} = r^{-2\ell-(n^2-1)/2}$$

where $\tilde{d} = \dim H$. Thus if π_y is injective on $B_{cr}^G(e)$, then

$$\left| I_X(u) - \int_X f dm_X \right| \ll R^{-\beta_1} r^{-p_1} \mathcal{S}_\ell(f) \tag{3.18}$$

where $p_1 = 2\ell + (n^2-1)/2$.

Step 3. However, as we have noted, $y = x_0 a_{\log R} u$ depends on u, which varies over $B_{T/R}$ in (3.9). While we cannot ensure that π_y is injective on $B_{cr}^G(e)$ for all $u \in B_{T/R}$, we can say that the set on which this does not occur has small measure.

Recall from Lemma 5 that $\pi_y : B_r^G(e) \to B_r^X(y)$ is injective for $y \in L_\varepsilon$ for r proportional to ε^n and for ε small enough. Furthermore, observe that condition (3.2c) is equivalent to the statement that for all $j \in \{1, \cdots, n-1\}$ and primitive $w \in \Lambda^j(\mathbb{Z}^d) \setminus \{0\}$, there exists $\mathbf{t} \in [0,1]^d$ such that $\left\|wg_0 a_{\log T} u(\mathbf{t}) a_{-\log T}\right\| \geq R^q$. Then by Corollary 7 in Section 2.8, we have that

$$\left|\{\mathbf{t} \in [0,1]^d \mid x_0 a_{\log T} u(\mathbf{t}) a_{-\log T} a_{\log R} \notin L_\varepsilon\}\right| \ll \varepsilon^{1/d(n-1)}.$$

From this we find that

$$\begin{aligned}
&\left|\{\mathbf{t} \in [0,1]^d \mid x_0 a_{\log T} u(\mathbf{t}) a_{-\log T} a_{\log R} \notin L_\varepsilon\}\right| \\
&\qquad = \left|\{\mathbf{t} \in [0,1]^d \mid x_0 a_{\log R} a_{\log T/R} u(\mathbf{t}) a_{-\log T/R} \notin L_\varepsilon\}\right| \\
&\qquad = m_U\left(\{u \in B_1 \mid x_0 a_{\log R} a_{\log T/R} u a_{-\log T/R} \notin L_\varepsilon\}\right) \\
&\qquad = m_U\left(\{u \in B_{T/R} \mid x_0 a_{\log R} u \notin L_\varepsilon\}\right) / m_U(B_{T/R})
\end{aligned}$$

where the last equality can be verified using a change of variables. That is, for x_0 satisfying condition (3.2c), we have

$$m_U\left(\{u \in B_{T/R} \mid x_0 a_{\log R} u \notin L_\varepsilon\}\right) \ll \varepsilon^{1/d(n-1)} m_U(B_{T/R}).$$

In other words, if we let $E := \{u \in B_{T/R} \mid x_0 a_{\log R} u \in L_\varepsilon\}$, then (3.18) holds for all $u \in E$ and

$m_U(B_{T/R} \setminus E) \ll \varepsilon^{1/d(n-1)} m_U(B_{T/R})$. Thus, from (3.9), (3.16), and (3.18), we find

$$\begin{aligned}
\left| I_{\mathrm{smth}} - \int_X f\, dm_X \right| \leq & \frac{1}{m_U(B_{T/R})} \int_{B_{T/R}} \left| I_U(u) - \int_X f\, dm_X \right| dm_U(u) \\
\leq & \frac{1}{m_U(B_{T/R})} \int_{B_{T/R}} |I_U(u) - I_X(u)|\, dm_U(u) \\
& + \frac{1}{m_U(B_{T/R})} \int_{B_{T/R}} \left| I_X(u) - \int_X f\, dm_X \right| dm_U(u) \\
\ll & \mathcal{S}_{\infty,1}(f) r + \frac{1}{m_U(B_{T/R})} \int_E \left| I_X(u) - \int_X f\, dm_X \right| dm_U(u) \\
& + \frac{1}{m_U(B_{T/R})} \int_{B_{T/R} \setminus E} \left| I_X(u) - \int_X f\, dm_X \right| dm_U(u) \\
\ll & \mathcal{S}_{\infty,1}(f) r + \frac{m_U(E)}{m_U(B_{T/R})} R^{-\beta_1} r^{-p_1} \mathcal{S}_\ell(f) \\
& + \frac{m_U(B_{T/R} \setminus E)}{m_U(B_{T/R})} \mathcal{S}_{\infty,0}(f) \\
\ll & \mathcal{S}_{\infty,1}(f) r + R^{-\beta_1} r^{-p_1} \mathcal{S}_\ell(f) + \varepsilon^{1/d(n-1)} \mathcal{S}_{\infty,0}(f).
\end{aligned}$$

Finally, from this and (3.8), we have

$$\begin{aligned}
\left| I_0 - \int_X f\, dm_X \right| & \leq |I_0 - I_{\mathrm{smth}}| + \left| I_{\mathrm{smth}} - \int_X f\, dm_X \right| \\
& \ll \left(\varepsilon^n + R^{-\beta_1} \varepsilon^{-p_1 n} + \varepsilon^{1/d(n-1)} \right) \mathcal{S}_{\infty,\ell}(f)
\end{aligned}$$

where we have used that r is proportional to ε^n, as well as Sobolev property (i).

Let $p_2 := 1/d(n-1)$. Since $n > p_2$, the ε^n term above decays more quickly than other terms and can be ignored. To optimize the rate of decay, we set

$$R^{-\beta_1} \varepsilon^{-p_1 n} = \varepsilon^{p_2}$$

which implies

$$\varepsilon = R^{-\beta_1/(p_1 n + p_2)}.$$

Then so long as R is chosen sufficiently large so that ε (and subsequently r) are small enough to make Corollary 7 and Lemma 5 true (along with several other statements we made regarding neighborhoods of the identity), then we have demonstrated (3.2a) in Theorem 11 with the rate

$$\left| I_0 - \int_X f dm_X \right| \ll R^{-\gamma} \mathcal{S}_{\infty,\ell}(f)$$

where

$$\gamma = \frac{\beta_1 p_2}{p_1 n + p_2} = \frac{2\beta_1}{nd(3n+1)(n-1)^2 + 2} \tag{3.19}$$

where we have written p_1, p_2, and ℓ in terms of n and d. □

3.2 Γ Cocompact

In the case of Γ cocompact, it follows directly from the above proof that we may remove dependence on the basepoint from our effective equidistribution statement. That is, for $X = \Gamma \backslash G$, $\Gamma \leq G$ a cocompact lattice, and $U \leq G$ a horospherical subgroup, we have that there exists $\gamma > 0$ (depending[3] only on n and β_1) such that for T large enough,

$$\left| \frac{1}{m_U(B_T)} \int_{B_T} f(x_0 u) dm_U(u) - \int_X f dm_X \right| \ll_\Gamma T^{-\gamma} \mathcal{S}_{\infty,\ell}(f) \tag{3.20}$$

for any $f \in C^\infty(X)$ and $x_0 \in X$. This is because we only make use of the basepoint condition in **Step 3**, where we need it to deal with the moving basepoint and the fact that the radius of

[3] In this case we have $\gamma = 2\beta_1/((3n+1)(n-1)+2)$.

injection depends on where we are in X. However, in the compact setting, we have a uniform injectivity radius, so we may avoid this step altogether.

Morally speaking, uniformity in the basepoint is due to the fact that in the compact setting the dynamics are minimal, so there are no proper invariant subspaces near which an orbit can become trapped for long periods of time.

Chapter 3, in part, has been submitted for publication of the material as it may appear in the Journal of Modern Dynamics, 2019, McAdam, Taylor, American Institute of Mathematical Sciences (AIMS), 2018. The dissertation author was the primary investigator and author of this material.

Chapter 4

Equidistribution for Arithmetic Sequences In Abelian Horospherical Flows

4.1 The Space of Lattices

Let $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, and $X = \Gamma\backslash G$. Let U be an upper triangular unipotent subgroup of the form

$$U = \left\{ \left(\begin{array}{c|c} I_m & * \\ \hline 0 & I_{n-m} \end{array} \right) \right\} \tag{4.1}$$

for $m < n$. Note that $U \cong \mathbb{R}^d$ as groups for $d = m(n-m)$ under the identification $u(\mathbf{t})$ which maps the coordinates of $\mathbf{t} \in \mathbb{R}^d$ to the matrix entries in the upper-right block of (4.1). Recall that the Haar measure on U is the Lebesgue measure on $\mathbb{R}^d$ under this identification, which we normalize so that $u([0,1]^d)$ has unit measure. Observe that U is horospherical with respect to the element

$$a_t = \mathrm{diag}(\underbrace{e^{t(n-m)/n}, \cdots, e^{t(n-m)/n}}_{m}, \underbrace{e^{-tm/n}, \cdots, e^{-tm/n}}_{n-m}) \tag{4.2}$$

for any $t > 0$ and that conjugation by a_t scales all entries in the upper-right block of U by $e^{t(n-m)/n}e^{tm/n} = e^t$. Hence, for this choice of a_t, we have $B_T = a_{\log T}u([0,1]^d)a_{-\log T} = u([0,T]^d)$. For this reason we will conflate the notation and write B_T for both $[0,T]^d \subseteq \mathbb{R}^d$ and $u([0,T]^d) \subseteq U$.

Let ψ be an additive character of U (so $\psi(\mathbf{t}) = e^{i\mathbf{a}\cdot\mathbf{t}}$ for some $\mathbf{a} \in \mathbb{R}^d$). Define measure ν_T and (complex) measure $\mu_{T,\psi}$ on X via duality: for $f \in C_c^\infty(X)$ let

$$\int_X f d\nu_T = \nu_T(f) := \frac{1}{|B_T|}\int_{B_T} f(x_0u(\mathbf{t}))d\mathbf{t}$$

and

$$\int_X f d\mu_{T,\psi} = \mu_{T,\psi}(f) := \frac{1}{|B_T|}\int_{B_T} \psi(\mathbf{t})\left(f(x_0u(\mathbf{t})) - \int_X f dm_X\right)d\mathbf{t}.$$

Our main goal in this chapter is to obtain an effective rate of equidistribution along (multivariate) arithmetic sequences of inputs for the right action of U on X. To do this, we first present the following lemma, the proof of which closely follows the proof of Lemma 3.1 in [Ven10] for the case of $G = \mathrm{SL}_2(\mathbb{R})$ and Γ cocompact.

Lemma 12. *Let $x_0 = \Gamma g_0 \in X$ satisfy (3.2c) for $T > R > C$. Then there exists $b > 0$ such that for any $f \in C_c^\infty(X)$ and additive character ψ,*

$$\left|\mu_{T,\psi}(f)\right| \ll R^{-b}\mathcal{S}_{\infty,\ell}(f)$$

where ℓ is as in Theorem 11.

Remark. As noted in [Ven10], the significance of this lemma is that the implicit constant is independent of choice of ψ. This can be shown for highly oscillatory ψ using integration by parts and for almost constant ψ using equidistribution of the horospherical flow directly, thus this lemma is most significant for ψ of moderate oscillation. The proof will use our effective equidistribution result as well as a variety of standard analytic techniques.

Proof. Let $1 \leq H \leq T$ and define a complex measure σ_H on U by

$$\int_U g d\sigma_H = \sigma_H(g) := \frac{1}{|B_H|} \int_{B_H} \psi(\mathbf{t}) g(u(\mathbf{t})) d\mathbf{t}$$

for $g \in C_c^\infty(U)$.

Let $f * \sigma_H$ be the right convolution of f by σ_H, i.e., for $x \in X$

$$\begin{aligned} f * \sigma_H(x) &= \int f(xu(\mathbf{t})) d\sigma_H(\mathbf{t}) \\ &= \frac{1}{|B_H|} \int_{B_H} \psi(\mathbf{t}) f(xu(\mathbf{t})) d\mathbf{t}. \end{aligned}$$

Notice that by switching the order of integration (one may verify that the conditions of Fubini's theorem are satisfied) and using invariance of the measure m_X, we have

$$\begin{aligned} \int_X f * \sigma_H dm_X &= \int_X \frac{1}{|B_H|} \int_{B_H} \psi(\mathbf{t}) f(xu(\mathbf{t})) d\mathbf{t}\, dm_X(x) \\ &= \frac{1}{|B_H|} \int_{B_H} \psi(\mathbf{t}) \left(\int_X f(xu(\mathbf{t})) dm_X(x) \right) d\mathbf{t} \\ &= \frac{1}{|B_H|} \int_{B_H} \psi(\mathbf{t}) \left(\int_X f dm_X \right) d\mathbf{t}. \end{aligned}$$

Hence,

$$\begin{aligned} \mu_{T,\psi}(f * \sigma_H) &= \frac{1}{|B_T|} \int_{B_T} \psi(\mathbf{t}) \left(f * \sigma_H(x_0 u(\mathbf{t})) - \int_X f * \sigma_H dm_X \right) d\mathbf{t} \\ &= \frac{1}{|B_T|} \int_{B_T} \psi(\mathbf{t}) \frac{1}{|B_H|} \int_{B_H} \psi(\mathbf{s}) \left(f(x_0 u(\mathbf{t}) u(\mathbf{s})) - \int_X f dm_X \right) d\mathbf{s} d\mathbf{t} \\ &= \frac{1}{|B_T||B_H|} \int_{B_T} \int_{B_H} \psi(\mathbf{t}+\mathbf{s}) \left(f(x_0 u(\mathbf{t}+\mathbf{s})) - \int_X f dm_X \right) d\mathbf{s} d\mathbf{t} \end{aligned}$$

since $U \cong R^d$. Now by switching the order of integration and applying a change of variables, we get

$$\mu_{T,\psi}(f*\sigma_H)=\frac{1}{|B_T||B_H|}\int_{B_H}\int_{B_T+\mathbf{s}}\psi(\mathbf{t})\left(f(x_0u(\mathbf{t}))-\int_X f\,dm_X\right)d\mathbf{t}d\mathbf{s}.$$

But we may also write

$$\begin{aligned}\mu_{T,\psi}(f)&=\frac{1}{|B_T|}\int_{B_T}\psi(\mathbf{t})\left(f(x_0u(\mathbf{t}))-\int_X f\,dm_X\right)d\mathbf{t}\\&=\frac{1}{|B_T||B_H|}\int_{B_H}\int_{B_T}\psi(\mathbf{t})\left(f(x_0u(\mathbf{t}))-\int_X f\,dm_X\right)d\mathbf{t}d\mathbf{s}.\end{aligned}$$

Thus

$$\begin{aligned}|\mu_{T,\psi}(f)-\mu_{T,\psi}(f*\sigma_H)|&\leq\frac{1}{|B_T||B_H|}\int_{B_H}\int_{B_T\triangle(B_T+\mathbf{s})}\left|f(x_0u(\mathbf{t}))-\int_X f\,dm_X\right|d\mathbf{t}d\mathbf{s}\\&\ll\frac{1}{|B_T||B_H|}\int_{B_H}|B_T\triangle(B_T+\mathbf{s})|\mathcal{S}_{\infty,0}(f)d\mathbf{s}.\end{aligned}$$

But notice that $B_T\triangle(B_T+\mathbf{s})$ is simply the symmetric difference of two shifted cubes, the measure of which will be maximized when $\mathbf{s}=(H,\cdots,H)$ (see Figure 4.1). Hence,

$$\begin{aligned}|B_T\triangle(B_T+\mathbf{s})|&\leq 2(T^d-(T-H)^d)\\&=2(dT^{d-1}H-\cdots\pm dTH^{d-1}\mp H^d)\\&\ll T^{d-1}H.\end{aligned}$$

since $H\leq T$ implies that the leading term dominates. It follows that

$$\int_{B_H}|B_T\triangle(B_T+\mathbf{s})|d\mathbf{s}\ll T^{d-1}H^{d+1}.$$

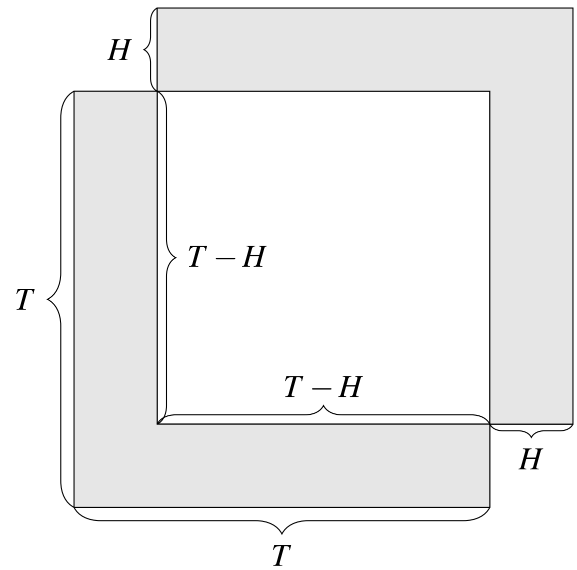

Figure 4.1: The symmetric difference between B_T and $B_T + (H, \cdots, H)$. The sets B_T form a family of Følner sets in U as $T \to \infty$.

Thus,

$$|\mu_{T,\psi}(f) - \mu_{T,\psi}(f * \sigma_H)| \ll \frac{T^{d-1}H^{d+1}}{|B_T||B_H|}\mathcal{S}_{\infty,0}(f) = \frac{H}{T}\mathcal{S}_{\infty,0}(f). \tag{4.3}$$

Now consider the squared quantity

$$\begin{aligned}
|\mu_{T,\psi}(f * \sigma_H)|^2 &= \left| \frac{1}{|B_T|} \int_{B_T} \psi(\mathbf{t}) \left(f * \sigma_H(x_0 u(\mathbf{t})) - \int_X f * \sigma_H dm_X \right) d\mathbf{t} \right|^2 \\
&\leq \frac{1}{|B_T|^2} \left(\int_{B_T} \left| f * \sigma_H(x_0 u(\mathbf{t})) - \int_X f * \sigma_H dm_X \right| d\mathbf{t} \right)^2 \\
&= \frac{1}{|B_T|^2} \left\langle 1, \left| f * \sigma_H(x_0 u(\cdot)) - \int_X f * \sigma_H dm_X \right| \right\rangle_{L^2(B_T)}^2 .
\end{aligned}$$

By Cauchy-Schwarz, we know that

$$|\mu_{T,\psi}(f * \sigma_H)|^2 \leq \frac{1}{|B_T|^2} \|1\|_{L^2(B_T)}^2 \left\| f * \sigma_H(x_0 u(\cdot)) - \int_X f * \sigma_H dm_X \right\|_{L^2(B_T)}^2 .$$

Now, $\|1\|^2_{L^2(B_T)} = \int_{B_T} 1^2 d\mathbf{t} = |B_T|$, and

$$\left\| f * \sigma_H(x_0 u(\cdot)) - \int_X f * \sigma_H dm_X \right\|^2_{L^2(B_T)} = \int_{B_T} \left| f * \sigma_H(x_0 u(\mathbf{t})) - \int_X f * \sigma_H dm_X \right|^2 d\mathbf{t}$$
$$= |B_T| \nu_T \left(\left| f * \sigma_H - \int_X f * \sigma_H dm_X \right|^2 \right)$$

which shows that

$$|\mu_{T,\psi}(f * \sigma_H)|^2 \leq \nu_T \left(\left| f * \sigma_H - \int_X f * \sigma_H dm_X \right|^2 \right). \tag{4.4}$$

Hence, by (4.3) and (4.4), we have

$$\begin{aligned} |\mu_{T,\psi}(f)| &\leq |\mu_{T,\psi}(f) - \mu_{T,\psi}(f * \sigma_H)| + |\mu_{T,\psi}(f * \sigma_H)| \\ &\ll \frac{H}{T} \mathcal{S}_{\infty,0}(f) + \nu_T \left(\left| f * \sigma_H - \int_X f * \sigma_H dm_X \right|^2 \right)^{1/2}. \end{aligned} \tag{4.5}$$

To estimate $\nu_T \left(|f * \sigma_H - \int_X f * \sigma_H dm_X|^2 \right)$, observe that

$$\begin{aligned} &\left| f * \sigma_H(x) - \int_X f * \sigma_H dm_X \right|^2 \\ &= \left| \frac{1}{|B_H|} \int_{B_H} \psi(\mathbf{s}) \left([u(\mathbf{s}) f](x) - \int_X f dm_X \right) d\mathbf{s} \right|^2 \\ &= \frac{1}{|B_H|^2} \left(\int_{B_H} \psi(\mathbf{s_1}) \left([u(\mathbf{s_1}) f](x) - \int_X f dm_X \right) d\mathbf{s_1} \right) \\ &\qquad \cdot \left(\int_{B_H} \overline{\psi}(\mathbf{s_2}) \left([u(\mathbf{s_2}) \overline{f}](x) - \int_X \overline{f} dm_X \right) d\mathbf{s_2} \right) \\ &= \frac{1}{|B_H|^2} \iint_{B_H \times B_H} \psi(\mathbf{s_1} - \mathbf{s_2}) \left([u(\mathbf{s_1}) f](x) - \int_X f dm_X \right) \left([u(\mathbf{s_2}) \overline{f}](x) - \int_X \overline{f} dm_X \right) d\mathbf{s_1} d\mathbf{s_2}. \end{aligned}$$

When we apply ν_T to this, we can change the order of integration so that the innermost integral is over B_T, with the character $\psi(\mathbf{s_1} - \mathbf{s_2})$ outside this integral. We may then integrate separately over the four terms we get by expanding the bracketed product above. That is,

$$\begin{aligned}
&\nu_T\left(\left|f*\sigma_H-\int_X f*\sigma_H dm_X\right|^2\right)\\
&=\frac{1}{|B_H|^2}\iint\limits_{B_H\times B_H}\psi(\mathbf{s_1}-\mathbf{s_2})\nu_T\left(\left(u(\mathbf{s_1})f-\int_X f dm_X\right)\left(u(\mathbf{s_2})\overline{f}-\int_X \overline{f} dm_X\right)\right)d\mathbf{s_1}d\mathbf{s_2}\\
&=\frac{1}{|B_H|^2}\iint\limits_{B_H\times B_H}\psi(\mathbf{s_1}-\mathbf{s_2})F(\mathbf{s_1},\mathbf{s_2})d\mathbf{s_1}d\mathbf{s_2} \qquad (4.6)
\end{aligned}$$

where

$$\begin{aligned}
F(\mathbf{s_1},\mathbf{s_2})=&\,\nu_T(u(\mathbf{s_1})f\cdot u(\mathbf{s_2})\overline{f})\\
&-\nu_T(u(\mathbf{s_1})f)\int_X \overline{f}dm_X\\
&-\nu_T(u(\mathbf{s_2})\overline{f})\int_X f dm_X\\
&+\left|\int_X f dm_X\right|^2.
\end{aligned} \qquad (4.7)$$

Now from Theorem 11 we know that for arbitrary $\tilde{f}\in C_c^\infty(X)$ and x_0 satisfying the Diophantine basepoint condition (3.2c) with $T>R>C$, we have

$$\left|\nu_T(\tilde{f})-\int_X \tilde{f}dm_X\right|=\left|\frac{1}{|B_T|}\int_{B_T}\tilde{f}(x_0u(\mathbf{t}))d\mathbf{t}-\int_X \tilde{f}dm_X\right|\ll R^{-\gamma}\mathcal{S}_{\infty,\ell}(\tilde{f}),$$

that is,

$$\nu_T(\tilde{f})=\int_X \tilde{f}dm_X+O(R^{-\gamma}\mathcal{S}_{\infty,\ell}(\tilde{f})). \qquad (4.8)$$

Applying this to the function $\tilde{f}=u(\mathbf{s_1})f$, we find that

$$\nu_T(u(\mathbf{s_1})f)=\int_X u(\mathbf{s_1})f dm_X+O(R^{-\gamma}\mathcal{S}_{\infty,\ell}(u(\mathbf{s_1})f)).$$

But by invariance of m_X, we have

$$\int_X u(\mathbf{s_1}) f dm_X = \int_X f(xu(\mathbf{s_1}))dm_X(x) = \int_X f dm_X.$$

Thus

$$\nu_T(u(\mathbf{s_1})f)\int_X \overline{f} dm_X = \left|\int_X f dm_X\right|^2 + O\left(R^{-\gamma}\mathcal{S}_{\infty,\ell}(u(\mathbf{s_1})f)\left|\int_X \overline{f} dm_X\right|\right).$$

Furthermore, from Sobolev norm property (iii), we know that for $f \in C_c^\infty(X)$ and $h \in G$, we have $\mathcal{S}_{\infty,\ell}(hf) \ll_\ell \|h\|^\ell \mathcal{S}_{\infty,\ell}(f)$, where $\|h\|$ is the operator norm of $\mathrm{Ad}_{h^{-1}}$. Since the entries of $u(\mathbf{s})$ are bounded by $\max(1,|\mathbf{s}|)$, we have $\|u(\mathbf{s})\| \ll \max(1,|\mathbf{s}|)^2$. Thus for $\mathbf{s_1} \in [0,H]$ with $H \geq 1$, $\mathcal{S}_{\infty,\ell}(u(\mathbf{s_1})f) \ll H^{2\ell}\mathcal{S}_{\infty,\ell}(f)$. Combining this with the bound $\left|\int \overline{f} dm_X\right| \ll \mathcal{S}_{\infty,0}(f) \ll \mathcal{S}_{\infty,\ell}(f)$, we find that

$$\nu_T(u(\mathbf{s_1})f)\int_X \overline{f} dm_X = \left|\int_X f dm_X\right|^2 + O(R^{-\gamma}H^{2\ell}\mathcal{S}_{\infty,\ell}(f)^2).$$

Likewise,

$$\nu_T(u(\mathbf{s_2})\overline{f})\int_X f dm_X = \left|\int_X f dm_X\right|^2 + O(R^{-\gamma}H^{2\ell}\mathcal{S}_{\infty,\ell}(f)^2).$$

Therefore, (4.7) becomes simply

$$F(\mathbf{s_1},\mathbf{s_2}) = \nu_T(u(\mathbf{s_1})f \cdot u(\mathbf{s_2})\overline{f}) - \left|\int_X f dm_X\right|^2 + O(T^{-\alpha\gamma}H^{2\ell}\mathcal{S}_{\infty,\ell}(f)^2).$$

Substituting this back into (4.6), we conclude that

$$\nu_T\left(\left|f*\sigma_H(x)-\int_X f*\sigma_H dm_X\right|^2\right)$$
$$\ll \frac{1}{|B_H|^2}\iint_{B_H\times B_H}\left|\nu_T(u(\mathbf{s_1})f\cdot u(\mathbf{s_2})\overline{f})-\left|\int_X f dm_X\right|^2\right| d\mathbf{s_1}d\mathbf{s_2}+R^{-\gamma}H^{2\ell}\mathcal{S}_{\infty,\ell}(f)^2. \quad (4.9)$$

But now notice that

$$\int_X u(\mathbf{s_1})f\cdot u(\mathbf{s_2})\overline{f}dm_X=\langle u(\mathbf{s_1})f,u(\mathbf{s_2})f\rangle_{L^2(X)}=\langle u(\mathbf{s_1}-\mathbf{s_2})f,f\rangle_{L^2(X)}$$

so by the triangle inequality, we can estimate

$$\left|\nu_T(u(\mathbf{s_1})f\cdot u(\mathbf{s_2})\overline{f})-\left|\int_X f dm_X\right|^2\right|$$
$$\leq \left|\nu_T(u(\mathbf{s_1})f\cdot u(\mathbf{s_2})\overline{f})-\int_X u(\mathbf{s_1})f\cdot u(\mathbf{s_2})\overline{f}dm_X\right|$$
$$+\left|\langle u(\mathbf{s_1}-\mathbf{s_2})f,f\rangle_{L^2(X)}-\left|\int_X f dm_X\right|^2\right|. \quad (4.10)$$

Again, by our equidistribution result in (4.8), we know that

$$\left|\nu_T(u(\mathbf{s_1})f\cdot u(\mathbf{s_2})\overline{f})-\int_X u(\mathbf{s_1})f\cdot u(\mathbf{s_2})\overline{f}dm_X\right|\ll R^{-\gamma}\mathcal{S}_{\infty,\ell}(u(\mathbf{s_1})f\cdot u(\mathbf{s_2})\overline{f}) \quad (4.11)$$

and by properties (ii) and (iii) of Sobolev norms, we have

$$\mathcal{S}_{\infty,\ell}(u(\mathbf{s_1})f\cdot u(\mathbf{s_2})\overline{f})\ll \mathcal{S}_{\infty,\ell}(u(\mathbf{s_1})f)\mathcal{S}_{\infty,\ell}(u(\mathbf{s_2})f)\ll H^{4\ell}\mathcal{S}_{\infty,\ell}(f)^2 \quad (4.12)$$

for $\mathbf{s_1}, \mathbf{s_2} \in [0, H]$. Thus, from (4.12), (4.11), and (4.10), equation (4.9) becomes

$$\nu_T\left(\left|f * \sigma_H(x) - \int_X f * \sigma_H dm_X\right|^2\right)$$
$$\ll \frac{1}{|B_H|^2} \iint\limits_{B_H \times B_H} \left| \langle u(\mathbf{s_1} - \mathbf{s_2}) f, f \rangle_{L^2(X)} - \left| \int_X f dm_X \right|^2 \right| d\mathbf{s_1} d\mathbf{s_2} + R^{-\gamma} H^{4\ell} \mathcal{S}_{\infty,\ell}(f)^2. \tag{4.13}$$

Now from Corollary 9 (ii), we know there exists $\beta_2 > 0$ such that for any $\mathbf{s} \in \mathbb{R}^d$,

$$\left| \langle u(\mathbf{s}) f, f \rangle_{L^2(X)} - \left| \int_X f dm_X \right|^2 \right| \ll \max(1, |\mathbf{s}|)^{-\beta_2} \mathcal{S}_{\infty,\ell}(f)^2. \tag{4.14}$$

Then for $\mathbf{s} = \mathbf{s_1} - \mathbf{s_2}$, we have the following problem: We want to bound the integral in (4.13) by a power of H, but for $(\mathbf{s_1}, \mathbf{s_2})$ close to the diagonal in $B_H \times B_H$ we cannot do better than a constant times the Sobolev norm of f in (4.14). We will address this by integrating separately over a neighborhood of the diagonal that has small measure (depending on H) and away from the diagonal where $\max(1, |\mathbf{s_1} - \mathbf{s_2}|)$ is dominated by H.

To make this precise, let $D := \{(\mathbf{s_1}, \mathbf{s_2}) \in B_H \times B_H \mid \mathbf{s_1} = \mathbf{s_2}\}$ be the diagonal of $B_H \times B_H$ and define $D_\varepsilon := \{(\mathbf{s_1}, \mathbf{s_2}) \in B_H \times B_H \mid |\mathbf{s_1} - \mathbf{s_2}| < \varepsilon\}$. Notice that D is a d-dimensional subset of $\mathbb{R}^{2d}$ with diameter $\sqrt{2d}H$. Furthermore, any point satisfying $|\mathbf{s_1} - \mathbf{s_2}| = \varepsilon$ is distance $\varepsilon/\sqrt{2}$ from the diagonal, so D_ε is an $(\varepsilon/\sqrt{2})$-neighborhood of D sitting inside $[0, H]^{2d}$. Thus D_ε is contained within a box in $\mathbb{R}^{2d}$ with d side-lengths of $\sqrt{2d}H$ and d side-lengths of $2\varepsilon/\sqrt{2}$, so

$$|D_\varepsilon| \ll H^d \varepsilon^d$$

(see Figure 4.2). In particular, if $\varepsilon = H^\zeta$ (for $0 < \zeta < 1$ to be determined), then

$$\left| \{(\mathbf{s_1}, \mathbf{s_2}) \in B_H \times B_H \mid |\mathbf{s_1} - \mathbf{s_2}| < H^\zeta\} \right| \ll H^{d(1+\zeta)}.$$

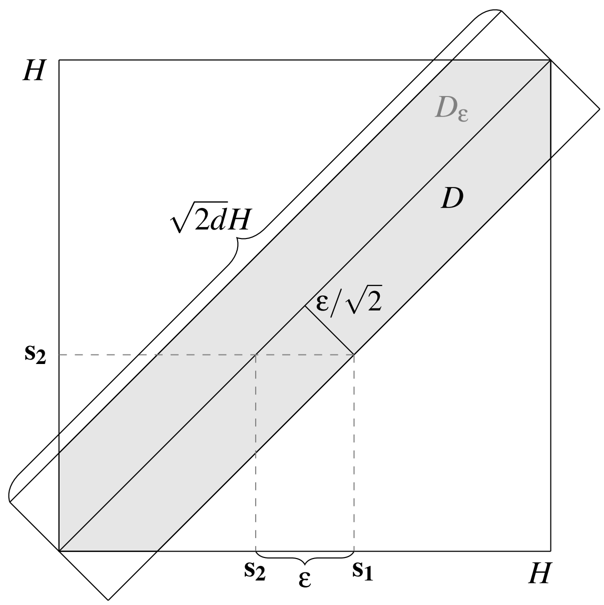

Figure 4.2: The measure of the set where $|\mathbf{s_1}-\mathbf{s_2}| < \varepsilon$ has measure bounded by $H^d\varepsilon^d$ in $B_H \times B_H$ (shown here for one dimensional U).

In this region, the integrand is dominated by 1, so when we integrate over this region and divide by $|B_H|^2 = H^{2d}$ (as we are doing in equation (4.13)), we get a term of order $H^d(\zeta-1)\mathcal{S}_{\infty,\ell}(f)^2$.

On the other hand, for $|\mathbf{s_1}-\mathbf{s_2}| \geq H^\zeta$, we can say that

$$\left|\langle u(\mathbf{s_1}-\mathbf{s_2})f,f\rangle_{L^2(X)} - \left|\int_X f dm_X\right|^2\right| \ll \max(1,|\mathbf{s_1}-\mathbf{s_2}|)^{-\beta_2}\mathcal{S}_{\infty,\ell}(f)^2$$
$$\leq H^{-\zeta\beta_2}\mathcal{S}_{\infty,\ell}(f)^2.$$

Hence,

$$\frac{1}{|B_H|^2}\iint_{B_H\times B_H}\left|\langle u(\mathbf{s_1}-\mathbf{s_2})f,f\rangle_{L^2(X)} - \left|\int_X f dm_X\right|^2\right| d\mathbf{s_1}d\mathbf{s_2}$$
$$\ll (H^{-\zeta\beta_2}+H^{d(\zeta-1)})\mathcal{S}_{\infty,\ell}(f)^2$$
$$= H^{-d\beta_2+2/(d+\beta_2)}\mathcal{S}_{\infty,\ell}(f)^2 \qquad (4.15)$$

where we have chosen $\zeta = d/(d+\beta_2)$ to optimize the error.

Together, the bounds in (4.13) and (4.15) imply that

$$\nu_T\left(\left|f*\sigma_H(x) - \int_X f*\sigma_H dm_X\right|^2\right) \ll \left(R^{-\gamma}H^{4\ell} + H^{-d\beta_2/(d+\beta_2)}\right)\mathcal{S}_{\infty,\ell}(f)^2. \tag{4.16}$$

Finally, from (4.5) and (4.16), we have

$$|\mu_{T,\psi}(f)| \ll \left(T^{-1}H + R^{-\gamma/2}H^{2\ell} + H^{-d\beta_2/(2d+2\beta_2)}\right)\mathcal{S}_{\infty,\ell}(f).$$

Since $\gamma < 1$ and $R < T$, the first term decays more quickly that the second, and can be ignored. Thus the decay is optimized when

$$H^{-d\beta_2/(2d+2\beta_2)} = R^{-\gamma/2}H^{2\ell}$$
$$H = R^{\gamma(d+\beta_2)/(4\ell d+4\ell\beta_2+d\beta_2)}.$$

This demonstrates the claim that

$$|\mu_{T,\psi}(f)| \ll R^{-b}\mathcal{S}_{\infty,\ell}(f)$$

where

$$b = \frac{d\beta_2\gamma}{8d\ell+8\ell\beta_2+2d\beta_2} = \frac{d\beta_1\beta_2}{(nd(3n+1)(n-1)^2+2)(2n(n-1)(d+\beta_2)+d\beta_2)} \tag{4.17}$$

where we have used the formula for γ in (3.19). □

We will now use this lemma to establish an effective equidistribution bound along multivariate arithmetic sequences.

Let $K_1, \ldots, K_d \geq 1$ and define K to be the diagonal matrix

$$K := \operatorname{diag}(K_1, \cdots, K_d) = \begin{pmatrix} K_1 & & \\ & \ddots & \\ & & K_d \end{pmatrix}$$

and $|K| = \det(K) = K_1 K_2 \cdots K_d$.

We want to understand the behavior of

$$S := \sum_{\substack{\mathbf{k} \in \mathbb{Z}^d \\ K\mathbf{k} \in B_T}} f(x_0 u(K\mathbf{k})). \tag{4.18}$$

For equidistribution, we want this to be close to $\#\{\mathbf{k} \in \mathbb{Z}^d | K\mathbf{k} \in B_T\} \int_X f dm_X \approx \frac{T^d}{|K|} \int_X f dm_X$. For x_0 satisfying a basepoint property, we have the following result.

Theorem 13. *Let $K = \operatorname{diag}(K_1, \cdots, K_d)$ with $T \geq K_1, \ldots, K_d \geq 1$ and determinant $|K|$. Then for all $x_0 \in X$ satisfying (3.2c) with $T > R > C_K$, we have*

$$\left| \sum_{\substack{\mathbf{k} \in \mathbb{Z}^d \\ K\mathbf{k} \in B_T}} f(x_0 u(K\mathbf{k})) - \frac{T^d}{|K|} \int_X f dm_X \right| \ll \left(T^d R^{-b/(d+1)} |K|^{-d/(d+1)} + \frac{T^{d-1} \max_i K_i}{|K|} \right) \mathcal{S}_{\infty,\ell}(f)$$

where $C_K := \max(C, (2/\min K_i)^{(d+1)/b} |K|^{1/b})$ with C and ℓ as in Theorem 11.

Proof. Let $\delta > 0$ be small (to be determined) and define the single-variable hat function

$$g_\delta(t) := \max(\delta^{-2}(\delta - |t|), 0)$$

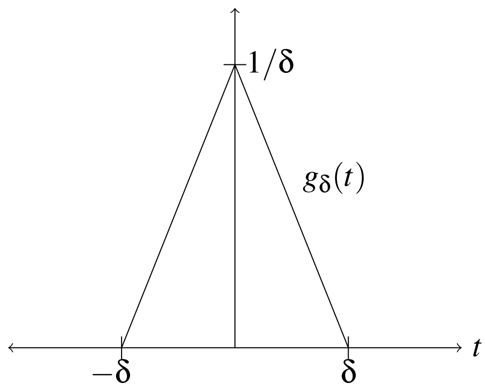

Figure 4.3: The single-variable hat function $g_\delta(t)$. The multivariate bump function we use is the product of functions of this form in each coordinate.

for $t \in \mathbb{R}$ and (through slight abuse of notation) the multivariable function

$$g_\delta(\mathbf{t}) := g_\delta(t_1)\cdots g_\delta(t_d)$$

for $\mathbf{t} = (t_1, \cdots, t_d) \in \mathbb{R}^d$. Notice that $\int_{\mathbb{R}^d} g_\delta(\mathbf{t})d\mathbf{t} = 1$ and supp $(g_\delta) \subseteq [-\delta, \delta]^d$.

Define an approximation to the sum S by

$$S_{\text{approx}} := \int_{B_T} \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} g_\delta(\mathbf{t} - K\mathbf{k}) \right) f(x_0 u(\mathbf{t}))d\mathbf{t}. \tag{4.19}$$

That is, instead of averaging f over the lattice points of $K\mathbb{Z}^d$, we average over small neighborhoods around the lattice points using the bump function g_δ, since $\sum_{\mathbf{k}} g_\delta(\mathbf{t} - K\mathbf{k})$ is supported on a disjoint union of δ-cubes centered around the points of $K\mathbb{Z}^d$ (that is, so long as $\delta < \min_i K_i/2$).

We want to show that S_{approx} can be written

$$S_{\text{approx}} = \left(\sum_{\substack{\mathbf{k}\in\mathbb{Z}^d \\ K\mathbf{k}\in B_T}} \int_{[-\delta,\delta]^d + K\mathbf{k}} g_\delta(\mathbf{t} - K\mathbf{k}) f(x_0 u(\mathbf{t}))d\mathbf{t} \right) + r(T, K, f, d) \tag{4.20}$$

where $r(T, K, f, d)$ is an error term depending on T, $K_1, \cdots, K_d$, f, and dimension d. To see this,

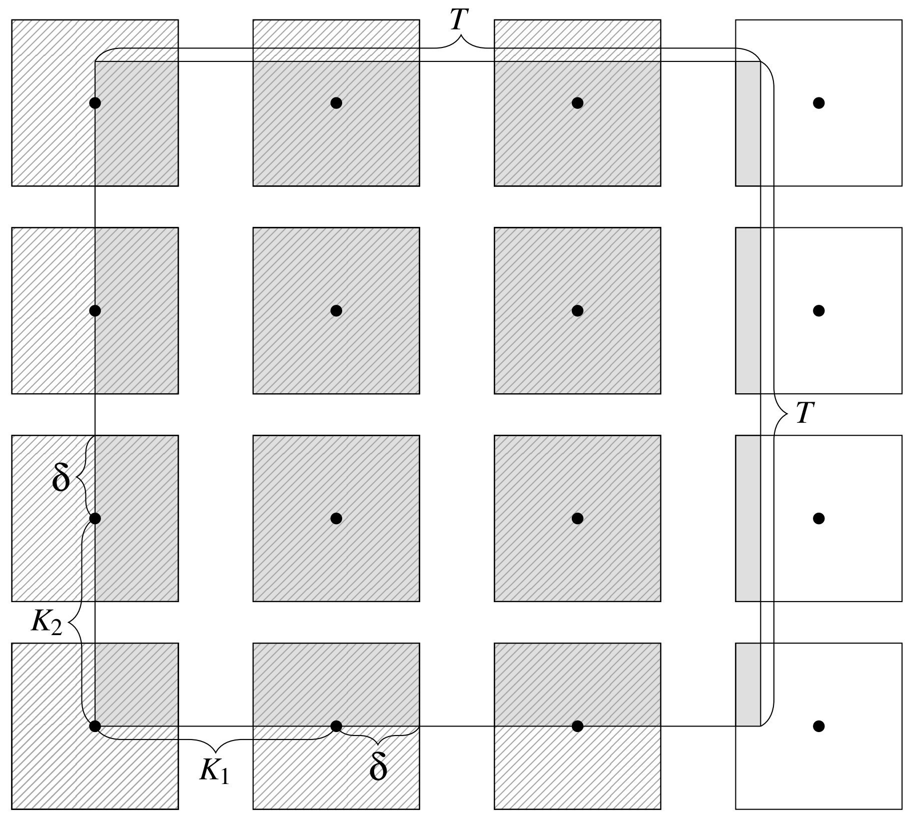

Figure 4.4: The error $r(T,K,f,d)$ can be bounded by the number of δ-cubes intersecting the boundary of B_T multiplied by the supremum of f. The area shaded in solid gray indicates the region over which we are integrating in the definition of S_{approx}, whereas the area shaded with diagonal lines represents the region over which we are integrating in our estimate of S_{approx} given in (4.20).

observe that in both (4.19) and (4.20) we are integrating f against a sum of bump functions supported on a disjoint union of δ-cubes centered at the lattice points of $K\mathbb{Z}^d$. However, in (4.19) we are integrating over the region shaded in solid gray in Figure 4.4, whereas in (4.20) we are integrating over the region shaded with diagonal lines (that is, we are only integrating against the hat functions whose centers intersect B_T).

Thus all of the possible error comes from integrating over those δ-cubes that intersect the boundary of B_T. Consider a face of B_T that is orthogonal to the i^{th} standard basis vector. It will intersect at most $T/K_j + O(1)$ of these cubes along an edge in the j^{th} direction for $j \neq i$. Hence, the total number of cubes that face intersects can be bounded by

$$\frac{T}{K_1}\cdots\frac{T}{K_{i-1}}\cdot\frac{T}{K_{i+1}}\cdots\frac{T}{K_d}=T^{d-1}\frac{K_i}{|K|}.$$

Since g_δ integrates to one, the error that results from integrating over one of these δ-cubes is bounded by $\mathcal{S}_{\infty,0}(f)$. Then considering all the faces of B_T, we see that the error satisfies

$$|r(T,K,f,d)|\ll\mathcal{S}_{\infty,0}(f)\sum_{i=1}^{d}T^{d-1}\frac{K_i}{|K|}\ll T^{d-1}\frac{\max_i K_i}{|K|}\mathcal{S}_{\infty,0}(f).$$

Then by a change of variables in (4.20), we have

$$S_{\text{approx}}=\left(\sum_{\substack{\mathbf{k}\in\mathbb{Z}^d\\ K\mathbf{k}\in B_T}}\int_{[-\delta,\delta]^d}g_\delta(\mathbf{s})f(x_0u(K\mathbf{k}+\mathbf{s}))d\mathbf{s}\right)+r(T,K,f,d). \tag{4.21}$$

Also, since $\int_{[-\delta,\delta]^d}g_\delta(\mathbf{s})d\mathbf{s}=1$, we may rewrite the definition of S in (4.18) as

$$S=\sum_{\substack{\mathbf{k}\in\mathbb{Z}^d\\ K\mathbf{k}\in B_T}}\int_{[-\delta,\delta]^d}g_\delta(\mathbf{s})f(x_0u(K\mathbf{k}))d\mathbf{s}$$

and combining this with (4.21), we obtain

$$\begin{aligned}|S_{\text{approx}}-S|\le&\left(\sum_{\substack{\mathbf{k}\in\mathbb{Z}^d\\ K\mathbf{k}\in B_T}}\int_{[-\delta,\delta]^d}g_\delta(\mathbf{s})|f(x_0u(K\mathbf{k}+\mathbf{s}))-f(x_0u(K\mathbf{k}))|d\mathbf{s}\right)\\&+|r(T,K,f,d)|.\end{aligned}$$

But note that from property (iv) of Sobolev norms, we have

$$|f(x_0u(K\mathbf{k}+\mathbf{s}))-f(x_0u(K\mathbf{k}))|\ll\mathcal{S}_{\infty,1}(f)|\mathbf{s}|\ll\mathcal{S}_{\infty,1}(f)\delta$$

for $\mathbf{s} \in [-\delta, \delta]^d$. Together with our error bound, this implies that

$$\begin{aligned}
|S_{\text{approx}} - S| &\ll \left(\sum_{\substack{\mathbf{k}\in\mathbb{Z}^d \\ K\mathbf{k}\in B_T}} \int_{[-\delta,\delta]^d} g_\delta(\mathbf{s}) d\mathbf{s} \right) \delta \mathcal{S}_{\infty,1}(f) + T^{d-1} \frac{\max_i K_i}{|K|} \mathcal{S}_{\infty,0}(f) \\
&= \left(\#\{\mathbf{k} \in \mathbb{Z}^d | K\mathbf{k} \in B_T\} \delta + T^{d-1} \frac{\max_i K_i}{|K|} \right) \mathcal{S}_{\infty,1}(f)
\end{aligned}$$

once again, because $\int_{[-\delta,\delta]^d} g_\delta(\mathbf{s}) d\mathbf{s} = 1$. But $\#\{\mathbf{k} \in \mathbb{Z}^d | K\mathbf{k} \in B_T\} \approx T^d/|K|$, also with an error of magnitude $\ll T^{d-1} \max_i K_i/|K|$ (for reasons analogous to those illustrated in Figure 4.4). Therefore

$$|S_{\text{approx}} - S| \ll \left(\frac{T^d}{|K|} \delta + \frac{T^{d-1} \max_i K_i}{|K|} \right) \mathcal{S}_{\infty,1}(f). \tag{4.22}$$

To show that S_{approx} and $\frac{T^d}{|K|} \int_X f dm_X$ are close, we observe that by Poisson summation,

$$\begin{aligned}
\sum_{\mathbf{k}\in\mathbb{Z}^d} g_\delta(\mathbf{t} - K\mathbf{k}) &= \sum_{\mathbf{k}\in\mathbb{Z}^d} g_\delta(\mathbf{t} + K\mathbf{k}) \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} \widetilde{g}_\delta(K^{-1}\mathbf{t} + \mathbf{k}) \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} \psi_{K^{-1}\mathbf{k}}(\mathbf{t}) \widehat{\widetilde{g}}_\delta(\mathbf{k})
\end{aligned} \tag{4.23}$$

where $\psi_{K^{-1}\mathbf{k}}(\mathbf{t}) = e^{2\pi i \mathbf{k}\cdot(K^{-1}\mathbf{t})} = e^{2\pi i (K^{-1}\mathbf{k})\cdot\mathbf{t}}$ and $\widehat{\widetilde{g}}_\delta$ is the multivariate Fourier transform of $\widetilde{g}_\delta(\mathbf{x}) = g_\delta(K\mathbf{x})$. When we substitute (4.23) into the definition of S_{approx} given in (4.19), we get

$$\begin{aligned}
S_{\text{approx}} &= \int_{B_T} \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} \psi_{K^{-1}\mathbf{k}}(\mathbf{t}) \widehat{\widetilde{g}}_\delta(\mathbf{k}) \right) f(x_0 u(\mathbf{t})) d\mathbf{t} \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} \widehat{\widetilde{g}}_\delta(\mathbf{k}) \left(\int_{B_T} \psi_{K^{-1}\mathbf{k}}(\mathbf{t}) f(x_0 u(\mathbf{t})) d\mathbf{t} \right)
\end{aligned}$$

where Fubini's Theorem allows us to switch the order of the sum and the integral. Similarly,

$$\begin{aligned}\frac{T^d}{|K|}\int_X f dm_X &= \left(\int_{B_T}\sum_{\mathbf{k}\in\mathbb{Z}^d} g_\delta(\mathbf{t}-K\mathbf{k})d\mathbf{t} + O\left(\frac{T^{d-1}\max_i K_i}{|K|}\right)\right)\int_X f dm_X \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^d}\widehat{\widetilde{g_\delta}}(\mathbf{k})\left(\int_{B_T}\psi_{K^{-1}\mathbf{k}}(\mathbf{t})\int_X f dm_X d\mathbf{t}\right) + O\left(\frac{T^{d-1}\max_i K_i}{|K|}\mathcal{S}_{\infty,0}(f)\right)\end{aligned}$$

where we have used that $|\int_X f dm_X| \leq \mathcal{S}_{\infty,0}(f)$. Thus

$$\begin{aligned}\left|S_{\text{approx}} - \frac{T^d}{|K|}\int_X f dm_X\right| &= \left|\sum_{\mathbf{k}\in\mathbb{Z}^d}\widehat{\widetilde{g_\delta}}(\mathbf{k})\int_{B_T} e^{2\pi i\mathbf{k}\cdot(K^{-1}\mathbf{t})}\left(f(x_0u(\mathbf{t})) - \int_X f dm_X\right)d\mathbf{t}\right| \\ &\quad + O\left(\frac{T^{d-1}\max_i K_i}{|K|}\mathcal{S}_{\infty,0}(f)\right) \\ &= \left|\sum_{\mathbf{k}\in\mathbb{Z}^d}\widehat{\widetilde{g_\delta}}(\mathbf{k})|B_T|\mu_{T,\psi_{K^{-1}\mathbf{k}}}(f)\right| + O\left(\frac{T^{d-1}\max_i K_i}{|K|}\mathcal{S}_{\infty,0}(f)\right). \quad (4.24)\end{aligned}$$

Then since $R > C$, we can apply Lemma 12 to obtain

$$\left|S_{\text{approx}} - \frac{T^d}{|K|}\int_X f dm_X\right| \ll_f T^d R^{-b}\mathcal{S}_{\infty,\ell}(f)\sum_{\mathbf{k}\in\mathbb{Z}^d}\widehat{\widetilde{g_\delta}}(\mathbf{k}) + \frac{T^{d-1}\max_i K_i}{|K|}\mathcal{S}_{\infty,0}(f)$$

(by direct computation we can see that $\widehat{\widetilde{g_\delta}}$ is positive). Observe how it was crucial here that the result in Lemma 12 was uniform over characters.

Finally, again by Poisson summation, we have

$$\begin{aligned}\sum_{\mathbf{k}\in\mathbb{Z}^d}\widehat{\widetilde{g_\delta}}(\mathbf{k}) &= \sum_{\mathbf{k}\in\mathbb{Z}^d}\widetilde{g_\delta}(\mathbf{k}) \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^d} g_\delta(K\mathbf{k}) \\ &= g_\delta(0,\ldots,0) = \delta^{-d}\end{aligned}$$

since supp $(g_\delta) \subseteq [-\delta,\delta]^d$ and $\delta < \min_i K_i/2$ implies that $g_\delta(K\mathbf{k}) = 0$ for $\mathbf{k} \neq (0,\ldots,0)$. Substituting this into equation (4.24), combining it with (4.22), and using property (i) of Sobolev norms, we get

$$\left| S - \frac{T^d}{|K|} \int_X f dm_X \right| \ll \left(T^d R^{-b} \delta^{-d} + \frac{T^d}{|K|} \delta + \frac{T^{d-1} \max_i K_i}{|K|} \right) \mathcal{S}_{\infty,\ell}(f).$$

We can optimize the first two terms by choosing $\delta = (|K|/R^b)^{1/(d+1)}$. Observe that our only restriction on δ was that $\delta < \min_i K_i/2$. This will be achieved with our choice of δ so long as $R > (2/\min K_i)^{(d+1)/b} |K|^{1/b}$. Thus, under these conditions,

$$\left| S - \frac{T^d}{|K|} \int_X f dm_X \right| \ll \left(T^d R^{-b/(d+1)} |K|^{-d/(d+1)} + \frac{T^{d-1} \max_i K_i}{|K|} \right) \mathcal{S}_{\infty,\ell}(f).$$

□

If K has all diagonal entries of equal weight (in abuse of notation, say all of weight K) then we get the following corollary which will be of use to us in the next chapter.

Corollary 14. *Let $T \geq K \geq 1$. There exists a constant $\tilde{C} > 0$ (depending only on n and d) such that for all $x_0 \in X$ satisfying (3.2c) with $T > R > \tilde{C}$, we have*

$$\left| \sum_{\substack{\mathbf{k} \in \mathbb{Z}^d \\ K\mathbf{k} \in B_T}} f(x_0 u(K\mathbf{k})) - \frac{T^d}{K^d} \int_X f dm_X \right| \ll T^d R^{-b/(d+1)} K^{-d^2/(d+1)} \mathcal{S}_{\infty,\ell}(f).$$

Proof. This is a straightforward application of the previous theorem, observing that in this case $(2/\min K_i)^{(d+1)/b} |K|^{1/b} = (2^{d+1}/K)^{1/b} \leq 2^{(d+1)/b}$ since $K \geq 1$. Thus the theorem holds with $\tilde{C} = \max(C, 2^{(d+1)/b})$. Moreover, the second error term in Theorem 13 in this case is simply $T^{d-1} K^{1-d}$, and since $K, R < T$ and $b < 1$, this term decays more quickly than the first and can be ignored. □

4.2 Γ Cocompact

For $X = \Gamma\backslash G$ where Γ is a cocompact lattice, we have the following basepoint-independent versions of Lemma 12, Theorem 13, and Corollary 14, where U is still an abelian horospherical flow as in (4.1). The proofs of these results are completely analogous to the corresponding proofs for $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$, but use the basepoint-independent equidistribution result stated in (3.20) instead of Theorem 11.

Lemma 15. *There exists $b > 0$ (depending[1] on n, d, and Γ) such that for all T large enough, we have*

$$\left|\mu_{T,\psi}(f)\right| \ll_\Gamma T^{-b}\mathcal{S}_{\infty,\ell}(f)$$

for any $f \in C^\infty(X)$, $x_0 \in X$, and additive character ψ.

Theorem 16. *Let $K = \mathrm{diag}(K_1,\cdots,K_d)$ with $T \geq K_1,\ldots,K_d \geq 1$ and determinant $|K|$. Then for all T large enough, we have*

$$\left|\sum_{\substack{\mathbf{k}\in\mathbb{Z}^d \\ K\mathbf{k}\in B_T}} f(x_0u(K\mathbf{k})) - \frac{T^d}{|K|}\int_X f\,dm_X\right| \ll_\Gamma \left(T^{d-b/(d+1)}|K|^{-d/(d+1)} + \frac{T^{d-1}\max_i K_i}{|K|}\right)\mathcal{S}_{\infty,\ell}(f)$$

for all $f \in C^\infty(X)$ and $x_0 \in X$.

[1] In this case, $b = d\beta_1\beta_2/((3n+1)(n-1)+2)(2n(n-1)(d+\beta_2)+d\beta_2)$. Since β_1 and β_2 depend on the spectral gap for the action of $\mathrm{SL}_n(\mathbb{R})$ on X we may remove dependence on Γ for $n \geq 3$ or for $n = 2$ if Γ is a congruence lattice.

Corollary 17. *Let $T \geq K \geq 1$. Then for all T large enough, we have*

$$\left| \sum_{\substack{\mathbf{k} \in \mathbb{Z}^d \\ K\mathbf{k} \in B_T}} f(x_0 u(K\mathbf{k})) - \frac{T^d}{K^d} \int_X f \, dm_X \right| \ll_\Gamma T^{d-b/(d+1)} K^{-d^2/(d+1)} \mathcal{S}_{\infty,\ell}(f)$$

for all $f \in C^\infty(X)$ and $x_0 \in X$.

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Chapter 5

Sieving and Orbits Along Almost-Primes

5.1 Γ Cocompact

Let Γ be a cocompact lattice in $G = \mathrm{SL}_n(\mathbb{R})$ and let $u(\mathbf{t})$ be an abelian horospherical flow on $X = \Gamma\backslash G$, as in Chapter 4. We know that the orbit of $u(\mathbf{t})$ equidistributes with a uniform rate for all $x_0 \in X$, and that as a consequence we have a uniform rate of equidistribution along multivariable arithmetic sequences of the form given in Corollary 17. Here and throughout this chapter, assume $\mathbf{k} = (k_1, \cdots, k_d) \in \mathbb{Z}^d$.

We want to understand the behavior of orbits at almost-prime entries of $u(\mathbf{t})$. More precisely, we want to understand averages of positive $f \in C_c^\infty(X)$ over points in B_T that have entries with fewer than a certain fixed number of primes in their prime factorization.

To investigate this question, we will use the following combinatorial sieve theorem (see [HR74] Theorem 7.4, [IK04] Sections 6.1-6.4, or [NS10] for a form more similar to that stated here).

Theorem 18 ([HR74, Theorem 7.4])**.** *Let $A = \{a_n\}$ be a sequence of nonnegative numbers and* *let* $P = P(z) = \prod_{p<z} p$ *be the product of primes less than* z. *Define*

$$S(A,P) = \sum_{(n,P)=1} a_n \qquad \text{and} \qquad S_K(A) = \sum_{n \equiv 0 \mod K} a_n.$$

Then suppose

(i) *There exists a multiplicative function* $g(K)$ *on* K *squarefree such that*

$$S_K(A) = g(K)\mathcal{X} + r_K(A)$$

and for some $c_1 > 1$, *we have* $0 \le g(p) < 1 - \frac{1}{c_1}$ *for all primes* p.

(ii) A *has level distribution* $D(\mathcal{X})$, *i.e. there is* $\varepsilon > 0$ *such that*

$$\sum_{\substack{K<D \\ \text{squarefree}}} |r_K(A)| \le C_\varepsilon \mathcal{X}^{1-\varepsilon}.$$

(iii) A *has sieve dimension* r, *i.e. there exists a constant* $c_2 > 0$ *such that for all* $2 \le w \le z$, *we have*

$$-c_2 \le \sum_{w \le p \le z} g(p) \log p - r \log \frac{z}{w} \le c_2.$$

Then for $s > 9r$, $z = D^{1/s}$, *and* $\mathcal{X}$ *large enough (depending on* ε, C_ε, *and* r*), we have*

$$S(A,P) \asymp_{c_1,c_2,r} \frac{\mathcal{X}}{(\log \mathcal{X})^r}.$$

In the context of our problem, we want to define

$$S(A,P) := \sum_{\substack{\mathbf{k}\in B_T \\ \gcd(k_1\cdots k_d,P)=1}} f(x_0u(\mathbf{k}))$$

where $f \in C_c^\infty(X)$, $f \geq 0$, and P is the product of primes less than z (to be determined). That is, we are summing over integer points in B_T with entries containing no primes smaller than z in their prime factorizations. Then let

$$A = \{a_n\} := \left\{ \sum_{\substack{\mathbf{k}\in B_T \\ k_1\cdots k_d=n}} f(x_0u(\mathbf{k})) \right\}$$

and observe that

$$S_K(A) := \sum_{n\equiv 0 \mod K} \sum_{\substack{\mathbf{k}\in B_T \\ k_1\cdots k_d=n}} f(x_0u(\mathbf{k})) = \sum_{\substack{\mathbf{k}\in \tilde{B}_T \\ K|k_1k_2\cdots k_d}} f(x_0u(\mathbf{k}))$$

where $\tilde{B}_T = (0,T]^d$ (since the index n starts at 1 we want to avoid counting terms of the form K divides 0).

Notice that $K|k_1\cdots k_d$ if and only if $K|k_1\cdots(k_i+K)\cdots k_d$, that is, the collection of points that we are summing over is periodic with period K in each coordinate. Thus we can rewrite $S_K(A)$ as a sum over cubic grids of side length K based at each point in the first box B_K:

$$\sum_{\substack{\mathbf{k}\in \tilde{B}_T \\ K|k_1\cdots k_d}} f(x_0u(\mathbf{k})) = \sum_{\substack{\tilde{\mathbf{k}}\in \tilde{B}_K \\ K|\tilde{k}_1\cdots\tilde{k}_d}} \left(\sum_{K\mathbf{k}\in B_T} f(x_0u(\tilde{\mathbf{k}})u(K\mathbf{k})) + O(T^{d-1}K^{1-d}\mathcal{S}_{\infty,0}(f)) \right) \tag{5.1}$$

where the error arises from the fact that a point $\tilde{\mathbf{k}}+K\mathbf{k}$ for $K\mathbf{k}\in B_T$ may, in fact, fall outside of B_T (see Figure 5.1).

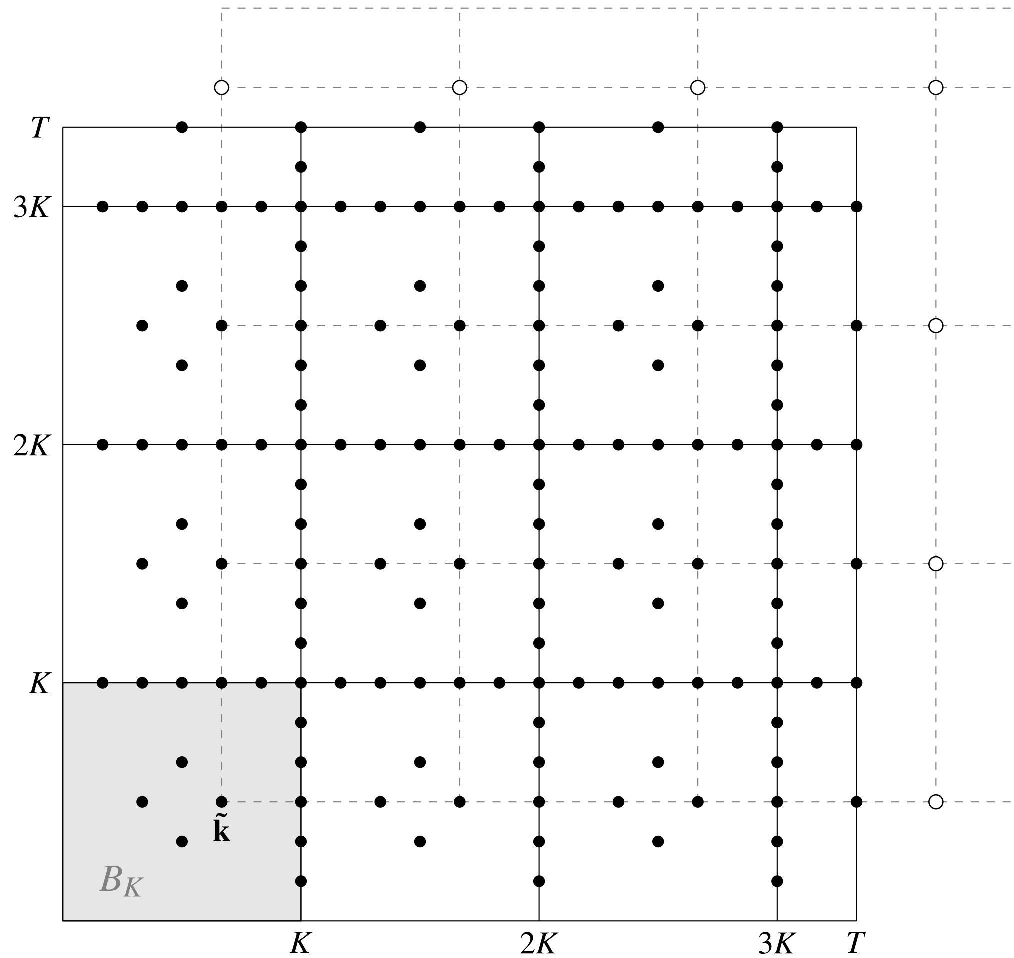

Figure 5.1: In $S_K(A)$ we are summing over the integer points in $\tilde{B}_T$ such that $K|k_1\cdots k_2$ (filled in black). We may do this by summing over shifted grids based at each of the points in the first box $\tilde{B}_K$ (filled in gray). However, this introduces an error determined by $\mathcal{S}_{\infty,0}(f)$ and the number of points in each of these shifted grids falling outside B_T (filled in white). The number of such points can be bounded by $T^{d-1}K^{1-d}$, as we have seen before.

From Corollary 17, we know that at each basepoint $x_{\tilde{\mathbf{k}}} = x_0 u(\tilde{\mathbf{k}})$, we have

$$\sum_{K\mathbf{k}\in B_T} f(x_{\tilde{\mathbf{k}}} u(K\mathbf{k})) = \frac{T^d}{K^d} \int f dm_X + O\left(T^{d-b/(d+1)} K^{-d^2/(d+1)} \mathcal{S}_{\infty,\ell}(f)\right). \tag{5.2}$$

If we let

$$G_d(K) := \#\{\mathbf{k} \in \tilde{B}_K | k_1 \cdots k_d \equiv 0 \mod K\} \tag{5.3}$$

then (5.1) together with (5.2) says that

$$S_K(A) = \sum_{\substack{\mathbf{k}\in\tilde{B}_T \\ K|k_1k_2\cdots k_d}} f(x_0 u(\mathbf{k})) = \frac{G_d(K)}{K^d} \mathcal{X} + r(f,K,T)$$

where $\mathcal{X} = T^d \int f dm_X$ and

$$|r(f,K,T)| \ll_\Gamma G_d(K) T^{d-b/(d+1)} K^{-d^2/(d+1)} \mathcal{S}_{\infty,\ell}(f).$$

This suggests that our function $g(K)$ in Theorem 18 should be $G_d(K)/K^d$. It remains to show that this function satisfies the sieve axioms (i) and (iii) and that the corresponding remainders satisfy axiom (ii) for appropriately chosen $D(\mathcal{X}) = D(T)$. Verifying these conditions gives us the following theorem.

Theorem 19. *Let u be a d-dimensional abelian horospherical flow on $X = \Gamma\backslash \mathrm{SL}_n(\mathbb{R})$ for Γ cocompact, and let P be the product of primes less than T^α for $\alpha < b/9d^2$, where b is the constant from Lemma 15. Then for any $x_0 \in X$, positive $f \in C^\infty(X)$, and T large enough (depending on α, n, d, Γ, and f), we have*

$$\sum_{\substack{\mathbf{k}\in B_T \\ \gcd(k_1\cdots k_d, P)=1}} f(x_0 u(\mathbf{k})) \asymp \left(\frac{T}{\log T}\right)^d \int f dm_X.$$

Remark. The implicit constants in the conclusion of this theorem depend only d, however b depends on n, d, and Γ, where dependence on Γ may be removed if $n \geq 3$ or if Γ is a congruence lattice.

Remark. Let $\phi(x,y)$ be the number of positive integers $\leq x$ not divisible by any prime $\leq y$ for $x \geq y \geq 2$. It is known that

$$\phi(x,y) = \frac{x\omega(\log x/\log y) - y}{\log y} + O\left(\frac{x}{(\log y)^2}\right)$$

where $\omega : [1,\infty) \to [1/2,1]$ is the Buchstab function. Thus, the number of integers in $[0,T]$ not divisible by any prime less than T^α for $\alpha < 1$ is given by

$$\phi(T,T^\alpha) = \frac{\omega(1/\alpha)T}{\alpha \log T} - \frac{T^\alpha}{\alpha \log T} + O\left(\frac{T}{(\alpha \log T)^2}\right).$$

Thus the number of points $\mathbf{k} \in B_T$ such that $\gcd(k_1 \cdots k_d, P) = 1$ where P is the product of primes less than T^α is $\phi(T,T^\alpha)^d$, which grows asymptotically like $(T/\log T)^d$ as $T \to \infty$. Although our result above only states that there is an upper and lower bound with respect to this quantity, it hints that there may be underlying equidistribution behavior.

Remark. Notice that $G_2(K) = \sum_{j=1}^{K} \gcd(K,j)$ is Pillai's arithmetical function,[1] a multiplicative function first considered by Cesàro and rediscovered by Pillai in [Pil33] which counts the number of non-congruent solutions to the equation $k_1k_2 \equiv 0 \mod K$. From the definition of G_d in (5.3), we can see that $G_d(K)$ counts the number of non-congruent solutions to $k_1k_2 \cdots k_d \equiv 0 \mod K$,

[1] This is a classical function that has been well studied. In terms of Dirichlet convolution, we have the useful identities $G_2 = \mathrm{Id} * \phi$ and $G_2 = \mu * (\mathrm{Id} \cdot \tau)$, where ϕ is Euler's totient function, μ is the Möbius function, and τ is the divisor function. In [Bro01], Broughan used this to derive a closed form for the Dirichlet series in terms of the Riemann zeta function, as well as an asymptotic formula for the partial sums of the Dirichlet series. The asymptotics for partial sums of the Dirichlet series were later refined by [Bor07a], [Bro07], and [TZ08]. The values of $G_2(K)$ for $K = 1,2,3,\ldots$ are given as sequence A018804 in the OEIS [OEIS].

so it can be considered a generalization of Pillai's arithmetical function.[2] The generalized Pillai's functions G_d have several interesting properties and interpretations that are not necessary for the proof of Theorem 19 which we have included in Appendix B for those interested.

Proof. We need to show that sieve axioms (i), (ii), and (iii) are satisfied for

$$S_K(A) := \sum_{\substack{\mathbf{k}\in\tilde{B}_T \\ K|k_1\cdots k_d}} f(x_0u(\mathbf{k})) = g(K)\mathcal{X} + r(f,K,T)$$

where $g(K) = G_d(K)/K^d$, $\mathcal{X} = T^d \int f dm_X$, and

$$|r(f,K,T)| \ll_\Gamma G_d(K)T^{d-b/(d+1)}K^{-d^2/(d+1)}\mathcal{S}_{\infty,\ell}(f).$$

For K_1 and K_2 coprime, the Chinese Remainder Theorem implies that there is a bijection between $(k_1,\cdots,k_d) \in \tilde{B}_{K_1K_2}$ such that $k_1\cdots k_d \equiv 0 \mod K_1K_2$ and $(\ell_1,\cdots,\ell_d,\ell'_1,\cdots,\ell'_d) \in \tilde{B}_{K_1}\times\tilde{B}_{K_2}$ such that $\ell_1\cdots\ell_d \equiv 0 \mod K_1$ and $\ell'_1\cdots\ell'_d \equiv 0 \mod K_2$, where k_i is the unique integer in $\{1,\cdots,d\}$ such that $k_i \equiv \ell_i \mod K_1$ and $k_i \equiv \ell'_i \mod K_2$. By counting the number of solutions in both settings, we have that $G_d(K_1K_2) = G_d(K_1)G_d(K_2)$, which shows that G_d (and hence g) is multiplicative.

For p prime and $\mathbf{k} \in \tilde{B}_p$, $k_1\cdots k_d \equiv 0 \mod p$ implies that $k_i = p$ for some i. Then the number of such solutions $G_d(p)$ will be the total number of $\mathbf{k} \in \tilde{B}_p$ except for those with $k_i \in \{1,\cdots,p-1\}$ for all $i \in \{1,\cdots,d\}$. Thus for p prime, we have

$$G_d(p) = p^d - (p-1)^d.$$

[2] Other generalizations of Pillai's arithmetical function have been studied. Examples include [CSR85], [Tót98], [Bor07b], [Hau08], and [Tót10], however none of these include the generalization given here. In [Tót13], Tòth considers a generalization that is very similar to ours, and in the notation of that paper, $G_d(K) = A_{d-1}(K)K^{d-1}$. Sieve axioms (i) and (iii) can thus be considered corollaries of results proved in [Tót13], but we prove them independently in order to keep this document self-contained. Tòth also gives a formula for the Dirichlet series of this generalization in terms of the Dirichlet series of a related arithmetic function, however we will need an explicit estimate for the partial sums of the Dirichlet series where it does not converge in order to verify sieve axiom (ii).

Therefore

$$
\begin{aligned}
0 < g(p) &= \frac{p^d - (p-1)^d}{p^d} \\
&= 1 - \left(\frac{p-1}{p}\right)^d \\
&\leq 1 - \left(\frac{2-1}{2}\right)^d = 1 - \frac{1}{2^d}
\end{aligned}
$$

since $p \geq 2$. So sieve axiom (i) is satisfied with, e.g., $c_1 = 2^{d+1}$.

For prime p, we have the bound $G_d(p) = p^d - (p-1)^d < dp^{d-1}$. By the multiplicativity of G_d, this implies that for arbitrary squarefree K, we have $G_d(K) < d^{\omega(K)} K^{d-1}$, where $\omega(K) = \Omega(K)$ is the number of (distinct) prime factors of K. Then

$$
\begin{aligned}
\sum_{\substack{K<D \\ \text{squarefree}}} |r(f,K,T)| &\ll_\Gamma \mathcal{S}_{\infty,\ell}(f) T^{d-b/(d+1)} \sum_{\substack{K<D \\ \text{squarefree}}} G_d(K) K^{-d^2/(d+1)} \\
&\ll_\Gamma \mathcal{S}_{\infty,\ell}(f) T^{d-b/(d+1)} \sum_{K<D} d^{\omega(K)} K^{-1/(d+1)}.
\end{aligned}
$$

Now observe that for any $\varepsilon_1 > 0$, $\omega(K) < \varepsilon_1 \log(K)$ for all but a finite number of squarefree K, so by appropriately adjusting the implicit constant (depending on ε_1) we may write

$$
\begin{aligned}
\sum_{K<D} |r(f,K,T)| &\ll_{\Gamma,\varepsilon_1} \mathcal{S}_{\infty,\ell}(f) T^{d-b/(d+1)} \sum_{K<D} K^{-1/(d+1)+\varepsilon_1} \\
&\ll_{\Gamma,\varepsilon_1} \mathcal{S}_{\infty,\ell}(f) T^{d-b/(d+1)} D^{d/(d+1)+\varepsilon_1}.
\end{aligned}
$$

Now if we let $D = T^\eta$ for any $\eta < b/d$, say $\eta = b/d - 2(d+1)\varepsilon$ for $\varepsilon > 0$, and set $\varepsilon_1 = d^2\varepsilon/b$, we get that

$$
\sum_{K<D} |r(f,K,T)| \ll_{\Gamma,\varepsilon} \mathcal{S}_{\infty,\ell}(f) T^{d(1-\varepsilon)} \ll_{\Gamma,f,\varepsilon} \mathcal{X}^{1-\varepsilon} \tag{5.4}
$$

which demonstrates sieve axiom (ii).

Finally, to verify sieve axiom (iii), notice that (by the binomial theorem)

$$g(p) = \frac{p^d - (p-1)^d}{p^d} = \frac{d}{p} - \sum_{i=2}^{d} \frac{a_i}{p^i} \tag{5.5}$$

where $a_i = (-1)^i \binom{d}{i}$. Since $\sum_{j=1}^{\infty} \log(j)/j^i$ converges for any $i > 1$, we have that

$$\left| \sum_{w \le p \le z} \sum_{i=2}^{d} \frac{a_i \log p}{p^i} \right| \le \sum_{i=2}^{d} |a_i| \sum_{j=1}^{\infty} \frac{\log j}{j^i} = C_2 \tag{5.6}$$

and by a corollary of the Prime Number Theorem, we know that

$$\sum_{p \le x} \frac{\log p}{p} = \log(x) + O(1).$$

Hence,

$$\begin{aligned} \sum_{p \le z} \frac{\log p}{p} - \sum_{p < w} \frac{\log p}{p} &= \log(z) - \log(w) + O(1) \\ \sum_{w \le p \le z} \frac{\log p}{p} &= \log \frac{z}{w} + O(1) \end{aligned}$$

i.e., there exists C_2' such that

$$\left| \sum_{w \le p \le z} \frac{\log p}{p} - \log \frac{z}{w} \right| \le C_2' \tag{5.7}$$

for all $2 \le w \le z$. Putting (5.5), (5.6), and (5.7) together, we see that

$$\begin{aligned} \left| \sum_{w \le p \le z} g(p) \log p - d \log \frac{z}{w} \right| &\le d \left| \sum_{w \le p \le z} \frac{\log p}{p} - \log \frac{z}{w} \right| + \left| \sum_{w \le p \le z} \sum_{i=2}^{d} \frac{a_i \log p}{p^i} \right| \\ &\le dC_2' + C_2 \end{aligned}$$

which shows that axiom (iii) is satisfied with sieve dimension $r = d$ and $c_2 = dC_2' + C_2$.

Since we have demonstrated that sieve axioms (i), (ii), and (iii) hold, we have the conclusion of Theorem 18. This, along with the various dependencies of the constants in that theorem on n, d, f, and Γ, implies our result. □

From this theorem we may easily deduce the density of almost-prime times in arbitrary horospherical flows.

Theorem 1. *Let $u(\mathbf{t})$ be a horospherical flow of dimension d on $X = \Gamma\backslash \mathrm{SL}_n(\mathbb{R})$ for Γ cocompact. Then there exists a constant M (depending only on n, d, and Γ) such that for any $x_0 \in X$, the set*

$$\{x_0 u(k_1, k_2, \cdots, k_d) \mid k_i \in \mathbb{Z} \text{ has fewer than } M \text{ prime factors}\}$$

is dense in X.

Remark. Explicitly, we may take M to be any integer satisfying

$$M > \frac{9d((3n+1)(n-1)+2)(2n(n-1)(d+\beta_2)+d\beta_2)}{\beta_1\beta_2} \tag{5.8}$$

where β_1, β_2 are as in Corollary 9 for the element a_t is as in (4.2). For example, considering an abelian horospherical flow in $\mathrm{SL}_3(\mathbb{R})$ ($n = 3$, $d = 2$, $\beta_1 = 1/2 + \varepsilon$, $\beta_2 = 1 - \varepsilon$) we may take $M = 30097$. In general, M is $O(n^4 d^2)$, and since $n - 1 \leq d \leq n(n-1)$, we have that M is $O(n^6)$ in the minimal case and $O(n^8)$ in the maximal case.

Proof. First consider $u(\mathbf{t})$ abelian. If an integer $k < T$ has no prime factors less than T^α, then it must have fewer than $1/\alpha$ prime factors total. Hence, if we take f to be a positive function supported on any small neighborhood, Theorem 19 tells us that we can take T large enough so that averaging f over integer points in B_T with no prime factors less than T^α has a positive lower bound. This means that the set of almost-prime times with fewer than M prime factors hitting any neighborhood is nonempty, where $M = 1/\alpha$. From Theorem 19, we have $\alpha < b/9d^2$, then substituting for b gives us the formula in (5.8).

To move from abelian to arbitrary horospherical flows, observe that any horospherical flow can be written as the product of a unipotent flow (not necessarily horospherical) and an abelian horospherical flow. Explicitly, an element of U given by

$$\left(\begin{array}{c|ccc} I_{m_1} & & a_{ij} & \\ \hline 0 & I_{m_2} & & b_{ij} \\ \vdots & & \ddots & \\ 0 & 0 & & I_{m_N} \end{array}\right)$$

can be expressed in the form

$$\left(\begin{array}{c|ccc} I_{m_1} & 0 & \cdots & 0 \\ \hline 0 & I_{m_2} & & b_{ij} \\ \vdots & & \ddots & \\ 0 & 0 & & I_{m_N} \end{array}\right)\left(\begin{array}{c|ccc} I_{m_1} & & a_{ij} & \\ \hline 0 & I_{m_2} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & I_{m_N} \end{array}\right).$$

Let $\tilde{d} = m_1(n - m_1)$, and let $\tilde{u}(t_1, \cdots, t_{\tilde{d}})$ represent the abelian horospherical flow on the right and $v(t_{\tilde{d}+1}, \cdots, t_d)$ represent the unipotent flow on the left, so that

$$u(t_1, \cdots, t_d) = v(t_{\tilde{d}+1}, \cdots, t_d)\tilde{u}(t_1, \cdots, t_{\tilde{d}}) \tag{5.9}$$

for any $t_1, \cdots, t_d \in \mathbb{R}^d$. Then from the density result in the abelian case, we know that the set

$$\{\tilde{x}\tilde{u}(k_1, k_2, \cdots, k_{\tilde{d}}) \mid k_i \in \mathbb{Z} \text{ has fewer than } M \text{ prime factors}\}$$

is dense for appropriately chosen[3] M and $\tilde{x} = x_0 v(k_{\tilde{d}+1}, \cdots, k_d)$, where $\{k_{\tilde{d}+1}, \cdots, k_d\}$ are any fixed almost-primes of order M. This is a subset of the almost-prime times in the larger horospherical flow, so the result follows. □

Remark. In fact, we can make Theorem 1 effective by examining the proof of Theorem 19. We will get a positive lower bound from the sieve in Theorem 18 so long as the main term surpasses the error term from sieve axiom (ii), i.e. so long as

$$C_\varepsilon \mathcal{X}^{1-\varepsilon} \ll \frac{\mathcal{X}}{(\log \mathcal{X})^r}.$$

From (5.4) we see that this means that for fixed ε (suppressing dependence on Γ) we require

$$\mathcal{S}_{\infty,\ell}(f) T^{d(1-\varepsilon)} \ll_\varepsilon \frac{T^d \int f dm_X}{(d \log T + \log \int f dm_X)^d}.$$

For any small ball of radius $r > 0$, choose f as in Lemma 4 supported on that ball. For such an f, we get a positive lower bound in Theorem 19 if

$$r^{-(\ell+(n^2-1)/2)} \ll_\varepsilon \frac{T^{d\varepsilon}}{(\log T)^d}.$$

Recall that $\ell = n(n-1)/2$. It is sufficient to require that

$$r^{-(2n^2-n-1)/2} \ll_\varepsilon T^{d\varepsilon/2}$$

$$T^{-d\varepsilon/(2n^2-n-1)} \ll_\varepsilon r.$$

[3]The dimension of the relevant abelian subgroup is $m_1(n-m_1)$, but we can always bound this in terms of d if desired.

This gives us the following effective density statement: For any $x_0 \in X$, the subset of its U-orbit consisting of integer times of norm less than T with fewer than M prime factors is $O_\varepsilon(T^{-d\varepsilon/(2n^2-n-1)})$-dense[4] in X, where $M = 9d^2/(b - 2d(d+1)\varepsilon)$.

5.2 The Space of Lattices

Now consider $X = \Gamma\backslash G$ for the non-cocompact lattice $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. Since we no longer have a uniform rate of equidistribution for our abelian horospherical flow $u(\mathbf{s})$, we will consider a basepoint $x_0 = \Gamma g_0 \in X$ satisfying a Diophantine condition of the following form.

Definition 2. *We say that $x = \Gamma g$ is* strongly polynomially δ-Diophantine *if there exists a sequence $T_i \to \infty$ as $i \to \infty$ such that*

$$\inf_{\substack{w\in\Lambda^j(\mathbb{Z}^n)\setminus\{0\}\\ j=1,\cdots,n-1}} \sup_{\mathbf{t}\in[0,T_i]^d} \|wgu(\mathbf{t})\| > T_i^\delta$$

for all $i \in \mathbb{N}$.

The motivation for this definition is that, as in the compact setting, we will want to apply sieving to learn about integer points having few prime factors. However, unlike in the compact case, we do not have a uniform rate of equidistribution, so we must consider the effect of the basepoint. For a given time-scale T, to obtain information about almost-primes of a certain order, we would want R in the basepoint condition (3.2c) to look like a small power of T (say T^δ). However, a theorem like that of Theorem 19 will require T be "large enough," which depends on the function f, and so any fixed time-scale T is insufficient. Moreover, the constant δ we are able to take at one time-scale may not work for a different time-scale, which affects the number of prime factors we allow for our almost-prime points. The condition given in Definition 2 ensures that for any function (hence any neighborhood in X) we will be able to find a time-scale large

[4] Recall that a subset is δ-dense if any ball of radius δ contains a point in the subset.

enough so that our sieving provides positive information about almost-primes of the same, fixed order.

Before moving on to the main theorem of this section, we briefly remark that this definition is a meaningful one. In view of results in [KM99], we see that not only do such points exist, but any generic point for the flow u will satisfy this definition for some positive δ.

Theorem 20. *Let u be a d-dimensional abelian horospherical flow on $X = \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$ and let $x_0 \in X$ be strongly polynomially δ-Diophantine. Let P be the product of primes less than T^α for $\alpha < \delta bn/9d(d^2+bn\kappa)$, where b is the constant from Lemma 12 and $\kappa = \min(m, n-m)$ for m as in (4.1). Then for any positive $f \in C_c^\infty(X)$ there exists a sequence $T_i \to \infty$ as $i \to \infty$ where*

$$\sum_{\substack{\mathbf{k}\in B_{T_i} \\ \gcd(k_1\cdots k_d, P)=1}} f(x_0u(\mathbf{k})) \asymp \left(\frac{T_i}{\log T_i}\right)^d \int f\, dm_X.$$

Proof. Let $f \in C_c^\infty(X)$, $f \geq 0$, and let u be an abelian horospherical flow as given in (4.1) of Chapter 4. As in the compact setting, we want to use our equidistribution theorem for arithmetic sequences to say that

$$\begin{aligned} S_K(A) :&= \sum_{\substack{\mathbf{k}\in B_{T_i} \\ K|k_1\cdots k_d}} f(x_0u(\mathbf{k})) \\ &= \sum_{\substack{\tilde{\mathbf{k}}\in \tilde{B}_K \\ K|\tilde{k}_1\cdots\tilde{k}_d}} \left(\sum_{K\mathbf{k}\in B_{T_i}} f(x_0u(\tilde{\mathbf{k}})u(K\mathbf{k})) + O(T_i^{d-1}K^{1-d}) \right) \\ &= g(K)\mathcal{X} + r(f, K, T_i) \end{aligned} \tag{5.10}$$

where $g(K) = G_d(K)/K^d$, $\mathcal{X} = T_i^d \int f\, dm_X$, and the error terms can be suitably controlled. Unfortunately, we cannot apply the same equidistribution result to the shifted basepoints $x_0u(\tilde{\mathbf{k}})$ since they will not necessarily satisfy the same Diophantine condition. However, since K is understood to be small in comparison to the T_i, all of the points in B_K lie comparatively close to x_0. Then since the Diophantine property varies continuously, we expect the points in this region to satisfy a Diophantine condition not much worse than that of x_0, and in fact we can make this quantitative.

Observe that if x_0 is strongly polynomially δ-Diophantine, it means that condition (3.2c) holds for the sequence of parameters $T = T_i$ and $R = T_i^{\delta/q}$, where $q = \sum_{\lambda_i<0} -m_i\lambda_i = d/n$ for abelian u of this form. That is, for all $j \in \{1,\cdots,n-1\}$ and $w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\}$, we have

$$\exists\, \mathbf{t} \in [0,T_i]^d \text{ s.t. } \|wg_0u(\mathbf{t})\| = \left\|wg_0u(\tilde{\mathbf{k}})u(\mathbf{t})u(-\tilde{\mathbf{k}})\right\| \geq T_i^{\delta}. \tag{5.11}$$

Recall that any $w \in \Lambda^j(\mathbb{R}^n)$ can be written as a sum $w = \sum_I w_I e_I$ over multi-indices $I = (i_1,\cdots,i_j)$ with $0 < i_j < \cdots < i_1 < n$, coefficients $w_I \in \mathbb{R}$, and basis elements $e_I = e_{i_1} \wedge \cdots \wedge e_{i_j}$ where $\{e_i\}_{1\leq i\leq n}$ is the standard basis on $\mathbb{R}^n$. Recall also that the norm above is defined by

$$\|w\| = \max_I |w_I|$$

and that G acts linearly on $\Lambda^j(\mathbb{R}^n)$ by sending a basis vector $e_{i_1} \wedge \cdots \wedge e_{i_j}$ to

$$(e_{i_1} \wedge \cdots \wedge e_{i_j})g = (e_{i_1}g) \wedge \cdots \wedge (e_{i_j}g).$$

Since our abelian horospherical subgroup has the form given in (4.1), we can write an arbitrary $u \in B_K^{-1} = u([-K,0]^d)$ as

$$u = \left(\begin{array}{c|ccc} & a_{1(m+1)} & \cdots & a_{1n} \\ I_m & \vdots & & \vdots \\ & a_{m(m+1)} & \cdots & a_{mn} \\ \hline 0 & & I_{n-m} & \end{array}\right) \tag{5.12}$$

where $a_{ij} \in [-K, 0]$ for all $1 \le i \le m$ and $m+1 \le j \le n$. One may verify that

$$e_i u = e_i + a_{i(m+1)} e_{m+1} + \cdots + a_{in} e_n$$

for $1 \le i \le m$, and

$$e_i u = e_i$$

for $m+1 \le i \le n$. Hence, when we take wedge products $(e_{i_1} u) \wedge \cdots \wedge (e_{i_j} u)$, we cannot get a coefficient of order greater than K^m, since only the first m transformed basis vectors have nontrivial coefficients and none of these coefficients have magnitude greater than K. On the other hand, we cannot get a coefficient of order larger than K^{n-m}, since only the basis vectors e_{m+1} through e_n carry nontrivial coefficients. Thus if we let $\kappa := \min\{m, n-m\}$, we find that

$$\left\|(e_{i_1} u) \wedge \cdots \wedge (e_{i_j} u)\right\| \ll K^{\kappa}.$$

Then for general $w \in \Lambda^j(\mathbb{R}^n)$ and $u \in B_K$, we have

$$\|wu\| \ll K^{\kappa} \|w\|.$$

Thus from (5.11), we can say that for any $w \in \Lambda^j(\mathbb{Z}^n) \setminus \{0\},\ j \in \{1, \cdots, n-1\}$, there exists $\mathbf{t} \in [0, T_i]^d$ such that

$$K^{\kappa} \left\|w g_0 u(\tilde{\mathbf{k}}) u(\mathbf{t})\right\| \gg \left\|w g_0 u(\tilde{\mathbf{k}}) u(\mathbf{t}) u(-\tilde{\mathbf{k}})\right\| \ge T_i^{\delta}$$

so

$$\left\|w g_0 u(\tilde{\mathbf{k}}) u(\mathbf{t})\right\| \gg T_i^{\delta} / K^{\kappa}$$

That is, for any $u(\tilde{\mathbf{k}}) \in B_K$, the shifted basepoint $x_0 u(\tilde{\mathbf{k}})$ satisfies a Diophantine condition of the form (3.2c) with new parameter proportional to $(T_i^{\delta}/K^{\kappa})^{1/q} = (T_i^{\delta}/K^{\kappa})^{n/d}$. From Corollary 14, this implies that for T_i large enough (i.e. for i large enough), we have equidistribution with

$$\sum_{K\mathbf{k}\in B_{T_i}} f(x_0 u(K\mathbf{k})) = \frac{T_i^d}{K^d}\int_X f\,dm_X + O_f(T_i^d (T_i^{\delta}/K^{\kappa})^{-nb/d(d+1)} K^{-d^2/(d+1)}) \tag{5.13}$$

for any $\mathbf{k} \in B_K$. Using this in (5.10), we find that

$$S_K(A) = g(K)\mathcal{X} + r(f,K,T_i)$$

where

$$|r(f,K,T_i)| \ll G_d(K) T_i^{d-\delta nb/d(d+1)} K^{(\kappa nb - d^3)/d(d+1)} \mathcal{S}_{\infty,\ell}(f).$$

Since we have already shown that the function $g(K) = G_d(K)/K^d$ satisfies sieve axioms (i) and (iii) with sieve dimension d, it remains to verify sieve axiom (ii). As before, we know that for any $\varepsilon_1 > 0$,

$$\begin{aligned}\sum_{K<D} |r(f,K,T_i)| &\ll_f T_i^{d-\delta nb/d(d+1)} \sum_{K<D} G_d(K)/K^{(d^3-\kappa nb)/d(d+1)} \\ &\ll_{f,\varepsilon_1} T_i^{d-\delta nb/d(d+1)} D^{(d^2+\kappa nb)/d(d+1)+\varepsilon_1}.\end{aligned}$$

Now let $D = T_i^{\eta}$ for $\eta < \delta bn/(d^2+\kappa bn)$, say, e.g.,

$$\eta = \delta bn/(d^2+\kappa bn) - 2\varepsilon d^2(d+1)/(d^2+\kappa bn)$$

for $\varepsilon > 0$, and set $\varepsilon_1 = (d^2+\kappa bn)\varepsilon/\delta bn$. Then

$$\sum_{K<D} |r(f,K,T_i)| \ll_{f,\varepsilon} T_i^{d(1-\varepsilon)} \ll_f \mathcal{X}^{1-\varepsilon}.$$

Thus for given f, the conclusion of Theorem 18 holds for all i large enough, which gives us Theorem 20. □

As before, if we consider positive f supported on a neighborhood of X, the above theorem tells us that we may take i large enough so that we have a positive lower bound on averages over almost-prime points with fewer than $1/\alpha > 9d(d^2+\kappa nb)/\delta bn$ prime factors, hence such points are dense in X. This gives us the theorem for the space of lattices from the introduction.

Theorem 3. *Let $u(\mathbf{t})$ be an abelian horospherical flow of dimension d on $X = \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$ and let $x_0 \in X$ be strongly polynomially δ-Diophantine for some $\delta > 0$. Then there exists a constant M_δ (depending[5] on δ, n, and d) such that*

$$\{x_0 u(k_1, k_2, \cdots, k_d) \mid k_i \in \mathbb{Z} \textit{ has fewer than } M_\delta \textit{ prime factors}\}$$

is dense in X.

Remark. Explicitly, we have

$$M_\delta > \frac{9d\left(d\cdot\left(nd(3n+1)(n-1)^2+2\right)\cdot(2n(n-1)(d+\beta_2)+d\beta_2)+n\kappa\beta_1\beta_2\right)}{\delta n\beta_1\beta_2}. \tag{5.14}$$

For an abelian horospherical flow in $\mathrm{SL}_3(\mathbb{R})$ ($n=3$, $d=2$, $\beta_1 = 1/2-\varepsilon$, $\beta_2 = 1-\varepsilon$, $\kappa = 1$) we may take $M_\delta = 220723\delta^{-1}$. In general, we have $M_\delta = O(n^5d^4\delta^{-1})$, so it is $O(n^9\delta^{-1})$ in the minimal case and $O(n^{13}\delta^{-1})$ in the maximal case.[6]

Remark. Unfortunately, we cannot easily generalize from abelian to arbitrary flows as we did in the compact setting. As before, we may write an arbitrary horospherical flow as

$$u(t_1, \cdots, t_d) = v(t_{\tilde{d}+1}, \cdots, t_d)\tilde{u}(t_1, \cdots, t_{\tilde{d}})$$

[5] Strictly speaking, M_δ also depends on κ, however $\kappa \le n/2$. Additionally, the constants β_1 and β_2 depend solely on n (for the a_t and $u(\mathbf{t})$ we fixed at the beginning of Chapter 4) and are $O(1)$ in any case.

[6] If we use the Remez inequality instead of Theorem 6, we have $M_\delta = O(n^5d^3\delta^{-1})$, and for the example of an abelian flow in $\mathrm{SL}_3(\mathbb{R})$, we may take $M_\delta = 111283\delta^{-1}$.

where v is unipotent, $\tilde{u}$ is abelian horospherical, and $\tilde{d} = m_1(n-m_1)$. However, it is not clear how the Diophantine condition for the flow u at the point x_0 relates to any sort of Diophantine condition for the flow $\tilde{u}$ at the point $x_0 v(k_{\tilde{d}+1}, \cdots, k_d)$ where $\{k_{\tilde{d}+1}, \cdots, k_d\}$ are almost-prime. Note that we do not consider 0 to be almost-prime, so the set of points of the form $x_0 \tilde{u}(k_1, \cdots, k_{\tilde{d}})$ is not a subset of the larger flow. Moreover, the Diophantine condition depends on the flow under consideration, and it is possible to be Diophantine for a horospherical flow but not for a horospherical subset of that flow.

Remark. As in the compact setting , it is possible to extract an effective density statement from the proof of Theorem 20 of the form: For any strongly polynomially δ-Diophantine $x_0 \in X$, there exists a sequence of times $T_i \to \infty$ as $i \to \infty$ such that the subset of its orbit consisting of integer times of norm less than T_i with fewer than M_δ prime factors is $O(T_i^{-\star})$-dense in $L_{T_i^{-\star}}$ (recall from Section 2.6 that L_ε is the compact subset of X consisting of lattices with no nonzero vectors of norm less than ε).

Chapter 5, in part, has been submitted for publication of the material as it may appear in the Journal of Modern Dynamics, 2019, McAdam, Taylor, American Institute of Mathematical Sciences (AIMS), 2018. The dissertation author was the primary investigator and author of this material.

Chapter 6

Conclusion

6.1 Directions for Future Work

There are numerous improvements and generalizations of this work that can be readily imagined, some of which are currently underway.

It seems likely that the methods used here can be generalized to quotients of connected, semisimple Lie groups by lattices. Moreover, the methods of Chapter 4 can be applied along a single (central) direction in such a way that in the other flow directions we may select any discrete set satisfying mild conditions, which in particular are satisfied for primes. For example, if $U < \mathrm{SL}_3(\mathbb{R})$ is the Heisenberg group,

$$U = \left\{ u(t_1, t_2, t_3) = \begin{pmatrix} 1 & t_1 & t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, t_1, t_2, t_3 \in \mathbb{R} \right\}$$

then the center of U is given by $\{u(0,0,t_3)\}_{t_3 \in \mathbb{R}}$, and we can show that there is an integer ℓ such that the set $\{xu(k_1,k_2,k_3) \mid k_1, k_2 \text{ prime; } k_3 \text{ has fewer than } \ell \text{ prime factors}\}$ is dense in $X = \Gamma\backslash \mathrm{SL}_3(\mathbb{R})$ for any $x \in X$ (for Γ cocompact) or for x satisfying a Diophantine condition (for

Γ non-uniform). This work is currently being prepared for publication with a collaborator.

One may also wish to remove dependence on the basepoint for the number of prime factors allowed in the almost-prime times when the lattice is non-uniform. Currently, dependence on the basepoint results from the fact that the number of prime factors depends on the rate of equidistribution for the continuous flow, which itself varies depending on the basepoint. In the noncompact setting, it is not possible to obtain a uniform rate of equidistribution for the full flow, since the rate becomes arbitrarily bad for points near a proper orbit closure. Nonetheless, Sarnak-Ubis are able to show in [SU15] that almost-prime times of a fixed order are dense in $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ for *any* nonperiodic horocycle orbit, independent of basepoint. It seems likely that the core ideas they use can be transported to a higher-dimensional setting. Namely, the "linearization" technique of Dani-Margulis [MD93] may be used to show that if a basepoint equidistributes too slowly, then it must spend a significant proportion of time near a proper orbit closure which is itself a lower-dimensional homogeneous space, such that the almost-prime times in the orbit we are considering are quantitatively close to the almost-prime times of a basepoint within this proper subspace. We hope to use an induction-type argument to show that the almost-prime times in this nearby orbit are dense in the proper subspace, and then use known results regarding the equidistribution of such proper subspaces within the whole space to complete the proof.

Some of the methods used in this work may be generalized or modified to study other interesting sequences in addition to almost-primes. For example, Venkatesh uses effective equidistribution for arithmetic sequences in the horocycle flow on compact quotients of $SL_2(\mathbb{R})$ to obtain equidistribution for sequences comprised of integer points raised to a small power [Ven10]. He does this using a Taylor series approximation, which suggests that the approach could be replicated for other sequences that can be similarly approximated. Additionally, the sieve methods used Chapter 5 can be modified to learn about averages over points $(k_1, \cdots, k_d) \in \mathbb{R}^d$ satisfying $\gcd(\mathcal{P}(k_1, \cdots, k_d), P) = 1$, where $\mathcal{P}$ is a suitably nice irreducible polynomial (note that we considered the case where $\mathcal{P}(k_1, \cdots, k_d) = k_1 k_2 \cdots k_d$). Such a result would yield a statement about integer points in horospherical flows which are not necessarily almost-prime, but which have other interesting arithmetic properties. More generally, one might hope to axiomatize the types of sets these methods work for, i.e. provide explicit conditions that guarantee density for a broad class of discrete subsets of orbits.

Another interesting extension of this work would be to try to find an analogous result for horospherical flows on the quotient of a semisimple Lie group by a discrete, geometrically finite, Zariski dense subgroup of infinite covolume, i.e. a thin group. Such spaces can be found throughout geometry and number theory and are an exciting area of current research. For example, the curvatures of circles in an Apollonian circle packing are described by the orbit of a thin group in the orthogonal group preserving a particular quadratic form, and any geometrically finite, infinite volume hyperbolic 3-manifold can be represented as the quotient of $\mathrm{Isom}^+(\mathbb{H}^3)$ by a thin Kleinian group. For these spaces, we may still define natural geometric flows such as geodesic and horospherical flows, but many basic dynamical tools break down in the infinite volume setting. In fact, in this setting neither the geodesic nor horospherical flows will even be recurrent (not to mention ergodic) with respect to the Haar measure, as most trajectories will disappear toward infinity. Nonetheless, it is possible to define a subset of the space and certain special measures supported on this subset with respect to which some of the previous dynamical methods can be applied (see, e.g., recent works of Kontorovich-Oh [KO11], Oh-Shah [OS13], Mohammadi-Oh [MO15; MO16], and McMullen-Mohammadi-Oh [MMO16; MMO17; MMO18]).

For example, let $G = \mathrm{SO}(2,1) \cong \mathrm{Isom}(\mathbb{H}^2)$, let $\Gamma < G$ be a discrete, finitely generated, Zariski-dense subgroup of infinite covolume, and let $A = \{a_t\}$ and $U = \{u_t\}$ be the usual geodesic and horocycle flows. Define the convex core to be the quotient by Γ of the smallest convex set in G containing all geodesics connecting points in the limit set corresponding to Γ. Using Patterson-Sullivan conformal densities developed in [Pat76] and [Sul79], one can define the Bowen-Margulis-Sullivan measure m^{BMS} and the Burger-Roblin measure m^{BR} on $\Gamma \backslash G$. For the above setting, m^{BMS} is a finite measure with support in the convex core of Γ that is invariant and mixing for the A-action. On the other hand, m^{BR} is an infinite measure which is the only locally-finite U-invariant ergodic measure on $\Gamma\backslash G$ that is not supported on a closed U-orbit. We may now ask questions of the form: For any U-orbit that is recurrent to the convex core of $\Gamma\backslash G$, will prime times return to the convex core infinitely often? Will they be dense in the convex core? To the author's knowledge, no attempt has yet been made to study these sorts of sparse equidistribution questions in the infinite volume setting.

6.2 Concluding Remarks

We gave an effective equidistribution result for horospherical flows on the space of lattices and an effective rate of equidistribution for arithmetic sequences of entries in abelian horospherical flows on both the space of lattices and compact quotients of $\mathrm{SL}_n(\mathbb{R})$. We then use sieve methods to derive an upper and lower bound for averages over almost-prime entries in abelian horospherical flows. In the compact setting, we have as a result the density of almost-prime times in arbitrary horospherical orbits, where the number of prime factors depends only on the dynamical system and not on the basepoint. In the space of lattices, we consider the orbits of points satisfying a Diophantine condition with parameter δ and we prove the density of almost-prime times where the number of prime factors depends on the system and on δ.

Of course, the more natural question is not what happens at almost-prime times, but what happens at prime times. It does not seem possible at present to use these methods to establish results about true primes, and additional ingredients or a wholly different approach may be required. However, the results presented here are significant in that they continue to lend support to the conjecture, already suggested by [SU15], that prime times in horospherical orbits are dense and possibly equidistributed.

Chapter 6, in part, has been submitted for publication of the material as it may appear in the Journal of Modern Dynamics, 2019, McAdam, Taylor, American Institute of Mathematical Sciences (AIMS), 2018. The dissertation author was the primary investigator and author of this material. Chapter 6, in part, is currently being prepared for submission for publication of the material. Luethi, Manuel; McAdam, Taylor. The dissertation author is one of the primary investigators and authors of this material.

Appendix A

Radius of Injection

Let $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, and $X = \Gamma \backslash G$ be the space of lattices. We want to prove the following lemma for the radius of injection.

Lemma 5. *There exist constants $c_1, c_2 > 0$ (depending only on n) such that for any $0 < \varepsilon < c_1$, the projection map*

$$\pi_x : B_r^G(e) \to B_r^X(x)$$
$$g \mapsto xg$$

is injective for all $x \in L_\varepsilon$, where $r = c_2\varepsilon^n$.

To do this, we will first need some background on Siegel sets for the action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathrm{SL}_n(\mathbb{R})$.

A.1 Siegel Sets

Let $K = \mathrm{SO}(n)$, let A be the positive diagonal subgroup, and let N be the subgroup of upper triangular unipotents. The Iwasawa decomposition of G is given by $G = NAK$. One can

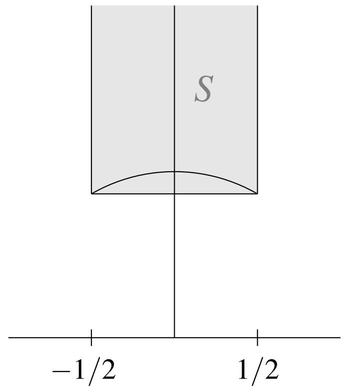

Figure A.1: The Siegel set $S = \Sigma_{1/2,2/\sqrt{3}}$ and the fundamental domain when $n = 2$, represented in the Poincaré upper half plane.

use reduction theory for arithmetic groups to find a convenient way of writing $x \in X$ in terms of particular subsets of these subgroups.

Given $\varepsilon > 0$, define

$$A_\varepsilon = \left\{ \operatorname{diag}(a_1, \cdots, a_n) \in A \,\middle|\, \frac{a_{i+1}}{a_i} \le \varepsilon \right\}$$
$$N_\varepsilon = \left\{ u \in N \,\middle|\, |u_{i,j}| \le \varepsilon \ \ \forall i < j \right\}.$$

A Siegel set for G is a set of the form $\Sigma_{s,t} := N_s A_t K$ for some $s, t > 0$.

Siegel sets can be thought of as a nice way of approximating a fundamental domain for the action of $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ on G (see Figure A.1). This approximation can be optimized in the following sense: For any $s \ge 1/2$ and $t \ge 2/\sqrt{3}$, $G = \mathrm{SL}_n(\mathbb{R})$ can be written as

$$G = \Gamma \Sigma_{s,t}$$

(for details and a proof, see [Rag72] Theorem 10.4 or [BM00] Theorem 5.1.7).

A.2 Proof of Radius of Injection

We start with the following well-known computation.

Lemma 21. *Let* $g \in \Sigma_{\frac{1}{2},\frac{2}{\sqrt{3}}}$ *satisfy* $\Gamma g \in L_\varepsilon$. *Then the operator norm of* $\mathrm{Ad}_g : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ *satisfies*

$$\left\|\mathrm{Ad}_g\right\| \ll \varepsilon^{-n}$$

where the implicit constant depends only on dimension n.

Proof. Let $g = uak$ where $u \in U_{1/2}$, $a = \mathrm{diag}(a_1, \cdots, a_n) \in A_{2/\sqrt{3}}$, and $k \in \mathrm{SO}(n)$. Let $\{e_i\}_{1\le i\le n}$ be the standard basis on $\mathbb{R}^n$ and fix $\|\cdot\|$ to be the max matrix norm on $\mathrm{Mat}_{n\times n}(\mathbb{R})$ (any other norm will work equally well).

Notice that $e_n u = e_n$ for any $u \in U$ and that $e_n a = a_n e_n$ for $a \in A$. Furthermore, since k is an orthogonal matrix, we have that $\|vk\| \le \sqrt{n}\|v\|$ for any $v \in \mathbb{R}^n$. Then, since $\Gamma uak \in L_\varepsilon$, we know that $\|vuak\| \ge \varepsilon$ for all $v \in \mathbb{Z}^n \setminus \{0\}$. In particular,

$$\varepsilon \le \|e_n uak\| \le \sqrt{n}\|e_n ua\| = \sqrt{n}\|e_n a\| = \sqrt{n}a_n\|e_n\| = \sqrt{n}a_n.$$

But since $a \in A_{2/\sqrt{3}}$, we can also say

$$\varepsilon/\sqrt{n} \le a_n \le (2/\sqrt{3})a_{n-1} \le (2/\sqrt{3})^2 a_{n-2} \le \cdots \le (2/\sqrt{3})^{n-1}a_1$$

which means that $a_i \ge C\varepsilon$ for all $1 \le i \le n$, where $C = (\sqrt{3}/2)^{n-1}/\sqrt{n}$. Moreover, since $\det a = a_1 a_2 \cdots a_n = 1$, we have that

$$a_i = \frac{1}{a_1 \cdots a_{i-1}a_{i+1}\cdots a_n} \le \frac{1}{C^{n-1}\varepsilon^{n-1}}$$

for any $1 \leq i \leq n$. This implies that for any $1 \leq i, j \leq n$, the ratio a_i/a_j can be bounded by

$$\frac{a_i}{a_j} \leq \frac{1}{C^n \varepsilon^n}.$$

But notice that for an arbitrary matrix $m \in \mathrm{Mat}_{n\times n}(\mathbb{R})$,

$$|(ama^{-1})_{ij}| = \frac{a_i}{a_j}|m_{ij}| \leq C^{-n}\varepsilon^{-n}|m_{ij}|.$$

Thus for $a \in A_{2/\sqrt{3}}$, under the max norm on matrices, we have

$$\left\|ama^{-1}\right\| \leq C^{-n}\varepsilon^{-n}\left\|m\right\|.$$

Furthermore, since $u \in U_{1/2}$, the magnitudes of all entries of u are bounded by 1. It is therefore relatively straightforward to see (via matrix multiplication) that $|(umu^{-1})_{ij}| \leq n^2 \max_{i,j}|m_{ij}|$, hence $\left\|umu^{-1}\right\| \leq n^2\left\|m\right\|$, and the same follows for $k \in K$. Thus for arbitrary $m \in \mathrm{Mat}_{n\times n}(\mathbb{R}^n)$,

$$\begin{aligned} \left\|gmg^{-1}\right\| &= \left\|uakmk^{-1}a^{-1}u^{-1}\right\| \\ &\ll \left\|akmk^{-1}a^{-1}\right\| \\ &\ll \varepsilon^{-n}\left\|kmk^{-1}\right\| \\ &\ll \varepsilon^{-n}\left\|m\right\| \end{aligned}$$

where all of the above constants depend solely on n. This implies that

$$\left\|\mathrm{Ad}(g)\right\| \ll \varepsilon^{-n}$$

as claimed. □

We may now prove our original lemma for the radius of injection.

Proof of Lemma 5. Let $x \in L_\varepsilon$. By Section A.1, we can write $x = \Gamma g$, for some $g \in \Sigma_{1/2,2/\sqrt{3}}$. Suppose $g_1, g_2 \in B_r^G(e)$ and $\pi_x(g_1) = \pi_x(g_2)$, i.e. $\Gamma g g_1 = \Gamma g g_2$. Then there exists $\gamma \in \Gamma$ such that $g g_1 = \gamma g g_2$, i.e. $g_1 = g^{-1}\gamma g g_2$. From this and left-invariance of the metric, we have that

$$\begin{aligned} d_G(e, g^{-1}\gamma g) &\leq d_G(e, g_1) + d_G(g_1, g^{-1}\gamma g) \\ &\leq r + d_G(g\gamma^{-1}g^{-1}g_1, e) \\ &\leq r + d_G(g\gamma^{-1}g^{-1}g_1, g_2) + d_G(g_2, e) \\ &\leq r + d_G(g_1, g^{-1}\gamma g g_2) + r \\ &= 2r. \end{aligned}$$

But recall that around every point in G there is a neighborhood on which the metric d_G and the metric derived from any matrix norm are Lipschitz equivalent. Hence, around the identity, for r less than some fixed value depending only on n, we have

$$\left\| e - g^{-1}\gamma g \right\| \ll d_G(e, g^{-1}\gamma g) \ll r$$

where $\|\cdot\|$ is the max norm. Finally, by Lemma 21,

$$\|e - \gamma\| = \left\| g g^{-1}(e-\gamma) g g^{-1} \right\| \ll \varepsilon^{-n} \left\| g^{-1}(e-\gamma) g \right\| = \varepsilon^{-n} \left\| e - g^{-1}\gamma g \right\| \ll r/\varepsilon^n.$$

Thus for a correctly chosen constant c_2, $r = c_2 \varepsilon^n$ implies that

$$\|e - \gamma\| < 1.$$

But since $\gamma \in \Gamma = \mathrm{SL}_n(\mathbb{Z})$ has integer entries, this can only happen if $\gamma = e$, which implies $g_1 = g_2$, so π_x is injective on $B_r^G(e)$. $\square$

Appendix A, in part, has been submitted for publication of the material as it may appear in the Journal of Modern Dynamics, 2019, McAdam, Taylor, American Institute of Mathematical Sciences (AIMS), 2018. The dissertation author was the primary investigator and author of this material.

Appendix B

Properties of the Function G_d

Recall that we defined the generalized Pillai's function $G_d : \mathbb{N} \to \mathbb{N}$ by

$$G_d(K) := \#\{\mathbf{k} \in \tilde{B}_K \mid k_1 \cdots k_d \equiv 0 \mod K\}.$$

We want to prove the following properties of this function.

Lemma 22. *For $d \geq 1$, the following hold:*

(i) *G_d is multiplicative.*

(ii) *(Behavior at primes) Let p be a prime. Then*

$$G_d(p) = p^d - (p-1)^d.$$

(iii) *(Squarefree bound) For K squarefree,*

$$G_d(K) < d^{\omega(K)} K^{d-1}.$$

(iv) *(Iterated sum formula)*

$$G_d(K) = \sum_{k_{d-1}=1}^{K} \cdots \sum_{k_1=1}^{K} \gcd(K, k_1 \cdots k_{d-1}).$$

(v) *(Recursive formula) Let* $\mathrm{Id}^d(K) = K^d$*. Then*

$$G_{d+1} = \mathrm{Id}^d * (\phi \cdot G_d).$$

(vi) *(Dirichlet series bound)*[1] *For real* $x > e$ *and* $s < d$*,*

$$\sum_{K \leq x} \frac{G_d(K)}{K^s} \ll_{s,d} x^{d-s} (\log x)^{d-1}.$$

To do this, let us first recall a few basic facts from number theory. For any function $f : \mathbb{N} \to \mathbb{R}$, we have

$$\sum_{i=1}^{K} f(\gcd(K,i)) = \sum_{j|K} f(j) \phi(K/j) \tag{B.1}$$

where ϕ is Euler's totient function, i.e. $\phi(n)$ is the number of positive integers less than n that are relatively prime with n. This formula dates back to the work of Cesàro and is sometimes referred to as Cesàro's formula (cf. [Ces85] or [Dic19]).

Recall also the definition of Dirichlet convolution: If f and g are functions on the natural numbers, then their convolution is defined by

$$(f * g)(K) := \sum_{j|K} f(j) g(K/j).$$

So, for example, (B.1) says that $\sum_{i=1}^{K} f(\gcd(K,i)) = (f * \phi)(K)$. Recall that the convolution of

[1] This property can be used as an alternative way to verify sieve axiom (ii) in the proofs of Theorems 19 and 20.

two multiplicative arithmetic functions is again multiplicative. We now have everything we need to complete the proof.

Proof. Properties (i)-(iii) were proved as part of the proof of Theorem 19. Observe that the multiplicatvity of G_d also follows directly from the recursive formula in (v) and the fact that the convolution of multiplicative functions is again multiplicative. Property (ii) can also be easily proved by induction using the iterated sum formula in (iv).

(iv) Notice that to specify a point $\mathbf{k} \in \tilde{B}_K$ such that $k_1 \cdots k_d \equiv 0 \mod K$, we can choose k_1 through k_{d-1} independently to be any integers between 1 and K, but then the remaining coordinate k_d must be a multiple of $K/\gcd(K, k_1 \cdots k_{d-1})$, that is, the last coordinate must contain all primes in K not contained in any of the previous coordinates. Since there are $\gcd(K, k_1 \cdots k_{d-1})$ multiples of $K/\gcd(K, k_1 \cdots k_{d-1})$ less than or equal to K, the total number of points counted in this way is given by

$$G_d(K) = \sum_{k_{d-1}=1}^{K} \cdots \sum_{k_1=1}^{K} \gcd(K, k_1 \cdots k_{d-1}).$$

(v) We will proceed by induction on d. For the base case, we have $G_2(K) = \mathrm{Id} * \phi = \mathrm{Id} * (\phi \cdot 1) = \mathrm{Id} * (\phi \cdot G_1)$, which is a well-known formula for Pillai's arithmetical function. Then suppose $G_d = \mathrm{Id}^{d-1} * (\phi \cdot G_{d-1})$ for $d \geq 2$ and consider G_{d+1}.

Notice that for any integers k, n, and m, we can write $\gcd(k, nm) = \gcd(k, n\gcd(k, m))$, that is, we can throw out all the primes in m that are not in k. Furthermore, since $\gcd(k, m)|k$, we can write

$$\gcd(k, n\gcd(k, m)) = \gcd(k, m)\gcd(k/\gcd(k, m), n).$$

Hence, G_{d+1} may be written

$$\begin{aligned} G_{d+1}(K) &= \sum_{k_d=1}^{K} \cdots \sum_{k_1=1}^{K} \gcd(K, k_1 \cdots k_d) \\ &= \sum_{k_d=1}^{K} \cdots \sum_{k_1=1}^{K} \gcd(K, k_d) \gcd(K/\gcd(K,k_d), k_1 \cdots k_{d-1}) \\ &= \sum_{k_d=1}^{K} \gcd(K, k_d) \left(\sum_{k_{d-1}=1}^{K} \cdots \sum_{k_1=1}^{K} \gcd(K/\gcd(K,k_d), k_1 \cdots k_{d-1}) \right) \end{aligned}$$

But now notice that the function $\gcd(K/\gcd(K,k_d), k_1 \cdots k_{d-1})$ is periodic with period $K/\gcd(K,k_d)$ in each coordinate k_i for $i = 1, \cdots, d-1$. Thus

$$\sum_{k_i=1}^{K} \gcd(K/\gcd(K,k_d), k_1 \cdots k_{d-1}) = \gcd(K, k_d) \sum_{k_i=1}^{K/\gcd(K,k_d)} \gcd(K/\gcd(K,k_d), k_1 \cdots k_{d-1})$$

for $i = 1, \cdots, d-1$. Therefore,

$$\begin{aligned} G_{d+1}(K) &= \sum_{k_d=1}^{K} \gcd(K, k_d)^d \left(\sum_{k_{d-1}=1}^{K/\gcd(K,k_d)} \cdots \sum_{k_1=1}^{K/\gcd(K,k_d)} \gcd(K/\gcd(K,k_d), k_1 \cdots k_{d-1}) \right) \\ &= \sum_{k_d=1}^{K} \gcd(K, k_d)^d G_d(K/\gcd(K,k_d)). \end{aligned}$$

But by Cesàro's formula, this is simply

$$G_{d+1}(K) = \sum_{j|K} j^d \phi(K/j) G_d(K/j)$$

Finally, we can express this in terms of Dirichlet convolution as

$$G_{d+1}(K) = (\mathrm{Id}^d * (\phi \cdot G_d))(K)$$

which completes our proof by induction.

(vi) Once again, we proceed by induction on d. Observe that for $d = 1$, we have

$$\sum_{K\leq x} \frac{G_1(K)}{K^s} = \sum_{K\leq x} \frac{1}{K^s}.$$

When $0 \leq s < 1$, we have that $1/K^s$ is decreasing, and $\sum_{K\leq x} 1/K^s \leq 1 + \int_1^x 1/t^s dt$. On the other hand, when $s < 0$, we have that $1/K^s$ is increasing, and $\sum_{K\leq x} 1/K^s \leq \int_1^{x+1} 1/t^s dt$. In either case, we have

$$\sum_{K\leq x} \frac{G_1(K)}{K^s} \ll_s x^{1-s}$$

which is the desired bound for $d = 1$.

Now suppose that for $d \geq 1$ we have $\sum_{K\leq x} G_d(K)/K^s \ll_{s,d} x^{d-s}(\log x)^{d-1}$ for all $x > e$ and $s < d$. By the complete multiplicativity of Id^{-s} and the recursive formula for G_d, we can write

$$G_{d+1}(K)/K^s = (\mathrm{Id}^{d-s} * (\mathrm{Id}^{-s} \cdot \phi \cdot G_d))(K).$$

Also note that a Dirichlet product $(f * g)(K) = \sum_{j|K} f(j)g(K/j)$ can be seen as a sum over pairs of positive integers (n,m) whose product is K, i.e.

$$(f * g)(K) = \sum_{\substack{n,m \\ nm=K}} f(n)g(m).$$

Hence, the sum

$$\sum_{K\leq x} (f * g)(K) = \sum_{\substack{n,m \\ nm\leq x}} f(n)g(m) = \sum_{n\leq x} f(n) \sum_{m\leq x/n} g(m)$$

is a sum over pairs of integers whose product is no greater than x. Also notice that $\phi(n) < n$ for any positive integer n. Thus for any $s < d+1$ and $x > e$, we may write

$$\begin{aligned}\sum_{K\le x}\frac{G_{d+1}(K)}{K^s} &= \sum_{K\le x}(\mathrm{Id}^{d-s}*(\mathrm{Id}^{-s}\cdot\phi\cdot G_d))(K)\\ &= \sum_{n\le x}\frac{1}{n^{s-d}}\sum_{m\le x/n}\frac{\phi(m)G_d(m)}{m^s}\\ &< \sum_{n\le x}\frac{1}{n^{s-d}}\sum_{m\le x/n}\frac{G_d(m)}{m^{s-1}}.\end{aligned}$$

Then since $s < d+1$, we have $s-1 < d$. Also, notice that for $n < x/e$, we have $x/n > e$, so the induction hypothesis applies to sums over $m \le x/n$ for n in this region. On the other hand, for $n \ge x/e$, we have $x/n \le e$, so a sum over $m \le x/n$ is only a sum over the first two terms, $m = 1$ and $m = 2$, and can thus be bounded by a constant (depending on s and d). Hence, we may write

$$\begin{aligned}\sum_{K\le x}\frac{G_{d+1}(K)}{K^s} &< \sum_{n\le x}\frac{1}{n^{s-d}}\sum_{m\le x/n}\frac{G_d(m)}{m^{s-1}}\\ &= \sum_{n<x/e}\frac{1}{n^{s-d}}\sum_{m\le x/n}\frac{G_d(m)}{m^{s-1}} + \sum_{x/e\le n\le x}\frac{1}{n^{s-d}}\sum_{m\le x/n}\frac{G_d(m)}{m^{s-1}}\\ &\ll_{s,d} \sum_{n<x/e}\frac{1}{n^{s-d}}(x/n)^{d+1-s}\log(x/n)^{d-1} + \sum_{x/e\le n\le x}\frac{1}{n^{s-d}}\\ &\ll_{s,d} x^{d+1-s}\sum_{n<x/e}\frac{\log(x/n)^{d-1}}{n} + \sum_{x/e\le n\le x}\frac{1}{n^{s-d}}.\end{aligned}$$

Observe that $\sum_{x/e\le n\le x}\frac{1}{n^{s-d}} \ll_{s,d} x^{d+1-s}$ (this can be seen with a calculation similar to that of the base case). On the other hand, the function $\log(x/t)^{d-1}/t$ is positive and decreasing in the region $(1, x/e)$, so we may bound the sum by the first term plus the corresponding integral:

$$\sum_{n<x/e}\frac{\log(x/n)^{d-1}}{n} \le (\log x)^{d-1} + \int_1^{x/e}\frac{\log(x/t)^{d-1}}{t}dt.$$

With the substitution $u = \log(x/t)$, we find that

$$\int_1^{x/e} \frac{\log(x/t)^{d-1}}{t} dt = \int_1^{\log x} u^{d-1} du = \frac{(\log x)^d - 1}{d}.$$

In total, we have that

$$\begin{aligned} \sum_{K \leq x} \frac{G_{d+1}(K)}{K^s} &\ll_{s,d} x^{d+1-s} \left(1 + (\log x)^{d-1} + (\log x)^d\right) \\ &\ll x^{d+1-s} (\log x)^d \end{aligned}$$

since $x > e$, and this completes the proof. $\square$

Appendix B, in part, has been submitted for publication of the material as it may appear in the Journal of Modern Dynamics, 2019, McAdam, Taylor, American Institute of Mathematical Sciences (AIMS), 2018. The dissertation author was the primary investigator and author of this material.

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