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FLUCTUATIONS OF A PLASMA (I)*

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We consider a fully ionized plasma. At time t the state of the system is represented by a point X in the phase space of all the particles. We define $D_s dX dX' \ldots dX^{(s)}$ as the joint probability that at time t the system will be in (X, dX), at time t' in (X', dX'), etc. A systematic procedure has been developed for calculating any desired moment of D_s as an expansion in the discreteness parameters e, m, and $1/n$. Spectral densities and autocorrelation functions can thus be obtained without any "Stoßzahlansatz" or Markoffian assumption. A comprehensive treatment of a plasma in thermal equilibrium has been carried out. A large class of non-equilibrium states may exist in a hot plasma for sufficient time to be considered stationary. Fluctuations have been calculated for the class of spatially homogeneous states of an infinite plasma. It is of some interest that thermal equilibrium relationships such as Kirchhoff's radiation law and the fluctuation-dissipation theorem survive. As an application we have calculated the degree of excitation of the collective modes such as plasma waves, ion oscillations, etc. For distribution functions which approach instability as some parameter is varied, the energy for some modes becomes very large and ultimately becomes infinite as instability is approached.

CONTENTS

1. Thermal equilibrium with Coulomb interactions

1.1 INTRODUCTION

The state of a plasma at time t is represented by a point in phase space $X = (\mathbf{x}_1, \mathbf{v}_1); (\mathbf{x}_2, \mathbf{v}_2) \ldots (\mathbf{x}_n, \mathbf{v}_n)$ where $\mathbf{x}_n, \mathbf{v}_n$ are position and velocity of the n-th particle. For an ensemble of systems $D_1 (X, t)\, dX$ means the probability that at time t a system will be in the volume element (X, dX) of phase space. $D_1 (X, t)$ determines the expectation value for the measurement of any observable at position $\mathbf{x}$ and time t, i. e.,

$$\langle O(\mathbf{x}, t)\rangle = \int D_1 (Xt)\, O(\mathbf{x}t)\, dX\,. \qquad (1)$$

For example $O(\mathbf{x}t) = \sum_n q_n\, \delta(\mathbf{x} - \mathbf{x}_n)$, the charge density; $\sum_n q_n \mathbf{v}_n\, \delta(\mathbf{x} - \mathbf{x}_n)$, the current density, etc. If a plasma is in thermal equilibrium, $\langle O(\mathbf{x}t)\rangle = 0$ for these quantities. However there are spontaneous fluctuations so that

$$\langle O^2(\mathbf{x}, t)\rangle = \int D_1 (X, t)\, O^2(\mathbf{x}t)\, dX \neq 0\,. \qquad (2)$$

It is possible to make more sophisticated measurements of fluctuating quantities whose expectation values are not determined by $D_1 (Xt)$. We shall be

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concerned in particular with the auto-correlation function, see, e. g. LAX [1]:

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} dt\, O(t)\, O(t+\tau) \qquad (3)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega\tau} S(\omega)\, d\omega$$

and the spectral density

$$S(\omega) = \lim_{T\to\infty} \frac{1}{T}\int\int_{-T/2}^{T/2} dt\, dt'\, e^{-i\omega(t'-t)} O(t)\, O(t') \qquad (4)$$

$$= \int_{-\infty}^{\infty} e^{-i\omega\tau} C(\tau)\, d\tau.$$

To determine the expectation value of quantities like $O(t)\,O(t')$, the state of the system at time t is insufficient. It is necessary to consider a more general description of the plasma that involves $D_2(Xt; X'\,t')\,dX\,dX'$, the probability that at time t the system will be in (X, dX), and at time t' in (X', dX'). In terms of this function the expectation value is

$$\langle O(t)\, O(t')\rangle = \int D_2(Xt; X'\,t')\, O(t)\, O(t')\, dX\, dX'. \qquad (5)$$

For a stationary random process this will depend only on $\tau = t' - t$ so that

$$\langle C(\tau)\rangle = \langle O(t)\, O(t+\tau)\rangle. \qquad (6)$$

Laplace transforms will be employed in most calculations. To express $S(\omega)$ and $C(\tau)$ in terms of Laplace transforms, consider the identity

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} dt\, e^{i(\omega'-\omega)t} S(\omega').$$

Now,

$$\int_{0}^{\infty} dt\, e^{i(\omega'-\omega)t} = \pi\,\delta(\omega'-\omega) + i\,\mathsf{P}\frac{1}{\omega'-\omega}$$

$$\int_{-\infty}^{0} dt\, e^{i(\omega'-\omega)t} = \pi\,\delta(\omega'-\omega) - i\,\mathsf{P}\frac{1}{\omega'-\omega},$$

where P means the principal part. Let

$$S^+(i\omega) = \frac{S(\omega)}{2} + \frac{i}{2\pi}\,\mathsf{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\omega'-\omega} S(\omega') \qquad (7)$$

$$S^-(i\omega) = \frac{S(\omega)}{2} - \frac{i}{2\pi}\,\mathsf{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\omega'-\omega} S(\omega');$$

then

$$S^+(i\omega) + S^-(i\omega) = S(\omega)$$

$$S^+(i\omega) - S^-(i\omega) = \frac{i}{\pi}\,\mathsf{P}\int_{-\infty}^{\infty} \frac{S(\omega')\, d\omega'}{\omega'-\omega}$$

$S^+(p)$ is the Laplace transform of the function $C^+(\tau)$ where

$$C^+(\tau) = C(\tau) \qquad (\tau > 0)$$
$$= 0 \qquad (\tau < 0).$$

$S^-(p)$ is the Laplace transform of the function $C^-(\tau)$ where

$$C^-(\tau) = C(\tau) \qquad (\tau < 0)$$
$$= 0 \qquad (\tau > 0).$$

For example;

$$\lim_{\gamma\to 0} \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} S^+(p)\, e^{p\tau} dp = \frac{1}{2\pi}\int S^+(i\omega)\, e^{i\omega\tau} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega\tau} \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega' \int_{0}^{\infty} d\tau'\, e^{i(\omega'-\omega)\tau'} S(\omega')$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega'\, S(\omega') \int_{0}^{\infty} d\tau'\, e^{i\omega'\tau'} \delta(\tau'-\tau)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega\tau} S(\omega) = C(\tau) \qquad (\tau > 0)$$

$$= 0 \qquad (\tau > 0).$$

Since $C(\tau)$ is an even function of τ,

$$S^-(p) = S^+(-p).$$

$S^+(p)$ is regular in the right half of the p-plane and $S^-(p)$ is regular in the left half.

According to Eq. (4), $S(\omega)$ must be real if ω is real. $S^+(i\omega)$ is, however, complex; the real and imaginary parts satisfy a dispersion relation.

$$\mathrm{Im}\,[S^+(i\omega)] = \frac{1}{\pi}\,\mathsf{P}\int \frac{d\omega'}{\omega'-\omega}\,\mathrm{Re}\,[S^+(i\omega)] \qquad (8)$$

or

$$\mathrm{Re}\,[S^+(i\omega)] = -\frac{1}{\pi}\,\mathsf{P}\int \frac{d\omega'}{\omega'-\omega}\,\mathrm{Im}\,[S^+(i\omega)].$$

The real and imaginary parts of $S^+(i\omega)$ are Hilbert transforms. An alternative way of writing Eqs. (8) is

$$\lim_{\lambda\to 0} \int_{-\infty}^{\infty} \frac{S^+(i\omega')\, d\omega'}{\omega'-\omega-i\lambda} = 0, \qquad (9)$$

in which it is clear that the equality exists because of the fact that $S^+(i\omega)$ is regular in the lower half of the ω-plane (or $S^+(p)$ is regular in the right half of the p-plane). Similarly

$$\lim_{\lambda\to 0} \int_{-\infty}^{\infty} \frac{S^-(i\omega)\, d\omega'}{\omega'-\omega+i\lambda} = 0 \qquad (10)$$

because $S^-(i\omega)$ is regular in the upper half of the ω-plane.

The spectral density and auto-correlation function can be generalized to include spatial fluctuations and also different components of a tensor. The spectral density is defined as

$$S_{\alpha\beta}(\mathbf{k},\omega)=\lim_{V,T\to\infty}\frac{1}{VT}\int_V\int_T d\mathbf{x}\,d\mathbf{x}'\,dt\,dt'\,e^{-i[\omega(t'-t)+\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})]}\times O_\alpha(\mathbf{x}\,t)O_\beta(\mathbf{x}'\,t')$$

$$=\lim\frac{1}{VT}\,O^*_\alpha(\mathbf{k},\omega)\,O_\beta(\mathbf{k},\omega)\,, \tag{11}$$

which is Hermitian. The auto-correlation function is

$$C_{\alpha\beta}(\mathbf{r},\tau)=\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\,e^{i(\omega\tau+\mathbf{k}\cdot\mathbf{r})}S_{\alpha\beta}(\mathbf{k},\omega)$$

$$=\lim_{V,T\to\infty}\frac{1}{VT}\int_{VT}d\mathbf{x}\,dt\,O_\alpha(\mathbf{x}\,t)\,O_\beta(\mathbf{x}+\mathbf{r},t+\tau)\,. \tag{12}$$

The symmetry properties of these quantities are

$$S_{\alpha\beta}(\mathbf{k},\omega)=S^*_{\beta\alpha}(\mathbf{k},\omega)=S^*_{\alpha\beta}(-\mathbf{k},-\omega)$$

$$C_{\alpha\beta}(\mathbf{r},\tau)=C_{\beta\alpha}(-\mathbf{r},-\tau)\,.$$

The total fluctuation is symmetric, i.e.,

$$C_{\alpha\beta}(0,0)=\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\,S_{\alpha\beta}(\mathbf{k},\omega)$$

$$=\lim\frac{1}{VT}\int_{VT}d\mathbf{x}\,dt\,O_\alpha(\mathbf{x}\,t)\,O_\beta(\mathbf{x}\,t)\,. \tag{13}$$

For a spatially homogeneous plasma and a stationary random process $\langle O_\alpha(\mathbf{x}t)\,O_\beta(\mathbf{x}+\mathbf{r},t+\tau)\rangle$ depends only on $\mathbf{r}$ and τ so that $C_{\alpha\beta}(\mathbf{r},\tau)=\langle O_\alpha(\mathbf{x}t)\,O_\beta(\mathbf{x}+\mathbf{r},t+\tau)\rangle$. A systematic procedure will be developed for calculating $C_{\alpha\beta}$. Fourier transforms will be employed for the spatial co-ordinates and Laplace transforms for the time. The result will be obtained in the form

$$C^+_{\alpha\beta}(\mathbf{r},\tau)=\int\frac{dp}{2\pi i}\,e^{p\tau}\int\frac{d\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{r}}S^+_{\alpha\beta}(\mathbf{k},p) \tag{14}$$

$$=C^+_{\alpha\beta}(\bar{\mathbf{r}},\tau)\quad(\tau>0)$$

$$=0\qquad(\tau<0)\,.$$

The previous discussion of the two-sided Laplace transforms may be applied to infer a Hermitian and an anti-Hermitian spectral density.

$$S^+_{\alpha\beta}(\mathbf{k},i\omega)=\frac{S_{\alpha\beta}(\mathbf{k},\omega)}{2}+\frac{i}{2\pi}\,\mathsf{P}\int\frac{d\omega'}{\omega'-\omega}\,S_{\alpha\beta}(\mathbf{k},\omega')$$

$$S^-_{\alpha\beta}(\mathbf{k},i\omega)=\frac{S_{\alpha\beta}(\mathbf{k},\omega)}{2}-\frac{i}{2\pi}\,\mathsf{P}\int\frac{d\omega'}{\omega'-\omega}\,S_{\alpha\beta}(\mathbf{k},\omega')$$

$$=S^+_{\beta\alpha}(-\mathbf{k},-i\omega)=[S^+_{\beta\alpha}(\mathbf{k},i\omega)]^*$$

The Hermitian spectral density is

$$S_{\alpha\beta}(\mathbf{k},\omega)=S^+_{\alpha\beta}(\mathbf{k},i\omega)+[S^+_{\beta\alpha}(\mathbf{k},i\omega)]^*$$

There is also an anti-Hermitian spectral density

$$A_{\alpha\beta}(\mathbf{k},\omega)=S_{\alpha\beta}(\mathbf{k},i\omega)-[S^+_{\beta\alpha}(\mathbf{k},i\omega)]^*$$

$$=\frac{i}{\pi}\,\mathsf{P}\int\frac{d\omega'}{\omega'-\omega}\,S_{\alpha\beta}(\mathbf{k},\omega')\,,$$

which is simply related to the Hilbert transform of the Hermitian spectral density. If $2\,S^+_{\alpha\beta}(\mathbf{k},i\omega)$ is symmetric, the real part will be the spectral density and the imaginary part will be its Hilbert transform.

We shall begin with a plasma consisting of electrons and randomly distributed positive ions of infinite mass. Only Coulomb forces will be considered. The calculations will be progressively generalized to include ions of finite mass, constant external magnetic field, the complete electromagnetic field and relativistic modifications. In Section 1 we shall consider only Coulomb interactions and thermal equilibrium.

1.2 Joint probability functions

$D_s(Xt;X'\,t';\cdots X^{(s)}t^{(s)})\,dX\,dX'\cdots dX^{(s)}$ means the joint probability that at time t the system will be in (X,dX), at time t' in (X',dX') etc. The entire system is trivially Markoffian so that all functions D_s can be expressed in terms of D_1 and D_2. $D_2(Xt;X'\,t')$ satisfies the Liouville equation in the variables $X'\,t'$,

$$\left\{\frac{\partial}{\partial t'}+\sum_{n=1}^{N}\mathbf{v}_{n'}\cdot\frac{\partial}{\partial\mathbf{x}_{n'}}-\frac{e^2}{m}\sum_{l\neq n}^{N}\frac{\partial}{\partial\mathbf{x}'_n}\frac{1}{|\mathbf{x}'_n-\mathbf{x}'_l|}\cdot\frac{\partial}{\partial\mathbf{v}_{n'}}\right\}D_2(Xt;X't')=0 \tag{15}$$

and the initial condition

$$D_2(Xt;X'\,t')=D_1(Xt)\,\delta(X'-X)$$

where

$$\delta(X'-X)=\prod_{n=1}^{N}\delta(\mathbf{x}'_n-\mathbf{x}_n)\,\delta(\mathbf{v}'_n-\mathbf{v}_n)\,.$$

Coulomb forces only are considered and the ions are omitted from the problem in the usual way. For present purposes it is sufficient to determine

$$W_{ij}(X_i\,t;X_j'\,t')=V^2\int D_2(Xt;X'\,t')\,(dX)^{N-1}\,(dX')^{N-1} \tag{16}$$

where all coordinates are integrated out except X_i, X_j'. The method consists of taking moments of the Liouville equation to produce chains of equations. The chains are solved by an expansion procedure in which the parameters of expansion are e, m or $1/n$ as discussed previously by Rostoker and Rosenbluth [2]. The determination of W_{ij} is very directly related to the previously discussed problem of test particles in a plasma [2].

Let

$$\psi(X_1\,t;X'\,t')=V\int D_2(Xt;X'\,t')\,(dX)^{N-1}. \tag{17}$$

ψ satisfies the Liouville equation in $(X';t')$ and the initial condition

$$\psi(X_1\,t;X'\,t')=V\,D_1(X'\,t)\,\delta(X'_1-X_1). \tag{18}$$

Assuming that $D_1(Xt)$ is symmetric with respect to the interchange of the co-ordinates of any two particles it follows that ψ is also symmetric except for particle one, i.e., particle one is a singled-out test particle. We have thus reduced the problem to the previously discussed test-particle problem except that we have different initial conditions for the present case.

The s-body functions may be defined as follows:

$$f_s(X_1 -- X_s; t) = V^s \int D_1(Xt)\,(dX)^{N-s}$$

$$F_s(X_1, t; X_2' -- X'_{s+1}, t') = V^s \int \psi(X_1 t; X' t')\, dX_1'\, dX'_{s+2} -- dX'_N$$

$$\Omega_s(X_1 t; X_1' -- X_s', t') = V^s \int \psi(X_1 t; X' t')\, dX'_{s+1} -- dX'_N. \quad (19)$$

We note that

$$W_{11}(X_1 t; X_1' t') = \Omega_1(X_1 t; X_1' t')$$

$$W_{12}(X_1 t; X_2' t') = F_1(X_1 t; X_2' t').$$

By taking moments of the Liouville equation, coupled chains of equations are obtained for F_s, Ω_s. These chains have previously been terminated by expanding in terms of the discreteness parameters [2]. For our present purposes we need to know $W_{11}^{(0)}$, $W_{12}^{(0)}$, $W_{12}^{(1)}$. The equations for these functions are as follows:

$$\left\{\frac{\partial}{\partial t'} + \mathbf{v}_1' \cdot \frac{\partial}{\partial \mathbf{x}_1'} - \frac{e}{m}\mathbf{E}_M^{(0)}(\mathbf{x}', t') \cdot \frac{\partial}{\partial \mathbf{v}_1'}\right\} W_{11}^{(0)}(X_1 t; X_1' t') = 0 \quad (20)$$

$$W_{11}^{(0)}(X, t; X_1' t) = V f(v_1)\, \delta(X_1' - X_1)$$

$$\left\{\frac{\partial}{\partial t'} + \mathbf{v}_2' \cdot \frac{\partial}{\partial \mathbf{x}_2'} - \frac{e}{m}\mathbf{E}_M^{(0)}(\mathbf{x}_2', t') \cdot \frac{\partial}{\partial \mathbf{v}_2'}\right\} W_{12}^{(0)}(X_1 t; X_2' t') = 0 \quad (21)$$

$$W_{12}^{(0)}(X_1 t; X_2' t) = f_2^{(0)}(X_1, X_2'; t) = f(v_1)\, f(v_2').$$

$\mathbf{E}_M^{(0)}$ means the macroscopic field

$$\mathbf{E}_M^{(0)}(\mathbf{x}', t') = n e \int \frac{\partial}{\partial \mathbf{x}'} \frac{1}{|\mathbf{x}' - \mathbf{x}_2'|} f(v_2')\, dX_2' = 0.$$

$$\left\{\frac{\partial}{\partial t'} + \mathbf{v}_2' \cdot \frac{\partial}{\partial \mathbf{x}_2'}\right\} W_{12}^{(1)}(X_1 t; X_2' t') - \frac{e^2}{m}\frac{\partial}{\partial \mathbf{v}_2'} f(v_2') \cdot \frac{\partial}{\partial \mathbf{x}_2'} \left\{\frac{f(v_1)}{|\mathbf{x}_2' - \mathbf{x}_1 - \mathbf{v}_1(t'-t)|} + n \int \frac{W_{12}^{(1)}(X_1 t; X_3' t')\, dX_3'}{|\mathbf{x}_2' - \mathbf{x}_3'|}\right\} = 0 \quad (22)$$

$$W_{12}^{(1)}(X_1 t; X_2' t) = -\frac{e^2}{\Theta} \frac{\exp[-|\mathbf{x}_1 - \mathbf{x}_2'|/L_D]}{|\mathbf{x}_1 - \mathbf{x}_2'|} f(v_1) f(v_2').$$

In the above equations

$$f(v) = \left(\frac{m}{2\pi\Theta}\right)^{\frac{3}{2}} \exp[-m v^2/2\Theta], \quad \frac{1}{L_D^2} = \frac{4\pi n e^2}{\Theta}.$$

The solutions of these equations are

$$W_{11}^{(0)}(X_1 t; X'_1 t') = V f(v_1)\, \delta[\mathbf{x}'_1 - \mathbf{x}_1 - \mathbf{v}_1(t'-t)]\, \delta(\mathbf{v}'_1 - \mathbf{v}_1)$$

$$W_{12}^{(0)}(X_1 t; X_2' t') = f(v_1)\, f(v_2') \quad (23)$$

$$W_{12}^{(1)}(X_1 t; X_2' t') = f(v_1)\, f(v_2') \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}_2' - \mathbf{x}_1)} \int \frac{dp}{2\pi i} e^{p(t'-t)}\, W_{pk}(\mathbf{v}_1, \mathbf{v}_2')$$

where

$$W_{pk}(\mathbf{v}_1, \mathbf{v}_2) = -\frac{1}{n(k L_D^2)\, \varepsilon(\mathbf{k}, 0)\, (p + i\mathbf{k}\cdot\mathbf{v}_2)} \times \left\{1 + \frac{i\mathbf{k}\cdot\mathbf{v}_2}{p + i\mathbf{k}\cdot\mathbf{v}_1}\left[1 - \frac{i(\mathbf{k}\cdot\mathbf{v}_1)(\pi/k)\, U(\mathbf{k}, p)}{(k L_D)^2\, \varepsilon(\mathbf{k}, p)}\right]\right\}$$

$$U(\mathbf{k}, p) = \frac{k}{\pi}\int \frac{f(v')\, d\mathbf{v}'}{p + i\mathbf{k}\cdot\mathbf{v}'}$$

$$\varepsilon(\mathbf{k}, p) = 1 - \frac{\omega_p^2}{k^2}\int \frac{i\mathbf{k}\cdot\partial f/\partial v'\, d\mathbf{v}'}{p + i\mathbf{k}\cdot\mathbf{v}'} = 1 + \frac{1}{(k L_D)^2}\left[1 - \frac{\pi p}{k} U(\mathbf{k}, p)\right].$$

1.3 Fluctuations of electric field

Consider first the total fluctuation

$$\langle E_\alpha(\mathbf{x}t)\, E_\beta(\mathbf{x}t)\rangle = \int D_1(Xt) \sum_{l,n} \frac{\partial}{\partial x_\alpha}\frac{e}{|\mathbf{x} - \mathbf{x}_l|}\frac{\partial}{\partial x_\beta}\frac{e}{|\mathbf{x} - \mathbf{x}_n|}\, dX$$

$$= n e^2 \int \frac{\partial}{\partial x_\alpha}\frac{1}{|\mathbf{x} - \mathbf{x}_1|}\frac{\partial}{\partial x_\beta}\frac{1}{|\mathbf{x} - \mathbf{x}_1|} f_1(v_1)\, dX_1 + n^2 e^2 \int \frac{\partial}{\partial x_\alpha}\frac{1}{|\mathbf{x} - \mathbf{x}_1|}\frac{\partial}{\partial x_\beta}\frac{1}{|\mathbf{x} - \mathbf{x}_2|} f_2(X_1, X_2)\, dX_1\, dX_2. \quad (24)$$

It is clear that to obtain the lowest order result consistently, f_1 is required to lowest order and f_2 to first order. Substituting the thermal equilibrium functions we obtain the result

$$\langle E_\alpha(\mathbf{x}t)\, E_\beta(\mathbf{x}t)\rangle = (4\pi e)^2 n \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{k_\alpha k_\beta}{k^4}\left\{1 - \frac{1}{[1 + (k L_D)^2]}\right\}. \quad (25)$$

The term $[1 + (k L_D)^2]^{-1}$ comes from the terms in Eq. (24) where $l \neq n$. These terms can be neglected for $k \gg 1/L_D$, but are quite important for $k \ll 1/L_D$. The energy associated with a given $\mathbf{k}$ can be obtained from

$$\frac{\langle \mathbf{E}\cdot\mathbf{E}\rangle}{8\pi} = \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{\Theta}{2}\frac{1}{1 + (k L_D)^2}. \quad (25.1)$$

The energy per degree of freedom in the electric field is evidently $\Theta/2$ for $kL_D \ll 1$ and much less for $kL_D \gg 1$.

To obtain the spectral density consider the ensemble average

$$\langle E_\alpha(\mathbf{x}t)\, E_\beta(\mathbf{x}'t')\rangle = \int D_2(Xt; X't') \sum_{l,n} \frac{\partial}{\partial x_\alpha}\frac{e}{|\mathbf{x} - \mathbf{x}_l|}\frac{\partial}{\partial \mathbf{x}_\beta'}\frac{e}{|\mathbf{x}' - \mathbf{x}_n'|}\, dX\, dX'$$

$$= \frac{ne^2}{V}\int \frac{\partial}{\partial x_\alpha}\frac{1}{|\mathbf{x} - \mathbf{x}_1|}\frac{\partial}{\partial x_\beta'}\frac{1}{|\mathbf{x}' - \mathbf{x}_1'|} W_{11}^{(0)}(X_1 t; X_1' t')\, dX_1\, dX_1' + n^2 e^2 \int \frac{\partial}{\partial x_\alpha}\frac{1}{|\mathbf{x} - \mathbf{x}_1|}\frac{\partial}{\partial \mathbf{x}_\beta'}\frac{1}{|\mathbf{x}' - \mathbf{x}_2'|} W_{12}^{(1)}(X_1 t; X_2' t_2')\, dX_1\, dX_2'.$$

After substituting from Eq. (23) and carrying out the integrations, the result is

$$\langle E_\alpha(\mathbf{x}, t)\, E_\beta(\mathbf{x}', t')\rangle = (4\pi e)^2 n \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dp}{2\pi i} \times \int \frac{dk}{(2\pi)^3} e^{p(t'-t)} e^{i\mathbf{k}\cdot(\mathbf{x}' - \mathbf{x})} \frac{k_\alpha k_\beta}{k^4}\frac{(\pi/k)\, U(\mathbf{k}, p)}{\varepsilon(k, 0)\, \varepsilon(\mathbf{k}, p)}.$$

According to the definitions of Eqs. (12.1) and (14)

$$S^+_{\alpha\beta}(k,p) = (4\pi e)^2 n \frac{k_\alpha k_\beta}{k^4} \frac{(\pi/k)\,U(\mathbf{k},p)}{\varepsilon(k,0)\,\varepsilon(\mathbf{k},p)}. \quad (26)$$

The spectral density is therefore

$$\begin{aligned} S_{\alpha\beta}(\mathbf{k},\omega) &= 2\,\mathrm{Re}\,[S^+_{\alpha\beta}(\mathbf{k},\mathrm{i}\,\omega)] \\ &= (4\pi e)^2 n \frac{k_\alpha k_\beta}{k^4} \frac{2\pi}{k} \frac{\mathrm{Re}\,[U(\mathbf{k},\mathrm{i}\,\omega)]}{|\varepsilon(\mathbf{k},\mathrm{i}\,\omega)|^2} \end{aligned} \quad (27)$$

$$\begin{aligned} U(\mathbf{k},\mathrm{i}\,\omega) &= \lim_{\lambda\to 0} \frac{k}{\pi\,\mathrm{i}} \int \frac{f(v')\,d\mathbf{v}'}{[\omega + \mathbf{k}\cdot\mathbf{v} - \mathrm{i}\,\lambda]} \\ &= \frac{k}{\pi\,\mathrm{i}} \int f(v') \Big[\pi\,\mathrm{i}\,\delta(\omega + \mathbf{k}\cdot\mathbf{v}') + \mathsf{P}\frac{1}{\omega + \mathbf{k}\cdot\mathbf{v}}\Big] d\mathbf{v}'. \end{aligned}$$

If we define $v_{\parallel} = \mathbf{k}\cdot\mathbf{v}/k$ and $m\,\bar{v}^2 = \Theta$,

$$F(v_{\parallel}) = \int f(v)\, v_\perp\, d\,v_\perp\, d\,\theta = \frac{1}{\sqrt{2\pi}\,\bar{v}} \exp\Big[-\frac{v_{\parallel}{}^2}{\bar{v}^2}\Big]$$

$$\begin{aligned} U(\mathbf{k},\mathrm{i}\,\omega) &= F(-\omega/k) + \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty} \frac{F(-\omega'/k)\,d\omega'}{(\omega' - \omega)} \\ &= \frac{1}{\bar{v}}\Big\{\frac{1}{\sqrt{2\pi}}\exp\Big[-\frac{\omega^2}{(k\bar{v})^2}\Big] - \frac{\mathrm{i}}{\pi}\Big(\frac{k\bar{v}}{\omega} + \Big(\frac{k\bar{v}}{\omega}\Big)^3 - - -\Big\}\quad (\omega \gg k\bar{v}) \\ &= \frac{1}{\bar{v}}\Big\{\frac{1}{\sqrt{2\pi}}\exp\Big[-\frac{\omega^2}{(k\bar{v})^2}\Big] - \frac{\mathrm{i}}{\pi}\Big(\frac{\omega}{k\bar{v}} + \frac{1}{3}\Big(\frac{\omega}{k\bar{v}}\Big)^3 - - -\Big\}\quad (\omega \ll k\bar{v}). \end{aligned} \quad (28)$$

When $kL_D \gg 1$, $|\varepsilon(\mathbf{k},\omega)|^2 \cong 1$; when $kL_D \ll 1$ and $\omega \gg k\,\bar{v}$

$$|\varepsilon(\mathbf{k},\mathrm{i}\omega)|^2 \cong \Big[1 - \Big(\frac{\omega_p}{\omega}\Big)^2\Big]^2 + \frac{\pi}{2}\frac{1}{(k\,L_D)^6}\exp\Big[-\frac{1}{(k\,L_D)^2}\Big]. \quad (29)$$

In this case the denominator exhibits a resonance at $\omega = \omega_p$. The spectral density $S_{\alpha\beta}(\mathbf{k},\omega)$ must satisfy the relation

$$\int_{-\infty}^{\infty} S_{\alpha\beta}(k,\omega)\frac{d\,\omega}{2\pi} = \langle E_\alpha(\mathbf{x}\,t)\,E_\beta(\mathbf{x}\,t)\rangle_k\,,$$

or

$$\frac{1}{k}\int_{-\infty}^{\infty} \frac{F(-\omega/k)}{|\varepsilon(\mathbf{k},\mathrm{i}\,\omega)|^2}\,d\omega = \frac{(k\,L_D)^2}{1 + (k\,L_D)^2}\,. \quad (30)$$

By using the asymptotic forms for $\varepsilon(\mathbf{k},\mathrm{i}\omega)$ the integration can easily be carried out and we obtain

$$\begin{aligned} \frac{1}{k}\int_{-\infty}^{\infty} \frac{F(-\omega/k)\,d\omega}{|\varepsilon(\mathbf{k},\mathrm{i}\,\omega)|^2} &\cong 1 \qquad (k\,L_D \gg 1) \\ &\cong (k\,L_D)^2 \qquad (k\,L_D \ll 1)\,. \end{aligned}$$

By carrying out the integration approximately, making use of asymptotic forms, it is apparent that only values of **k** for which weakly damped plasma oscillations exist are fully excited to the energy $\Theta/2$ per mode. It is possible to carry out the integration exactly by a contour method, but this gives less physical insight. From Eq. (27) it is apparent that $S_{\alpha\beta}(\mathbf{k},\omega)$ must be an even function of ω. Since

$$2\,S^+_{\alpha\beta}(\mathbf{k},\mathrm{i}\,\omega) = S_{\alpha\beta}(\mathbf{k},\omega) + \frac{\mathrm{i}}{\pi}\,\mathsf{P}\int_{-\infty}^{\infty}\frac{d\,\omega'}{\omega' - \omega} S_{\alpha\beta}(\mathbf{k},\omega')\,,$$

and the Hilbert transform of $S_{\alpha\beta}$ is an odd function of ω,

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{d\,\omega}{2\pi} S_{\alpha\beta}(\mathbf{k},\omega) &= \int_{-\infty}^{\infty}\frac{d\,\omega}{2\pi}\, 2\,S^+_{\alpha\beta}(\mathbf{k},\mathrm{i}\,\omega) \\ &= (4\pi e)^2 n \frac{k_\alpha k_\beta}{k^4}\frac{2\pi}{k}\frac{1}{\varepsilon(k,0)}\int_{-\infty}^{\infty}\frac{d\omega\, U(\mathbf{k},\mathrm{i}\omega)}{2\pi\,\varepsilon(\mathbf{k},\mathrm{i}\,\omega)}\,. \end{aligned}$$

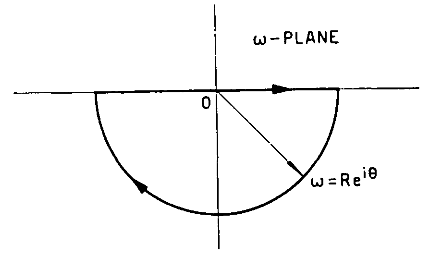

Fig. 1 Contour of integration used in evaluating the integral of Eq. (30).

The most convenient contour of integration is illustrated in Fig. 1. In the lower half of the ω-plane $U(\mathbf{k},\mathrm{i}\omega)/\varepsilon(\mathbf{k},\mathrm{i}\omega)$ has no poles. On the boundary circle $\omega = R\,e^{i\theta}$

$$\lim_{R\to\infty} U(\mathbf{k},\mathrm{i}\,\omega) = \frac{k}{\mathrm{i}\,\pi}\frac{1}{\omega}$$

$$\lim_{R\to\infty} \varepsilon(\mathbf{k},\mathrm{i}\,\omega) = 1$$

$$\int_{-\infty}^{\infty} d\omega \frac{U(k,\mathrm{i}\omega)}{\varepsilon(k,\mathrm{i}\omega)} + \frac{k}{\pi}\int_{\theta=0}^{-\pi} d\,\theta = 0\,. \quad (31)$$

Therefore

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi} S_{\alpha\beta}(k,\omega) = (4\pi e)^2 n \frac{k_\alpha k_\beta}{k^4}\frac{1}{\varepsilon(k,0)}\,,$$

which agrees with Eq. (25) since $\varepsilon(k,0)$ is equal to $(k\,L_D)^2/[1 + (k\,L_D)^2]$.

For a plasma consisting of electrons and ions, Eqs. (26) and (27) apply if we define U and ε as follows:

$$U(\mathbf{k},p) = \frac{k}{\pi}\sum_j \int \frac{f_j(v')\,d\,v'}{p + \mathrm{i}\,\mathbf{k}\cdot\mathbf{v}'} \quad (32)$$

$$\varepsilon(\mathbf{k},p) = 1 - \sum_j \frac{\omega_{pj}{}^2}{k^2}\int\frac{(\mathrm{i}\,\mathbf{k}\cdot\partial f_j/\partial\,\mathbf{v}')\,d\,\mathbf{v}'}{p + \mathrm{i}\,\mathbf{k}\cdot\mathbf{v}'}$$

$$f_j(v') = \Big(\frac{m_j}{2\pi\Theta}\Big)^{\frac{3}{2}}\exp\Big(-\frac{m_j v^2}{2\Theta}\Big)$$

$$\omega_{pj}{}^2 = 4\pi\, n e^2/m_j\,.$$

1.4 Superposition of dressed particles

Consider the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial\mathbf{x}} + \frac{e}{m}\frac{\partial\Phi}{\partial\mathbf{x}}\cdot\frac{\partial f}{\partial\mathbf{v}} = 0 \quad (33)$$

where

$$\nabla^2 \Phi = -4\pi\left[\varrho_{\text{ext}} - n e \int f d\mathbf{v} + n e\right].$$

Suppose at $t = -\infty$, $f^{(0)} = (m/2\pi\,\Theta)^{\frac{3}{2}} \exp\left[-m\,v^2/2\Theta\right]$ and an external charge density of order e is switched on adiabatically; i.e.,

$$\varrho_{\text{ext}} = \lim_{\lambda \to 0} \varrho(\mathbf{k}, \omega)\, e^{i(\omega - i\lambda)t} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

If Eqs. (33) are solved in the usual way, the result is

$$\Phi(\mathbf{x}, t) = \frac{4\pi\varrho(\mathbf{k}, \omega)}{k^2 \varepsilon(\mathbf{k}, i\omega)}\, e^{i(\omega t + \mathbf{k}\cdot\mathbf{x})} \tag{34}$$

and

$$f(\mathbf{x}, \mathbf{v}; t) = f^{(0)}(v) - \frac{e}{m}\frac{\mathbf{k}\cdot\partial f^{(0)}/\partial\mathbf{v}}{\omega + \mathbf{k}\cdot\mathbf{v} - i\lambda}\,\Phi(\mathbf{x}, t).$$

It is therefore clear that $\varepsilon(\mathbf{k}, i\omega)$ may be interpreted as a dielectric constant.

For a test charge $\varrho_{\text{ext}} = -e\,\delta(\mathbf{x} - \mathbf{x}_0 - \mathbf{v}_0 t)$ and $\varrho(\mathbf{k}, \omega) = -e\, e^{-i\mathbf{k}\cdot\mathbf{x}_0}\,\delta(\omega + \mathbf{k}\cdot\mathbf{v}_0)$. The electric field at a point $\mathbf{x}$ due to a fully "dressed" test particle at $\mathbf{x}' = \mathbf{x}_0 + \mathbf{v}_0 t$ with velocity $\mathbf{v}' = \mathbf{v}_0$ is

$$\mathbf{E}(\mathbf{x}, X') = 4\pi e \int \frac{dk}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\, \frac{i\mathbf{k}}{k^2 \varepsilon(\mathbf{k}, -i\mathbf{k}\cdot\mathbf{v}')}. \tag{35}$$

If we imagine the particles of the plasma immersed in a dielectric medium characterized by $\varepsilon(\mathbf{k}, i\omega)$, then the Coulomb electric field due to a particle is effectively replaced by Eq. (35). If this is done the particles can then be regarded as statistically independent in the following sense:

$$\begin{aligned}
&\langle E_\alpha(\mathbf{x}t)\, E_\beta(\mathbf{x}'t')\rangle \\
&= \frac{n}{V}\int E_\alpha(\mathbf{x}, X_1)\, E_\beta(\mathbf{x}', X_1')\, W_{11}^{(0)}(X_1 t; X_1' t')\, dX_1\, dX_1' \\
&= (4\pi e)^2 n \int \frac{dp}{2\pi i}\int\frac{d\mathbf{k}}{(2\pi)^3}\, e^{p(t'-t)}\, e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}\,\frac{k_\alpha k_\beta}{k^4} \\
&\quad \times \int \frac{d\mathbf{v}_1 f(v_1)}{(p + i\mathbf{k}\cdot\mathbf{v}_1)|\varepsilon(\mathbf{k}, -i\mathbf{k}\cdot\mathbf{v}_1)|^2}.
\end{aligned} \tag{36}$$

Therefore

$$\begin{aligned}
2 S_{\alpha\beta}^{+}(\mathbf{k}, i\omega) &= (4\pi e)^2 n \frac{k_\alpha k_\beta}{k^4}\frac{2\pi}{k} \\
&\times \left\{\frac{F(-\omega/k)}{|\varepsilon(\mathbf{k}, i\omega)|^2} + \frac{i}{\pi}\,\mathrm{P}\!\int_{-\infty}^{\infty}\frac{d\omega' F(-\omega'/k)}{(\omega' - \omega)\,|\varepsilon(\mathbf{k}, i\omega')|^2}\right\}.
\end{aligned}$$

This is the same as the previous result and was obtained by "dressing the particles" and neglecting the contribution from $W_{12}^{(1)}(X_1 t; X_2' t')$, the correlation of different particles.* Similarly in the calculation of

$$\langle E_\alpha(\mathbf{x}t)\, E_\beta(\mathbf{x}t)\rangle = \int D_1(Xt) \sum_{ln} E_\alpha(\mathbf{x}, \mathbf{x}_l)\, E_\beta(\mathbf{x}, \mathbf{x}_n)\, dX$$

we can neglect the terms $l \neq n$, or assume $f_2(X_1 X_2 t) = f(v_1)\, f(v_2)$ provided $\mathbf{E}(\mathbf{x}, \mathbf{x}_n)$ is replaced by $\mathbf{E}(\mathbf{x}, X_n)$.

* This method of obtaining the spectral density of electric field fluctuation was first pointed out to the author by W. B. Thompson of the Atomic Energy Research Establishment, Harwell, United Kingdom, in a lecture given at General Atomic in January 1960.

Thus

$$\begin{aligned}
\langle E_\alpha(\mathbf{x}t)\, E_\beta(\mathbf{x}t)\rangle &= n\int f(v_1)\, E_\alpha(\mathbf{x}, X_1)\, E_\beta(\mathbf{x}, X_1)\, d\mathbf{x}_1 d\mathbf{v}_1 \\
&= (4\pi e)^2 n \int\frac{dk}{(2\pi)^3}\int\frac{f(v_1)\, d\mathbf{v}_1\, k_\alpha k_\beta}{k^4 \varepsilon(\mathbf{k}, -i\mathbf{k}\cdot\mathbf{v}_1)\,\varepsilon(-\mathbf{k}, i\mathbf{k}\cdot\mathbf{v}_1)}.
\end{aligned}$$

With the change of variable $\omega = \mathbf{k}\cdot\mathbf{v}_1$ this becomes

$$\langle E_\alpha(\mathbf{x}t)\, E_\beta(\mathbf{x}t)\rangle = (4\pi e)^2 n\int\frac{dk\, d\omega}{(2\pi)^4}\frac{k_\alpha k_\beta}{k^4}\frac{2\pi}{k}\frac{F(-\omega/k)}{|\varepsilon(\mathbf{k}, i\omega)|^2}$$

in agreement with Eq. (27).

1.5 Fluctuation-dissipation theorem

Consider the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \frac{e}{m}\mathbf{E}\cdot\frac{\partial f}{\partial \mathbf{v}} = 0.$$

Suppose that at $t = -\infty$, $f = f^{(0)} = (m/2\pi\,\Theta)^{\frac{3}{2}} \exp(-m\,v^2/2\,\Theta)$ and the electric field is switched on adiabatically, i.e.,

$$\mathbf{E} = \lim_{\lambda\to 0} E_{\parallel}(\mathbf{k}/k)\, e^{i(\omega - i\lambda)t} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

It is assumed that E is of order e so that $f = f^{(0)} + f^{(1)}$ where

$$\frac{\partial f^{(1)}}{\partial t} + \mathbf{v}\cdot\frac{\partial f^{(1)}}{\partial \mathbf{x}} - \frac{e}{m}\mathbf{E}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}} = 0.$$

After solving for $f^{(1)}$, the total current density is determined

$$\begin{aligned}
\mathbf{j}(xt) &= -ne\int \mathbf{v}\, f^{(1)}(\mathbf{x}\,\mathbf{v}; t)\, d\mathbf{v} + \frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t} \\
&= j_{\parallel}(\mathbf{k}/k)\, e^{i(\omega t + \mathbf{k}\cdot\mathbf{x})}.
\end{aligned}$$

The result for the current amplitude is

$$j_{\parallel} = (1/z_{\parallel})\, E_{\parallel}$$

where

$$\begin{aligned}
z_{\parallel} = r_{\parallel} + i\,x_{\parallel} &= \frac{4\pi}{i\omega\varepsilon(\mathbf{k}, i\omega)} \\
&= \frac{4\pi}{\omega}\frac{1}{|\varepsilon(\mathbf{k}, i\omega)|^2}\left[\operatorname{Im}\varepsilon + i\operatorname{Re}\varepsilon\right].
\end{aligned}$$

Since $-\operatorname{Im}\varepsilon = (k L_D)^{-2}(\pi/k)\,\omega\, F(-\omega/k)$, the spectral density of the electric field fluctuations may be expressed as

$$S_{\alpha\beta}(\mathbf{k}, \omega) = 2\,\Theta\, r_{\parallel}\, k_\alpha k_\beta/k^2. \tag{37}$$

1.6 Kirchhoff's Radiation Law

The energy density of the electrostatic field is

$$\frac{\langle \mathbf{E}(\mathbf{x}t)\cdot\mathbf{E}(\mathbf{x}t)\rangle}{8\pi} = \int\frac{d\mathbf{k}\, d\omega}{(2\pi)^4}\, W(\mathbf{k}, \omega)$$

where

$$W(\mathbf{k}, \omega) = \frac{4\pi n e^2}{k^2}\frac{\pi}{k}\frac{F(-\omega/k)}{|\varepsilon(\mathbf{k}, i\omega)|^2}.$$

From the previous problem, the absorption coefficient may be defined as follows: the power absorption is

$$\mathfrak{P} = \frac{1}{2}\operatorname{Re} j_{\|} E_{\|}^{*} = \mathfrak{a}(\mathbf{k},\omega)\,\frac{1}{2}\operatorname{Re}\frac{E_{\|}E_{\|}^{*}}{8\pi}.$$

The absorption coefficient is

$$\mathfrak{a}(\mathbf{k},\omega) = 8\pi \operatorname{Re}(1/z_{\|}) = -2\omega \operatorname{Im}\varepsilon(\mathbf{k}, \mathrm{i}\omega) = \frac{1}{(kL_D)^2}\,\frac{2\pi}{k}\,\omega^2 F(-\omega/k).$$

According to Kirchhoff's law, we should expect that the emission per unit volume from the plasma would be

$$\mathfrak{e}(\mathbf{k},\omega) = W(\mathbf{k},\omega)\,\mathfrak{a}(\mathbf{k},\omega) = 2\pi^2 \frac{\Theta}{(kL_D)^4}\,\frac{\omega^2}{k^2}\,\frac{F^2(-\omega/k)}{|\varepsilon(\mathbf{k},\mathrm{i}\omega)|^2}. \quad (38)$$

That this is the case can be seen by a direct calculation of the emission. The force on a test particle of velocity $\mathbf{v}'$ is from Eq. (35),

$$\mathbf{F}(\mathbf{v}') = -e\mathbf{E}(\mathbf{x}'X') = -4\pi e^2 \int \frac{d\mathbf{k}}{(2\pi)^3}\,\frac{\mathrm{i}\mathbf{k}}{k^2\varepsilon(\mathbf{k},-\mathrm{i}\mathbf{k}\cdot\mathbf{v}')} = -4\pi e^2 \int \frac{d\mathbf{k}}{(2\pi)^3}\,\frac{\mathbf{k}}{k^2}\,\frac{\operatorname{Im}\varepsilon(\mathbf{k},-\mathrm{i}\mathbf{k}\cdot\mathbf{v}')}{|\varepsilon(\mathbf{k},-\mathrm{i}\mathbf{k}\cdot\mathbf{v}')|^2}.$$

The rate at which the particle loses energy is $\mathbf{v}'\cdot\mathbf{F}(v')$ which is therefore the rate of emission of energy from one particle. For a plasma in equilibrium there are $nf(v')$ particles in (X', dX') so that the total emission per unit volume is

$$4\pi n e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int d\mathbf{v}' f(v')\,\frac{\mathbf{k}\cdot\mathbf{v}'}{k^2}\,\frac{\operatorname{Im}\varepsilon(\mathbf{k},-\mathrm{i}\mathbf{k}\cdot\mathbf{v}')}{|\varepsilon(\mathbf{k},-\mathrm{i}\mathbf{k}\cdot\mathbf{v}')|^2}.$$

If we change the variable of integration to $\omega = -\mathbf{k}\cdot\mathbf{v}'$ the result is

$$\int \frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\,\mathfrak{e}(\mathbf{k},\omega)$$

where $\mathfrak{e}(\mathbf{k},\omega)$ is given by Eq. (38). Kirchhoff's law has previously been stated for plasma waves [2] in the form

$$\mathfrak{e}(k) = \Theta\, 2\,\varepsilon_L(k) \quad (39)$$

where $\varepsilon_L(k)$ is Landau's damping coefficient. This result is recovered if we integrate Eq. (38) over ω for $kL_D \ll 1$. Eq. (38) is more general than Eq. (39) in that it applies to all wavelengths including $kL_D \gg 1$ in which case plasma waves are very strongly damped.

1.7 Fluctuations of current density

The current density is $\mathbf{j} = -e\sum_n \mathbf{v}_n\,\delta(\mathbf{x}-\mathbf{x}_n)$. The ensemble average of $\langle j_\alpha(\mathbf{x}\,t)\, j_\beta(\mathbf{x}'\,t')\rangle$ is calculated making use of Eq. (23). The result is

$$S^{+}_{\alpha\beta}(\mathbf{k},\mathrm{i}\omega) = ne^2\left\{\frac{\Theta}{m}\,\frac{\pi}{k}\,U(\mathbf{k},\mathrm{i}\omega)\left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right] + \frac{k_\alpha k_\beta}{k^4}\,\mathrm{i}\omega\,(kL_D)^2\left[1 - \frac{1}{\varepsilon(\mathbf{k},\mathrm{i}\omega)}\right]\right\}. \quad (40)$$

The real part of $2\,S^{+}_{\alpha\beta}(\mathbf{k},\mathrm{i}\omega)$ is the spectral density

$$S_{\alpha\beta}(\mathbf{k},\omega) = ne^2\left\{\frac{\Theta}{m}\,\frac{2\pi}{k}\,F(-\omega/k)\left[\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right] + \frac{2\pi}{k}\,\frac{\omega^2 F(-\omega/k)}{|\varepsilon(\mathbf{k},\mathrm{i}\omega)|^2}\,\frac{k_\alpha k_\beta}{k^4}\right\}. \quad (41)$$

We note that $S_{\|}(\mathbf{k},\omega)$ differs from the corresponding quantity for the electric field by a factor of $(\omega/4\pi)^2$. This could have been anticipated because the spectral density is essentially an ensemble average of the square of Fourier components. According to Maxwell's equations $\mathrm{i}\omega\,\varrho(\mathbf{k},\omega) + \mathrm{i}\,\mathbf{k}\cdot\mathbf{j}(\mathbf{k},\omega) = 0$, $\mathrm{i}\,\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega) = 4\pi\,\varrho(\mathbf{k},\omega)$, or $\mathbf{k}\cdot\mathbf{j}(\mathbf{k},\omega) = -(\mathrm{i}\omega/4\pi)\,[\mathbf{k}\cdot\mathbf{E}(\mathbf{k},\omega)]$. The relationship between the spectral densities is thus apparent.

The above result can also be obtained by a superposition of independent dressed test particles as in Section 1.4. The current density at x due to a dressed test particle with $X_1 = (\mathbf{x}_1, \mathbf{v}_1)$ is

$$\mathbf{j}(\mathbf{x}, X_1) = -e\,\mathbf{v}_1\,\delta(\mathbf{x}-\mathbf{x}_1) - ne\int \mathbf{v}\,\delta f(\mathbf{x},\mathbf{v},t)\,d\mathbf{v}$$

where [2]

$$\delta f(\mathbf{x},\mathbf{v},t) = \frac{4\pi e^2}{m}\int \frac{dk}{(2\pi)^3}\,e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_1)} \times \left[\frac{\mathbf{k}\cdot\partial f^{(0)}/\partial\mathbf{v}}{\mathbf{k}\cdot(\mathbf{v}-\mathbf{v}_1) - \mathrm{i}\lambda}\right]\left[\frac{1}{k^2\,\varepsilon(\mathbf{k},-\mathrm{i}\mathbf{k}\cdot\mathbf{v}_1)}\right].$$

Eq. (41) may be obtained as follows:

$$\langle j_\alpha(\mathbf{x}t)\, j_\beta(\mathbf{x}',t')\rangle = \frac{n}{V}\int j_\alpha(\mathbf{x}, X_1)\, j_\beta(\mathbf{x}', X_1')\, W_{\|}^{(0)}(X_1 t; X_1' t_1)\, dX_1\, dX'_1 = \int \frac{d\mathbf{k}}{(2\pi)^3}\int \frac{dp}{2\pi\mathrm{i}}\, e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}\, e^{p(t'-t)}\, S^{+}_{\alpha\beta}(\mathbf{k},p).$$

Substituting $W_{\|}^{(0)}$ from Eq. (23) we obtain the same result for $2\operatorname{Re} S^{+}_{\alpha\beta}(\mathbf{k},\mathrm{i}\omega)$ as Eq. (41).

A fluctuation dissipation theorem exists for the current density that involves a different dissipation tensor from that previously employed for the electric field fluctuations. It is defined as follows:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial\mathbf{x}} - \frac{e}{m}\left[\mathbf{E}_{\text{ext}} - \nabla\Phi\right]\cdot\frac{\partial f}{\partial\mathbf{v}} = 0$$

$$\nabla^2\Phi = 4\pi e n\left[\int f(\mathbf{x},\mathbf{v},t)\,d\mathbf{v} - 1\right].$$

At $t = -\infty$, $f = f^{(0)}(v)$ and an external electric field of order e is switched on adiabatically, i.e.,

$$\mathbf{E}_{\text{ext}}(\mathbf{x}t) = \lim_{\lambda\to 0}\,\mathbf{E}(\mathbf{k}\,\omega)\, e^{(\mathrm{i}\omega - \mathrm{i}\lambda t)}\, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}.$$

We can calculate the conduction current as

$$\mathbf{j}(\mathbf{x}t) = -ne\int \mathbf{v}\, f^{(1)}(\mathbf{x}\,\mathbf{v},t)\,d\mathbf{v} = \mathbf{j}(\mathbf{k},\omega)\, e^{\mathrm{i}(\omega t + \mathbf{k}\cdot\mathbf{x})}.$$

where

$$j_\alpha(\mathbf{k},\omega) = \sigma_{\alpha\beta}(\mathbf{k},\mathrm{i}\omega)\, E_\beta(\mathbf{k},\omega)$$

and

$$\sigma_{\alpha\beta}(\mathbf{k},\mathrm{i}\omega) = \frac{ne^2}{\Theta}\left\{\frac{\Theta}{m}\,\frac{\pi}{k}\,U(\mathbf{k},\mathrm{i}\omega)\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) + \frac{k_\alpha k_\beta}{k^4}\,\mathrm{i}\omega\,(kL_D)^2\left[1 - \frac{1}{\varepsilon(\mathbf{k},\mathrm{i}\omega)}\right]\right\}.$$

The theorem for current fluctuations is

$$S^{+}_{\alpha\beta}(\mathbf{k},\mathrm{i}\omega) = \Theta\,\sigma_{\alpha\beta}(\mathbf{k},\mathrm{i}\omega) \quad (42)$$

or

$$S_{\alpha\beta}(\mathbf{k},\omega) = 2\,\Theta\operatorname{Re}\left[\sigma_{\alpha\beta}(\mathbf{k},\mathrm{i}\omega)\right].$$

1.8 Fluctuations with a constant magnetic field

The calculations in this case follow the same pattern as in the case of zero magnetic field. The only new feature is the addition of the term $-(e/m\,c)\,(\mathbf{v}\times\mathbf{B})\cdot(\partial f/\partial \mathbf{v})$ to the Vlasov equations. The resultant spiral unperturbed orbits make the calculations considerably more involved. However, no new techniques are required so that we shall simply discuss the results.

The dielectric constant is

$$\varepsilon(\mathbf{k},p) = 1-\frac{\omega_p^2}{k^2}\int d\mathbf{v}\sum_n \frac{J_n^2(k_\perp a)\,\mathrm{i}\,[\mathbf{k}\cdot\partial f/\partial \mathbf{v}]_n}{p+\mathrm{i}\,[\mathbf{k}\cdot\mathbf{v}]_n} \quad (43)$$

where

$$\left[\mathbf{k}\cdot\frac{\partial f}{\partial \mathbf{v}}\right]_n = k_z\frac{\partial f}{\partial v_z}+\frac{n}{a}\frac{\partial f}{\partial v_\perp}$$

$$[\mathbf{k}\cdot\mathbf{v}]_n = k_z v_z+\frac{n}{a}v_\perp$$

$$a = v_\perp/\omega_c$$

$$\omega_c = e\,B/mc.$$

We note that

$$\varepsilon(\mathbf{k},0) = 1+\frac{1}{(k L_D)^2}$$

as in the case of zero magnetic field. The other function required to express the results is

$$U(\mathbf{k},p) = \frac{k}{\pi}\int d\mathbf{v} f(\mathbf{v})\sum_n \frac{J_n^2(k_\perp a)}{p+\mathrm{i}\,[\mathbf{k}\cdot\mathbf{v}]_n}$$

with this definition

$$\varepsilon(\mathbf{k},p) = 1+(k L_D)^{-2}\,[1-(\pi\,p/k)\,U(\mathbf{k},p)].$$

The joint probability functions are as follows:

$$W_{\|}^{(0)}(X_1 t; X_1' t') = V f(v_1)\,\delta[\mathbf{x}_1'-\mathbf{x}(t')]\,\delta[\mathbf{v}_1'-\mathbf{v}(t')] \quad (44)$$

where $\tau = t'-t$ and

$$\mathbf{v}(t') = -v_{1\perp}\sin(\beta_1+\omega_c\tau)\,\mathbf{e}_x + v_{1\perp}\cos(\beta_1+\omega_c\tau)\,\mathbf{e}_y + v_{1z}\,\mathbf{e}_z$$

$$\mathbf{x}(t') = \mathbf{x}_1 + a\,[\cos(\beta_1+\omega_c\tau)-\cos\beta_1])\,\mathbf{e}_x + a\,[\sin(\beta_1+\omega_c\tau)-\sin\beta_1]\,\mathbf{e}_y + v_{1z}\,\tau\,\mathbf{e}_z.$$

$$W_{12}^{(0)}(X_1 t; X_2' t') = f(v_1)\,f(v_2')$$

$$W_{12}^{(1)}(X_1 t; X_2' t') = f(v_1)\,f(v_2')\int\frac{d\mathbf{k}}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}_2'-\mathbf{x}_1)} \times\int\frac{dp}{2\pi\mathrm{i}}\,\mathrm{e}^{p(t'-t)}\,W_{pk}(\mathbf{v}_1,\mathbf{v}_2')$$

$$W_{pk}(\mathbf{v}_1,\mathbf{v}_2) = \frac{\exp\{-i\,[k_\perp a_2\cos(\beta_2-\alpha)-k_\perp a_1\cos(\beta_1-\alpha)]\}}{n\,(k L_D)^2\,\varepsilon(k,0)} \times\sum_{n_1,n_2}\frac{\mathrm{i}^{n_2-n_1}J_{n_1}(k_\perp a_1)\,J_{n_2}(k_\perp a_2)}{[p+\mathrm{i}\,(\mathbf{k}\cdot\mathbf{v}_2)_{n_2}]}\exp\{i\,[n_2(\beta_2-\alpha)-n_1(\beta_1-\alpha)]\} \times\left\{1+\frac{\mathrm{i}\,(\mathbf{k}\cdot\mathbf{v}_2)_{n_2}}{p+\mathrm{i}\,(\mathbf{k}\cdot\mathbf{v}_1)_{n_1}}\left[1-\frac{\mathrm{i}\,(\mathbf{k}\cdot v_1)_{n_1}\,(\pi/k)\,U(\mathbf{k},p)}{(k L_D)^2\,\varepsilon(\mathbf{k},p)}\right]\right\}.$$

Most of the previous results for zero magnetic field are recovered in the following sense; results that depend only on $\mathbf{k}$ and p remain formally the same, but with the new definitions of $U(\mathbf{k},p)$ and $\varepsilon(\mathbf{k},p)$. The expressions that contain position and velocity coordinates are formally similar with the exception of the Bessel-function sums and the angular factors produced by the spiral orbits. For example, the electric field at $\mathbf{x}$ due to a fully dressed particle at position $\mathbf{x}$ and velocity $\mathbf{v}'$ is

$$\mathbf{E}(\mathbf{x},X') = 4\pi e\int\frac{d\mathbf{k}}{(2\pi)^3}\,\frac{\mathrm{i}\mathbf{k}}{k^2}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \sum_{n,n'}\frac{J_n(k_\perp a')\,J_{n'}(k_\perp a')\,\mathrm{i}^{n-n'}\mathrm{e}^{\mathrm{i}(n-n')(\beta'-\alpha)}}{\varepsilon(\mathbf{k},-\mathrm{i}\,[\mathbf{k}\cdot\mathbf{v}']_n)}. \quad (45)$$

This is to be compared with Eq. (35). The spectral density of electric field fluctuations can be calculated in the manner of Eq. (36).

$$\langle E_\alpha(\mathbf{x}t)\,E_\beta(\mathbf{x}'t')\rangle = \frac{n}{V}\int E_\alpha(x,X_1)\,E_\beta(x';X_1')\,W_{11}^{(0)}(X_1t;X_1't')\,dX_1\,dX_1' = \int\frac{dp}{2\pi\mathrm{i}}\int\frac{d\mathbf{k}}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,\mathrm{e}^{p\tau}\,S_{\alpha\beta}^{+}(\mathbf{k},p).$$

The result for $S_{\alpha\beta}(\mathbf{k},\omega) = 2\,\mathrm{Re}\,S_{\alpha\beta}^{+}(\mathbf{k},i\omega)$ is

$$S_{\alpha\beta}(\mathbf{k},\omega) = (4\pi e)^2\,n\,\frac{k_\alpha k_\beta}{k^4}\,\frac{2\pi}{k}\,\frac{\mathrm{Re}\,U(\mathbf{k},\mathrm{i}\,\omega)}{|\,\varepsilon(\mathbf{k},\mathrm{i}\,\omega)\,|^2}$$

which is formally the same as Eq. (27).

As in Section 1.5, a resistance can be defined. The only modification is the addition of the Lorentz force term to the Vlasov equation. The result is formally the same, i.e.,

$$r_{\|} = \frac{(4\pi e)^2\,n}{k^2\,\Theta}\,\frac{\pi}{k}\,\mathrm{Re}\,\frac{U(\mathbf{k},\mathrm{i}\,\omega)}{|\,\varepsilon(\mathbf{k},\mathrm{i}\,\omega)\,|^2}$$

so that, as before,

$$S_{\alpha\beta}(\mathbf{k},\omega) = 2\,\Theta\,r_{\|}\,k_\alpha\,k_\beta/k^2.$$

Similarly the formal expressions for Kirchhoff's law in Section 1.6 are unaltered.

The current density fluctuations are somewhat more involved so that a more detailed discussion will be given. The ensemble average is

$$\langle \mathbf{j}(\mathbf{x},t)\,\mathbf{j}(\mathbf{x}',t')\rangle = (ne^2/V)\int W_{11}^{(0)}(X,t;X_1't)\,\delta(\mathbf{x}-\mathbf{x}_1)\,\delta(\mathbf{x}'-\mathbf{x}_1')\,\mathbf{v}_1\,\mathbf{v}_1'\,dX_1\,dX_1' + n^2e^2\int W_{12}^{(1)}(X_1t;X_1't')\,\delta(\mathbf{x}-\mathbf{x}_1)\,\delta(\mathbf{x}'-\mathbf{x}_2')\,\mathbf{v}_1\,\mathbf{v}_2'\,dX_1\,dX_2'. \quad (46)$$

$W_{11}^{(0)}$ and $W_{12}^{(1)}$ are given by Eq. (44). It is convenient to express all vectors and tensors in terms of the unit vectors

$$\mathbf{e}_1 = \mathbf{k}/k,\quad \mathbf{e}_2 = \mathbf{k}\times\mathbf{B}/(k_\perp B)\quad\text{and}\quad \mathbf{e}_3 = \mathbf{k}\times\mathbf{B}\times\mathbf{k}/(kk_\perp B).$$

The cartesian components of $\mathbf{k}$, $\mathbf{v}$ and $\mathbf{B}$ are $(k_\perp\cos\alpha, k_\perp\sin\alpha, k_z)$; $(-v_\perp\sin\beta, v_\perp\cos\beta, v_z)$; and $(0,0,B)$.

Therefore $\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3$

where $v_1 = \mathbf{v} \cdot \mathbf{e}_1 = \frac{1}{k} [k_z v_z - k_\perp v_\perp \sin(\beta - \alpha)]$

$$v_2 = -v_\perp \cos(\beta - \alpha)$$

$$v_3 = \frac{k_\perp}{k} v_z + \frac{k_z}{k} v_\perp \sin(\beta - \alpha).$$

We can associate with these components certain symbolic components

$$v_1^{(n)} = (k_z v_z + n\omega_c)/k$$

$$v_2^{(n)} = \mathrm{i}\, v_\perp J_n'(k_\perp a)/J_n(k_\perp a)$$

$$v_3^{(n)} = \frac{k_\perp}{k} v_z - \frac{k_z}{k k_\perp} n\omega_c.$$

The purpose of the symbolic components is to express quantities like $v_\alpha \exp[\mathrm{i}\, k_\perp a \cos(\beta - \alpha)]$ in terms of Bessel-function sums. For example,

$$v_1 \exp[\mathrm{i}\, k_\perp a \cos(\beta - \alpha)]$$
$$= \sum_n J_n(k_\perp a)\, \mathrm{e}^{\mathrm{i}\, n(\beta - \alpha + \pi/2)} [k_z v_z - k_\perp v_\perp \sin(\beta - \alpha)]/k$$
$$= \sum_n J_n(k_\perp a)\, \mathrm{e}^{\mathrm{i}\, n(\beta - \alpha + \pi/2)} v_1^{(n)}.$$

The result for Eq. (46) is

$$\langle \mathbf{j}(\mathbf{x}\, t)\, \mathbf{j}(\mathbf{x}', t') \rangle = \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{dp}{2\pi \mathrm{i}} \mathrm{e}^{\mathrm{i}\, \mathbf{k} \cdot \mathbf{r}} \mathrm{e}^{p\tau} \times \sum_{\alpha\beta} S^+_{\alpha\beta}(k, p)\, \mathbf{e}_\alpha \mathbf{e}_\beta$$

where

$$S^+_{\alpha\beta}(k, p) = n e^2 \int f^{(0)}(v)\, d\mathbf{v} \sum_n \frac{J_n^2(k_\perp a)\, v_\alpha^{(n)} v_\beta^{(n)*}}{p + \mathrm{i}\, k v_1^{(n)}} + \frac{n e^2 p}{(k L_D)^2 \varepsilon(\mathbf{k}, p)} \int f^{(0)}(v)\, d\mathbf{v} \sum_n \frac{J_n^2(k_\perp a)\, v_\alpha^{(n)}}{p + \mathrm{i}\, k v_1^{(n)}} \times \int f^{(0)}(v)\, d\mathbf{v} \sum_n \frac{J_n^2(k_\perp a)\, v_\beta^{(n)*}}{p + \mathrm{i}\, k v_1^{(n)}}. \quad (47)$$

A conductivity tensor is defined as follows:

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{e}{mc} \mathbf{v} \times \mathbf{B} \cdot \frac{\partial f}{\partial \mathbf{v}} - \frac{e}{m} (\mathbf{E} - \nabla \Phi) \cdot \frac{\partial f}{\partial v} = 0$$

$$\nabla^2 \Phi = 4\pi\, ne \left[\int f\, dv - 1 \right].$$

At $t = -\infty$, $f = f^{(0)}(v)$ and the external field

$$\mathbf{E}(\mathbf{x}, t) = \lim_{\lambda \to 0} \mathbf{E}(\mathbf{k}, \omega)\, \mathrm{e}^{\mathrm{i}(\omega - \mathrm{i}\lambda) t} \mathrm{e}^{\mathrm{i}\, \mathbf{k} \cdot \mathbf{x}}$$

is switched on adiabatically, E is of order [e]. The linear response is calculated and

$$j_\alpha(\mathbf{x}, t) = -ne \int v_\alpha\, \delta f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v} = j_\alpha(\mathbf{k}, \omega)\, \mathrm{e}^{\mathrm{i}(\omega t + \mathbf{k} \cdot \mathbf{x})};$$

$$j_\alpha(\mathbf{k}, \omega) = \sigma_{\alpha\beta}(\mathbf{k}, \mathrm{i}\omega)\, E_\beta(\mathbf{k}, \omega).$$

Placing $p = \mathrm{i}\omega$, the result for $\sigma_{\alpha\beta}$ is

$$\sigma_{\alpha\beta}(\mathbf{k}, p) = \frac{ne^2}{\Theta} \Big\{ \int d\mathbf{v}\, f^{(0)}(v)\, v_\alpha \int_0^\infty v_\beta'(-\tau)\, \mathrm{e}^{\mathrm{i}\, \mathbf{k} \cdot [\mathbf{x}'(-\tau) - \mathbf{x}]} \mathrm{e}^{-p\tau} d\tau - \frac{1}{(k L_D)^2 \varepsilon(\mathbf{k}, p)} \int d\mathbf{v}\, f^{(0)}(v)\, v_\alpha \times \int_0^\infty \mathrm{i}\, \mathbf{k} \cdot \mathbf{v}'(-\tau)\, \mathrm{e}^{\mathrm{i}\, \mathbf{k} \cdot [\mathbf{x}'(-\tau) - \mathbf{x}]} \mathrm{e}^{-p\tau} d\tau \times \int_0^\infty d\mathbf{v}\, f^{(0)}(v) \int v_\beta'(-\tau)\, \mathrm{e}^{\mathrm{i}\, \mathbf{k} \cdot [\mathbf{x}'(-\tau) - \mathbf{x}]} \mathrm{e}^{-p\tau} d\tau \Big\}. \quad (48)$$

The orbit functions $\mathbf{x}'(\tau)$, $\mathbf{v}'(\tau)$ are given by Eq. (44). After making Bessel function expansions and carrying out the angular integrations, we find that $S^+_{\alpha\beta}(\mathbf{k}, p) = \Theta\, \sigma_{\alpha\beta}(\mathbf{k}, p)$ which is the fluctuation dissipation theorem.

In order to establish a superposition principle, consider the Hermitian spectral density

$$S_{\alpha\beta}(\mathbf{k}, \omega) = S^+_{\alpha\beta}(\mathbf{k}, \mathrm{i}\omega) + [S^+_{\beta\alpha}(\mathbf{k}, \mathrm{i}\omega)]^*$$

and substitute

$$\frac{1}{\omega + k v_1^{(n)}} = \pi \mathrm{i}\, \delta(\omega + k v_1^{(n)}) + \frac{\mathsf{P}}{\omega + k v_1^{(n)}}$$

into Eq. (47). The result is

$$S_{\alpha\beta}(\mathbf{k}, \omega) = 2\pi n e^2 \int d\mathbf{v}\, f^{(0)}(v) \sum_n J_n^2(k_\perp a)\, v_\alpha^{(n)} v_\beta^{(n)*}\, \delta[\omega + k v_1^{(n)}] + \frac{ne^2}{(k L_D)^2} \Big\{ \frac{\mathrm{i}\omega}{\varepsilon(\mathbf{k}, \mathrm{i}\omega)} (\xi_\alpha - \mathrm{i}\, \eta_\alpha)(\xi_\beta^* - \mathrm{i}\, \eta_\beta^*) - \frac{\mathrm{i}\omega}{\varepsilon^*(\mathbf{k}, \mathrm{i}\omega)} (\xi_\alpha + \mathrm{i}\, \eta_\alpha)(\xi_\beta^* + \mathrm{i}\, \eta_\beta^*) \Big\}$$

where

$$\xi_\alpha(\mathbf{k}, \omega) = \pi \sum_n \int d\mathbf{v}\, f^{(0)}(v)\, J_n^2(k_\perp a)\, v_\alpha^{(n)}\, \delta[\omega + k v_1^{(n)}]$$

$$\eta_\alpha(\mathbf{k}, \omega) = \sum_n \mathsf{P} \int d\mathbf{v} \frac{f^{(0)}(v)\, J_n^2(k_\perp a)\, v_\alpha^{(n)}}{[\omega + k v_1^{(n)}]}.$$

This can be put in a more suggestive form by introducing the effective velocity

$$\delta V_\alpha(\mathbf{k}, \mathrm{i}\omega) = -\frac{1}{(k L_D)^2} \frac{\mathrm{i}\omega}{\varepsilon(\mathbf{k}, \mathrm{i}\omega)} [\xi_\alpha - \mathrm{i}\, \eta_\alpha],$$

in terms of which

$$S_{\alpha\beta}(k, \omega) = 2\pi n e^2 \int d\mathbf{v}\, f^{(0)}(v) \sum_n J_n^2(k_\perp a)\, \delta[\omega + k v_1^{(n)}] \times [v_\alpha^{(n)} - \delta V_\alpha]\, [v_\beta^{(n)} - \delta V_\beta]^*. \quad (49)$$

We shall now obtain this result by superposing independent dressed test particles. The current density at x_0 due to a dressed test particle at X can be found by solving the test particle problem [2].

$$j_\alpha(\mathbf{x}_0, X) = -ev_\alpha \delta(\mathbf{x}_0 - \mathbf{x}) - ne\int v_\alpha \delta f \, d\mathbf{v} = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}_0 - \mathbf{x})} I_\alpha(\mathbf{k}, v)$$

where

$$I_\alpha(\mathbf{k}, v) = -e\, e^{-ik_\perp a\cos(\beta-\alpha)} \sum_n J_n(k_\perp a)\, e^{in(\beta-\alpha+\pi/2)} \times \left\{ v_\alpha^{(n)} - \frac{v_1^{(n)}}{(kL_D)^2 \varepsilon(\mathbf{k}, -i\mathbf{k}v_1^{(n)})} \sum_m \int d\mathbf{v}' \cdot \frac{f^{(0)}(v') J_m^2(k_\perp a') v_\alpha'^{(m)}}{[v_1'^{(m)} - v_1^{(n)} - i\lambda]} \right\}$$

$$= -e\, e^{-ik_\perp a\cos(\beta-\alpha)} \sum_n J_n(k_\perp a)\, e^{in(\beta-\alpha+\pi/2)} \times \{ v_\alpha^{(n)} - \delta V_\alpha(\mathbf{k}, -i\mathbf{k}v_1^{(n)})\}. \quad (50)$$

We can now compute

$$\langle \mathbf{j}(\mathbf{x}, t)\, \mathbf{j}(\mathbf{x}' t')\rangle = \frac{n}{V}\int \mathbf{j}(\mathbf{x}, X_1)\, \mathbf{j}(\mathbf{x}', X'_1)\, W_{11}^{(0)}(X_1 t; X_1' t')\, dX_1\, dX_1' = \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{dp}{2\pi i} e^{i\mathbf{k}\cdot\mathbf{r}} e^{p\tau} \sum_{\alpha\beta} S^+_{\alpha\beta}(\mathbf{k}, p)\, \mathbf{e}_\alpha \mathbf{e}_\beta .$$

The result is

$$S^+_{\alpha\beta}(\mathbf{k}, p) = ne^2 \int f(v)\, d\mathbf{v} \sum_n \frac{J_n^2(k_\perp a)}{p + ikv_1^{(n)}} \times [v_\alpha^{(n)} - \delta V_\alpha(\mathbf{k}, -ikv_1^{(n)})] \times [v_\beta^{(n)} - \delta V_\beta(\mathbf{k}, -ikv_1^{(n)})]^* .$$

The Hermitian spectral density clearly agrees with Eq. (49). The anti-Hermitian spectral density is the Hilbert transform of the Hermitian spectral density so that the superposition of dressed test particles gives completely equivalent results.

2. Non-equilibrium states with Coulomb forces

2.1 Introduction

A hot plasma may exist in a state quite different from thermodynamic equilibrium for a substantial length of time. Indeed, it is upon this fact that the hope for fusion power is based. Such states are approximately stationary and it is of some interest to consider fluctuations.

The states about which we shall examine fluctuations are stationary in the sense of our expansion. They will be adequately described for our present purposes by specifying the one-body distribution function to lowest order. This in turn uniquely determines the two-body correlation function. For example the one-body distribution function will be a solution of

$$\mathbf{v} \cdot \frac{\partial f^{(0)}}{\partial \mathbf{x}} - \frac{ne^2}{m} \int \frac{\partial}{\partial \mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} f^{(0)}(X')\, dX' \cdot \frac{\partial f^{(0)}}{\partial \mathbf{v}} = 0 . \quad (1)$$

We shall consider only spatially homogeneous solutions $f^{(0)} = f^{(0)}(\mathbf{v})$. There are two restrictions on the function $f^{(0)}(\mathbf{v})$. First of all, there must be no current density

$$\mathbf{j} = -ne\int f^{(0)}(\mathbf{v})\, \mathbf{v}\, d\mathbf{v} = 0,$$

because the magnetic field term is absent in Eq. (1). Second, the secular equation

$$\varepsilon(\mathbf{k}, p) = 1 - \frac{\omega_p^2}{k^2} \int \frac{i\mathbf{k} \cdot \partial f_1^{(0)}/\partial \mathbf{v}}{p + i\mathbf{k} \cdot \mathbf{v}}\, d\mathbf{v} = 0 ,$$

must have no roots in which p has a positive real part, i.e., the state $f^{(0)}(\mathbf{v})$ must be stable.

The two-body correlation function satisfies the equation

$$\left(\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + \mathbf{v}' \cdot \frac{\partial}{\partial \mathbf{x}'}\right) P(X, X') - \frac{ne^2}{m} \frac{\partial f_1^{(0)}}{\partial \mathbf{v}} \int \frac{\partial}{\partial \mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}''|} P(X', X'')\, dX'' - \frac{ne^2}{m} \frac{\partial f_1^{(0)}}{\partial \mathbf{v}'} \int \frac{\partial}{\partial \mathbf{x}'} \frac{1}{|\mathbf{x}' - \mathbf{x}''|} P(X, X'')\, dX'' = \frac{e^2}{m} \frac{\partial}{\partial \mathbf{x}} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \cdot \left[f_1^{(0)}(\mathbf{v}') \frac{\partial f_1^{(0)}}{\partial \mathbf{v}} - f_1^{(0)}(\mathbf{v}) \frac{\partial f_1^{(0)}}{\partial \mathbf{v}'} \right] . \quad (2)$$

This equation determines $P(X, X')$ when $f_1^{(0)}(\mathbf{v})$ is given [3]. This information is sufficient to determine the auto-correlation function and spectral density for any quantity such as electric field or current density. Calculations will be made for a plasma consisting of electrons and infinite-mass ions, electrons and finite-mass ions and then a constant magnetic field will be added.

2.2 Joint probability functions

This treatment will be quite similar to Section 1.2. It is however more convenient to introduce conditional probability functions for our present purposes, so that we shall repeat some of the previous discussion. The function $D_2(Xt; X't')$ satisfies the Liouville equation in the co-ordinates X', t' and the initial conditions

$$D_2(Xt; X't) = D_1(Xt)\, \delta(X' - X).$$

The function $C(Xt \mid X't')$ is defined by integrating out all initial co-ordinates but one, i.e.,

$$f_1(X_1 t)\, C(X_1 t \mid X' t') = V\int D_2(Xt; X't')\, (dX)^{N-1} \quad (3)$$

where

$$f_1(X_1 t) = V\int D_1(Xt)\, (dX)^{N-1}$$

$C(X_1 t \mid X' t')$ satisfies the Liouville equation in (X', t') and the initial condition

$$C(X_1 t \mid X' t) = V D_1(X' t)\, \delta(X'_1 - X_1)/f_1(X_1, t). \quad (4)$$

The s-body functions are defined as in Section 1:

$$F_s(X_1 t \mid X_2' \cdots X'_{s+1}, t') = V^s \int C(X_1 t \mid X' t')\, dX_1'\, dX'_{s+2} \cdots dX_N'$$
$$\Omega_s(X_1 t \mid X_1' \cdots X_s', t') = V^s \int C(X_1 t \mid X' t')\, dX'_{s+1} \cdots dX_N'$$
$$f_s(X_1 \cdots X_s; t) = V^s \int D_1(X t)\, (dX)^{N-s}$$

The initial conditions for one- and two-body functions are as follows

$$\begin{aligned} &\Omega_1(X_1' t) = V\, \delta(X_1' - X_1) \\ &\Omega_2(X_1', X_2'; t) = V\, \delta(X_1' - X_1)\, f_2(X_1, X_2'; t)/f_1(X_1 t) \\ &F_1(X_2', t) = f_2(X_1 X_2'; t)/f_1(X_1 t) \quad (5) \\ &F_2(X_2' X_3'; t) = f_3(X_1, X_2', X_3'; t)/f_1(X_1 t). \end{aligned}$$

(The abbreviated notation $\Omega_1(X_1' t')$ will be employed instead of $\Omega_1(X_1 t \mid X_1' t')$ wherever this can be accomplished without confusion.)

F_s and Ω_s are determined by taking moments of the Liouville equation and then expanding; i.e.,

$$F_s = F_s^{(0)} + F_s^{(1)} + \ldots,$$

where the parameter of expansion is e, m, or $1/n$. This procedure has been carried out in detail for the test-particle problem [2]. Essentially the same equations apply to the present problem. However the initial conditions are different in the present case; in particular we do not wish to assume that the field particles are initially in thermal equilibrium. It is assumed that a partial specification of the initial density in phase space $D_1(Xt)$ is given in terms of its moments. For a spatially homogeneous plasma

$$\begin{aligned} &f_1(X_1 t) = f_1^{(0)}(\mathbf{v}_1) + f_1^{(1)}(\mathbf{v}_1 t) = f_1(\mathbf{v}_1 t) \\ &f_2(X_1 X_2 t) = f_1(\mathbf{v}_1 t)\, f_1(\mathbf{v}_2 t) + P(X_1, X_2) \\ &f_3(X_1 X_2 X_3; t) = f_1(\mathbf{v}_1 t)\, f_1(\mathbf{v}_2 t)\, f_1(\mathbf{v}_3 t) \\ &\quad + f_1^{(0)}(\mathbf{v}_1)\, P(X_2 X_3) + f_1^{(0)}(\mathbf{v}_2)\, P(X_1, X_3) \\ &\quad + f_1^{(0)}(\mathbf{v}_3)\, P(X_1, X_2). \quad (6) \end{aligned}$$

$P(X_1, X_2)$ is first order in the expansion parameter, symmetric with respect to the interchange of X_1 and X_2, and depends on the spatial co-ordinates only as $\mathbf{x}_2 - \mathbf{x}_1$. It is determined by $f_1^{(0)}(\mathbf{v}_1)$ according to Eq. (1). It is convenient to introduce a conditional probability defined by

$$P(X_1, X_2) = f_1^{(0)}(v_1)\, G(X_1 \mid X_2) \quad (7)$$

For present purposes it is necessary to calculate Ω_1 to zero order and F_1 to first order. The required initial conditions are from Eqs. (5) and (6)

$$\begin{aligned} &\Omega_1^{(0)}(X_1' t) = V\, \delta(X_1' - X_1) \\ &F_1^{(0)}(X_2' t) = f_1^{(0)}(\mathbf{v}_2') \quad (8) \\ &F_1^{(1)}(X_2' t) = f_1^{(1)}(v_2' t) + G(X_1 \mid X_2'). \end{aligned}$$

The zero-order functions satisfy the differential equation

$$\left\{\frac{\partial}{\partial t'} + \mathbf{v}_1' \frac{\partial}{\partial \mathbf{x}_1'} - \frac{e}{m} \mathbf{E}_M^{(0)}(\mathbf{x}_1', t') \cdot \frac{\partial}{\partial \mathbf{v}_1'}\right\} \frac{\Omega_1^{(0)}(X_1', t')}{F_1^{(0)}(X_1', t')} = 0, \quad (9)$$

where

$$\mathbf{E}_M^{(0)}(x_1' t') = n e \int \frac{\partial}{\partial \mathbf{x}_1'} \frac{1}{|\mathbf{x}_1' - \mathbf{x}_2'|} F_1^{(0)}(X_2' t')\, dX_2'.$$

The solutions are

$$\begin{aligned} &\Omega_1^{(0)}(X_1' t') = V\, \delta[\mathbf{x}_1' - \mathbf{x}_1 - \mathbf{v}_1 (t' - t)]\, \delta(\mathbf{v}_1' - \mathbf{v}_1) \\ &F_1^{(0)}(X_2' t') = f_1^{(0)}(\mathbf{v}_2'). \quad (10) \end{aligned}$$

The equation for the first order contribution to F_1 is

$$\left\{\frac{\partial}{\partial t'} + \mathbf{v}_2' \cdot \frac{\partial}{\partial \mathbf{x}_2'}\right\} F_1^{(1)}(X_2' t') - \frac{e^2}{m} \frac{\partial f_1^{(0)}}{\partial v_2'} \cdot \frac{\partial}{\partial \mathbf{x}_2'} \times \left\{\frac{1}{|\mathbf{x}_2' - \mathbf{x}_1 - \mathbf{v}_1(t' - t)|} + n \int \frac{F_1^{(1)}(X_3' t')}{|\mathbf{x}_2' - \mathbf{x}_3'|} dX_3'\right\} = \mathsf{St}\{F_1\}. \quad (11)$$

This is the usual equation for the field particle distribution [2] except for the term $\mathsf{St}\{F_1\}$. The form of this collision operator is

$$\mathsf{St}\{F_1\} = \frac{ne^2}{m} \int \frac{\partial}{\partial \mathbf{x}_2'} \frac{1}{|\mathbf{x}_2' - \mathbf{x}_3'|} \cdot \frac{\partial}{\partial \mathbf{v}_2'} P(X_2', X_3'; t')\, dX_3'.$$

$P(X_2', X_3'; t')$ depends only on $\mathbf{x}_2' - \mathbf{x}_3'$ so that $\mathsf{St}\{F_1\}$ is independent of spatial co-ordinates.

The solution of Eq. (11) subject to the initial condition of Eq. (8) is obtained in the usual way [2] by integrating along the characteristic or unperturbed orbits and making use of Fourier and Laplace transforms. The result is

$$\begin{aligned} F_1^{(1)}(X_2' t') = {} & f_1^{(1)}(v_2' t) + \int_{t''=t}^{t'} \mathsf{St}\{F_1\}\, dt'' \\ & + \frac{1}{n} \int \frac{d^3 k}{(2\pi)^3} e^{\mathrm{i}\mathbf{k} \cdot (\mathbf{x}_2' - \mathbf{x}_1)} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \frac{dp}{2\pi \mathrm{i}} \frac{e^{p(t' - t)}}{p + \mathrm{i}\mathbf{k} \cdot \mathbf{v}_2'} \Big\{ n G_k(\mathbf{v}_1 \mid \mathbf{v}_2') \\ & + \frac{\mathrm{i}\mathbf{k} \cdot \partial f_1^{(0)}/\partial \mathbf{v}_2'}{\varepsilon(\mathbf{k}, p)} \frac{\omega_p^2}{k^2} \left[\frac{1}{p + \mathrm{i}(\mathbf{k} \cdot \mathbf{v}_1)} + n \int \frac{G_k(\mathbf{v}_1 \mid \mathbf{v})}{(p + \mathrm{i}\mathbf{k} \cdot \mathbf{v})} d\mathbf{v}\right]\Big\} \end{aligned} \quad (12)$$

In this expression,

$$\varepsilon(\mathbf{k}, p) = 1 - \frac{\omega_p^2}{k^2} \int \frac{\mathrm{i}\mathbf{k} \cdot \partial f^{(0)}/\partial \mathbf{v}}{p + \mathrm{i}\mathbf{k} \cdot \mathbf{v}} d\mathbf{v}$$

and $G_k(\mathbf{v}_1 \mid \mathbf{v}_2)$ is defined such that

$$G(X_1 \mid X_2) = \frac{1}{(2\pi)^3} \int d\mathbf{k}\, e^{\mathrm{i}\mathbf{k} \cdot (\mathbf{x}_2 - \mathbf{x}_1)} G_k(\mathbf{v}_1 \mid \mathbf{v}_2).$$

The particular moments of $D_2(Xt; X't')$ that are usually required are

$$W_{ij}(X_i t; X_j' t') = V^2 \int D_2(Xt; X't')\, (dX)^{N-1} (dX')^{N-1}.$$

The only two independent moments are

$$\begin{aligned} &W_{11}(X_1 t; X_1' t') = f_1(X_1 t)\, \Omega_1(X_1 t \mid X_1' t'), \\ &W_{12}(X_1 t; X_2' t_2) = f_1(X_1 t)\, F_1(X_1 t \mid X_2' t'). \end{aligned}$$

We shall require W_{11} only to zero order and W_{12} to first order. The results for these quantities are as follows

$$W_{11}^{(0)}(Xt_1;\,X_1{'}t') = V\,f_1^{(0)}(\mathbf{v}_1)\,\delta\,[\mathbf{x}_1{'}-\mathbf{x}_1-\mathbf{v}_1\,(t'-t)]\,\delta\,[\mathbf{v}_1{'}-\mathbf{v}_1] \qquad (13)$$

$$W_{12}^{(0)}(X_1 t;\,X_1{'}\,t') = f_1^{(0)}(\mathbf{v}_1)\,f_1^{(0)}(\mathbf{v}_2{'})$$

$$W_{12}^{(1)}(Xt_1;X_1{'}t') = f_1^{(0)}(\mathbf{v}_1)f_1^{(1)}(\mathbf{v}_2{'}t') + f_1^{(1)}(\mathbf{v}_1 t)f_1^{(0)}(\mathbf{v}_2{'}) + f_1^{(0)}(\mathbf{v}_1)\int_{t''=t}^{t'} \mathsf{St}\,\{F_1\}\,dt'' + \int\frac{d\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot(\mathbf{x}_2{'}-\mathbf{x}_1)}\int\frac{dp}{2\pi i}\,e^{p(t'-t)}W_{pk}(\mathbf{v}_1,\mathbf{v}_2{'}),$$

where

$$W_{pk}(\mathbf{v}_1,\mathbf{v}_2) = \left(\frac{1}{n}\right)\frac{1}{(p+i\mathbf{k}\cdot\mathbf{v}_2)}\left\{n\,P_k(\mathbf{v}_1,\mathbf{v}_2) + \frac{\mathbf{k}\cdot\partial f^{(0)}/\partial\mathbf{v}_2}{\varepsilon(\mathbf{k},p)}\;i\frac{\omega_p{}^2}{k^2}\left[\frac{f^{(0)}\,\mathbf{v}_1}{p+i\mathbf{k}\cdot\mathbf{v}_1} + n\int\frac{P_k(\mathbf{v}_1,\mathbf{v})}{(p+i\mathbf{k}\cdot\mathbf{v})}\,d\mathbf{v}\right]\right\}$$

and $P_k(\mathbf{v}_1,\mathbf{v}_2) = f_1^{(0)}(\mathbf{v}_1)\,G_k(\mathbf{v}_1\,|\,\mathbf{v}_2)$.
With the present restriction to a spatially homogeneous plasma, there is never any contribution from $W_{12}^{(0)}$ or the first three terms of $W_{12}^{(1)}$. They will henceforth be simply omitted.

For some applications additional moments are required of $D_2\,(Xt;\,X't')$ such as

$$W_{ijk}(X_i,X_j t;X_k{'}t') = V^3\int D_2(Xt,X't')(dX)^{N-2}(dX')^{N-1} \qquad (14)$$

The independent functions are W_{121}, W_{122} and W_{123} which are obtained by a simple generalization of the procedure employed to calculate W_{11} and W_{12}. The problem is one of two singled-out test particles, which to the order considered in the present calculations, do not interact. The results are as follows

$$W_{121}^{(0)}(X_1,X_2,t;\,X_1{'}\,t') = Vf_1^{(0)}(\mathbf{v}_1)\,f_1^{(0)}(\mathbf{v}_2)\,\delta\,[\mathbf{x}_1{'}-\mathbf{x}_1-\mathbf{v}_1\,(t'-t)]\,\delta\,[\mathbf{v}_1{'}-\mathbf{v}_1]$$

$$W_{122}^{(0)}(X_1,X_2,t;X_2{'}t') = Vf_1^{(0)}(\mathbf{v}_1)f_1^{(0)}(\mathbf{v}_2)\,\delta[\mathbf{x}_2{'}-\mathbf{x}_2-\mathbf{v}_2\,(t'-t)]\,\delta\,(\mathbf{v}_2{'}-\mathbf{v}_2)$$

$$W_{123}^{(1)}(X_1,X_2,t;\,X_3{'}\,t') = f_1^{(0)}(\mathbf{v}_3{'})\,P\,(X_1,X_2) + f_1^{(0)}(\mathbf{v}_1)\,W_{23}^{(1)}(X_2 t;\,X_3{'}\,t') + f_1^{(0)}(\mathbf{v}_2)\,W_{13}^{(1)}(X_1 t;\,X_3{'}\,t')\ldots \qquad (15)$$

In the latter expression we have omitted terms such as $f_1^{(0)}(\mathbf{v}_3{'})\,f_1^{(0)}(\mathbf{v}_2)\,f_1^{(1)}(\mathbf{v}_1 t)$, which give no contribution for a spatially homogeneous plasma.

2.3. Electric field fluctuations

The ensemble average is calculated as in Section 1.3:

$$\langle\mathbf{E}\,(\mathbf{x}t)\,\mathbf{E}\,(\mathbf{x}'t')\rangle = \int D_2(Xt;\,X'\,t')\sum_{l,n}\frac{\partial}{\partial x}\frac{e}{|\mathbf{x}-\mathbf{x}_l|}\frac{\partial}{\partial\mathbf{x}'}\frac{e}{|\mathbf{x}'-\mathbf{x}_n{'}|}\,dX\,dX'$$

$$= \frac{ne^2}{V}\int\frac{\partial}{\partial\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}_1|}\frac{\partial}{\partial\mathbf{x}'}\frac{1}{|\mathbf{x}'-\mathbf{x}_1{'}|}\,W_{11}^{(0)}(X_1 t;X_1{'}\,t')\,dX_1\,dX_1{'} + n^2e^2\int\frac{\partial}{\partial\mathbf{x}}\frac{1}{|\mathbf{x}-\mathbf{x}_1|}\frac{\partial}{\partial\mathbf{x}'}\frac{1}{|\mathbf{x}'-\mathbf{x}_2{'}|}\,W_{12}^{(1)}(X_1\,t;\,X_2{'}\,t')\,dX_1\,dX_2{'}.$$

After carrying out the integrations as far as possible,

$$\langle\mathbf{E}\,(\mathbf{x}t)\,\mathbf{E}\,(\mathbf{x}'t')\rangle = \int\frac{dp}{2\pi i}\,e^{p\tau}\int\frac{d\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{S}^+(\mathbf{k},p)$$

where $\tau = t'-t$, $\mathbf{r} = \mathbf{x}'-\mathbf{x}$ and

$$\mathbf{S}^+(\mathbf{k},p) = (4\pi)^2\,ne^2\,\frac{\mathbf{k}\mathbf{k}}{k^4}\times\left\{\int\frac{d\mathbf{v}_1\,f_1^{(0)}(\mathbf{v}_1)}{p+i\mathbf{k}\cdot\mathbf{v}_1} + \int\frac{d\mathbf{v}_1\,d\mathbf{v}_2}{p+i\mathbf{k}\cdot\mathbf{v}_2}\left(n\,P_k(\mathbf{v}_1,\mathbf{v}_2) + i\,\frac{\omega_p{}^2}{k^2}\,\frac{\mathbf{k}\cdot\partial f_1^{(0)}/\partial\mathbf{v}_2}{\varepsilon(\mathbf{k},p)}\left[\frac{f_1^{(0)}(\mathbf{v}_1)}{p+i\mathbf{k}\cdot\mathbf{v}_1} + n\int\frac{P_k(\mathbf{v}_1,\mathbf{v})\,d\mathbf{v}}{p+i\mathbf{k}\cdot\mathbf{v}}\right]\right)\right\}.$$

To achieve a more manageable form the following quantities are introduced

$$U\,(\mathbf{k},p) = \frac{k}{\pi}\int\frac{f_1^{(0)}(\mathbf{v})\,d\mathbf{v}}{p+i\mathbf{k}\cdot\mathbf{v}}\qquad h_k(\mathbf{v}) = n\int P_k(\mathbf{v},\mathbf{v}')\,d\mathbf{v}'\,.$$

In terms of these quantities

$$\mathbf{S}^+(\mathbf{k},p) = (4\pi)^2\,ne^2\,\frac{\mathbf{k}\mathbf{k}}{k^4}\,\frac{1}{\varepsilon(\mathbf{k},p)}\times\left\{\frac{\pi}{k}\,U\,(\mathbf{k},p) + \int\frac{h_k{}^*(\mathbf{v}_2)\,d\mathbf{v}_2}{p+i\mathbf{k}\cdot\mathbf{v}_2}\right\}, \qquad (16)$$

and

$$\mathbf{S}^+(\mathbf{k},i\omega) = (4\pi)^2\,ne^2\,\frac{\mathbf{k}\mathbf{k}}{k^4}\,\frac{1}{\varepsilon(\mathbf{k},i\omega)}\left\{\frac{\pi}{k}\,U\,(\mathbf{k},i\omega) + \lim_{\lambda\to 0}\int\frac{h_k{}^*(\mathbf{v}_2)\,d\mathbf{v}_2}{i\,(\omega+\mathbf{k}\cdot\mathbf{v}_2-i\lambda)}\right\}$$

where the interpretation of the integral is such that

$$\lim_{\lambda\to 0}\frac{1}{(\omega+\mathbf{k}\cdot\mathbf{v}_2-i\lambda)} = \pi\,i\,\delta\,(\omega+\mathbf{k}\cdot\mathbf{v}_2) + \frac{\mathsf{P}}{\omega+\mathbf{k}\cdot\mathbf{v}_2}\,.$$

To make any further progress we must make use of some of the properties of the pair distribution function $P_k(\mathbf{v}_1,\mathbf{v}_2)$. The Fourier transform of Eq. (2) is

$$\lim_{\lambda\to 0}\,[\mathbf{k}\cdot(\mathbf{v}_2-\mathbf{v}_1)-i\,\lambda]\,n\,P_k(\mathbf{v}_1,\mathbf{v}_2) = \frac{\omega_p{}^2}{k^2}\left\{\left[f_1^{(0)}(\mathbf{v}_1) + h_k(\mathbf{v}_1)\right]\frac{\mathbf{k}\cdot\partial f_1^{(0)}}{\partial\mathbf{v}_2} - [f_1^{(0)}(\mathbf{v}_2) + h_k{}^*(\mathbf{v}_2)]\,\frac{\mathbf{k}\cdot\partial f_1^{(0)}}{\partial\mathbf{v}_1}\right\}\,.$$

After dividing by $\mathbf{k}\cdot(\mathbf{v}_2-\mathbf{v}_1)-i\lambda$ and integrating over $\mathbf{v}_2$ we obtain an integral equation for $h_k(\mathbf{v}_1)$

$$h_k(\mathbf{v}_1) = \frac{1}{\varepsilon(\mathbf{k},-i\mathbf{k}\cdot\mathbf{v}_1)}\left\{[1-\varepsilon(\mathbf{k},-i\mathbf{k}\cdot\mathbf{v}_1)]\,f^{(0)}(\mathbf{v}_1) + \frac{i\,\mathbf{k}\cdot\partial f_1^{(0)}}{\partial\mathbf{v}_1}\,\frac{\omega_p{}^2}{k^2}\left[\frac{\pi}{k}\,U\,(\mathbf{k},-i\mathbf{k}\cdot\mathbf{v}_1) + \int\frac{h_k{}^*(\mathbf{v}_2)\,d\mathbf{v}_2}{i\,[\mathbf{k}\cdot(\mathbf{v}_2-\mathbf{v}_1)\,i\,\lambda]}\right]\right\}. \qquad (17)$$

Let $H_k(u) = \int d\mathbf{v}\,h_k(\mathbf{v})\,\delta\,(u-\mathbf{k}\cdot\mathbf{v}/k)$, multiply Eq. (17) by $\delta\,(\omega+\mathbf{k}\cdot\mathbf{v}_1)$ and integrate over $\mathbf{v}_1$. The result is

$$H_k(-\omega/k) = \frac{1}{\varepsilon\,(\mathbf{k},i\,\omega)}\left\{[1-\varepsilon\,(\mathbf{k},i\,\omega)]\,\mathrm{Re}\,U\,(\mathbf{k},i\omega) - \frac{i\,k}{\pi}\,\mathrm{Im}\,\varepsilon\,(\mathbf{k},i\,\omega)\left[\frac{\pi}{\mathbf{k}}\,U\,(\mathbf{k},i\,\omega) + \int\frac{h_k{}^*(\mathbf{v}_2)\,d\mathbf{v}_2}{i\,(\omega+\mathbf{k}\cdot\mathbf{v}_2-i\,\lambda)}\right]\right\}. \qquad (18)$$

We can now substitute this result into Eq. (16) and obtain:

$$\mathbf{S}^+(\mathbf{k}, \mathrm{i}\,\omega) = (4\pi)^2 n e^2 \frac{\mathbf{k}\mathbf{k}}{k^4}\frac{\pi \mathrm{i}}{k} \times \left\{\frac{[1-\varepsilon(\mathbf{k},\mathrm{i}\,\omega)]}{\varepsilon\,\mathrm{Im}\,\varepsilon}\mathrm{Re}\,U(\mathbf{k},\mathrm{i}\,\omega) - \frac{H_k(-\omega/k)}{\mathrm{Im}\,\varepsilon}\right\}.$$

Since $\mathbf{S}^+$ is a symmetric dyadic, the spectral density is

$$\mathbf{S}(\mathbf{k},\omega) = 2\,\mathrm{Re}\,\mathbf{S}^+(\mathbf{k},\mathrm{i}\,\omega) = (4\pi)^2 n e^2 \frac{\mathbf{k}\mathbf{k}}{k^4}\frac{2\pi}{k}\left\{\frac{\mathrm{Re}\,U(\mathbf{k},\mathrm{i}\,\omega)}{|\varepsilon(\mathbf{k},\mathrm{i}\,\omega)|^2} + \frac{\mathrm{Im}\,H_k(-\omega/\mathbf{k})}{\mathrm{Im}\,\varepsilon}\right\}.$$

It has been previously established by LENARD [3] that $\mathrm{Im}\,H_k(-\omega/k) = 0$. The final result is therefore

$$\mathbf{S}(\mathbf{k},\omega) = (4\pi)^2 n e^2 \frac{\mathbf{k}\mathbf{k}}{k^4}\frac{2\pi}{k}\frac{\mathrm{Re}\,U(\mathbf{k},\mathrm{i}\omega)}{|\varepsilon(\mathbf{k},\mathrm{i}\,\omega)|^2}. \tag{19}$$

This is formally the same as Eq. (27) in Section 1. The previous derivation can easily be generalized to apply to a plasma consisting of electrons and ions. Eq. (19) still applies with the following new definitions

$$U(\mathbf{k},p) = \frac{k}{\pi}\sum_j \int \frac{f_j^{(0)}(\mathbf{v})\,d\mathbf{v}}{p + \mathrm{i}\mathbf{k}\cdot\mathbf{v}} \tag{20}$$

$$\varepsilon(\mathbf{k},p) = 1 - \sum_j \frac{\omega_{pj}^2}{k^2}\int \frac{\mathrm{i}\mathbf{k}\cdot\partial f_j^{(0)}/\partial\mathbf{v}}{p+\mathrm{i}\mathbf{k}\cdot\mathbf{v}}\,d\mathbf{v},$$

where $\omega_{pj}^2 = 4\pi\,ne^2/m_j$ and the summation is over particle species.

To indicate some of the features of non-equilibrium states, the electrostatic energy per degree of freedom will be calculated. That is

$$\frac{\langle \mathbf{E}(\mathbf{x},t)\cdot\mathbf{E}(\mathbf{x},t)\rangle}{8\pi} = \int \frac{d\mathbf{k}\,d\omega}{(2\pi)^4} W(\mathbf{k},\omega)$$

where

$$W(\mathbf{k},\omega) = \frac{4\pi n e^2}{k^2}\frac{\pi}{k}\frac{\mathrm{Re}\,U(\mathbf{k},\mathrm{i}\,\omega)}{|\varepsilon(\mathbf{k},\mathrm{i}\,\omega)|^2}, \tag{21}$$

and the energy per degree of freedom is defined as

$$\frac{\Theta(\mathbf{k})}{2} = \int \frac{d\omega}{2\pi} W(\mathbf{k},\omega). \tag{22}$$

Consider for example, a plasma in which the electron and ion distribution functions are

$$f_e^{(0)}(\mathbf{v}) = \left(\frac{m_e}{2\pi\Theta_e}\right)^{\frac{3}{2}} \exp[-m_e v^2/2\Theta_e]$$

$$f_i^{(0)}(\mathbf{v}) = \left(\frac{m_i}{2\pi\Theta_i}\right)^{\frac{3}{2}} \exp[-m_i v^2/2\Theta_i],$$

where $\Theta_i \ll \Theta_e$. Asymptotic forms for U and ε can be employed in various regions of ω, k space as illustrated in Fig. 2. The following definitions are employed

$$m_e v_e^2 = \Theta_e \qquad \frac{1}{L_e^2} = \frac{4\pi n e^2}{\Theta_e}$$

$$m_i v_i^2 = \Theta_i \qquad \frac{1}{L_i^2} = \frac{4\pi n e^2}{\Theta_i}.$$

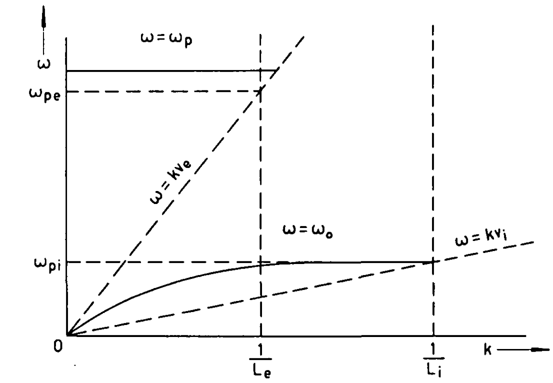

Fig. 2 Asymptotic forms for U and ε in various regions of ω, k space.

i) *Plasma wave region*: If $k < 1/L_e$ and $\omega > kv_e$, the asymptotic forms are as follows:

$$\mathrm{Re}\,U(k,\mathrm{i}\omega) \cong \frac{1}{\sqrt{2\pi v_e^2}} \mathrm{e}^{-\frac{1}{2}\left(\frac{\omega}{kv_e}\right)^2} + \frac{1}{\sqrt{2\pi v_i^2}} \mathrm{e}^{-\frac{1}{2}\left(\frac{\omega}{kv_i}\right)^2}$$

$$\cong \frac{1}{\sqrt{2\pi v_e^2}} \mathrm{e}^{-\frac{1}{2}\left(\frac{\omega}{kv_e}\right)^2}, \qquad \text{since } \frac{\omega}{kv_i} \gg \frac{\omega}{kv_e}.$$

$$|\varepsilon(k,\mathrm{i}\omega)|^2 \cong \left[1 - \frac{\omega_p{}^2}{\omega}\right]^2 + \frac{\pi}{2}\left(\frac{\omega}{kv_e}\right)^2 \frac{1}{(kL_e)^4}\mathrm{e}^{-\left(\frac{\omega}{kv_e}\right)^2},$$

where $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$. The denominator in Eq. (21) has a resonance at $\omega = \omega_p$. The result obtained from integrating across this resonance is

$$\frac{\Theta(k)}{2} = \frac{\Theta_e}{2} \text{ for } k < \frac{1}{L_e}. \tag{23}$$

ii) *Ion wave region*: If $kL_i < 1$ and $kv_e > \omega > kv_i$, the asymptotic forms are as follows

$$\mathrm{Re}\,U(k,\mathrm{i}\omega) = \frac{1}{\sqrt{2\pi v_e^2}} + \frac{1}{\sqrt{2\pi v_i^2}}\mathrm{e}^{-\frac{1}{2}\left(\frac{\omega}{kv_i}\right)^2}$$

$$|\varepsilon(k,\mathrm{i}\omega)|^2 = \left[1 + \frac{1}{(kL_e)^2} - \left(\frac{\omega_{pi}}{\omega}\right)^2\right]^2 + \frac{\pi}{2}\frac{\omega^2}{k^6}\left[\frac{1}{v_e L_e^2} + \frac{1}{v_i L_i^2}\mathrm{e}^{-\frac{1}{2}\left(\frac{\omega}{kv_i}\right)^2}\right]^2.$$

The resonance now takes place at

$$\omega_0^2 = \omega_{pi}^2\,(k\,L_e)^2 / [1 + (k\,L_e)^2] \tag{24}$$

The resonant width is

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{2}\sqrt{\frac{\pi}{2}}[1+(kL_e)^2]^{-\frac{3}{2}} \times \left\{\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} + \left(\frac{\Theta_e}{\Theta_i}\right)^{\frac{3}{2}}\exp\left[-\frac{1}{2}\frac{\Theta_e}{\Theta_i}\frac{1}{1+(kL_e)^2}\right]\right\},$$

which remains sufficiently small for approximate integration as long as $\Theta_e \gg \Theta_i$ and $kL_i < 1$. The result obtained from integrating across the ion resonance is

$$\frac{\Theta(k)}{2} = \frac{\Theta_e}{2}\,\frac{(kL_e)^2}{1+(kL_e)^2} \times \frac{\left\{\sqrt{\frac{m_e}{m_i}} + \sqrt{\frac{\Theta_e}{\Theta_i}}\exp\left[-\frac{1}{2}\frac{\Theta_e}{\Theta_i}\frac{1}{1+(kL_e)^2}\right]\right\}}{\left\{\sqrt{\frac{m_e}{m_i}} + \left(\frac{\Theta_e}{\Theta_i}\right)^{\frac{3}{2}}\exp\left[-\frac{1}{2}\frac{\Theta_e}{\Theta_i}\frac{1}{1+(kL_e)^2}\right]\right\}}\,. \tag{25}$$

If $kL_e < 1$, then $\Theta(k) \simeq \Theta_e (kL_e)^2$, provided that $m_e/m_i > (\Theta_e/\Theta_i)\exp(-\Theta_e/\Theta_i)$ and $\Theta(k) \simeq \Theta_i (kL_e)^2$ for the other direction of the inequality. For $kL_e \sim 1$, $\Theta(k) \simeq \frac{1}{2}\Theta_e$ or $\frac{1}{2}\Theta_i$ according to the sign of the same inequality. For $kL_e > 1$ the exponential terms will eventually dominate and $\Theta(k) \simeq \Theta_i$. For moderately high electron temperature, the energy/degree of freedom increases monotonically with k from zero to $\frac{1}{2}\Theta_i$. For very high electron temperature there is a maximum in the neighborhood of $kL_e \sim 1$ which is about $\frac{1}{4}\Theta_e$.

Another case of interest is where there is a small number of runaway electrons. For example,

$$f_e^{(0)}(v) = \frac{[1-(\Delta n/n)]}{(2\pi v_0^2)^{\frac{3}{2}}}\exp\left[-\frac{1}{2}\left(\frac{v}{v_0}\right)^2\right] + \frac{\Delta n}{n}\frac{1}{(2\pi v_1^2)^{\frac{3}{2}}}\exp\left[-\frac{1}{2}\frac{(\mathbf{v}+\mathbf{V}_e)^2}{v_1^2}\right]$$

$$f_i^{(0)}(v) = \delta(\mathbf{v}+\mathbf{V}_i)\,. \tag{26}$$

For simplicity the ions are assumed to have infinite mass. If $n\mathbf{V}_i = \mathbf{V}_e \Delta n$ there will be no current as required for a spatially homogeneous plasma. The requirement for stability is

$$\frac{\Delta n}{n} < \left(\frac{v_1}{v_0}\right)^2\left(\frac{V_e}{v_0}\right)\exp\left[-\frac{1}{2}\left(\frac{V_e}{v_0}\right)^2\right]. \tag{27}$$

The validity of the present calculations is restricted to cases where Eq. (27) is satisfied. However we can consider the energy per mode as Δn increases up to the limit given by Eq. (27). Asymptotic forms for U and ε can be employed for various regions of ω, k_z space as indicated in Fig. 3. The z-axis is taken to be in the direction of $\mathbf{V}_e$, and we consider only modes for which $k_x = k_y = 0$.

$$\operatorname{Re} U(k_z, i\omega) = \frac{1}{\sqrt{2\pi v_0^2}}\exp\left[-\frac{1}{2}\left(\frac{\omega}{k_z v_0}\right)^2\right] + \frac{\Delta n}{n}\frac{1}{\sqrt{2\pi v_1^2}}\exp\left[-\frac{1}{2}\frac{(\omega-k_z V_e)^2}{(k_z v_1)^2}\right]$$

For $\omega > k_z v_0$ and $k_z < \omega_p/v_0$

$$|\varepsilon(k, i\omega)|^2 = \left[1-\frac{\omega_p^2}{\omega^2}\right]^2 + \frac{\pi}{2}\left\{\frac{\omega}{k_z v_0}\left(\frac{\omega_p}{k_z v_0}\right)^2\exp\left[-\frac{1}{2}\left(\frac{\omega}{k_z v_0}\right)^2\right] + \frac{\omega-k_z V_e}{k_z v_1}\frac{\omega_p^2 \Delta n}{n(k_z v_1)^2}\exp\left[-\frac{1}{2}\left(\frac{\omega-k_z V_e}{k_z v_1}\right)^2\right]\right\}^2.$$

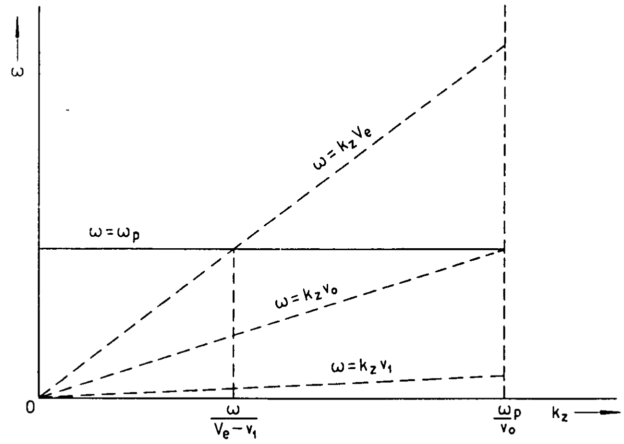

Fig. 3 Asymptotic forms for U and ε in various regions of ω, k_z space.

A resonance takes place at $\omega = \omega_p$ of width

$$\frac{\Delta\omega}{\omega_p} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\left\{\left(\frac{\omega_p}{k_z v_0}\right)^3\exp\left[-\frac{1}{2}\left(\frac{\omega_p}{k_z v_0}\right)^2\right] + \frac{(\omega_p - k_z V_e)}{k_z v_1}\frac{\omega_p^2 \Delta n}{n(k_z v_1)^2}\exp\left[-\frac{1}{2}\left(\frac{\omega_p - k_z V_e}{k_z v_1}\right)^2\right]\right\}.$$

Provided $k_z < \omega_p/v_0$ and $\Delta n/n \ll (v_1/V_e)^2$, then $\Delta\omega/\omega_p \ll 1$ so that the resonance is sharp. The result obtained from integrating across the resonance is given in Eq. (28), at bottom of page. The latter term of Eq. (28) comes from the region of $k_z > 0$, $\omega < 0$ where the first exponential always dominates. As long as $k_z \gg \omega_p/V_e$ or $k_z \ll \omega_p/V_e$ the first exponential dominates and $\Theta(k_z) \simeq \Theta_e$. However when $k_z = \omega_p/(V_e - v_1) \simeq \omega_p/V_e$, then

$$\frac{\Theta(k_z)}{2} \simeq \frac{\Theta_e}{4} \times \left\{1 + \frac{\exp\left[-\frac{1}{2}(V_e/v_0)^2\right] + (v_0 \Delta n / v_1 n)}{\exp\left[-\frac{1}{2}(V_e/v_0)^2\right] - (\Delta n/n)(v_0/v_1)^2 (v_0/V_e)}\right\}. \tag{29}$$

As long as Eq. (27) is satisfied, this result remains finite. However the energy for modes in the neighborhood of $k_z \simeq \omega_p/V_e$ becomes very large and ultimately infinite as

$$\frac{\Delta n}{n} \to \left(\frac{v_1}{v_0}\right)^2\left(\frac{V_e}{v_0}\right)\exp\left[-\frac{1}{2}\left(\frac{V_e}{v_0}\right)^2\right].$$

2.4. Theorems relating to fluctuations

The electric field due to a test charge is defined as follows:

$$\frac{\partial \delta f}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\delta f + \frac{e}{m}\frac{\partial \Phi}{\partial \mathbf{x}}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}} = \mathrm{St}\{f\} \tag{30}$$

$$\nabla^2 \Phi = 4\pi e\left[\delta(\mathbf{x}-\mathbf{x}_1-\mathbf{v}_1 t) + n\int \delta f\, d\mathbf{v}\right].$$

$$\frac{\Theta(k_z)}{2} = \frac{\Theta_e}{4}\left\{\frac{\exp\left(-\frac{\omega_p^2}{2k_z^2 v_0^2}\right) + \left(\frac{v_0 \Delta n}{n v_1}\right)\exp\left[-\frac{(\omega_p - k_z V_e)^2}{2(k_z v_1)^2}\right]}{\exp\left(-\frac{\omega_p^2}{2k_z^2 v_0^2}\right) + \left(\frac{\omega_p - k_z V_e}{k_z v_1}\right)\left(\frac{k_z v_0}{\omega_p}\right)\frac{\Delta n}{n}\left(\frac{v_0}{v_1}\right)^2\exp\left[-\frac{(\omega_p - k_z V_e)^2}{2k_z^2 v_1^2}\right]}\right\} + \frac{\Theta_e}{4}\,. \tag{28}$$

This differs from the thermal equilibrium problem in that St $\{f\}$, the collision term for the field particles does not vanish. However, for a spatially homogeneous plasma it is independent of position, and can be neglected because it only drives the $\mathbf{k}=0$ modes. The calculation is therefore identical to the thermal equilibrium problem formally. The electric field at a point $\mathbf{x}$ due to a fully dressed test particle at $\mathbf{x}_1$ with velocity $\mathbf{v}_1$ is

$$\mathbf{E}(\mathbf{x}, X_1) = 4\pi e \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}_1)} \frac{i\mathbf{k}}{k^2 \varepsilon(\mathbf{k}, -i\mathbf{k}\cdot\mathbf{v})}$$

and the superposition theorem is retained i.e.,

$$\langle \mathbf{E}(\mathbf{x}, t)\, \mathbf{E}(\mathbf{x}' t')\rangle = \frac{n}{V}\int \mathbf{E}(\mathbf{x}, X_1)\, \mathbf{E}(\mathbf{x}', X_1')\, W_{11}^{(0)}(X_1 t; X_1' t')\, dX_1\, dX_1'$$

leads to Eq. (19) for $\mathbf{S}(\mathbf{k}, \omega)$.

For the same reason the calculation of $r_{\|} = -[4\pi/(\omega\,|\,\varepsilon\,|^2)]\, \mathrm{Im}\, \varepsilon(\mathbf{k}, i\omega)$ in Section 1.5 remains formally correct. Therefore the fluctuation-dissipation theorem takes the form

$$\mathbf{S}(\mathbf{k}, \omega) = 2\Theta(\mathbf{k}, \omega)\, r_{\|}(\mathbf{k}, \omega)\, \mathbf{k}\,\mathbf{k}/k^2 \qquad (31)$$

where

$$\Theta(k,\omega) = -\frac{4\pi^2 n e^2}{k^2}\frac{\omega}{k}\frac{\mathrm{Re}\, U(\mathbf{k}, i\omega)}{\mathrm{Im}\, \varepsilon(\mathbf{k}, i\omega)}.$$

Consider for example, the case of electrons and ions at different temperatures:

$$\Theta(\mathbf{k}, \omega) = \Theta_e \times \frac{\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{\omega}{k v_e}\right)^2\right] + \left(\frac{\Theta_e}{\Theta_i}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{\omega}{k v_i}\right)^2\right]}{\left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{\omega}{k v_e}\right)^2\right] + \left(\frac{\Theta_e}{\Theta_i}\right)^{\frac{3}{2}} \exp\left[-\frac{1}{2}\left(\frac{\omega}{k v_i}\right)^2\right]}. \qquad (32)$$

If $\Theta_e = \Theta_i$ it is clear that $\Theta(\mathbf{k}, \omega) = \Theta_e = \Theta_i$. Many limiting cases are possible for $\Theta_e \neq \Theta_i$. For example, if $\Theta_e > \Theta_i$, then $\Theta(k, \omega) \cong \Theta_i$ for $\omega < k v_i$ and $\Theta(k, \omega) \cong \Theta_e$ for $\omega > k v_e$. It is clear that for non-equilibrium states the fluctuation-dissipation relation is not very useful.

The previous calculation of the absorption coefficient in Section 1.6 also remains formally correct, i.e.,

$$\mathfrak{a}(\mathbf{k}, \omega) = -2\omega\, \mathrm{Im}\, \varepsilon(\mathbf{k}, i\omega).$$

According to Kirchhoff's law we should expect that the emission per unit volume from the plasma would be

$$\mathfrak{e}(\mathbf{k}, \omega) = \mathfrak{a}(\mathbf{k}, \omega)\, W(\mathbf{k}, \omega) = -\frac{4\pi n e^2}{k^2}\frac{2\pi\omega}{k}\frac{\mathrm{Re}\, U(\mathbf{k}, i\omega)\, \mathrm{Im}\, \varepsilon(\mathbf{k}, i\omega)}{|\,\varepsilon(\mathbf{k}, i\omega)\,|^2}. \qquad (33)$$

The force on a test particle is $-eE(\mathbf{x}_1, X_1)$ so that the spontaneous emission from $n f^{(0)}(\mathbf{v}_1)$ test particles per unit volume is

$$4\pi n e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int d\mathbf{v}'\, f(\mathbf{v}')\, \frac{\mathbf{k}\cdot\mathbf{v}'}{k^2}\frac{\mathrm{Im}\, \varepsilon(\mathbf{k}, i\omega)}{|\,\varepsilon(\mathbf{k}, i\omega)\,|^2}.$$

Since $\mathrm{Re}\, U(\mathbf{k}, i\omega) = \int f(\mathbf{v}')\, d\mathbf{v}'\, \delta(\omega + \mathbf{k}\cdot\mathbf{v}'/k)$, this reduces to

$$\int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, \mathfrak{e}(\mathbf{k}, \omega)$$

where $\mathfrak{e}(\mathbf{k}, \omega)$ is given by Eq. (33). In a situation where instability is approached, $\mathrm{Re}\, \varepsilon \to 0$, $\mathrm{Im}\, \varepsilon \to 0$ so that $\mathfrak{e}(\mathbf{k}, \omega) \to \infty$. However, the emission

$$\mathfrak{e}(\mathbf{k}) = \int \frac{d\omega}{2\pi}\, \mathfrak{e}(\mathbf{k}, \omega)$$

remains finite. For example if $f_j^{(0)}(v)$ is given by Eqs. (26), the result from integrating across the resonance at $\omega = \omega_p$ is

$$\mathfrak{e}(0, 0, k_z) = \Theta_e \sqrt{\frac{\pi}{2}}\frac{\omega_p^4}{(k_z v_0)^3}\left\{\exp\left[-\frac{1}{2}\left(\frac{\omega_p}{k_z v_0}\right)^2\right] + \frac{\Delta n}{n}\frac{v_0}{v_1}\left[1 + \left(\frac{v_0}{v_1}\right)^2\left(1 - \frac{k_z V_e}{\omega_p}\right)\right] \times \exp\left[-\frac{1}{2}\left(\frac{\omega_p - k_z V_e}{k_z v_1}\right)^2\right]\right\} \qquad (34)$$

which remains finite when $k_z \cong \omega_p / V_e$ and

$$\frac{\Delta n}{n} \to \left(\frac{v_1}{v_0}\right)^2 \frac{V_e}{v_0} \exp\left[-\frac{1}{2}\left(\frac{V_e}{v_0}\right)^2\right].$$

2.5. Fokker-Planck equations

The fact that $\mathrm{Im}\, H_k = 0$ is sufficient to determine $\mathrm{Im}\, h_k(\mathbf{v}_1)$ in Eq. (17) and the collision operator

$$\mathrm{St}\,\{f\} = -\frac{e^2}{m}\frac{\partial}{\partial \mathbf{v}} \cdot \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{4\pi\mathbf{k}}{k^2}\, \mathrm{Im}\, h_k(\mathbf{v}), \qquad (35)$$

is therefore determined as shown by LENARD [3]. It is however instructive to obtain this result by the present methods.

The number of particles in (X', dX') at time t is

$$f(X', t') = \sum_n \delta[\mathbf{x}' - \mathbf{x}_n'(t')]\, \delta[\mathbf{v}' - \mathbf{v}_n'(t')] \qquad (36)$$

where $\mathbf{x}_n'(t')$, $\mathbf{v}_n'(t')$ describe the orbit of the n^{th} particle. $f(X', t')$ satisfies the equation

$$\left\{\frac{\partial}{\partial t'} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}'} - \frac{e}{m}\mathbf{E}(\mathbf{x}', t')\cdot\frac{\partial}{\partial \mathbf{v}}\right\} f(X', t') = 0 \qquad (37)$$

with

$$\mathbf{E}(\mathbf{x}', t') = e\sum_{l\neq n}\frac{\partial}{\partial \mathbf{x}'}\frac{1}{|\mathbf{x}' - \mathbf{x}_l'|}.$$

Eq. (37) can be integrated along the unperturbed linear orbits and the result can then be substituted back into Eq. (37) to obtain

$$\left\{\frac{\partial}{\partial t'} + \mathbf{v}'\cdot\frac{\partial}{\partial \mathbf{x}'}\right\} f(X', t') = \frac{e}{m}\frac{\partial}{\partial \mathbf{v}'}\cdot\{\mathbf{E}(\mathbf{x}', t')\, f[\mathbf{x}' - \mathbf{v}'(t' - t), \mathbf{v}'; t]\} + \left(\frac{e}{m}\right)^2 \frac{\partial}{\partial \mathbf{v}'}\cdot\mathbf{E}(\mathbf{x}', t')\int_{t''=t}^{t'} dt''\, \mathbf{E}[\mathbf{x}' + \mathbf{v}'(t'' - t'); t''] \times \frac{\partial}{\partial \mathbf{v}'} f[\mathbf{x}' + \mathbf{v}'(t'' - t'), \mathbf{v}'; t''] \qquad (38)$$

The next step is to take the ensemble average of each term in the equation. Thus

$$\langle f(\mathbf{x}',\mathbf{v}';t')\rangle = \int\sum_n \delta(\mathbf{x}'-\mathbf{x}_n')\,\delta(\mathbf{v}'-\mathbf{v}_n')\,D(Xt;X''t'';X't')\,dX\,dX''\,dX' = n f_1(X',t')$$

where f_1 is the usual one-body function.

$$\begin{aligned}\langle \mathbf{E}(\mathbf{x}',t')\,f[\mathbf{x}'-\mathbf{v}'(t'-t),\mathbf{v}';t]\rangle &= \int e\sum_{l\neq n}\frac{\partial}{\partial\mathbf{x}'}\frac{1}{|\mathbf{x}'-\mathbf{x}_l'|}\,\delta[\mathbf{x}'-\mathbf{v}'(t'-t)-\mathbf{x}_n]\\ &\quad\times\delta(\mathbf{v}'-\mathbf{v}_n)\,D(Xt;X''t'';X't')\,dX\,dX''\,dX'\\ &= n^2 e\int\frac{\partial}{\partial\mathbf{x}'}\frac{1}{|\mathbf{x}'-\mathbf{x}_2'|}\,\delta[\mathbf{x}'-\mathbf{v}'(t'-t)-\mathbf{x}_1]\\ &\quad\times\delta(\mathbf{v}'-\mathbf{v}_1)\,W_{12}(X_1t;X_2't')\,dX_1\,dX_2'\end{aligned}$$

Making use of Eq. (13) we finally get the result

$$n^2 e f_1^{(0)}(\mathbf{v}')\int\frac{d\mathbf{k}}{(2\pi)^3}\int\frac{dp}{2\pi i}\frac{e^{p(t'-t)}4\pi i\mathbf{k}}{k^2\varepsilon(\mathbf{k},p-i\mathbf{k}\cdot\mathbf{v}')}\times\left\{\int\frac{d\mathbf{v}'\,P_k(\mathbf{v},\mathbf{v}')}{p+i\mathbf{k}\cdot(\mathbf{v}'-\mathbf{v})}+\frac{1}{np}(1-\varepsilon)\right\}.$$

As usual, it is assumed that this expression goes to its asymptotic form, determined by the pole at $p=0$, in a time sufficiently short compared to observable times, that the asymptotic form is always a good approximation. Thus

$$\begin{aligned}\langle \mathbf{E}(\mathbf{x}'t')f[\mathbf{x}'-\mathbf{v}'(t'-t),\mathbf{v}';t]\rangle &\cong nef_1^{(0)}(\mathbf{v}')\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{4\pi\mathbf{k}}{k^2}\frac{\operatorname{Im}\varepsilon(\mathbf{k};-i\mathbf{k}\cdot\mathbf{v}')}{|\varepsilon|^2}\\ &= n\mathbf{E}(\mathbf{x}',X')\,f_1^{(0)}(\mathbf{v}').\end{aligned}$$

In the second term on the right hand side of Eq. (38), it is necessary to make use of Eqs. (15), the probability distributions for two singled-out particles. The result is

$$\langle \mathbf{E}(\mathbf{x}'t')\,\mathbf{E}[\mathbf{x}'+\mathbf{v}'(t''-t');t'']\,f[x'+\mathbf{v}'(t''-t'),\mathbf{v}';t'']\rangle = n\mathbf{C}^+[v'(t'-t''),\ t'-t'']\,f_1^{(0)}(\mathbf{v}')$$

where

$$\mathbf{C}^+(\mathbf{r},\tau)=\int\frac{d\mathbf{k}}{(2\pi)^3}\int\frac{dp}{2\pi i}e^{p\tau+i\mathbf{k}\cdot\mathbf{r}}\,\mathbf{S}^+(\mathbf{k},p)$$

and $\mathbf{S}^+(\mathbf{k},p)$ is given by Eq. (16). The form of the Fokker-Planck equation is therefore

$$\left\{\frac{\partial}{\partial t'}+\mathbf{v}'\cdot\frac{\partial}{\partial\mathbf{x}'}\right\}f(\mathbf{x}'t')=\mathsf{St}\{f\}$$

where

$$\mathsf{St}\{f\}=\frac{e}{m}\frac{\partial}{\partial\mathbf{v}'}\cdot\mathbf{E}(\mathbf{x}',X')\,f_1^{(0)}(\mathbf{v}')+\left(\frac{e}{m}\right)^2\frac{\partial}{\partial\mathbf{v}'}\cdot\int_0^{t'-t}d\tau\,\mathbf{C}^+(\mathbf{v}'\tau,\tau)\cdot\frac{\partial}{\partial\mathbf{v}'}f^{(0)}(\mathbf{v}') \quad (39)$$

$$\begin{aligned}\mathbf{C}^+(\mathbf{r},\tau)&=\mathbf{C}(\mathbf{r},\tau) && (\tau>0)\\ &=0 && (\tau<0)\end{aligned}$$

so that $\mathbf{C}^+$ can be replaced by $\mathbf{C}$ in the integration. $\mathbf{C}$ is a symmetric dyadic in the present case so that $\mathbf{C}(\mathbf{r},\tau)=\mathbf{C}(-\mathbf{r},-\tau)$. If most of the integral comes from values of τ less than any observable time, we can use the asymptotic value; i.e.,

$$\int_0^{t'-t}d\tau\,\mathbf{C}^+(\mathbf{v}'\tau,\tau)\cong\frac{1}{2}\int_{-\infty}^{\infty}\mathbf{C}(\mathbf{v}'\tau,\tau)\,d\tau=\frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^3}\mathbf{S}(\mathbf{k},-i\mathbf{k}\cdot\mathbf{v}').$$

Now, substituting from Eq. (19), the final result is

$$\mathsf{St}\{f\}=\frac{4\pi e^2}{m}\frac{\partial}{\partial\mathbf{v}'}\cdot\int\frac{d\mathbf{k}}{(2\pi)^3}\left\{\frac{\operatorname{Im}\varepsilon}{|\varepsilon(\mathbf{k},-i\mathbf{k}\cdot\mathbf{v}')|^2}\frac{\mathbf{k}}{k^2}f_1^{(0)}(\mathbf{v}')+\frac{\pi\omega_p^2}{k^3}\frac{\operatorname{Re}U}{|\varepsilon|^2}\frac{\mathbf{k}\mathbf{k}}{k^2}\cdot\frac{\partial f_1^{(0)}}{\partial\mathbf{v}'}\right\}. \quad (40)$$

Although this is in a different form, the result is identical to that of Lenard. The purpose of the present calculation has been simply to express the Fokker-Planck coefficients in terms of the electric field fluctuations.

There is another problem in which at $t=0$ all particles but one have the distribution function $f^{(0)}(\mathbf{v})$ which is spatially homogeneous. One particle is singled out and initially has the arbitrary distribution function $\Omega(X)$. The lowest order one-body function for the singled-out particle is

$$W^{(0)}(X,t)=\int\Omega(X_0)\,\delta[X-X_0(t)]\,dX_0 \quad (41)$$

where $\delta[X-X_0(t)]=\delta(\mathbf{x}-\mathbf{x}_0-\mathbf{v}_0t)\,\delta(\mathbf{v}-\mathbf{v}_0)$. The first order equation for W is

$$\left\{\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial x}\right\}W(Xt)=\mathsf{St}\{W\}$$

The determination of $\mathsf{St}\{W\}$ has previously been accomplished in the case where the field particle distribution is Maxwellian [2]. The equations previously employed are applicable with some alterations that will be cited. The collision operator takes the form

$$\mathsf{St}\{W\}=-\frac{1}{m}\frac{\partial}{\partial\mathbf{v}}\cdot\left\{\mathbf{F}W^{(0)}+\mathbf{T}\cdot\frac{\partial}{\partial\mathbf{v}}W^{(0)}\right\} \quad (42)$$

with

$$\mathbf{F}=-ne\int\mathbf{E}(\mathbf{x},\mathbf{x}')\,\delta f(X,X';t)\,dX'$$

$$\mathbf{T}=-ne\int\mathbf{E}(\mathbf{x},\mathbf{x}')\,\mathbf{G}(X,X';t)\,dX'.$$

δf is determined by the equations

$$\left\{\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}}+\mathbf{v}'\cdot\frac{\partial}{\partial\mathbf{x}'}\right\}\delta f(X,X';t)=-\frac{e}{m}\frac{\partial f^{(0)}}{\partial\mathbf{v}'}\cdot\frac{\partial\Phi(X,\mathbf{x}',t,)}{\partial\mathbf{x}'}+\mathsf{St}\{f^{(0)}(\mathbf{v}')\} \quad (43)$$

$$\nabla'^2\Phi(X,\mathbf{x}',t)=4\pi e\,\delta(\mathbf{x}'-\mathbf{x})+4\pi ne\int\delta f(X,X',t)\,d\mathbf{v}'.$$

In this case the only additional term is $\mathsf{St}\{f^{(0)}(\mathbf{v}')\}$ given by Eq. (40). Since it only drives the $\mathbf{k}=0$ modes, it can be omitted and the result is formally the same as before.

G is determined by the equations:

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}'} + \mathbf{v}'\cdot\frac{\partial}{\partial \mathbf{x}'}\right\} G(X, X'; t) = -\frac{e}{m}\frac{\partial f^{(0)}}{\partial \mathbf{v}'}\cdot\frac{\partial}{\partial \mathbf{x}'}\mathbf{A}(X, x'; t) - \frac{e}{m}\frac{\partial}{\partial \mathbf{x}}\psi(x, X'), \tag{44}$$

$$\nabla'^2 \mathbf{A}(X, \mathbf{x}'; t) = 4\pi n e\int G(X, X'; t)\, d\mathbf{v}',$$

and

$$\psi(x, X') = -4\pi e\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}}{k_2}[f^{(0)}(\mathbf{v}') + h_k(\mathbf{v}')]$$

where $h_k(\mathbf{v}')$ is given by Eq. (17). It is a straightforward matter to solve these equations and show that

$$\mathsf{St}\{W_i\} = \frac{e}{m}\frac{\partial}{\partial \mathbf{v}}\cdot\Big\{\mathbf{E}(\mathbf{x}, X)\, W^{(0)}(X t) + \frac{1}{2}\frac{e}{m}\int\frac{d\mathbf{k}}{(2\pi)^3}\mathbf{S}(\mathbf{k}, -i\mathbf{k}\cdot\mathbf{v})\cdot\frac{\partial}{\partial \mathbf{v}}W^{(0)}(X t)\Big\}. \tag{45}$$

The coefficients are the same as before. This is a Fokker-Planck equation of the classic type where the coefficients do not contain the dependent variable. $f^{(0)}(\mathbf{v})$ is required to be spatially homogeneous, but $W^{(0)}(Xt)$ is not.

2.6 Fluctuations with a constant magnetic field

The calculations follow the same pattern as in the case of zero magnetic field. The details of the calculations will be omitted here. The velocity coordinates $(\mathbf{v}_\perp, \beta, v_z)$ will be employed where the magnetic field is taken to be in the z-direction. The spatially homogeneous one-body function $f^{(0)}(v_\perp, v_z)$ is independent of β.

The joint probability functions are as follows:

$$W^{(0)}_{11}(Xt; X't') = V f^{(0)}(\mathbf{v})\,\delta[\mathbf{x}'-\mathbf{x}(\tau)]\frac{\delta(v_\perp{}'-v_\perp)}{v_\perp{}'}\delta(v_z{}'-v_z)\delta(\beta'-\beta-\omega_c\tau) \tag{46}$$

where $\tau = t' - t$ and

$$\mathbf{x}(\tau) = \mathbf{x} + a[\cos(\beta+\omega_c\tau) - \cos\beta]\,\mathbf{e}_x + a[\sin(\beta+\omega_c\tau) - \sin\beta]\,\mathbf{e}_y + v_z\tau\,\mathbf{e}_z$$

$$W_{12}(Xt; X't') = f^{(0)}(\mathbf{v})\, f^{(0)}(\mathbf{v}') + \int\frac{d\mathbf{k}}{(2\pi)^3}\int\frac{dp}{2\pi i}e^{p\tau + i\mathbf{k}\cdot\mathbf{r}}\,W_{pk}(\mathbf{v}, \mathbf{v}') \tag{47}$$

where $\mathbf{r} = \mathbf{x}' - \mathbf{x}$, $\mathbf{k} = (k_\perp, \alpha, k_z)$,

$$W_{pk}(\mathbf{v}, \mathbf{v}') = (1/n)\exp[-i k_\perp a'\cos(\beta'-\alpha)]\sum_{n'}\frac{J_{n'}(k_\perp a')}{p + i[\mathbf{k}\cdot\mathbf{v}']_{n'}}\, i^{n'}e^{in'(\beta'-\alpha)}$$
$$\times\Big\{n P_k(\mathbf{v}, \mathbf{v}') + i\frac{\omega_p{}^2}{\mathbf{k}^2}\frac{[\mathbf{k}\cdot\partial f/\partial \mathbf{v}']_{n'}}{\varepsilon(\mathbf{k}, p)}\times\Big[f^{(0)}(\mathbf{v}_1)e^{-ik_\perp a\cos(\beta-\alpha)}\sum_n\frac{J_n(k_\perp a)i^n e^{in(\beta-\alpha)}}{p + i[\mathbf{k}\cdot\mathbf{v}]_n}$$
$$+\int d\mathbf{v}''\, n P_k(\mathbf{v}, \mathbf{v}'')\, e^{-ik_\perp a''\cos(\beta''-\alpha)}\sum_{n''}\frac{J_n(k_\perp a'')\, i^{n''}e^{in''(\beta''-\alpha)}}{p + i[\mathbf{k}\cdot\mathbf{v}'']_{n''}}\Big]\Big\},$$

$$[\mathbf{k}\cdot\mathbf{v}]_n = k_z v_z + n\omega_c$$

$$\left[\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right]_n = k_z\frac{\partial f^{(0)}}{\partial v_z} + \frac{n}{a}\frac{\partial f^{(0)}}{\partial v_\perp}$$

$$\varepsilon(\mathbf{k}, p) = 1 - \frac{\omega_p{}^2}{k^2}\int d\mathbf{v}\sum_n\frac{J_n{}^2(k_\perp a)\, i[\mathbf{k}\cdot\partial f/\partial \mathbf{v}]_n}{p + i[\mathbf{k}\cdot\mathbf{v}]_n}.$$

Quantities analogous to $h_k(\mathbf{v}_1)$ and $H_k(u)$ in Section 2.3 are introduced as follows:

$$h_k(\mathbf{v}) = \int n P_k(\mathbf{v}, \mathbf{v}')\, d\mathbf{v}' = e^{-ik_\perp a\cos(\beta-\alpha)}\sum_n h_n(v_\perp, v_z)\, J_n(k_\perp a)\, i^n e^{in(\beta-\alpha)}$$

$$H(u) = \sum_n\int h_n(v_\perp, v_z)\, J_n{}^2(k_\perp a)\,\delta(u - [\mathbf{k}\cdot\mathbf{v}]_n/k)\, d\mathbf{v}.$$

The results for the electric field fluctuations are then formally the same as the zero magnetic field case including the theorems discussed in Section 2.4. For example,

$$\mathbf{S}^+(\mathbf{k}, i\omega) = (4\pi)^2 n e^2\frac{\mathbf{k}\mathbf{k}}{k^4}\frac{\pi i}{k}\left\{\frac{1-\varepsilon(\mathbf{k}, i\omega)}{\varepsilon\,\mathrm{Im}\,\varepsilon}\mathrm{Re}\, U - \frac{H(-\omega/k)}{\mathrm{Im}\,\varepsilon}\right\} \tag{48}$$

where the present definitions of ε and H apply and

$$U(\mathbf{k}, p) = \frac{k}{\pi}\int d\mathbf{v}\, f^{(0)}(\mathbf{v})\sum_n\frac{J_n{}^2(k_\perp a)}{p + i[\mathbf{k}\cdot\mathbf{v}]_n}.$$

It has been established, that $\mathrm{Im}\, H(u) = 0$ in the presence of a magnetic field, Rostoker [4]. Therefore

$$\mathbf{S}(\mathbf{k}, \omega) = (4\pi)^2 n e^2\frac{\mathbf{k}\mathbf{k}}{k^4}\frac{2\pi}{k}\frac{\mathrm{Re}\, U(\mathbf{k}, i\omega)}{|\varepsilon(\mathbf{k}, i\omega)|^2} \tag{49}$$

and

$$\mathbf{S}^+(\mathbf{k}, i\omega) = \frac{\mathbf{S}(\mathbf{k}, \omega)}{2} + \frac{i}{2\pi}\mathsf{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'-\omega}\mathbf{S}(\mathbf{k}, \omega'). \tag{50}$$

It should be noted that Eqs. (48), (49) and (50) imply an explicit expression for $\mathrm{Re}\, H(u)$

$$\frac{\mathrm{Re}\, H(u)}{\mathrm{Im}\,\varepsilon} = \left[1 - \frac{\mathrm{Re}\,\varepsilon(\mathbf{k}, -iku)}{|\varepsilon|^2}\right]\frac{\mathrm{Re}\, U}{\mathrm{Im}\,\varepsilon} - \frac{1}{\pi}\mathsf{P}\int_{-\infty}^{\infty}\frac{du'}{u'-u}\frac{\mathrm{Re}\, U(\mathbf{k}, -iku')}{|\varepsilon(\mathbf{k}, -iku')|^2}. \tag{51}$$

By means of the procedure employed in Section 2.5, the Fokker-Planck equation for a spatially homogeneous plasma with a constant magnetic field is obtained. The result is

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} - \frac{e}{mc}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial}{\partial \mathbf{v}}\right\} f(Xt) = \mathsf{St}\{f\} \tag{52}$$

$$\mathsf{St}\{f\} = \frac{e}{m}\frac{\partial}{\partial \mathbf{v}}\mathbf{E}(\mathbf{x}, X) f^{(0)}(\mathbf{v}) + \left(\frac{e}{m}\right)^2\frac{\partial}{\partial \mathbf{v}}\cdot\int_0^\infty d\tau\,\mathbf{C}(\mathbf{r}(\tau); \tau)\cdot\frac{\partial}{\partial \mathbf{v}}f^{(0)}(\mathbf{v})$$

$$\mathbf{E}(\mathbf{x}, X) = 4\pi e\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{i\mathbf{k}}{k^2}e^{-ik_\perp a\cos(\beta-\alpha)}\sum_n\frac{J_n(k_\perp a)\, i^n e^{in(\beta-\alpha)}}{\varepsilon(\mathbf{k}, -i[\mathbf{k}\cdot\mathbf{v}]_n)} \tag{53}$$

and $\mathbf{r}(\tau) = \mathbf{x}(\tau) - \mathbf{x}$ where $\mathbf{x}(\tau)$ is given in Eq. (47). It should be noted that $\mathbf{C}(\mathbf{r}(\tau); \tau) \neq \mathbf{C}(\mathbf{r}(-\tau); -\tau)$ so that the limits of the τ-integration in Eq. (52) may not be changed from $(0, \infty)$ to $(-\infty, \infty)$. Instead we have

$$\int_0^\infty d\tau \, \mathbf{C}(\mathbf{r}(\tau);\tau) = \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_{n,n'} J_n(k_\perp a) J_{n'}(k_\perp a) \, \mathrm{i}^{n-n'} \mathrm{e}^{\mathrm{i}(n-n')(\beta-\alpha)} \times \mathbf{S}^+(\mathbf{k}, -i[\mathbf{k}\cdot\mathbf{v}]_n). \quad (54)$$

where $\mathbf{S}^+(\mathbf{k}, \mathrm{i}\omega)$ is given by Eq. (48) or Eqs. (49) and (50). The only dependence on α is through the operators

$$\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{v}} = -\frac{k_\perp}{v_\perp}\cos(\beta-\alpha)\frac{\partial}{\partial\beta} - k_\perp \sin(\beta-\alpha)\frac{\partial}{\partial v_\perp} + k_z \frac{\partial}{\partial v_z}, \quad (55)$$

so that the α-integrations can easily be carried out. This accomplishes the following reduction of the Fokker-Planck equation

$$\mathrm{St}\{f\} = -\frac{4\pi e^2}{m}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{\mathrm{Im}}{k^2}\sum_n\left[\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right]_n J_n^2(k_\perp a) \times\left\{\frac{f^{(0)}}{\varepsilon(k,-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)} - \frac{\pi\mathrm{i}\,\omega_p^2}{k^3}\left[\frac{\mathrm{Re}\,U(k,-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)}{|\varepsilon(k,-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)|^2} + \frac{\mathrm{i}}{\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega'+[\mathbf{k}\cdot\mathbf{v}]_n}\frac{\mathrm{Re}\,U(\mathbf{k},\mathrm{i}\omega')}{|\varepsilon(\mathbf{k},\mathrm{i}\omega')|^2}\right]\left[\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right]_n f^{(0)}\right\} \quad (56)$$

where

$$\left[\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right]_n = k_z\frac{\partial}{\partial v_z} + \frac{n}{a}\frac{\partial}{\partial v_\perp} + \mathrm{i}\frac{k_\perp}{v_\perp}\frac{J_n'(k_\perp a)}{J_n(k_\perp a)}\frac{\partial}{\partial\beta}.$$

Since $\partial f^{(0)}/\partial\beta = 0$ for a spatially homogeneous plasma, the result is simply

$$\mathrm{St}\{f\} = -\frac{4\pi e^2}{m}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{k^2}\sum_n\left[\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right]_n J_n^2(k_\perp a)\,\mathrm{Im}\,h_n(v_\perp, v_z)$$

where

$$-\mathrm{Im}\,h_n(v_\perp,v_z) = \frac{1}{|\varepsilon(k,-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)|^2}\left\{f^{(0)}(\mathbf{v})\,\mathrm{Im}\,\varepsilon + \frac{\pi\,\omega_p^2}{k^3}\mathrm{Re}\,U\left[\mathbf{k}\cdot\frac{\partial f^{(0)}}{\partial\mathbf{v}}\right]_n\right\}. \quad (57)$$

This result has been obtained previously [4]. We note that the terms involving the Hilbert transform of the spectral density have all dropped out, not because of symmetry of $\mathbf{C}[\mathbf{r}(\tau), \tau]$, but because $f^{(0)}(\mathbf{v})$ is independent of β. In the case of a test-particle problem the Fokker-Planck equation takes the form

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial\mathbf{x}} - \frac{e}{mc}\mathbf{v}\times\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{v}}\right\} w(Xt) = \mathrm{St}\{w\}$$

where

$$\mathrm{St}\{w\} = \frac{e}{m}\frac{\partial}{\partial\mathbf{v}}\mathbf{E}(\mathbf{x},X)w^{(0)}(X,t) + \left(\frac{e}{m}\right)^2\frac{\partial}{\partial\mathbf{v}}\cdot\int_0^\infty d\tau\,\mathbf{C}(\mathbf{r}(\tau),\tau)\cdot\frac{\partial}{\partial\mathbf{v}}w^{(0)}(Xt) \quad (58)$$

and $\mathbf{E}(\mathbf{x}, X)$, $\mathbf{C}[\mathbf{r}(\tau), \tau]$ are the same as in Eq. (52). In this case however $w^{(0)}(X, t)$, the lowest order one-body function for the singled-out particle is arbitrary so that $\partial w^{(0)}/\partial\beta \neq 0$.

The collision operator can be expressed as

$$\mathrm{St}\{w\} = \frac{1}{m}\left\{\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp F_\beta w^{(0)} + \frac{\partial}{\partial v_z}F_z w^{(0)} - \frac{1}{v_\perp}F_\varrho\frac{\partial w^{(0)}}{\partial\beta} + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp T_{\beta\beta}\frac{\partial w^{(0)}}{\partial v_\perp} + \frac{\partial}{\partial v_z}T_{zz}\frac{\partial}{\partial v_z}w^{(0)} + \frac{T_{\varrho\varrho}}{v_\perp^2}\frac{\partial^2 w^{(0)}}{\partial\beta^2} + \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp T_{\beta z}\frac{\partial w^{(0)}}{\partial v_z} + \frac{\partial}{\partial v_z}T_{\beta z}\frac{\partial w^{(0)}}{\partial v_\perp} + \frac{1}{v_\perp}\left(\frac{\partial T_{\varrho\beta}}{\partial v_\perp} - \frac{\partial T_{z\varrho}}{\partial v_z}\right)\frac{\partial w^{(0)}}{\partial\beta}\right\}.$$

The coefficients in this collision operator are obtained from Eq. (56).

$$F_z = 4\pi e^2\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_z}{k^2}\sum_n\frac{J_n^2(k_\perp a)\,\mathrm{Im}\,\varepsilon}{|\varepsilon(k,-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)|^2}$$

$$F_\beta = 4\pi e^2\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{k^2}\sum_n\frac{n}{a}\frac{J_n^2(k_\perp a)\,\mathrm{Im}\,\varepsilon}{|\varepsilon(k,-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)|^2}$$

$$T_{zz} = (2\pi e\omega_p)^2\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_z^2}{k^5}\sum_n\frac{J_n^2(k_\perp a)\,\mathrm{Re}\,U(\mathbf{k},-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)}{|\varepsilon(\mathbf{k},-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)|^2}$$

$$T_{\beta\beta} = (2\pi e\omega_p)^2\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{1}{k^5}\sum_n\left(\frac{n}{a}\right)^2\frac{J_n^2(k_\perp a)\,\mathrm{Re}\,U}{|\varepsilon(\mathbf{k},-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)|^2}$$

$$T_{\beta z} = (2\pi e\omega_p)^2\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_z}{k^5}\sum_n\left(\frac{n}{a}\right)\frac{J_n^2(k_\perp a)\,\mathrm{Re}\,U}{|\varepsilon(\mathbf{k},-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)|^2}.$$

These coefficients are the same as in the spatially homogeneous case. The additional coefficients are

$$F_\varrho = 4\pi e^2\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_\perp}{k^2}\sum_n\frac{J_n(k_\perp a)J_n'(k_\perp a)\,\mathrm{Re}\,\varepsilon}{|\varepsilon(k,-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)|^2}$$

$$T_{\varrho\varrho} = (2\pi e\omega_p)^2\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_\perp^2}{k^5}\sum_n\frac{[J_n'(k_\perp a)]^2\,\mathrm{Re}\,U}{|\varepsilon(\mathbf{k},-\mathrm{i}[\mathbf{k}\cdot\mathbf{v}]_n)|^2}$$

$$T_{\varrho\beta} = -\frac{(2\pi e\omega_p)^2}{\pi}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_\perp}{k^5}\sum_n\frac{n}{a}J_n(k_\perp a)J_n'(k_\perp a) \times \mathrm{P}\int_{-\infty}^{\infty}\frac{d\omega}{\omega+[\mathbf{k}\cdot\mathbf{v}]_n}\frac{\mathrm{Re}\,U(\mathbf{k},\mathrm{i}\omega)}{|\varepsilon(\mathbf{k},\mathrm{i}\omega)|^2}$$

$$T_{z\varrho} = \frac{(2\pi e\omega_p)^2}{\pi}\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_\perp k_z}{k^5}\sum_n J_n(k_\perp a)J_n'(k_\perp a) \times \mathrm{P}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega+[\mathbf{k}\cdot\mathbf{v}]_n}\frac{\mathrm{Re}\,U(\mathbf{k},\mathrm{i}\omega)}{|\varepsilon(\mathbf{k},\mathrm{i}\omega)|^2}.$$

2.7 Diffusion in the presence of a constant magnetic field

Consider the test-particle problem where the lowest order one-body function is

$$w^{(0)}(X' t') = \delta\,[\mathbf{x}' - \mathbf{x}(\tau)]\,\frac{\delta(v_\perp{}' - v_\perp)}{v_\perp{}'}\,\delta(v_z{}' - v_z)\,\delta(\beta' - \beta + \omega_c \tau)$$

and $\mathbf{x}(\tau)$ is given in Eq. (43). This function simply describes the motion of the test particle on its unperturbed orbit. We shall be interested in the quantity

$$\langle \mathbf{r}_\perp{}^2 \rangle = \int w(X' t')\, d X'\, r_\perp{}^2$$

where

$$\mathbf{r}_\perp{}^2 = [x' - x(\tau)]^2 + [y' - y(\tau)]^2 .$$

If $w(X' t') = w^{(0)}(X' t')$, it is clear that $\langle r_\perp{}^2 \rangle = 0$. The collision operator, however, has the effect of spreading out the distribution function so that if $w(X' t) = w^{(0)} + w^{(1)}$, then $\langle \mathbf{r}_\perp{}^2 \rangle \neq 0$. To calculate $w^{(1)}$, we integrate Eq. (58) along the unperturbed orbits:

$$w^{(1)}(X' t') = \int_{t''=t}^{t'} d t''\, \mathsf{St}\,\{w^{(0)}(t'')\}$$

where

$$\begin{aligned} w^{(0)}(t'') = \delta\,\{ & x' + a'(\cos[\beta' + \omega_c(t'' - t')] - \cos\beta') - \\ & - x - a(\cos[\beta + \omega_c(t'' - t)] - \cos\beta)\} \\ \times\, \delta\,\{ & y' + a'(\sin[\beta' + \omega_c(t'' - t')] - \sin\beta') - y \\ & - a(\sin[\beta' + \omega_c(t'' - t)] - \sin\beta)\} \\ \times\, \delta\,[& z' + v_z{}'(t'' - t') - z - v_z(t'' - t)] \\ \times\, & \frac{\delta(v_\perp{}' - v_\perp)}{v_\perp{}'}\,\delta(v_z{}' - v_z)\,\delta[\beta' - \beta - \omega_c(t' - t)] . \end{aligned} \tag{60}$$

Eq. (59) is employed for $\mathsf{St}\,\{w^{(0)}\}$. To calculate $\langle \varrho_\perp{}^2 \rangle$ we first carry out the coordinate integration, then the velocity integration and finally the time integration. It will be apparent that only two terms in $\mathsf{St}\,\{w^{(0)}\}$ produce anything so that the others will be omitted.

$$\begin{aligned} \langle r_\perp{}^2 \rangle = & \int_t^{t'} d t'' \int_0^{2\pi} d\beta' \int_0^{\infty} v_\perp{}'\, d v_\perp{}' \\ & \times \Big[\{a'(\cos[\beta' + \omega_c(t'' - t')] - \cos\beta') \\ & - a(\cos[\beta + \omega_c(t'' - t)] - \cos[\beta + \omega_c(t' - t)])\}^2 \\ & + \{a'(\sin[\beta' + \omega_c(t'' - t)] - \sin\beta') \\ & - a(\sin[\beta + \omega_c(t'' - t)] - \sin[\beta + \omega_c(t' - t)])\}^2 \Big] \\ & \times \frac{1}{m} \left\{ \frac{T_{\varrho\varrho}}{v'_\perp{}^2} \frac{\partial}{\partial \beta'^2} + \frac{1}{v'_\perp} \frac{\partial}{\partial v'_\perp} v'_\perp\, T_{\beta\beta} \frac{\partial}{\partial v'_\perp} \right\} \\ & \frac{\delta(v'_\perp - v_\perp)}{v'_\perp}\,\delta[\beta' - \beta - \omega_c(t' - t)] . \end{aligned}$$

Now integrate by parts twice. It is apparent that the only contributions obtain when $r_\perp{}^2$ is differentiated twice. The result is therefore

$$\begin{aligned} \langle r_\perp{}^2 \rangle & = \int_{t''=t}^{t'} d t''\, \frac{4 a^2}{m v_\perp{}^2} (T_{\varrho\varrho} + T_{\beta\beta})\,[1 - \cos\omega_c(t'' - t')] \\ & \cong \frac{4}{m \omega_c{}^2} \{T_{\varrho\varrho} + T_{\beta\beta}\}\,(t' - t) \end{aligned} \tag{61}$$

for $t' - t \gg 1/\omega_c$. In terms of the electric field fluctuations

$$\mathsf{T} = \frac{e^2}{m} \int_0^{\infty} d\tau\, \mathsf{C}(\mathbf{r}(\tau); \tau)\, d\tau ,$$

where

$$\mathsf{C}(\mathbf{r}(\tau), \tau) = \langle \mathbf{E}(\mathbf{x} t)\, \mathbf{E}(\mathbf{x} + \mathbf{r}(\tau), t + \tau) \rangle .$$

Therefore $\langle r_\perp{}^2 \rangle = D(t' - t)$, where the diffusion coefficient is

$$D = \frac{4 c^2}{B^2} \int_0^{\infty} d\tau\, \langle \mathbf{E}_\perp(\mathbf{x} t) \cdot \mathbf{E}_\perp(\mathbf{x} + \mathbf{r}(\tau), t + \tau) \rangle \tag{62}$$

and

$$\mathbf{E}_\perp = \mathbf{E} - \frac{\mathbf{B} \cdot \mathbf{E}}{B^2} \mathbf{B} .$$

An elementary derivation of a formula similar to Eq. (62) has previously been given by Spitzer [5]. Eq. (62) is, in fact, the same as Spitzer's formula for a zero-velocity test charge; i.e., $\mathbf{r}(\tau) = 0$.

Spitzer has discussed the effect of an instability of the collective modes of oscillation on the diffusion coefficient D. It is a qualitative discussion because his formula is not explicit and because non-linear effects are considered. The present calculations do not include non-linear effects since they are restricted to stable distribution functions. However we may consider a distribution function that approaches instability when some parameter is varied such as that given by Eq. (26). Only a zero velocity test charge will be considered so that

$$D = \frac{4 n e^2 c^2}{B^2} \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{k_\perp{}^2}{k^5} \left\{ \frac{\operatorname{Re} U(\mathbf{k}, -\mathrm{i}\,\omega_c)}{|\varepsilon(\mathbf{k}, -\mathrm{i}\,\omega_c)|^2} + \frac{\operatorname{Re} U(\mathbf{k}, \mathrm{i}\,\omega_c)}{|\varepsilon(\mathbf{k}, \mathrm{i}\,\omega_c)|^2} \right\} \tag{63}$$

where

$$\begin{aligned} \operatorname{Re} U(\mathbf{k}, \pm \mathrm{i}\omega_c) = \frac{k}{2\pi} \int_{-\infty}^{\infty} d\tau \Big\{ & \exp\Big[\mp \mathrm{i}\omega_c \tau - \frac{(k_z v_0 \tau)^2}{2} \\ & - (k_\perp a_0)^2 (1 - \cos\omega_c \tau) \Big] \\ & + \frac{\Delta n}{n} \exp\Big[\mp \mathrm{i}\omega_c \tau + \mathrm{i} k_z V_e \tau - \frac{(k_z v_1 \tau)^2}{2} \\ & - (k_\perp a_1)^2 (1 - \cos\omega_c \tau) \Big] \end{aligned} \tag{64}$$

$$a_0 = v_0/\omega_c , \quad a_1 = v_1/\omega_c ;$$

$$\begin{aligned} \varepsilon(\mathbf{k}, \pm \mathrm{i}\omega_c) = 1 - \frac{1}{(k L_0)^2} \Big\{ & \int_0^{\infty} d\tau\, \exp[\mp \mathrm{i}\,\omega_c \tau] \\ & \frac{\partial}{\partial \tau} \exp\Big[- \frac{(k_z v_0 \tau)^2}{2} - (k_\perp a_0)^2 (1 - \cos\omega_c \tau) \Big] \\ & + \frac{\Delta n}{n} \left(\frac{v_0}{v_1} \right)^2 \int_0^{\infty} d\tau \exp[\mp \mathrm{i}\omega_c \tau + \mathrm{i} k_z V_e \tau] \\ & \frac{\partial}{\partial \tau} \exp\Big[- \frac{(k_z v_1 \tau)^2}{2} - (k_\perp a_1)^2 (1 - \cos\omega_c \tau) \Big] \Big\} \end{aligned} \tag{65}$$

where $1/L_0{}^2 = \omega_p{}^2/v_0{}^2$.

We assume that $\omega_p > \omega_c$ and $(\Delta n/n)(v_0/v_1)^2 \ll 1$. Δn is the parameter to be varied to ultimately produce instability. Eq. (63) can be evaluated by considering various regions of **k**-space in which U and ε have simple asymptotic forms. For example if $k > 1/L_0$, $\varepsilon \cong 1$, $\operatorname{Re} U \cong 1/\sqrt{2\pi v_0^2}$ and

$$D_0 \cong \frac{4\sqrt{2}}{3\pi^{5/2}} \frac{a_0^2 e^2}{L_0^2} \frac{\ln(k_{\max} L_0)}{m v_0}, \tag{66}$$

where $k_{\max}$ is the usual cut-off at the inverse of the closest distance of approach. This is the usual classical result. Now consider the contribution from the region $ka_0 < 1$ where

$$\operatorname{Re} U(k, \pm i\omega_c) \cong \frac{k}{|k_z|}\left\{\frac{1}{\sqrt{2\pi v_0^2}} \exp\left[-\frac{1}{2}\left(\frac{\omega_c}{k_z v_0}\right)^2\right] + \frac{1}{\sqrt{2\pi v_1^2}} \frac{\Delta n}{n} \exp\left[-\frac{1}{2}\frac{(\pm\omega_c - k_z V_e)^2}{(k_z v_1)^2}\right]\right\}$$

$$|\varepsilon(k, \pm i\omega_c)|^2 \cong \left[1 - \left(\frac{k_z \omega_p}{k\omega_c}\right)^2\right]^2 + \frac{\pi}{2}\frac{1}{(k L_0)^4}\left\{\frac{\omega_c}{|k_z| v_0}\exp\left[-\frac{1}{2}\left(\frac{\omega_c}{k_z v_0}\right)^2\right] + \frac{(\mp\omega_c - k_z V_e)}{|k_z| v_1}\frac{\Delta n}{n}\left(\frac{v_0}{v_1}\right)^2 \exp\left[-\frac{1}{2}\left(\frac{\mp\omega_c - k_z V_e}{k_z v_1}\right)^2\right]\right\}^2.$$

It is more convenient to carry out the integration in spherical coordinates i.e., $k_z = k\mu$, $k_\perp = k\sqrt{1-\mu^2}$, $d\mathbf{k} = 2\pi k^2\, dk\, d\mu$. There is a resonance for $\mu^2 = \omega_c^2/\omega_p^2$ that is sufficiently narrow to permit approximate methods of integration as long as $ka_0 \ll 1$ and $(\Delta n/n)(v_0/v_1)^2 \ll 1$. After integrating across the resonance

$$D = \frac{4}{(2\pi)^3}\frac{a_0^2 e^2}{L_0^2}\frac{1}{m v_0}\left(\frac{\omega_c}{\omega_p}\right)^3 a_0^3 \int_0^{1/a_0} k^2\, dk \times \tag{67}$$

$$\left\{\frac{\exp\left[-\frac{1}{2}\left(\frac{\omega_p}{k v_0}\right)^2\right] + \frac{v_0}{v_1}\frac{\Delta n}{n}\exp\left[-\frac{1}{2}\left(\frac{\omega_p - k V_e}{k v_1}\right)^2\right]}{\exp\left[-\frac{1}{2}\left(\frac{\omega_p}{k v_0}\right)^2\right] + \frac{\omega_p - k V_e}{\omega_p}\left(\frac{v_0}{v_1}\right)^3\frac{\Delta n}{n}\exp\left[-\frac{1}{2}\left(\frac{\omega_p - k V_e}{k v_1}\right)^2\right]}\right\}$$

If $\Delta n/n = 0$, the result is small compared to Eq. (66). However, if

$$\frac{\Delta n}{n} \to \left(\frac{v_1}{v_0}\right)^2 \frac{V_e}{v_1}\exp\left[-\frac{1}{2}\frac{V_e}{v_0}\right]^2,$$

the denominator in Eq. (67) becomes infinite for $k \cong \omega_p / V_c$. We may therefore expect significant contributions to the diffusion coefficient from the collective modes when they become unstable. The present formalism is not suitable for calculating the diffusion coefficient under these circumstances, since it diverges. It should be possible to treat the linear phase of instabilities by an extension of the present formalism in which the time-dependence of the Fokker-Planck coefficients is retained.

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