REPORT NO.
UCB/SEMM-88/06

STRUCTURAL ENGINEERING,
MECHANICS AND MATERIALS

MEASURES OF STRUCTURAL SAFETY UNDER IMPERFECT STATES OF KNOWLEDGE

by

ARMEN DER KIUREGHIAN

Report to Sponsor:
National Science Foundation

MARCH 1988

DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA AT BERKELEY
BERKELEY, CALIFORNIA

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Armen Der Kiureghian
Professor of Civil Engineering

A report on research sponsored by
the National Science Foundation
under Grant No. CES-8618905

Report No. UCB/SEMM-88/06
Structural Engineering, Mechanics, and Materials
Department of Civil Engineering
University of California, Berkeley

March 1988

ABSTRACT

In assessing structural safety, the state of knowledge is imperfect when uncertainties exist due to estimation and modeling errors. The nature of these uncertainties is fundamentally different from the uncertainties arising from inherent variabilities. A reliability index defined under such conditions of uncertainty is a point estimator of safety.

Motivated by needs in structural code development, a set of fundamental requirements on the point-estimator reliability index are formulated. Existing reliability indices are examined in light of these requirements and are found to be lacking with regard to one or more of the requirements. Based on concepts in Bayesian statistical decision theory, a new index is introduced which is shown to satisfy all the stipulated requirements. The index recognizes the fundamental difference between the sources of uncertainty and provides a rational basis for the assessment of structural safety and for development of reliability-based codes under arbitrary states of knowledge.

Methods are developed for quantifying the uncertainty in the measure of safety arising from the imperfect state of knowledge. It is shown that existing reliability methods can be used to compute the probability distribution or variance of the safety measure. A simple example, showing the uncertainty in the reliability index as a function of the sample size, is used to illustrate the main concepts of the paper.

ACKNOWLEDGEMENT

The work presented in this report has been supported in part by the National Science Foundation under Grant No. CES-8618905, with Dr. S. C. Liu as Program Director. This support is gratefully acknowledged. Opinions, findings, and conclusions expressed in this publication are those of the author and do not necessarily reflect the views of the National Science Foundation.

Special thanks are due Peter Bjerager for insightful discussions during the course of this study and for his critical comments on an early version of the manuscript.

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INTRODUCTION

In assessing structural safety, four broad sources of uncertainty are relevant: (1) *inherent variability*, (2) *estimation error*, (3) *model imperfection*, and (4) *human error*. The first kind of uncertainty (often called randomness) arises from inherent variability in the characteristics of the structure itself, such as variability in its material properties and member strengths, or inherent variability in the environment to which the structure is exposed, such as variability in loads and support movements. The second source of uncertainty, estimation error, arises from incompleteness of statistical data and our inability to accurately predict probability laws governing the inherent variabilities. An example is the error in predicting the parameters of a probability distribution due to the limited size of the available sample. The third kind of uncertainty, model imperfection, arises from our use of idealized mathematical models to describe complex phenomena. Model imperfection has two components: one due to our lack of understanding of the phenomenon itself, which may be denoted *ignorance*, and the other due our use of simplified models, which may be denoted *error of simplification*. Imperfections in both mechanical models (e.g., a model describing the flexural strength of a reinforced concrete beam) and probabilistic models (e.g., the choice of a parameterized distribution model) give rise to this kind of uncertainty. Finally, the human error uncertainty arises from errors made by engineers or operators in the design, construction, or operation phases of a structure. Examples may include calculational errors or omissions in the design phase, errors in the placement of rebars in reinforced concrete construction, and errors in the operation of a structure which result in its exposure to overloads.

There is a fundamental difference between the first and the next two sources of uncertainty. Namely, whereas inherent variability is intrinsic to nature and beyond our control, the uncertainties due to estimation error and model imperfection are extrinsic and to some extent reducible. For example, a reliability analyst may choose to obtain additional information to improve the accuracy of estimation, or use more refined models to reduce errors of simplification. He/she may even choose to conduct experiments or analyses to gain a better understanding of the relevant phenomena, and thereby reduce the model uncertainty due to ignorance. Such improvements, however, usually entail an investment in time and money which the analyst, or his/her client, may not be willing to undertake. Nevertheless, the sheer possibility of influencing the uncertainties due to estimation error and model imperfection signifies a fundamental difference between them and the uncertainty due to inherent variability. The uncertainty due to human error may also be reduced by implementing rigorous quality control measures in the design, construction, and operation phases of a structure. Such measures tend to reduce the rate of occurrence of human errors and/or the magnitudes of their consequences. However, within a specified quality control program, human errors tend to occur inherently randomly.

The available statistical information (objective and subjective) on relevant variables and the set of mechanical and probabilistic models and their associated error estimates constitute the *state of knowledge* in a reliability problem. The state of knowledge is said to be *perfect* when complete statistical information (i.e., probability laws governing inherent variabilities) and perfect models are available, i.e., when there is no uncertainty due to estimation error or model imperfection; otherwise, the state of knowledge is said to be *imperfect*. Most real engineering problems deal with imperfect states of knowledge.

In an ideal situation where the state of knowledge is perfect and uncertainties arise only from inherent variabilities, a strict measure of structural safety is the probability of failure, denoted P_F. An alternative measure is the reliability index, β, conventionally defined by (Madsen et al. 1986)

$$\beta = \Phi^{-1}(1 - P_F) \tag{1}$$

where $\Phi^{-1}(.)$ is the inverse of the standard normal probability. This index, which is also a strict measure of safety under a perfect state of knowledge, offers a more convenient range for analysis: For problems of structural engineering interest P_F usually ranges from 10^{-1} to 10^{-7}, while β ranges from 1 to 5. It is important to note that P_F and β are intrinsic statistical properties of the structure and its environment; these properties cannot be influenced without physically changing either the structure or its environment.

When the state of knowledge is imperfect, i.e, when the statistical information is incomplete and/or the employed models are imperfect, a strict assessment of structural safety is impossible. Because of the uncertainties arising from estimation error and model imperfection, P_F and β are themselves uncertain and can only be assessed in a probabilistic sense, i.e., through probability distributions. The dispersions in these distributions may be considered as measures of the quality of knowledge. In particular, one may expect that when the state of knowledge is improved (by either increasing the amount of statistical information, using more refined models, or both), the dispersions in the distributions of P_F and β will decrease. Thus, the dispersions in the distributions of P_F and β are not intrinsic properties of the structure or its environment. Rather, they are due to *our* use of incomplete statistical information or imperfect models.

Several measures of structural safety under imperfect states of knowledge have been introduced in the past two decades. These include: (1) rule-based reliability indices; (2) estimates of probability of failure or reliability index based on predictor models; (3) confidence bounds on the probability of failure; and (4) fuzzy measures of safety.

Among the above measures, rule-based reliability indices have received the widest attention and popularity. The word "rule-based" is used here, since these indices are based on reasonable rules rather than derived from basic principles. These measures may be regarded as point-estimators of β. A review of these indices is given later in this paper. In particular, the first-order, second-moment reliability index, which employs only means and variance/covariances of random variables (a case of incomplete statistical information), has played a central role in the development of modern structural design codes (e.g., CEB 1976, Ravindra et al. 1978, Ellingwood et al. 1982). In such codes, partial factors (such as load and resistance factors in AISC's LRFD code, Ravindra et al. 1978) are determined in terms of a target reliability index, which itself is determined by calibration to the accepted practice. In this context, the role of the rule-based reliability index is a relative measure of safety for comparison.

The second measure described above is based on predictor distributions of uncertain variables, which combine the uncertainties due to inherent variability, estimation error, and model imperfection. No distinction between the natures of these uncertainties is made in this approach. This measure is examined later in this paper.

Methods for developing confidence intervals or probability distributions on the probability of failure have received limited attention in the theoretical reliability literature, but have been used extensively in application, particularly in studies dealing with probabilistic risk analysis of nuclear power plants. In the context of code development, such an approach is not suitable as it does not provide a convenient means for calibration or decision making. Nevertheless, there are situations where an explicit measure of confidence in computed probabilities is required. Methods for such analyses are described in this paper.

Finally, the fuzzy safety measure (Brown 1979) is based on fuzzy set theory, which emphasizes the subjective nature of uncertainties. This paper will not address this approach.

The main focus of this paper is on the analysis and development of point-estimators of β for use as measures of structural safety under imperfect states of knowledge. Motivated by needs and requirements in structural code development, a set of fundamental requirements on the point-estimator are formulated. These include requirements previously introduced by others (Hasofer and Lind 1974, Veneziano 1979, Ditlevsen 1979, Der Kiureghian and Liu 1986), which are extended or modified here, as well as new requirements that together form a complete set that is relevant to a broad spectrum of reliability problems. Existing definitions of rule-based reliability indices are reviewed and examined in light of these requirements. Then, using elementary concepts from optimal statistical decision theory (Ang and Tang, 1984), a new point-estimate reliability index is introduced. It is shown that the new index is a generalization of some previously defined indices and that it is superior to all previous indices in satisfying the stated requirements. Methods for computing the distribution of β and its confidence intervals are also described. The paper concludes with a numerical example that illustrates the main concepts of the paper. For convenience, in the following the term "reliability index" is used to denote a point-estimator of β, while β itself is denoted "the strict reliability index."

The approach undertaken in this paper is consistent with the Bayesian thinking in its treatment of objective and subjective information, and in using probability theory for the modeling and analysis of all uncertainties. However, unlike previous works in structural reliability that have employed the Bayesian method, the present work explicitly distinguishes the uncertainties arising from inherent variability from those arising from estimation error or model imperfection, and thereby relates the measure of safety to the quality of the state of knowledge.

FUNDAMENTAL REQUIREMENTS ON THE RELIABILITY INDEX

As stated in the introduction, under an imperfect state of knowledge, the strict reliability index, β, is uncertain and is described in terms of a probability distribution. The reliability index under such a condition is defined as a point estimator, $\hat{\beta}$, of β. As the state of knowledge improves, the dispersion in the distribution of β decreases. As a matter of consistency, we shall require that when the state of knowledge approaches the perfect state and the dispersion in the distribution of β vanishes, $\hat{\beta}$ coincide with β defined by Eq. 1. We denote this as the *consistency* requirement.

The second requirement that we impose on the reliability index is that it shall incorporate the entire information available for each reliability problem. Thus, for example, if information on higher moments or distributions are available, the reliability index shall incorporate such information and not be restricted to the second moments only. This also requires that the reliability index fully account for the uncertainties arising from estimation error and model imperfection. We denote this as the *completeness* requirement. It is important to note that in real applications of structural reliability one is required to deal with a broad spectrum of knowledge states. Therefore, the reliability index shall be capable of incorporating arbitrary states of knowledge.

The third requirement that we impose on the reliability index is that, for a given state of knowledge, it shall be invariant with respect to mutually consistent formulations of the reliability problem. In particular, the computed reliability index shall be the same for all mutually consistent formulations of the safety criterion. This requirement, denoted *invariance*, is well known for the second-moment reliability index (Ditlevsen 1973, Hasofer and Lind 1974). However, here, it is stated in a broader sense for reliability problems under arbitrary states of knowledge.

As stated earlier, the nature of uncertainties due to estimation error and model imperfection is such that these uncertainties can be reduced by improving the state of knowledge. The resulting reduction in the dispersion of β leads to a reliability index estimate which is closer to a strict measure of safety and, therefore, is of improved quality. However, improvements in the state of knowledge normally require an investment in time and money which may not be undertaken if there is no clear incentive. Assuming that it is desirable to have as strict a measure of safety as possible, we require that the reliability index penalize poor states of knowledge and, thereby, provide an incentive for improving the state of knowledge. We denote this as the *remunerability* requirement.

As described earlier, an important role of the reliability index is to serve as a relative measure of safety. For this purpose, it is necessary that an ordering of reliability indices for any group of structures be consistent with the corresponding ordering of their strict safeties. However, under an imperfect state of knowledge, a strict measure of safety and, therefore, an ordering thereof is not available. Thus, in lieu of a strict ordering of safeties, we require that the ordering of the reliability indices be consistent with the ordering of the corresponding safeties at a prescribed probability level. We denote this as the requirement of *orderability*. This requirement is analogous to a comparativeness requirement previously defined for a second-moment reliability index (Ditlevsen 1979).

The sixth and last requirement that we impose on the reliability index is *simplicity*. Since the reliability index is merely a rule-based, point estimate of safety, extensive computations for its determination are not justified. Therefore, the index shall be such that the required effort for its computation is commensurate with its approximate nature.

In summary, the fundamental requirements on the reliability index are stated as follows:

1. *Consistency*: For a perfect state of knowledge the reliability index shall coincide with the strict reliability index in Eq. 1
2. *Completeness*: The reliability index shall incorporate all available information and account for all uncertainties.
3. *Invariance*: For a given state of knowledge, the reliability index shall be invariant for mutually consistent formulations of the reliability problem.
4. *Remunerability*: The reliability index shall provide an incentive for improving the state of knowledge.
5. *Orderability*: Any ordering of reliability indices shall be consistent with the corresponding ordering of the strict safeties at a prescribed probability level.
6. *Simplicity*: The required effort for computing the reliability index shall be commensurate with its approximate nature.

HISTORICAL BACKGROUND

The structural reliability problem is usually formulated in terms of a limit-state function $g(\mathbf{X})$ of random variables $\mathbf{X}^T = (X_1, \cdots, X_n)$ defined such that

$$g(\mathbf{x}) \begin{cases} \leq 0 & \text{failure set} \\ = 0 & \text{limit-state surface} \\ > 0 & \text{safe set} \end{cases} \tag{2}$$

The failure probability is given by

$$P_F = \int_{g(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} \tag{3}$$

in which $f_{\mathbf{X}}(\mathbf{x})$ denotes the joint probability density function (PDF) of $\mathbf{X}$.

From the earliest studies in structural reliability, the need to work with incomplete statistical information was realized, as $f_{\mathbf{X}}(\mathbf{x})$ is seldom completely known. Indeed, the bulk of the early work assumed knowledge of only the first two moments of **X**, i.e., the mean vector $\mathbf{M} = \{\mu_i\}$ and the covariance matrix $\Sigma = [\rho_{ij}\sigma_i\sigma_j]$, where μ_i, σ_i, and ρ_{ij} respectively denote the mean, standard deviation, and correlation coefficient values of the elements of **X**. Reliability methods using only this information are known as second-moment reliability methods.

The earliest formal definition of a second-moment reliability index, introduced by Cornell (1969) and further formalized by Ang and Cornell (1974), is the ratio of the mean, μ_g, to the standard deviation, σ_g, of the limit-state function $g(\mathbf{X})$. Since for nonlinear $g(\mathbf{X})$ a first-order, Taylor-series approximation around the mean point is used to compute the mean and standard deviation, the index is known as the *mean-value, first-order, second-moment (mvfosm) reliability index*:

$$\beta_{mvfosm} = \frac{\mu_g}{\sigma_g} \tag{4}$$

The motivation behind this definition is the fact that the probability of failure generally can be written in the form $P_F = F_U(-\mu_g/\sigma_g)$, where $F_U(.)$ is the cumulative distribution function (CDF) of the standard variate $U = (g(\mathbf{X}) - \mu_g)/\sigma_g$. In the special case where the variables are normal (a case of complete statistical information) and $g(\mathbf{X})$ is linear, the probability of failure is given exactly by $P_F = \Phi(-\beta_{mvfosm})$ and β_{mvfosm} coincides with the strict reliability index in Eq. 1.

A problem with β_{mvfosm} that was soon realized (Ditlevsen 1973, Hasofer and Lind 1974) was that it depended on the formulation of the limit-state function. For example, β_{mvfosm} values computed for two equivalent limit-state functions $g = X_1 - X_2$ and $g = X_1/X_2 - 1$ are different. This problem was resolved by Hasofer and Lind (1974), who suggested expanding the Taylor series around a point on the limit-state surface with minimum distance from the origin in a transformed standard space. The standard variates, **U**, having zero means and unit covariance matrix, are defined by a linear transformation of **X**, e.g.,

$$\mathbf{U} = \mathbf{L}^{-1}(\mathbf{X} - \mathbf{M}) \tag{5}$$

where $\mathbf{L}$ is a decomposition of the covariance matrix such that $\Sigma = \mathbf{L}\mathbf{L}^T$. The corresponding index, denoted *first-order, second-moment (fosm) reliability index*, is

$$\beta_{fosm} = \min_{G(\mathbf{u})=0} |\mathbf{u}| \tag{6}$$

where $G(\mathbf{u}) = g(\mathbf{x}(\mathbf{u}))$ is the limit-state function in the transformed space. β_{fosm} is invariant of the formulation of $g(\mathbf{X})$ since the linearization point on the limit-state surface remains invariant. For linear $g(\mathbf{X})$, β_{fosm} coincides with β_{mvfosm}.

A fundamental shortcoming of β_{fosm} is that it lacks orderability. That is, an ordering of β_{fosm} values for a group of structures may not be consistent with the ordering of the corresponding safeties. This is obvious since all limit-state surfaces having equal minimum distances from the origin in the standard space have identical β_{fosm} values, regardless of the shapes of the respective safe sets. To overcome this, Ditlevsen (1979) introduced the *generalized (second-moment) reliability index*

$$\beta_{gsm} = \Phi^{-1}\left[\int_{G(\mathbf{u})>0} \phi_n(\mathbf{u})\,d\mathbf{u}\right] \tag{7}$$

where $\phi_n(\mathbf{u})$ is a normalized weight function over the n-dimensional safe set. Based on the rotational symmetry of the standard space and a requirement of simplicity, $\phi_n(\mathbf{u})$ was taken to be the n-dimensional standard normal density, $\phi_n(\mathbf{u}) = (2\pi)^{-n/2}\exp(-\mathbf{u}^T\mathbf{u}/2)$. In a second-moment context, this index is orderable since it accounts for the entire safe set. However, β_{gsm} does not satisfy the orderability requirement as defined in this paper, since it lacks a precise relationship with the strict reliability index. For linear $g(\mathbf{X})$, β_{gsm}, β_{fosm} and β_{mvfosm} coincide.

The preceding indices are strictly applicable when the available information is limited to the first and second moments. Thus, they do not satisfy the completeness requirement whenever information beyond the second moments is available. For the same reason they fail to satisfy the consistency requirement. It can be shown that in the presence of information beyond the second moments, the standard space defined by Eq. 5 loses its rotational symmetry. To account for such information, therefore, Ditlevsen (1979) and Der Kiureghian and Liu (1986) suggested using the beyond-the-second-moment information to construct a nonlinear transformation which produces a rotationally symmetric standard space. For example, when a variable is known to be positive, Ditlevsen (1979) suggests using the logarithmic transformation. Der Kiureghian and Liu (1986) have formulated such transformations for different states of information, including the knowledge of marginal distributions and covariances. An essentially similar approach has been suggested by Winterstein and Bjerager (1987) to account for knowledge of higher than second moments. The resulting reliability indices computed with these methods and Eq. 7, although satisfying the completeness requirement, cannot satisfy the requirements of remunerability and orderability as defined in this paper. This is because Eq. 7 does not account for the extent of completeness of the statistical information. For example, with these methods one would obtain identical results for a case with only the second-moments known and a case with known normal distributions having the same second moments.

To account for information beyond the second moments, Veneziano (1979) suggested a definition of the reliability index in terms of the upper bound of failure probability,

$$\beta_{pu} = (P_F^u)^{-1/2} \tag{8}$$

where the upper bound P_F^u is computed as a generalized Tchebysheff bound, including the information beyond the second moments. (This definition coincides with β_{fosm} in the special case of a single variable with known mean and variance, with the safe set defined symmetrically around the mean point.) Solutions of β_{pu} for some idealized limit-state functions, assuming knowledge of various statistical moments, are reported by Veneziano (1979). Theoretically, this index can incorporate any information on the random variables. However, in practice, the generalized Tchebysheff bound is extremely difficult or impossible to compute for arbitrary statistical information and for general limit-state functions. Hence, this index fails to satisfy the simplicity requirement. It also fails the consistency requirement, since the definition adopted by Veneziano is not consistent with the definition in Eq. 1. Furthermore, it is not clear how uncertainties due to model imperfection can be incorporated in this formulation. On the other hand, this is the only existing reliability index that, at least theoretically, satisfies the remunerability and orderability requirements. Specifically, it satisfies the remunerability requirement since β_{pu} normally increases with increasing statistical information, and it satisfies the orderability requirement since β_{pu} may be regarded as the lower reliability bound at 100 percent probability level. It is shown in the next section that the new reliability index to be proposed in a special case reduces to an index analogous to β_{pu} but consistent with the definition in Eq. 1.

Uncertainties due to estimation error and model imperfection have received little attention in the definitions of the preceding reliability indices. The suggested second-moment methods by Ang and Cornell (1974) and by Ditlevsen (1982) essentially amount to a modification of the second-moment statistics of the basic variables to account for these uncertainties. Another suggestion has been to consider such uncertainties as additional random variables and to incorporate them as a part of the vector of variables **X** (Hohenbichler and Rackwitz 1981). Both methods are based on the Bayesian thinking. However, these methods do not differentiate between the uncertainties arising from inherent variabilities from those arising from estimation errors and model imperfections. Further examination of this approach is given in the following section. An approach based on confidence intervals analogous to classical statistical estimation has also been suggested with, however, no clear definition of a reliability index (Greimann 1984).

As is clear from the above review, the prevailing notion in the reliability literature regarding incomplete statistical information is the exclusive knowledge of a number of lower statistical moments (e.g., the first two, in second-moment methods) and nothing else. This notion, however, is far from reality. In actual practice, one usually deals with observed samples of random variables, from which statistical moments of arbitrary order may be estimated, with decreasing accuracy for higher moments. Techniques are available for examining the relative fitness of various theoretical distribution models to such data. Furthermore, the Bayesian statistical method offers a valuable framework for incorporating subjective information, such as an expert's opinion regarding the parameters of a distribution, which is an indispensable source of information in many engineering applications. Second-moment reliability methods are ill-suited to make use of these techniques. The formulation presented in the following section provides a framework for reliability analysis which is consistent with this more realistic and practical viewpoint.

BASIC FORMULATION

As defined in the introduction, two types of uncertainties enter into a reliability problem: uncertainties which are irreducible, such as those arising from inherent variabilities, and uncertainties which are reducible, such as those arising from estimation error and model imperfection. Let the vector of random variables **X** describe the first set of uncertainties and the vector of random variables Θ describe the second set of uncertainties. Θ includes the set of parameters that define the probability model of **X** and the limit-state function $g(\mathbf{X})$. To explicitly delineate this dependence, we express the distribution of **X** as a conditional distribution, $f_{\mathbf{X}|\Theta}(\mathbf{x},\theta)$, and the limit-state function as $g(\mathbf{X},\Theta)$. It should be clear that the elements of Θ which represent estimation errors only appear in the distribution function, whereas the elements which represent modeling imperfections may appear in both functions. When the state of knowledge is perfect, complete statistical information and perfect models are available and Θ is deterministically known. Otherwise, the state of knowledge is imperfect and its degree of imperfection is characterized by the distribution of Θ, denoted $f_{\Theta}(\theta)$.

The Bayesian statistical method provides a rational framework for constructing the distribution of Θ based on both objective and subjective information, through the updating rule

$$f_{\Theta}(\theta) = c\,L(\theta) f'_{\Theta}(\theta) \tag{9}$$

where $f'_{\Theta}(\theta)$ is the prior distribution, which is usually based on subjective information such as the opinion of an expert, $L(\theta)$ is the likelihood function of the observed data (objective information), c is a normalizing factor, and $f_{\Theta}(\theta)$ is the posterior distribution which incorporates both sets of information (Ang and Tang 1975). The likelihood function is given by

$$L(\theta) = \prod_i f_{\mathbf{X}|\Theta}(\mathbf{x}_i, \theta) \tag{10}$$

where $\mathbf{x}_i$ is the i-th observation of $\mathbf{X}$ and the product is over the observed population. Furthermore, the predictor distribution of $\mathbf{X}$, which combines the inherent variability of $\mathbf{X}$ with the uncertainty in Θ by the total probability rule, is given by

$$\tilde{f}_{\mathbf{X}}(\mathbf{x}) = \int f_{\mathbf{X}|\Theta}(\mathbf{x}, \theta) f_{\Theta}(\theta)\, d\theta \tag{11}$$

Illustrative examples later in this paper will make use of the preceding three formulas.

For a given $\Theta = \theta$, the conditional probability of failure is

$$P_{F|\Theta}(\theta) = \int_{g(\mathbf{x},\theta)\le 0} f_{\mathbf{X}|\Theta}(\mathbf{x}, \theta)\, d\mathbf{x} \tag{12}$$

Using this value, a conditional reliability index consistent with the definition in Eq. 1 is introduced:

$$\beta_{|\Theta}(\theta) = \Phi^{-1}[1 - P_{F|\Theta}(\theta)] \tag{13}$$

For uncertain Θ, $P_{F|\Theta}(\Theta)$ and $\beta_{|\Theta}(\Theta)$ are also uncertain. Distributions expressing these uncertainties may be obtained, at least theoretically, by using well known techniques for functions of random variables. For convenience, let $B = \beta_{|\Theta}(\Theta)$ and denote $f_B(\beta)$ and $F_B(\beta)$ as its PDF and CDF.

The distribution $f_B(\beta)$ expresses our uncertainty with respect to the strict reliability index, as arisen from the uncertainties due to estimation error and model imperfection. The dispersion indicated by this distribution, as measured for example by its standard deviation σ_β, is a measure of the quality of the state of knowledge. As more information is gathered and refined models are used, this dispersion decreases. In the limit, when the state of knowledge approaches the perfect state, Θ becomes deterministic and the probability mass under $f_B(\beta)$ coalesces at a point β, which coincides with the strict reliability index in Eq. 1.

The reliability index under an imperfect state of knowledge is defined as a point-estimator of B. In particular, the following point estimators may be considered:

Mean-Value Estimator:

$$\mu_\beta = \int_{\beta_{min}}^{\beta_{max}} \beta f_B(\beta)\, d\beta \tag{14}$$

Median-Value Estimator:

$$\bar{\beta}: \quad F_B(\bar{\beta}) = 0.5 \tag{15}$$

Maximum-Likelihood Estimator:

$$\beta_{ml}: \quad \max_{\beta} f_B(\beta) = f_B(\beta_{ml}) \tag{16}$$

Lower-Bound Estimator:

$$\beta_{min} = \max_{F_B(\beta)=0} \beta \tag{17}$$

Predictor Estimator:

$$\tilde{\beta} = \Phi^{-1}(1 - \tilde{P}_F) \tag{18a}$$

$$\tilde{P}_F = \int P_{F\,|\Theta}(\theta) f_{\Theta}(\theta)\, d\theta = E[P_{F\,|\Theta}(\Theta)] \tag{18b}$$

where $E[.]$ denotes the expectation. The first three estimators are central measures of safety and disregard the dispersion in the distribution of B. The lower-bound estimator, β_{min}, defines the reliability index as the lower bound of the distribution of B. This is analogous to the definition by Veneziano in Eq. 8. This estimator clearly accounts for the dispersion in the distribution of B. The predictor estimator, $\tilde{\beta}$, is defined in terms of the predictor failure probability, $\tilde{P}_F$, which is obtained by the total probability rule and is identical to the expected value of the conditional failure probability. This estimator makes no distinction between the uncertainty in **X**, which is due to the inherent variabilities, and the uncertainty in Θ, which is due to estimation errors and model imperfections. As indicated earlier, this approach has been suggested by Hohenbichler and Rackwitz (1981) and is often employed in practice (Madsen et al. 1986). Using a first-order approximation of Eq. 13a, $\mu_\beta \approx \tilde{\beta}$. Thus, the predictor reliability index is nearly equal to the mean of B.

The preceding point-estimator reliability indices satisfy the consistency, completeness and invariance requirements. However, μ_β, $\bar{\beta}$ and β_{ml} clearly violate the requirement of remunerability, as they are independent of the dispersion in the distribution of B and, hence, of the quality of the state of knowledge. Furthermore, the estimators μ_β and β_{ml} may not satisfy the orderability requirement, as they do not represent consistent probability levels of B. The same shortcomings can be stated for $\tilde{\beta}$, since $\tilde{\beta} \approx \mu_\beta$, as just mentioned. Thus, the point-estimators μ_β, $\bar{\beta}$, β_{ml}, and $\tilde{\beta}$ are not satisfactory candidates for the reliability index. The estimator β_{min}, on the other hand, satisfies the remunerability and orderability requirements in the same manner as Veneziano's index, β_{pu}, does. Therefore, β_{min} is a better choice for the reliability index than all the other indices introduced this far. It will be shown shortly that this definition of the reliability index is a special case of the new reliability index introduced below.

THE MINIMUM-PENALTY RELIABILITY INDEX

The new reliability index is based on elementary concepts from statistical decision theory (Ang and Tang 1984). Consider a penalty function $p(B - \hat{\beta})$ in terms of the deviation of the point-estimator reliability index, $\hat{\beta}$, from the uncertain reliability index B. The expected penalty for a choice of $\hat{\beta}$ is

$$E[p(B - \hat{\beta})] = \int_{\beta_{min}}^{\beta_{max}} p(\beta - \hat{\beta}) f_B(\beta) \, d\beta \tag{19}$$

The new reliability index, denoted β_{mp} (subscript *mp* for minimum penalty), is defined as the value of the estimator that minimizes the expected penalty, i.e.,

$$\beta_{mp}: \ \min_{\hat{\beta}} E[p(B - \hat{\beta})] = E[p(B - \beta_{mp})] \tag{20}$$

To make the formulation manageable, we consider simple forms of the penalty function. The penalty function clearly should be zero when $B = \hat{\beta}$ and it should increase with increasing $|B - \hat{\beta}|$. In general, one may expect different penalty values for positive and negative deviations of equal magnitude, i.e., the penalty function may lack symmetry with respect to the point $B = \hat{\beta}$. In particular, for a fixed $|B - \hat{\beta}|$, normal engineering practice would dictate a greater penalty for $B < \hat{\beta}$, i.e., for an overestimation of safety, than for $B > \hat{\beta}$, i.e., for an underestimation of safety. This is because an overestimation of safety normally entails tangible or intangible consequences which are far greater in value than the purely economic consequences of underestimating the safety. With these in mind, the following two forms of the penalty function are considered:

Linear Penalty Function:

$$p(B - \hat{\beta}) = \begin{cases} a\,(B - \hat{\beta}), & \hat{\beta} \le B \\ ka\,(\hat{\beta} - B), & \hat{\beta} > B \end{cases} \tag{21a}$$

Quadratic Penalty Function:

$$p(B - \hat{\beta}) = \begin{cases} a\,(B - \hat{\beta})^2, & \hat{\beta} \le B \\ ka\,(\hat{\beta} - B)^2, & \hat{\beta} > B \end{cases} \tag{21b}$$

where a and k are deterministic coefficients. The parameter k represents a measure of the asymmetry in the penalty function, with $k > 1$ indicating a higher penalty for overestimation than for underestimation of safety. Qualitative plots of these functions are shown in Fig. 1.

Minimizing the expected penalty for the above functions results in the following solutions for the minimum-penalty reliability index:

$$\beta_{mp}^{L} = F_B^{-1}\left(\frac{1}{1+k}\right) \tag{22a}$$

$$\beta_{mp}^{Q} = \frac{\mu_\beta + (k-1)\,E_{\beta_{min}}^{\beta_{mp}^{Q}}[B]}{1 + (k-1)\,F_B(\beta_{mp}^{Q})} \tag{22b}$$

in which the superscripts L and Q respectively denote solutions based on the linear and quadratic penalty functions, and

$$E_{\beta_{min}}^{\beta_{mp}^{Q}}[B] = \int_{\beta_{min}}^{\beta_{mp}^{Q}} \beta f_B(\beta)\, d\beta \tag{22c}$$

is the incomplete expectation of B. Note that the scale factor a does not appear in the solutions and that in Eq. 22b β_{mp}^{Q} appears on both sides and the solution is in a transcendental form. The derivation of Eqs. 22 is given in Appendix I.

Before proceeding further, it is useful to examine the preceding results for the limiting values of k. For $k = 1$, it is easy to see that $\beta_{mp}^{L} = \bar{\beta}$ and $\beta_{mp}^{Q} = \mu_\beta$. Thus, $\bar{\beta}$ and μ_β are appropriate reliability indices only when the penalty function is symmetric, i.e., when there is no difference in the penalty for underestimating or overestimating the safety. One could easily show that the maximum likelihood estimator, β_{ml}, results when the penalty function is a constant for all deviations of $\hat{\beta}$ from B. Such a penalty function is unreasonable and, therefore, β_{ml} is not an appropriate choice for the reliability index. For $k = \infty$, i.e., when there is no penalty for underestimating the safety, it is easy to verify that both solutions in Eqs. 22 reduce to the lower-bound estimator, i.e., $\beta_{mp}^{L} = \beta_{mp}^{Q} = \beta_{min}$. Thus, the reliability index introduced by Veneziano (Eq. 8) is appropriate when there is a penalty for overestimating the safety, but not for underestimating it.

The above results for the limiting values of k are independent of the distribution of B. More generally, the solutions for β_{mp}^{L} and β_{mp}^{Q} depend on the form of this distribution. Herein, we first examine the case where B has the normal distribution, as this assumption leads to simple results. The effect of the distribution is examined next.

It is shown in Appendix I that for a normal distribution of B the solutions in Eqs. 22 both reduce to the form

$$\beta_{mp} = \mu_\beta (1 - \delta_\beta u) \tag{23}$$

where $\delta_\beta = \sigma_\beta / \mu_\beta$ is the coefficient of variation of B and $u = u(k)$ is a function of k which depends on the choice of the penalty function. For the linear penalty function,

$$u = \Phi^{-1}\left(\frac{k}{k+1}\right) \tag{24a}$$

and for the quadratic penalty function u is obtained as the solution to the transcendental equation

$$ku - (k-1)[u\,\Phi(u) + \phi(u)] = 0 \tag{24b}$$

in which $\Phi(.)$ and $\phi(.)$ are the standard normal CDF and PDF, respectively. Plots of $u(k)$ for the two penalty functions are shown in Fig. 2. Note that $u = 0$ for $k = 1$ for both cases, and that $u > 0$ for $k > 1$. Plots of β_{mp}/μ_β as a function of δ_β and k for the two penalty functions are shown in Fig. 3. Observe that β_{mp}/μ_β decreases with increasing δ_β (i.e., decreasing quality of the state of knowledge) and with increasing k (i.e., increasing asymmetry of the penalty function).

The formula for the minimum-penalty reliability index in Eq. 23 has three components, μ_β, δ_β, and $u(k)$. The first term provides a central measure of safety, the second term accounts for the uncertainty in estimation of safety, and the third term accounts for the asymmetry in the penalty function. When the state of knowledge is perfect and Θ is deterministic, δ_β is zero and β_{mp} coincides with the strict reliability index in Eq. 1. Thus, β_{mp} satisfies the consistency requirement. When the state of knowledge is imperfect, δ_β is non-zero and β_{mp} deviates from the central measure of safety by u units of the standard deviation σ_β. As the state of knowledge improves, δ_β decreases and β_{mp}/μ_β increases for $k > 1$. Thus, for $k > 1$, β_{mp} satisfies the remunerability requirement. The requirement of orderability is also satisfied for a fixed u, since β_{mp} is then a fixed number of standard deviations from the mean, which corresponds to a fixed probability level for the assumed normal distribution.

The significance of the assumed distribution of B can be readily examined in Eq. 22a for the linear penalty function. This equation defines the minimum-penalty reliability index as the $1/(1+k)$ cumulative probability level of B. Obviously this value will be sensitive to the choice of the distribution when k is large. Figure 4 examines this sensitivity as a function of k for three selected distributions with $\delta_\beta = 0.3$. Observe that for values of k less than around 10 the choice of the distribution is not essential. Thus, for such values of k, Eq. 23, which requires only the mean and standard deviation of B, provides a good approximation of the minimum-penalty reliability index regardless of the actual distribution of B. For larger values of k, the distribution of B and the minimization in Eq. 20 are required to make an accurate estimate of the minimum-penalty reliability index. Methods for computing $f_B(\beta)$, μ_β, and σ_β are presented in the following section.

COMPUTATIONAL METHODS

Probability integrals of the type in Eqs. 3 or 12 have been of interest to structural reliability analysts for a long time. In the past decade a number of approximate techniques for their evaluation have been developed. These include first and second-order reliability methods (FORM and SORM) (Hohenbichler and Rackwitz 1981, Madsen et al. 1986, Der

Kiureghian et al. 1987), various simulation techniques (e.g., Shinozuka 1983, Ang and Tang 1984, Schueller et al. 1987, Bjerager 1988), and hybrid methods which combine FORM/SORM with simulation (e.g., Schueller et al. 1987, Fujita et al. 1987, Bjerager 1988). In addition to providing an estimate of the failure probability and the associated reliability index, methods based on FORM (Madsen et al. 1986) and the directional simulation method (Ditlevsen and Bjerager 1987) also readily provide the sensitivities (i.e., the partial derivatives) of the failure probability or the reliability index with respect to any parameter of the probability distribution or the limit-state function. For the conditional probability of failure and reliability index in Eqs. 12 and 13, these sensitivities are the partial derivatives $\partial/\partial\theta_i\,[P_{F\,|\Theta}(\theta)]$ and $\partial/\partial\theta_i\,[\beta_{\,|\Theta}(\theta)]$. The reader is referred to the cited literature for a review of these methods. It is shown in this section that these same methods can be used to compute the distribution $f_B(\beta)$, or approximations to μ_β and σ_β which define the reliability index in Eq. 23.

To compute the distribution of B, we write the CDF of B as

$$F_B(\beta) = P(B - \beta \leq 0) = \int_{\beta_{|\Theta}(\theta) - \beta \leq 0} f_{\Theta}(\theta)\, d\theta \qquad (25)$$

The probability integral on the right-hand side is in the same form as the integrals in Eqs. 3 and 12. This suggests that the same techniques can be used to compute this integral. One should note, however, that the dependence of $\beta_{\,|\Theta}(\theta)$ on θ is itself in terms of the integral in Eq. 12. Thus, the computation of the integral in Eq. 25 requires a nested application of the reliability methods described above.

As an example, consider the computation of the integral in Eq. 25 using the FORM. This would require finding the minimum-distance point from the origin to the surface $\beta_{\,|\Theta}(\theta) - \beta = 0$ in a transformed standard normal space of the random variables Θ, and the linearization of the surface at that point. Standard optimization algorithms for solving this problem are available (Liu and Der Kiureghian, 1986). These algorithms typically require repeated computations of the function $\beta_{\,|\Theta}(\theta) - \beta$ and its gradient with respect to θ, for values of θ selected in accordance to a search rule. For the present case, for each value of θ, the function and its gradient (i.e., the sensitivities) are computed through an application of FORM to Eqs. 12 and 13. (Note that for this FORM, the random variables are $\mathbf{X}$ and $\Theta = \theta$ is fixed.) Thus, the complete solution requires repeated FORM solutions of Eqs. 12 and 13 for selected θ until convergence in the computation of Eq. 25 is achieved. This approach also readily provides the PDF of B, as $f_B(\beta) = d/d\beta[F_B(\beta)]$ is the sensitivity of the computed probability with respect to the parameter β.

Similarly, the solution of Eq. 25 by a simulation method requires solution of the integral in Eq. 12 (by simulation or other means) for each single simulation of Θ. One may also consider various combinations of hybrid methods where two different techniques are used to compute the inner (Eq. 12) and outer (Eq. 25) integrals. These computations of course need to be repeated for a range of values of β to provide a sufficient description of $f_B(\beta)$ to be used in computing β_{mp} from Eqs. 19 and 20.

It is clear that the computation of $f_B(\beta)$ and the minimum-penalty reliability index from Eq. 20, although theoretically possible, can be cumbersome and inconsistent with the simplicity requirement. After all, β_{mp} is merely a point estimator of safety and may not justify extensive computations, particularly when it is used in the context of code calibration. With this in mind, it is proposed to use the reliability index in Eq. 23 for all distributions of B. As discussed earlier, this would be entirely appropriate as long as k is not much larger than 10.

The definition of β_{mp} in Eq. 23 requires knowledge of the mean and standard deviation of B, which are also difficult to compute exactly. However, if in the spirit of simplicity first-order approximations are used, these values are easily computed from

$$\mu_\beta \approx \beta_{|\Theta}(\mathbf{M}_\Theta) \tag{26a}$$

$$\sigma_\beta^2 \approx \nabla_\Theta\beta\, \Sigma_{\Theta\Theta} \nabla_\Theta\beta^T \tag{26b}$$

in which $\mathbf{M}_\Theta$ and $\Sigma_{\Theta\Theta}$ are the mean vector and covariance matrix of Θ, and $\nabla_\Theta\beta$ is the row vector of partial derivatives $\partial/\partial\theta_i[\beta_{|\Theta}(\theta)]$ evaluated at the mean point. Note that one solution of Eqs. 12 and 13 with $\theta = \mathbf{M}_\Theta$, together with the partial derivatives with respect to θ (which are readily available in FORM and directional simulation) provide the necessary information to compute the above approximations of μ_β and σ_β.

Computation of the minimum-penalty reliability index from Eq. 23 also requires knowledge of the parameter u, which depends on the form of the penalty function and the parameter k. For a given class of structures, these may be selected by examining the appropriate penalties for underestimating or overestimating the safety. This task may not be simple. Alternatively, the proper value for u may be selected by calibration to the accepted practice, i.e., by adjusting u such that structures having different states of knowledge and perceived to be equally "safe" have the same β_{mp} values. For now, purely on intuitive grounds, the use of $u = 1$ in Eq. 23 is suggested. This value, which conveniently defines the minimum-penalty reliability index as the mean minus one standard deviation of the conditional reliability index, $\beta_{|\Theta}(\Theta)$, corresponds to k around 5 to 10, as can be seen in Fig. 2.

The computed distribution $f_B(\beta)$ or the mean and standard deviation, μ_β and σ_β, can be used to compute confidence intervals on the strict reliability index or the probability of failure. This and other concepts are illustrated in the following section.

ILLUSTRATIVE EXAMPLE

The example presented in this section illustrates the various notions of the reliability index that were described in the preceding section. The emphasis is on investigating the effect of uncertainties due to estimation error and model imperfection on the reliability index. For the sake of clarity, therefore, a simple limit-state function and only two random variables are considered; the extension to more elaborate problems is straightforward. Initially, two simple cases are considered for which closed form solutions are derived. For a more elaborate case, numerical solutions such as those suggested in the preceding section are employed.

Consider the limit-state function

$$g = X_1 - X_2 \tag{27}$$

where X_i, $i = 1,2$, are independent random variables, respectively representing the resistance of and the applied load on a structure. First assume X_i are both normal with known standard deviations, σ_i, and unknown means μ_i, the latter to be estimated from existing objective and subjective information. Thus, this example represents a case where perfect models are available, but uncertainty may arise from errors in the estimation of μ_i.

It is well known that if the prior distribution of the unknown mean, μ, of a normal random variable X with known variance σ^2 is taken to be normal with mean μ'_μ and standard deviation σ'_μ, the posterior distribution obtained from Eq. 9 will also be normal with the mean and variance given by (Ang and Tang 1975)

$$\mu_\mu = \frac{\bar{x}\sigma'^2_\mu + \mu'_\mu \dfrac{\sigma^2}{n}}{\sigma'^2_\mu + \dfrac{\sigma^2}{n}} \tag{28a}$$

$$\sigma_\mu^2 = \frac{\sigma'^2_\mu \dfrac{\sigma^2}{n}}{\sigma'^2_\mu + \dfrac{\sigma^2}{n}} \tag{28b}$$

where n denotes the sample size and $\bar{x}$ denotes the sample mean. It is also known (Ang and Tang 1975) that the predictor distribution of X from Eq. 11, which combines the inherent variability in X with the uncertainty in the estimation of μ, is also normal with mean μ_μ and variance $\sigma^2 + \sigma_\mu^2$. These results are used in the following to derive closed-form expressions for the reliability index.

The conditional reliability index (Eq. 13) given the unknown parameters μ_1 and μ_2 is

$$B = \frac{\mu_1 - \mu_2}{(\sigma_1^2 + \sigma_2^2)^{1/2}} \tag{29}$$

Since μ_i are normal, it follows that B is also normal with the mean and variance

$$\mu_\beta = \frac{\mu_{\mu_1} - \mu_{\mu_2}}{(\sigma_1^2 + \sigma_2^2)^{1/2}} \tag{30a}$$

$$\sigma_\beta^2 = \frac{\sigma_{\mu_1}^2 + \sigma_{\mu_2}^2}{\sigma_1^2 + \sigma_2^2} \tag{30b}$$

The variance σ_β^2 is clearly a measure of the influence of the estimation error on the reliability index. In particular, with a noninformative prior (i.e., with $\sigma'_{\mu_i} = \infty$) and with equal sample size $n_1 = n_2 = n$, one obtains

$$\mu_\beta = \frac{\bar{x}_1 - \bar{x}_2}{(\sigma_1^2 + \sigma_2^2)^{1/2}} \tag{31a}$$

$$\sigma_\beta^2 = \frac{1}{n} \tag{31b}$$

which shows a simple relation between the variance of the reliability index and the size of the observed sample.

For the present example, owing to the normal distribution of B, the median and most likely reliability indices, $\bar{\beta}$ and β_{ml} respectively, are identical to the mean reliability index, μ_β. These indices are influenced by the quality of the information (as measured by the sample size and the prior variance $\sigma'^2_{\mu_i}$) only through the posterior means μ_{μ_i}, which in the case of noninformative priors are the sample means $\bar{x}_i$. This influence generally will be insignificant. The lower bound reliability index, β_{min}, is $-\infty$ because of the unbounded nature of the normal distribution. Therefore, this index is an inappropriate measure for the present example. The minimum-penalty reliability index is given exactly by Eq. 23, which for the case of noninformative priors and equal sample size reduces to

$$\beta_{mp} = \frac{\bar{x}_1 - \bar{x}_2}{(\sigma_1^2 + \sigma_2^2)^{1/2}} - \frac{u}{\sqrt{n}} \tag{32}$$

Thus, the uncertainty arising from the estimation of μ_i influences the minimum-penalty reliability index in terms of a subtractive factor inversely proportional to the square root of the sample size. For the present case a closed form expression for the predictor reliability index, $\tilde{\beta}$, is also possible. Since the predictor distributions of X_i are normal, it follows that

$$\tilde{\beta} = \frac{\mu_{\mu_1} - \mu_{\mu_2}}{(\sigma_1^2 + \sigma_{\mu_1}^2 + \sigma_2^2 + \sigma_{\mu_2}^2)^{1/2}} \tag{33}$$

In particular, with noninformative priors and equal sample size

$$\tilde{\beta} = \frac{\bar{x}_1 - \bar{x}_2}{(\sigma_1^2 + \sigma_2^2)^{1/2}} \left(\frac{n}{1+n} \right)^{1/2} \tag{34}$$

The predictor reliability index is also influenced by the sample size and the quality of the prior information. However, from Eq. 34, it is clear that unless n is very small the influence will be insignificant.

Now assume the standard deviations σ_i are also unknown and are to be estimated from the available information. The conditional reliability index given the unknown parameters μ_1, σ_1, μ_2, σ_2 is the same as in Eq. 29. However, its distribution is no longer normal.

The joint distribution of the unknown mean, μ, and variance, σ^2, of a normal random variable X can be obtained by use of Eqs. 9 and 10. In particular, for a noninformative prior (which should be taken proportional to σ^{-1}, see Box and Tiao 1973), the joint distribution is such that μ is conditionally normal with mean $\bar{x}$ and variance σ^2/n, and $1/\sigma^2$ is gamma distributed with parameters $(n-1)/2$ and $s^2(n-1)/2$, where n is the sample size and $s^2 = 1/(n-1)\sum_k (x_k - \bar{x})^2$ is the sample variance. Based on these, the second moments of μ and σ^2 are: $\mu_\mu = \bar{x}$, $\sigma_\mu^2 = \mu_{\sigma^2}/n$, $\mu_{\sigma^2} = s^2(n-1)/(n-3)$, $\sigma_{\sigma^2}^2 = 2\mu_{\sigma^2}^2/(n-5)$, and $\rho_{\mu\sigma^2} = 0$. Using these results and the partial derivatives from Eq. 29 in

Eqs. 26, the first order approximations of μ_β and σ_β^2 for the case of equal sample size are

$$\mu_\beta \approx \frac{\bar{x}_1 - \bar{x}_2}{(s_1^2 + s_2^2)^{1/2}} \left(\frac{n-3}{n-1} \right)^{1/2} \tag{35a}$$

$$\sigma_\beta^2 \approx \frac{1}{n} + \mu_\beta^2 \frac{1}{2(n-5)} \frac{s_1^4 + s_2^4}{(s_1^2 + s_2^2)^2} \tag{35b}$$

The first term on the right-hand side of Eq. 35b is the same as that in Eq. 31b and is due to the uncertainty in μ_i. The second term, which includes the factor μ_β^2, therefore, represents the influence of the uncertainty in σ_i. In fact, one could show that when μ_i are known and only σ_i are to be estimated, μ_β remains the same as in Eq. 35a with only the term $n-3$ replaced by $n-2$, and σ_β^2 equals the second term on the right-hand side of Eq. 35b with the term $n-5$ replaced by $n-4$. It is interesting to see that the influences of the uncertainties in the estimation of μ_i and σ_i on the reliability index are distinctly different in form; namely, the former is independent of the magnitude of the reliability index, while the latter is not. It is also worthwhile to note that the term involving the sample size in the expression for μ_β is a consequence of the definition of the sample variance. Specifically, if the sample variance is defined by $s^2 = 1/(n-3)\sum_k (x_k - \bar{x})^2$, then $\mu_{\sigma^2} = s^2$ and the term in Eq. 35a involving n drops.

It can be shown that the predictor distribution of X derived from Eq. 11 is such that $[n/(n+1)]^{1/2}(X - \bar{x})/s$ has the t-distribution with $n-1$ degrees of freedom. Based on this, the mean and variance of the limit-state function (Eq. 27) for the case of equal sample size are: $\mu_g = \bar{x}_1 - \bar{x}_2$ and $\sigma_g^2 = (s_1^2 + s_2^2)\,[(n-1)(n+1)]/[n(n-3)]$. An approximation to the predictor reliability index is obtained as the ratio μ_g/σ_g (which is the exact result for normal distributions),

$$\tilde{\beta} \approx \frac{\bar{x}_1 - \bar{x}_2}{(s_1^2 + s_2^2)^{1/2}} \left[\frac{n(n-3)}{(n+1)(n-1)} \right]^{1/2} \tag{36}$$

This approximation is expected to be accurate because of the proximity of the t and normal distributions. Comparing the expressions in Eqs. 34 and 36, the influence of the uncertainty in σ_i on the predictor reliability index is the factor $[(n-3)/(n-1)]^{1/2}$. This factor is a consequence of the definition of the sample variance and drops when the alternative definition mentioned above is used. Thus, in reality, $\tilde{\beta}$ is not influenced by the uncertainty in the estimation of the variances σ_i^2.

The preceding results are summarized in Fig. 5. The results in this figure are for noninformative priors, equal sample sizes ($n_1 = n_2 = n$), and the following mean values of the parameters: $\mu_{\mu_1} = 100$, $\mu_{\sigma_1^2} = 400$, $\mu_{\mu_2} = 40$, and $\mu_{\sigma_2^2} = 100$. (Note that it is more appropriate to fix $\mu_{\sigma_i^2}$ than s_i^2 when the sample size is varied. This assumption is consistent with the alternative definition of the sample variance mentioned above.) Shown in the figure are the mean reliability index, μ_β, the predictor reliability index, $\tilde{\beta}$, and the one standard deviation intervals, $\mu_\beta \pm \sigma_\beta$, for the cases with unknown μ_i (dashed lines) and unknown μ_i and σ_i (solid lines), all plotted against the common sample size n. Note that, with the mean values of the parameters fixed, the mean and predictor reliability indices for the two cases are identical. Also note that the $\mu_\beta - \sigma_\beta$ values are the same as the minimum-penalty reliability index β_{mp} for $u = 1$. Other results presented in Fig. 5 will be discussed shortly.

Several interesting observations can be made in Fig. 5. We first note that the $\pm\, \sigma_\beta$ band gradually narrows as the sample size increases and the estimation uncertainty decreases. For the selected mean parameters, the influence of the uncertainty arising from the estimation of the variances appears to be more significant than that from the estimation of the means. Second, we observe that $\tilde{\beta}$ virtually coincides with μ_β and is independent of the sample size, except for very small sample sizes. This index is also independent of the uncertainty in the estimation of the variances, as mentioned earlier. These results, and the fact that $\tilde{\beta}$ is entirely independent of any penalty function, show that the predictor reliability index is a poor choice as a measure of safety. On the other hand, the minimum-penalty reliability index (for $u = 1$) steadily increases with n and only asymptotically coincides with μ_β as n approaches infinity, i.e., when the statistical information becomes complete. This index clearly accounts for the cumulative influence of the uncertainties in the estimation of the means and variances.

As a further item of interest, Fig. 6 shows plots of the PDF $f_B(\beta)$ for different sample sizes for the case where both μ_i and σ_i are unknown. These results, which were obtained by repeated FORM analyses, clearly show the decreasing dispersion in the reliability index as the quality of information (i.e., the sample size) is increased. Mean and one standard deviation bands numerically computed from these PDF's are shown in Fig. 5 as open circles. The discrepancy between the circles and the solid lines, which is more pronounced at small values of n, is due to the approximations employed both in Eqs. 26 and in in computation of the PDF's.

In the preceding analysis, the distribution of each X_i was known to be normal, so that uncertainty arose only from the estimation of parameters. Now suppose that there is also uncertainty in the distribution model itself. One way to account for the distribution model uncertainty is to choose a parametrized family of distributions and allow the the parameter to be estimated through the Bayesian updating formula, Eq. 9. For the present example, an appropriate choice is the family of exponential power distributions (Box and Tiao 1973):

$$f_{X|\mu,\sigma,\gamma}(x,\mu,\sigma,\gamma) = \frac{p(\gamma)}{\sigma}\exp\left[-q(\gamma)\left|\frac{x-\mu}{\sigma}\right|^{r(\gamma)}\right], \quad 0<\sigma, \ -1\le\gamma\le 1 \quad (37a)$$

with

$$r = \frac{2}{1+\gamma}, \quad p = \frac{r}{2}\frac{[\Gamma(3/r)]^{1/2}}{[\Gamma(1/r)]^{3/2}}, \quad q = \left[\frac{\Gamma(3/r)}{\Gamma(1/r)}\right]^{r/2} \quad (37b)$$

Of the three parameters of this distribution, μ denotes the mean, σ denotes the standard deviation, and γ is a measure of flatness of the distribution. For $\gamma = 0$ the distribution reduces to the normal distribution, for $\gamma = -1$ it reduces to the uniform distribution in the interval $\mu \pm \sqrt{3}\,\sigma$, and for $\gamma = 1$ it reduces to the double exponential distribution. Thus, the model in Eq. 37 with a variable γ represents a large family of distributions. It is important to note, however, that all these distributions are symmetric. Therefore, the choice of this family is appropriate when it is known with certainty that X is symmetrically distributed. When this is not the case, other families suggested below can be used.

The likelihood function for the above distribution involves the r-th absolute moment of the observed sample, which cannot be expressed simply in terms of the sample statistics. Therefore, a rigorous comparison with the previous results is not possible without having the entire sample of observations for each X_i. For the sake of a reasonable comparison, it is assumed here that μ_i and σ_i^2 have the same moments as in the preceding case (see the paragraph preceding Eqs. 35), and that γ_i is uncorrelated of μ_i and σ_i and has a zero mean and variance $\sigma_{\gamma_i}^2 = 1/n_i$. (This last assumption implies that with increasing sample size there is increasing evidence that the distribution is normal.) A FORM analysis with the mean values of the distribution parameters yields $\mu_\beta \approx \beta_{|\Theta}(M_\Theta) = 2.683$ and the sensitivities $\partial\beta/\partial\mu_1 = -\partial\beta/\partial\mu_2 = 0.0447$, $\partial\beta/\partial\sigma_1^2 = \partial\beta/\partial\sigma_2^2 = -0.00268$, $\partial\beta/\partial\gamma_1 = -0.558$, and $\partial\beta/\partial\gamma_2 = 0.0665$. These sensitivity estimates together with the parameter variances are used in Eq. 26b to compute approximations of σ_β as a function of n. The results are plotted in Fig. 5 as dash-dotted lines. With the assumed statistics of γ_i, the added effect of the distribution model uncertainty appears not to be significant for the present problem.

As mentioned earlier, the choice of the family of distributions in Eq. 37 assumes knowledge of the symmetry of the unknown distribution. This, in fact, might be the reason for the relative insignificance of the uncertainty in the distribution model. If this assumption is not valid, then one should select a family which includes member distributions with skewed shapes. Many such families can be constructed. One possible example is the family of mixed distributions

$$f_X(x) = \sum_i \frac{\theta_i}{\sum_k \theta_k} f_i(x) \tag{38}$$

where θ_i are positive-valued parameters and $f_i(x)$ are member distributions with skewed or symmetric shapes. For any observed sample, the joint distribution of θ_i and the parameters defining each member distribution can be obtained from Eqs. 9 and 10.

Finally, imperfections in the mechanical model can be analyzed by introducing uncertainty parameters in the limit-state function. For example, if X represents the capacity of a structural member, then it may be represented as $\theta\hat{X}$, where $\hat{X}$ is the predictive model of X and represents the inherent variability in X, and θ represents the uncertainty in the predictive model. The statistics of θ are usually obtained by comparing predictions by the model with measured results in carefully controlled experiments. Examples for such analysis are widely available in the literature (e.g., see Ang and Tang 1984).

SUMMARY AND CONCLUSIONS

The nature of uncertainties in structural reliability is examined and two fundamentally distinct sources are identified: uncertainties due to inherent variabilities, which are irriducible, and uncertainties due to estimation error and model imperfection, which are reducible. The reliability index under such conditions of uncertainty is defined as a point estimator of safety. Motivated by needs in probabilistic structural code development, a set of fundamental requirements on the reliability index are formulated. These include the requirements of consistency, completeness, invariance, remunerability, orderability, and simplicity. The existing reliability indices are examined in this light and are invariably found to be deficient in satisfying several of the requirements. More specifically, second-moment reliability indices fail to satisfy at least the consistency and completeness requirements; Veneziano's reliability index fails to satisfy the consistency and simplicity requirements; and the mean, median, maximum likelihood, and predictor reliability indices fail to satisfy the remunerability and (with the exception of the median) orderability requirements.

A new index of reliability based on minimizing a penalty function is introduced. The index, denoted minimum-penalty reliability index β_{mp}, is shown to satisfy all the stipulated requirements. This new index recognizes the fundamental difference between the two sources of uncertainty and provides a rational basis for reliability analysis and code development under arbitrary states of knowledge.

Methods for computing the distribution or variance of the safety measure for uncertainties arising from estimation error and model imperfection are developed in this paper. The existing first and second-order reliability methods, or various simulation and hybrid methods can be used for this purpose. In particular, a simple approximation to the variance of the reliability index is obtained with a single FORM analysis. An example is used to illustrate the main concepts of the paper.

APPENDIX I -- DERIVATION OF EQUATIONS 22 - 24

The minimum-penalty reliability index is the solution to the equation

$$\frac{d}{d\hat{\beta}} \int_{\beta_{min}}^{\beta_{max}} p(\beta - \hat{\beta}) f_B(\beta)\, d\beta = 0 \tag{39}$$

For an m-th order penalty function of the form in Eqs. 21, this gives

$$\frac{d}{d\hat{\beta}} \left[ka \int_{\beta_{min}}^{\hat{\beta}} (\hat{\beta} - \beta)^m f_B(\beta)\, d\beta + a \int_{\hat{\beta}}^{\beta_{max}} (\beta - \hat{\beta})^m f_B(\beta)\, d\beta \right] =$$

$$-kam \int_{\beta_{min}}^{\hat{\beta}} (\hat{\beta} - \beta)^{m-1} f_B(\beta)\, d\beta + am \int_{\hat{\beta}}^{\beta_{max}} (\beta - \hat{\beta})^{m-1} f_B(\beta)\, d\beta = 0 \tag{40}$$

For $m = 1$, the preceding equation reduces to

$$-k F_B(\hat{\beta}) + [1 - F_B(\hat{\beta})] = 0 \tag{41}$$

which has the solution given in Eq. 22a. For $m = 2$, Eq. 40 reduces to

$$-k \left\{ \hat{\beta} F_B(\hat{\beta}) - E_{\beta_{min}}^{\hat{\beta}} [B] \right\} + \mu_\beta - E_{\beta_{min}}^{\hat{\beta}} [B] - \hat{\beta} [1 - F_B(\hat{\beta})] = 0 \tag{42}$$

in which the incomplete expectation is as defined in Eq. 22c. Equation 22b is obtained by rearranging the terms in the preceding equation.

For a normal distribution of B with mean μ_β and standard deviation σ_β, one has

$$F_B(\hat{\beta}) = \Phi \left(\frac{\hat{\beta} - \mu_\beta}{\sigma_\beta} \right) \tag{43}$$

$$E_{\beta_{min}}^{\hat{\beta}} [B] = \mu_\beta \Phi \left(\frac{\hat{\beta} - \mu_\beta}{\sigma_\beta} \right) - \sigma_\beta \phi \left(\frac{\hat{\beta} - \mu_\beta}{\sigma_\beta} \right) \tag{44}$$

Denoting $u = (\mu_\beta - \hat{\beta})/\sigma_\beta$ and substituting Eqs. 43 and 44 in Eqs. 22, after rearranging the terms solutions in Eqs. 24 are obtained.

APPENDIX II -- REFERENCES

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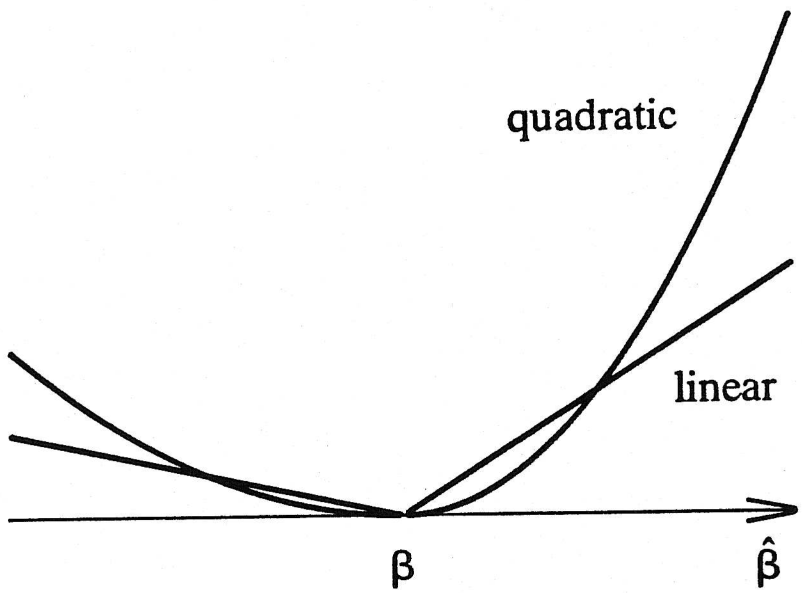

Figure 1. Penalty Functions

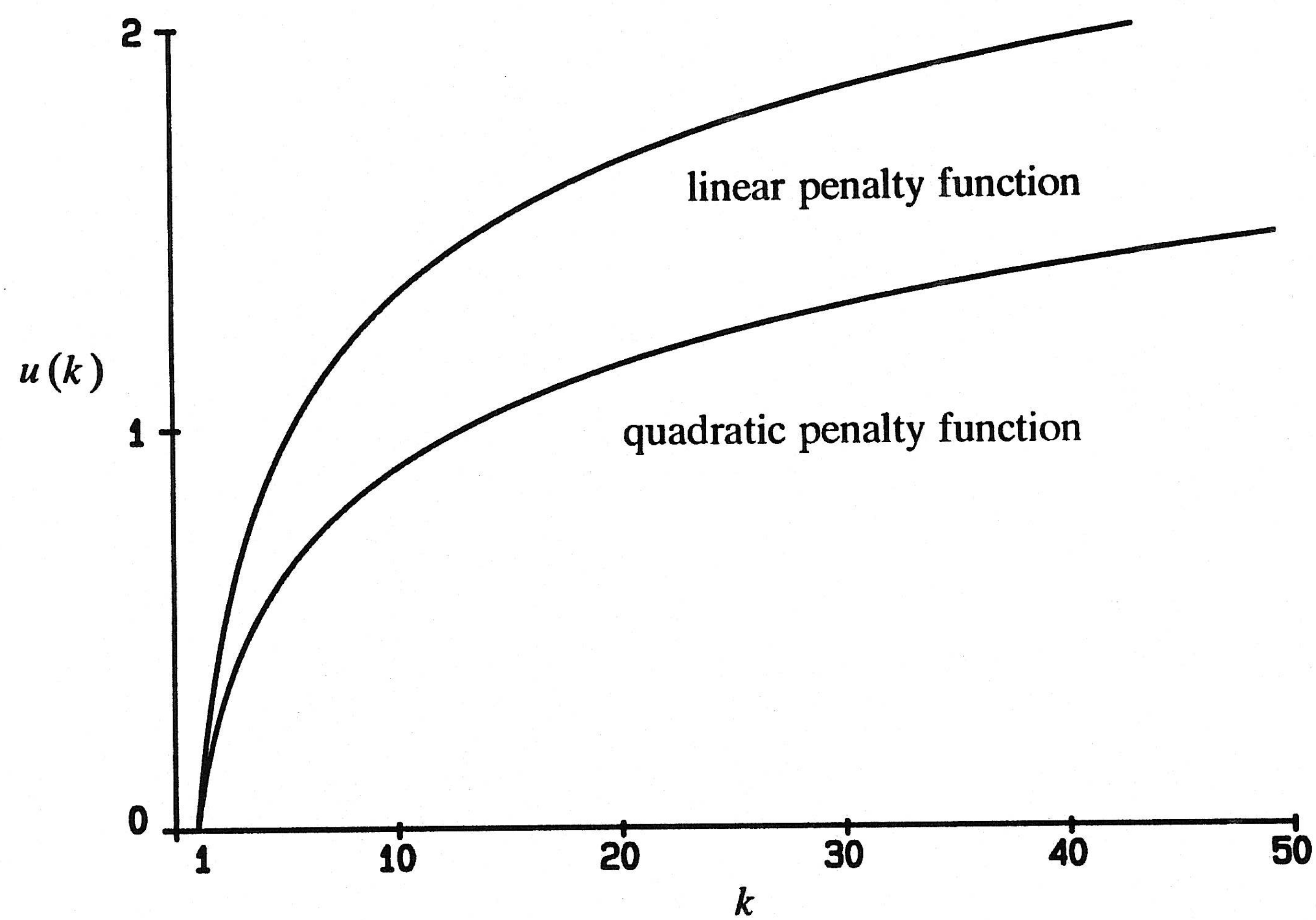

Figure 2. Function $u(k)$

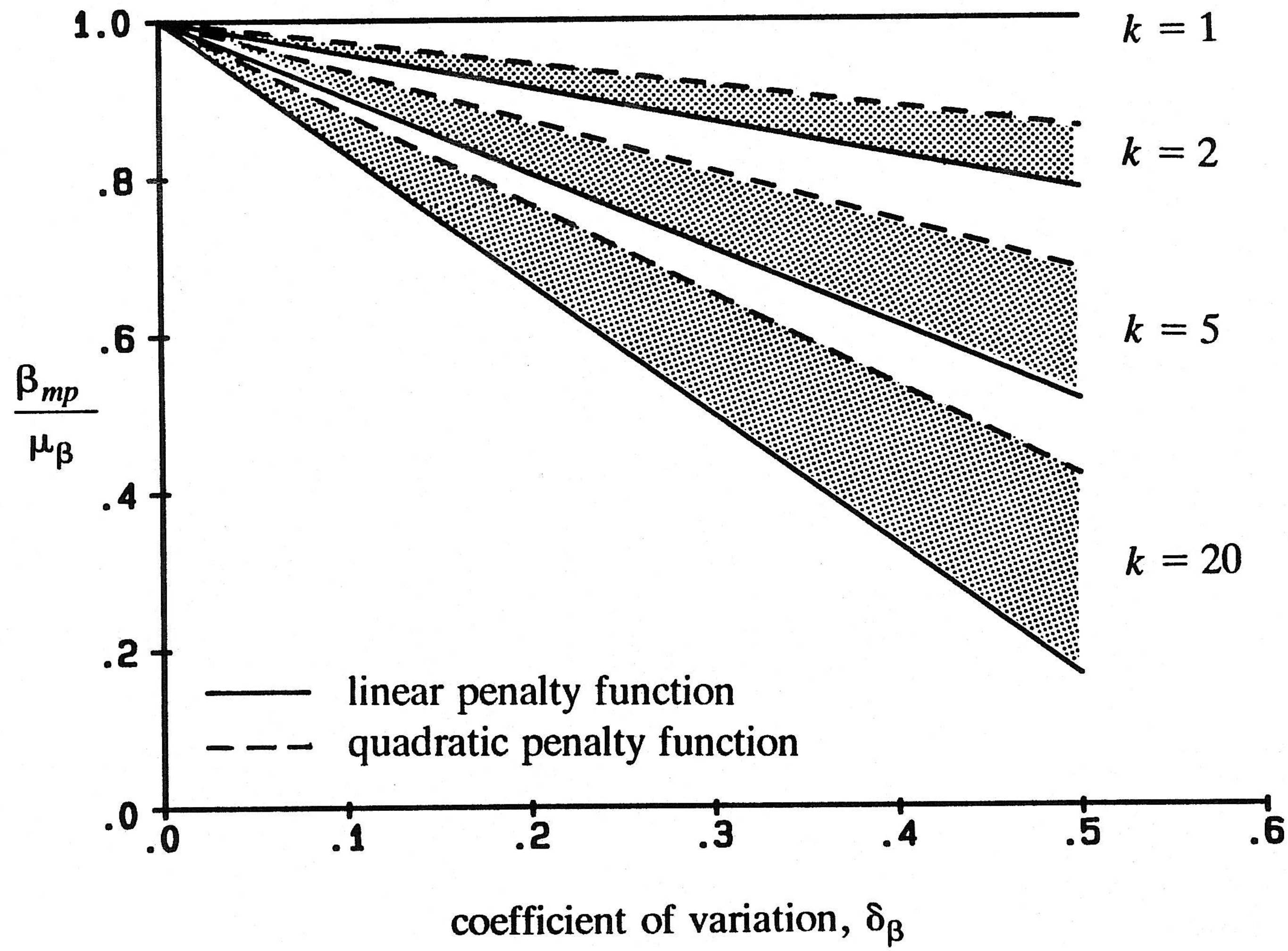

Figure 3. The Minimum-Penalty Reliability Index as a Function of μ_β, δ_β, and k

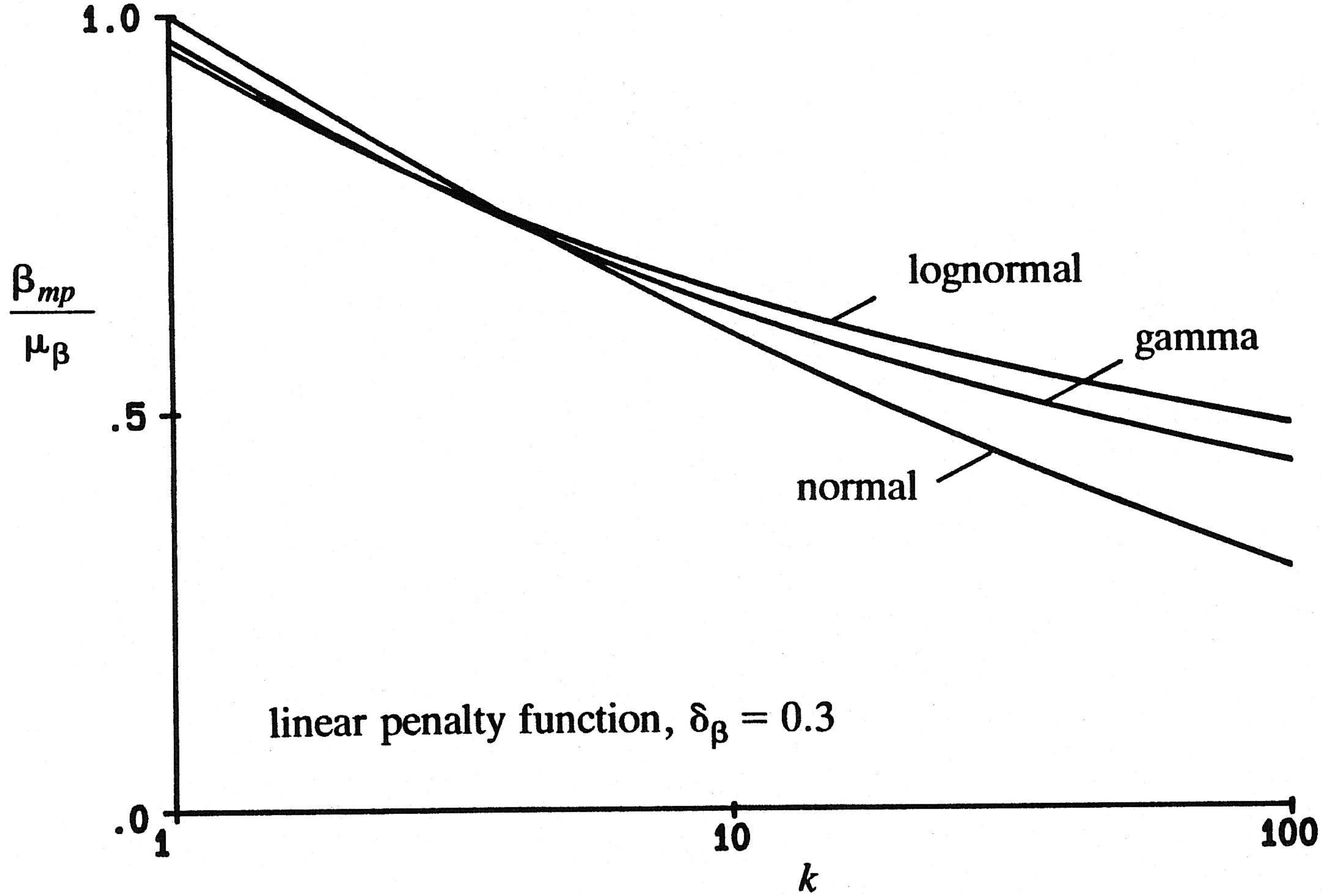

Figure 4. Influence of the Distribution Type on the Minimum-Penalty Reliability Index

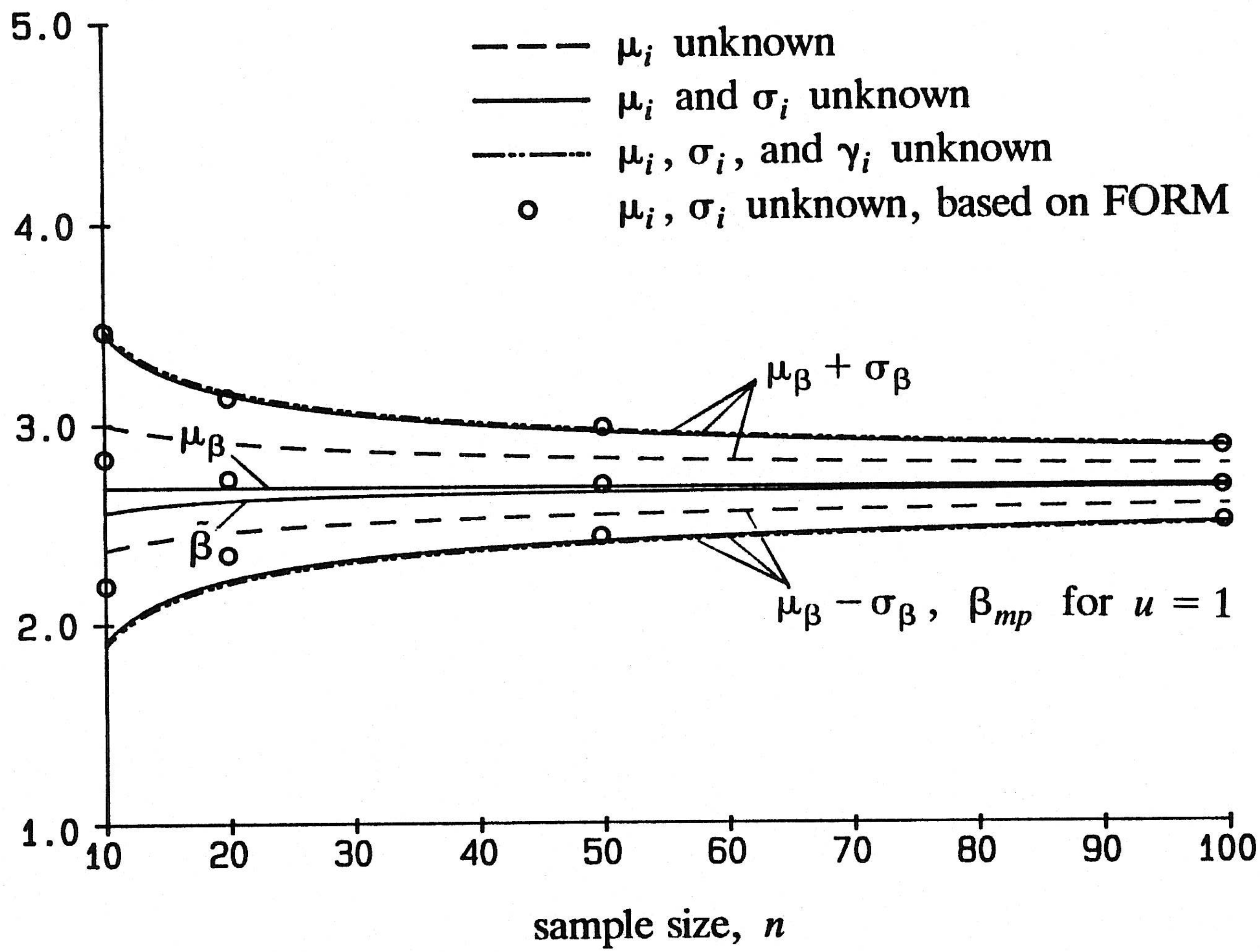

Figure 5. Measures of Safety as Functions of Sample Size

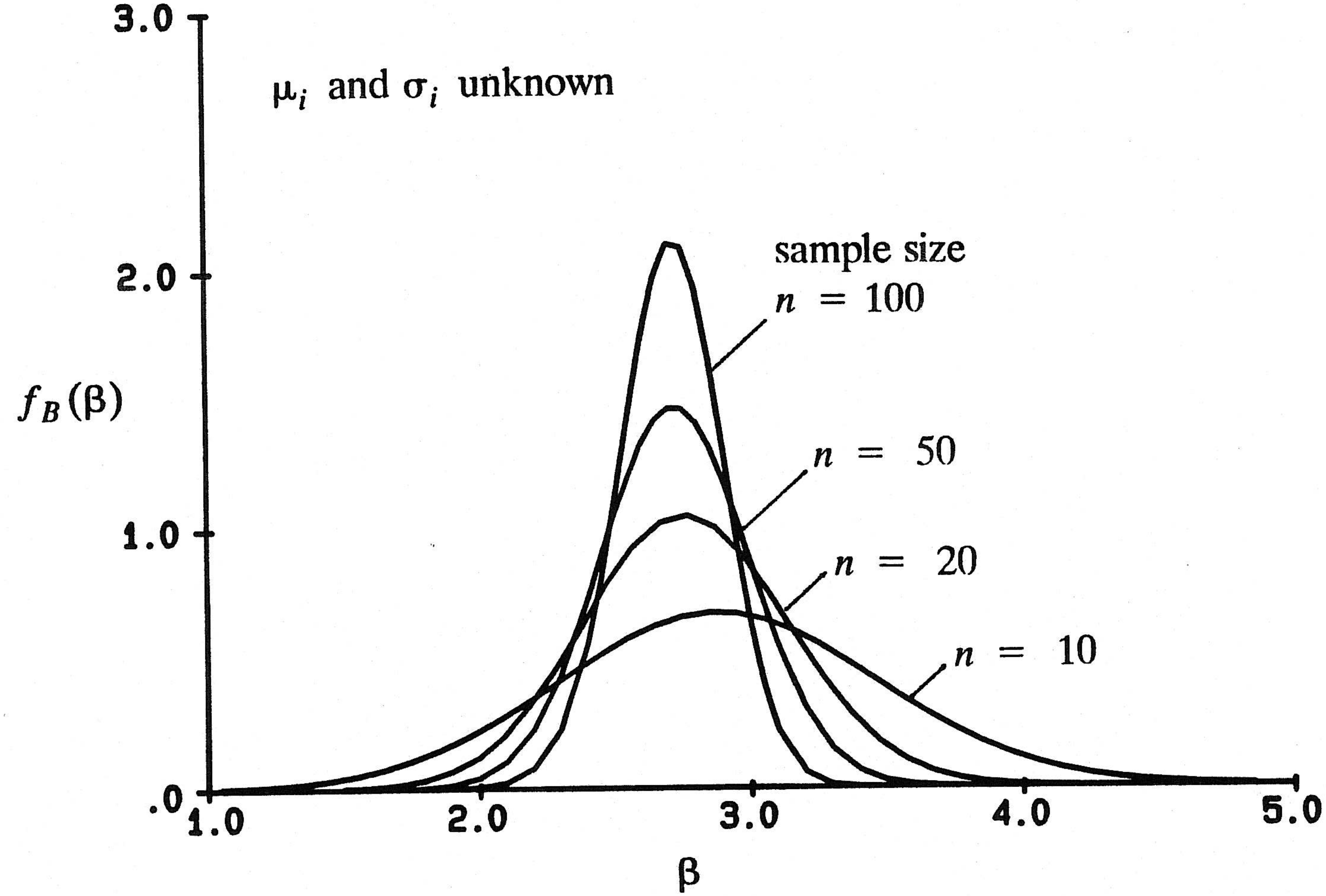

Figure 6. PDF $f_B(\beta)$ Based on FORM Analysis