Title

Properties of Gamma factors for GSp(4) x GL(r) with r = 1, 2 /

Author

Townsend, Nelson J.

Publication Date

2013

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Properties of Gamma factors for $GSp(4) \times GL(r)$ with $r = 1, 2$.

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Nelson J. Townsend

Committee in charge:

Professor Nolan Wallach, Chair
Professor Wee Teck Gan, Co-Chair
Professor Roland Graham
Professor Cristian D. Popescu
Professor Laurie Smith

2013

The dissertation of Nelson J. Townsend is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Co-Chair

Chair

University of California, San Diego

2013

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

Thanks to my advisor Wee Teck Gan for his guidance, patience and support while completing this work. I would also like to thank my committee chair Nolan Wallach for his advice and many helpful conversations.

Thanks to the members of my committee Christian Popescu, Laurie Smith, and Ronald Graham for agreeing to participate. To former vice chair Jim Lin and professor Lance Small for all their help along the way. To all the mathematics department faculty and staff.

VITA

2006	B.S. in Mathematics, California State University, East Bay
2007-2012	Graduate Teaching Assistant, University of California, San Diego
2013	Ph. D. in Mathematics, University of California, San Diego

ABSTRACT OF THE DISSERTATION

Properties of Gamma factors for $GSp(4) \times GL(r)$ with $r = 1, 2$.

by

Nelson J. Townsend

Doctor of Philosophy in Mathematics

University of California, San Diego, 2013

Professor Nolan Wallach, Chair
Professor Wee Teck Gan, Co-Chair

We show several analytic and LLC functorial properties of the local Gamma factors for non-generic representations of $GSp(4) \times GL(r)$ with $r = 1, 2$. In both cases the Gamma factors are obtained using the Rankin-Selberg integrals of [PS] and [MOR]. We also include a discussion of Bessel models and the asymptotics expansions of Bessel functions.

1 Introduction

Let F be a non-archimedian local field of characteristic 0 and residue characteristic p. Let W_F be the Weil group for F and $WD_F = W_F \times SL_2(\mathbb{C})$ be the Weil-Deligne group. The Langlands dual group of $G = GSp_4(F)$ is $G^\vee = GSp_4(\mathbb{C})$. We let $\Pi(G)$ be the set of equivalence classes of irreducible smooth representations of G and $\Phi(G)$ be the set of equivalence classes of admissible homomorphisms

$$WD_F \to G^\vee.$$

Then the local Langlands correspondence asserts that there is a finite-to-one surjection $\Pi(G) \to \Phi(G)$ which preserves certain invariants, including γ factors. This mapping was uniquely characterized by Gan-Takeda[GT], but without a complete theory of local γ, L and ϵ factors for non-generic representations of G, a coarser invariant known as the Plancherel measure was used.

The purpose of this paper is to expand the theory for local γ factors for representations of $GSp_4 \times GL_r$ with $r = 1, 2$.

Main Theorem

Given admissible irreducible representations π and σ of $GSp_4(F)$ and $GL_r(F)$ respectively and a character ψ of F, we define a meromorphic function $\gamma(s, \pi\times\sigma, \psi)$ satisfying the following properties:

(i) *Unramified Factors.* When the representations π and σ are unramified and the character ψ has conductor 0,

$$\gamma(s, \pi \times \sigma, \psi) = \frac{L(1-s, \pi \times \sigma)}{L(s, \pi \times \sigma)}.$$

(ii) *Unramified Twisting.* Given $s_0 \in \mathbb{C}$

$$\gamma(s + s_0, \pi \times \sigma, \psi) = \gamma(s, \pi \times \sigma \cdot |\cdot|^{s_0}, \psi).$$

(iii) *Dependence on* ψ. Let $a \in F^\times$, set $\psi_a(x) = \psi(ax)$. Then

$$\gamma(s, \pi \times \sigma, \psi_a) = \omega_\pi^{2r}(a)\omega_\sigma^4(a)|a|^{4rs}\gamma(s, \pi \times \sigma, \psi).$$

where ω_π and ω_σ denote the central characters for Π and σ.

(iv) *Functional Equation.*

$$\gamma(s, \pi \times \sigma, \psi) \cdot \gamma(1 - s, \tilde{\pi} \times \tilde{\sigma}, \psi^{-1}) = 1.$$

where $\tilde{\pi}$ and $\tilde{\sigma}$ are the contragredient representations of π and σ.

(v) *Global Property* Let $\Pi = \otimes_v' \pi_v$ and $\Sigma = \otimes_v' \sigma_v$ be automorphic cuspidal representations of $GSp_4(\mathbb{A})$ and $GL_r(\mathbb{A})$ respectively. Let S be a set containing all archimedean and 2-adic places and those places where any of the data π_v, σ_v and ψ_v are ramified. Let $L_S(s, \Pi \times \Sigma) = \prod_{v \notin S} L_v(s, \pi_v \times \sigma_v)$ be the partial L function, then

$$L_S(s, \Pi \times \Sigma) = \prod_{v \in S} \gamma_v(s, \pi_v \times \sigma_v, \psi_v) \cdot L_S(1 - s, \tilde{\Pi} \times \tilde{\Sigma}).$$

We follow the methodology of [TA] for defining the γ factor as a constant of proportionality for a zeta integral. We want to consider the general case where the representation of G may not be generic. Thus we use a Bessel model to capture this and spend sometime elaborating on the asymptotics of Bessel functionals in order to prove important analytic properties of the zeta integrals we will use.

We break up our analysis into the two cases, $r = 1$ and $r = 2$. In the first case we extend the work of [PS] using a slightly modified version of his zeta integral to define the γ factor. This integral is of Rankin-Selberg type for an Eisenstien series over a certain subgroup of GSp_4.

In the second case we look at the zeta integral defined in [MOR]. This again is of Rankin-Selberg type, but in this case makes use of an Eisenstien series defined over the larger group $GU(3, 3)$ that contains a subgroup of $GSp_4 \times GL_2$ which we will integrate over.

In both cases, we first define the global Rankin Selberg type integral which will have an integrand consisting of one or more cusp forms and an Eisenstien series.

$$Z(s, f, \phi) = \int E(g, f, s)\phi(g)dg.$$

Next we must produce a so called 'Basic Identity' which will allow us to write this global integral as a product over all places of local zeta integrals.

$$Z(s, f, \phi) = \prod_v Z_v(s, f_v, B_v)$$

where $\phi = \otimes_v \phi_v$, the restricted tensor product over all places v of F.

At this point we do some local analysis of these local zeta integrals including the definition of our gamma factor

$$Z(1 - s, Mf, \tilde{B}) = \gamma(s, \Pi \times \sigma, \psi) Z(s, f, B)$$

then we proceed to the 'Main Theorem' in both cases.

1.1 Notation

Let F be a number field and let $\mathbb{A}_{\mathbb{F}}$ or $\mathbb{A}$ denote the ring of adeles of F. The completion of F at a place v will be denoted F_v.

For a non-Archimedian completion F_v of F, let $\mathfrak{o}_{F_v}$ be the ring of integers, $\mathfrak{p}_{F_v}$ the unique maximal ideal with generator π_{F_v} and set $q = q_{F_v} = |\mathfrak{o}_{F_v}/\mathfrak{p}_{F_v}|$. Let E be a separable quadratic algebra over F. If E is a field, then for E_v define $\mathfrak{o}_{E_v}$, $\mathfrak{p}_{E_v}$, π_{E_v} and q_{E_v} analogusly. Let $\overline{a}$ denote the action of $Gal(E/F)$ for any $a \in E$ and let δ be an element of $E^\times$ such that $\overline{\delta} = -\delta$ and $\Delta = \delta^2 \in F$. If $E = F \oplus F$, then we take $\mathfrak{o}_{E_v} = (\mathfrak{o}_{F_v}, \mathfrak{o}_{F_v})$ and $\pi_{E_v} = (\pi_{F_v}, 1)$. Let $\psi : F \to \mathbb{C}^\times$ be a non-trivial additive character and set $\psi_E(a) = \psi \circ Tr_{E/F}(a)$.

If G is an algebraic group over F, we write $G(F)$, $G(F_v)$ and $G(\mathbb{A})$ respectively for the points of G over F, F_v and $\mathbb{A}$. Z_G will denote the center of G. For a representation π of G, we denote the central character of π by ω_π.

Let n be a positive integer. The *unitary similitude* group $G_n = GU(n, n)$ and the *symplectic similitude* group $H_n = GSp_{2n}$ are defined by

$$G_n = \{g \in GL_{2n}(E) \mid gJ_n{}^t g^\sigma = \lambda(g)J_n, \lambda(g) \in F^\times\} \tag{1.1}$$

and

$$H_n = \{h \in GL_{2n}(F) \mid hJ_n{}^t h = \lambda(h)J_n, \lambda(h) \in F^\times\} \tag{1.2}$$

where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

1.2 Preliminaries

1.2.1 Measure

Let ψ_v be an additive character of F_v. To simplify the notation of this section we will simply write ψ. We choose our Haar measure dy_ψ on the field F_v to be self-dual with respect to ψ. By which we mean the Fourier transform

$$FT_\psi(f)(x) = \int_{F_v} f(y)\psi(xy)dy_\psi$$

satisfies the Fourier inversion formula

$$FT_\psi(FT_\psi(f))(x) = f(-x).$$

For $a \in F_v^\times$ we define $\psi_a(x) = \psi(ax)$, we can define dy_{ψ_a} in terms of dy_ψ so that it is self-dual with respect to ψ_a. We consider the a-twisted Fourier transform

$$\begin{aligned} FT_{\psi_a}(f)(x) &= \int_{F_v} f(y)\psi_a(xy)dy_{\psi_a} \\ &= \int_{F_v} f(y)\psi(axy)dy_\psi \end{aligned}$$

On the one hand this is simply

$$\begin{aligned} \int_{F_v} f(y)\psi(axy)dy_\psi &= FT_\psi(f)(ax) \\ &= l_a^* FT_\psi(f)(x) \end{aligned}$$

where l_a denotes the action of left translation by a. At the same time we can make the change of variables $y \mapsto a^{-1}y$

$$\begin{aligned} \int_{F_v} f(y)\psi(axy)dy_\psi &= \int_{F_v} f(y/a)\psi(xy)|a|^{-1}dy_\psi \\ &= |a|^{-1}\int_{F_v} l_{a^{-1}}^* f(y)\psi(xy)dy_\psi \\ &= |a|^{-1}FT_\psi(l_{a^{-1}}^* f)(x) \end{aligned}$$

Now we look at applying the a-twisted Frouier transform twice to see what normailization will be needed.

$$\begin{aligned} FT_{\psi_a}(FT_{\psi_a}(f))(x) &= FT_{\psi_a}(l_a^* FT_\psi(f))(x) \\ &= |a|^{-1} FT_\psi(l_{a^{-1}}^* l_a^* FT_\psi(f))(x) \\ &= |a|^{-1} FT_\psi(FT_\psi(f))(x) \\ &= |a|^{-1} f(-x) \end{aligned}$$

Thus we set $dy_{\psi_a} = |a|^{1/2} dy_\psi$ to obtain a self-dual measure with respect to the a-twisted character ψ_a. Lastly, we note that the choice of measure dy_ψ has the benifit of giving volume 1 to the ring of integers $\mathfrak{o}_{F_v}$ when ψ is unramified.

1.2.2 Subgoups

Let P denote the Siegal parabolic supgroup of H_2 which consists of 2x2 block matrices of the form $\begin{pmatrix} * & * \\ & * \end{pmatrix}$. Let $M \cdot N$ be the Levi decomposition of P, with M the reductive part and N the unipotent radical of P. Explicitly,

$$M = \left\{ m(A,x) = \begin{pmatrix} A & \\ & x \cdot {}^t A^{-1} \end{pmatrix} | x \in F^\times, A \in GL_2(F) \right\}$$

$$N = \left\{ n(B) = \begin{pmatrix} I_2 & B \\ & I_2 \end{pmatrix} | {}^t B = B \right\}$$

For $n(B) \in N$ and any symmetric matrix β we can define the linear form $n(B) \mapsto \mathrm{Tr}(\beta \cdot B)$, furthermore all linear maps $N \to F$ are of this form. We call such a form non-degenerate if $\det(\beta) \neq 0$. Let us now fix a non-singular 2x2 symmetric matrix,

$$\beta = \begin{pmatrix} \beta_1 & \beta_2/2 \\ \beta_2/2 & \beta_3 \end{pmatrix}$$

and refer to the corresponding linear form as l_β. M acts on N by conjugation and thus on any linear form on N. Identifying $N \cong \mathrm{Sym}_2(F)$, the set of symmetric 2x2 matrices over F, the action of M is

$$m(A,x) \cdot B = x^{-1} A \cdot B \cdot {}^t A$$

Let $N_0 = \{n \in N | l_\beta(n) = 0\}$. Denote by T_β the connected component of the stabilizer of l_β in M. Set $d = -4\det(\beta)$ and let $[d]$ denote the square-class of d in $F^\times/F^{\times^2}$. If $[d] = 1$, then we choose $D_\beta = \begin{pmatrix} & \sqrt{d} \\ \sqrt{d} & \end{pmatrix}$ in the orbit of β under the action of $GL_2(F)$, we set $E = F \oplus F$ and $A(x, y) = \begin{pmatrix} x & \\ & y \end{pmatrix}$. Similarly, if $[d] \neq 1$, then we may choose $D_\beta = \begin{pmatrix} -d & \\ & 1 \end{pmatrix}$ and set $E = F(\sqrt{d})$ and $A(x + y\sqrt{d}) = \begin{pmatrix} x & yd \\ y & x \end{pmatrix}$. Hence $A : E^\times \to GL_2(F)$. Lastly we map $E^\times \to T_{D_\beta}$ by

$$t \mapsto \begin{pmatrix} A(t) & \\ & \det(A(t))^t A(t)^{-1} \end{pmatrix} \in T_{D_\beta}.$$

Therefore $E^\times \cong T_{D_\beta}$ and since T_β and T_{D_β} are conjugate we have:

Lemma 1.2.1. *There exists a unique up to isomorphism quadratic F-algebra, E, such that $T_\beta \cong E^\times$. If $[d] \neq 1$, then E is a field.*

Definition 1.2.2. The subgroup $R_\beta = T_\beta \cdot N$ is called a Bessel subgroup of H_2.

If we compose our additive character $\psi : F \to \mathbb{C}^\times$, with our linear form l_β, we get a character on N, we will denote this by ψ_β. Now let ν is a character on $T_\beta \cong E^\times$. Since T stabilizes l_β, it stabilizes ψ_β, thus we can form a character on R, $t \cdot n \mapsto \nu(t)\psi_\beta(n)$.

For elements in $x \in E$ write $\overline{x}$ for the Galois action($\overline{(x_1, x_2)} = (x_2, x_1)$ in the split case), then $\mathrm{Tr}_{E/F}(x) = x + \overline{x}$ and we may write $E = F + F\delta$ for some trace zero $\delta \in E$(e.g. $\sqrt{d}$ or $(1, -1)$). Let $V_E = Ev_1 \oplus Ev_2$ and let $<,>_E$ be the symplectic form on V_E with $< v_1, v_2 >= 1$. Let

$$GSp(V_E, <,>_E) = \{g \in GL(V_E) | < xg, yg >_E = \lambda_g < x, y >_E \text{ for some } \lambda_g \in E^\times\}$$

and let G° denote the subgroup with $\lambda_g \in F^\times$. Now we can view V_E as 4-dimensional F-vector space via the restriction of scalars map $\mathrm{Res}_{E/F}$. If we let

$<,>= Tr_{E/F}(<,>_E)$, then

$$G^\circ \subset GSp(V_E, <,>_E) \hookrightarrow GSp(V, <,>) \cong GSp_4(F)$$

If we write $V = X \oplus Y$, $X = \mathrm{Res}_{E/F}(Ev_1)$ and $Y = \mathrm{Res}_{E/F}(Ev_2)$ isotypic subspaces, then $P \cong P(X)$ the stabilizer of X in $GSp(V, <,>)$. Write $P(X) = M(X) \cdot U(X)$ for the Levi decomposition and $M(X) = GL(X) \times F^\times$. We can embed $i : E^\times \hookrightarrow M(X)$ via $e \mapsto (e, \mathrm{N}_{E/F}(e))$ noting this is just the coordinate free version of the mapping described in Lemma 1.2.1 above. Then the Bessel subgroup is $R = i(E^\times) \cdot N(X)$. Now since $R \subset P(X)$, $G^\circ \cap R$ is contained in the stabilizer of Ev_1 so we must have $G^\circ \cap R = B^\circ(v_1) \cap R$ where $B^\circ = B^\circ(v_1)$ is the Borel. For an element $b \in B^\circ$ write

$$b = \begin{pmatrix} x & \\ & 1 \end{pmatrix} \begin{pmatrix} t & \\ & \bar{t} \end{pmatrix} \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \quad x \in F^\times, t \in E^\times, \text{and } n \in E.$$

and denote the corresponding subgroups as H°, T° and N° respectively. Noting that $N^\circ = N_0$ we have $B^\circ \cap R = i(E^\times) \cdot N^\circ$. Hence

$$G^\circ \cap R = TN_0 = T^\circ N^\circ$$

We can be explicit with the embedding $G^\circ \hookrightarrow GSp_4(F)$ by writing down a basis for V. Recall we have basis $\{v_1, v_2\}$ and $E = F + F\delta$ where δ is trace 0. We take basis $\{e_1, e_2, f_1, f_2\} = \{v_1, \delta v_1, \frac{1}{2}v_2, \frac{1}{2\delta}v_2\}$, then $< e_i, f_j >= \delta_{ij}, i \leqslant j$ and $< e_i, e_j >= 0 =< f_i, f_j >, i, j = 1, 2$. With respect to this basis we get the following embedding,

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} x_1 & x_2 d & y_1/2 & y_2/2 \\ x_2 & x_1 & y_2/2 & y_1/2d \\ 2z_1 & 2z_2 d & w_1 & w_2 \\ 2z_2 d & 2z_1 d & w_2 d & w_1 \end{pmatrix} \text{ or } \begin{pmatrix} x_1 & & y_1 & \\ & x_2 & & y_2 \\ z_1 & & w_1 & \\ & z_2 & & w_2 \end{pmatrix}$$

if E is a field or split respectively.

Here the groups and subgroups were algebraic over F, but the same constructions work equally well over F_v or $\mathbb{A}_F$.

2 Bessel Models

Let E be a quadratic extension of F with associated Bessel subgroup 1.2.2 $R \cong E^{\times} \cdot N \subset GSp(4)$. Let π be an irreducible smooth representation of $GSp_4(F)$. For a character $\chi = \mu \otimes \psi$ of R, a Bessel functional of π with respect to (E, μ) is a linear functional

$$B : \pi \longrightarrow \mathbb{C}$$

such that

$$B(\pi(r)v) = \chi(r) \cdot B(v).$$

We let $\mathrm{Hom}_R(\pi, \chi)$ be the space of such Bessel functionals.

2.1 Uniqueness.

The following is a basic result of Novodrovsky in the p-adic case.

Theorem 2.1.1. *One has:*

$$\dim \mathrm{Hom}_R(\pi, \chi) \leqslant 1.$$

2.2 Local existence.

We would like to show:

Proposition 2.2.1. *Let π be an infinite-dimensional irreducible representation of $\mathrm{GSp}_4(F)$. Then π has nonzero Bessel functionals with respect to some (E, μ).*

Proof. By a result of Howe[**?**], one knows that there is a nondegenerate character ψ of N such that $\pi_{N,\psi} \neq 0$. Such a ψ is associated to a quadratic étale F-algebra E, and its stabilizer is isomorphic to $E^\times$. Thus we need to show that the $E^\times$-module $\pi_{N,\psi}$ has an irreducible quotient.

Assume first that E is a field. Note that $F^\times \subset E^\times$ acts as ω_π on $\pi_{N,\psi}$. Twisting $\pi_{N,\psi}$ by a character of $E^\times$ whose restriction to $F^\times$ is ω_π, one may assume without loss of generality that $F^\times$ acts trivially on $\pi_{N,\psi}$, i.e. $\pi_{N,\psi}$ is a nonzero module for the compact group $E^\times/F^\times$. It follows that $\pi_{N,\psi}$ has an irreducible quotient.

On the other hand, if $\pi_{N,\psi} = 0$ for all ψ associated to quadratic fields, then π is a so-called distinguished representation in the sense of J.S. Li [**?**], in the sense that $\pi_{N,\psi}$ is nonzero with respect to a unique M-orbit of nondegenerate ψ (associated to $E = F^2$). In this case, a result of J.S. Li implies that π is obtained as a local theta lift from the split orthogonal group $\mathrm{GSO}_{1,1} \cong E^\times$. In this case, one can show that $\pi_{N,\psi}$ is finite -dimensional, and so has a nonzero irreducible quotient. □

2.3 Global Bessel models.

We now consider the global analog of the above discussion. Thus let k be a number field with ring of adeles $\mathbb{A}$. Let $\psi : N(k)\backslash N(\mathbb{A}) \longrightarrow \mathbb{C}^\times$ be a nondegenerate automorphic character of N, whose stabiliser is isomorphic to $\mathbb{A}_{E^\times}$ for some quadratic étale k-algebra E. For a Hecke character μ of $\mathbb{A}_{E^\times}$, one has the automorphic character $\chi = \mu \otimes \psi$ of R.

If $\mathcal{A}_{cusp}(GSp_4)$ denotes the space of cusp forms of GSp_4 with central character $\mu|_{\mathbb{A}^\times}$, then the global Bessel integral with respect to (E, μ) is the linear functional on $\mathcal{A}_{cusp}(GSp_4)$ defined by

$$\mathcal{B}(f) = \int_{R(k)Z(\mathbb{A})\backslash R(\mathbb{A})} f(r) \cdot \overline{\chi(r)}\, dr.$$

We say that $\pi \subset \mathcal{A}_{cusp}(GSp_4)$ has nonzero global Bessel period with respect to (E, μ) if $\mathcal{B}$ is nonzero when restricted to π.

2.4 Global existence.

We have the following global existence result:

Proposition 2.4.1. *Let π be a cuspidal representation of GSp_4. Then there exists (E, μ) with E a quadratic field and $\mu|_Z = \omega_\pi$ such that π has nonzero Bessel period with respect to (E, μ).*

Proof. By Howe [H1] and J.S. Li [L3], one knows that π has nonzero Fourier coefficient with respect to some nondegenerate character ψ of N. Suppose that ψ corresponds to a quadratic field extension E of k. Then for some $f \in \pi$, $f_{N,\psi}$ is a nonzero function on $E^\times \backslash \mathbb{A}_E^\times$ on which $\mathbb{A}^\times$ acts by ω_π. It then follows that there is a Hecke character μ of $\mathbb{A}_E^\times$ with $\mu|_{\mathbb{A}^\times} = \omega_\pi$ such that

$$\int_{E^\times \mathbb{A}^\times \backslash \mathbb{A}_E^\times} f_{N,\psi}(t) \cdot \overline{\mu(t)}\, dt \neq 0.$$

Suppose that the only nondegenerate Fourier coefficient along N supported by π is the one corresponding to the split algebra k^2. Then by a result of J. S. Li, π has nonzero global theta lift to the split orthogonal group $\mathrm{GO}_{1,1}$. There is no cuspidal representation of GSp_4 which could participate in the global theta correspondence with $\mathrm{GO}_{1,1}$.

□

2.5 Asymptotics of Bessel functions

Let E be a quadratic extension of F with associated Bessel subgroup $R \cong E^\times \cdot N \subset GSp(4)$. Let π be an irreducible smooth representation of $GSp_4(F)$. For a character $\chi = \mu \otimes \psi$ of R, let

$$B \in \mathrm{Hom}_R(\pi, \chi).$$

For fixed $v \in \pi$, we would like to investigate the asymptotic behaviour of the function on $H^\circ \cong F^\times$ defined by

$$t \mapsto B(t \cdot v).$$

2.5.1 Non-archimedean case.

Assume first that F is nonarchimedean. We have:

Lemma 2.5.1. *The function $B(tv)$ vanishes if $|t|_F$ is sufficiently large.*

Proof. Since π is smooth, there is an open compact subgroup $C \subset N$ such that $n \cdot v = v$ for all $n \in C$. Thus,

$$B(tv) = B(tn \cdot v) = \psi(tnt^{-1}) \cdot B(tv).$$

Thus, if $B(tv) \neq 0$, we must have

$$\psi(tnt^{-1}) = 1 \quad \text{for all } n \in C.$$

In other words, $tCt^{-1} \subset \mathrm{Ker}(\psi)$ if $B(tv) \neq 0$. Since $Ker(\psi)$ is a compact subgroup of N, and tCt^{-1} is unbounded as $|t| \to \infty$, we see that when $|t|$ is sufficiently large, $B(tv) = 0$. □

Now we want to examine the behaviour of $B(tv)$ as $|t| \to 0$. We shall see that this behaviour is controlled by the Jacquet module π_N, which is a finite length representation of the Levi subgroup M of the Siegel parabolic $P = MN$. Note that $F^{\times} \cong H^{\circ}$ is contained in the center of M. Regarded as a representation of $F^{\times}$, one has a finite decomposition

$$\pi_N = \oplus_{\chi} \pi_N[\chi]$$

into generalized eigenspaces for $F^{\times}$. We first note:

Lemma 2.5.2. *If $v \in \pi[N] = \mathrm{Ker}(p : \pi \to \pi_N)$, then $B(tv)$ vanishes near $0 \in F^{\times}$.*

Proof. Note that $\pi[N]$ is spanned by elements of the form $v = nw - w$ for $n \in N$ and $w \in V_{\pi}$. Then

$$B(t(nw - w)) = (\psi(tnt^{-1}) - 1) \cdot B(tw).$$

But $\psi(tnt^{-1}) = 1$ for t sufficiently close to 0. This proves the lemma. □

Next, suppose $v \in V$ is such that the image of v is in π_N belongs to $\pi_N[\chi]$. The space $\pi_N[\chi]$ has an increasing filtration $(\pi_N[\chi]_n)$, with $n \geqslant 0$, and with $\pi_N[\chi]_n$ consisting of those $w \in \pi_N[\chi]$ such that

$$[\prod_{i=1}^{n}(t_i - \chi(t_i))] \cdot w = 0$$

for all $t_i \in F^\times$. The case $n = 0$ is interpreted to mean $w \in \mathrm{Ker}(p)$. We shall analyze the behaviour of $B(tv)$ for v such that $p(v) \in \pi_N[\chi]_n$. More precisely, we shall show:

Lemma 2.5.3. *For $v \in \pi$ such that $p(v) \in \pi_N[\chi]_n$, $B(tv) = \chi(t) \cdot f(\log|t|)$ for some polynomial f of degree $\leqslant n-1$ when t is sufficiently close to 0. Here, for $n = 0$, f is interpreted to be 0, and* log *refers to* $\log_q$.

Proof. The base case $n = 0$ is the previous lemma. Now we deal with the inductive step. By the hypothesis on v,

$$\prod_{i=1}^{n}(t_i - \chi(t_i))v \in \mathrm{Ker}(p).$$

So the previous lemma implies that for all $|a| < \epsilon_{t_1 \cdots, t_n}$,

$$B(a \cdot \prod_{i=1}^{n}(t_i - \chi(t_i))v) = 0.$$

Since π and χ are smooth, the number $\epsilon_{t_1, \cdots, t_n}$ is locally constant in $t_1, \cdots, t_n$. In particular, if the t_is vary over a compact set C, one can pick an ϵ which works for all choices of t_i in C. In particular, one picks an ϵ which works for all $q^{-1} \leqslant |t_i| \leqslant 1$. Then we claim that this same ϵ works for all $|t_i| \leqslant 1$. To see this, let us replace t_1 by $t_1 t$ with $|t| = q^{-1}$, so that $|t_1 t| = q^{-2}$. We want to show the vanishing of

$$B(a \cdot (t_1 t - \chi(t_1 t)) \cdot \prod_{i>1}(t_i - \chi(t_i))v), \quad \text{when } |a| < \epsilon.$$

Then with $w = \prod_{i>1}(t_i - \chi(t_i))v)$, we see that for $|a| < \epsilon$,

$$B(a \cdot (t_1 t - \chi(t_1 t)) \cdot w) = B(at \cdot t_1 \cdot w) - \chi(t_1 t) B(aw)$$

$$= \chi(t_1) \cdot B(atw) - \chi(t_1 t) B(aw) = \chi(t_1) \cdot (B(a \cdot (t - \chi(t)) \cdot w)) = 0$$

as desired. Repeating this argument establishes our claim.

Thus we now have: whenever $|a| < \epsilon$ and $|t_i| \leqslant 1$,

$$\sum_S (-1)^{\#S} \chi(t_S)^{-1} \cdot B(at_S v) = 0 \tag{2.1}$$

where the sum runs over all subset S of $\{1, , n\}$ and $t_S = \prod_{i \in S} t_i$. For fixed a with $|a| < \epsilon$, if we set

$$f(t) = \chi(t)^{-1} B(atv) - B(av),$$

then $f(1) = 0$ and (2.1) can be rewritten as:

$$\sum_S (-1)^{\#S} \cdot f(t_S) = 0.$$

We claim that a function f satisfying this must be a polynomial in $\log |t|$ of degree $\leqslant n-1$ and constant term 0.

We shall proceed by induction on n. For t_0 fixed, consider

$$F_{t_0}(t) = f(t \cdot t_0) - f(t) - f(t_0).$$

Then F_{t_0} satisfies $F_{t_0}(1) = 0$ and

$$\sum_{S'} (-1)^{\#S'} f(t_{S'}) = 0$$

as S' ranges over all subsets of $\{1, ..., n-1\}$. By induction hypothesis, $F_{t_0}(t) = P_{t_0}(\log |t|)$ for a polynomial P_{t_0} of degree $\leqslant n-2$. Moreover, the constant term of P_{t_0} is 0 since $F_{t_0}(1) = 0$. So we have

$$f(tt_0) - f(t) = f(t_0) + P_{t_0}(\log |t|).$$

Now, if we assume that t is a unit, then

$$f(tt_0) = f(t) + f(t_0),$$

and so the continuity of f thus implies that $f(t) = 0$ if $|t| = 1$ (else $f(t^n t_0) = n \cdot f(t) + f(t_0) \to \infty$ as $n \to \infty$). In other words, f is a function of $\mathrm{ord}(t) = -\log |t|$, say $f(t) = Q(-\log |t|)$. Then, taking t_0 to be a uniformizer, we deduce that

$$Q(x+1) - Q(x) - Q(1) = P(x),$$

for $x \in \mathbb{Z}$, with P a polynomial of degree $\leqslant n-2$. This implies that Q can be taken to be a polynomial of degree $\leqslant n-1$ (with constant term 0), as desired.

We have thus shown that there is an $\epsilon > 0$ such that for a fixed $|a| \leqslant \epsilon$ and all $|t| \leqslant 1$,

$$B(tav) = \chi(t) \cdot q_a(\log|t|)$$

for some polynomial q_a of degree $\leqslant n-1$. The lemma is proved.

□

2.5.2 Archimedean case.

Suppose now that F is archimedean. In this case, π is a smooth Frechét representation of moderate growth (a Casselman-Wallach representation) and the Bessel funcitonal

$$B : \pi \longrightarrow \mathbb{C}$$

is continuous, so that there is a semi-norm ν on π with

$$B(v) \leqslant \nu(v) \quad \text{for all } v \in \pi.$$

Thus

$$B(tv) \leqslant \nu(tv) \leqslant \max(|t|, |t|^{-1})^k \cdot \mu(v)$$

for some $k \in \mathbb{N}$, and some seminorm μ on π and for all $v \in \pi$. Thus, $B(tv)$ grows like a polynomial in $|t|$ as $|t| \to \infty$, and it grows like a polynomial in $|t|^{-1}$ as $|t| \to 0$.

Lemma 2.5.4. *As $|t| \to \infty$, $B(tv)$ is rapidly decreasing.*

Proof. By the Dixmier-Malliavin theorem, we can express $v \in \pi$ as

$$v = f * v_0 := \int_N f(n)\pi(n)v_0 \, dn$$

for some $v_0 \in \pi$ and some function $f \in C_c^\infty(N)$. Then

$$B(tv) = \int_N f(n) \cdot B(tnv_0)dn = \widehat{f}(t) \cdot B(tv_0)$$

where $\widehat{f}$ denotes the Fourier transform of f. Thus, $\widehat{f}$ is rapidly decreasing as $|t| \to \infty$, whereas $B(tv_0)$ is of polynomial-growth. Hence, $B(tv)$ behaves like a Schwarz function as $|t| \to \infty$. □

The above lemma implies that the linear functional B is *tame* in the sense of Wallach [Wa]. Then [Wa] gives an asymptotic expansion for $B(tv)$ as $|t| \to 0$.

Lemma 2.5.5. *As $|t| \to 0$, one has*

$$B(tv) \sim \sum_{\chi \in E(P,\pi)} \chi(t) \cdot \sum_{k \geqslant 0} t^k \cdot q_{\chi,k}(\log|t|, v)$$

where

- *$E(P,\pi)$ denotes the finite set of leading exponents of π along P*
- *$q_{\chi,k}(\log|t|, v)$ is a polynomial in $\log|t|$ for fixed $v \in \pi$, and $v \mapsto q_{\chi,k}(\log|t|, v)$ is a continuous functional of π.*

Moreover, the meaning of $\sim$ is: for any $n \in \mathbb{N}$,

$$|B(tv) - \sum_{\chi} \sum_{k=0}^{n} t^k \cdot q_{\chi,k}(\log|t|, v)| \leqslant C_n(v) \cdot |t|^{A+n}$$

for some constant A independent of n, and for $|t|$ sufficiently small.

3 $GSp_4 \times GL_1$

This chapter develops the theory of local γ-factors in the case when $r = 1$. Essentially a refinement and extension of the work [PS]. First we establish the global zeta integral and show a 'Basic Identity' which will allow us to factor into local zeta integrals. After verifying analytic properties we define the γ-factor in the usual way as the a constant of proportionality. Explicit calculations are given for the unramified case and the 'Main Theorem' in this case is stated and proved. We conclude the chapter with a discussion of Multiplicativity, which we would like to prove at a future time, and the immediate consequences of this property.

3.1 Global Integral

For the algebraic group $H_2(\mathbb{A}) = GSp_4(\mathbb{A}_F)$ let the subgroups: $M_{\mathbb{A}}$, $T_{\mathbb{A}}$, $N_{\mathbb{A}}$, $R_{\mathbb{A}}$, $G^{\circ}_{\mathbb{A}}$, $B^{\circ}_{\mathbb{A}}$, $T^{\circ}_{\mathbb{A}}$, $N^{\circ}_{\mathbb{A}}$ be the adelic analogues defined in the previous chapter. Here we use F, E, $F^{\times}$, $E^{\times}$ to denote the diagonal embeddings into $\mathbb{A}_F$, $\mathbb{A}_E$, $\mathbb{A}_F^{\times}$, $\mathbb{A}_E^{\times}$ repectively. Note that here $T_{\mathbb{A}} \cong I_E$ where I_E are the ideles of our quadratic F-algebra E.

Let (π, V) be an automorphic cuspidal representation of $GSp_4(\mathbb{A}_F)$ with central character ϖ_π and let $\nu_{\mathbb{A}}$ be a Hecke character of $E^{\times}$ such that $\nu_{\mathbb{A}} = \varpi_\pi$. Let $\psi_{\mathbb{A}}$ a non-degenerate Hecke character on $\mathbb{A}_F/F$ and as before denote the extended character on $U_{\mathbb{A}}$ by $\psi_{l,\mathbb{A}}$. Let $\mu_{\mathbb{A}}$ be a character on I_F and define a character on $B^{\circ}_{\mathbb{A}}$ by

$$\chi_s\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}\begin{pmatrix} \bar{t} & \\ & t \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\right) = \mu_{\mathbb{A}}(x)|x|^s_{I_F}\nu^{-1}_{\mathbb{A}}(t).$$

Let f be a flat section of the family of induced normalized representations

$$I(s,\nu_{\mathbb{A}},\mu_{\mathbb{A}}) = \mathrm{Ind}_{B^\circ_{\mathbb{A}}}^{G^\circ_{\mathbb{A}}}\chi_s.$$

Hence, f is a smooth function on G° such that

$$f\left(\begin{pmatrix} x & \\ & 1\end{pmatrix}\begin{pmatrix} \bar{t} & \\ & t\end{pmatrix}\begin{pmatrix} 1 & n \\ & 1\end{pmatrix} g\right) = \mu_{\mathbb{A}}(x)|x|_{I_F}^{s+1}\nu_{\mathbb{A}}^{-1}(t)f(g)$$

from which we can form the Eisenstien series

$$E(s,g,f,\nu_{\mathbb{A}},\mu_{\mathbb{A}}) = \sum_{\alpha\in B^\circ_F\backslash G^\circ_F} f(\alpha g).$$

This series is known to be meromorphic function of $s \in \mathbb{C}$ and satisfies a functional equation

$$E(s,g,f,\nu_{\mathbb{A}},\mu_{\mathbb{A}}) = E(-s,g,M(s,\nu,\mu,\psi)f,\overline{\nu}_{\mathbb{A}},\nu_{\mathbb{A}_F}^{-1}\mu_{\mathbb{A}}^{-1})$$

where $M(s,\nu_{\mathbb{A}},\mu_{\mathbb{A}},\psi_{\mathbb{A}}) : I(s,\nu_{\mathbb{A}},\mu_{\mathbb{A}}) \mapsto I(-s,\overline{\nu}_{\mathbb{A}},\mu_{\mathbb{A}}^{-1}\nu_{\mathbb{A}}|_{\mathbb{A}_F^\times}^{-1})$ is the standard global intertwining operator defined as

$$M(s,\nu_{\mathbb{A}},\mu_{\mathbb{A}},\psi_{\mathbb{A}})f(g) = \int\limits_{\mathbb{A}_E} f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1\end{pmatrix} g\right) dn$$

for $\mathrm{Re}(s) >> 0$ and has meromorphic continuation to all of $\mathbb{C}$.

We say that π has a non trivial Bessel model with respect to $\psi_{\mathbb{A}}$ and $\nu_{\mathbb{A}}$ if there exists a cusp form $\varphi \in V$ such that for the character $\chi(r) = \chi(tn) = \nu_{\mathbb{A}}(t)\psi_{\mathbb{A}}(n)$ on $R_{\mathbb{A}} = T_{\mathbb{A}}N_{\mathbb{A}}$ the following global Bessel period is nonzero,

$$\int_{Z_{\mathbb{A}}R_F\backslash R_{\mathbb{A}}} \varphi(r)\chi^{-1}(r)dr. \qquad (*)$$

By proposition 2.4.1 we know π has a non-trivial bessel model and we fix a cusp form φ such that (*) does not vanish, we then define for each $g \in GSp(4)$

$$B^\varphi(g) = \int_{Z_{\mathbb{A}}R_F\backslash R_{\mathbb{A}}} \varphi(rg)\chi^{-1}(r)dr$$

For $r \in R_{\mathbb{A}}$

$$B^\varphi(rg) = \chi(r)B^\varphi(g)$$

We denote the space of such functions $\mathcal{B}_\pi(E, \nu_{\mathbb{A}}, \psi_{\mathbb{A}})$ and via right translation gives a representation of $GSp_4(\mathbb{A}_F)$ equivalent to π. The global Bessel model will factor as a restricted tensor product of local Bessel models and uniqueness follows from the uniqueness of the local models, Theorem 2.1.1.

We may now define our global zeta integral

$$Z(s, \varphi, f, \nu_{\mathbb{A}}, \mu_{\mathbb{A}}) = \int_{Z_{\mathbb{A}} G_F^\circ \backslash G_{\mathbb{A}}^\circ} \varphi(g) E(s, g, f, \nu_{\mathbb{A}}, \mu_{\mathbb{A}}) dg.$$

The convergence and meromorphic continuation of such an integral is well know. The functional equation for the Eisenstien series gives us a functional equation for the zeta integral for free

$$Z(s, f, \varphi, \nu_{\mathbb{A}}, \mu_{\mathbb{A}}) = Z(-s, M(s, \nu_{\mathbb{A}}, \mu_{\mathbb{A}}, \psi_{\mathbb{A}}) f, \varphi, \overline{\nu}_{\mathbb{A}}, \nu_{\mathbb{A}}^{-1} \mu_{\mathbb{A}}^{-1}). \tag{3.1}$$

3.1.1 Basic Identity

$$\begin{aligned} Z(s, f, \varphi, \nu_{\mathbb{A}}, \mu_{\mathbb{A}}) &= \int\limits_{Z_{\mathbb{A}} G_F^\circ \backslash G_{\mathbb{A}}^\circ} \varphi(g) E(s, g, f, \nu_{\mathbb{A}}, \mu_{\mathbb{A}}) dg \\ &= \int\limits_{Z_{\mathbb{A}} G_F^\circ \backslash G_{\mathbb{A}}^\circ} \varphi(g) \sum_{\gamma \in B_F^\circ \backslash G_F^\circ} f(\gamma g) dg \\ &= \int\limits_{Z_{\mathbb{A}} G_F^\circ \backslash G_{\mathbb{A}}^\circ} \varphi(\gamma g) \sum_{\gamma \in B_F^\circ \backslash G_F^\circ} f(\gamma g) dg \\ &= \int\limits_{Z_{\mathbb{A}} B_F^\circ \backslash G_{\mathbb{A}}^\circ} \varphi(g) f(g) dg \end{aligned}$$

We have the Fourier expansion

$$\varphi(g) = \sum_{\psi \in \widehat{N_F \backslash N_{\mathbb{A}}}, \psi \not\equiv 1} \varphi_\psi(g), \quad \varphi_\psi(g) = \int_{N_F \backslash N_{\mathbb{A}}} \varphi(ng) \psi^{-1}(n) dn$$

Continuing to unfold

$$Z(s,f,\varphi,\nu_{\mathbb{A}},\mu_{\mathbb{A}}) = \int\limits_{Z_{\mathbb{A}}B_F^\circ\backslash G_{\mathbb{A}}^\circ} \sum_{\substack{\psi\in\widehat{N_F\backslash N_{\mathbb{A}}}\\ \psi\not\equiv 1}} \varphi_\psi(g)f(g)dg$$
$$= \int\limits_{Z_{\mathbb{A}}H_F^\circ T_F^\circ N_{\mathbb{A}}^\circ\backslash G_{\mathbb{A}}^\circ} \int\limits_{N_F^\circ\backslash N_{\mathbb{A}}^\circ} \sum_{\substack{\psi\in\widehat{N_F\backslash N_{\mathbb{A}}}\\ \psi\not\equiv 1}} \varphi_\psi(ng)f(ng)dn\ dg$$

Note that here since f is invariant under $N_{\mathbb{A}}^\circ$ the integration will kill terms with characters ψ which are non-trivial on $N_{\mathbb{A}}^\circ$. Thus we need only consider those characters, which are nontrivial on $N_F\backslash N_{\mathbb{A}}$ and trivial on $N_{\mathbb{A}}^\circ$. Since H_F° acts simply transitive on this set of characters we reduce to

$$= \int\limits_{Z_{\mathbb{A}}H_F^\circ T_F^\circ N_{\mathbb{A}}^\circ\backslash G_{\mathbb{A}}^\circ} \sum_{h\in H_F^\circ} \varphi_\psi(hg)f(g)dg$$
$$= \int\limits_{Z_{\mathbb{A}}T_F^\circ N_{\mathbb{A}}^\circ\backslash G_{\mathbb{A}}^\circ} \varphi_\psi(g)f(g)dg$$
$$= \int\limits_{T_{\mathbb{A}}^\circ N_{\mathbb{A}}^\circ\backslash G_{\mathbb{A}}^\circ} \left(\int\limits_{Z_{\mathbb{A}}T_F\backslash T_{\mathbb{A}}} \varphi_\psi(tg)f(tg)dt \right) dg$$
$$= \int\limits_{T_{\mathbb{A}}^\circ N_{\mathbb{A}}^\circ\backslash G_{\mathbb{A}}^\circ} \left(\int\limits_{Z_{\mathbb{A}}T_F^\circ\backslash T_{\mathbb{A}}^\circ} \varphi_\psi(tg)\nu^{-1}(t)dt \right) f(g)dg$$
$$= \int\limits_{T_{\mathbb{A}}^\circ N_{\mathbb{A}}^\circ\backslash G_{\mathbb{A}}^\circ} \left(\int\limits_{Z_{\mathbb{A}}T_F^\circ\backslash T_{\mathbb{A}}^\circ} \int_{N_F\backslash N_{\mathbb{A}}} \varphi(tng)\psi^{-1}(n)\nu^{-1}(t)dndt \right) f(g)dg$$
$$= \int\limits_{T_{\mathbb{A}}^\circ N_{\mathbb{A}}^\circ\backslash G_{\mathbb{A}}^\circ} \left(\int\limits_{Z_{\mathbb{A}}R_F\backslash R_{\mathbb{A}}} \varphi(rg)\alpha_{\nu,\psi}^{-1}(r)dr \right) f(g)dg$$
$$= \int\limits_{T_{\mathbb{A}}^\circ N_{\mathbb{A}}^\circ\backslash G_{\mathbb{A}}^\circ} B^\varphi(g)f(g)dg,$$

Thus now if we assume $f=\prod_v f_v$ then we get the desired Eulerian property,

$$Z(s,f,\varphi,\nu_{\mathbb{A}},\mu_{\mathbb{A}}) = \int\limits_{T_{\mathbb{A}}^\circ N_{\mathbb{A}}^\circ\backslash G_{\mathbb{A}}^\circ} B^\varphi(g)f(g)dg = \prod_v \int\limits_{T_{F_v}^\circ N_{F_v}^\circ\backslash G_{F_v}^\circ} B_v^\varphi(g)f_v(g)dg.$$

for Re(s)>> 0.

Definition 3.1.1. For a fixed place v of F, the local zeta integral is defined as

$$Z_v(s, f_v, B_v^{\varphi}) = \int\limits_{T^{\circ}_{F_v} N^{\circ}_{F_v} \backslash G^{\circ}_{F_v}} B_v^{\varphi}(g) f_v(g) dg.$$

3.2 Local Integral

For this section we fix a place v of F and by abuse of notation drop the subscript and simply write F for F_v. We similarly drop the subscript v for $\mathfrak{p}_{F_v}$, $\mathfrak{o}_{F_v}$, π_{F_v} and p_{F_v}. We do the same for the F_v-algebra E_v. Our usual notion resumes in the next section.

3.2.1 Preliminaries

Let $s \in \mathbb{C}$. For characters $\mu : F^{\times} \to \mathbb{C}$ and $\nu : E^{\times} \to \mathbb{C}$ define on B° the quasicharacter

$$\chi_s\left(\begin{pmatrix} xt & \\ & \bar{t} \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}\right) = \mu(x)\nu^{-1}(t)|x|_F^s, x \in F^{\times}, t \in E^{\times}, n \in E.$$

We consider the family of induced normalized representations

$$I(s, \nu, \mu) = \mathrm{Ind}_{B^{\circ}}^{G^{\circ}} \chi_s$$

and we will always take $f \in I(s, \nu, \mu)$ to be a flat section of this family relative to the standard compact $K^{\circ} = \{k \in GL_2(\mathfrak{o}_E) | \det(k) \in F^{\times}\}$ of G°.

We have the standard intertwining operator $M(s, \nu, \mu, \psi) : I(s, \nu, \mu) \to I(-s, \overline{\nu}, \nu_F^{-1}\mu^{-1})$ defined as

$$M(s, \nu, \mu, \psi) f(g) = \int\limits_E f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} g\right) dy_{\psi_E}$$

which we sometimes simply denote by M. Note that we are using the measure self dual to $\psi_E = \psi \circ tr_{E/F}$. Twisting by a is simple

$$M(s, \nu, \mu, \psi_a) = |a|_E^{1/2} M(s, \nu, \mu, \psi).$$

Lemma 3.2.1. *[BU, Proposition 4.5.10] Composing twice* $M(-s,\psi)\circ M(s,\psi):$ $I(s,\nu,\mu)\to I(s,\nu,\mu)$ *scales by*

$$\gamma(1-s,(\mu^{-1}\circ N_{E/F})\cdot\overline{\nu}^{-1},\psi^{-1})\cdot\gamma(1+s,(\nu_F\mu\circ N_{E/F})\cdot\nu^{-1},\psi)$$

Let (π,V) be an irreducible smooth representation of $GSp_4(F)$, $\mu:F^\times\to\mathbb{C}^\times$ a character and let π have Bessel model $\mathcal{B}(E,\nu,\psi)$. Let $B_\psi\in\mathcal{B}(E,\nu,\psi)$ and let f be a flat section of $\mathrm{Ind}_{B^\circ}^{G^\circ}\chi_s$. Then define

$$\begin{aligned}Z(s,B_\psi,f_s,\nu,\mu)&=\int\limits_{T^\circ N^\circ\backslash G^\circ}B_\psi(g)f_s(g)dg\\&=\int\limits_K\int\limits_{F^\times}B_\psi\left(\begin{pmatrix}x&0\\&1\end{pmatrix}k\right)f_s\left(\begin{pmatrix}x&0\\&1\end{pmatrix}k\right)\delta_{B^\circ}(x)^{-1}dxdk\end{aligned}$$

Proposition 3.2.2. *The local zeta integral*

$$Z(s,B,f_s)=\int_{N^0T^0\backslash G^0}B(gv)\cdot f_s(g)\,dg$$

converges absolutely when $Re(s)\gg 0$*. Moreover, it admits a meromorphic continuation to* $\mathbb{C}$*.*

The absolute convergence of $Z(s)$ when $Re(s)\gg 0$ follows immediately from the asymptotic behaviour of $B(tv)$ discussed in 1.3. Now we consider the question of meromorphic continuation of $Z(s)$. When F is nonarchimedean, $Z(s)$ can be expressed as an integral over $|t|>\epsilon$ (which converges for all s), and a finite linear combination of integrals of the form

$$\int_{|t|<\epsilon}\chi(t)\cdot|t|^s\cdot(\log|t|)^k\,dt$$

It is easy to see that such an integral is a rational function in q^{-s} and this provides the meromorphic continuation of $Z(s)$ to $\mathbb{C}$.

Suppose now that F is archimedean.

$$Z(s,B,f)=\int_T\int_K B(tkv)\cdot|t|^s\cdot f_s(k)\cdot\delta_{B^0}(t)^{-1}\,dk\,dk=\int_T B(t\cdot(f*v))\cdot|t|^s\cdot\delta_B(t)^{-1}\,dt$$

Again, we may split the integral into $\int_{|t|>\epsilon} + \int_{|t|<\epsilon}$. The former integral converges since $B(t \cdot f * v)$ behaves like a Schwarz function as $|t| \to \infty$. For the second integral, using the asymptotic expansion given in Lemma 2.5.5, we see that, for any $n \in \mathbb{N}$,

$$B(t \cdot (f * v)) = \left(\sum_{\chi} \chi(t) \cdot \sum_{k=0}^{n} t^k \cdot q_{\chi,k}(\log |t|, f * v) \right) + E_n(t)$$

where

$$|E_n(t)| \leqslant C_n(f * v) \cdot |t|^{A+n}$$

for some A independent of n. Now the integral

$$\int_{|t|<\epsilon} \chi(t) \cdot t^{k+s} \cdot \log |t|^r \, dt$$

is easily seen to have a meromorphic continuation to all of $\mathbb{C}$. On the other hand, the integral

$$\int_{F^\times} E_n(t) \cdot |t|^s \, dt$$

is convergent when $Re(s) > -A-n$. Thus, we see that $Z(s)$ admits a meromorphic continuation to $Re(s) > -A - n$. Since n is arbitrary, we deduce that $Z(s)$ has a meromorphic continuation to $\mathbb{C}$, as desired.

Proposition 3.2.3. *There is a meromorphic function* $\Gamma^{E,\nu}(s, \pi \times \mu, \psi)$ *such that*

$$Z(-s, \overline{B}_\psi, Mf_s, \overline{\nu}, \nu^{-1}\mu^{-1}) = \Gamma^{E,\nu}(s + \frac{1}{2}, \pi \times \mu, \psi) Z(s, B_\psi, f_s, \nu, \mu)$$

Proof. To show the functional equation for the local zeta integral, we need to consider the abstract Hom space

$$V_s = \mathrm{Hom}_H(\pi \otimes I(s, \mu, \nu), \mathbb{C}) = \mathrm{Hom}_H(\pi, I(-s, \mu^{-1}, \nu^{-1})$$

where $H = \mathrm{GL}_2(E)^0$.

Let us write

$$I(-s, \mu^{-1}, \nu^{-1}) = \mathrm{Ind}_{B^0}^{H} \chi_s.$$

By Frobenius reciprocity, we see that

$$V_s = \mathrm{Hom}_{T^0}(\pi_{N^0}, \chi_s).$$

On the other hand, we have a short exact sequence of T^0-modules

$$0 \longrightarrow \mathrm{ind}_{E^\times}^{T^0}(\pi_{N,\psi}) \longrightarrow \pi_{N^0} \longrightarrow \pi_N \longrightarrow 0$$

For generic s,

$$\mathrm{Hom}_{T^0}(\pi_N, \chi_s) = 0,$$

so that

$$V_s = \mathrm{Hom}_{T^0}(\mathrm{ind}_{E^\times}^{T^0}(\pi_{N,\psi}), \chi_s).$$

By Frobenius reciprocity, this is equal to

$$\mathrm{Hom}_{E^\times}(\pi_{N,\psi}, \chi_s)$$

which is the space of Bessel functionals of π with respect to $\mu^{-1}|-|^{-s}$. Thus, we have shown that, for generic s, V is isomorphic to a space of Bessel functionals on π and Theorem 2.1.1 says that this space is 1-dimensional. This then implies the local functional equaton for local zeta integrals, since both sides of the functional equation defines elements of V_s. $\square$

3.2.2 Unramified Calculation

Let (π, V_π) be an unramified smooth representation of $GSp_4(F)$ and μ an unramified quasicharacter of $F^\times$. Let π have Bessel model $\mathcal{B}(E, \nu, \psi)$. Since π is spherical, ν is unramified. Note that when $E = F \oplus F$ characters on E have the form $\chi(x, y) = (\chi_1 \oplus \chi_2)(x, y) = \chi_1(x)\chi_2(y)$ where χ_1, χ_2 are characters of F.

Since the data: ν and μ are unramified, $I(s, \nu, \mu)$ is unramified. Let $K^\circ = \{k \in GL_2(\mathfrak{o}_E) \mid \det(k) \in F^\times\}$ and let ϕ_{K° and ϕ'_{K° be the normalized K°-fixed vectors for $I(s, \nu, \mu)$ and $I(-s, \overline{\nu}, \nu^{-1}\mu^{-1})$ respectively, i.e. $\phi_{K^\circ}(bk) = (\delta_{B^\circ}^{1/2}\chi_s)(b)$, where δ_{B° is the modular character for the Borel subgroup in G°.

Lemma 3.2.4. *[BU] With ϕ_{K° defined above and our Haar measure chosen to give volume 1 on $\mathfrak{o}_E$ we have*

$$M(s, \psi)\phi_{K^\circ} = \frac{L_E(s, (\mu \circ N_{E/F}) \cdot \overline{\nu})}{L_E(s+1, (\mu \circ N_{E/F}) \cdot \overline{\nu})}\phi'_{K^\circ}$$

where L_E *is the standard abelian L-factor over* E*, i.e.* $L_E(s,\chi) = (1-\chi(\pi_E)q_E^{-s})^{-1}$ *when* E *is a field and* $L_E(s,\chi) = L_F(s,\chi_1 \oplus \chi_2) = L_F(s,\chi_1)L_F(s,\chi_2)$ *when* E *is split.*

We now evaluate our zeta integral

$$Z(s, B_\psi, f_s, \nu, \mu) = \int\limits_{TN\backslash G^\circ} B_\psi(g) f_s(g) dg,$$

with unramified data and use the funtional equation

$$Z(-s, \overline{B}_\psi, Mf_s, \overline{\nu}, \nu^{-1}\mu^{-1}) = \Gamma^{E,\nu}(s, \pi \times \mu, \psi) Z(s, B_\psi, f_s, \nu, \mu)$$

to compute the constant of proportionality.

$$\begin{aligned}
&\int\limits_{TN\backslash G^\circ} B_\psi(g)\phi_s^{K^\circ}(g)dg \\
&= \int\limits_K \int\limits_{F^\times} B_\psi\left(\begin{pmatrix} x & 0 \\ & 1 \end{pmatrix} k\right) \phi_s^{K^\circ}\left(\begin{pmatrix} x & 0 \\ & 1 \end{pmatrix} k\right) \delta_B(x)^{-1} dx dk \\
&= \int\limits_{F^\times} B_\psi\left(\begin{pmatrix} x & 0 \\ & 1 \end{pmatrix}\right) \phi_s^{K^\circ}\left(\begin{pmatrix} x & 0 \\ & 1 \end{pmatrix}\right) |x|_F^{-2} dx \\
&= \int\limits_{F^\times} B_\psi\left(\begin{pmatrix} x & 0 \\ & 1 \end{pmatrix}\right) \mu(x)|x|_F^{1+s}|x|_F^{-2} dx \\
&= \sum_{n\geqslant 0} B_\psi\left(\begin{pmatrix} \pi^n & 0 \\ & 1 \end{pmatrix}\right) \mu(\pi)^n q^{-n(s-1)} \\
&= \sum_{n\geqslant 0} B_\psi\left(\begin{pmatrix} \pi^n & 0 \\ & 1 \end{pmatrix}\right) y_1^n,\ y_1 = \mu(\pi)q^{1-s}
\end{aligned}$$

Thus we have,

$$Z(s, B_\psi, \phi_s^{K^\circ}, \nu, \mu) = \sum_{n\geqslant 0} B_\psi\left(\begin{pmatrix} \pi^n & 0 \\ & 1 \end{pmatrix}\right) y_1^n,\ y_1 = \mu(\pi)q^{1-s}.$$

A simular calculation for the other side gives

$$Z(-s,\overline{B}_\psi,M\phi_s^{K^\circ},\overline{\nu},\nu^{-1}\mu^{-1}) =$$
$$\frac{L_E(s,(\mu\circ N_{E/F})\cdot\overline{\nu})}{L_E(s+1,(\mu\circ N_{E/F})\cdot\overline{\nu})}\sum_{n\geqslant 0}\overline{B}\left(\begin{pmatrix}\pi^n & 0\\ & 1\end{pmatrix}\right)y_2^n,\ y_2=(\nu^{-1}\mu^{-1})(\pi)q^{s+1}$$

In order to evaluate the remaining summation, we make use of a formula due to Sugano[SU] which we restate here. Let

$$M_0=\left\{\begin{pmatrix}x_1 & & & \\ & x_2 & & \\ & & x_3 & \\ & & & x_4\end{pmatrix}\Big| x_1x_3=x_2x_4\right\},$$

the Levi factor for the Borel subgroup of H_2. Since π_v is spherical, there exists a unramified character ρ_v of $M_0(F_v)$ such that $\pi_v = Ind_{M_0(F_v)}^{H_2(F_v)}$. Define characters $\rho_v^{(i)}(i=1,2,3,4)$ on F_v as

$$\rho_v^{(1)}(x)=\rho_v\begin{pmatrix}x & & & \\ & x & & \\ & & 1 & \\ & & & 1\end{pmatrix},\ \rho_v^{(2)}(x)=\rho_v\begin{pmatrix}x & & & \\ & 1 & & \\ & & 1 & \\ & & & x\end{pmatrix}$$

$$\rho_v^{(3)}(x)=\rho_v\begin{pmatrix}1 & & & \\ & 1 & & \\ & & x & \\ & & & x\end{pmatrix},\ \rho_v^{(4)}(x)=\rho_v\begin{pmatrix}1 & & & \\ & x & & \\ & & x & \\ & & & x\end{pmatrix}.$$

It is clear that

$$\rho_v^{(1)}\rho_v^{(3)}=\rho_v^{(2)}\rho_v^{(4)}=\omega_\pi.$$

For $v\notin S$, set

$$\epsilon_v=\begin{cases}0, & \text{if } \varepsilon_{E/F}=1,\\ \nu(\pi_E) & \text{if } \varepsilon_{E/F}=1,\ \cdot\\ \nu(\pi_E)+\nu_v(\pi_F\cdot(\pi_E)^{-1}) & \text{if } \varepsilon_{E/F}=1.\end{cases}$$

Let

$$h_v(l,m) = \begin{pmatrix} \pi_F^{m+l}\begin{pmatrix}\pi_F^m & 0\\ 0 & 1\end{pmatrix} & \\ & \begin{pmatrix}1 & 0\\ 0 & \pi_F^m\end{pmatrix}\end{pmatrix} \in H_2(F_v)$$

and define

$$C_v(x,y) = \sum_{l\geqslant 0}\sum_{m\geqslant 0} B_v(h_v(l,m))x^m y^l$$

Theorem 3.2.5. *[SU] For $v \notin S$,*

$$C_v(x,y) = \frac{H_v(x,y)}{P_v(x)Q_v(y)},$$

where

$$\begin{aligned} P_v(x) =&(1-\rho_v^{(1)}\rho_v^{(2)}(\pi_F)q_v^{-2}x)(1-\rho_v^{(1)}\rho_v^{(4)}(\pi_F)q_v^{-2}x)\\ &\cdot(1-\rho_v^{(2)}\rho_v^{(3)}(\pi_F)q_v^{-2}x)(1-\rho_v^{(3)}\rho_v^{(4)}(\pi_F)q_v^{-2}x), \end{aligned}$$

$$Q_v(y) = \prod_{i=1}^{4}(1-\rho_v^{(i)}(\pi_F)q_v^{-3/2}y),$$

$$\begin{aligned} H_v(x,y) =&(1+A_2A_3xy^2)\{M_1(x)(1+A_2x)+A_2A_5A_1^{-1}\alpha x^2\}\\ &-A_2xy\{\alpha M_1(x)-A_5M_2(x)\}-A_5P_v(x)y-A_2A_4P_v(x)y^2, \end{aligned}$$

$$M_1(x) = 1-A_1^{-1}(A_1+A_4)^{-1}(A_1A_5\alpha+A_4\beta-A_1A_5^2-2A_1A_2A_4)x+A_1^{-1}A_2^2A_4x^2$$

$$M_2(x) = 1+A_1^{-1}(A_1A_2-\beta)x+A_1^{-1}A_2(A_1A_2-\beta)x^2+A_2^3x^3,$$

$$\alpha = q_v^{3/2}\sum_{i=1}^{4}\rho_v^{(i)}(\pi_F),\ \beta = q_v^{-3}\sum_{1\leqslant i<j\leqslant 4}\rho_v^{(i)}\rho_v^{(j)},$$

$$A_1 = q_v^{-1},\ A_2 = q_v^{-2}\nu_v(\pi_F),\ A_3 = q_v^{-3}\nu_v(\pi_F),$$

$$A_4 = -q_v^{-2}\varepsilon_{E_v/F_v},\ A_5 = q_v^{-2}\epsilon_v.$$

We now use the following special case of Sugano's formula to evaluate the sums.

$$\sum_{n\geqslant 0} B\left(\begin{pmatrix}\pi^n & 0\\ & 1\end{pmatrix}\right)y^n = \frac{H(y)}{\prod_{i=1}^{4}(1-\gamma^{(i)}(\pi)q^{-\frac{3}{2}}y)}$$

where $H(y)$ is a polynomial in y described above. Our Zeta integrals becomes

$$Z(s, B_\psi, \phi_s^{K^\circ}, \nu, \mu) = H(y_1) \cdot L(\frac{1}{2} + s, \pi \times \mu)$$
$$Z(-s, \overline{B}_\psi, M\phi_s^{K^\circ}, \overline{\nu}, \nu^{-1}\mu^{-1}) = \frac{L_E(s, (\mu \circ N_{E/F}) \cdot \overline{\nu})}{L_E(s+1, (\mu \circ N_{E/F}) \cdot \overline{\nu})} H(y_2) \cdot L(\frac{1}{2} - s, \tilde{\pi} \times \mu^{-1})$$

Were $H(y)$ depends on E/F and is described above. Combining this with our functional equation gives

$$\Gamma^{E,\nu}(s, \pi \times \mu, \psi) = \frac{L_E(s, (\mu \circ N_{E/F}) \cdot \overline{\nu})}{L_E(s+1, (\mu \circ N_{E/F}) \cdot \overline{\nu})} \frac{H(y_2)}{H(y_1)} \frac{L(\frac{1}{2} - s, \tilde{\pi} \times \mu^{-1})}{L(\frac{1}{2} + s, \pi \times \mu)}$$

Next we compute $H(y_1)$ and $H(y_2)$ in each case. What we show is that $H(y_1)$ is the inverse of $L_E(s+1, (\mu \circ N_{E/F}) \cdot \overline{\nu})$ and $H(y_2)$ is at least consistently described in all cases.

Unramified Case

In this case $H(y) = 1 - q_F^{-4}\nu(\pi_F)y^2$ and we note that $q_E = q_F^2$. Thus we have

$$\begin{aligned} H(y_1) &= 1 - \nu(\pi_F)\mu^2(\pi_F)q_F^{-2(1+s)} \\ &= 1 - \overline{\nu}(\pi_E)(\mu \circ N_{E/F})(\pi_E)q_E^{-(1+s)} \\ &= L_E(s+1, (\mu \circ N_{E/F}) \cdot \overline{\nu})^{-1} \end{aligned}$$

and

$$\begin{aligned} H(y_2) &= 1 - \nu^{-1}(\pi)\mu^{-2}(\pi)q^{-2(1-s)} \\ &= 1 - \overline{\nu}^{-1}(\pi_E)(\mu^{-1} \circ N_{E/F})(\pi_E)q^{-(1-s)} \\ &= L_E(1-s, (\mu^{-1} \circ N_{E/F}) \cdot \overline{\nu}^{-1})^{-1} \end{aligned}$$

Ramified Case

Here $H(y) = 1 - q_F^{-2}\nu(\pi_E)y$ and since the residue fields are isomorphic we know $q_E = q_F$.

$$\begin{aligned} H(y_1) &= 1 - \nu(\pi_E)\mu(\pi_F)q_F^{-(1+s)} \\ &= 1 - \nu(\pi_E)(\mu \circ N_{E/F})(\pi_E)q_E^{-(1+s)} \\ &= 1 - \overline{\nu}(\pi_E)(\mu \circ N_{E/F})(\pi_E)q_E^{-(1+s)} \\ &= L_E(s+1, (\mu \circ N_{E/F}) \cdot \overline{\nu})^{-1} \end{aligned}$$

where the last simplification follows form the fact that conjugates have the same valuation and ν is unramified, i.e. $\nu|_{\mathfrak{o}_E} = 1$. Hence

$$\overline{\nu}(\pi_E) = \nu(\overline{\pi}_E) = \nu(\pi_E \cdot u) = \nu(\pi_E)$$

where $u \in \mathfrak{o}_E^\times$. Simularly we have

$$\begin{aligned} H(y_2) &= 1 - \overline{\nu}(\pi_E)\nu^{-1}(\pi_F)\mu^{-1}(\pi_F)q_F^{-(1-s)} \\ &= 1 - \overline{\nu}(\pi_E)\nu^{-1}(\pi_F)(\mu^{-1} \circ N_{E/F})(\pi_E)q_E^{-(1-s)} \\ &= 1 - \overline{\nu}(\pi_E)\nu^{-1}(\pi_E\overline{\pi}_E)(\mu^{-1} \circ N_{E/F})(\pi_E)q_E^{-(1-s)} \\ &= 1 - \nu^{-1}(\pi_E)(\mu^{-1} \circ N_{E/F})(\pi_E)q_E^{-(1-s)} \\ &= L_E(1-s, (\mu^{-1} \circ N_{E/F}) \cdot \overline{\nu}^{-1})^{-1} \end{aligned}$$

Split Case

In this final case $H(y) = (1 - q^{-2}\nu(\pi_E)y)(1 - q^{-2}\nu(\pi\pi_E^{-1})y)$

$$\begin{aligned} H(y_1) &= \left[1 - \nu(\pi_E)\mu(\pi_F)q_F^{-(1+s)}\right]\left[1 - \nu(\pi_F\pi_E^{-1})\mu(\pi_F)q_F^{-(1+s)}\right] \\ &= \left[1 - \nu_1(\pi_F)\mu(\pi_F)q_F^{-(1+s)}\right]\left[1 - \nu_2(\pi_F)\mu(\pi_F)q_F^{-(1+s)}\right] \\ &= L_F(1+s, \nu_1 \cdot \mu)^{-1} L_F(1+s, \nu_2 \cdot \mu)^{-1} \\ &= L_F(1+s, (\nu_1 \oplus \nu_2) \cdot \mu)^{-1} \\ &= L_E(1+s, (\mu \circ N_{E/F}) \cdot \overline{\nu})^{-1} \end{aligned}$$

and

$$\begin{aligned}
H(y_2) &= \left[1 - \overline{\nu}(\pi_E)\nu^{-1}(\pi_F)\mu^{-1}(\pi_F)q_F^{-(1-s)}\right]\left[1 - \overline{\nu}(\pi_F\pi_E^{-1})\nu^{-1}(\pi_F)\mu^{-1}(\pi_F)q_F^{-(1-s)}\right] \\
&= \left[1 - \nu_2(\pi_F)\nu^{-1}(\pi_F)\mu^{-1}(\pi_F)q_F^{-(1-s)}\right]\left[1 - \nu_1(\pi_F)\nu^{-1}(\pi_F)\mu^{-1}(\pi_F)q_F^{-(1-s)}\right] \\
&= \left[1 - \nu_1^{-1}(\pi_F)\mu^{-1}(\pi_F)q_F^{-(1-s)}\right]\left[1 - \nu_2^{-1}(\pi_F)\mu^{-1}(\pi_F)q_F^{-(1-s)}\right] \\
&= L_F(1-s, \nu_1^{-1}\cdot\mu^{-1})^{-1}L_F(1-s, \nu_2^{-1}\cdot\mu^{-1})^{-1} \\
&= L_E(1-s, (\mu^{-1}\circ N_{E/F})\cdot\overline{\nu}^{-1})^{-1}
\end{aligned}$$

We end this section with a theorem summarizing the results of this calculation.

Theorem 3.2.6.

$$Z(s, B_\psi, \phi_s^{K^\circ}, \nu, \mu) = L_E(s+1, (\mu\circ N_{E/F})\cdot\overline{\nu})^{-1}\cdot L(\frac{1}{2}+s, \pi\times\mu)$$

and

$$Z(-s, \overline{B}_\psi, M(s,\psi)\phi_s^{K^\circ}, \overline{\nu}, \nu^{-1}\mu^{-1}) =$$
$$L_E(s+1, (\mu\circ N_{E/F})\cdot\overline{\nu})^{-1}\cdot\gamma(s, (\mu\circ N_{E/F})\cdot\overline{\nu}, \psi)^{-1}\cdot L(\frac{1}{2}-s, \tilde{\pi}\times\mu^{-1})$$

3.3 γ-factor

Definition 3.3.1.

$$\gamma^{E,\nu}(s+1/2, \pi\times\mu, \psi) = \Gamma^{E,\nu}(s, \pi\times\mu, \psi)\cdot\gamma(s, (\mu\circ N_{E/F})\cdot\overline{\nu}) \tag{3.2}$$

3.3.1 Main Theorem

Theorem

(1) Unramified factors:

$$\gamma^{E,\nu}(s+1/2, \pi\times\mu, \psi) = \frac{L(\frac{1}{2}-s, \tilde{\pi}\times\mu^{-1})}{L(\frac{1}{2}+s, \pi\times\mu)}$$

(2) Dependence on ψ: Let $a\in F^\times$ and set $\psi_a(x) = \psi(ax)$, then

$$\gamma^{E,\nu}(s+1/2, \pi\times\mu, \psi_a) = \mu^4(a)\omega_\pi^2(a)|a|_F^{4s}\gamma^{E,\nu}(s+1/2, \pi\times\mu, \psi)$$

(3) Unramified Twisting: Let $s_0 \in \mathbb{C}$ be fixed then,

$$\gamma^{E,\nu}(s+s_0, \pi \times \mu, \psi) = \gamma^{E,\nu}(s, \pi \times \mu_{s_0}, \psi)$$

(4) Functional equation:

$$\gamma^{E,\nu}(s, \pi \times \mu, \psi)\gamma^{E,\nu}(1-s, \tilde{\pi} \times \mu^{-1}, \psi^{-1}) = 1$$

(5) Global property: Let π be a cuspidal representation of $G(\mathbb{A})$, μ a Hecke character of $F^\times \backslash \mathbb{I}_F$, and $\psi_F = \otimes_v \psi_{F_v}$ a non-trivial character of $F\backslash \mathbb{A}_F$. Let S be a finite set of places containing all the archimedean ones and the places where either π, μ or ψ is ramified. Then,

$$L^S(s, \pi \times, \mu) = \prod_{v \in S} \gamma_v^{E,\nu}(s, \pi_v \times \mu_v, \psi_v) \cdot L^S(1-s, \tilde{\pi} \times \mu^{-1}).$$

Proof. (1) follows from immediately from Theorem 3.2.6 and Definition 3.2.
For (2) let $a \in F^\times$ and set $\psi_a(x) = \psi(ax)$, then

$$\gamma^{E,\nu}(s+1/2, \pi \times \mu, \psi_a) = \mu^4(a)\omega_\pi^2(a)|a|_F^{4s}\gamma^{E,\nu}(s+1/2, \pi \times \mu, \psi)$$

To see this result we will demonstrate the effect of twisting ψ by a on the local zeta integrals

$$Z(s, B_\psi, f, \nu, \mu) = \int_{TN\backslash G^\circ} B_\psi(g) f_s(g) dg$$

then take the ratio.

Since the only part of the integral that depends on ψ is the Bessel function B_ψ, we must see how B_{ψ_a} should be defined. That is, it should be defined in terms of B_ψ. We let

$$B_{\psi_a}(g) = B_\psi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right)$$

then $B_{\psi_a} \in \mathcal{B}(E, \nu, \psi_a)$.

Now we consider the zeta integrals on both sides of the functional equation but with B_ψ replaced with B_{ψ_a}. The idea being that since B_{ψ_a} is defined in terms of

B_ψ we will be able to reduce back to the original zeta after pulling out some factors.

Recall our local functional equation

$$Z(-s, \overline{B}_\psi, M * f, \overline{\nu}, \nu^{-1}\mu^{-1}) = \Gamma^{E,\nu}(s, \pi \times \mu, \psi) Z(s, B_\psi, f, \nu, \mu)$$

To simplify notation we will supress the domain of integration $TN\backslash G^\circ$.

$$\begin{aligned}
Z(s, B_{\psi_a}, f, \nu, \mu) &= \int B_{\psi_a}(g) f_s(g) dg \\
&= \int B_\psi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right) f_s(g) dg \\
&= \int B_\psi(g) f_s\left(\begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix} g\right) d(a^{-1}g) \\
&= \int B_\psi(g)\mu(a^{-1})|a^{-1}|_F^{s+1} f_s(g)|a^{-1}|_F^{-2} dg \\
&= \mu^{-1}(a)|a|_F^{1-s} \cdot Z(s, B_\psi, f, \nu, \mu)
\end{aligned}$$

Now consider,

$$\begin{aligned}
& Z(-s, \overline{B}_{\psi_a}, M(s,\psi_a) f, \overline{\nu}, \nu^{-1}\mu^{-1}) \cdot \gamma(s, (\mu \circ N_{E/F}) \cdot \overline{\nu}, \psi_a) \\
&= \gamma(s, (\mu \circ N_{E/F}) \cdot \overline{\nu}, \psi_a)|a|_E^{1/2} \cdot \int \overline{B}_\psi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} g\right) M(s,\psi) f_s(g) dg \\
&= [(\mu \circ N_{E/F})(a)\overline{\nu}(a)|a|_E^s] \int \overline{B}_\psi(g)(\nu_F^{-1}\mu^{-1})(a^{-1})|a^{-1}|_F^{-s+1} M(s,\psi) f_s(g)|a^{-1}|_F^{-2} dg \\
&= \mu^2(a) \cdot \nu_F(a)|a|_F^{2s} \cdot \nu_F(a) \cdot \mu(a)|a|_F^{s-1} \cdot |a|_F^2 \cdot Z(-s, \overline{B}_\psi, M(s,\psi) f, \overline{\nu}, \nu^{-1}\mu^{-1}) \\
&= \mu^3(a) \cdot \omega_\pi^2(a) \cdot |a|_F^{3s+1} \cdot Z(-s, \overline{B}_\psi, M(s,\psi) f, \overline{\nu}, \nu^{-1}\mu^{-1})
\end{aligned}$$

Now we take the ratio to see the effect of the a twist on our $\gamma^{E,\nu}$-factor.

$$\begin{aligned}
\gamma^{E,\nu}(s+1/2, \pi \times \mu, \psi_a) &= \frac{Z(-s, \overline{B}_{\psi_a}, M^* f, \overline{\nu}, \nu^{-1}\mu^{-1}) \cdot \gamma(s, (\mu \circ N_{E/F}) \cdot \overline{\nu}, \psi_a)}{Z(s, B_{\psi_a}, f, \nu, \mu)} \\
&= \frac{\mu^3(a) \cdot \omega_\pi^2(a) \cdot |a|_F^{3s+1}}{\mu^{-1}(a)|a|_F^{1-s}} \cdot \gamma^{E,\nu}(s+1/2, \pi \times \mu, \psi) \\
&= \mu^4(a)\omega_\pi^2(a)|a|_F^{4s} \cdot \gamma^{E,\nu}(s+1/2, \pi \times \mu, \psi)
\end{aligned}$$

(3) Let $s_0 \in \mathbb{C}$ be fixed. We must show

$$\gamma^{E,\nu}(s+s_0, \pi\times\mu, \psi) = \gamma^{E,\nu}(s, \pi\times\mu_{s_0}, \psi).$$

We proceed by evaluating the effect of this shifting on both sides of the local functional equation, then take the ratio of the two sides.

$$\begin{aligned}
Z(s+s_0, B_\psi, f, \nu, \mu) &= \int_{TN\backslash G^\circ} B(g) f_s(g) dg \\
&= \int_K \int_{F^\times} B\left(\begin{pmatrix} x & \\ & 1\end{pmatrix} k\right) f_{s+s_0}\left(\begin{pmatrix} x & \\ & 1\end{pmatrix} k\right) |x|_F^{-2} dx dk \\
&= \int_K \int_{F^\times} B\left(\begin{pmatrix} x & \\ & 1\end{pmatrix} k\right) \mu(x)|x|_F^{s+s_0+1} f_s\left(k\right) |x|_F^{-2} dx dk \\
&= \int_K \int_{F^\times} B\left(\begin{pmatrix} x & \\ & 1\end{pmatrix} k\right) \mu_{s_0}(x)|x|_F^{s+1} f_s\left(k\right) |x|_F^{-2} dx dk \\
&= Z(s, B_\psi, f, \nu, \mu_{s_0})
\end{aligned}$$

Note that $I(s+s_0, \mu, \nu) = I(s, \mu_{s_0}, \nu)$ and $M(s+s_0, \mu, \nu, \psi) = M(s, \mu_{s_0}, \nu, \psi)$. A similar calculation for the other side of the functional equation gives

$$\begin{aligned}
&Z(-s-s_0, \overline{B}_\psi, M^*(s+s_0, \mu, \nu, \psi) f, \overline{\nu}, \nu^{-1}\mu^{-1}) \\
&= Z(-s, \overline{B}_\psi, M^*(s+s_0, \mu, \nu, \psi) f, \overline{\nu}, (\nu^{-1}\mu^{-1})_{s_0}).
\end{aligned}$$

Finally taking ratios we get (3).

(4)

We know previously that

$$M(-s,\psi)\circ M(s,\psi) = \gamma(1-s, (\mu^{-1}\circ N_{E/F})\cdot\overline{\nu}^{-1}, \psi^{-1})\gamma(1+s, (\nu_F\mu\circ N_{E/F})\cdot\nu^{-1}, \psi^{-1})$$

Now we have our functional equation

$$Z(-s, \overline{B}_\psi, M(s,\psi)f, \overline{\nu}, \nu_F^{-1}\mu^{-1}) = \Gamma^{E,\nu}(s, \pi\times\mu, \psi) Z(s, B_\psi, f, \nu, \mu)$$

which we may apply again to get

$$\begin{aligned}
&Z(s, B_\psi, M(-s,\psi)\circ M(s,\psi) f, \nu, \nu_F^{-1}(\nu_F\mu)) = \\
&\Gamma^{E,\nu}(-s, \pi\times\nu_F^{-1}\mu^{-1}, \psi)\Gamma^{E,\nu}(s, \pi\times\mu, \psi) Z(s, B_\psi, f, \nu, \mu)
\end{aligned}$$

Combining and simplifying we get

$$\begin{aligned}&\gamma(1-s,(\mu^{-1}\circ N_{E/F})\cdot\overline{\nu}^{-1},\psi^{-1})\gamma(1+s,(\nu_F\mu\circ N_{E/F})\cdot\nu^{-1},\psi^{-1})\\&=\Gamma^{E,\nu}(-s,\tilde{\pi}\times\mu^{-1},\psi)\Gamma^{E,\nu}(s,\pi\times\mu,\psi)\end{aligned}$$

By the definition of $\gamma^{E,\nu}$ 3.2 , we get

$$\gamma^{E,\nu}(s,\pi\times\mu,\psi)\gamma^{E,\nu}(1-s,\tilde{\pi}\times\mu^{-1},\psi)=1$$

By the 'dependence on ψ' result we may freely replace $\psi(x)$ with

$$\psi^{-1}(x)=\psi(-1\cdot x)=\mu(-1)^4\omega(-1)^2|-1|^{4s}\psi(x)=\psi(x).$$

Thus,

$$\gamma^{E,\nu}(s,\pi\times\mu,\psi)\gamma^{E,\nu}(1-s,\tilde{\pi}\times\mu^{-1},\psi^{-1})=1$$

(5) Let π be a cuspidal representation of $G(\mathbb{A})$, μ a Hecke character of $F^\times\backslash\mathbb{I}_F$, and $\psi_F=\otimes_v\psi_{F_v}$ a non-trivial character of $F\backslash\mathbb{A}_F$. Let S be a finite set of places containing all the archimedean ones and the places where either π,μ or ψ is ramified. Then we would like to show:

$$L^S(s,\pi\times,\mu)=\prod_{v\in S}\gamma_v^{E,\nu}(s,\pi_v\times\mu_v,\psi_v)\cdot L^S(1-s,\tilde{\pi}\times\mu^{-1}).$$

By 2.4.1 we know π has a Bessel model with respect to some (E,ν) and by 3.1.1 we may write

$$Z(s,f,\phi,\nu,\mu)=Z(s,B,f,\nu,\mu)$$

Under the statement assumptions, for Re$(s)>>0$ the global zeta integral has an Euler product,

$$\begin{aligned}Z(s,B,f,\nu,\mu)&=\prod_v Z_v(s,B_v,f_v,\nu_v,\mu_v)\\&=\prod_{v\notin S}Z_v(s,B_v,f_v,\nu_v,\mu_v)\cdot\prod_{v\in S}Z_v(s,B_v,f_v,\nu_v,\mu_v).\end{aligned}$$

For $v\notin S$ by Theorem 3.2.6,

$$Z_v(s,B_v,f_v,\nu_v,\mu_v)=\frac{L_v(1/2+s,\pi_v\times\mu_v)}{L_{E,v}(s+1,(\mu_v\circ N_{E_v/F_v})\cdot\overline{\nu}_v)}.$$

Therefore the following identity

$$Z(s,B,f,\nu,\mu) = \frac{L^S(1/2+s,\pi\times\mu)}{L^S_E(s+1,(\mu\circ N_{E/F})\cdot\overline{\nu})}\cdot\prod_{v\in S} Z_v(s,B_v,f_v,\nu_v,\mu_v) \tag{3.3}$$

holds for all s and hence shows that the partial L-function L^S is meromorphic.

We will need the following identity about the global intertwining operator. For $\mathrm{Re}(s) >> 0$ the intertwining operator has an Euler product,

$$\begin{aligned} M(s,\nu,\mu,\psi)f^{(s)} &= \prod_v M_v(s,\nu_v,\mu_v,\psi_v)f_v^{(s)} \\ &= \prod_{v\notin S} M_v(s,\nu_v,\mu_v,\psi_v)f_v^{(s)}\cdot\prod_{v\in S} M_v(s,\nu_v,\mu_v,\psi_v)f_v^{(s)}. \end{aligned}$$

By Lemma 3.2.4, for $v\notin S$,

$$M_v(s,\nu_v,\mu_v,\psi_v)f_v^{(s)} = \frac{L_{E,v}(s,(\mu_v\circ N_{E_v/F_v})\cdot\overline{\nu}_v)}{L_{E,v}(s+1,(\mu_v\circ N_{E_v/F_v})\cdot\overline{\nu}_v)} f_v^{(-s)}$$

We get the following identity which holds for all s,

$$M(s,\nu,\mu,\psi)f^{(s)} = \frac{L^S_E(s,(\mu\circ N_{E/F})\cdot\overline{\nu})}{L^S_E(s+1,(\mu\circ N_{E/F})\cdot\overline{\nu})}\prod_{v\notin S} f_v^{(-s)}\prod_{v\in S} M_v(s,\nu_v,\mu_v,\psi_v)f_v^{(s)}. \tag{3.4}$$

We now use 3.3 with $-s, M(s,\nu,\mu,\psi)f^{(s)},\overline{\nu}$ and $\omega_\pi^{-1}\mu^{-1}$ while applying 3.4

$$\begin{aligned} &Z(-s,M(s,\nu,\mu,\psi)f^{(s)},\overline{B},\overline{\nu},\omega_\pi^{-1}\mu^{-1}) = \\ &\frac{L^S(1/2-s,\pi\times\omega_\pi^{-1}\mu^{-1})}{L^S_E(1-s,(\omega_\pi^{-1}\mu^{-1}\circ N_{E/F})\cdot\nu)}\cdot\frac{L^S_E(s,(\mu\circ N_{E/F})\cdot\overline{\nu})}{L^S_E(s+1,(\mu\circ N_{E/F})\cdot\overline{\nu})} \\ &\cdot\prod_{v\in S} Z_v(-s,\overline{B}_v,M_v(s,\nu_v,\mu_v,\psi_v)f_v^{(s)},\overline{\nu}_v,\omega_{\pi_v}^{-1}\mu_v^{-1}). \end{aligned}$$

Now we make use of the global functional equation 3.1, the definition on γ_v, the fact that $\tilde{\pi} = \pi\otimes\omega_\pi^{-1}$ and the analogous statement for Hecke γ-factors to get

$$L^S(1/2+s,\pi\times\mu) = \left\{\prod_{v\in S}\gamma_v^{E,\nu}(s,\pi_v\times\mu_v,\psi_v)\right\}\cdot L^S(1-s,\tilde{\pi}\times\mu^{-1}).$$

$\square$

3.4 Multiplicativity

In this section, we describe the property of "multipictivity" for the local γ-factor $\gamma^{E,\nu}(s,\pi\times\mu,\psi)$ defined using the local zeta integral. For more details, see [Sh]. We then deduce some consequence of this property.

3.4.1 Multiplicativity.

Suppose that π is an irreducible subquotient of an induced representation of GSp_4. Since there are 3 conjugacy classes of parabolic subgroups of GSp_4, we have the following 3 cases:

- if B is the Borel subgroup of GSp_4, then suppose that π is a subquotient of

  $$I_B(\chi_1,\chi_2;\chi) = \mathrm{Ind}_B^{GSp_4}\chi_1\otimes\chi_2\otimes\chi.$$

  In this case, multiplicativity is the identity

  $$\gamma^{E,\nu}(s,\pi\times\mu,\psi) = \gamma(s,\chi\mu,\psi)\cdot\gamma(s,\chi_1\chi\mu,\psi)\cdot\gamma(s,\chi_2,\chi\mu,\psi)\cdot\gamma(s,\chi_1\chi_2\chi\mu,\psi). \tag{3.5}$$

- suppose that P is the Siegel parabolic subgroup of GSp_4, so that its Levi factor is $M\cong \mathrm{GL}_2\times\mathrm{GL}_1$, and π is a subquotient of

  $$I_P(\tau,\chi) = \mathrm{Ind}_P^{GSp_4}\tau\boxtimes\chi,$$

  where $\tau\boxtimes\chi$ is an irreducible representation of M. In this case, multiplicativity is the identity

  $$\gamma^{E,\nu}(s,\pi\times\mu,\psi) = \gamma(s,\chi\mu,\psi)\cdot\gamma(s,\tau\times\chi\mu,\psi)\cdot\gamma(s,\chi\omega_\tau\mu,\psi). \tag{3.6}$$

- if Q is the Heisenberg parabolic subgroup of GSp_4, so that its Levi factor is $L\cong\mathrm{GL}_1\times GSp_2\cong\mathrm{GL}_1\times\mathrm{GL}_2$, and π is a subquotient of

  $$I_Q(\chi,\sigma) = \mathrm{Ind}_Q^{GSp_4}\chi\boxtimes\sigma,$$

  In this case, multipicativity is the identity

  $$\gamma^{E,\nu}(s,\pi\times\mu,\psi) = \gamma(s,\chi\mu,\psi)\cdot\gamma(s,\sigma\times\mu,\psi)\cdot\gamma(s,\chi^{-1}\omega_\sigma\cdot\mu,\psi). \tag{3.7}$$

We would like to prove the identities (3.5), (3.6) and (3.7), but we are not able to do so.

3.4.2 Consequences.

In the following, we deduce some consequences of multipicativity. The first obvious consequence is:

Proposition 3.4.1. *Assuming multiplicativity, we have:*

(i) If π is a non-supercuspidal representation of a p-adic field, then the local γ-factor $\gamma^{E,\nu}(s, \pi \times \mu, \psi)$ is independent of the choice of the data (E, ν) with respect to which π has a (E, ν)-Bessel model.

(ii) If $F = \mathbb{R}$ or $\mathbb{C}$, and π has L-parameter ϕ_π, then $\gamma^{E,\nu}(s, \pi \times \mu, \psi)$ is independent of (E, ν) and

$$\gamma^{E,\nu}(s, \pi \times \mu, \psi) = \gamma(s, \phi_\pi \otimes \mu, \psi).$$

3.5 Independence of (E, ν)

In the previous section, we have seen that the local γ-factor is independent of the choice of the data (E, ν) with respect to which π has a Bessel model, when π is non-supercuspidal. In this section, we address this issue of independence when π is supercuspidal.

Hence, suppose that π is a supercuspidal representation which supports nonzero Bessel functionals with respect to (E, ν). From [GT], we know that there are two types of supercuspidal representations:

(a) π is the local theta lift of a supercuspidal representation $\tau_1 \boxtimes \tau_2$ of

$$\mathrm{GSO}_4 \cong (B^\times \times B^\times)/\{(t, t^{-1} : t \in \mathbf{G}_1\},$$

where B is a quaternion F-algebra (possibly split).

(b) π is the local theta lift of a supercuipidal representation $\sigma \boxtimes \omega_\pi$ of

$$\mathrm{GSO}_6 = (\mathrm{GL}_4 \times \mathrm{GL}_1)/\{(t, t^{-2}) : t \in \mathrm{GL}_1\}.$$

We shall treat the two cases in turn.

3.5.1 Case (a).

Since $\pi = \Theta(\tau_1 \boxtimes \tau_2)$ has nonzero Bessel functional with respect to (E, ν), it follows by [PT] that

$$\mathrm{Hom}_{E^\times}(\tau_1, \nu) \neq 0 \quad \text{and} \quad \mathrm{Hom}_{E^\times}(\tau_2, \nu^{-1}) \neq 0.$$

Now choose

- a number field $\mathbb{F}$ which has 2 places v_1 and v_2 such that $\mathbb{F}_{v_1} = \mathbb{F}_{v_2} = F$,
- a quaternion $\mathbb{F}$-algebra $\mathbb{D}$ such that $\mathbb{D}_{v_1} \cong \mathbb{D}_{v_2} \cong B$ and which is split at all other places of $\mathbb{F}$.
- a quadratic field extension $\mathbb{E}$ of $\mathbb{F}$ such that $\mathbb{E} \hookrightarrow \mathbb{D}$ and $\mathbb{E}_{v_1} = \mathbb{E}_2 = E$;
- a Hecke character $\daleth$ of $\mathcal{A}_{\mathbb{E}}^\times$ such that $\daleth_{v_1} = \daleth_{v_2} = \nu$.

By a result of Prasad-Schule-Pillot [PSP], we may find a global cuspidal representation $\mathfrak{T}_1$ and $\mathfrak{T}_2$ such that

- $\mathfrak{T}_{i,v_1} = \mathfrak{T}_{i,v_2} = \tau_i$ for $i = 1$ or 2;
- $\mathfrak{T}_i$ is unramified at all finite places outside v_1 and v_2;
- $\mathfrak{T}_1$ has nonzero global period with respect to $(\mathbb{E}, \daleth)$;
- $\mathfrak{T}_2$ has nonzero global period with respect to $(\mathbb{E}, \daleth^{-1})$

Then by [PT], the global theta lift

$$\Pi = \Theta(\mathfrak{T}_1 \boxtimes \mathfrak{T}_2)$$

of $\mathfrak{T}_1 \boxtimes \mathfrak{T}_2$ to GSp_4 is an irreducible cuspidal representation which has nonzero global Bessel period with respect to $(\mathbb{E}, \daleth)$. Moreover, for $i = 1$ or 2,

$$\Pi_{v_i} = \pi.$$

Finally, one chooses a Hecke character $\mu_{\mathbb{F}}$ of $\mathcal{A}^\times$ such that $\mu_{\mathbb{F},v_i} = \mu$ for $i = 1$ or 2.

By the global functional equation, one has

$$L^S(1-s, \Pi^\vee \times \mu_{\mathbb{F}}^{-1}) = \left(\prod_{v \in S} \gamma_{\mathbb{E}_v, \daleth_v}(s, \Pi_v \times \mu_{\mathbb{F},v}, \Psi_v)\right) \cdot L^S(s, \Pi \times \mu_{\mathbb{F}}).$$

and

$$L^S(1-s, \mathfrak{T}_i^\vee \times \mu_{\mathbb{F}}^\vee) = \left(\prod_{v \in S} \gamma(s, \mathfrak{T}_i \times \mu_{\mathbb{F}}, \Psi_v)\right) \cdot L(s, \mathfrak{T}_i \times \mu_{\mathbb{F}})$$

for $I = 1$ or 2. Since

$$L^S(s, \Pi \times \mu_{\mathbb{F}}) = L(s, \mathfrak{T}_1 \times \mu_{\mathbb{F}}) \cdot L(s, \mathfrak{T}_2 \times \mu_{\mathbb{F}}),$$

and

$$L^S(1-s, \Pi^\vee \times \mu_{\mathbb{F}}^{-1}) = L^S(1-s, \mathfrak{T}_1^\vee \times \mu_{\mathbb{F}}^\vee) \cdot L^S(1-s, \mathfrak{T}_1^\vee \times \mu_{\mathbb{F}}^\vee),$$

we deduce that

$$\prod_{v \in S} \gamma_{\mathbb{E}_v, \daleth_v}(s, \Pi_v \times \mu_{\mathbb{F},v}, \Psi_v) = \prod_{v \in S} (\gamma(s, \mathfrak{T}_1 \times \mu_{\mathbb{F}}, \Psi_v) \cdot \gamma(s, \mathfrak{T}_2 \times \mu_{\mathbb{F}}, \Psi).)$$

But by Proposition 3.4.1, one knows that for all $v \neq v_1$ or v_2, one has

$$\gamma_{\mathbb{E}_v, \daleth_v}(s, \Pi_v \times \mu_{\mathbb{F},v}, \Psi_v) = \gamma(s, \mathfrak{T}_1 \times \mu_{\mathbb{F}}, \Psi_v) \cdot \gamma(s, \mathfrak{T}_2 \times \mu_{\mathbb{F}}, \Psi_v). \tag{3.8}$$

Thus, we conclude that

$$\gamma^{E,\nu}(s, \pi \times \mu, \psi)^2 = \gamma(s, \tau_1 \times \mu, \psi)^2,$$

so that

$$\gamma^{E,\nu}(s, \pi \times \mu, \psi) = \pm\gamma(s, \tau_1 \times \mu, \psi),$$

for some sign $\pm$ which may a priori depend on (E, ν).

In fact, one can show that the sign is +. In the argument given above, instead of globalising so that the local situations at v_1 and v_2 are the same, one may globalize so that at v_2, one has

$$\mathfrak{T}_{1,v_2} = \mathfrak{T}_{v_2} = \tau_1.$$

One of course needs to adjust $\mathbb{E}$ and $\daleth$ at the place $v_2$appropriately; we leave the details to the reader. Then Π_{v_2} will be a non-supercuspidal representation, so that one has the equality (3.8) at all places outside v_1 by Proposition 3.4.1. Then one deduces the desired identity

$$\gamma^{E,\nu}(s, \pi \times \mu, \psi) = \gamma(s, \tau_1 \times \mu, \psi)$$

at the place v_1.

3.5.2 Case (b).

The argument in Case (b) is similar, so we shall be brief. Suppose that $\pi = \Theta(\sigma \boxtimes \omega_\pi)$ for a supercuspidal representation σ of GL_4. As above, since π has nonzero Bessel period with respect to (E, ν), it follows by [PT] that

$$\mathrm{Hom}_{\mathrm{GL}_2(E)}(\sigma, \nu) \neq 0.$$

Now one globalises τ to a cuspidal representation $\mathfrak{T}$ so that $\mathfrak{T}_{v_0} = \sigma$ for some place v_0, and $\mathfrak{T}_v$ is unramified for all other finite places of v. Then the global theta lift $\Pi = \Theta(\mathfrak{T})$ is a nonzero cuspidal representation of GSp_4 such that $\Pi_{v_0} = \pi$ and Π has nonzero global Bessel period with respect to some $(\mathbb{E}, \daleth)$ with $\mathbb{E}_{v_0} = E$ and $\daleth_{v_0} = \nu$. Then using the global functional equations as we did above, and using the fact that one understands the local gamma factors at all places out side v_0 (by Proposition 3.4.1), we deduce that

$$\gamma^{E,\nu}(s, \pi \times \mu, \psi) = \gamma(s, \tau \times \mu, \psi).$$

To summarise, we have shown:

Proposition 3.5.1. *Assume that multiplicativity holds. For an irreducible representation π of GSp_4 with L-parameter ϕ, $\gamma^{E,\nu}(s, \pi \times \mu, \psi)$ is independent of the data (E, ν) with respect to which π has nonzero Bessel functional. Moreover,*

$$\gamma^{E,\nu}(s, \pi\chi\mu, \psi) = \gamma(s, \phi \otimes \mu, \psi).$$

4 $GSp_4 \times GL_2$

This chapter develops the theory for γ-factors in the $r = 2$ case. We proceed as we did in chapter 3 by first laying out the global zeta integral then passing to the local Eulerian factors. Once we have the local functional equation and hence the definition of γ we proceed to the 'Main Theorem' and conclude, as before, with a discussion of the Multiplicativity property and it's consequences.

4.1 Global Integral

Let $G = G_3$ and $G_{1,2} = \{(g_1, g_2) \in G_1 \times G_2 \mid \lambda(g_2) = \lambda(g_2)\}$ be thought of as a subgroup of G via the injection

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) \hookrightarrow \begin{pmatrix} a & 0 & b & 0 \\ 0 & A & 0 & B \\ c & 0 & d & 0 \\ 0 & C & 0 & D \end{pmatrix}$$

Let H be the supgroup of $G_{1,2}$ defined by

$$H = \{(g_1, h_2) \in G_1 \times H_2 \mid \lambda(g_1) = \lambda(h_2)\}$$

and thus may be thought of as a subgroup of G as well.

Set $\Delta G_m = \{(a \cdot I_2, a) \in GL_2(F) \times E^\times \mid a \in G_m(F) = F^\times\}$. We have the following exact sequence

$$1 \to \Delta G_m \to GL_2(F) \times E^\times \to G_1 \to 1$$

where the first map is clear from the definition of ΔG_m and the second map is given by $(g, z) \mapsto z^{-1}g$.

Zeta Integral

Let $P = MN$ be the Siegal parabolic subgroup of G where

$$M = \left\{ m(g,\lambda) = \begin{pmatrix} g & 0 \\ 0 & \lambda \cdot {}^t g^{-\sigma} \end{pmatrix} \mid g \in GL_3(E), \lambda \in F^\times \right\}$$

$$N = \left\{ n(X) = \begin{pmatrix} 1_3 & X \\ 0 & 1_3 \end{pmatrix} \mid X \in Herm_3(E) \right\}$$

Let ν and τ be characters of $\mathbb{A}_E/E^\times$ and $\mathbb{A}_F/F^\times$ respectively. We may regard $\nu \otimes \tau$ as a character of $P(\mathbb{A}_F)$ by

$$\nu \otimes \tau \left[\begin{pmatrix} g & 0 \\ 0 & \lambda \cdot {}^t g^{-\sigma} \end{pmatrix} \begin{pmatrix} 1_3 & X \\ 0 & 1_3 \end{pmatrix} \right] = \nu(\det g)\tau(\lambda)$$

Let δ_P denote the modulus character of $P(\mathbb{A}_F)$ given by

$$\delta_P \left[\begin{pmatrix} g & 0 \\ 0 & \lambda \cdot {}^t g^{-\sigma} \end{pmatrix} \begin{pmatrix} 1_3 & X \\ 0 & 1_3 \end{pmatrix} \right] = |N_{E/F}(\det g)|^3 \cdot |\lambda|^{-9}$$

Let $I(s, \nu \otimes \tau) = \mathrm{Ind}\,_{P(\mathbb{A}_F)}^{H(\mathbb{A}_F)} \nu \otimes \tau \cdot \delta_P^s$ where we take the induction to be normalized. Thus $I(s, \nu \otimes \tau)$ is the space of locally constant functions f on $H(\mathbb{A}_F)$ such that

$$f(m(g,\lambda)n(X)g) = \nu(\det g)\tau(\lambda)|N_{E/F}(\det g)|^{3(s+1/2)} \cdot |\lambda|^{-9(s+1/2)} f(g).$$

Definition 4.1.1. For a section $f^{(s)} \in I(s, \nu \otimes \tau)$ we define the standard Eisenstien series:

$$E(f^{(s)}, h) = \sum_{\alpha \in P(F) \backslash H(F)} f^{(s)}(\alpha h),$$

which converges for $\mathrm{Re}(s) >> 0$ and has meromorphic continuation to the whole complex plane.

Let Π and σ be cuspidal representations of $GSp_4(\mathbb{A}_F)$ and $GL_2(\mathbb{A}_F)$ respectively with central characters ω_Π and ω_σ. Let χ be a character of $E^\times \backslash \mathbb{A}_E^\times$ such that $\chi|_{\mathbb{A}_F} = \omega_\sigma$. Let $\pi = \sigma \otimes \chi$ be the representation of $G_1(\mathbb{A}_F)$ defined by the exact sequence above. That is, suppose $g \in G_1$ and $(g', z) \mapsto z^{-1}g' = g$, then $\pi(g) = \chi(z)^{-1}\sigma(g')$. Let V_π and V_Π be the respective spaces of cusp forms. Finally for $\Phi \in V_\Pi$ and $\phi \in V_\pi$ define the global zeta integral

$$Z(f^{(s)}, \phi, \Phi) = \int\limits_{Z(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)} E(f^{(s)}, h)\phi(g_1)\Phi(h_2)dh,\ (g_1, h_2) \in H(\mathbb{A}_F),$$

which makes sense if $\omega_\Pi \cdot \omega_\sigma \cdot \tau^2 \cdot (\nu^3|_{\mathbb{A}_F}) = 1$.

We now define our intertwining operator. Let ψ be a character on $\mathbb{A}_F$ then we define the map

$$M(s, \nu \otimes \tau, \psi) : I(s, \nu \otimes \tau) \to I(-s, \breve{\nu} \otimes \tau \cdot \nu_0^3)$$

where $\breve{\nu} = \nu^{-\sigma}$ and $\nu_0 = \nu|_{F^\times}$ as follows. Given $f^{(s)} \in I(s, \nu \otimes \tau)$ set

$$M(s, \nu \otimes \tau, \psi)f^{(s)}(g) = \int\limits_N f^{(s)}(wng)\, dn$$

where the measure dn is self dual with respect to the form $(x, y) \mapsto \psi(Tr(xy))$ for $x, y \in N$ and

$$w = \begin{pmatrix} & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \\ & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \end{pmatrix}.$$

This integral converges for $\text{Re}(s) >> 0$ and has meromorphic continuation to the complex plane. Furthermore, the well known functional equation for Eisenstien series gives a functional equation for our global zeta integral,

$$Z(s, f^{(s)}, \phi, \Phi) = Z(-s, M(s, \nu \otimes \tau, \psi)f^{(s)}, \phi, \Phi) \tag{4.1}$$

4.1.1 Basic Identity

Recall from 1.2.2 the Bessel subgroup for H_2, $R = T \cdot N$ where R is the stabilizer of a non-degenerate linear form l in Levi component M of the Siegal parabolic P, $R = \mathrm{stab}_M l$. Further recall that this linear form corresponds to a non-degenerate symmetric matrix β. Let us fix such a matrix and specify coordinates

$$\beta = \begin{pmatrix} \beta_1 & \beta_2/2 \\ \beta_2/2 & \beta_3 \end{pmatrix}$$

such that $-4\det\beta = \beta_2^2 - 4\beta_1\beta_3 = \Delta$. Also, fix the matrix

$$A_\delta = \begin{pmatrix} \beta_2 & 2\beta_3 \\ -2\beta_1 & -\beta_2 \end{pmatrix}.$$

Let μ be a character of $\mathbb{A}_E^\times / E^\times$ defined as

$$\mu = \{\chi\nu^2\nu^\sigma(\tau \circ N_{E/F})\}^{-1}.$$

We define the Bessel function B_Φ on $H_2(\mathbb{A}_F)$ of type (β, μ) associated to Φ to be

$$B_\Phi(h') = \int\limits_{Z_2(\mathbb{A}_F)R(F)\backslash R(\mathbb{A}_F)} \Phi(tuh')\mu^{-1}(t)\psi_\beta^{-1}(u)dtdu$$

The Whittaker function W_ϕ on $G_1(\mathbb{A}_F)$ for ϕ is defined by

$$W_\phi(g) = \int\limits_{F\backslash\mathbb{A}_F} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\psi^{-1}(\beta_3 x)dx.$$

Finally define a homomorphism $\varphi_b : R \to G_1$:

$$\varphi_b\left[\begin{pmatrix} x\cdot 1_2 + y\cdot A_\delta & 0 \\ 0 & x\cdot 1_2 - y\cdot {}^t A_\delta \end{pmatrix}\begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix}\right]$$
$$= \begin{pmatrix} x + y\delta & 0 \\ 0 & x + y\delta \end{pmatrix}\begin{pmatrix} 1 & -\frac{tr(\beta X)}{\beta_3} \\ 0 & 1 \end{pmatrix}.$$

Since φ_b preserves similitudes, we can define a subgroup S of H by

$$S = \{(\varphi_b(r), r) \mid r \in R\}.$$

Proposition 4.1.2. *[MOR, Proposition 2.1]*

$$Z(f^{(s)}, \phi, \Phi) = \int\limits_{S(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} f^{(s)}(\eta h)W_\phi(g_1)B_\Phi(h_2)dh$$

We take everything to be factorable here and we get the following local Zeta integral

$$Z(f^{(s)}, W, B) = \int\limits_{S(F)\backslash H(F)} f^{(s)}(\eta h)W(g_1)B(h_2)dh \tag{4.2}$$

where

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\alpha^{\sigma} \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \alpha = \frac{\beta_2 + \delta}{2\beta_3} \in E^{\times}$$

4.2 Local Integral

As in Chapter 1, we will define and establish technical properties of the local zeta integral. For this section we fix a place v of F and refer to the completion of F at this place as simply F. We also similarly abuse notation for the $2-$dimensional F-algebra E.

4.2.1 Preliminaries

Proposition 4.2.1. *The local zeta integral*

$$Z(f^{(s)}, W, B) = \int\limits_{S(F)\backslash H(F)} f^{(s)}(\eta h)W(g_1)B(h_2)dh$$

converges absolutely when $Re(s) \gg 0$. Moreover, it admits a meromorphic continuation to $\mathbb{C}$.

Next we establish a local functional equation for the local zeta integral (4.2). With respect to the intertwining operator,

$$\nu \mapsto \check{\nu}, \;\; \tau \mapsto \tau \cdot \nu_0^3$$

If we let $\check{B}$ be the corresponding Bessel function in $\mathcal{B}(\check{\mu}, \psi)$, where

$$\begin{aligned}
\breve{\mu} &= \left[\chi\breve{\nu}^2\breve{\nu}^\sigma(\tau\cdot\nu_0^3\circ N_{E/F}\right]^{-1}\\
&= \left[\chi\cdot(\nu^{-\sigma})^2\nu^{-1}(\tau\cdot\nu_0^3\circ N_{E/F})\right]^{-1}\\
&= \left[\chi\cdot(\nu^{-\sigma})^2\nu^{-1}\nu^3(\nu^\sigma)^3(\tau\cdot\circ N_{E/F})\right]^{-1}\\
&= \mu.
\end{aligned}$$

Therefore, $\breve{B} = B$ and we assert:

Proposition 4.2.2. *There is a meromorphic function $\Gamma^{E,\nu}(s,\pi\times\mu,\psi)$ such that*

$$Z(M(s,\psi)f^{(s)},W,B) = \Gamma^{E,\nu}(3s+1/2,\Pi\times\sigma\times\nu_0^2\times\tau,\psi)Z(f^{(s)},W,B) \tag{4.3}$$

4.2.2 Unramified Calculation

In this section we assume both Π and σ are unramified representations of H_2 and GL_2 respectively over a local non-archimedian field F. We also assume the characters ν, τ and χ are unramified quasi-characters and our additive character ψ has conductor $\mathfrak{o}_F$.

Let $\varepsilon_{E/F}$ be the quadratic character obtained via class field theory. Let Φ_0 be the normalized spherical vector and $\nu_0 = \nu|_{F^\times}$.

Theorem 4.2.3. *[MOR, Theorem 8.1]*

$$Z(s) = Z(s,\Phi_0^{(s)},W,B) = \prod_{i=1}^{3} L\left(6s+i,\nu_0\cdot\varepsilon_{E/F}^{i+3}\right)^{-1}\cdot L\left(3s+\frac{1}{2},\pi\times\sigma\times\nu_0^2\times\tau\right).$$

4.3 Normalization

Recall the definition for the intertwing operator, for $f_s \in I(s,\nu\otimes\tau)$

$$M(s,\nu\otimes\tau,\psi)f^{(s)}(g) = \int_N f^{(s)}(wng)\,dn$$

We need to know the result of this operator when applied to our spherical vector Φ_0. We have the following result

Lemma 4.3.1. *[HKS, eq. 6.14]*

$$M(s,\nu)\Phi_0^s = \frac{a_3(s,\nu)}{b_3(s,\nu)}\Phi_0^{(-s)}$$

where

$$a_3(s,\nu) = \prod_{i=1}^{3} L(6s-i+1, \nu_0 \cdot \varepsilon_{E/F}^{i+3})$$

$$b_3(s,\nu) = \prod_{i=1}^{3} L(6s+i, \nu_0 \cdot \varepsilon_{E/F}^{i+3})$$

Now applying (4.2.3) and (4.3.1) we can directly compute

$$\begin{aligned}
&Z(s, M(s,\nu)\Phi_0^{(s)}, W, B) \\
&= \frac{a_3(s,\nu)}{b_3(s,\nu)} \prod_{i=1}^{3} L\left(-6s+i, \nu_0^{-1} \cdot \varepsilon_{E/F}^{i+3}\right)^{-1} \cdot L\left(-3s+\frac{1}{2}, \tilde{\pi} \times \tilde{\sigma} \times \nu_0^{-2} \times \tau^{-1}\right) \\
&= \prod_{i=1}^{3} \frac{L(6s-i+1, \nu_0 \cdot \varepsilon_{E/F}^{i+3})}{L(6s+i, \nu_0 \cdot \varepsilon_{E/F}^{i+3}) L(-6s+i, \nu_0^{-1} \cdot \varepsilon_{E/F}^{i+3})} L\left(-3s+\frac{1}{2}, \tilde{\pi} \times \tilde{\sigma} \times \nu_0^{-2} \times \tau^{-1}\right) \\
&= \prod_{i=1}^{3} L\left(6s+i, \nu_0 \cdot \varepsilon_{E/F}^{i+3}\right)^{-1} \gamma(6s-i+1, \nu_0 \cdot \varepsilon_{E/F}^{i+3}, \psi)^{-1} \\
&\cdot L\left(-3s+\frac{1}{2}, \tilde{\pi} \times \tilde{\sigma} \times \nu_0^{-2} \times \tau^{-1}\right)
\end{aligned}$$

We conclude that in the unramified context, the factor $\Gamma^{E,\nu}(3s+1/2, \Pi \times \sigma \times \nu_0^2 \times \tau, \psi)$ defined in (4.3) is exactly

$$\prod_{i=1}^{3} \gamma(6s-i+1, \nu_0 \cdot \varepsilon_{E/F}^{i+3}, \psi)^{-1} \cdot \frac{L\left(-3s+\frac{1}{2}, \tilde{\pi} \times \tilde{\sigma} \times \nu_0^{-2} \times \tau^{-1}\right)}{L\left(3s+\frac{1}{2}, \pi \times \sigma \times \nu_0^{2} \times \tau\right)}. \tag{4.4}$$

Let μ_n denote the n-th roots of unity. For a character η of F and $a \in F^\times$ let

$$W_F(a,\eta) = \frac{W_F(\eta_a)}{W_F(\eta)} \in \mu_4,$$

where $W_F(\eta) \in \mu_8$ denotes the Weil index of the character of second degree $x \mapsto \eta(x^2)$. Note

$$W_F(\Delta, \psi_a) = \varepsilon_{E/F}(a) W_F(\Delta, \psi)$$

and in the unramified context: the conductor of ψ is $\mathfrak{o}_F$, Δ is a unit and the residue characteristic p is not 2,

$$W_F(\Delta, \psi) = 1$$

Let

$$\kappa(s,\nu,\psi) = W_F(\Delta,\psi)^3 \prod_{i=0}^{2} \gamma(6s-i,\nu^o \cdot \varepsilon_{E/F}^i,\psi)$$

then

$$\begin{aligned}\kappa(s,\nu,\psi_a) &= W_F(\Delta,\psi_a)^3 \prod_{i=0}^{2} \gamma(6s-i,\nu^o \cdot \varepsilon_{E/F}^i,\psi_a)\\ &= \varepsilon_{E/F}(a)\prod_{i=0}^{2}\left[\nu^o(a)\varepsilon_{E/F}^i|a|^{6s-i-1/2}\right] W_F(\Delta,\psi)^3 \prod_{i=0}^{2} \gamma(6s-i,\nu^o \cdot \varepsilon_{E/F}^i,\psi)\\ &= \nu(a)^3|a|^{18s-9/2}\kappa(s,\nu,\psi)\end{aligned}$$

Lastly, note that $M(s,\psi_a) = |a|_F^{9/2}M(s,\psi)$

4.4 γ-factor

Definition 4.4.1.

$$\gamma^{E,\nu}(3s+1/2,s,\Pi\times\sigma\times\nu_0^2\times\tau,\psi) = \kappa(s,\nu,\psi)\cdot\Gamma^{E,\nu}(3s+1/2,\Pi\times\sigma\times\nu_0^2\times\tau,\psi) \quad (4.5)$$

4.4.1 Main Theorem

Theorem

(1) Unramified factors:

$$\gamma^{E,\nu}(s,\Pi\times\sigma\times\nu_0^2\times\tau,\psi) = \frac{L(1-s,\tilde{\Pi}\times\tilde{\sigma})}{L(s,\Pi\times\mu)}$$

(2) Dependence on ψ: Let $a \in F^\times$ and set $\psi_a(x) = \psi(ax)$, then

$$\begin{aligned}\gamma^{E,\nu}(3s+1/2,\Pi\times\sigma\times\nu_0^2\times\tau,\psi_a) &= \omega_\Pi^4(a)\omega_\sigma^4(a)\nu_0^{16}(a)\tau^8(a)|a|^{8(s-1/2)}\\ &\quad\cdot\gamma^{E,\nu}(3s+1/2,\Pi\times\sigma\times\nu_0^2\times\tau,\psi)\end{aligned}$$

(3) Unramified Twisting: Let $s_0 \in \mathbb{C}$ be fixed then,

$$\gamma^{E,\nu}(3(s+s_0)+1/2,\Pi\times\sigma\times\nu|_{F^\times}^2\times\tau,\psi) = \gamma^{E,\nu}(3s+1/2,\Pi\times\sigma\times(\nu_{3s_0}|_{F^\times})^2\times\tau_{-9s_0},\psi)$$

(4) Functional equation:

$$\gamma^{E,\nu}(3s+1/2,\Pi\times\sigma\times\nu_0^2\times\tau,\psi)\gamma^{E,\nu}(1/s-3s,\tilde{\pi}\times\tilde{\sigma}\times\nu_0^{-2}\times\tau^{-1},\psi^{-1}) = 1$$

(5) Global property: Let Π and σ be a cuspidal representation of $H_2(\mathbb{A})$ and GL_2 respectively and assume Π has a global bessel period with respect to (E,μ). τ a Hecke character of $F^\times\backslash\mathbb{I}_F$, and $\psi_F=\otimes_v\psi_{F_v}$ a non-trivial character of $F\backslash\mathbb{A}_F$. Let S be a finite set of places containing all the archimedean, 2-adic and places where any of π,μ or ψ is ramified. Then,

$$L^S(3s+\frac{1}{2},\Pi\times\sigma\times\nu_0^2\times\tau)=\prod_{v\in S}\gamma_v^{E,\nu}(3s+\frac{1}{2},\Pi_v\times\sigma_v\times\nu_{v,0}^2\times\tau_v,\psi_v)$$
$$\cdot L^S(\frac{1}{2}-3s,\tilde{\Pi}\times\tilde{\sigma}\times\nu_0^{-2}\times\tau^{-1})$$

proof.

(2) We must show

$$\gamma^{E,\nu}(3s+1/2,\Pi\times\sigma\times\nu_0^2\times\tau,\psi_a)=\nu_0^4(a)|a|^{24s}\gamma^{E,\nu}(3s+1/2,\Pi\times\sigma\times\nu_0^2\times\tau,\psi).$$

Since both the Whittaker and Bessel functions used in our zeta integral depend on the character ψ, we must replace these with suitable functions in the spaces $\mathcal{W}(\psi_a)$ and $\mathcal{B}(E,\mu,\psi_a)$.

$$B_{\psi_a}(g)=B_\psi\left(\begin{pmatrix}a&&&\\&a&&\\&&1&\\&&&1\end{pmatrix}g\right)$$

Now we check that this belongs to the proper space by calculating the action of R.

$$B_{\psi_a}(tug) = B_\psi\left(t\begin{pmatrix}a&&&\\&a&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}1&&X&\\&1&&\\&&1&\\&&&1\end{pmatrix}g\right)$$
$$= \mu(t)B_\psi\left(\begin{pmatrix}1&&aX&\\&1&&\\&&1&\\&&&1\end{pmatrix}\begin{pmatrix}a&&&\\&a&&\\&&1&\\&&&1\end{pmatrix}g\right)$$
$$= \mu(t)\psi^*(u(aX))B_\psi\left(\begin{pmatrix}a&&&\\&a&&\\&&1&\\&&&1\end{pmatrix}g\right)$$
$$= \mu(t)\psi_a^*(u(X))B_{\psi_a}(g).$$

therefore $B_{\psi_a} \in \mathcal{B}(E, \mu, \psi_a)$.

Similarly we consider the following candidate for a twisted Whittaker function,

$$W_{\psi_a}(g) = W_\psi\left(\begin{pmatrix}a&\\&1\end{pmatrix}g\right).$$

We then observe that

$$W_{\psi_a}\left(\begin{pmatrix}1&x\\&1\end{pmatrix}g\right) = W_\psi\left(\begin{pmatrix}a&\\&1\end{pmatrix}\begin{pmatrix}1&x\\&1\end{pmatrix}g\right)$$
$$= W_\psi\left(\begin{pmatrix}1&ax\\&1\end{pmatrix}\begin{pmatrix}a&\\&1\end{pmatrix}g\right)$$
$$= \psi(ax)W_\psi\left(\begin{pmatrix}a&\\&1\end{pmatrix}g\right)$$
$$= \psi_a(x)W_{\psi_a}(g),$$

therefore $W_{\psi_a} \in \mathcal{W}(\psi_a)$.

As before, we evaluate both sides of the local functional equation then take the ratio to see the effect on $\gamma^{E,\nu}$.

$$Z(s,f,B_{\psi_a},W_{\psi_a}) = \int\limits_{S\backslash H} f(\eta h,s)B_{\psi_a}(h_2)W_{\psi_a}(h_1)dh$$

$$= \int\limits_{S\backslash H} f(\eta(h_1,h_2),s)B_\psi\left(\begin{pmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{pmatrix}h_2\right)W_\psi\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}h_1\right)dh$$

$$h_1 \mapsto \begin{pmatrix} a^{-1} & \\ & 1\end{pmatrix}h_1 \;\; h_2 \mapsto \begin{pmatrix} a^{-1} &&& \\ & a^{-1} && \\ && 1 & \\ &&& 1\end{pmatrix}h_2$$

$$\text{Let } h_a = \left(\begin{pmatrix} a^{-1} & \\ & 1\end{pmatrix}, \begin{pmatrix} a^{-1} &&& \\ & a^{-1} && \\ && 1 & \\ &&& 1\end{pmatrix}\right) \hookrightarrow \begin{pmatrix} a^{-1} &&&&& \\ & a^{-1} &&&& \\ && a^{-1} &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1\end{pmatrix}$$

$$Z(s,f,B_{\psi_a},W_{\psi_a}) = \int\limits_{S\backslash H} f(\eta h_a h,s)B_\psi(h_2)W_\psi(h_1)d(h_a h)$$

$$\eta h_a = \begin{pmatrix} \begin{pmatrix} 1 && \\ & a^{-1} & \\ && a^{-1}\end{pmatrix} & \\ & a^{-1}\begin{pmatrix} 1 && \\ & a & \\ && a\end{pmatrix}\end{pmatrix}\eta = h_a'\eta, \text{ Let } A = \begin{pmatrix} 1 && \\ & a^{-1} & \\ && a^{-1}\end{pmatrix}$$

$$Z(s,f,B_{\psi_a},W_{\psi_a}) = \int\limits_{S\backslash H} f(h'_a\eta h,s)B_\psi(h_2)W_\psi(h_1)d(h_ah)$$
$$= \nu(\det A)\tau(a^{-1})\left[|N_{E/F}(\det A)|^3|a^{-1}|^{-9}\right]^{s+\frac{1}{2}} \cdot \int\limits_{S\backslash H} f(\eta h,s)B_\psi(h_2)W_\psi(h_1)d(h_ah)$$
$$= \nu_0(a)^{-2}\tau(a)^{-1}|a|^{-3s-\frac{3}{2}} \int\limits_{S\backslash H} f(\eta h,s)B_\psi(h_2)W_\psi(h_1)d(h_ah).$$

Now we perform a similar calculation on the other side:

$$Z(-s,M(s,\psi_a)f,B_{\psi_a},W_{\psi_a}) = \int\limits_{S\backslash H} M(s,\psi_a)f(\eta h,s)B_{\psi_a}(h_2)W_{\psi_a}(h_1)dh$$
$$= |a|^{9/2}\int\limits_{S\backslash H} M(s,\psi)f(h'_a\eta h,s)B_\psi(h_2)W_\psi(h_1)d(h_ah)$$
$$= |a|^{9/2}\nu^{-\sigma}(\det A)\tau\nu_0^3(a^{-1})\delta_P(h'_a)^{-s+\frac{1}{2}} \cdot \int\limits_{S\backslash H} M(s,\psi)f(\eta h,s)B_\psi(h_2)W_\psi(h_1)d(h_ah)$$
$$= |a|^{9/2}\nu_0(a)^{-1}\tau(a)^{-1}|a|^{3s-\frac{3}{2}}\int\limits_{S\backslash H} M(s,\psi)f(\eta h,s)B_\psi(h_2)W_\psi(h_1)d(h_ah)$$
$$= \nu_0(a)^{-1}\tau(a)^{-1}|a|^{3s+3}\int\limits_{S\backslash H} M(s,\psi)f(\eta h,s)B_\psi(h_2)W_\psi(h_1)d(h_ah).$$

Therefore,

$$\gamma^{E,\nu}(s,\Pi\times\sigma\times\nu_0^2\times\tau,\psi_a) = \frac{Z(-s,Mf,B_{\psi_a},W_{\psi_a})\cdot\kappa(s,\nu,\psi_a)}{Z(s,f,B_{\psi_a},W_{\psi_a})}$$
$$= \frac{\nu_0(a)^2\tau(a)^{-1}|a|^{21s-3/2}}{\nu_0(a)^{-2}\tau(a)^{-1}|a|^{-3s-\frac{3}{2}}}\gamma^{E,\nu}(s,\Pi\times\sigma\times\nu_0^2\times\tau,\psi)$$
$$= \nu_0^4(a)|a|^{24s}\gamma^{E,\nu}(s,\Pi\times\sigma\times\nu_0^2\times\tau,\psi).$$

(3) Given a field k and a quasi-character η of k, let $\eta_t = \eta\cdot|\cdot|_k^t$. We will show

$$\gamma(3(s+s_0)+1/2,\Pi\times\sigma\times\nu|_{F^\times}^2\times\tau) = \gamma(3s+1/2,\Pi\times\sigma\times(\nu_{3s_0}|_{F^\times})^2\times\tau_{-9s_0}).$$

Again we proceed by computing both sides of the functional equation.

$$Z(s+s_0, f, B, W, \nu\otimes\tau) = \int\limits_{S\backslash H} f(\eta h, s+s_0)B(h_2)W(h_1)dh$$

$$= \int\limits_{S\backslash H} f(p_h k, s+s_0)B(h_2)W(h_1)dh, \text{ where } p_h = m(a,\lambda)n \in P_{G_3} \text{ and } k \in K_{G_3}$$

$$= \int\limits_{S\backslash H} \nu(\det\ a)|(\det\ a)|_E^{3(s+s_0)}\tau(\lambda)|\lambda|^{-9(s+s_0)} f(k, s+s_0)B(h_2)W(h_1)dh$$

$$= \int\limits_{S\backslash H} \nu_{3s_0}(\det\ a)\tau_{-9s_0}(\lambda)|(\det\ a)|_E^{3s}|\lambda|^{-9s} f(k,s)B(h_2)W(h_1)dh$$

$$= Z(s, f, B, W, \nu_{3s_0}\otimes\tau_{-9s_0})$$

For the other side:

$$Z(-s-s_0, M^*(s+s_0)f, B, W, \breve{\nu}\otimes\tau\nu_0^3)$$

$$= \int M^*(s+s_0)f(\eta h, s+s_0)B(h_2)W(h_1)dh$$

$$= \int\limits_{S\backslash H} \breve{\nu}(\det\ a)\cdot(\tau\nu_0^3)(\lambda)|(\det\ a)|_E^{3(-s-s_0)}|\lambda|^{-9(-s-s_0)}$$

$$\cdot M^*(s+s_0)f(k, s+s_0)B(h_2)W(h_1)dh.$$

We can absorb the powers containing s_0 as follows. $\breve{\nu}(\text{det a})|\text{det } a|_E^{-3s_0} = (\breve{\nu}\cdot|\cdot|_E^{-3s_0})(\det\ a) = \widecheck{(\nu_{3s_0})}(\det\ a)$. Since we replaced ν with ν_{3s_0}, we need to replace $\nu_0 = \nu|_{F^\times}$ with $\nu_{3s_0}|_{F^\times}$. Now $\tau_{-9s_0}\nu_{3s_0}|_{F^\times}^3(\lambda) = \tau(\lambda)|\lambda_F|^{-9s_0}\nu|_{F^\times}^3(\lambda)|\lambda|_E^{9s_0} = \tau(\lambda)\nu_0^3(\lambda)|\lambda|^{9s_0}$ which is exactly what we need to absorb the $|\lambda|^{9s_0}$ term. Therefore,

$$Z(-s-s_0, M^*(s+s_0)f, B, W, \breve{\nu}\otimes\tau\nu_0^3)$$

$$= \int\limits_{S\backslash H} \widecheck{\nu_{3s_0}}(\det\ a)(\tau_{-9s_0}\nu_{3s_0}|_{F^\times}^3)(\lambda)|(\det\ a)|_E^{-3s}|\lambda|^{9s}$$

$$\cdot M^*(s+s_0)f(k, s+s_0)B(h_2)W(h_1)dh$$

$$= Z(-s, M^*f, B, W, \widecheck{\nu_{3s_0}}\otimes\tau_{-9s_0}\nu_{3s_0}|_{F^\times}^3)$$

(4) We must show:

$$\gamma^{E,\nu}(3s+1/2, \Pi\times\sigma\times\nu_0^2\times\tau, \psi)\gamma^{E,\nu}(1/2-3s, \tilde{\pi}\times\tilde{\sigma}\times\nu_0^{-2}\times\tau^{-1}, \psi^{-1}) = 1$$

We start with the local functional equation for Z

$$Z(-s, M(s, \nu\otimes\tau, \psi)f^{(s)}, B, W) = \Gamma^{E,\nu}(3s+1/2, \Pi\times\sigma\times\nu_0^2\times\tau, \psi)Z(s, f^{(s)}, B, W)$$

and apply it twice.

$$\begin{aligned}
&Z(s, M(-s, \check{\nu}\otimes\tau\nu_0^3, \psi)\circ M(s, \nu\otimes\tau, \psi)f^{(s)}, B, W)\\
&= \Gamma^{E,\nu}(1/2-3s, \tilde{\pi}\times\tilde{\sigma}\times\nu_0^{-2}\times\tau^{-1}, \psi)Z(-s, M(s, \nu\otimes\tau, \psi)f^{(s)}, B, W)\\
&= \Gamma^{E,\nu}(1/2-3s, \tilde{\pi}\times\tilde{\sigma}\times\nu_0^{-2}\times\tau^{-1}, \psi)\Gamma^{E,\nu}(3s+1/2, \Pi\times\sigma\times\nu_0^2\times\tau, \psi)\\
&\qquad\cdot Z(s, f^{(s)}, B, W)
\end{aligned}$$

By a result of [HKS]

$$M(s, \nu\otimes\tau, \psi)M(-s, \check{\nu}\otimes\tau\nu_0^3, \psi) = \kappa(-s, \check{\nu}\otimes\tau\nu_0^3, \psi)^{-1}\kappa(s, \nu\otimes\tau, \psi)^{-1}.$$

Putting this all together we have

$$\begin{aligned}
&\kappa(-s, \check{\nu}\otimes\tau\nu_0^3, \psi)\kappa(s, \nu\otimes\tau, \psi)\\
&\quad\cdot\Gamma^{E,\nu}(1/2-3s, \tilde{\pi}\times\tilde{\sigma}\times\nu_0^{-2}\times\tau^{-1}, \psi)\Gamma^{E,\nu}(3s+1/2, \Pi\times\sigma\times\nu_0^2\times\tau, \psi) = 1
\end{aligned}$$

(5) We must now show that:

$$\begin{aligned}
&L_S(3s+\frac{1}{2}, \Pi\times\sigma\times\nu_0^2\times\tau)\\
&= \prod_{v\in S}\gamma_v^{E,\nu}(3s+\frac{1}{2}, \Pi_v\times\sigma_v\times\nu_{v,0}^2\times\tau_v, \psi_v)\cdot L_s(\frac{1}{2}-3s, \tilde{\Pi}\times\tilde{\sigma}\times\nu_0^{-2}\times\tau^{-1})
\end{aligned}$$

By (2.4.1), Π has a Bessel model with respect to some Hecke character ν of $\mathbb{A}_E^\times$ and some character ψ of $F\backslash\mathbb{A}_F$. Furthermore, by well known results, σ has a Whittaker model with respect to ψ. Let W and B be the corresponding vectors for Φ and ϕ. Recall the 'Basic Identity' (4.1.2)

$$Z(s, f^{(s)}, \phi, \Phi) = Z(s, f^{(s)}, B, W)$$

We assume that f, B,and W factor into restricted products over all places v, then for Re(s)>> 0 we have an Euler product,

$$\begin{aligned}Z(s,f^{(s)},B,W) &= \prod_v Z_v(s,f_v^{(s)},B_v,W_v)\\ &= \prod_{v\in S} Z_v(s,f_v^{(s)},B_v,W_v)\cdot\prod_{v\notin S} Z_v(s,f_v^{(s)},B_v,W_v).\end{aligned}$$

For $v \notin S$ by Theorem 4.2.3,

$$\begin{aligned}&Z_v(s,f_v^{(s)},B_v,W_v)\\ &= \prod_{i=1}^{3} L_v\left(6s+i,\nu_{v,0}\cdot\varepsilon_{E_v/F_v}^{i+3}\right)^{-1}\cdot L_v\left(3s+\frac{1}{2},\Pi_v\times\sigma_v\times\nu_{v,0}^2\times\tau_v\right).\end{aligned}$$

We end up with the following identity which is valid for all $s\in\mathbb{C}$,

$$Z(s,f^{(s)},B,W) = \frac{L^S\left(3s+\frac{1}{2},\Pi\times\sigma\times\nu_0^2\times\tau\right)}{\prod_{i=1}^{3} L^S\left(6s+i,\nu_0\cdot\varepsilon_{E/F}^{i+3}\right)}\cdot\prod_{v\in S} Z_v(s,f_v^{(s)},B_v,W_v). \tag{4.6}$$

Next we need another identity for our global intertwining operator. For Re(s)>> 0 the operator factors over all places,

$$\begin{aligned}M(s,\nu\otimes\tau,\psi)f^{(s)} &= \prod_v M_v(s,\nu_v\otimes\tau_v,\psi_v)f_v^{(s)}\\ &= \prod_{v\in S} M_v(s,\nu_v\otimes\tau_v,\psi_v)f_v^{(s)}\cdot\prod_{v\notin S} M_v(s,\nu_v\otimes\tau_v,\psi_v)f_v^{(s)}.\end{aligned}$$

By Lemma (4.3.1)

$$M_v(s,\nu_v\otimes\tau_v,\psi_v)f_v^{(s)} = \frac{\prod_{i=1}^{3} L_v(6s-i+1,\nu_{v,0}\cdot\varepsilon_{E_v/F_v}^{i+3})}{\prod_{i=1}^{3} L_v(6s+i,\nu_{v,0}\cdot\varepsilon_{E_v/F_v}^{i+3})}f_v^{(-s)}.$$

Therefore, for all $s\in\mathbb{C}$,

$$M(s,\nu\otimes\tau,\psi)f^{(s)} = \prod_{i=1}^{3}\frac{L^S(6s-i+1,\nu_0\cdot\varepsilon_{E/F}^{i+3})}{L^S(6s+i,\nu_0\cdot\varepsilon_{E/F}^{i+3})}\prod_{v\notin S} f_v^{(-s)}\prod_{v\in S} M_v(s,\nu_v\otimes\tau_v,\psi_v)f_v^{(s)}. \tag{4.7}$$

Now we evaluate (4.6) at $-s, M(s,\nu\otimes\tau,\psi)f^{(s)}, \nu\mapsto\check{\nu}$ and $\tau\mapsto\tau\nu_0^3$ while making use of (4.7)

$$\begin{aligned}&Z(-s,M(s,\nu\otimes\tau,\psi)f^{(s)},B,W) =\\ &\frac{L^S\left(-3s+\frac{1}{2},\Pi\times\sigma\times\nu_0^{-2}\times\tau\nu_0^3\right)}{\prod_{i=1}^{3} L^S\left(-6s+i,\nu_0^{-1}\cdot\varepsilon_{E/F}^{i+3}\right)}\frac{\prod_{i=1}^{3} L^S(6s-i+1,\nu_0\cdot\varepsilon_{E/F}^{i+3})}{\prod_{i=1}^{3} L^S(6s+i,\nu_0\cdot\varepsilon_{E/F}^{i+3})}\\ &\cdot\prod_{v\in S} Z_v(-s,M(s,\nu\otimes\tau,\psi)f_v^{(s)},B_v,W_v)\end{aligned}$$

Since $\tau\nu_0^3 = \tau(tau^{-2}\tau^2)\nu_0^3 = \tau^{-1}\omega_\Pi^{-1}\omega_\sigma^{-1}$, $\tilde{\Pi} = \Pi \otimes \omega_\Pi^{-1}$, and $\tilde{\sigma} = \sigma \otimes \omega_\sigma^{-1}$, we see that

$$L^S\left(-3s + \frac{1}{2}, \Pi \times \sigma \times \nu_0^{-2} \times \tau\nu_0^3\right) = L^S\left(\frac{1}{2} - 3s, \tilde{\Pi} \times \tilde{\sigma} \times \nu_0^{-2} \times \tau^{-1}\right).$$

By now using the global functional equation (4.1), (4.6), the definition of $\gamma_v^{E,\nu}$ and the analogous 'global property' for Hecke L-functions, we get

$$\begin{aligned} &L_S(3s + \frac{1}{2}, \Pi \times \sigma \times \nu_0^2 \times \tau) \\ &= \prod_{v \in S} \gamma_v^{E,\nu}(3s + \frac{1}{2}, \Pi_v \times \sigma_v \times \nu_{v,0}^2 \times \tau_v, \psi_v) \cdot L_s(\frac{1}{2} - 3s, \tilde{\Pi} \times \tilde{\sigma} \times \nu_0^{-2} \times \tau^{-1}). \end{aligned}$$

4.5 Multiplicativity

Suppose that Π is an irreducible subquotient of an induced representation of GSp_4 and σ is an irreducible representation of $\mathrm{GL}_2(F)$. Since there are 3 conjugacy classes of parabolic subgroups of GSp_4, we have the following 3 cases:

- if B is the Borel subgroup of GSp_4, then suppose that Π is a subquotient of

  $$I_B(\chi_1, \chi_2; \chi) = \mathrm{Ind}_B^{GSp_4} \chi_1 \otimes \chi_2 \otimes \chi.$$

  In this case, multiplicativity is the identity

  $$\begin{aligned} \gamma_{E,\nu}(s, \pi \times \sigma, \psi) = \gamma(s, \sigma \times \chi, \psi) \cdot \gamma(s, \sigma \times \chi_1\chi, \psi) \\ \cdot \gamma(s, \sigma \times \chi_2\chi, \psi) \cdot \gamma(s, \sigma \times \chi_1\chi_2\chi, \psi). \end{aligned} \tag{4.8}$$

- suppose that P is the Siegel parabolic subgroup of GSp_4, so that its Levi factor is $M \cong \mathrm{GL}_2 \times \mathrm{GL}_1$, and Π is a subquotient of

  $$I_P(\tau, \chi) = \mathrm{Ind}_P^{GSp_4} \tau \boxtimes \chi,$$

  where $\tau \boxtimes \chi$ is an irreducible representation of M. In this case, multiplicativity is the identity

  $$\gamma_{E,\nu}(s, \Pi \times \sigma, \psi) = \gamma(s, \sigma \times \chi, \psi) \cdot \gamma(s, \sigma \times \tau \cdot \chi, \psi) \cdot \gamma(s, \sigma \times \chi\omega_\tau, \psi). \tag{4.9}$$

  Here, the gamma factors on the RHS are the Rankin-Selberg gamma factors for $\mathrm{GL}(2) \times \mathrm{GL}(1)$ or $\mathrm{GL}(2) \times \mathrm{GL}(2)$.

- if Q is the Heisenberg parabolic subgroup of GSp_4, so that its Levi factor is $L \cong \mathrm{GL}_1 \times GSp_2 \cong \mathrm{GL}_1 \times \mathrm{GL}_2$, and Π is a subquotient of

$$I_Q(\chi, \tau) = \mathrm{Ind}_Q^{GSp_4} \chi \boxtimes \tau,$$

In this case, multipicativity is the identity

$$\gamma_{E,\nu}(s, \Pi \times \sigma, \psi) = \gamma(s, \tau \times \sigma, \psi) \cdot \gamma(s, \tau\chi \times \sigma, \psi). \tag{4.10}$$

Here the gamma factors on the RHS are $\mathrm{GL}(2) \times \mathrm{GL}(2)$ gamma factors.

In addition, suppose that σ is a constituent of a principal series representation $\pi(\chi_1, \chi_2)$ of $\mathrm{GL}_2(F)$. In this case, multiplicativity says that

$$\gamma(s, \Pi \times \sigma, \psi) = \gamma(s, \Pi \times \chi_1, \psi) \cdot \gamma(s, \Pi \times \chi_2, \psi), \tag{4.11}$$

where the gamma factors on the RHS are the $\mathrm{GSp}_4 \times \mathrm{GL}_1$ gamma factors defined in (4.3).

4.5.1 Consequences of multiplicativity.

If we assume that we have the identities (4.8), (4.9), (4.10) and (4.11), the the consequences are similar to those for the $\mathrm{GSp}_4 \times \mathrm{GL}_1$ case. We first have:

Proposition 4.5.1. *Assuming multiplicativity for both* $\mathrm{GSp}_4 \times \mathrm{GL}_2$ *and* $\mathrm{GSp}_4 \times \mathrm{GL}_1$ *context, we have:*

(i) If π or σ is a non-supercuspidal representation of a p-adic field, then the local γ-factor $\gamma_{E,\nu}(s, \pi \times \mu, \psi)$ is independent of the choice of the data (E, ν) with respect to which π has a (E, ν)-Bessel model.

(ii) If $F = \mathbb{R}$ or $\mathbb{C}$, and π has L-parameter ϕ_π, then $\gamma_{E,\nu}(s, \pi \times \mu, \psi)$ is independent of (E, ν) and

$$\gamma_{E,\nu}(s, \pi \times \mu, \psi) = \gamma(s, \phi_\pi \otimes \mu, \psi).$$

4.5.2 Independence of (E, ν).

In the previous subsection, we have seen that the local γ-factor is independent of the choice of the data (E, ν) with respect to which π has a Bessel model, when π or σ is non-supercuspidal. In this section, we address this issue of independence when π and σ are both supercuspidal.

Hence, suppose that π is a supercuspidal representation which supports nonzero Bessel functionals with respect to (E, ν). As we noted in §3.5, there are two cases to consider:

(a) π is the local theta lift of a supercuspidal representation $\tau_1 \boxtimes \tau_2$ of

$$\mathrm{GSO}_4 \cong (B^\times \times B^\times)/\{(t, t^{-1} : t \in \mathrm{GL}_1\},$$

where B is a quaternion F-algebra (possibly split).

(b) π is the local theta lift of a supercuipidal representation $\tau \boxtimes \omega_\pi$ of

$$\mathrm{GSO}_6 = (\mathrm{GL}_4 \times \mathrm{GL}_1)/\{(t, t^{-2}) : t \in \mathrm{GL}_1\}.$$

We shall treat the two cases in turn.

For Case (a), we shall use the global cuspidal representation Π constructed in 3.5.1, together with all the auxiliary data there (i.e. $\mathbb{E}/\mathbb{F}$, $\daleth$, $\mu_{\mathbb{F}}$. Let Σ be a cuspidal representation of $\mathrm{GL}_2(\mathcal{A}_{\mathbb{E}})$ such that Σ_v is unramified for all $v \neq v_1$ and $\Sigma_{v_1} = \sigma$. Then the same argument as in 3.5.1, using the global functional equation, shows that

$$\gamma_{E,\nu}(s, \pi \times \sigma, \psi) = \gamma(s, \tau_1 \times \sigma, \psi) \cdot \gamma(s, \tau_2 \times \sigma, \psi).$$

In particular, the RHS is independent of (E, ν) and hence so is the LHS.

For Case (b), we use the global cuspidal representation Π constructed in 3.5.2 and the cuspidal representation Σ of GL_2 as in the previous paragraph. Then the global fictional equation , together with the fact that we understand the local gamma factor at all places $v \neq v_1$ (as a consequence of multipicativity), implies that

$$\gamma_{E,\nu}(s, \pi \times \sigma, \psi) = \gamma(s, \tau \times \sigma, \psi).$$

Hence the LHS is independent of (E, ν).

To summarise, we have shown:

Proposition 4.5.2. *Assume that multiplicativity holds for $GSp_4 \times \mathrm{GL}_r$ with $r = 1$ or 2. For an irreducible representation π of GSp_4 with L-parameter ϕ, and an irreducible representation σ of GL_2 with L-parameter ϕ_σ, the local gamma factor $\gamma_{E,\nu}(s, \pi \times \sigma, \psi)$ is independent of the data (E, ν) with respect to which π has nonzero Bessel functional. Moreover,*

$$\gamma_{E,\nu}(s, \pi \otimes \sigma, \psi) = \gamma(s, \phi \otimes \phi_\sigma, \psi).$$

Bibliography

[BU] D. Bump, *Automorphic Forms and Representations*, Cambridge University Press (1997)

[FUR] M. Furusawa, *On L-functions for $GSp(4) \times GL(2)$ and their special values*, Journal fur die Reine und Angewandte Mathematik, Vol 483, pp. 187-218, (1993)

[GT] W. T. Gan and S. Takeda, *The local Langlands conjecture for* GSp(4), Annals of Math, Vol 173, issue 3 (2011), 1841-1882.

[HKS] M. Harris, S. Kudla and W. Sweet, *Theta Dichotomy For Unitary Groups*, Journal of the American Mathematical Society, Vol 4 no. 4, pp. 941-1004, 1996

[H1] R. Howe, *Automorphic forms of low rank*, in *Noncommutative harmonic analysis and Lie groups (Marseille, 1980)*, pp. 211-248, Lecture Notes in Math., 880, Springer, Berlin-New York, 1981.

[H2] R. Howe, *On a notion of rank for unitary representations of the classical groups*, in *Harmonic analysis and group representations*, 223-331, Liguori, Naples, 1982.

[LR] E. Lapid and S. Rallis, *On the local factors of representations of classical groups, Automorphic representations, L-functions and applications: progress and prospects 305-359*, Ohio State Univ. Math. Res. Inst. Publ. 11, ,de Gruyter, Berlin (2005)

[L1] J. S. Li, *Singular unitary representations of classical groups*, Invent. Math. 97 (1989), no. 2, 237-255.

[L2] J. S. Li, *Distinguished cusp forms are theta series*, Duke Math. J. 59 (1989), no. 1, 175-189.

[L3] J.S. Li, *Nonexistence of singular cusp forms*, Compositio Math. 83 (1992), no. 1, 43-51.

[MOR] K. Morimoto, *On L-Functions for Quaternion Unitary Groups of Degree 2 and $GL(2)$*, Int Math Res Notices, Vol 2013, Issue 24 (2013).

[PS] L. Piatetski-Shapiro, *L-Functions for GSp_4*, Pacific Journal of Mathematics, Vol 181 no. 3 (1997).

[PSP] D. Prasad and R. Schulze-Pillot, *Generalized form of a conjecture of Jacquet and a local consequence*, J. Reine Angew Math. 616 (2007), 219-236.

[PT] D.Prasad and R. Takloo-Bighash, *Bessel models for* GSp(4), J. Reine Angew. Math. Vol. 655 (2011), 189-243.

[Sh] F. Shahidi, *Langlands-Shahidi method*, in *Automorphic forms and applications*, IAS/Park City Math. Series Vol. 12, AMS (2007), 297-330.

[SU] , T. Sugano, *On holomorphic cusp forms on quaternion unitary groups of degree 2*, J. Fac. Sci. Univ. Tokyo Sect. IA. Math., vol 31, pp. 521-568 1984.

[TA] J. Tate, *Fourier analysis in number fields, and Hecke's zeta-functions(Cassels and Frohlich, ed.)*, Academic Press 1968.