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# Properties of the $A_\infty$-Structure on Primitive Forms and its Cohomology

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for the degree of

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by

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# CURRICULUM VITAE

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# ABSTRACT OF THE DISSERTATION


Properties of the $A_\infty$-Structure on Primitive Forms and its Cohomology

By

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Doctor of Philosophy in Mathematics

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We study a symplectic cohomology, $PH^*_\pm(X,\omega)$, defined on any symplectic manifold $(X,\omega)$, introduced by Tseng and Yau. As a main application, we analyze two different fibrations of a link complement $M^3$ constructed by McMullen-Taubes, and studied further by Vidussi. These examples lead to inequivalent symplectic forms $\omega_1$ and $\omega_2$ on $X = S^1 \times M^3$, which can be distinguished by the dimension of the primitive cohomologies of differential forms. We provide a general algorithm for computing the monodromies of the fibrations explicitly, which are needed to determine the primitive cohomologies. We also investigate a similar phenomenon coming from fibrations of a class of graph links, whose primitive cohomology provides information about the fibration structure. We then study the $A_\infty$-structure on the differential forms underlying $PH^*_\pm(X,\omega)$. We use this $A_\infty$-structure to generalize classic notions such as Massey products and twisted differentials. These tools capture more information on certain symplectic 4-manifolds compared to the DGA structure on $H^*(X)$.

# Chapter 1

# Symplectic and Cohomological Background

## 1.1 Introduction

In this chapter, we review the necessary basics in symplectic geometry and cohomology. We begin by introducing the $\mathfrak{sl}(2)$-representation on the space of differential forms on a symplectic manifold. The highest weight vectors under this representation form an important sub-algebra known as primitive forms. This algebra is used to construct a symplectic cohomology. We cover the construction of the differentials and properties of this algebra, investigating several examples. We end with a discussion of Massey products and $A_\infty$-algebras. In particular, we recap the underlying $A_3$-structure on primitive forms given in [15]. This $A_3$-structure will be used in Chapter 4, when we introduce primitive Massey products.

## 1.2 Primitive Forms and $\mathfrak{sl}_2$-Representation

Let $(M^{2n}, \omega, g)$ be a symplectic manifold. We define the following operators on $\Omega^*(M)$, its space of differential forms:

$$\begin{aligned} L :& \Omega^k(M) \to \Omega^{k+2}(M) \\ & A_k \mapsto \omega \wedge A_k \\ \Lambda :& \Omega^k(M) \to \Omega^{k-2}(M) \\ & A_k \mapsto \frac{1}{2}(\omega^{-1})^{ij} \iota_{e_i} \iota_{e_j} A_k \\ H :& \Omega^k(M) \to \Omega^k(M) \\ & A_k \mapsto (n-k)A_k \end{aligned}$$

where $\{e_i\}$ is an orthonormal basis for $T^*M$ with respect to $g$. Here, $\Lambda$ is the formal adjoint of $L$.

**Proposition 1.2.1.** *$\Omega^*(M)$ is an $\mathfrak{sl}_2$-module with respect to the operators $(L, \Lambda, H)$. That is, the following identities hold*

$$[H, \Lambda] = 2\Lambda, \tag{1.1}$$

$$[H, L] = -2L, \tag{1.2}$$

$$[\Lambda, L] = H. \tag{1.3}$$

*Proof.* Identities (1.1) and (1.2) follow easily from degree considerations. For equation (1.3), choose local Darboux coordinates $(p_1, \cdots, p_n, q_1, \cdots, q_n)$. It follows that $L = \sum_k (dp_k \wedge dq_k) \wedge$

and $\Lambda = \sum_k \iota_{\frac{\partial}{\partial q_k}} \iota_{\frac{\partial}{\partial p_k}}$. Using these formulas, and the interior product Leibniz rule, shows

$$\begin{aligned}
\Lambda L &= \sum_{k,i} \iota_{\frac{\partial}{\partial q_k}} \iota_{\frac{\partial}{\partial p_k}} dp_i \wedge dq_i \wedge = \iota_{\frac{\partial}{\partial q_k}} \left[ \delta_{ik} dq_i \wedge + dp_i \wedge dq_i \wedge \iota_{\frac{\partial}{\partial p_k}} \right] \\
&= \sum_{i,k} \delta_{ik} I - \delta_{ik} dp_i \wedge \iota_{\frac{\partial}{\partial p_k}} - \delta_{ik} dq_i \wedge \iota_{\frac{\partial}{\partial q_k}} + dp_i \wedge dq_i \wedge \iota_{\frac{\partial}{\partial q_k}} \iota_{\frac{\partial}{\partial p_k}} \\
&= L\Lambda + \sum_k I - dp_k \wedge \iota_{\frac{\partial}{\partial p_k}} - dq_k \wedge \iota_{\frac{\partial}{\partial q_k}} \\
\implies [\Lambda, L] &= nI - \sum_k dp_k \wedge \iota_{\frac{\partial}{\partial p_k}} + dq_k \wedge \iota_{\frac{\partial}{\partial q_k}}.
\end{aligned}$$

Now for an s-form $A$, write $A = \sum_{|I|+|J|=s} A_{I,J} dp_I \wedge dq_J$. Then

$$\begin{aligned}
\sum_k (dp_k \wedge \iota_{\frac{\partial}{\partial p_k}} + dq_k \wedge \iota_{\frac{\partial}{\partial q_k}}) A &= \sum |I| A_{I,J} dp_I \wedge dq_J + |J| A_{I,J} dp_I \wedge dq_J \\
&= \sum (|I| + |J|) A_{I,J} dp_I \wedge dq_J = sA.
\end{aligned}$$

Combining the above computations yields $[\Lambda, L]A = (n - s)A = H(A)$, as required. □

The $\mathfrak{sl}_2$-representation given in Proposition 1.2.1 leads to the following definition.

**Definition 1.2.1.** A k-form ($k \leq n$) $B_k$ is called *primitive* if $\Lambda B_k = 0$.

We denote the space of all primitive forms on $M$ by $\mathcal{P}^*(M)$. Standard representation theory applied to the $\mathfrak{sl}_2$-module $\Omega^*(M)$ gives the *Lefschetz decomposition* by primitive forms:

$$\Omega^k(M) = \bigoplus_p L^p \mathcal{P}^{k-2p}(M).$$

Hence, every $k$-form $A_k$ admits a decomposition $A_k = B_k + \omega \wedge B_{k-2} + \omega^2 \wedge B_{k-4} + \cdots$ where each $B_i$ is primitive. This expression furnishes two more operators

$$\begin{aligned} L^{-p} &: \Omega^k(M) \to \Omega^{k-2p}(M) \\ &A_k \mapsto B_{k-2p} + \omega \wedge B_{k-2p-2} + \omega^2 \wedge B_{k-2p-4} + \cdots \\ \Pi^p &: \Omega^k(M) \to \Omega^k(M) \\ &A_k \mapsto B_k + \omega \wedge B_{k-2} + \cdots + \omega^p B_{k-2p} \end{aligned}$$

Intuitively, $L^{-p}$ removes $\omega^p$ from the decomposition of $A_k$ and $\Pi^p$ project onto the first $p+1$ factors. This primitive decomposition provides a useful characterization of $\mathcal{P}^*(M)$.

**Proposition 1.2.2.** *Let $B_k \in \mathcal{P}^k(M)$. The following statements are equivalent:*

*(i)* $\Lambda(B_k) = 0$,

*(ii)* $L^{n-k+1}B_k = 0$ *and* $L^s(B_k) \neq 0$ *for* $s < n-k+1$.

*Proof.* Using identity (1.3) of Proposition 1.2.1, and an easy induction argument, we have

$$\begin{aligned} \Lambda L(B_k) &= L\Lambda(B_k) + (n-k)B_k, \\ \Lambda L^2(B_k) &= L\Lambda(LB_k) + (n-k-2)LB_k = (L^2\Lambda + [2(n-k)-2]L)B_k, \\ \Lambda L^3(B_k) &= (L^2\Lambda + [2(n-k)-6]L)LB_k = (L^3\Lambda + [3(n-k)-6]L^2)B_k \\ &\vdots \\ \Lambda L^p(B_k) &= L^p\Lambda(B_k) + (p(n-k) - p(p-1))L^{p-1}B_k = L^p\Lambda(B_k) + p(n-k+1-p)L^{p-1}B_k. \end{aligned} \tag{1.4}$$

Now, suppose $\Lambda B_k = 0$ and $L^s(B_k) \neq 0$, $L^{s+1}(B_k) = 0$. Setting $p = s+1$, with our assumption on $B_k$, reduces the last equation in (1.4) to

$$0 = (s+1)(n-k-s)L^s B_k.$$

Since $L^s(B_k) \neq 0$, this implies $s+1 = n-k+1$, as desired.

For the other direction, expand $\Lambda B_k = B_{k-2} + \omega \wedge B_{k-4} + \cdots$. Using equation (1.4) with $p = n - k + 1$ yields

$$0 = L^{n-k+1}(\Lambda B_k) = L^{n-k+1} B_{k-2} + L^{n-k+2} B_{k-4} + \cdots + L^{n-k+i} B_{k-2i} + \cdots \tag{1.5}$$

We have already established above that $\Lambda(B_{k-2i}) = 0$ implies $L^{n-k+2i+1} B_{k-2i} = 0$ and is non-zero for any smaller power. Consequently, the only way for Equation (1.5) to hold is if each $B_i = 0$. Hence $\Lambda B_k = 0$, completing the proof. □

## 1.3 Primitive Differentials and Cohomology

Having established the existence of primitive forms, we now review the differential $m_1$ on $\mathcal{P}^*(M)$. Its explicit definition will depend on the grading of the form in $\mathcal{P}^*(M)$. Given $A_k \in \Omega^k(M)$, we may expand $dA_k = B_{k+1} + \omega \wedge B_{k-1} + \omega^2 \wedge B_{k-3} + \cdots = B_{k+1} + \omega \wedge (B_{k-1} + \omega \wedge B_{k-3} + \cdots)$ and define operators

$$\begin{aligned}
&\partial_+ : \Omega^k(M) \to \Omega^{k+1}(M) \\
&\qquad A_k \mapsto B_{k+1} \\
&\partial_- : \Omega^k(M) \to \Omega^{k-1}(M) \\
&\qquad A_k \mapsto B_{k-1} + \omega \wedge B_{k-3} + \cdots
\end{aligned}$$

If $A_k$ is primitive, then

$$\begin{aligned}
dL^{n-k+1} A_k = 0 = L^{n-k+1} dA_k& \\
&= \omega^{n-k+1} \wedge B_{k+1} + \omega^{n-k+2} \wedge (B_{k-1} + \omega \wedge B_{k-3} + \cdots) \\
&= \omega^{n-k+3} \wedge B_{k-3} + \omega^{n-k+4} \wedge B_{k-5} + \cdots ,
\end{aligned}$$

and so by the Lefschetz decomposition, we have $dA_k = B_{k+1} + \omega \wedge B_{k-1}$. Thus when restricted to primitive forms, the above operators simplify to $\partial_\pm : \mathcal{P}^k(M) \to \mathcal{P}^{k\pm1}(M)$. By construction, note that in general $d = \partial_+ + \omega \wedge \partial_-$. This observation, with the fact that $d^2 = 0$, leads to the following identities.

**Proposition 1.3.1.** *The operators $\partial_\pm$ satisfy*

*(i)* $\partial_+^2 = 0 = \partial_-^2$

*(ii)* $L(\partial_+\partial_- + \partial_-\partial_+) = 0$

Note that as a corollary of Proposition 1.3.1, $\partial_\pm$ are in fact differentials on $\mathcal{P}^*(M)$. These differentials fit together in one chain complex, but with two copies of $\mathcal{P}^*(M)$. To do so, we introduce another copy $\bar{\mathcal{P}}^*(M)$ with grading $\left|\bar{\mathcal{P}}^k(M)\right| = 2n-k+1$. Thus $\partial_+$ and $\partial_-$ increase the degree on $\mathcal{P}^*(M)$ and $\bar{\mathcal{P}}^*(M)$, respectively. We connect the two complexes with $\partial_+\partial_-$ to obtain the chain complex

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{P}^0 & \xrightarrow{\partial_+} & \mathcal{P}^1 & \xrightarrow{\partial_+} & \mathcal{P}^2 & \xrightarrow{\partial_+} & \cdots & \xrightarrow{\partial_+} & \mathcal{P}^n \\ & & & & & & & & & & \big\downarrow{\scriptstyle \partial_+\partial_-} \\ 0 & \longleftarrow & \bar{\mathcal{P}}^0 & \xleftarrow{\partial_-} & \bar{\mathcal{P}}^1 & \xleftarrow{\partial_-} & \bar{\mathcal{P}}^2 & \xleftarrow{\partial_-} & \cdots & \xleftarrow{\partial_-} & \bar{\mathcal{P}}^n \end{array}$$

which satisfies $(\partial_+\partial_-) \circ \partial_+ = 0 = \partial_- \circ (\partial_+\partial_-)$, by a careful application of Proposition 1.3.1. Its cohomologies, called the primitive cohomologies, are denoted by

$$PH_+^k(M,\omega) = \frac{\ker\left(\partial_+ : \mathcal{P}^k \to \mathcal{P}^{k+1}\right)}{\operatorname{Im}\left(\partial_+ : \mathcal{P}^{k-1} \to \mathcal{P}^k\right)}, \quad PH_-^k(M,\omega) = \frac{\ker\left(\partial_- : \bar{\mathcal{P}}^k \to \bar{\mathcal{P}}^{k-1}\right)}{\operatorname{Im}\left(\partial_- : \bar{\mathcal{P}}^{k+1} \to \bar{\mathcal{P}}^k\right)} \tag{1.6}$$

for $k < n$ and

$$PH_+^n(M,\omega) = \frac{\ker\left(\partial_+\partial_- : \mathcal{P}^n \to \bar{\mathcal{P}}^n\right)}{\operatorname{Im}\left(\partial_+ : \mathcal{P}^{n-1} \to \mathcal{P}^n\right)}, \quad PH_-^n(M,\omega) = \frac{\ker\left(\partial_- : \bar{\mathcal{P}}^n \to \bar{\mathcal{P}}^{n-1}\right)}{\operatorname{Im}\left(\partial_+\partial_- : \mathcal{P}^n \to \bar{\mathcal{P}}^n\right)} \tag{1.7}$$

This notation will simply be abbreviated to $PH_\pm^*(M)$ when the choice of symplectic structure is clear. We now consider some examples of $PH^*(M,\omega)$, which illustrate key differences between de Rham and primitive cohomology.

**Example 1.3.1.** Let $\Sigma_g$ be a closed surface of genus $g$ with symplectic form $\omega_\Sigma$. We note that in general, all 0-forms and 1-forms are automatically primitive since $L^{n+1}A_0$ and $L^n A_1$ are $2n+2$ and $2n+1$ forms, respectively. Furthermore on 0-forms, $\partial_+ B_0 = dB_0$. Thus the relevant chain complex is

$$\begin{array}{ccccc} 0 & \longrightarrow & \Omega^0(\Sigma_g) & \xrightarrow{d} & \Omega^1(\Sigma_g) \\ & & & & \big\downarrow \partial_+\partial_- \\ 0 & \longleftarrow & \bar{\Omega}^0(\Sigma_g) & \xleftarrow{\partial_-} & \bar{\Omega}^1(\Sigma_g) \end{array}$$

It follows immediately that $PH^0_+(\Sigma_g) = \ker\left(d : \Omega^0(\Sigma_g) \to \Omega^1(\Sigma_g)\right) = H^0(\Sigma_g)$. Moving on to $PH^1_+(\Sigma_g)$, consider $B_1 \in \ker\left(\partial_+\partial_- : \Omega^1(\Sigma_g) \to \Omega^1(\Sigma_g)\right)$. Writing $dB_1 = B_0\omega_\Sigma$, we have $\partial_+ B_0 = dB_0 = 0$ so that $B_0 \in H^0(\Sigma_g)$. But this implies $\omega_\Sigma$ is exact unless $B_0 = 0$. Since $\Sigma_g$ is compact, we conclude $dB_1 = 0$. Hence

$$\ker\left(\partial_+\partial_- : \Omega^1(\Sigma_g) \to \Omega^1(\Sigma_g)\right) = \ker\left(d : \Omega^1(\Sigma_g) \to \Omega^2(\Sigma_g)\right),$$

and so $PH^1_+(\Sigma_g) = H^1(\Sigma_g)$. Similar considerations show

$$\begin{aligned} PH^1_-(\Sigma_g) &= \frac{\ker(\partial_- : \Omega^1(\Sigma_g) \to \Omega^0(\Sigma_g))}{\mathrm{Im}(\partial_+\partial_- : \Omega^1(\Sigma_g) \to \Omega^1(\Sigma_g))} \\ &= \frac{\ker(d : \Omega^1(\Sigma_g) \to \Omega^2(\Sigma_g))}{\mathrm{Im}(d : \Omega^0(\Sigma_g) \to \Omega^1(\Sigma_g))} \\ &= H^1(\Sigma_g). \end{aligned}$$

Finally,

$$\begin{aligned} PH^0_-(\Sigma_g) &= \frac{\Omega^0(\Sigma_g)}{Im(\partial_- : \Omega^1(\Sigma_g) \to \Omega^0(\Sigma_g))} \\ &= \frac{\Omega^2(\Sigma_g)}{Im(d : \Omega^1(\Sigma_g) \to \Omega^2(\Sigma_g))} \\ &= H^2(\Sigma_g). \end{aligned}$$

We summarize the groups below:

$$\begin{aligned} &PH^0_+(\Sigma_g) = H^0(\Sigma_g), \quad PH^1_+(\Sigma_g) = H^1(\Sigma_g), \\ &PH^1_-(\Sigma_g) = H^1(\Sigma_g), \quad PH^0_-(\Sigma_g) = H^2(\Sigma_g) \cong H^0(\Sigma_g). \end{aligned}$$

Hence in this case, the primitive cohomology is two copies of the de Rham cohomology, with grading given by $|PH^k_+(\Sigma_g)| = k$, $|PH^k_-(\Sigma_g)| = 3-k$. We also note that this conclusion does not depend on the choice of symplectic form $\omega_\Sigma$. The next example will show that this occurrence does not always happen.

As with de Rham cohomology, $PH^*(M, \omega)$ can become quite cumbersome to compute directly. Below, we provide a useful theorem from [15] which decomposes primitive cohomology in terms of kernels and cokernels of the Lefschetz maps, $L$. We omit the proof, but interested readers may consult [15] for details of the long-exact sequence. This theorem will be crucial in many computations moving forward.

**Theorem 1.3.1** (Tsai, Tseng, Yau). *Let $(M^{2n}, \omega)$ be a symplectic manifold. For integers $k \leq n$, the following group isomorphisms hold:*

$$\begin{aligned} &PH^k_+(M, \omega) = \ker\left(L : H^{k-1}(M) \to H^{k+1}(M)\right) \oplus \operatorname{coker}\left(L : H^{k-2}(M) \to H^k(M)\right), \\ &PH^k_-(M, \omega) = \ker\left(L : H^{2n-k}(M) \to H^{2n-k+2}(M)\right) \oplus \operatorname{coker}\left(L : H^{2n-k-1}(M) \to H^{2n-k+1}(M)\right). \end{aligned}$$

**Example 1.3.2.** Let $\mathbb{T}^4$ denote the 4-torus and fix some symplectic form $\omega$. Using Theorem

1.3.1, we know immediately

$$PH^0_+(\mathbb{T}^4,\omega) = H^0(\mathbb{T}^4), \quad PH^0_-(\mathbb{T}^4,\omega) = H^4(\mathbb{T}^4),$$
$$PH^1_+(\mathbb{T}^4,\omega) = H^1(\mathbb{T}^4), \quad PH^1_-(\mathbb{T}^4,\omega) = H^3(\mathbb{T}^4).$$

Furthermore,

$$PH^2_+(\mathbb{T}^4,\omega) = \ker\left(L : H^1(\mathbb{T}^4) \to H^3(\mathbb{T}^4)\right) \oplus \operatorname{coker}\left(L : H^0(\mathbb{T}^4) \to H^2(\mathbb{T}^4)\right),$$
$$PH^2_-(\mathbb{T}^4,\omega) = \ker\left(L : H^2(\mathbb{T}^4) \to H^4(\mathbb{T}^4)\right) \oplus \operatorname{coker}\left(L : H^1(\mathbb{T}^4) \to H^3(\mathbb{T}^4)\right).$$

To get a more concrete representation, choose coordinates $(x_i, y_i)$ and write $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. By the Kunneth formula it follows,

$$H^1(\mathbb{T}^4) = \langle dx_1, dx_2, dy_1, dy_2\rangle,$$
$$H^2(\mathbb{T}^4) = \langle dx_i \wedge dx_j, dx_i \wedge dy_j, dy_i \wedge dy_j\rangle_{1\le i,j\le 2},$$
$$H^3(\mathbb{T}^4) = \langle dx_i \wedge dx_j \wedge dy_k, dx_i \wedge dy_j \wedge dy_k\rangle_{1\le i,j,k\le 2},$$
$$H^4(\mathbb{T}^4) = \langle \omega^2\rangle.$$

These formulas lead to the simplifications

$$PH^2_+(\mathbb{T}^4,\omega) = H^2(\mathbb{T}^4)/\langle\omega\rangle,$$
$$PH^2_-(\mathbb{T}^4,\omega) = \ker\left(L : H^2(\mathbb{T}^4) \to H^4(\mathbb{T}^4)\right) \cong H^2(\mathbb{T}^4)/\langle\omega\rangle,$$
$$H^2(\mathbb{T}^4)/\langle\omega\rangle = \langle x_1 \wedge x_2, y_1 \wedge y_2, x_1 \wedge y_2, x_2 \wedge y_1, x_1 \wedge y_1 - x_2 \wedge y_2\rangle.$$

We note that all the elements of $PH^*(\mathbb{T}^4,\omega)$ are still $d$-closed, but are a proper subset of $H^*(\mathbb{T}^4)$. Furthermore, this example illustrates that an obvious Kunneth formula fails for primitive cohomology. If we write $\mathbb{T}^4 = \Sigma_1 \times \Sigma_1$, we would expect such a formula should give

$$PH^2_+(\mathbb{T}^4) \cong PH^1_+(\Sigma_1) \otimes PH^1_+(\Sigma_1) \oplus PH^1_-(\Sigma_1) \oplus PH^1_-(\Sigma_1).$$

However, applying Example 1.3.1 gives

$$PH^1_\pm(\Sigma_1) = H^1(\Sigma_1) = \mathbb{R}^2,$$

but

$$PH^2_+(\mathbb{T}^4) = H^2(\mathbb{T}^4)/\langle\omega\rangle \neq \mathbb{R}^2 \otimes \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2.$$

**Example 1.3.3.** We let $X$ be the Kodaira-Thurston manifold $KT^4$, a classic example of a non-Kahler, symplectic manifold. $X$ can be realized as $\mathbb{R}^4$ under the identification $(x_1, y_1, x_2, y_2) \sim (x_1 + a, y_1 + b, x_2 + c, y_2 + d - bx_2)$, $a, b, c, d \in \mathbb{Z}$. One can also view $X$ as $S^1$ times a mapping torus, with monodromy given by a Dehn twist along the meridian of the 2-torus. We take the following basis of 1-forms:

$$e_1 = dx_1, \quad e_2 = dx_2, \quad e_3 = dx_3, \quad e_4 = dx_4 + x_2 dx_3,$$

and define the symplectic form $\omega = e_1 \wedge e_2 + e_3 \wedge e_4$. Using the Wang exact sequence (see Chapter 2) and the Kunneth formula, we can compute the de Rham cohomology to be

$$H^1(X) = \langle e_1, e_2, e_3\rangle,$$

$$H^2(X) = \langle e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\rangle,$$

$$H^3(X) = \langle e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\rangle,$$

$$H^4(X) = \langle\omega^2\rangle.$$

Finally, applying Theorem 1.3.1 yields,

$$PH^0_+(X,\omega) = \mathbb{R},$$
$$PH^1_+(X,\omega) = H^1(X),$$
$$PH^2_+(X,\omega) = H^2(X)/\langle\omega\rangle \oplus \langle e_3\rangle,$$
$$PH^2_-(X,\omega) = \langle e_1\wedge e_3, e_2\wedge e_4, e_1\wedge e_2 - e_3\wedge e_4\rangle \oplus \langle e_1\wedge e_2\wedge e_4\rangle,$$
$$PH^1_-(X,\omega) = H^3(X),$$
$$PH^0_-(X,\omega) = H^4(X).$$

In practice, one often must realize these isomorphisms in terms of explicit primitive forms of appropriate degree. We demonstrate the process on the $e_3$ term appearing in $PH^2_+(X,\omega)$. Notice $\omega\wedge e_3 = d(e_4\wedge e_1)$. We define $B_2 = e_4\wedge e_1$, which indeed is a primitive 2-form since $\omega\wedge B_2 = 0$. Furthermore, $\partial_-(B_2) = e_3$ so that $B_2$ is $\partial_+\partial_-$-closed. Thus $e_3$ corresponds to the explicit form $B_2$ in $PH^2_+(X)$. We point out that $B_2$ is a NON d-closed element, showing $PH^*(X)$ and $H^*(X)$ truly differ. See [17] for more details on this manifold and its various cohomologies.

**Example 1.3.4.** As a final example, consider $\mathbb{CP}^n$ with any symplectic form $\omega$. We may express its de Rham cohomology as $H^{2k}(\mathbb{CP}^n) = \langle\omega^k\rangle$ and $H^{2k+1}(\mathbb{CP}^n) = 0$. It's easy to see $L$ is an isomorphism on cohomology, so that each component of Theorem 1.3.1 is trivial. Hence,

$$PH^0_+(\mathbb{CP}^n) = PH^0_-(\mathbb{CP}^n) = \mathbb{R},$$
$$PH^k_+(\mathbb{CP}^n) = PH^k_-(\mathbb{CP}^n) = 0, \quad 0 < k \leq n.$$

## 1.4 Massey Products and $A_\infty$-structure on $\mathcal{P}^*(M)$

In this section, we review the construction of classic Massey products on de Rham cohomology in order to set conventions on the defining systems and signs. We then provide the definitions of $A_\infty$-algebras and formality, motivated from the perspective of the DGA structure on differential forms. We end with a discussion of the $A_\infty$-structure on $\mathcal{P}^*(M)$, whose explicit maps will be used in later chapters.

### 1.4.1 Classic Massey Products

For our purposes, we need only consider the Massey product $\langle a_1, \cdots, a_k \rangle$ where each $a_i \in H^1(M)$. However, if needed, the below system can be generalized appropriately.

**Definition 1.4.1.** A *defining system* $(a_{ij})$ for the $k-$fold Massey product is an upper-triangular collection of 1-forms satisfying the following properties:

1. $a_{i,j} = 0$ for $i < j$,
2. $a_{i,i}$ is a representative of the cohomology class $[a_i]$,
3. $da_{i,j} = \sum_{r=i}^{j-1} a_{i,r} \wedge a_{r+1,j}$, $(i,j) \neq (1,k)$.

If such a defining system exists, the *Massey product* is the collection of all representatives given by $\sum_{r=1}^{k-1} a_{1,r} \wedge a_{r+1,k}$.

By abuse of notation, when clear, $\langle a_1, \cdots, a_k \rangle$ will be used to denote a specific representative. The above conditions intuitively measure the exactness of consecutive $n$-fold Massey products $(n < k)$. That is, $da_{i,j} = \langle a_i, a_{i+1}, \cdots, a_j \rangle$.

This construction is best illustrated through the 3-point Massey product. This product requires closed 1-forms $a_1, a_2, a_3$ such that $a_1 \wedge a_2 = da_{12}$ and $a_2 \wedge a_3 = da_{23}$. The defining system is summarized by the following matrix

$$\begin{pmatrix} a_1 & a_{12} & * \\ 0 & a_2 & a_{23} \\ 0 & 0 & a_3 \end{pmatrix},$$

where the upper-right entry is the Massey product representative given by $a_1 \wedge a_{23} + a_{12} \wedge a_3$. Given another defining system $(a'_{ij})$ with $a'_{ii} = a_i$, we have $d(a_{ij} - a'_{ij}) = 0$. Hence $a_{12} - a'_{12}$ and $a_{23} - a'_{23}$ descend to representatives in $H^1(M)$. It follows that the difference between the two Massey product representatives satisfies

$$(a_1 \wedge a_{23} + a_{12} \wedge a_3) - (a_1 \wedge a'_{23} + a'_{12} \wedge a_3) = a_1 \wedge (a_{23} - a'_{23}) + (a_{12} - a'_{12}) \wedge a_3 \in \langle a_1 \rangle \wedge H^1(M) + H^1(M) \wedge \langle a_3 \rangle.$$

Therefore, in the case of the 3-point Massey product, we can define the representative $\langle a_1, a_2, a_3 \rangle$ to be an element in $H^2(M) / [\langle a_1 \rangle \wedge H^1(M) + H^1(M) \wedge \langle a_3 \rangle]$. In general, the higher ($k > 3$) Massey products don't have such a quotient space.

For a concrete example of the 3-fold Massey product, we turn to the symplectic manifold in Ex 1.3.3.

**Example 1.4.1** ($KT^4$)**.** The only elements in the kernel of the wedge product are $e_2$ and $e_3$, given by $e_2 \wedge e_3 = de_4$. Hence we may consider the product $\langle e_3, e_2, e_3 \rangle$, where $a_{12} = -e_4$ and $a_{23} = e_4$. After considering the quotient, the above formulation gives a non-trivial representative $\langle e_3, e_2, e_3 \rangle = 2e_{34}$. In particular, the Kodaira-Thurston manifold has a non-zero Massey product.

To see the importance of these products, we take a digression into $A_\infty$ algebras.

### 1.4.2 $A_\infty$-Algebras and Formality

Given a differential graded algebra (DGA) $(A, d, \cdot)$, the multiplication $\cdot$ is assumed to be associative. A familiar example of a DGA is differential forms on a manifold, $(\Omega^*(M), d, \wedge)$.

The idea of an $A_\infty$-algebra is to generalize this structure to the case where the multiplication is not associative, introducing higher maps to measure the failure. We give the formal definition below.

**Definition 1.4.2** ($A_\infty$-Algebra)**.** An $A_\infty$-algebra over a field $k$ is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p\in\mathbb{Z}} A^p$$

with graded maps

$$m_k : A^{\otimes k} \to A, \quad k \geq 1,$$

of degree $2-k$ satisfying

$$\sum_{k=r+s+t} (-1)^{r+st} m_{1+r+t}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0.$$

To be clear, we use the Koszul sign rule, stating $(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b)$. The first three maps are given by the defining equations

$$\begin{aligned}
m_1 m_1(a) &= 0,\\
m_1 m_2(a,b) &= m_2(m_1(a), b) + (-1)^{|a|} m_2(a, m_1(b)),\\
m_2(a, m_2(b,c)) - m_2(m_2(a,b),c) &= m_1 m_3(a,b,c) + m_3(m_1(a),b,c)\\
&\quad + (-1)^{|a|} m_3(a, m_1(b), c) + (-1)^{|a|+|b|} m_3(a,b,m_1(c)).
\end{aligned}$$

The first equation says $m_1$ is a differential, the second equation says $m_2$ satisfies the Leibniz rule, and the last equation states that the associator of $m_2$ is homotopic to 0 given by the differential of $m_3$ in the morphism complex. In this thesis, we focus on $A_3$-algebras, $A_\infty$-algebras where $m_k = 0$ for $k > 3$. See [5] for a more in depth discussion of $A_\infty$-algebras in general.

Given a *DGA* $(A, d, m_2)$, a theorem of Kadeishvili says there is a unique $A_\infty$-structure on $H^*(A)$ making $A$ and $H^*(A)$ *quasi-isomorphic* as $A_\infty$-algebras. We omit the precise definition, but informally, one may think of a quasi-isomorphism as a morphism of $A_\infty$-algebras inducing an isomorphism on cohomology. If this structure on the cohomology remains a $DGA$, then $A$ is called *formal*.

**Definition 1.4.3** (Formal)**.** A $DGA$ $(A, d, m_2)$ is called *formal* if the $A_\infty$-structure on $H^*(A)$ making $A$ and $H^*(A)$ quasi-isomorphic is still a $DGA$. A manifold $M$ is called formal if $(\Omega^*(M), d, \wedge)$ is formal.

A folklore theorem states that a formal manifold $M$ has all Massey products trivial on $H^*(M)$. Let $(m_k)$ denote the $A_\infty$-model on $H^*(M)$. This fact follows from showing that when $\langle a_1, a_2, \cdots, a_k\rangle$ is defined, then a cohomology representative of it is given by $m_k(a_1, a_2, \cdots, a_k)$. Since $m_k = 0$ for $k > 2$, all 3-point and higher products must vanish. A famous paper of Deligne, Griffiths, Morgan, and Sullivan proves that every Kahler manifold is formal. Using this Massey product theorem shows that the Kodaira-Thurston manifold of Example 1.3.3 is NOT formal and therefore cannot be Kahler.

### 1.4.3 $A_3$-structure on $\mathcal{P}^*(M)$

In [15], an $A_\infty$ structure $(m_1, \times, m_3)$ was introduced on $\mathcal{P}^*(M)$. We already saw the construction of $m_1$ in Section 1.3. This differential is given by

$$m_1(B_k) = \begin{cases} \partial_+(B_k), & 0 \leq k < n \\ -\partial_+\partial_-(B_k), & k = n \\ -\partial_-(B_k), & n+1 \leq k \leq 2n+1 \end{cases}$$

The addition of minus signs is to account for the Leibniz rule required on $m_2$ in Definition 1.4.2. For the multiplication, we omit the derivation and simply give the formulas. Recall our grading on $\mathcal{P}^*(M)$ given by $|\mathcal{P}^k(M)| = k$ and $|\overline{\mathcal{P}}^k(M)| = 2n+1-k$ for $0 \leq k \leq n$.

Furthermore we introduce the map (see [15] for more motivation)

$$\begin{aligned} *_r :&\Omega^k(M) \to \Omega^k(M) \\ &A_k \mapsto L^{n-k} A_k. \end{aligned}$$

Then the product $\times : \mathcal{P}^i(M) \otimes \mathcal{P}^j(M) \to \mathcal{P}^{i+j}(M)$ is defined as

$$A_j \times A_k = \begin{cases} \Pi^0(A_j \wedge A_k), & j+k \leq n \\ \Pi^0 *_r \left[-dL^{-1}(A_j \wedge A_k) + (L^{-1}dA_j) \wedge A_k + (-1)^j A_j \wedge (L^{-1}dA_k)\right], & j+k > n \end{cases} \tag{1.8}$$

$$A_j \times \overline{A}_k = (-1)^j *_r (A_j \wedge (*_r \overline{A}_k)), \tag{1.9}$$

$$\overline{A_j} \times A_k = *_r((*_r \overline{A}_j) \wedge A_k), \tag{1.10}$$

$$\overline{A}_j \times \overline{A}_k = 0. \tag{1.11}$$

**Example 1.4.2.** To demonstrate that $\times$ is not associative we quickly revisit the Kodaira-Thurston manifold. We compute,

$$\begin{aligned} (e_1 \times e_2) \times e_4 &= \frac{1}{2}(e_1 \wedge e_2 - e_3 \wedge e_4) \times e_4 \\ &= \Pi^0 *_r \left[-\frac{1}{2} dL^{-1}(e_1 \wedge e_2 \wedge e_4) + \frac{1}{2}(e_1 \wedge e_2 - e_3 \wedge e_4) \wedge L^{-1}(e_2 \wedge e_3)\right] \\ &= -\frac{1}{2} e_2 \wedge e_3, \\ e_1 \times (e_2 \times e_4) &= e_1 \times (e_2 \wedge e_4) = \Pi^0 *_r \left[-dL^{-1}(e_1 \wedge e_2 \wedge e_4)\right] \\ &= -e_2 \wedge e_3. \end{aligned}$$

As it turns out, we need only introduce one higher map $m_3$ to measure the failure of associativity. Again omitting details, we provide the formulas below. The $m_3$ will only be non-trivial on gradings $(i, j, k)$ with $i, j, k < n$ and $i + j + k > n$. It is given by,

$$m_3(A_i, A_j, A_k) = \begin{cases} 0, & i + j + k < n + 2 \\ \Pi^0 *_r \left[A_i \wedge L^{-1}(A_j \wedge A_k) - L^{-1}(A_i \wedge A_j) \wedge A_k\right], & i + j + k \geq n + 2. \end{cases}$$

A quick check on Example 1.4.2 reveals

$$\begin{aligned} e_1 \times (e_2 \times e_4) - (e_1 \times e_2) \times e_4 &= -\frac{1}{2} e_2 \wedge e_3 \\ &= m_3(e_1, e_2, de_4), \end{aligned}$$

as expected. This $m_3$ map will reappear in Chapter 4 when we introduce primitive Massey products and investigate the structure on some symplectic 4-manifolds.

# Chapter 2

# Fibered 3-Manifold Background

In this chapter, we apply the theory from Chapter 1 to a symplectic 4-manifold associated to surface bundles. We consider different symplectic forms and determine its effect on the primitive cohomology. We conclude with the necessary theory of fibered 3-manifolds, discussing the mapping class group and its generators for a four-times punctured torus.

## 2.1 de Rham and Primitive Cohomologies

In this section, we briefly review the basics of the de Rham cohomology of surface bundles over a circle. We then apply primitive cohomology studied in Chapter 1 to a symplectic 4-manifold associated to surface bundles.

Let $\Sigma_{g,n} = \Sigma_g - \{y_1, \cdots, y_n\}$ be a Riemann surface of genus $g$ with $n$ points removed. When clear, the surface will simply be abbreviated by $\Sigma$. Moreover, when convenient, $P := \{y_1, \cdots, y_n\}$ may be thought of as marked points. We endow $\Sigma$ with a symplectic form $\omega_\Sigma$ and let $f : \Sigma \to \Sigma$ be any symplectic diffeomorphism preserving $P$ setwise. Form the 3-dimensional mapping torus $Y_f = \Sigma \times [0, 1]/(x, 1) \sim (f(x), 0)$. It follows that $Y_f$ has a $\Sigma$-bundle structure over $S^1$ with the projection given by $\pi : Y_f \to S^1$, $\pi([x, t]) = t$. The associated map $f$ is called the *monodromy* of the bundle and determines the de Rham cohomology according to the Wang exact sequence

$$\cdots \longrightarrow H^0(\Sigma) \longrightarrow H^1(Y_f) \longrightarrow H^1(\Sigma) \xrightarrow{f^*-1} H^1(\Sigma) \longrightarrow H^2(Y_f) \longrightarrow \cdots$$

This sequence yields

$$\begin{aligned}
&H^0(Y_f) = \mathbb{R}, \\
&H^1(Y_f) = \ker(f^* - 1 : H^1(\Sigma) \to H^1(\Sigma)) \oplus \langle d\pi \rangle, \\
&H^2(Y_f) = \langle d\pi \rangle \wedge \operatorname{coker}(f^* - 1 : H^1(\Sigma) \to H^1(\Sigma)), \\
&H^3(Y_f) = 0,
\end{aligned}$$

where $d\pi = \pi^*(d\theta)$ is the pullback under $\pi$ of the volume form on $S^1$.

Next we construct a symplectic manifold $X = S^1 \times Y_f$ with symplectic form $\omega = dt \wedge d\pi + \omega_\Sigma$. Here, $dt$ is the volume form on the second $S^1$ factor and $\omega_\Sigma$ (by abuse of notation) is a global closed 2-form on $Y_f$ which restricts to the symplectic form on each fiber. The Kunneth formula easily shows

$$\begin{aligned}
&H^0(X) = \mathbb{R}, \\
&H^1(X) = \langle dt, d\pi \rangle \oplus \ker(f^* - 1 : H^1(\Sigma) \to H^1(\Sigma)), \\
&H^2(X) = \langle dt \wedge d\pi \rangle \oplus d\pi \wedge \operatorname{coker}(f^* - 1 : H^1(\Sigma) \to H^1(\Sigma)) \oplus dt \wedge \ker(f^* - 1 : H^1(\Sigma) \to H^1(\Sigma)), \\
&H^3(X) = \langle dt \wedge d\pi \rangle \wedge \operatorname{coker}(f^* - 1 : H^1(\Sigma) \to H^1(\Sigma)), \\
&H^4(X) = 0.
\end{aligned}$$

Let us first discuss the case where $\omega$ is chosen so that $[\omega]_{dR} = [dt \wedge d\pi]_{dR}$, the more general case will be treated at the end of the section. Applying Theorem 1.3.1 to the 4-manifold

$X = S^1 \times Y_f$, along with the computations from above, yields

$$PH^0_+(X) \cong \mathbb{R},$$

$$PH^1_+(X) \cong H^1(X),$$

$$PH^2_+(X) \cong H^2(X)/\langle dt \wedge d\pi \rangle \oplus \langle dt, d\pi \rangle \oplus [\ker(f^* - 1) \cap \operatorname{Im}(f^* - 1)],$$

$$PH^2_-(X) \cong H^2(X) \oplus [\langle dt \wedge d\pi \rangle \wedge \operatorname{coker}(f^* - 1)] \,/\, [\langle dt \wedge d\pi \rangle \wedge \ker(f^* - 1)] \,,$$

$$PH^1_-(X) \cong H^3(X),$$

$$PH^0_-(X) \cong 0.$$

Let $b_i$ denote the Betti numbers of $X$ and $p_i^{\pm}(X, \omega)$ denote the dimensions of $PH^i_{\pm}(X, \omega)$. When the choice of the underlying symplectic structure is clear, we simply write $p_i^{\pm}$. Then,

$$p_0^+ = 1,$$

$$p_1^+ = b_1,$$

$$p_2^+ = b_2 + 1 + \dim [\ker(f^* - 1) \cap \operatorname{Im}(f^* - 1)] \,,$$

$$p_2^- = b_2 + \dim [\ker(f^* - 1) \cap \operatorname{Im}(f^* - 1)] \,,$$

$$p_1^- = b_3,$$

$$p_0^- = 0,$$

where we have used the fact that $\dim [\ker(f^* - 1) \cap \operatorname{Im}(f^* - 1)]$ and $\dim [(dt \wedge d\pi \wedge \operatorname{coker}(f^* - 1))/(dt \wedge d\pi \wedge \ker(f^* - 1))]$ are equal by realizing that both quantities count the number of Jordan blocks of $f^* - 1$ of size strictly greater than 1 (see discussion below). We note that the *primitive Euler characteristic* $\chi_p(X) = \sum(-1)^i p_i^+ - \sum(-1)^i p_i^- = 2 - b_1 + b_3$ is fixed under homeomorphism type. However, the primitive Betti numbers $p_2^{\pm}$ may vary in general.

Let us explain how this dimension relates to the Jordan blocks of $f^* - 1$. For brevity we write $\nu_2 := \dim [\ker(f^* - 1) \cap \operatorname{Im}(f^* - 1)]$. Now if $\alpha \in \ker(f^* - 1) \cap \operatorname{Im}(f^* - 1)$, then

$(f^* - 1)\alpha = 0$ and $(f^* - 1)\beta = \alpha$ for some $\beta$. That is, $\alpha$ is an eigenvector in a Jordan chain of length at least 2. It follows that $\nu_2$ counts the number of Jordan blocks corresponding to eigenvalue $\lambda = 1$ of size *at least* 2. More generally there is a descending filtration of subgroups $PH^2_+(M) \supset J_1(M) \supset J_2(M) \supset \cdots$ where $J_k(M) = \ker(f^* - 1) \cap \mathrm{Im}(f^* - 1)^k$. If $\alpha \in J_k(M)$, then it is the eigenvector in a Jordan chain of length at least $k+1$ given by $x_1 = \alpha$, $x_2 = (f^* - 1)^{k-1}\beta$, $x_3 = (f^* - 1)^{k-2}\beta, \cdots$, $x_k = (f^* - 1)\beta$, $x_{k+1} = \beta$. Thus the dimension of the filtered quotient $J_{k-1}/J_k$ counts the number of Jordan blocks of size *exactly* $k$.

We now consider the case where $[\omega] \neq [dt \wedge d\pi]$. Let $i : \Sigma \hookrightarrow Y_f$ be the inclusion map of the fiber and choose $\tilde{\omega}_f \in \Omega^2(Y_f)$ such that $i^*(\tilde{\omega}_f) = \omega_\Sigma$. Furthermore, assume $\tilde{\omega}_f$ can be chosen so that $[\omega_0] := [dt \wedge d\pi + \tilde{\omega}_f] = [dt \wedge d\pi]$. Then $PH^*(X, \omega_0)$ is given by the above computations. Given $\eta \in \Omega^1(Y_f)$ such that $d(\eta \wedge d\pi) = 0$, we can define a new symplectic form, $\omega_\eta := \omega_0 + \eta \wedge d\pi = (dt + \eta) \wedge d\pi + \tilde{\omega}_f$. We wish to choose $\eta$ so that $[\omega_\eta] \neq [\omega_0]$, which holds precisely when $[d\pi \wedge \eta] \in H^2(Y_f)$ is non-trivial. Choose a Jordan basis $\{x_{i,0}\}_{i=1}^k$ for $\ker(f^* - 1)$ and denote the corresponding Jordan chain of $x_{i,0}$ by $\{x_{i,0}, x_{i,1}, \cdots, x_{i,n_i}\}$. Rearranging if necessary, we assume $n_i = 0$ for $1 \leq i \leq s$. Thus $\{x_{i,0}\}_{i=1}^s$ are the Jordan blocks of size exactly 1. Then, we can write

$$H^1(Y_f) = \langle d\pi \rangle \oplus \langle x_{i,0} \rangle_{i=1}^k,$$
$$H^2(Y_f) = \langle d\pi \wedge x_{i,n_i} \rangle_{i=1}^k,$$

and express $[d\pi \wedge \eta] = \sum_{i=1}^k \lambda_i [d\pi \wedge x_{i,n_i}]$. We may write $PH^2_+(X, \omega_\eta) = H^2(X)/\langle[\omega_\eta]\rangle \oplus K_\eta$ where $K_\eta = \ker(\omega_\eta \wedge : H^1(X) \to H^3(X))$. Then

$$[\omega_\eta \wedge d\pi] = [0],$$
$$[\omega_\eta \wedge dt] = [\eta \wedge d\pi \wedge dt] = -[dt \wedge d\pi \wedge \eta],$$
$$[\omega_\eta \wedge x_{i,0}] = [dt \wedge d\pi \wedge x_{i,0}].$$

We see that $[\omega_\eta \wedge (\sum_{i=1}^{s} \lambda_i x_{i,0} + dt)] = [dt \wedge d\pi \wedge \sum_{i=s+1}^{k} \lambda_i x_{i,n_i}]$, which is trivial if and only if $\eta \in \ker(f^* - 1)$. Similarly, denote by $C_\eta = \mathrm{coker}(\omega_\eta \wedge : H^1(X) \to H^3(X))$. The above computations show $C_\eta \cong \langle dt \wedge d\pi \wedge x_{i,n_i} \rangle_{i=s+1}^{k} / \langle dt \wedge d\pi \wedge \eta \rangle$. The quotient by the $\eta$ term will be extraneous in the case that $\eta \in \ker(f^* - 1)$. The groups $PH^*(X, \omega_\eta)$ are recorded below.

$$
\begin{aligned}
&PH^0_+(X, \omega_\eta) \cong H^0(X),\\
&PH^1_+(X, \omega_\eta) \cong H^1(X),\\
&PH^2_+(X, \omega_\eta) \cong H^2(X)/\langle [\omega_\eta] \rangle \oplus K_\eta,\\
&PH^2_-(X, \omega_\eta) \cong H^2(X) \oplus C_\eta,\\
&PH^1_-(X, \omega_\eta) \cong H^3(X),\\
&PH^0_-(X, \omega_\eta) \cong \langle 0 \rangle,
\end{aligned}
$$

where

$$
K_\eta \cong \begin{cases} \langle d\pi \rangle \oplus \langle x_{i,0} \rangle_{i=s+1}^{k}, & \lambda_i \neq 0 \text{ for some } i > s \\ \langle d\pi, dt + \eta \rangle \oplus \langle x_{i,0} \rangle_{i=s+1}^{k}, & \lambda_i = 0 \text{ for all } i > s \end{cases}
$$

$$
C_\eta \cong \begin{cases} \langle dt \wedge d\pi \wedge x_{i,n_i} \rangle_{i=s+1}^{k} / \langle dt \wedge d\pi \wedge \eta \rangle, & \lambda_i \neq 0 \text{ for some } i > s \\ \langle dt \wedge d\pi \wedge x_{i,n_i} \rangle_{i=s+1}^{k}, & \lambda_i = 0 \text{ for all } i > s \end{cases}
$$

Regardless of the class of $\eta$, we see $PH^k_\pm(X, \omega_\eta)$ are isomorphic to de Rham cohomologies for $0 \leq k \leq 1$. Furthermore, in the case that $\eta$ descends to a cohomology class $[\eta] \in H^1(Y_f)$, the above computations show $\dim PH^*(X, \omega_\eta) = \dim PH^*(X, \omega_0)$. Unless otherwise stated in the thesis, we assume $[\omega] = [dt \wedge d\pi]$.

## 2.2 Mapping Class Groups

In this section, we review some of the necessary topics from mapping class group theory. We focus mainly on the mapping class group of $\Sigma_{1,4}$, detailing a set of generators given in [1]. We wish to study the diffeomorphisms of $\Sigma_{g,n}$ up to an equivalence. We define the *mapping class group*, denoted by $\mathcal{M}(\Sigma_{g,n})$, as the group of diffeomorphisms fixing $P$ setwise, up to isotopies fixing $P$ setwise. We define the *pure mapping class group*, $\mathcal{PM}(\Sigma_{g,n})$, as the subset of elements from $\mathcal{M}(\Sigma_{g,n})$ fixing $P$ pointwise. Since the majority of the next chapter takes place in $\mathcal{PM}(\Sigma_{1,4})$ we briefly discuss the diffeomorphisms generating this subgroup for the torus with four marked points. We define $\tau_i$ as the longitudinal curve which passes above $y_1, y_2, \cdots, y_{i-1}$, through $y_i$, and below $y_{i+1}, \cdots, y_n$. Denote by $\rho_i$ the meridian curve passing through $y_i$.

From these curves we define homeomorphisms $\mathcal{P}ush(\tau_i)$ and $\mathcal{P}ush(\rho_i)$, called the point-pushing maps. These are classical maps in mapping class group theory. They may be loosely visualized as follows: $\mathcal{P}ush(\tau_i)$ is the map which pushes the point $x_i$ around the curve $\tau_i$, "dragging" the rest of the surface $\Sigma_{1,4}$ with it. $\mathcal{P}ush(\rho_i)$ has a similar interpretation. In [1], Birman showed that the push maps generate the mapping class group:

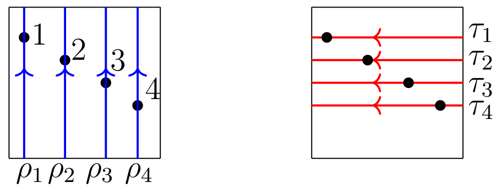
$$\mathcal{PM}(\Sigma_{1,4}) = \langle \mathcal{P}ush(\tau_i), \mathcal{P}ush(\rho_i) \rangle, i = 1, 2, 3, 4.$$

It turns out that these maps can be realized in terms of Dehn twists along homology generators for $H_1(\Sigma_{1,4})$. These explicit expressions are worked out in Section 3.3. (The curves $\rho_i$ and $\tau_i$ are pictured in Figure 2.1, drawn on the square representing $\Sigma_{1,4}$.)

Another important subgroup of the mapping class group is the *Torelli group*, $\mathcal{I}(\Sigma)$, consisting of diffeomorphisms acting trivially on (co)homology. Thus,

$$\mathcal{I}(\Sigma) = \{f \in \mathcal{M}(\Sigma) : 1 = f^* : H^1(\Sigma) \to H^1(\Sigma)\}.$$

Figure 2.1: $\rho_i$ and $\tau_i$ paths on $\Sigma_{1,4}$




Calculations in Section 2.1 show that if $f \in \mathcal{I}(\Sigma)$ then $H^*(S^1 \times Y_f) = H^*(T^2 \times \Sigma)$ and $PH^*(S^1 \times Y_f) = PH^*(T^2 \times \Sigma)$ as groups. Thus two Torelli-bundles cannot be distinguished from their primitive cohomology groups alone. However, by the same reasoning, $f \in \mathcal{I}$ and $g \notin \mathcal{I}$ can *always* be distinguished by the dimension of the cohomology groups.

# Chapter 3

# Homeomorphic 4-folds with Non-Isomorphic Primitive Cohomology

We analyze two classes of fibered 3-manifolds and study the effect of different symplectic structures on the primitive cohomology of the associated symplectic 4-fold. The first class of examples comes from fibrations given in [7] and [20]. Studying the primitive cohomologies of these fibrations requires knowledge of the monodromies explicitly. We provide a detailed algorithm for constructing monodromies coming from fibrations of the type in [7]. Using these calculations, we show a pair of inequivalent symplectic structures are distinguished by their primitive cohomologies. The second class of examples arise from a graph link provided in [19]. In this class, the primitive cohomology provides information about the fibration structure of the graph link.

## 3.1 McMullen-Taubes Type 4-manifolds

In this section, we will discuss different presentations of a 3–manifold, the complement of a link in $S^3$, as fibration with fiber a punctured torus or sphere. All the torus fiber examples will induce symplectic structures with identical primitive cohomologies but the sphere fibration will be shown to give primitive cohomology of different dimension.

We quickly review the examples constructed in [7] and [20]. In [7], McMullen and Taubes considered a 3-manifold $M$ which is a link complement $S^3 \backslash K$. Here, $K$ is the Borromean rings $K_1 \cup K_2 \cup K_3$ plus $K_4$, the axis of symmetry of the rings. By performing 0-surgery along the Borromean rings we obtain a presentation of $M$ as $\mathbb{T}^3 \backslash L$ where:

- $L \subset \mathbb{T}^3$ is a union of four disjoint, closed geodesics $L_1, L_2, L_3, L_4$,
- $H_1(\mathbb{T}^3) = \langle L_1, L_2.L_3 \rangle$,
- $L_4 = L_1 + L_2 + L_3$.

The fiber of $M$ is the 2-torus with four punctures coming from the $L_i$. The different fibration structures are captured by the Thurston ball. In [7], this ball is computed as the dual of the Newton polytope of the Alexander polynomial. Endow the ball with coordinates $\phi = (x, y, z, t)$ as in [7]. Then, the Thurston unit ball has 16 top–dimensional faces (each fibered) coming in 8 pairs under the symmetry $(\phi, -\phi)$. Furthermore, restricting to faces that are dual to those vertices of the Newton polytope with no $t$–component, we get 14 faces, that come in two types; quadrilateral and triangular. It is shown in [7] that there exist a pair of inequivalent symplectic forms on a 4-manifold coming from different fibrations of $\mathbb{T}^3 \backslash L$. These fibrations correspond to points lying on two distinct types of faces. In [20], it is shown that the remaining pair of $16 - 14 = 2$ faces (with a non-zero $t$-component) yield a third symplectic structure which is inequivalent to the two found by McMullen and Taubes.

We will investigate the monodromy of the fibration given in [20], in which it is observed that $M$ admits a fibration with fiber the four-punctured 2-sphere. Table 3.1 summarizes the conclusions of the examples to follow. Determining these monodromy formulas explicitly is a crucial step in computing the dimension of $PH^2_{\pm}(X, \omega)$.

The first example is the fibration with fiber $\Sigma_{0,4}$, hence 'spherical' type. The other two examples are of 'toroidal' type with fiber $\Sigma_{1,4}$. In the spherical example, the given projection vector is the cohomology class in $H^1(M^3)$ corresponding to a point on the Thurston ball. The projection vectors of the 'toroidal' type examples refer to the vector used in its fiber

Table 3.1: Monodromies

| Type of Face | Projection Vector $v_1$ | Monodromy |
|---|---|---|
| Spherical | (0,0,0,1) | $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$ |
| Toroidal | $(-1,-1,1)$ | $\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}\rho_1^{-1}\rho_2^{-1}\tau_1^{-1}\rho_2\tau_4^{-1}\rho_4^{-1}\tau_3^{-1}$ |
| Toroidal | $(-1,1,1)$ | $\rho_2^{-1}\tau_1\rho_2^{-1}\tau_1^{-1}\tau_4^{-1}\rho_3^{-2}\tau_2^{-1}\rho_4^{-1}\rho_1^{-1}$ |

bundle construction and not the point on the Thurston ball. These details are elaborated on in Section3.3. For notational simplicity, in Table 3.1, $\mathcal{P}ush(\rho_i)$ and $\mathcal{P}ush(\tau_i)$ are abbreviated to $\rho_i$ and $\tau_i$, respectively.

**Spherical Example.** In this Example, we take the fibration from [20] obtained by performing 0-surgery along the $K_4$ axis. The fiber is the 2-sphere punctured four-times, with monodromy given by the braid word corresponding to the Borromean rings. Let $\sigma_i$ denote the half-Dehn twist which switches marked points $i$ and $i+1$. This homeomorphism can be viewed similar to the push map, where we "push" the surface through the arc connecting the $i$th and $(i+1)$th points. As a braid it is the element which passes the $i$th string over the $(i+1)$th string. Under this identification, the monodromy is given by

$$\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2.$$

The derivation of the toroidal type monodromies is much more involved. We carefully work out these formulas in the next section. For now, we take the monodromies from Table 3.1 as true and examine their cohomological implications.

### 3.1.1 Cohomological Analysis

Let $f$ denote any monodromy coming with the four–punctured torus fiber $\Sigma_{1,4}$. Similarly, denote by $g$ the monodromy with fiber four–punctured 2-sphere $\Sigma_{0,4}$. By choosing any of the monodromy $f$ we can compute its action on $H^1(\Sigma_{1,4})$ (either by hand or with the help of software) to conclude that $\dim\ker(f^*-1) = b_1(Y_f) - 1 = 3$ in both cases. Let $X_f = S^1 \times Y_f$

and $X_g = S^1 \times Y_g$. By the previous discussions, these manifolds are diffeomorphic, and we will compute the primitive cohomology of the symplectic structures naturally associated to the fibrations, determined by the monodromy $f$ and $g$.

With respect to the ordering $(a_0, a_1, a_2, a_3, b_0)$ of basis vectors for $H^1(\Sigma_{1,4})$, computation shows the action on $H^1(\Sigma_{1,4})$ is given by

$$f^* - 1 = \begin{pmatrix} -1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for all $f$. Here $J$ is the Jordan matrix for $f^* - 1$. We note it has two blocks of size 2 and one of size 1. It follows that

$$\ker(f^* - 1) = \langle (1,0,0,-1,0), (0,1,0,-1,0), (0,0,1,-1,0) \rangle,$$
$$\operatorname{Im}(f^* - 1) = \langle (-1,0,1,0,0), (1,-1,1,-1,0) \rangle.$$

A quick check shows

$$(f^* - 1)(-1,0,1,0,0) = 0 = (f^* - 1)(1,-1,1,-1,0).$$

Hence we conclude

$$\dim \ker(f^* - 1) \cap \operatorname{Im}(f^* - 1) = \dim \operatorname{Im}(f^* - 1) = 2.$$

Notice this dimension agrees with the number of blocks from $J$ of size at least 2. Computations from Section 2 show

$$p_2^+(X_f, \omega_\eta) = \begin{cases} 9, & \lambda_i \neq 0 \text{ for some } i > s \\ 10, & \lambda_i = 0 \text{ for all } i > s \end{cases}$$

We now turn to $X_g$. Since $X_f$ is diffeomorphic to $X_g$, we must have

$$b_1(X_f) = b_1(X_g) \implies \dim\ker(g^* - 1) = \dim\ker(f^* - 1) = 3.$$

Moreover using the formula $\chi(\Sigma_{g,n}) = 2 - 2g - n$, it follows $\chi(\Sigma_{0,4}) = -2 = 1 - b_1(\Sigma_{0,4})$, and so $b_1(\Sigma_{0,4}) = 3$. But by Rank-Nullity, $3 = 3 + \dim \mathrm{Im}(g^* - 1)$, from which it follows $\dim\ker(g^* - 1) \cap \mathrm{Im}(g^* - 1) = 0$. Thus $p_2^+(X_g, \omega_\eta) = b_2(X_g) + 1 = 8 \neq p_2^+(X_f, \omega_\eta)$.

We point out that from the Jordan form of the $f$, these monodromies are not Torelli elements of $\mathcal{M}(\Sigma_{1,4})$. However by dimension considerations, we saw $\dim \mathrm{Im}(g^* - 1) = 0$ and so $g$ *is* a Torelli element of $\mathcal{M}(\Sigma_{0,4})$. Moreover even though each $f$, $f'$ coming from fiber $\Sigma_{1,4}$ are not Torelli, $f^* = f'^*$ and so it follows that $f'f^{-1}$ is a Torelli element.

These calculations give the following theorem.

**Theorem 3.1.1.** *There exist fibrations $Y_f$ and $Y_g$ of the 3-manifold $M$ with inequivalent associated symplectic 4-manifolds $(X_f, \omega_1)$, $(X_g, \omega_2)$, which can be distinguished by primitive cohomologies. In particular,*

$$p_2^+(X_f, \omega_1) \neq p_2^+(X_g, \omega_2).$$

To establish Theorem 3.1.1, it only remains to verify the toroidal type monodromies in Table 3.1.

## 3.2 Construction of Monodromies

In this section, we provide details for the construction of the toroidal mondromies in Table 3.1. Section 3.3 gives an even more specific outline of the procedure that follows. In the examples to come, we take different bases $v_1 = (a_1, a_2, a_3)$, $v_2 = (1, 1, 0)$, $v_3 = (0, 1, 1)$ and fiber along $v_1$ so that the fiber at time $t$ looks like $\Sigma_{t,4} = tv_1 + \langle v_2, v_3 \rangle$ with marked points

$$y_1(t) = (-4\epsilon, 3\epsilon) + (a_3 - a_2, -a_3)t,$$

$$y_2(t) = (-\epsilon, 2\epsilon) + (-a_1, a_1 - a_2)t,$$

$$y_3(t) = (0, 0) + (a_3 - a_2, a_1 - a_2)t,$$

$$y_4(t) = (\epsilon, -3\epsilon) + (-a_1, -a_3)t.$$

Here, $\epsilon$ is some small fixed constant used to shift the coordinate axes away from the origin. The vector $v_1$ is the projection vector given in column 2 of Table 3.1. The general idea is as follows,

1. Using the paths of the punctures $y_i$, find relative locations to determine if $y_i$ passes above or below $y_j$.
2. Express $\mathcal{P}ush(y_i(t))$ of the $y_i$ path in terms of generators $\mathcal{P}ush(\rho_i)$, $\mathcal{P}ush(\tau_i)$.
3. Calculate the intersection points of punctures $(y_i(t), y_j(t))$ at times $(t_i, t_j)$. If $t_i > t_j$ then $y_i$ crosses over $y_j$. If $t_i < t_j$ then $y_j$ crosses over $y_i$.
4. Use the crossings information to determine the order of $\mathcal{P}ush(y_i(t))$ maps in the final monodromy.

The procedure is best demonstrated through examples. As before, we drop the push notation so that $\mathcal{P}ush(\rho_2)\mathcal{P}ush(\tau_1)^{-1}\mathcal{P}ush(\tau_3)$ is simply denoted by $\rho_2\tau_1^{-1}\tau_3$. We also use function notation right to left so that the previous word indicates $y_3$ travels along $\tau_3$ then $y_1$

along the inverse of $\tau_1$ then finally $y_2$ along $\rho_2$. Homeomorphism type of the below examples was confirmed with SnapPea ([2]).

**Toroidal Example 1.** $v_1 = (-1, -1, 1)$

The paths of the corresponding marked points are

$$
\begin{aligned}
y_1(t) &= (-4\epsilon, 3\epsilon) + (2, -1)t, \\
y_2(t) &= (-\epsilon, 2\epsilon) + (1, 0)t, \\
y_3(t) &= (0, 0) + (2, 0)t, \\
y_4(t) &= (\epsilon, -3\epsilon) + (1, -1)t.
\end{aligned}
$$

Thus $y_2$ and $y_3$ travel in a parallel horizontal direction. $y_1$ and $y_4$ travel downwards and to the right and so will intersect both $y_2$ and $y_3$. We first find these intersection times. We illustrate the process for $y_1$ and $y_3$ and summarize the other points in Table 3.2. We need times $t_1$ and $t_3$ so that $y_1(t_1) = y_3(t_3)$. In other words, we seek a solution to the system

$$
\begin{aligned}
-4\epsilon + 2t_1 &= 2t_3, \\
3\epsilon - t_1 &= 0,
\end{aligned}
$$

which gives $(t_1, t_3) = (3\epsilon, \epsilon + \frac{n}{2})$, $n = 0, 1$. Hence $y_1$ and $y_3$ intersect twice. The first time $y_1$ passes over $y_3$. Then at $t_3 = \epsilon + \frac{1}{2}$, $y_3$ crosses $y_1$. At $t_2 = \frac{5}{8}\epsilon + \frac{1}{2}$, $y_2$ passes over $y_1$. Similarly solving the corresponding system for $y_2$ and $y_3$ yields $(t_2, t_3) = (\frac{2}{3}\epsilon + \frac{n}{2}, 1 - \frac{1}{3}\epsilon)$, $n = 0, 1$. Both $y_2$ times occur before $y_3$, hence we conclude $y_3$ passes over $y_2$ twice. The remaining points of intersection are given in Table 3.2. The times specified are the later of the two crossing times and the points have been listed in order of intersection occurrence, from first to last.

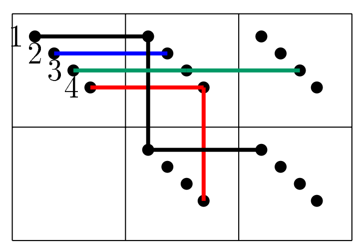
Pictured in Figure 3.1 are the paths of the $y_i$ drawn in the plane (up to identification),

Table 3.2: Toroidal Example 1 Intersections

| Points | Time | Crossing |
|---|---|---|
| $(y_1, y_3)$ | $3\epsilon$ | $y_1$ over $y_3$ |
| $(y_1, y_3)$ | $\epsilon + \frac{1}{2}$ | $y_3$ over $y_1$ |
| $(y_2, y_4)$ | $1 - 3\epsilon$ | $y_2$ over $y_4$ |
| $(y_3, y_4)$ | $1 - 3\epsilon$ | $y_4$ over $y_3$ |
| $(y_1, y_2)$ | $1 - \epsilon$ | $y_2$ over $y_1$ |
| $(y_1, y_4)$ | $1 - \epsilon$ | $y_1$ over $y_4$ |
| $(y_3, y_4)$ | $1 - \epsilon$ | $y_3$ over $y_4$ |

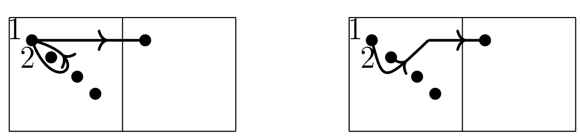
where we have decomposed the "diagonal" paths of $y_1$ and $y_4$ into a combination of basis curves $\rho_i$ and $\tau_i$. To find the path of $y_1$, for example, we must use its velocity vector $(2, -1)$ as well as the relative locations of $y_1$ with respect to the start points of $y_2$, $y_3$, and $y_4$. Given that point $y_2$ starts at $(-\epsilon, 2\epsilon)$, we have $y_1(\frac{3}{2}\epsilon) = (-\epsilon, \frac{3}{2}\epsilon)$ and so $y_1$ travels 'below' the $y_2$ start point. Similar computations show $y_1$ travels above both the $y_3$ and $y_4$ start points. As illustrated in Figure 3.1, the velocity vector $(2, -1)$ suggests $y_1$ has a path given by $\tau_1^{-1}\rho_1^{-1}\tau_1^{-1}$. However the diagonal path homotopic to this combination will not preserve the condition that $y_1$ travels below the $y_2$ start point. To remedy this situation, we must begin the $y_1$ monodromy with the loop $C_{12}$. This curve travels counterclockwise from $y_1$, enclosing $y_2$. Figure 3.2 illustrates the $\tau_1^{-1}C_{12}$ portion of the monodromy.

Figure 3.1: Example 1 Marked Point Paths




$y_4$ is the only other diagonal path. We can easily check that it travels above the $y_1$,

Figure 3.2: $C_{12}$ Path in Example 1



$y_2$, and $y_3$ start points. Hence its path is simply given by $\tau_4^{-1}\rho_4^{-1}$, indicated by the $(1,-1)$ velocity vector.

Summarizing, the monodromies of the punctures are given by

$$\begin{aligned}
&y_1(t) : \tau_1^{-1}\rho_1^{-1}\tau_1^{-1}C_{12} = \tau_1^{-1}\rho_1^{-1}\rho_2^{-1}\tau_1^{-1}\rho_2,\\
&y_2(t) : \tau_2^{-1},\\
&y_3(t) : \tau_3^{-2},\\
&y_4(t) : \tau_4^{-1}\rho_4^{-1}.
\end{aligned}$$

Now, we must determine the order of these individual monodromies in the final map. Using the above formulas, it's clear $y_2(t)$ and $y_3(t)$ are parallel so their relative order to each other in the final monodromy doesn't matter. From Table 1, we see every other point crosses over $y_3$ first, but then $y_3$ crosses over $y_1$ and $y_4$ again later. Thus we should put one $\tau_3^{-1}$ at the beginning of the monodromy and the other $\tau_3^{-1}$ at the end. Next, both $y_1$ and $y_2$ cross over $y_4$ so the $y_4$ term should come next.

It only remains to determine the order of $y_1$ and $y_2$, which is given by Table 1 as $y_1$ then $y_2$. Therefore our monodromy has the formula $y_3 \circ y_2 \circ y_1 \circ y_4 \circ y_3$, where the first and last $y_3$ terms are each a $\tau_3^{-1}$. This ordering gives 10 possible crossings, but $y_2$ and $y_3$ are parallel and $y_3$ appears twice. Hence the number reduces to $10 - 3 = 7$, matching the occurrences in Table 3.2.

Piecing all the arguments together shows the final monodromy is isotopic to

$$\tau_3^{-1}\tau_2^{-1}(\tau_1^{-1}\rho_1^{-1}\tau_1^{-1}C_{12})\tau_4^{-1}\rho_4^{-1}\tau_3^{-1} = \tau_3^{-1}\tau_2^{-1}(\tau_1^{-1}\rho_1^{-1}\rho_2^{-1}\tau_1^{-1}\rho_2)\tau_4^{-1}\rho_4^{-1}\tau_3^{-1}.$$

**Toroidal Example 2.** $v_1 = (-1, 1, 1)$

The paths of the punctures are given by

$$y_1(t) = (-4\epsilon, 3\epsilon) + (0, -1)t,$$

$$y_2(t) = (-\epsilon, 2\epsilon) + (1, -2)t,$$

$$y_3(t) = (0, 0) + (0, -2)t,$$

$$y_4(t) = (\epsilon, -3\epsilon) + (1, -1)t.$$

Implementing the techniques from the previous example, we obtain the intersections in Table 3.3. There is only one non-trivial diagonal path, given by $y_2$. Evaluating this path at the appropriate times yields

$$y_2(-3\epsilon) = (-4\epsilon, 8\epsilon),$$

$$y_2(\epsilon) = (0, 0),$$

$$y_2(2\epsilon) = (\epsilon, -2\epsilon).$$

We see that $y_2$ travels above $y_1$ and $y_4$ start points and through $y_3$ at the origin. We note at $t = \epsilon$, $y_3(\epsilon) = (0, -2\epsilon)$ has traveled away from the origin and so $y_2(t)$ and $y_3(t)$ do not actually collide. Thus, in between $\rho_2^{-1}\rho_2^{-1}\tau_2^{-1}$, we must insert a loop traveling counterclockwise starting at $y_2$ and enclosing $y_1$. It turns out this curve is also homotopic to $C_{12}$ (see [1] for more discussion). By drawing a diagram similar to Figure 3.1 one can see the correct placement should be $\rho_2^{-1}C_{12}\rho_2^{-1}\tau_2^{-1}$. The paths of the other points are straightforward, given by

$$y_1 : \rho_1^{-1},$$

$$y_2 : \rho_2^{-1}C_{12}\rho_2^{-1}\tau_2^{-1} = \rho_2^{-1}\tau_1\rho_2^{-1}\tau_1^{-1}\tau_2^{-1},$$

$$y_3 : \rho_3^{-2},$$

$$y_4 : \tau_4^{-1}\rho_4^{-1}.$$

The ordering for this example is similar to that of Example 1; this time we need to split both of the paths $y_2$ and $y_4$ into two parts each. Notice from the individual monodromies that $y_1$ and $y_3$ are parallel so their relative order doesn't matter. We proceed by considering the remaining interactions separately. Since $y_1$ passes under for all its crossings, it appears first. Then $y_3$ over $y_2$ and $y_2$ over $y_4$ suggests the ordering $y_3 \circ y_2 \circ y_4$. However, we need $y_4$ to cross over $y_3$ and this current arrangement does the opposite. Hence we must split the $y_4$ monodromy into two components: $y_4 \circ y_3 \circ y_2 \circ y_4$. Finally, if we leave $y_2$ together, we will have both $y_4$ and $y_2$ crossing over one another at different times. Consequently, we also split $y_2$ for the ultimate ordering given by $y_2 \circ y_4 \circ y_3 \circ y_2 \circ y_4 \circ y_1$. The final monodromy pieces together as

$$y_2 \circ \tau_4^{-1} \circ \rho_3^{-2} \circ y_2 \circ \rho_4^{-1} \circ \rho_1^{-1}.$$

To reiterate, we are required to separate $y_2$ such that the $\tau_4^{-1}$ does not intersect the first term. This obstruction suggests the first $y_2$ part is $\tau_2^{-1}$ and the second term is the remaining $\rho_2^{-1} C_{12} \rho_2^{-1}$. This construction yields the desired map

$$\rho_2^{-1} C_{12} \rho_2^{-1} \tau_4^{-1} \rho_3^{-2} \tau_2^{-1} \rho_4^{-1} \rho_1^{-1}.$$

Table 3.3: Toroidal Example 2 Intersections

| Points | Time | Crossing |
|---|---|---|
| $(y_2, y_4)$ | $3\epsilon$ | $y_2$ over $y_4$ |
| $(y_2, y_3)$ | $\frac{1}{2}$ | $y_3$ over $y_2$ |
| $(y_1, y_4)$ | $1 - 5\epsilon$ | $y_4$ over $y_1$ |
| $(y_1, y_2)$ | $1 - 3\epsilon$ | $y_2$ over $y_1$ |
| $(y_3, y_4)$ | $1 - \epsilon$ | $y_4$ over $y_3$ |

## 3.3 Further Details on Fibration Construction

We now provide the details of setting up the fibration structure and converting monodromies appropriately so that they can be entered into SnapPea ([2]). Let $\mathbb{T}^3$ denote the 3-torus. We view it as the cube $[0,1]^3$ under the identification $(x,y,z) \sim (x+p, y+q, z+r)$ for integers $p,q,r$. The axes $i,j,k$ and their sum $i+j+k$ form four lines in the cube $L_1, L_2, L_3, L_4$, respectively. By choosing different bases $(v_1, v_2, v_3)$ for the cube and displacing the four lines we may fiber $\mathbb{T}^3 - \{L_1, L_2, L_3, L_4\}$ in different ways as follows. First we shift the four lines from the origin by

$$L_1 = (x, -\epsilon, 3\epsilon),$$

$$L_2 = (\epsilon, y, -3\epsilon),$$

$$L_3 = (-\epsilon, \epsilon, z),$$

$$L_4 = (x = y = z).$$

Next we choose a basis $v_1 = (a_1, a_2, a_3)$, $v_2 = (1,1,0)$, $v_3 = (0,1,1)$. Initially $v_1$ may be any vector which gives a non-zero determinant, specifically, $a_1 - a_2 + a_3 \neq 0$. For brevity, let us denote $A := det(v_1, v_2, v_3) = a_1 - a_2 + a_3$. Choosing to fiber along $v_1$, each fiber has the form $\Sigma_t = tv_1 + \alpha v_2 + \beta v_3$ for $t \in [0,1]$. $\Sigma_t$ is $\mathbb{T}^2$ with four punctures denoted $x_1(t), x_2(t), x_3(t), x_4(t)$ coming from the respective lines $L_i$. To verify that each line $L_i$

intersects the fiber exactly once we must solve the following system of equations:

$$L_1 : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\epsilon - ta_2 \\ 3\epsilon - ta_3 \end{pmatrix},$$

$$L_2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \epsilon - ta_1 \\ -3\epsilon - ta_3 \end{pmatrix},$$

$$L_3 : \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\epsilon - ta_1 \\ \epsilon - ta_2 \end{pmatrix},$$

$$L_4 : \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} t(a_2 - a_1) \\ t(a_3 - a_1) \end{pmatrix}.$$

Solving these systems for the $(\alpha, \beta)$ coordinates of the marked points $x_i(t)$ yields

$$x_1(t) = (-4\epsilon, 3\epsilon) + (a_3 - a_2, -a_3)t,$$

$$x_2(t) = (\epsilon, -3\epsilon) + (-a_1, -a_3)t,$$

$$x_3(t) = (-\epsilon, 2\epsilon) + (-a_1, a_1 - a_2)t,$$

$$x_4(t) = (0, 0) + (a_3 - a_2, a_1 - a_2)t.$$

To align with the notation of [1], we relabel the points with respect to their first coordinate position, in increasing order, as $y_1(t) = x_1(t)$, $y_2(t) = x_3(t)$, $y_3(t) = x_4(t)$, $y_4(t) = x_2(t)$.

Under this new setting the formulas for the points become

$$y_1(t) = (-4\epsilon, 3\epsilon) + (a_3 - a_2, -a_3)t,$$

$$y_2(t) = (-\epsilon, 2\epsilon) + (-a_1, a_1 - a_2)t,$$

$$y_3(t) = (0, 0) + (a_3 - a_2, a_1 - a_2)t,$$

$$y_4(t) = (\epsilon, -3\epsilon) + (-a_1, -a_3)t.$$

Next we verify that none of the $y_i(t)$ intersect for any value of $t$. Notice $y_2$ and $y_3$ have the same second component in the $t$ variable but differ by the $\epsilon$-term constant so they will never intersect. We can apply a similar argument to the pairs $(y_1, y_3), (y_1, y_4)$, and $(y_2, y_4)$. Lastly, by considering the (separate) systems of equations $y_1(t) = y_2(t)$ and $y_3(t) = y_4(t)$, one can easily see no solutions exist.

Let $\Sigma_{1,4}$ be the 2−torus with four punctures and $\text{Mod}(\Sigma_{1,4})$ its mapping class group (which fixes the punctures setwise). Furthermore let $\mathcal{P}\text{Mod}(\Sigma_{1,4})$ denote the *pure* mapping class group, the set of mapping class elements fixing the punctures pointwise. We set

$$H_1(\Sigma) = \langle a_0, a_1, a_2, a_3, b_0 \rangle, \tag{3.1}$$

where $a_i$ is the homology curve between punctures $i$ and $i + 1$ for $i > 0$ and $a_0$ is between marked point 1 and 4. $b_0$ is the homology longitudinal curve, not enclosing any punctures. These curves have algebraic intersection numbers $a_i \cdot a_j = 0$ for $i \neq j$ and $a_i \cdot b_0 = 1$. [1] introduces the following elements (pictured below) and show Dehn twists along them generate the pure mapping class group. In our setting we have $\mathcal{P}\text{Mod}(\Sigma_{1,4}) = \langle \mathcal{P}ush(\rho_i), \mathcal{P}ush(\tau_i) \rangle$, $1 \leq i \leq 4$. Here, $\mathcal{P}ush(\gamma)$ is the point pushing map along $\gamma$. We also summarize some of the important relations to be used later:

$$[\tau_i, \tau_j] = [\rho_i, \rho_j] = 1,$$

$$A_{ij} = \rho_i \tau_j^{-1} \rho_i^{-1} \tau_j, \quad C_{ij} = \tau_i \rho_j^{-1} \tau_i^{-1} \rho_j,$$

for $1 \leq i < j < k \leq 4$.

For a more in depth discussion and outline of a proof for these identities, see [1]. We note

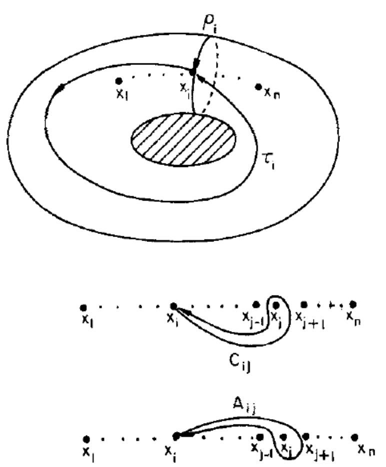



Figure 3.3: Diagram of generators taken from [1]

that the formulas here differ slightly from [1] as our choice of orientation is not the same. Moreover, we use functional composition, (right to left) as opposed to algebraic. In order to use SnapPea ([2]), we need to express $\mathcal{P}ush(\rho_i)$ and $\mathcal{P}ush(\tau_i)$ in terms of Dehn twists along the curves in (3.1). The trick is to use the following fact (4.7 proven in [4]), which states

**Fact.** Let $\alpha$ be a simple loop in a surface $S$ representing an element of $\pi_1(S, x)$, Then $\mathcal{P}ush([\alpha]) = T_a T_b^{-1}$, where $a$ and $b$ are isotopy classes of the simple closed curves in $S - x$ obtained by pushing $\alpha$ off itself to the left and right, respectively.

That is, we take an annular neighborhood of $\alpha$ bounded by curves $a$ and $b$ and then take the product of their Dehn and inverse Dehn twists, respectively. From this construction, we can immediately obtain that

$$\mathcal{P}ush(\rho_i) = T_{a_{i-1}} T_{a_i}^{-1}. \tag{3.2}$$

For the $\tau_i$ curves we need to find an annular boundary to work with. We introduce the longitudinal homology curves $b_i$, which enclose the punctures $1, 2, \cdots, i$ ["over" $1, 2, \cdots, i$ and "under" $i + 1, \cdots, 4$]. Thus $b_0$ agrees with the previous homology generator introduced, $b_1$ passes over puncture 1 and misses 2,3,4, and so on. The point of introducing these curves is that now $\tau_i$ has an annular neighborhood bounded by $b_{i-1}$ and $b_i$. By consulting the diagrams to determine proper orientation it follows that

$$\mathcal{P}ush(\tau_i) = T_{b_i} T_{b_{i-1}}^{-1}. \tag{3.3}$$

Next we need to convert Equation 3.3 into Dehn twists only involving the homology generators given in 3.1. First we observe that we may express $[b_i] = [a_0] + [b_0] - [a_i]$, which can be verified by constructing the fundamental square for the torus with the relevant curves. An example diagram in Figure 3.4 is given for the $[b_1]$ case. One can straightforwardly check

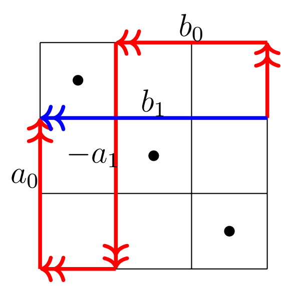



Figure 3.4: Diagram for $b_1$ Expression

that $T_{a_i} T_{b_0}([a_0]) = [a_0] + [b_0] - [a_i] = [b_i]$. Fact 3.7 in [4] states $T_{f(a)} = f T_a f^{-1}$, which we can apply to our situation by setting $a = a_0$ and $f = T_{a_i} T_{b_0}$. This fact then yields

$$T_{b_i} = T_{a_i} T_{b_0} T_{a_0} T_{b_0}^{-1} T_{a_i}^{-1}. \tag{3.4}$$

Finally, substituting formula 3.4 into equation 3.3 leads to our desired expression

$$\mathcal{P}ush(\tau_i) = T_{a_{i-1}} T_{b_0} T_{a_0}^{-1} T_{b_0}^{-1} T_{a_{i-1}}^{-1} T_{a_i} T_{b_0} T_{a_0} T_{b_0}^{-1} T_{a_i}^{-1}. \tag{3.5}$$

## 3.4 Another Example Using Graph Links

Here, we give another example of fibrations of a 3-manifold giving inequivalent symplectic structures on its associated (symplectic) 4-manifold $S^1 \times Y_f$. Let $M^{(2n)} = S^3 \backslash K^{(2n)}$, where $K^{(2n)}$ is the graph link pictured in Figure 3.5 below. The details of this diagram are given in [19], where the third author showed the existence of $n+1$ inequivalent symplectic structures coming from different fibrations of $M^{(2n)}$. A fibration of $M^{(2n)}$ is given by a choice of

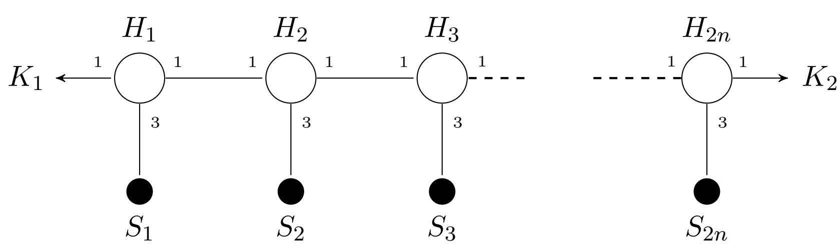



Figure 3.5: Diagram of $K^{(2n)}$

$(m_1, m_2) \in H^1(S^3 \backslash K^{(2n)}), \mathbb{Z}) \cong \mathbb{Z}^2$ satisfying the equations

$$3^i m_1 + 3^{2n-i+1} m_2 \neq 0, \text{ for all } 1 \leq i \leq 2n.$$

Details for such a fibration (and graph link theory in general) are worked out in [3]. In particular, let $h$ denote the monodromy and $h_*$ the induced map on homology of the fiber. [3, Theorem 13.6] shows there is an integer $q$ such that $(h_*^q - 1)^2 = 0$. Thus the Jordan decomposition of $h_*$ only has blocks of size 1 or 2. Furthermore, with the same $q$, [3] computes the characteristic polynomial of $h_*|_{Im(h_*^q - 1)}$, denoted $\Delta'(t)$. It turns out that the roots of $\Delta'(t)$ correspond to the eigenvalues of $h_*$ with size 2 Jordan blocks. Moreover the multiplicity of each root $\lambda_i$ in $\Delta'(t)$ gives the number of size 2 blocks for $\lambda_i$.

We first introduce some notation which will be used in the definition of $\Delta'(t)$. Fix a fibration $(m_1, m_2)$. Let $\mathcal{E} = \{E_1, \cdots, E_{2n-1}\}$ be the set of edges connecting the white nodes in Figure 3.5. Specifically, edge $E_i$ connects nodes labeled $H_i$ and $H_{i+1}$. For each $E_i \in \mathcal{E}$, we define an integer $d_{E_i}$ as follows. Take the path in $K^{(2n)}$ from the arrowhead of $K_1$ to halfway through edge $E_i$ (passing through nodes $H_1, H_2, \cdots, H_i$). Let $\ell_{E_i,1}$ denote the product of all weights on edges not contained in the path but are adjacent to vertices in the path. Similarly we can take the path from the arrowhead of $K_2$ to halfway through edge $E_i$ and define $\ell_{E_i,2}$ analogously. Set

$$d_{E_i} = \gcd(m_1 \ell_{E_i,1}, m_2 \ell_{E_i,2}).$$

Using Figure 3.5 as reference, we can easily compute that $\ell_{E_i,1} = 3^i$ and $\ell_{E_i,2} = 3^{2n-i}$. This simplifies the formula for $d_E$ to

$$d_{E_i} = \gcd(3^i m_1, 3^{2n-i} m_2). \tag{3.6}$$

For each vertex $H_i$, we define an integer $d_{V_i}$ by the formula

$$d_{V_i} = \begin{cases} \gcd(d_{E_{i-1}}, d_{E_i}), & 1 < i < 2n \\ \gcd(m_1, d_{E_1}), & i = 1 \\ \gcd(m_2, d_{E_{2n-1}}), & i = 2n \end{cases} \tag{3.7}$$

With these definitions in place, the (restricted) characteristic polynomial takes the form

$$\Delta'(t) = (t^d - 1) \prod_{i=1}^{2n-1} (t^{d_{E_i}} - 1) / \prod_{i=1}^{2n} (t^{d_{V_i}} - 1),$$

where $d = \gcd(m_1, m_2)$. To obtain a more concrete equation, we analyze several fibrations of $K^{(4)}$. Figure 3.6 demonstrates how $d_{E_1} = \gcd(3m_1, 3^3 m_2)$ is calculated. In particular, define $X^{(4)} = S^1 \times M^{(4)}$ and let $\deg \Delta'(t)$ denote the degree of the restricted characteristic polynomial $\Delta'(t)$. Since $\deg \Delta'(t)$ is the number of Jordan blocks of size 2, which equals the

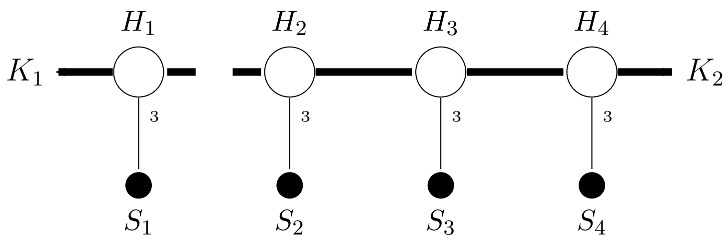



Figure 3.6: Paths $\ell_{E_1,1}$ and $\ell_{E_1,2}$ of $d_{E_1}$

number of blocks of size *at least* 2, it follows

$$p_2^+ = b_2(X^{(4)}) + 1 + \deg \Delta'(t),$$
$$p_2^- = b_2(X^{(4)}) + \deg \Delta'(t).$$

In the case of a fibration represented by coprime $(m_1, m_2)$, there are two possibilities: 3 divides exactly one of $m_1$ or $m_2$, or 3 neither divides $m_1$ nor $m_2$. It turns out $p_2^+$ can distinguish these two possibilities and in the first case provides information about the power of 3 dividing $m_1$ or $m_2$. We give the exact statement below.

**Theorem 3.4.1.** *Let* $(m_1, m_2)$ *be coprime, representing a fibration of* $M^{(4)}$*. By reversing the roles of* $m_1$ *and* $m_2$ *if necessary, we write* $m_1 = 3^k q$ *with* $\gcd(q, 3) = 1$ *and assume* $\gcd(3, m_2) = 1$*. It follows that*

$$p_2^+ = \begin{cases} b_2(X^{(4)}) + 9, & k = 0 \\ b_2(X^{(4)}) + 7, & k = 1 \\ b_2(X^{(4)}) + 19, & k = 2 \\ b_2(X^{(4)}) + 1, & k \geq 3 \end{cases}$$

*Proof.* We proceed by cases, treating $k = 0$ and $k > 0$ separately.

**Case 3.4.1.** $(k > 0)$

Using formulas (3.6) and (3.7) we compute

$$
\begin{aligned}
d_{E_1} &= \gcd(3^{k+1}q, 3^3 s) = \min(3^{k+1}, 3^3),\\
d_{E_2} &= \gcd(3^{k+2}q, 3^2 s) = 3^2,\\
d_{E_3} &= \gcd(3^{3+k}q, 3s) = 3,\\
d_{V_1} &= \gcd(3^k q, \min(3^{k+1}, 3^3)) = \min(3^k, 3^3),\\
d_{V_2} &= \gcd(\min(3^{k+1}, 3^3), 3^2) = \min(3^{k+1}, 3^2) = 3^2,\\
d_{V_3} &= \gcd(3^2, 3) = 3,\\
d_{V_4} &= \gcd(s, 3) = 1.
\end{aligned}
$$

from which it follows

$$
\begin{aligned}
\Delta'(t) &= \frac{(t-1)(t^3-1)(t^9-1)(t^{\min(3^{k+1},3^3)}-1)}{(t-1)(t^3-1)(t^{\min(3^k,3^3)}-1)(t^9-1)}\\
&= \frac{t^{3^2\min(3^{k-1},3)}-1}{t^{3\min(3^{k-1},3^2)}-1}\\
&= \begin{cases} t^6+t^3+1, & k=1\\ t^{18}+t^9+1, & k=2\\ 1, & k\geq 3 \end{cases}
\end{aligned}
$$

**Case 3.4.2.** $\gcd(m_1, m_2) = \gcd(m_1, 3) = \gcd(m_2, 3) = 1$. Applying a similar analysis as in

Case 1 shows

$$
\begin{aligned}
d_{E_1} &= \gcd(3, 3^3) = 3, \\
d_{E_2} &= \gcd(3^2, 3^2) = 3^2, \\
d_{E_3} &= \gcd(3^3, 3) = 3, \\
d_{V_1} &= \gcd(m_1, 3) = 1, \\
d_{V_2} &= \gcd(3, 3^2) = 3, \\
d_{V_3} &= \gcd(3^2, 3) = 3, \\
d_{V_4} &= \gcd(3, m_2) = 1,
\end{aligned}
$$

$$
\begin{aligned}
\Delta'(t) &= \frac{(t-1)(t^3-1)^2(t^9-1)}{(t-1)^2(t^3-1)^2} \\
&= \frac{t^9-1}{t-1} = (t^2+t+1)(t^6+t^3+1)
\end{aligned}
$$

Using the formula for $p_2^+$ and $\deg \Delta'(t)$ for each $k$ from the above cases produces the claimed dimensions. □

We conclude with some remarks. Theorem 3.4.1 uses $K^{(4)}$ as a matter of explicitness for factoring $\Delta(t)$ and $\Delta'(t)$. One could also consider other $K^{(2n)}$ to reach similar conclusions.

# Chapter 4

# Examples of the $m_2$-Structure and Symplectic Massey Products

In this chapter, we analyze the $A_3$-structure on primitive forms of $X = S^1 \times Y_f$ for a mapping torus $Y_f$. We compute the ring structure of $H^*(X)$ and work out some classical Massey products. Then, we move on to $PH^*(X, \omega)$ and show how its product reveals information about the Jordan blocks of the monodromy $f^* - 1$. We also construct a 3-fold and 4-fold symplectic Massey product. Unless otherwise stated, in this chapter, we reserve the notation $X$ for the 4-manifold $S^1 \times Y_f$ and $M$ for a general symplectic manifold.

## 4.1 Ring Structure and Massey Products on $H^*(X)$

We begin this section by calculating $\wedge : H^*(X) \times H^*(X) \to H^*(X)$ explicitly. For convenience, we restate the de Rham cohomology of $X$ below. Note, if the fiber of $Y_f$ is closed then $H^3(Y_f) = \langle dt \wedge d\pi \wedge \omega_\Sigma \rangle$. Otherwise, $H^3(Y_f) = 0$. We keep the same notation as before, where $x_{i,0} \in \ker(f^* - 1)$ is in a Jordan block of size $n_i + 1$.

$$H^1(X) = \langle dt, d\pi, x_{i,0} \rangle_{i=1}^k,$$

$$H^2(X) = \langle d\pi \wedge x_{i,n_i} \rangle_{i=1}^k \oplus \langle dt \wedge d\pi, dt \wedge x_{i,0} \rangle_{i=1}^k,$$

$$H^3(X) = \langle dt \wedge d\pi \wedge x_{i,n_i} \rangle_{i=1}^k \oplus H^3(Y_f),$$

$$H^4(X) = \langle dt \rangle \wedge H^3(Y_f).$$

Below we give some of the important (non-zero) entries of the ring structure on $H^*(X)$.

### $H^1(X) \wedge H^1(X) \to H^2(X)$:

| $H^1(X)$ | $H^1(X)$ | $H^2(X)$ |
|---|---|---|
| dt | $d\pi$ | $dt \wedge d\pi$ |
| | $x_{i,0}$ | $dt \wedge x_{i,0}$ |
| $x_{i,0}$ | $x_{j,0}$ | $d\pi \wedge F(x_{i,0}, x_{j,0})$ |
| $d\pi$ | $x_{i,0}$ | $d\pi \wedge x_{i,0}, \quad n_i = 0$ <br> $0, \quad n_i > 0$ |

where $F : \Omega^1(Y_f) \otimes \Omega^1(Y_f) \to \Omega^1(Y_f)$ is the map determined by the wedge product on $Y_f$. One possible trivial product from the table above is given by $d\pi$ with an element $x_{i,0}$ in a Jordan block of size greater than one. This combination will lead to an important Massey product determining the size of the block that $x_{i,0}$ comes from.

$\underline{H^1(X) \wedge H^2(X) \to H^3(X)}$:

| $H^1(X)$ | $H^2(X)$ | $H^3(X)$ |
|---|---|---|
| $dt$ | $d\pi \wedge x_{i,n_i}$ | $dt \wedge d\pi \wedge x_{i,n_i}$ |
| $d\pi$ | $dt \wedge x_{i,0}$ | $-dt \wedge d\pi \wedge x_{i,0}, \quad n_i = 0$ <br> $0, \quad n_i > 0$ |
| $x_{i,0}$ | $dt \wedge x_{j,0}$ | $-dt \wedge d\pi \wedge F(x_{i,0}, x_{j,0})$ |

We see that the standard product on $H^*(X)$ can tell if a Jordan block is of size 1 or greater than 1, but in the latter case does not provide any more information on the size. For this further refinement, we turn to a more specialized product.

Suppose $x_0$ is in a Jordan block $\mathcal{J} = \{x_0, x_1, \cdots, x_\ell\}$. As elements of $H^1(Y_f)$, the $(x_k)$ satisfy the formula (see [15])

$$dx_k = d\pi \wedge \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} x_{k-j}, \quad k = 0, 1, \cdots, \ell. \tag{4.1}$$

For concreteness, let us consider the case where $\ell = 2$. Then $\langle d\pi, d\pi, x_0 \rangle$ is defined since $d\pi \wedge d\pi = 0$ and $d\pi \wedge x_0 = dx_1$. Using this defining system yields $d\pi \wedge x_1$ as a representative for this 3-point Massey product. However this (and any other) representative is trivial in $H^2(X)$ since the formula $dx_2 = d\pi \wedge (x_1 - \frac{1}{2}x_0)$ implies $d\pi \wedge x_1 = d(x_2 + \frac{1}{2}x_1)$. Hence, we can turn to the 4-point Massey $\langle d\pi, d\pi, d\pi, x_{i,0} \rangle$ since $\langle d\pi, d\pi, d\pi \rangle = 0$ and $\langle d\pi, d\pi, x_0 \rangle = d(x_2 + \frac{1}{2}x_1)$ are both trivial. Computing this product gives a representative cohomologous to $d\pi \wedge x_2$ which is *non*-trivial in $H^2(X)$ since $x_2$ corresponds to the last vector in the Jordan basis. Thus it took a Massey product with three $d\pi$ terms to achieve a non-trivial representative. This motivating example leads to the following proposition.

**Proposition 4.1.1.** *For a general Jordan block of length $\ell+1$, the Massey product $\langle d\pi, \cdots, d\pi, x_{i,0} \rangle$ with the first $\ell + 1$ terms consisting of all $d\pi$, is defined. Furthermore, it has a (non-trivial) representative $[d\pi \wedge x_\ell]$.*

*Proof.* A defining system $(a_{ij})$ will be quite sparse since any $(n < \ell+1)$-fold Massey product not including the last term $(x_0)$ will look like $\langle d\pi, \cdots, d\pi \rangle$ and so has representative 0. Specifically, this means $a_{i,j} = 0$ for $1 \leq i \neq j < \ell + 2$. The only non-zero terms will be the diagonal ones and $a_{i,\ell+2}$ which satisfy $da_{i,\ell+2} = \langle d\pi, d\pi, \cdots, d\pi, x_0 \rangle$, the $(\ell + 3 - i)$-fold Massey product with $(\ell + 2 - i)$ $d\pi$ terms. At this point, our defining system looks like,

$$\begin{bmatrix} d\pi & 0 & \dots & 0 & * \\ 0 & d\pi & \dots & 0 & a_{2,\ell+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d\pi & a_{\ell+1,\ell+2} \\ 0 & 0 & \dots & 0 & x_0 \end{bmatrix}.$$

We work backwards, using equation (4.1), to compute the $a_{i,\ell+2}$ by the defining equations,

$$\begin{aligned} &da_{\ell+1,\ell+2} = d\pi \wedge x_0 = dx_1, \\ &da_{\ell,\ell+2} = \langle d\pi, d\pi, x_0 \rangle = d\pi \wedge x_1 = d(x_2 + \frac{1}{2}x_1), \\ &da_{\ell-1,\ell+2} = \langle d\pi, d\pi, d\pi, x_0 \rangle = d\pi \wedge x_2 = d(x_3 + \frac{1}{2}x_2 - \frac{1}{12}x_1), \\ &\vdots \\ &da_{2,\ell+2} = \langle d\pi, d\pi, \cdots, d\pi, x_0 \rangle = d\pi \wedge x_{\ell-1} = d(x_\ell + \frac{1}{2}x_{\ell-1} - \frac{1}{12}x_{\ell-2} + \frac{3}{8}x_{\ell-3} + \cdots). \end{aligned}$$

Plugging this system into the Massey product formula yields,

$$\begin{aligned} \langle d\pi, d\pi, \cdots, d\pi, x_0 \rangle &= [a_{11} \wedge a_{2,\ell+2} + a_{12} \wedge a_{3,\ell+2} + \cdots a_{1,\ell+1} \wedge a_{\ell+2,\ell+2}] \\ &= [d\pi \wedge (x_\ell + \frac{1}{2}x_{\ell-1} - \frac{1}{12}x_{\ell-2} + \frac{3}{8}x_{\ell-3} + \cdots)] \\ &= [d\pi \wedge x_\ell] + [d\pi \wedge (\frac{1}{2}x_{\ell-1} - \frac{1}{12}x_{\ell-2} + \frac{3}{8}x_{\ell-3} + \cdots)] \\ &= [d\pi \wedge x_\ell], \end{aligned}$$

where the last equality follows from the fact that $d\pi \wedge x_k$ is exact for all $0 < k < \ell$. □

**Remark 4.1.1.** After completing this proposition, the author later found a more general argument made by Pajitnov, where certain Massey products are computed to count lengths of Jordan blocks from cohomology and twisted cohomology. See [9] and [10] for such detail.

## 4.2 Primitive Cohomology and Explicit Generators

Next, we explore the product $m_2$ on $PH^*(X, \omega)$, where $X = S^1 \times Y_f$ and $\omega = dt \wedge d\phi + d\alpha$. To do so, we first construct explicit primitive forms that represent the isomorphisms in Theorem 1.3.1. From this point on, we use the notation consistent with [15]. Let each $[\gamma_{i,0}] \in \ker(f^* - 1 : H^1(\Sigma) \to H^1(\Sigma)) \subset PH^1(X)$ be in a Jordan block $\{\gamma_{i,0}, \gamma_{i,1}, \cdots, \gamma_{i,\ell_i}\}$. Then, there is some function $g_{i,0}$ on $\Sigma$ such that $f^*(\gamma_{i,0}) = \gamma_{i,0} + dg_{i,0}$. Let $\chi$ be a cutoff function on a neighborhood of $[0, 1]$ which is 0 near 0 and 1 near 1. Define the 1-form on $\tilde{\gamma}_{i,0} \in \Omega^*(\Sigma \times [0, 1])$ by

$$\tilde{\gamma}_{i,0}(x, t) = \gamma_{i,0}(x) + d(\chi(t)g_{i,0}(x)) = \gamma_{i,0}(x) + \chi(t)dg_{i,0}(x) + \chi'(t)g_{i,0}(x)dt.$$

Then $f^*(\tilde{\gamma}_{i,0}) = \gamma_{i,0}(x) + dg_{i,0}(x) + \chi(t)f^*(dg_{i,0}(x)) + \chi'(t)f^*(g_{i,0}(x))dt$. Let $N_0 = (-\epsilon, \epsilon)$ and $N_1 = (1 - \epsilon, 1 + \epsilon)$ be small neighborhoods of 0 and 1, respectively. It follows that

$$f^*(\tilde{\gamma}_{i,0})|_{t \in N_0} = \gamma_{i,0}(x) + dg_{i,0}(x) = \tilde{\gamma}_{i,0}(x)|_{t \in N_1}$$

and so $\tilde{\gamma}_{i,0}$ descends to a global one-form on $Y_f$. We still denote by $\tilde{\gamma}_{i,0}$ this one-form but use the coordinate $d\phi$ instead of $dt$. In a similar manner, we can construct global forms $\tilde{\gamma}_{i,k}$ for each $k = 1, 2, \cdots, \ell_i$ (of course, these won't be $d$-closed, in general). Consult [15] for this construction.

Letting $k = \dim \ker(f^* - 1)$, we have $PH^+_1(X) = \langle dt, d\phi, \tilde{\gamma}_{1,0}, \cdots, \tilde{\gamma}_{k,0} \rangle$ for each $\gamma_{i,0} \in \ker(f^* - 1)$. Moving on to $PH^2_+(X)$, we have one-forms $\langle dt, d\phi \rangle$ which need primitive two-form representatives which are $\partial_+\partial_-$-closed. Since $[\omega_\Sigma] \in H^2(Y_f)$ is trivial, there is some $\alpha \in \Omega^1(Y_f)$ such that $\omega_\Sigma = d\alpha$. Consider the element $d\phi \wedge \alpha$. Then $\omega \wedge (d\phi \wedge \alpha) = \omega_\Sigma \wedge d\phi \wedge \alpha = 0$, since it is a 4-form on $Y_f$. Moreover $d(d\phi \wedge \alpha) = -d\phi \wedge \omega_\Sigma = -d\phi \wedge \omega$. Therefore $\partial_-(d\phi \wedge \alpha) = -d\phi$ and it follows that $\partial_+\partial_-(d\phi \wedge \alpha) = -\partial_+(d\phi) = 0$. Thus $d\phi$ corresponds to the explicit primitive element $d\phi \wedge \alpha$ in $PH^2_+(X)$.

We claim $dt$ corresponds to the element $dt \wedge \alpha - \frac{1}{2}\omega \wedge \Lambda(dt \wedge \alpha)$. To see this element is primitive, recall the $\mathfrak{sl}(2)$ identity $[\Lambda, L] = H$. Hence for a 0-form $B_0$,

$$\Lambda(\omega \wedge B_0) - \omega \wedge \Lambda(B_0) = \Lambda(\omega \wedge B_0) = 2B_0.$$

In particular $\Lambda(\omega) = 2$. Similarly, for a 1-form $B_1$,

$$\Lambda(\omega \wedge B_1) - \omega \wedge \Lambda(B_1) = \Lambda(\omega \wedge B_1) = B_1.$$

It now follows immediately that $\Lambda(dt \wedge \alpha - \frac{1}{2}\omega \wedge \Lambda(dt \wedge \alpha)) = \Lambda(dt \wedge \alpha) - \Lambda(dt \wedge \alpha) = 0$ and so indeed the described element is primitive. It remains to show this element is $\partial_+\partial_-$-closed. To do so, we use the fact that $\partial_+\partial_-$ acting on primitive 2-forms takes the form $d\Lambda d$ (see [18] for a proof). Thus

$$\begin{aligned}
dB_2 &:= d(dt \wedge \alpha - \frac{1}{2}\omega \wedge \Lambda(dt \wedge \alpha)) \\
&= -dt \wedge d\alpha - \frac{1}{2}\omega \wedge d\Lambda(dt \wedge \alpha) \\
&= \omega \wedge (-dt - \frac{1}{2}d\Lambda(dt \wedge \alpha)) \\
\Lambda dB_2 &= -dt - \frac{1}{2}d\Lambda(dt \wedge \alpha),
\end{aligned}$$

and taking $d$ of the expression in the last equality clearly results in 0.

We summarize the generators in the table below. For the elements listed, but not discussed above, we refer the reader to [15].

Table 4.1: $PH^*_+(X)$ Elements

| k | dim $PH^k_+(X)$ | Generators for $PH^k_+(X)$ |
|---|---|---|
| 0 | 1 | 1 |
| 1 | $b_1(X)$ | $dt$, $d\phi$, $\tilde{\gamma}_{i,0}$, $i = 1, \cdots, k$ |
| 2 | $1 + b_2(X) + \nu_2(X)$ | $d\phi \wedge \tilde{\gamma}_{i,\ell_i}$, $\ell_i + 1$ size of corresponding Jordan block,<br>$dt \wedge \tilde{\gamma}_{i,0} - d(\chi' \mu_{i,0})$,<br>$d\phi \wedge \alpha$,<br>$dt \wedge \alpha - \frac{1}{2}\omega \wedge \Lambda(dt \wedge \alpha)$,<br>$dt \wedge \tilde{\gamma}_{i,1} + \chi' d\phi \wedge \mu_{i,0} - d(\chi' \mu_{i,1} + \chi'(\phi - 1)\mu_{i,0})$. |

Table 4.2: $PH^*_-(X)$ Elements

| k | dim $PH^k_-(X)$ | Generators for $PH^k_-(X)$ |
|---|---|---|
| 0 | 0 | $\emptyset$ |
| 1 | $b_3(X)$ | $\tilde{\gamma}_{i,\ell_i}$ |
| 2 | $b_2(X) + \nu_2(X)$ | $d\phi \wedge \tilde{\gamma}_{i,\ell_i}$,<br>$dt \wedge \tilde{\gamma}_{i,0} - d(\chi' \mu_{i,0})$,<br>$dt \wedge d\phi - \omega_\Sigma$,<br>$d\phi \wedge \sum_{j=1}^{\ell_k} \frac{(-1)^{j+1}}{j} \tilde{\gamma}_{k,\ell_k - j}$, for each $\ell_k > 0$. |

## 4.3 Primitive Massey Products

Fix a symplectic manifold $(M, \omega)$. We introduce a Massey product on $PH^*(M, \omega)$, denoted $\langle \cdot, \cdot, \cdot \rangle_s$. Motivated by the classic framework, suppose we have $m_1$-closed primitive forms $a_1, a_2, a_3$ such that

$$a_1 \times a_2 = m_1(a_{12}), \tag{4.2}$$

$$a_2 \times a_3 = m_1(a_{23}). \tag{4.3}$$

If we attempt to mimic the classic Massey product by $a_{12} \times a_3 - (-1)^{|a_1|} a_1 \times a_{23}$, unfortunately we have

$$m_1(a_{12} \times a_3 - (-1)^{|a_1|} a_1 \times a_{23}) = (a_1 \times a_2) \times a_3 - a_1 \times (a_2 \times a_3) \neq 0.$$

But by the $A_\infty$-relations, we know this associator term equals $-m_1 m_3(a_1, a_2, a_3)$ (since the $a_i$ are $m_1$-closed). Thus we can add a correction term, leading to the following definition.

**Definition 4.3.1** (primitive Massey product)**.** Let $a_1, a_2, a_3$ be $m_1$-closed primitive forms of degrees $k_1, k_2, k_3$, satisfying equations (4.2) and (4.3). The degree $-1$ *primitive Massey product* is given by

$$\langle a_1, a_2, a_3 \rangle_s = a_{12} \times a_3 - (-1)^{|a_1|} a_1 \times a_{23} + m_3(a_1, a_2, a_3).$$

As in the de Rham cohomology case, this product will have indeterminacy and therefore be a subset of elements in $PH^{k_1+k_2+k_3-1}(M)$. Like before, we can choose a representative in the quotient $PH^{k_1+k_2+k_3-1}(M)/(a_1 \times PH^{k_2+k_3-1} + PH^{k_1+k_2-1} \times a_3)$. Moreover, the definition of $\langle a_1, a_2, a_3 \rangle_s$ only depends on the primitive cohomology classes $[a_1], [a_2], [a_3]$.

**Proposition 4.3.1.** *The primitive Massey product $\langle a_1, a_2, a_3 \rangle_s$ is independent of each cohomology representative of $[a_i]$.*

*Proof.* By linearity of the product, it suffices to verify the three cases

1. $\langle a_1 + m_1 B, a_2, a_3 \rangle_s = \langle a_1, a_2, a_3 \rangle_s$,

2. $\langle a_1, a_2 + m_1 B, a_3 \rangle_s = \langle a_1, a_2, a_3 \rangle_s$,

3. $\langle a_1, a_2, a_3 + m_1 B \rangle_s = \langle a_1, a_2, a_3 \rangle_s$,

where $B$ is a primitive form of appropriate degree. For 1., suppose we have

$$a_{12} \times a_3 - (-1)^{|a_1|} a_1 \times a_{23} + m_3(a_1, a_2, a_3) \in \langle a_1, a_2, a_3 \rangle_s.$$

Then $(a_1 + m_1 B) \times a_2 = m_1(a_{12} + B \times a_2)$ and it follows that

$$(a_{12} + B \times a_2) \times a_3 - (-1)^{|a_1|}(a_1 + m_1 B) \times a_{23} + m_3(a_1 + m_1 B, a_2, a_3) \tag{4.4}$$

$$= \big(a_{12} \times a_3 - (-1)^{|a_1|} a_1 \times a_{23} + m_3(a_1, a_2, a_3)\big) \tag{4.5}$$

$$+ (B \times a_2) \times a_3 - (-1)^{|a_1|} m_1 B \times a_{23} + m_3(m_1 B, a_2, a_3) \tag{4.6}$$

is a representative of $\langle a_1 + m_1 B, a_2, a_3 \rangle_s$. Using the Leibniz rule on $m_1$ we have that

$$m_1(B \times a_{23}) = m_1 B \times a_{23} + (-1)^{|a_1|-1} B \times (a_2 \times a_3)$$

and so equation (4.6) becomes,

$$\begin{aligned}
&(B \times a_2) \times a_3 - B \times (a_2 \times a_3) - (-1)^{|a_1|} m_1(B \times a_{23}) + m_3(m_1 B, a_2, a_3) \\
&= -m_1 m_3(B, a_2, a_3) - m_3(m_1 B, a_2, a_3) - (-1)^{|a_1|} m_1(B \times a_{23}) + m_3(m_1 B, a_2, a_3) \\
&= -m_1 \left[ m_3(B, a_2, a_3) + (-1)^{|a_1|} B \times a_{23} \right].
\end{aligned}$$

The second equality follows from the $m_3$-relation and the fact that the $a_i$ are $m_1$-closed. This shows $\langle a_1, a_2, a_3 \rangle \subseteq \langle a_1 + m_1 B, a_2, a_3 \rangle$, since varying $a_1$ by an $m_1$-exact term only changes the representative by an $m_1$-exact term. By reversibility of the argument, we see the other inclusion follows similarly.

For 2., again suppose $a_{12} \times a_3 - (-1)^{|a_1|} a_1 \times a_{23} + m_3(a_1, a_2, a_3) \in \langle a_1, a_2, a_3 \rangle_s$. Then

$$\begin{aligned}
&a_1 \times (a_2 + m_1 B) = m_1(a_{12} + (-1)^{|a_1|} a_1 \times B), \\
&(a_2 + m_1 B) \times a_3 = m_1(a_{23} + B \times a_3).
\end{aligned}$$

This construction yields

$$(a_{12} + (-1)^{|a_1|} a_1 \times B) \times a_3 - (-1)^{|a_1|} a_1 \times (a_{23} + B \times a_3) + m_3(a_1, a_2 + m_1 B, a_3) \tag{4.7}$$

$$= (a_{12} \times a_3 - (-1)^{|a_1|} a_1 \times a_{23} + m_3(a_1, a_2, a_3)) \tag{4.8}$$

$$+ (-1)^{|a_1|}(a_1 \times B) \times a_3 - (-1)^{|a_1|} a_1 \times (B \times a_3) + m_3(a_1, m_1 B, a_3) \tag{4.9}$$

as a representative of $\langle a_1, a_2 + m_1 B, a_3 \rangle_s$. Using the fact that $a_1$ and $a_3$ are $m_1$-closed, the $m_3$- relation on $a_1 \otimes B \otimes a_3$ says that

$$a_1 \times (B \times a_3) - (a_1 \times B) \times a_3 = m_1 m_3(a_1, B, a_3) + (-1)^{|a_1|} m_3(a_1, m_1 B, a_3).$$

Applying this equality to term (4.9), we obtain

$$= -m_1 m_3(a_1, B, a_3) - m_3(a_1, m_1 B, a_3) + m_3(a_1, m_1 B, a_3)$$
$$= -m_1 m_3(a_1, B, a_3).$$

This establishes the inclusion $\langle a_1, a_2, a_3 \rangle_s \subseteq \langle a_1, a_2 + m_1 B, a_3 \rangle$ and the reverse follows from symmetry. Finally, 3. follows the same argument as in case 1., after taking into account our sign convention. □

### 4.3.1 Higher Primitive Massey Products

Next, we extend the 3-fold primitive Massey product to a 4-fold product. To do so, however, requires some additional setup compared to the 3-fold product. Like in the previous case, let $[a_1], [a_2], [a_3], [a_4] \in PH^*(M, \omega)$ such that

$$a_1 \times a_2 = m_1 a_{12}, \tag{4.10}$$

$$a_2 \times a_3 = m_1 a_{23}, \tag{4.11}$$

$$a_3 \times a_4 = m_1 a_{34} \tag{4.12}$$

for some choice of representatives $a_1, a_2, a_3, a_4$ and $a_{12}, a_{23}, a_{34}$. Additionally, we will require that the two 3-fold products $\langle [a_1], [a_2], [a_3] \rangle_s$ and $\langle [a_2], [a_3], [a_4] \rangle_s$ contain the cohomology element 0 in a compatible way. That is, choose representatives

$$x = a_{12} \times a_3 - (-1)^{|a_1|} a_1 \times a_{23} + m_3(a_1, a_2, a_3), \tag{4.13}$$

$$y = a_{23} \times a_4 - (-1)^{|a_2|} a_2 \times a_{34} + m_3(a_2, a_3, a_4), \tag{4.14}$$

of $\langle [a_1], [a_2], [a_3] \rangle_s$ and $\langle [a_2], [a_3], [a_4] \rangle_s$, respectively. We define $(\langle [a_1], [a_2], [a_3] \rangle_s, \langle [a_2], [a_3], [a_4] \rangle_s)$ to be the tuple of cohomology elements $([x], [y])$ that can be constructed in this way. The point is that the Massey product representatives in this set come from the *same* elements $a_{ij}$. Then our second requirement, the simultaneous vanishing of triple Massey products, is given by the exactness of equations (4.13) and (4.14). Thus, there also exist forms $c_{123}$ and $c_{234}$ so that

$$x = m_1 c_{123}, \tag{4.15}$$

$$y = m_1 c_{234}. \tag{4.16}$$

The forms $a_{ij}$ and $c_{ijk}$ provide the defining system for the 4-fold Massey product introduced below.

**Definition 4.3.2.** Let $a_1, a_2, a_3, a_4, a_{12}, a_{23}, a_{34}, c_{123}, c_{234}$ be chosen to satisfy equations (4.10)-(4.16). We define the 4-fold primitive Massey product $\langle a_1, a_2, a_3, a_4 \rangle_s$ to be the set of all representatives of the form

$$z = c_{123}\times a_4-(-1)^{|a_{12}|}a_{12}\times a_{34}+a_1\times c_{234}+m_3(a_{12},a_3,a_4)-(-1)^{|a_1|}m_3(a_1,a_{23},a_4)+(-1)^{|a_1|+|a_2|}m_3(a_1,a_2,a_{34}).$$

**Proposition 4.3.2.** *The Massey product introduced in Definition 4.3.2 is $m_1$-closed, and so descends to a representative in $PH^*(M,\omega)$.*

*Proof.* To prove this claim, we investigate $m_1$ of the two parts of $z$ separately; the terms involving $m_3$, and those not. We begin by calculating $m_1$ of the first three terms of $z$ in Definition 4.3.2,

$$\begin{aligned}
&m_1(c_{123}\times a_4-(-1)^{|a_{12}|}a_{12}\times a_{34}+a_1\times c_{234})\\
=&(a_{12}\times a_3-(-1)^{|a_1|}a_1\times a_{23}+m_3(a_1,a_2,a_3))\times a_4\\
&-(-1)^{|a_{12}|}(a_1\times a_2)\times a_{34}-a_{12}\times(a_3\times a_4)\\
&+(-1)^{|a_1|}a_1\times(a_{23}\times a_4-(-1)^{|a_2|}a_2\times a_{34}+m_3(a_2,a_3,a_4))
\end{aligned}$$

$$\begin{aligned}
=&((a_{12}\times a_3)\times a_4-a_{12}\times(a_3\times a_4)) &(4.17)\\
&+(-1)^{|a_1|}(a_1\times(a_{23}\times a_4)-(a_1\times a_{23})\times a_4) &(4.18)\\
&+(-1)^{|a_1|+|a_2|}((a_1\times a_2)\times a_{34}-a_1\times(a_2\times a_{34})) &(4.19)\\
&+m_3(a_1,a_2,a_3)\times a_4+(-1)^{|a_1|}a_1\times m_3(a_2,a_3,a_4). &(4.20)
\end{aligned}$$

Using the $m_3$-relation, we can transform each of the terms in lines (4.17)-(4.19) into expressions involving $m_1$ and $m_3$ to get

$$\begin{aligned}
&- m_1m_3(a_{12}, a_3, a_4) - m_3(a_1 \times a_2, a_3, a_4) \\
&- (-1)^{|a_1|+|a_2|}m_1m_3(a_1, a_2, a_{34}) - m_3(a_1, a_2, a_3 \times a_4) \\
&+ (-1)^{|a_1|}m_1m_3(a_1, a_{23}, a_4) + m_3(a_1, a_2 \times a_3, a_4) \\
&+ m_3(a_1, a_2, a_3) \times a_4 + (-1)^{|a_1|}a_1 \times m_3(a_2, a_3, a_4).
\end{aligned}$$

Now, adding $m_1$ of the remaining three terms in $z$ to the above sum leaves only

$$\begin{aligned}
&- m_3(a_1 \times a_2, a_3, a_4) - m_3(a_1, a_2, a_3 \times a_4) + m_3(a_1, a_2 \times a_3, a_4) \\
&+ m_3(a_1, a_2, a_3) \times a_4 + (-1)^{|a_1|}a_1 \times m_3(a_2, a_3, a_4).
\end{aligned}$$

However, using the fact that $m_4 = 0$ and the $a_i$ are $m_1$-closed, the above expression is precisely the $m_4$-relation equaling zero. Thus, $z$ is closed under $m_1$ and defines a class in $PH^*(M, \omega)$. □

We note that, with some work, one can generalize the methods of sections 4.3 to primitive Massey products of any length. Moreover, we remark that the calculations in this section are not unique to $PH^*(M)$, and in fact extend to any $A_3$-algebra.

## 4.4 Sphere Bundle Perspective

In [13], Tanaka and Tseng show the existence of a circle bundle $E$ over any symplectic manifold $(M, \omega)$ such that $(\Omega^*(E), d, \wedge)$ is quasi-isomorphic to $(\mathcal{P}^*(M), m_1, m_2, m_3)$. Consequently, $H^*(E) \cong PH^*(M)$. Moreover, the provide an explicit quasi-isomorphism $(f_n) : \Omega^*(E) \to \mathcal{P}^*(M)$ with $f_i = 0$ for $i \geq 3$. We won't go into the details, but the important properties of the $(f_1, f_2)$ are

$$f_1(dA) = m_1 f_1(A), \tag{4.21}$$

$$f_1(a \wedge b) = f_1(a) \times f_1(b) + m_1 f_2(a,b) + f_2(da,b) + (-1)^{|a|} f_2(a,db), \tag{4.22}$$

$$f_2(a \wedge b, c) - f_2(a, b \wedge c) = m_3(f_1(a), f_1(b), f_1(c)) + (-1)^{|a|} f_1(a) \times f_2(b,c) - f_2(a,b) \times f_1(c). \tag{4.23}$$

In particular, when $a$ and $b$ are d-closed, identity 4.22 implies $[f_1(a \wedge b)] = [f_1(a) \times f_1(b)]$, so that to study the product structure on $PH^*(M, \omega)$, it suffices to evaluate the (usual) wedge product on $H^*(E)$. Furthermore, $f_1$ also preserves Massey products, so that a Massey product on $H^*(E)$ is sent to a (primitive) Massey product on $PH^*(M, \omega)$. We prove this statement before proceeding.

**Lemma 4.4.1.** *Let* $\langle [a_1], [a_2], [a_3] \rangle \in H^*(E)$. *Then* $f_1(\langle [a_1], [a_2], [a_3] \rangle) \in \langle [f_1(a_1)], [f_1(a_2)], [f_1(a_3)] \rangle_s$.

*Proof.* Suppose $a_1 \wedge a_2 = da_{12}$ and $a_2 \wedge a_3 = da_{23}$. Applying identities (4.21) and (4.22) to these equations yield

$$\begin{aligned} m_1 f_1(a_{12}) &= f_1(a_1) \times f_1(a_2) + m_1 f_2(a_1, a_2), \\ m_1 f_1(a_{23}) &= f_1(a_2) \times f_1(a_3) + m_1 f_2(a_2, a_3), \\ \implies f_1(a_1) \times f_1(a_2) &= m_1(f_1(a_{12}) - f_2(a_1, a_2)), \\ f_1(a_2) \times f_1(a_3) &= m_1(f_1(a_{23}) - f_2(a_2, a_3)). \end{aligned}$$

Then using the appropriate identities (4.21)-(4.23), a representative of $\langle f_1(a_1), f_1(a_2), f_1(a_3) \rangle_s$

is given by

$$
\begin{aligned}
&(f_1(a_{12}) - f_2(a_1, a_2)) \times f_1(a_3) - (-1)^{|a_1|} f_1(a_1) \times (f_1(a_{23}) - f_2(a_2, a_3)) + m_3(f_1(a_1), f_1(a_2), f_1(a_3)) \\
=&f_1(a_{12}) \times f_1(a_3) - (-1)^{|a_1|} f_1(a_1) \times f_1(a_{23}) + m_3(f_1(a_1), f_1(a_2), f_1(a_3)) \\
&- f_2(a_1, a_2) \times f_1(a_3) + (-1)^{|a_1|} f_1(a_1) \times f_2(a_2, a_3) \\
=&f_1(a_{12}) \times f_1(a_3) - (-1)^{|a_1|} f_1(a_1) \times f_1(a_{23}) + f_2(a_1 \wedge a_2, a_3) - f_2(a_1, a_2 \wedge a_3) \\
&- (-1)^{|a_1|} f_1(a_1) \times f_2(a_2, a_3) + f_2(a_1, a_2) \times f_1(a_3) - f_2(a_1, a_2) \times f_1(a_3) + (-1)^{|a_1|} f_1(a_1) \times f_2(a_2, a_3) \\
=&f_1(a_{12}) \times f_1(a_3) - (-1)^{|a_1|} f_1(a_1) \times f_1(a_{23}) + f_2(a_1 \wedge a_2, a_3) - f_2(a_1, a_2 \wedge a_3).
\end{aligned}
$$

On the other hand, using identity (4.22),

$$
\begin{aligned}
&f_1(a_{12} \wedge a_3 - (-1)^{|a_1|} a_1 \wedge a_{23}) \\
=&f_1(a_{12}) \times f_1(a_3) + m_1 f_2(a_{12}, a_3) + f_2(a_1 \wedge a_2, a_3) - (-1)^{|a_1|} f_1(a_1) \times f_1(a_{23}) \\
&- (-1)^{|a_1|} m_1 f_2(a_1, a_{23}) - f_2(a_1, a_2 \wedge a_3).
\end{aligned}
$$

Thus, the two representatives of $f_1(\langle a_1, a_2, a_3 \rangle)$ and $\langle f_1(a_1), f_1(a_2), f_1(a_3) \rangle_s$ only differ by an $m_1$-exact term and so are equal in $PH^*(M, \omega)$. □

With the necessary propositions established, we can (justifiably) move forward in computing the product and Massey structures on $PH^*(X, \omega)$ through the aid of $H^*(E)$. We let $\theta$ denote the connection 1-form on $E$, which satisfies the property $d\theta = \omega$.

We summarize the de Rham cohomology of $E^5$ for $X = S^1 \times Y_f$ with an *open* fiber. As explained above, these groups are isomorphic to $PH^*(X)$ and so the generators below should be reminiscent of those given in Section 4.2

| k | Generators for $H^k(E)$ |
|---|---|
| 0 | 1 |
| 1 | $dt$, $d\phi$, $\tilde{\gamma}_{i,0}$ |
| 2 | $d\phi \wedge \tilde{\gamma}_{i,\ell_i}$, $dt \wedge \tilde{\gamma}_{i,0}$, $d\phi \wedge (\theta - \alpha)$, $dt \wedge (\theta - \alpha)$, $\theta \wedge \tilde{\gamma}_{i,0} + dt \wedge \tilde{\gamma}_{i,1} + \chi' d\phi \wedge \mu_{i,0}$ |
| 3 | $d\phi \wedge \tilde{\gamma}_{i,\ell_i} \wedge \theta$, $(dt \wedge \tilde{\gamma}_{i,0} - d(\chi' \mu_{i,0})) \wedge \theta$, $dt \wedge d\phi \wedge (\theta - \alpha)$, $\theta \wedge d\tilde{\gamma}_{i,\ell_i}$ |
| 4 | $dt \wedge d\phi \wedge \tilde{\gamma}_{i,\ell_i} \wedge \theta$ |
| 5 | $\emptyset$ |

Next, we compute the wedge product structure on $H^*(E)$ for most of the non-trivial pairings. Before beginning, we cover some useful observations in the computations to follow.

**Lemma 4.4.2.** *The following identities hold in* $H^*(E)$ *for* $X = S^1 \times Y_f$,

$$[\theta \wedge d\tilde{\gamma}_{i,k}] = \begin{cases} 0, & k < \ell_i \\ [dt \wedge d\phi \wedge \tilde{\gamma}_{i,k}], & k = \ell_i \end{cases} \tag{4.24}$$

$$[d\phi \wedge \tilde{\gamma}_{i,\ell_i} \wedge (\theta - \alpha)] = [d\phi \wedge \tilde{\gamma}_{i,\ell_i} \wedge \theta]. \tag{4.25}$$

*Proof.* We begin with observation (4.24). First, notice

$$d(\theta \wedge \tilde{\gamma}_{i,k}) = \omega \wedge \tilde{\gamma}_{i,k} - \theta \wedge d\tilde{\gamma}_{i,k},$$

which implies

$$[\omega \wedge \tilde{\gamma}_{i,k}] = [\theta \wedge d\tilde{\gamma}_{i,k}].$$

Furthermore,

$$
\begin{aligned}
\omega_\Sigma \wedge \tilde{\gamma}_{i,k} &= \omega_\Sigma \wedge \left( \sum_{j=0}^{k} f_j(\phi)\gamma_{i,k-j} + f_j(\phi-1)(\chi dg_{i,k-j} + g_{i,k-j}\chi' d\phi) \right) \\
&= \sum_{j=0}^{k} \chi' f_j(\phi-1) g_{i,k-j} d\phi \wedge \omega_\Sigma \\
&= \sum_{j=0}^{k} d(\chi' f(\phi-1)\mu_{i,k-j} \wedge d\phi) := dU_{i,k}, \\
U_{i,k} &= \sum_{j=0}^{k} \chi' f(\phi-1)\mu_{i,k-j} \wedge d\phi.
\end{aligned}
$$

Combining the above two computations shows $[\theta \wedge d\tilde{\gamma}_{i,k}] = [\omega \wedge \tilde{\gamma}_{i,k}] = [dt \wedge d\phi \wedge \tilde{\gamma}_{i,k}]$. Moreover, if $k < \ell_i$, $d\phi \wedge \tilde{\gamma}_{i,k}$ is $d-$exact. In particular, for a Jordan block of size at least three $\{\gamma_{i,0}, \gamma_{i,1}, \gamma_{i,2}, \cdots\}$ we have

$$
\begin{aligned}
&dt \wedge d\phi \wedge \tilde{\gamma}_{i,0} = d(-dt \wedge \tilde{\gamma}_{i,1}), \\
&dt \wedge d\phi \wedge \tilde{\gamma}_{i,1} = d(-dt \wedge (\tilde{\gamma}_{i,2} + \frac{1}{2}\tilde{\gamma}_{i,1})).
\end{aligned}
$$

Turning to (4.25), we expand

$$
\begin{aligned}
&d\phi \wedge \tilde{\gamma}_{i,\ell_i} \wedge (\theta - \alpha) - d\phi \wedge \tilde{\gamma}_{i,\ell_i} \wedge \theta = d\phi \wedge \tilde{\gamma}_{i,\ell_i} \wedge \alpha \\
&= d\phi \wedge \left( \sum_{j=0}^{\ell_i} f_j(\phi)\gamma_{\ell_i-j} + f_j(\phi-1)\chi(\phi) dg_{i,\ell_i-j} \right) \wedge \alpha \\
&= d\phi \wedge \sum_{j=0}^{\ell_i} f_j(\phi) dA_{\ell_i-j} + f_j(\phi-1)\chi(\phi) dB_{i,\ell_i-j} \\
&= d\left( -d\phi \wedge \sum_{j=0}^{\ell_i} f_j(\phi) A_{\ell_i-j} + f_j(\phi-1)\chi(\phi) B_{i,\ell_i-j} \right),
\end{aligned}
$$

where in the third line we have used the fact that $\alpha \wedge \gamma_{i,k}$ and $\alpha \wedge dg_{i,k}$ are exact in $\Omega^2(\Sigma)$. □

**Theorem 4.4.1.** *For the symplectic manifold $X = S^1 \times Y_f$ with open fiber and symplectic* *form $\omega = dt \wedge d\phi + d\alpha$, the $m_2$-structure on $PH^*(X, \omega)$ is summarized in Tables 4.3 - 4.6 below, in terms of the wedge product on $H^*(E)$.*

| $H^1(E)$ | $H^1(E)$ | $H^2(E)$ |
|---|---|---|
| $dt$ | $dt$ | $[0]$ |
| | $d\phi$ | $[0]$ |
| | $\tilde{\gamma}_{i,0}$ | $dt \wedge \tilde{\gamma}_{i,0}$ |
| $d\phi$ | $d\phi$ | $[0]$ |
| | $\tilde{\gamma}_{i,0}$ | $\begin{cases} [d\phi \wedge \tilde{\gamma}_{i,0}]\,, & \ell_i = 0 \\ [0]\,, & \ell_i > 0 \end{cases}$ |
| $\tilde{\gamma}_{i,0}$ | $\tilde{\gamma}_{j,0}$ | $[d\phi \wedge F(\gamma_{i,0}, \gamma_{j,0})]$ |

Table 4.3: $H^1(E) \wedge H^1(E) \to H^2(E)$

| $H^1(E)$ | $H^2(E)$ | $H^3(E)$ |
|---|---|---|
| $dt$ | $d\phi \wedge \tilde{\gamma}_{i,\ell_i}$ | $\theta \wedge d\tilde{\gamma}_{i,\ell_i}$ |
| | $dt \wedge \tilde{\gamma}_{i,0}$ | $[0]$ |
| | $d\phi \wedge (\theta - \alpha)$ | $dt \wedge d\phi \wedge (\theta - \alpha)$ |
| | $dt \wedge (\theta - \alpha)$ | $[0]$ |
| | $\theta \wedge \tilde{\gamma}_{i,0} + dt \wedge \tilde{\gamma}_{i,1} + \chi' d\phi \wedge \mu_{i,0}$ | $-[(dt \wedge \tilde{\gamma}_{i,0} - d(\chi' \mu_{i,0})) \wedge \theta]$ |
| $d\phi$ | $d\phi \wedge \tilde{\gamma}_{i,\ell_i}$ | $[0]$ |
| | $dt \wedge \tilde{\gamma}_{i,0}$ | $[0]$ |
| | $d\phi \wedge (\theta - \alpha)$ | $[0]$ |
| | $dt \wedge (\theta - \alpha)$ | $-[dt \wedge d\phi \wedge (\theta - \alpha)]$ |
| | $\theta \wedge \tilde{\gamma}_{i,0} + dt \wedge \tilde{\gamma}_{i,1} + \chi' d\phi \wedge \mu_{i,0}$ | $\begin{cases} -[2\theta \wedge d\tilde{\gamma}_{i,1}]\,, & \ell_i = 1 \\ [0]\,, & \ell_i > 1 \end{cases}$ |
| $\tilde{\gamma}_{i,0}$ | $d\phi \wedge \tilde{\gamma}_{i,\ell_i}$ | $[0]$ |
| | $dt \wedge \tilde{\gamma}_{j,0}$ | $[0]$ |

| | | |
|---|---|---|
| | $d\phi \wedge (\theta - \alpha)$ | $\begin{cases} -[d\phi \wedge \tilde{\gamma}_{i,0} \wedge (\theta - \alpha)], & \ell_i = 0 \\ -[dt \wedge d\phi \wedge \tilde{\gamma}_{i,1}], & \ell_i = 1 \\ [0]\,, & \ell_i > 1 \end{cases}$ |
| | $dt \wedge (\theta - \alpha)$ | $-[(dt \wedge \tilde{\gamma}_{i,0} - d(\chi' \mu_{i,0})) \wedge \theta]$ |

Table 4.4: $H^1(E) \wedge H^2(E) \to H^3(E)$

| $H^1(E)$ | $H^3(E)$ | $H^4(E)$ |
|---|---|---|
| $dt$ | $d\phi \wedge \tilde{\gamma}_{i,\ell_i} \wedge \theta$ | $[dt \wedge d\phi \wedge \tilde{\gamma}_{i,\ell_i} \wedge \theta]$ |
| | $(dt \wedge \tilde{\gamma}_{i,0} - d(\chi' \mu_{i,0})) \wedge \theta$ | $[0]$ |
| | $dt \wedge d\phi \wedge (\theta - \alpha)$ | $[0]$ |
| | $\theta \wedge d\tilde{\gamma}_{i,\ell_i}$ | $[0]$ |
| $d\phi$ | $d\phi \wedge \tilde{\gamma}_{i,\ell_i} \wedge \theta$ | $[0]$ |
| | $(dt \wedge \tilde{\gamma}_{i,0} - d(\chi' \mu_{i,0})) \wedge \theta$ | $\begin{cases} -[dt \wedge d\phi \wedge \tilde{\gamma}_{i,0} \wedge \theta], & \ell_i = 0 \\ [0]\,, & \ell_i > 0 \end{cases}$ |
| | $dt \wedge d\phi \wedge (\theta - \alpha)$ | $[0]$ |
| | $\theta \wedge d\tilde{\gamma}_{i,\ell_i}$ | $[0]$ |
| $\tilde{\gamma}_{i,0}$ | $d\phi \wedge \tilde{\gamma}_{j,\ell_j} \wedge \theta$ | $[0]$ |
| | $(dt \wedge \tilde{\gamma}_{j,0} - d(\chi' \mu_{j,0})) \wedge \theta$ | $-[dt \wedge d\phi \wedge \tilde{f}(\gamma_{i,0}, \gamma_{j,0}) \wedge \theta]$ |
| | $dt \wedge d\phi \wedge (\theta - \alpha)$ | $\begin{cases} [dt \wedge d\phi \wedge \tilde{\gamma}_{i,0} \wedge \theta], & \ell_i = 0 \\ [0]\,, & \ell_i > 0 \end{cases}$ |
| | $\theta \wedge d\tilde{\gamma}_{j,\ell_j}$ | $[0]$ |

Table 4.5: $H^1(E) \wedge H^3(E) \to H^4(E)$

| $H^2(E)$ | $H^2(E)$ | $H^4(E)$ |
|---|---|---|
| $d\phi \wedge \tilde{\gamma}_{i,\ell_i}$ | $d\phi \wedge \tilde{\gamma}_{j,\ell_j}$ | $[0]$ |
| | $dt \wedge \tilde{\gamma}_{j,0}$ | $[0]$ |

| | | |
|---|---|---|
| | $d\phi \wedge (\theta - \alpha)$ | $[0]$ |
| | $dt \wedge (\theta - \alpha)$ | $[dt \wedge d\phi \wedge \tilde{\gamma}_{j,\ell_j} \wedge \theta]$ |
| | $\theta \wedge \tilde{\gamma}_{j,0} + dt \wedge \tilde{\gamma}_{j,1} + \chi' d\phi \wedge \mu_{j,0}$ | $[0]$ |
| $dt \wedge \tilde{\gamma}_{i,0}$ | $dt \wedge \tilde{\gamma}_{j,0}$ | $[0]$ |
| | $d\phi \wedge (\theta - \alpha)$ | $\begin{cases} -[dt \wedge d\phi \wedge \tilde{\gamma}_{i,0} \wedge \theta], & \ell_i = 0 \\ [0]\,, & \ell_i > 0 \end{cases}$ |
| | $dt \wedge (\theta - \alpha)$ | $[0]$ |
| | $\theta \wedge \tilde{\gamma}_{j,0} + dt \wedge \tilde{\gamma}_{j,1} + \chi' d\phi \wedge \mu_{j,0}$ | $-[dt \wedge d\phi \wedge f(\gamma_{i,0}, \gamma_{j,0}) \wedge \theta]$ |
| $d\phi \wedge (\theta - \alpha)$ | $d\phi \wedge (\theta - \alpha)$ | $[0]$ |
| | $dt \wedge (\theta - \alpha)$ | $[0]$ |
| | $\theta \wedge \tilde{\gamma}_{i,0} + dt \wedge \tilde{\gamma}_{i,1} + \chi' d\phi \wedge \mu_{i,0}$ | $\begin{cases} [-dt \wedge d\phi \wedge \tilde{\gamma}_{i,1} \wedge \theta], & \ell_i = 1 \\ [0]\,, & \ell_i > 1 \end{cases}$ |
| $dt \wedge (\theta - \alpha)$ | $dt \wedge (\theta - \alpha)$ | $[0]$ |
| | $\theta \wedge \tilde{\gamma}_{i,0} + dt \wedge \tilde{\gamma}_{i,1} + \chi' d\phi \wedge \mu_{i,0}$ | $[0]$ |

Table 4.6: $H^2(E) \wedge H^2(E) \to H^4(E)$

*Proof.* Many of these computations are quite long and tedious. We provide the proof of only a few below.

$\underline{\tilde{\gamma}_{i,0} \wedge (d\phi \wedge \tilde{\gamma}_{j,\ell_j} \wedge \theta) = [0] :}$

Using Lemma 4.4.2,

$$\begin{aligned} \tilde{\gamma}_{i,0} \wedge (d\phi \wedge \tilde{\gamma}_{j,\ell_j} \wedge \theta) &= -d\phi \wedge \tilde{\gamma}_{i,0} \wedge \tilde{\gamma}_{j,\ell_j} \wedge \theta \\ &= d\left( d\phi \wedge \theta \wedge \sum_{k=0}^{\ell_j} f_k(\phi) A_{i,j,k} - f_k(\phi - 1)\chi g_{j,\ell_j - k}\gamma_{i,0} + f_k(\phi)\chi g_{i,0}\gamma_{j,\ell_j} \right), \end{aligned}$$

where the last equality follows since

$$\omega \wedge d\phi \wedge \sum_{k=0}^{\ell_j} f_k(\phi) A_{i,j,k} - f_k(\phi - 1)\chi g_{j,\ell_j - k}\gamma_{i,0} + f_k(\phi)\chi g_{i,0}\gamma_{j,\ell_j} = 0.$$

$\underline{dt \wedge (\theta - \alpha) \wedge (\theta \wedge \tilde{\gamma}_{i,0} + dt \wedge \tilde{\gamma}_{i,1} + \chi' d\phi \wedge \mu_{i,0}) = [0] :}$

$$\begin{aligned}
dt \wedge (\theta - \alpha) \wedge (\theta \wedge \tilde{\gamma}_{i,0} + dt \wedge \tilde{\gamma}_{i,1} + \chi' d\phi \wedge \mu_{i,0}) &= -dt \wedge \alpha \wedge \theta \wedge \tilde{\gamma}_{i,0} + dt \wedge (\theta - \alpha) \wedge \chi' d\phi \wedge \mu_{i,0} \\
&= -dt \wedge \alpha \wedge \theta \wedge \tilde{\gamma}_{i,0} + d(\chi dt \wedge (\theta - \alpha) \wedge \mu_{i,0}) - \chi g_{i,0} dt \wedge (\theta - \alpha) \wedge d\alpha \\
&= d(\chi dt \wedge (\theta - \alpha) \wedge \mu_{i,0}) - dt \wedge (\alpha \wedge \theta \wedge \tilde{\gamma}_{i,0} + \chi g_{i,0}\theta \wedge d\alpha) \\
&= d(\chi dt \wedge (\theta - \alpha) \wedge \mu_{i,0}) - dt \wedge (\alpha \wedge \theta \wedge \tilde{\gamma}_{i,0} + d(\chi g_{i,0}) \wedge \theta \wedge \alpha + \chi g_{i,0} dt \wedge d\phi \wedge \alpha - d(\chi g_{i,0}\theta \wedge \alpha)) \\
&= d(\chi dt \wedge (\theta - \alpha) \wedge \mu_{i,0} - \chi g_{i,0} dt \wedge \theta \wedge \alpha) - dt \wedge (\alpha \wedge \theta \wedge \tilde{\gamma}_{i,0} + d(\chi g_{i,0}) \wedge \theta \wedge \alpha) \\
&= d(\chi dt \wedge (\theta - \alpha) \wedge \mu_{i,0} - \chi g_{i,0} dt \wedge \theta \wedge \alpha) - dt \wedge \alpha \wedge \theta \wedge \gamma_{i,0} \\
&= d(\chi dt \wedge (\theta - \alpha) \wedge \mu_{i,0} - \chi g_{i,0} dt \wedge \theta \wedge \alpha - dt \wedge A_i \wedge \theta),
\end{aligned}$$

where we have written $\alpha \wedge \tilde{\gamma}_{i,0} = dA_i$.

$\underline{\tilde{\gamma}_{i,0} \wedge d\phi \wedge (\theta - \alpha) :}$

We note that if $\ell_i = 0$ then by Lemma 4.4.2, $H^3(E)$ contains the non-trivial element

$$[d\phi \wedge \tilde{\gamma}_{i,0} \wedge \theta] = [d\phi \wedge \tilde{\gamma}_{i,0} \wedge (\theta - \alpha)].$$

Otherwise, if $\ell_i \geq 1$, again Lemma 4.4.2 gives us

$$[\tilde{\gamma}_{i,0} \wedge d\phi \wedge (\theta - \alpha)] = [\tilde{\gamma}_{i,0} \wedge d\phi \wedge \theta] = -[d\phi \wedge \tilde{\gamma}_{i,0} \wedge \theta] = -[d\tilde{\gamma}_{i,1} \wedge \theta]$$

By considering whether $\ell_i = 1$ or $\ell_i > 1$ on the above equality, the result follows. $\square$

By recalling $\ell_i = 0$ is Jordan block of size 1, $\ell_i = 1$ is a Jordan block of size 2, and $\ell_i > 1$ is a Jordan block of size at least 3, we obtain the following important corollary.

**Corollary 4.4.1.** *Let $X = S^1 \times Y_f$ and $\omega = dt \wedge d\phi + d\alpha$. The $m_2$-structure $\times$ on $PH^2(X, \omega)$ can determine whether the Jordan blocks of $f^* - 1$ are of size 1,2, or at least 3.*

We remark that, in terms of Jordan block size, the structure on $PH^*(X, \omega)$ is 'one step ahead' of the structure on $H^*(X)$; the dimension of $PH^*(X)$ determines size 2 blocks whereas the wedge product on $H^*(X)$ does so. Similarly, the $\times$ product on $PH^*(X)$ determines size 3 or greater Jordan blocks whereas Massey products on $H^*(X)$ are needed for such conclusions. Following this line of reasoning, we can also show that a block size of *exactly* 3 is determined by a 3-fold Massey product on $H^*(E)$. By Lemma 4.4.1, this 3-fold product corresponds to a 3-fold primitive product on $PH^*(X)$. To determine such a block size on $H^*(X)$ would require a 4-fold Massey product.

**Proposition 4.4.1.** *If $\ell_i = 3$, then $\langle d\phi, \tilde{\gamma}_{i,0}, d\phi \wedge (\theta - \alpha)\rangle = -3[\theta \wedge d\tilde{\gamma}_{i,2}] \neq 0$.*

*Proof.* We may write $d\phi \wedge \tilde{\gamma}_{i,0} = d\tilde{\gamma}_{i,1}$. Moreover,

$$\begin{aligned} d(\tilde{\gamma}_{i,1} \wedge (\theta - \alpha)) &= d\tilde{\gamma}_{i,1} \wedge (\theta - \alpha) - \tilde{\gamma}_{i,1} \wedge dt \wedge d\phi \\ &= d\tilde{\gamma}_{i,1} \wedge (\theta - \alpha) - dt \wedge (d\phi \wedge \tilde{\gamma}_{i,1}) \\ &= d\tilde{\gamma}_{i,1} \wedge (\theta - \alpha) + d(dt \wedge (\tilde{\gamma}_{i,2} + \frac{1}{2}\tilde{\gamma}_{i,1})). \end{aligned}$$

Hence,

$$\tilde{\gamma}_{i,0} \wedge d\phi \wedge (\theta - \alpha) = -d\tilde{\gamma}_{i,1} \wedge (\theta - \alpha) = d\left(dt \wedge (\tilde{\gamma}_{i,2} + \frac{1}{2}\tilde{\gamma}_{i,1}) - \tilde{\gamma}_{i,1} \wedge (\theta - \alpha)\right). \tag{4.26}$$

Therefore, a representative of $\langle d\phi, \tilde{\gamma}_{i,0}, d\phi \wedge (\theta - \alpha)\rangle$ is given by

$$
\begin{aligned}
&[d\phi \wedge dt \wedge (\tilde{\gamma}_{i,2} + \frac{1}{2}\tilde{\gamma}_{i,1}) - d\phi \wedge \tilde{\gamma}_{i,1} \wedge (\theta - \alpha) + \tilde{\gamma}_{i,1} \wedge d\phi \wedge (\theta - \alpha)] \\
&= [-dt \wedge d\phi \wedge \tilde{\gamma}_{i,2} - \frac{1}{2} dt \wedge d\phi \wedge \tilde{\gamma}_{i,1} - 2d\phi \wedge \tilde{\gamma}_{i,1} \wedge (\theta - \alpha)] \\
&= [-dt \wedge d\phi \wedge \tilde{\gamma}_{i,2} - 2d(\tilde{\gamma}_{i,2} + \frac{1}{2}\tilde{\gamma}_{i,1}) \wedge (\theta - \alpha) + \frac{1}{2} d(dt \wedge (\tilde{\gamma}_{i,2} + \frac{1}{2}\tilde{\gamma}_{i,1}))] \\
&= -[dt \wedge d\phi \wedge \tilde{\gamma}_{i,2} + 2d(\tilde{\gamma}_{i,2} + \frac{1}{2}\tilde{\gamma}_{i,1}) \wedge (\theta - \alpha)] \\
&= -[dt \wedge d\phi \wedge \tilde{\gamma}_{i,2} + 2d(\tilde{\gamma}_{i,2}) \wedge (\theta - \alpha)],
\end{aligned}
$$

where the last equality follows from the previous calculation, in equation (4.26), showing $d\tilde{\gamma}_{i,1} \wedge (\theta - \alpha)$ is exact. To finish, we use the fact that $d\tilde{\gamma}_{i,2} \wedge \alpha$ is exact and apply Lemma 4.4.2 to the last line above, to yield the Massey product representative

$$-[dt \wedge d\phi \wedge \tilde{\gamma}_{i,2} + 2d(\tilde{\gamma}_{i,2}) \wedge (\theta - \alpha)] = -3[\theta \wedge d\tilde{\gamma}_{i,2}].$$

□

## 4.5 Twisted Primitive Cohomology

Recall that given a manifold $M$ and its D.G.A. of differential forms, $(\Omega^*(M), d, \wedge)$ we may define a new twisted map $\tilde{d} = d + \alpha\wedge : \Omega^*(M) \to \Omega^{*+1}(M) \oplus \Omega^{*+k}(M)$, for a fixed $\alpha \in \Omega^k(M)$. It is natural to ask, when is $\tilde{d}$ also a differential? We must ensure $(\tilde{d})^2 = 0$. If we require $\alpha$ is of odd degree then $(\tilde{d})^2 = 0$ precisely when $d\alpha = 0$. Indeed,

$$
\begin{aligned}
(\tilde{d})^2 &= d^2 + d(\alpha\wedge) + \alpha \wedge d + \alpha \wedge (\alpha\wedge) \\
&= d\alpha \wedge + (-1)^{|\alpha|}\alpha \wedge d + \alpha \wedge d + \alpha \wedge (\alpha\wedge) \\
&= d\alpha \wedge - \alpha \wedge d + \alpha \wedge d + (\alpha \wedge \alpha)\wedge \\
&= d\alpha\wedge = 0 \iff d\alpha = 0.
\end{aligned}
$$

Consequently, if $\alpha$ is such that $\tilde{d}$ is again a differential we define the twisted de Rham cohomology $H^*(M, \tilde{d}) = H^*(\Omega^*(M), \tilde{d})$. Using this situation as motivation, we wish to define a twisted primitive differential and cohomology. However, unlike above, $m_2$ is no longer associative and so our twisted differential will have to involve all the maps $m_1, m_2, m_3$. The following conditions involving $\alpha$ and the $(m_i)$ will guarantee the map squares to zero.

**Proposition 4.5.1.** *Let $\alpha \in \mathcal{P}^k(M)$ be of odd degree and $m_1$-closed. Define*

$$\tilde{m}_1 = m_1 + m_2(\alpha \otimes \mathbf{1}) - m_3(\alpha \otimes \alpha \otimes \mathbf{1}).$$

*Then $(\tilde{m}_1)^2 = 0$.*

*Proof.* To compute $(\tilde{m}_1)^2$, we recall the following $A_\infty$-identities, simplified to our algebra $(P^*(M), m_1, \times, m_3)$.

[Leibniz Rule] $m_1 m_2 = m_2(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$,

[$m_3$ Identity] $m_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}) = m_1 m_3 + m_3(m_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes m_1)$,

[$m_4$ Identity] $m_3(\mathbf{1}^{\otimes 2} \otimes m_2) - m_3(\mathbf{1} \otimes m_2 \otimes \mathbf{1}) - m_2(\mathbf{1} \otimes m_3) + m_3(m_2 \otimes \mathbf{1}^{\otimes 2}) - m_2(m_3 \otimes \mathbf{1}) = 0$,

[$m_5$ Identity] $m_3(\mathbf{1}^{\otimes 2} \otimes m_3) + m_3(\mathbf{1} \otimes m_3 \otimes \mathbf{1}) - m_3(m_3 \otimes \mathbf{1}^{\otimes 2}) = 0$.

Moreover in the $\mathcal{P}^*(M^{2n})$ $A_\infty$-algebra, for $\alpha$ an odd element we claim $m_2(\alpha, \alpha) = 0 = m_3(\alpha, \alpha, \alpha)$. The first equality follows immediately from the graded commutativity of $m_2$ combined with the fact $|\alpha|$ is odd. For the second equality, we apply the definition of $m_3$ directly:

$$m_3(\alpha, \alpha, \alpha) = \begin{cases} 0, & 3|\alpha| < n+2 \\ \Pi^0 *_r \left[\alpha \wedge L^{-1}(\alpha \wedge \alpha) - L^{-1}(\alpha \wedge \alpha) \wedge \alpha\right], & 3|\alpha| \geq n+2 \end{cases}$$

By graded commutativity of the wedge product and the fact that $|\alpha|$ is odd, this quantity will always vanish. Using these two properties as well as the $A_\infty$-identities listed above, we compute:

$$
\begin{aligned}
(\tilde{m}_1)^2 = & m_1^2 + m_1 m_2(\alpha \otimes \mathbf{1}) - m_1 m_3(\alpha \otimes \alpha \otimes \mathbf{1}) + m_2(\alpha \otimes m_1) + m_2(\alpha \otimes m_2(\alpha \otimes \mathbf{1})) \\
& - m_2(\alpha \otimes m_3(\alpha \otimes \alpha \otimes \mathbf{1})) - m_3(\alpha \otimes \alpha \otimes m_1) - m_3(\alpha \otimes \alpha \otimes m_2(\alpha \otimes \mathbf{1})) \\
& + m_3(\alpha \otimes \alpha \otimes m_3(\alpha \otimes \alpha \otimes \mathbf{1})) \\
= & m_2(m_1\alpha \otimes \mathbf{1}) + (-1)^{|\alpha|} m_2(\alpha \otimes m_1) + m_2(\alpha \otimes m_1) + m_3((m_1\alpha) \otimes \alpha \otimes \mathbf{1}) \\
& + (-1)^{|\alpha|} m_3(\alpha \otimes (m_1\alpha) \otimes \mathbf{1}) + m_2(m_2(\alpha \otimes \alpha) \otimes \mathbf{1}) - m_3(\alpha \otimes m_2(\alpha \otimes \alpha) \otimes \mathbf{1}) \\
& - (-1)^{|\alpha|} m_2(\alpha \otimes m_3(\alpha \otimes \alpha \otimes \mathbf{1})) + m_3(m_2(\alpha \otimes \alpha) \otimes \alpha \otimes \mathbf{1}) - m_2(m_3(\alpha \otimes \alpha \otimes \alpha) \otimes \mathbf{1}) \\
& - m_2(\alpha \otimes m_3(\alpha \otimes \alpha \otimes \mathbf{1})) + m_3(m_3(\alpha \otimes \alpha \otimes \alpha) \otimes \alpha \otimes \mathbf{1}) \\
& - (-1)^{|\alpha|} m_3(\alpha \otimes m_3(\alpha \otimes \alpha \otimes \alpha) \otimes \mathbf{1}). \\
= & m_2((m_1\alpha) \otimes \mathbf{1}) + m_3((m_1\alpha) \otimes \alpha \otimes \mathbf{1}) - m_3(\alpha \otimes (m_1\alpha) \otimes \mathbf{1}) = 0.
\end{aligned}
$$

□

**Definition 4.5.1.** Let $\alpha \in P^*(M)$ satisfy the conditions of Proposition 4.5.1. We define the *twisted primitive cohomology* $PH^*(M, \tilde{m}_1) := H^*(\mathcal{P}^*(M), \tilde{m}_1)$.

If $\alpha \in PH^1_+(M)$ then it follows that $\alpha \in H^1(M)$ as well. Hence both $PH^*(M, \tilde{m}_1)$ and $H^*(M, \tilde{d})$ are defined and one may wonder if there is a relationship between the two. Thus we construct twisted versions of $L$ and $\Pi^0$. For $\alpha \in PH^1_+(M)$ define

$$
\begin{aligned}
L_\alpha : & H^*(M, \tilde{d}) \to H^*(M, \tilde{d}) \\
& [A_k] \mapsto [\omega \wedge A_k]
\end{aligned}
$$

This map is well-defined since $\tilde{d}A_k = 0 = dA_k + \alpha \wedge A_k$ and so

$$\begin{aligned}
\tilde{d}(\omega \wedge A_k) &= d(\omega \wedge A_k) + \alpha \wedge (\omega \wedge A_k) \\
&= \omega \wedge dA_K + \omega \wedge (\alpha \wedge A_K) \\
&= \omega \wedge (dA_k + \alpha \wedge A_k) \\
&= \omega \wedge \tilde{d}A_k = 0.
\end{aligned}$$

**Proposition 4.5.2.** *The map $\Pi^0 : H^k(M, \tilde{d}) \to PH^k_+(M, \tilde{m}_1)$ given by $\Pi^0([A_k]) = [\Pi^0(A_k)]$ is well-defined for all $k \leq n$.*

*Proof.* Let $A_k$ be $\tilde{d}$-closed and $A_k = B_k + \omega \wedge B_{k-2} + \omega^2 \wedge B_{k-4} + \cdots$ denote its Lefschetz decomposition. We must show $B_k$ is $\tilde{m}_1$-closed. Consider first the case of $k < n$.

$$\begin{aligned}
\tilde{m}_1 B_k &= \partial_+ B_k + \Pi^0(\alpha \wedge B_k), \\
dA_k &= \partial_+ B_K + \omega \wedge (\partial_- B_k + dB_{k-2} + \cdots), \\
\alpha \wedge A_k &= \alpha \wedge B_k + \omega \wedge (\alpha \wedge B_{k-2} + \cdots), \\
\tilde{d}A_k = 0 &\implies \partial_+(B_k) + \Pi^0(\alpha \wedge B_k) = 0 = \tilde{m}_1 B_k
\end{aligned}$$

Finally, we handle the case $k = n$.

$$\tilde{m}_1 B_n = -\partial_+\partial_- B_n - \Pi^0\left[dL^{-1}(\alpha \wedge B_n) + \alpha \wedge L^{-1}(dB_n) + \alpha \wedge L^{-1}(\alpha \wedge B_n)\right] \tag{4.27}$$

$$dA_n + \alpha \wedge A_n = 0 = \alpha \wedge A_n + \omega \wedge (\partial_- B_n + \partial_+ B_{n-2}) + \omega^2 \wedge (\partial_- B_{n-2} + \partial_+ B_{n-4}) + \cdots \tag{4.28}$$

Focusing on equation (4.28), we expand $\alpha \wedge A_n = \alpha \wedge B_n + \omega \wedge (\alpha \wedge B_{n-2}) + \cdots$. Write $\alpha \wedge B_n = \omega \wedge B'_{n-1} + \omega^2 \wedge B'_{n-3} + \cdots$. Then by primitivity conditions on $B_n$ and $B'_i$ it follows that $\omega \wedge (\alpha \wedge B_n) = 0 = \omega^3 \wedge B'_{n-3} + \omega^4 \wedge B'_{n-5} + \cdots$. Thus we conclude $\alpha \wedge B_n = \omega \wedge B'_{n-1} = \omega \wedge L^{-1}(\alpha \wedge B_n)$. This observation allows us to rewrite equation (4.28) as

$$0 = \omega \wedge (\partial_- B_n + \partial_+ B_{n-2} + L^{-1}(\alpha \wedge B_n) + \Pi^0(\alpha \wedge B_{n-2})) + \cdots$$

and so

$$\partial_- B_n + \partial_+ B_{n-2} + L^{-1}(\alpha \wedge B_n) + \Pi^0(\alpha \wedge B_{n-2}) = 0.$$

Taking $\partial_+$ of this equation shows $\partial_+\partial_- B_n + \partial_+ L^{-1}(\alpha \wedge B_n) + \partial_+ \Pi^0(\alpha \wedge B_{n-2}) = 0$. Moreover, by degree considerations and the Leibniz rule for $m_2$, we have

$$\partial_+ \Pi^0(\alpha \wedge B_{n-2}) = \partial_+(\alpha \times B_{n-2}) = -\alpha \times \partial_+ B_{n-2}$$

Hence,

$$\begin{aligned} -\partial_+\partial_- B_n &= \partial_+ L^{-1}(\alpha \wedge B_n) - \alpha \times \partial_+ B_{n-2}, \\ \partial_+ B_{n-2} &= -(\partial_- B_n + L^{-1}(\alpha \wedge B_n) + \alpha \times B_{n-2}). \end{aligned}$$

Plugging these into equation (4.27) yields

$$\begin{aligned} \tilde{m}_1 B_n &= \partial_+ L^{-1}(\alpha \wedge B_n) - \alpha \times \partial_+ B_{n-2} - \Pi^0 \left[ dL^{-1}(\alpha \wedge B_n) + \alpha \wedge L^{-1}(dB_n) + \alpha \wedge L^{-1}(\alpha \wedge B_n) \right] \\ &= -\alpha \times \partial_+ B_{n-2} - \Pi^0 \left[ \alpha \wedge L^{-1}(dB_n) + \alpha \wedge L^{-1}(\alpha \wedge B_n) \right] \\ &= -\Pi^0 \left[ \alpha \wedge \partial_+ B_{n-2} + \alpha \wedge \partial_- B_n + \alpha \wedge L^{-1}(\alpha \wedge B_n) \right] \\ &= -\Pi^0 \left[ -\alpha \wedge (\partial_- B_n + L^{-1}(\alpha \wedge B_n) + \alpha \times B_{n-2}) + \alpha \wedge \partial_- B_n + \alpha \wedge L^{-1}(\alpha \wedge B_n) \right] \\ &= \Pi^0(\alpha \wedge (\alpha \times B_n)) = \alpha \times (\alpha \times B_n) = 0, \end{aligned}$$

where the second to last equality follows from the fact that by degree considerations, $\alpha \times (\alpha \times B_n) = (\alpha \times \alpha) \times B_n = 0$. □

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