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by

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Abstract

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Using the combinatorial formula for the transformed Macdonald polynomials of Haglund, Haiman, and Loehr, we investigate the combinatorics of the symmetry relation $\widetilde{H}_\mu(\mathbf{x}; q, t) = \widetilde{H}_{\mu^*}(\mathbf{x}; t, q)$. We provide a purely combinatorial proof of the relation in the case of Hall-Littlewood polynomials ($q = 0$) when μ is a partition with at most three rows, and for the coefficients of the square-free monomials in $\mathbf{x}$ for all shapes μ. We also provide a proof for the full relation in the case when μ is a hook shape, and for all shapes at the specialization $t = 1$. Our work in the Hall-Littlewood case reveals a new recursive structure for the cocharge statistic on words.

To my father, for guiding me through the beginnings of my lifelong mathematical journey.

To my mother, for her loving support and encouragement.

To my brothers, for making me smile.

Contents

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Chapter 1

Introduction

The study of Macdonald polynomials lies at the intersection of several different areas of mathematics: symmetric function theory, the combinatorics of q- and q,t-analogs, the ring of diagonal harmonics and other bi-graded S_n-representations, and the geometry of the Hilbert scheme of n points in the plane. These symmetric functions were first defined by Ian Macdonald in [30], and have been the subject of much recent study. In this work, we study a certain symmetry relation exhibited by the Macdonald polynomials, and present progress towards understanding the associated combinatorics from a more elementary point of view.

The Macdonald polynomials are a class of elements of the ring $\Lambda_{\mathbb{Q}(q,t)}(x_1, x_2, \ldots)$ of symmetric functions in the variables x_i with coefficients in the field $\mathbb{Q}(q,t)$ of rational functions in q and t with rational coefficients. There is one (transformed) Macdonald polynomial for every partition μ, written $\widetilde{H}_\mu(X;q,t)$, and the collection of all Macdonald polynomials forms a basis of $\Lambda_{\mathbb{Q}(q,t)}(x_1, x_2, \ldots)$ as a vector space over $\mathbb{Q}(q,t)$. They have the specializations

$$\widetilde{H}_\mu(x;0,1) = h_\mu \qquad \text{and} \qquad \widetilde{H}_\mu(x;1,1) = e_1^{|\mu|},$$

where h_λ and e_λ are the homogeneous and elementary symmetric functions, respectively.

As we will describe in more detail in Section 2.8, the Macdonald polynomials were originally defined by Ian Macdonald [30] as a two-parameter deformation of the Schur functions that extends both the well-known Hall-Littlewood polynomials (obtained by setting $q = 0$) as well as the Jack polynomials (obtained by setting $q = t^\alpha$ and letting $t \to 1$). In particular, the Macdonald polynomials can be defined as the unique collection of symmetric functions satisfying certain orthogonality and triangularity conditions with respect to a specific q,t-deformation of the Hall inner product.

In [30], Macdonald conjectured that the polynomials $\widetilde{H}_\mu(X;q,t)$ are Schur positive, in the sense that

$$\widetilde{H}_\mu(X;q,t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q,t) s_\lambda$$

where the coefficients $\widetilde{K}_{\lambda\mu}(q,t)$ are polynomials in q and t with positive integer coefficients, and where the s_λ's form the classical Schur basis for the ring of symmetric functions. Since

Schur functions are in one-to-one correpondence with the irreducible representations of the symmetric group S_n via the Frobenius characteristic map, the problem of proving Schur positivity is equivalent to finding a doubly graded S_n-module whose Frobenius characteristic is $\widetilde{H}_\mu(X;q,t)$.

In [12], Garsia and Haiman defined a class of doubly graded S_n-modules R_μ and conjectured that the bi-graded Frobenius characteristic of R_μ is $\widetilde{H}_\mu(X;q,t)$. The modules R_μ are defined as certain quotients the "diagonal action" of S_n on $\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ in which a permutation π acts by permuting the x_i's and y_i's simultaneously:

$$\pi\cdot x_i = x_{\pi(i)} \qquad \text{and} \qquad \pi\cdot y_j = y_{\pi(j)}.$$

(See Section 2.8 for details on the modules R_μ.)

Garsia and Haiman reduced their conjecture to the claim that R_μ has dimension $n!$ as a $\mathbb{C}$-vector space, which became known as the $n!$ ***conjecture***. In order to prove the $n!$ conjecture, Haiman [22] established a link between the symmetric functions $\widetilde{H}_\mu(X;q,t)$ and the geometry of the Hilbert scheme of n points in the plane. The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ is a moduli space which, as a set, consists of all ideals $I\subset\mathbb{C}[x,y]$ for which $\dim_{\mathbb{C}}\mathbb{C}[x,y]/I=n$. Haiman showed that the "isospectral" Hilbert scheme (see Section 2.8) is Cohen-Macaulay and Gorenstein, and from this inferred that the modules R_μ, which appear in the sheaf structure of the zero fiber of the map from the isospectral Hilbert scheme, have the correct dimension $n!$.

This geometric proof left open several other natural questions. First, if the spaces R_μ have dimension $n!$, can we find an explicit basis of $n!$ polynomials that generate it? In particular, can we find a combinatorial proof of positivity, in the sense that

$$\widetilde{K}_{\lambda\mu}(q,t)=\sum_T q^{s(T)}t^{r(T)},$$

where T ranges over an appropriate set of Young tableaux and r and s are combinatorial statistics? Assaf investigated these questions via the theory of dual equivalence in [1]. An explicit basis for two-column partitions was found by Assaf and Garsia in [2], and Reiner found a basis for "generalized hooks" (partitions of the form $(a,2,1)$) in [32]. In [17], Haiman, Haglund, and Loehr give a formula for $\widetilde{K}_{\lambda\mu}(q,t)$ for μ having two rows, and Fishel [10] found a formula for two-column shapes μ in terms of the Kirillov and Reshetikhin's *rigged configurations* [27, 25]. However, much remains to be understood.

The $n!$ conjecture is related to the similar-sounding $(n+1)^{n-1}$ conjecture. Define $R_n=\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]/I_n$, where I_n is the ideal generated by the positive-degree homogeneous invariants under the diagonal action of S_n. Then R_n is often called the ring of diagonal coinvariants, and is naturally isomorphic to the ring of diagonal harmonics $DH_n\subset\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ consisting of the functions f that are killed by all the associated polynomial differential operators of the elements of I_n. Each R_μ is in fact a quotient of this S_n-module R_n. In [21], Haiman conjectured that the ring R_n has dimension $(n+1)^{n-1}$ as a $\mathbb{C}$-vector space.

The $(n+1)^{n-1}$ conjecture follows from the more refined fact, proven by Haiman in [23], that the bigraded Frobenius character of R_n is ∇e_n. Here ∇ is the linear operator on symmetric functions given by

$$\nabla \widetilde{H}_\mu(X;q,t) = q^{n(\mu^*)}t^{n(\mu)}\widetilde{H}_\mu(X;q,t)$$

for all μ, where μ^* is the conjugate partition and $n(\lambda) = \sum_i (i-1)\lambda_i$. This proof too relied on geometric methods and was decidedly lacking in an elegant combinatorial or algebraic explanation.

More recently, combinatorial formulas for both the Macdonald polynomials and for ∇e_n have been discovered. In [19], Haiman, Haglund, Loehr, Remmel, and Ulyanov put forth the "shuffle conjecture", which states that the Frobenius characteristic of the ring R_n has the combinatorial formula

$$\nabla e_n = \sum_{P \in \mathrm{PF}_n} q^{\mathrm{area}(P)}t^{\mathrm{dinv}(P)}x^P$$

where PF_n is the set of word parking functions (associated to some Dyck path) of length n and area and dinv are certain statistics on word parking functions. Here x^P denotes the monomial $x_1^{m_1}x_2^{m_2}\cdots$ where m_i is the number of i's occuring in the parking function P. In the same year, Haiman, Haglund and Loehr [17] discovered and proved a similar combinatorial formula for Macdonald polynomials, showing that

$$\widetilde{H}_\mu(X;q,t) = \sum_{\sigma \in \mathcal{F}_\mu} q^{\mathrm{inv}(\sigma)}t^{\mathrm{maj}(\sigma)}x^\sigma$$

where $\mathcal{F}_\mu$ is the set of fillings of the Young diagram of μ with positive integers, and inv and maj are statistics on these objects. Here, similarly, $x^\sigma = x_1^{m_1}x_2^{m_2}\cdots$ where m_i is the number of i's occuring in the filling σ. Haglund described the genesis of these statistics in [15].

In a very recent preprint by Carlsson and Mellit [6], the shuffle conjecture was proven. In fact, the authors proved the more refined "compositional shuffle conjecture" of Haglund, Morse, and Zabrocki [18], which breaks down the shuffle conjecture in terms of the possible choices of points at which the Dyck paths in the summation touch the main diagonal. However, there are many generalizations of the shuffle conjecture which remain open, such as a conjectured formula for $\nabla^m e_n$ for all m [19], or the *rational shuffle conjecture*, involving north/east paths that remain above the diagonal in a $k \times n$ grid. [14] In 2013, Bergeron, Garsia, Leven, and Xin [4] devised the *compositional rational shuffle conjecture*, which generalizes all the other variants by constructing the operators $E_{k,n}^\alpha$, where α is a composition of $\gcd(k,n)$, and conjecturing that

$$E_{k,n}^\alpha \cdot 1 = \sum q^{\mathrm{dinv}(P)}t^{\mathrm{area}(P)}s_{\mathrm{co}(P)}.$$

In this formula, dinv and area are generalizations of the statistics to paths above the diagonal in a $k \times n$ grid, the sum is over all paths P that return to the diagonal according to $c\alpha$ where

$c = n/\gcd k, n$, and $s_{\mathrm{co(P)}}$ is a generalization of the Schur functions indexed by a certain composition associated to P.

The combinatorial formulas mentioned above raise an interesting question. Based on their connections to diagonal harmonics and the Hilbert scheme, the formulas should exhibit a certain q, t-symmetry. In the case of ∇e_n, the formula should actually be symmetric in q and t (see [29] for a potential combinatorial approach to this symmetry), and in the case of Macdonald polynomials we must have conjugate symmetry:

$$\widetilde{H}_\mu(X; q, t) = \widetilde{H}_{\mu^*}(X; t, q).$$

However, the combinatorial formulas as stated use different statistics for the q and t exponents, and it is not immediately obvious that, combinatorially, they should exhibit such symmetry.

In this work, we investigate the combinatorics of the conjugate q, t-symmetry relation in the Macdonald polynomials. In terms of the inv and maj statistics, this relation becomes

$$\sum_{\sigma:\mu \to \mathbb{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma = \sum_{\rho:\mu^* \to \mathbb{Z}_+} q^{\mathrm{maj}(\rho)} t^{\mathrm{inv}(\rho)} x^\rho. \tag{1.1}$$

Note that if we set $t = 1$ and $\mu = (n)$ and take the coefficient of $x_1 \cdots x_n$ on both sides, this reduces to

$$\sum_{w \in S_n} q^{\mathrm{inv}(w)} = \sum_{w \in S_n} q^{\mathrm{maj}(w)},$$

which turns out to be the well-known equidistribution of the Mahonian statistics inv and maj on permutations. There are several known bijective proofs of this simpler identity (see [5, 11, 34]).

In light of this, it is natural to ask if there is an elementary combinatorial proof of (2.5), in the sense of Problem 1.0.1 below. Here $\mathcal{F}_\mu$ again denotes the set of fillings of the Young diagram of μ with positive integers (see Section 2.5 below).

Problem 1.0.1. *Find an explicit bijection*

$$\varphi : \mathcal{F}_\mu \to \mathcal{F}_{\mu^*}$$

which interchanges inv *and* maj*, i.e.*

$$\mathrm{inv}(\varphi(\sigma)) = \mathrm{maj}(\sigma) \qquad \textit{and} \qquad \mathrm{maj}(\varphi(\sigma)) = \mathrm{inv}(\sigma)$$

for all $\sigma \in \mathcal{F}_\mu$.

In this work, we provide explicit bijections φ for several infinite families of fillings. Our bijections naturally extend Carlitz's bijection [5] that shows the equidistribution of inv and maj on permutations. We begin with some necessary background on tableaux, combinatorial statistics, symmetric functions, and Hall-Littlewood and Macdonald polynomials in Chapter 2.

Chapters 3 and 4 focus on the specialization of this symmetry relation at $q = 0$, at which the Macdonald polynomials specialize to the Hall-Littlewood polynomials. Combinatorially, this specialization involves only the fillings σ of μ or ρ of μ^* having $\mathrm{inv}(\sigma) = 0$ or $\mathrm{maj}(\rho) = 0$ respectively. In Chapter 3, we extend the Carlitz codes used in the Carlitz bijection to this setting using the theory of the Garsia-Procesi modules studied in [13]. We also demonstrate a bijection invcode from the fillings of μ^* having $\mathrm{maj} = 0$ to the generalized Carlitz codes.

In Chapter 4, we first complete the Hall-Littlewood specialization of the symmetry problem for fillings with distinct entries in section 4.1, by finding a bijection majcode (for fillings of μ having $\mathrm{inv} = 0$) which arises from a bijection of Killpatrick given in [26]. Since this bijection does not naturally standardize to repeated entries, we use a different approach in the remaining sections of Chapter 4 in order to obtain a full proof of symmetry for three-row shapes, as well as "fat hooks", the shapes $\mu = (a, b, 1, 1, 1, \ldots, 1)$ consisting of a single column plus two rows.

In essence, we construct a recursive procedure that is defined only on rectangular and two-row shapes, and can iterate this procedure to form our bijection only when every shape contained in μ is a union of a rectangle and a two-row shape. This condition is equivalent to the statement that μ is either a three-row shape or a fat hook.

In Section 4.6, we state some applications of the results on the Hall-Littlewood case to understanding the rings R_μ, in particular regarding the *cocharge* statistic of Lascoux and Schutzenberger (see [13] or [20], for instance). In particular, we demonstrate a new recursive structure exhibited by the cocharge statistic on words. Finally, in Section 5.1 we give a combinatorial proof of the symmetry relation for the specialization $t = 1$, and in Section 5.2 we give an explicit bijection φ in the case that μ is a hook shape.

The following theorem summarizes our results.

Theorem 1.0.2. *The bijective maps* invcode *and* majcode *that comprise the classical Carlitz bijection* $\mathrm{majcode} \circ \mathrm{invcode}^{-1} : S_n \to S_n$ *can be extended to give bijections on fillings that interchange* inv *and* maj *in the following cases:*

1. *In the Hall-Littlewood specialization $q = 0$, i.e. when one of the statistics is zero, for all partitions $\mu = (\mu_1, \mu_2, \mu_3)$ having at most three parts, and when $\mu = (a, b, 1, 1, \ldots, 1)$ is the union of a column and a two-row shape.*

2. *In the Hall-Littlewood specialization $q = 0$ (for all shapes) when we restrict to the fillings having distinct entries.*

3. *In the specialization $t = 1$, i.e. when one of the statistics is ignored.*

4. *When μ is a hook shape.*

Remark 1.0.3. The first item in the list above gives the first combinatorial results towards understanding the q, t-Kostka polynomials for these shapes. As mentioned above, the only shapes μ for which $\widetilde{K}_{\lambda\mu}(q, t)$ is currently understood via tableaux statistics are two-row shapes [2, 10] and generalized hooks [32].

Chapter 2

Background

2.1 Combinatorial statistics and q-analogs

We will be dealing throughout with a number of ***combinatorial statistics***, or combinatorially defined maps from a set C of combinatorial objects to $\mathbb{Z}$. We will also be working with maps that preserve these statistics. In order to simplify our notation, we define a category called CStat, the category of combinatorial statistics.

Objects: An object of CStat, called a ***weighted set*** and written

$$(C; \mathrm{stat}_1, \mathrm{stat}_2, \ldots),$$

consists of a countable set C and a countably infinite collection of functions $\mathrm{stat}_i : C \to \mathbb{Z}$, called *statistics*, such that:

1. For all $(n_1, n_2, \ldots) \in \mathbb{Z}^\infty$, the set $\{c \in C \mid \mathrm{stat}_i(c) = n_i \text{ for all } i\}$ is a *finite* subset of C, and
2. All but a finite number of the statistics stat_i are the zero map.

If only a finite number of statistics is specified, it is assumed that the rest of the statistics are the zero map. We also sometimes refer to a weighted set by its underlying set C if the statistics are understood.

Morphisms: A morphism of weighted sets $\phi : (C; s_1, s_2, \ldots) \to (D; t_1, t_2, \ldots)$ is a map $\phi : C \to D$ of sets such that for each i,

$$t_i \circ \phi = s_i.$$

It is not hard to see that this collection of objects and morphisms forms a category. We also use the following constructions on objects of CStat.

q-series: The finiteness conditions on the objects imply that we can associate a generating function, or *q-series*, to each object. In particular, let $q_1, q_2, \ldots$ be a countable collection of indeterminates, and define the q-series of a CStat object C to be the formal sum

$$C(q_1, q_2, \ldots) = \sum_{c \in C} q_1^{\mathrm{stat}_1(c)} q_2^{\mathrm{stat}_2(c)} \cdots$$

This is a well-defined formal series in the variables q_i since each monomial has a finite number of factors by the second condition, and the coefficient of $q_1^{a_1} q_2^{a_2} \cdots$ is finite by the first condition.

Restrictions: We can restrict a CStat object along any finite number of statistics as follows. Given an object $C = (C; \mathrm{stat}_1, \mathrm{stat}_2, \ldots)$, and any constant $s \in \mathbb{Z}$, we define the restriction object $C\,|_{\mathrm{stat}_i = s}$ by

$$C|_{\mathrm{stat}_i = s} = (\{a \in C \mid \mathrm{stat}_i(a) = s\}; \mathrm{stat}_1, \mathrm{stat}_2, \ldots, \widehat{\mathrm{stat}_i}, \ldots).$$

Its statistics are stat_j for $j \neq i$.

Note that restricting to $\mathrm{stat}_i = 0$ corresponds to setting $q_i = 0$ in the corresponding q-analog. Setting $q_i = 1$ corresponds to simply dropping the statistic stat_i from the list of statistics.

Remark 2.1.1. Notice that the q-series of a weighted set uniquely determines it up to isomorphism.

To illustrate weighted sets, we recall the classical Mahonian inv and maj statistics on permutations. (See [34] or [35] for a more complete introduction.) Let $[n] = \{1, 2, 3, \ldots, n\}$ throughout. We write S_n to be the set of permutations π of $[n]$, that is, bijections $\pi : [n] \to [n]$. We often write a permutation π as a list, $\pi_1 \cdots \pi_n$, where $\pi_i = \pi(i)$.

Definition 2.1.2. For a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$,

$$\mathrm{inv}(\pi) = |\{(i, j) | i < j, \pi_i > \pi_j\}|.$$

A pair (i, j) satisfying $i < j$ and $\pi_i > \pi_j$ is called an ***inversion*** of π.

Remark 2.1.3. The value of $\mathrm{inv}(\pi)$ is equal to the minimum length of π as a product of adjacent transpositions $s_i = (i \quad i+1)$.

The weighted set $(S_n; \mathrm{inv})$ is an object in CStat, with q-series $\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}$. Notice that this weighted set has a nontrivial automorphism, namely, the map sending $\pi \mapsto \pi^{-1}$, since any inversion of π corresponds to a unique inversion of π^{-1}.

As another example of a morphism, consider the weighted set $(S_n; \overline{\mathrm{inv}})$ where

$$\overline{\mathrm{inv}}(\pi) = \binom{n}{2} - \mathrm{inv}(\pi).$$

Then the "reversing" map that sends $\pi_1, \ldots, \pi_n$ to $\pi_n, \ldots, \pi_1$ is a bijective morphism in CStat, since (i, j) is an inversion of π if and only if it is not an inversion of the reverse of π.

Definition 2.1.4. The ***reverse*** map

$$r : (S_n; \mathrm{inv}) \to (S_n; \overline{\mathrm{inv}})$$

is given by $r(\pi) = \pi_n \cdots \pi_1$.

Finally, we will also use the "flip" morphism on permutations in which we replace any entry π_i with $n+1-\pi_i$, to take us back to inv from $\overline{inv}$.

Definition 2.1.5. The ***flip*** map flip : $(S_n; \overline{\mathrm{inv}}) \to (S_n; \mathrm{inv})$ is given by $\mathrm{flip}(\pi) = \rho$ where $\rho_i = n+1-\pi_i$ for all i.

It is well known (see [5], [35]) that the q-series $\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}$ factors as the *q-factorial*

$$(n)_q! = 1(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1}),$$

and we will recall a proof of this fact in the next section. A combinatorial statistic stat : $S_n \to \mathbb{Z}$ is called a *Mahonian* statistic if it has this exact q-series, i.e.

$$\sum_{\pi \in S_n} q^{\mathrm{stat}(\pi)} = (n)_q!.$$

There are many known families of Mahonian statistics (see e.g. [3], [24], [31], and [38]), and in this paper we will be considering one additional Mahonian statistic, the ***major index***.

Definition 2.1.6. The ***major index*** of a permutation $\pi = \pi_1 \cdots \pi_n$ is given by

$$\mathrm{maj}(\pi) = \sum_{\pi_d > \pi_{d+1}} d.$$

An entry π_d for which $\pi_d > \pi_{d+1}$ is called a ***descent*** of π.

2.2 The Carlitz bijection

We now construct the ***Carlitz bijection***, a bijection $\phi : (S_n; \mathrm{inv}) \to (S_n; \mathrm{maj})$, and show that inv and maj are both Mahonian. This shows in particular that inv and maj are ***equidistributed*** on S_n, meaning that the associated CStat objects are isomorphic.

Definition 2.2.1. A ***Carlitz code*** of length n is a sequence $c = c_1, \cdots, c_n$ consisting of nonnegative integers such that $c_{n-i} \le i$ for all i. Let C_n denote the set of all Carlitz codes of length n, equipped with the combinatorial statistic $\Sigma : C_n \to \mathbb{Z}$ defined by $\Sigma(c) = \sum_i c_i$.

Notice that there are $n!$ Carlitz codes of length n. We will make use of the weighted set $(C_n; \Sigma)$ as an intermediate CStat object connecting $(S_n; \mathrm{inv})$ to $(S_n; \mathrm{maj})$. In particular, the Carlitz bijection is the composite

$$(S_n; \mathrm{inv}) \xrightarrow{\mathrm{invcode}} (C_n; \Sigma) \xrightarrow{\mathrm{majcode}^{-1}} (S_n; \mathrm{maj})$$

of two simple isomorphisms of weighted sets. The existence of these isomorphisms implies that the q-series of $(S_n; \mathrm{inv})$ and of $(S_n; \mathrm{maj})$ are equal to that of $(C_n; \Sigma)$. The latter clearly has q-series $(n)_q! = (1 + q + q^2 + \cdots + q^{n-1}) \cdots (1 + q + q^2)(1 + q)(1)$, since the choice of monomial from the i-th factor corresponds to the value of c_i.

Definition 2.2.2. The ***inversion code*** of a permutation π, denoted $\mathrm{invcode}(\pi)$, is the sequence $c_1, c_2, \ldots, c_n$ where c_i is the number of inversions of the form $(\pi^{-1}(j), \pi^{-1}(i))$ for some j, i.e. where $i < j$ and i is to the right of j in π.

Example 2.2.3. We have $\mathrm{invcode}(4132) = 1210$, because the 1 is the smaller entry of one inversion $(4, 1)$, the 2 is the smaller entry of the two inversions $(3, 2)$ and $(4, 2)$, the 3 is the smaller entry of the inversion $(4, 3)$, and the 4 is not the smaller entry of any inversion.

Clearly invcode is a map $(S_n; \mathrm{inv}) \to (C_n; \Sigma)$, and it is not hard to see that it is bijective. Indeed, given a Carlitz code $c = c_1, \ldots, c_n$, the unique permutation π having $\mathrm{invcode}(\pi) = c$ can be constructed as follows. The entry $c_n = 0$ gives us no information, but c_{n-1} is either 0 or 1, and respectively determines whether the $n - 1$ is to the left or to the right of the n. The entry c_{n-2} then determines where $n - 2$ occurs relative to the positions of $n - 1$ and n, and so on. It is also clear that invcode is an isomorphism of weighted sets, sending $\mathrm{inv}(\pi)$ to $\Sigma(\{c_i\})$.

Definition 2.2.4. The ***major index code*** of a permutation π is defined as follows. For each k define $\psi_k : S_k \to S_{k-1}$ to be the restriction map that deletes the k from a permutation $\rho_1, \ldots, \rho_k$ of $\{1, 2, \ldots, k\}$. Define

$$\pi|_k = \psi_{k+1}(\psi_{k+2}(\cdots \psi_n(\pi) \cdots))$$

for each k, and let $c_i = \mathrm{maj}(\pi|_{n-i+1}) - \mathrm{maj}(\pi|_{n-i})$ for each i. Then we define $\mathrm{majcode}(\pi) = c_1, c_2, \ldots, c_n$.

Example 2.2.5. Let $\pi = 3241$. Its major index is $1 + 3 = 4$. Removing the 4 results in the permutation 321, which has major index 3, so the major index has decreased by 1 and we set $c_1 = 1$. Removing the 3 results in 21, which decreased the major index by 2. Hence $c_2 = 2$. Removing the 2 decreases the major index by $c_3 = 1$, and removing the 1 decreases it by $c_4 = 0$, so $\mathrm{majcode}(\pi) = 1210$.

As in the case of invcode above, it is not hard to construct an inverse for majcode, making it an isomorphism of weighted sets $(S_n; \mathrm{maj}) \to (C_n, \Sigma)$. Indeed, there is a unique way to reverse the restriction maps ψ_k given a code c_i, so as to increase the maj by c_{n-k+1} when inserting the entry k.

A full proof of this fact can be found in Carlitz's original paper [5], or in a somewhat cleaner form in [34]. We give a sketch of the proof here. Given a permutation ρ of $\{1, \ldots, k-1\}$, consider the k possible positions in which to insert a k to form a permutation of $\{1, \ldots, k\}$. Let p_0 be the rightmost such position and let $p_1, \ldots, p_r$ be the positions just after a descent of ρ, from right to left. Then, label the remaining positions $p_{r+1}, \ldots, p_k$. Then it is not hard to show that inserting the k at position p_d has the effect of increasing the major index by d. Therefore, the code entry $c_{n-k+1} \in \{0, 1, \ldots, k\}$ determines the position at which we must insert the k to reverse the restriction map ψ_k.

We now compose these bijections to form the Carlitz bijection.

Definition 2.2.6. The *Carlitz bijection* is the isomorphism

$$\mathrm{majcode}^{-1} \circ \mathrm{invcode} : (S_n; \mathrm{inv}) \to (S_n; \mathrm{maj}).$$

Example 2.2.7. We have $\mathrm{majcode}^{-1} \circ \mathrm{invcode}(4132) = \mathrm{majcode}^{-1}(1210) = 3241$ by the examples above.

2.3 Words and the Foata bijection

There is another classical bijection that demonstrates the equidistribution of inv and maj directly, called the ***Foata bijection*** ϕ. In order to make the definition cleaner we introduce a new operation on permutations, and more generally on ***words***.

Throughout this paper a ***word*** is any sequence of positive integers. The ***content*** of a word $w = w_1, \ldots, w_n$ is the tuple $(\alpha_1, \alpha_2, \ldots)$ where α_i is the number of occurrences of i among the entries of w. A permutation in S_n, then, can be thought of as a word with content (1^n).

Definition 2.3.1. If $w = w_1 w_2 \cdots w_n$ is a word, we define

$$\mathrm{cyc}(w) = w_2 w_3 \cdots w_n w_1$$

to be the word formed by moving the first letter to the end.

The cycling operation cyc is useful in defining the Foata bijection ϕ:

Definition 2.3.2. Let $v = v_1, \ldots, v_n$ be a word. We define $w = \phi(v)$ by recursively constructing a sequence of partial permutations $w^{(1)}, \ldots, w^{(n)}$, with $w = w^{(n)}$, as follows.

Let $w^{(1)} = v_1$. For $1 \leq i < n$, suppose $w^{(i)} = b_1, \ldots, b_i$. If $b_i \leq v_{i+1}$ (respectively if $b_i > v_{i+1}$), consider the unique factorization of $w^{(i)}$ into subwords $f^{(1)} \cdots f^{(k)}$ such that the rightmost element of each $f^{(j)}$ is less than or equal to (resp. greater than) v_{i+1}, and all other elements are greater than (resp. less than or equal to) v_{i+1}. Define

$$w^{(i+1)} = g_1 \cdots g_k v_{i+1},$$

where $g_j = \mathrm{cyc}^{-1}(f_j)$ for all j.

The restriction of ϕ to S_n is a bijection $\phi : S_n \to S_n$ called the ***Foata bijection***.

Foata and Schützenberger [11] showed that this bijection sends maj to inv, and so we have the following.

Proposition 2.3.3. *The Foata bijection is an isomorphism of weighted sets* $(S_n; \mathrm{maj}) \to (S_n; \mathrm{inv})$.

For a full proof, we refer the reader to [11] or [34].

Example 2.3.4. Let $\pi = 31254$. Then $\phi(\pi)$ is the last entry in this sequence of partial permutations:

$$\begin{array}{l} 3 \\ |3|1 \\ |31|2 \\ |1|3|2|5 \\ |1325|4 \\ 51324 \end{array}$$

At each step, the vertical lines divide the blocks that are cycled to obtain the next step, and the next entry in the permutation is added to the end at each step. Notice that $\mathrm{maj}(31254) = \mathrm{inv}(51324) = 5$.

Example 2.3.5. A similar process can be used on words, as described in the definition. For instance, starting with the word 21132, we have the sequence

$$\begin{array}{l} 2 \\ |2|1 \\ |21|1 \\ |1|2|1|3 \\ |1213|2 \\ 31212 \end{array}$$

Thus $\phi(21132) = 31212$.

Example 2.3.5 suggests extending the definitions of inv and maj to words.

Definition 2.3.6. Given a word $w = w_1, \ldots, w_n$, we define

$$\mathrm{inv}(w) = |\{(i, j)|i < j, w_i > w_j\}|$$

and

$$\mathrm{maj}(w) = \sum_{w_d > w_{d+1}} d.$$

Definition 2.3.7. Let W_n^α denote the set of all words of length n and content α.

Foata's map is also an isomorphism $\phi : (W_n^\alpha; \mathrm{maj}) \to (W_n^\alpha; \mathrm{inv})$ for any n and α.

Unlike the Carlitz bijection, the Foata bijection does not immediately give rise to a proof of the factorization of the q-series of $(S_n; \mathrm{inv})$. However, it has the advantage of preserving the "inverse descent set" of a permutation.

Definition 2.3.8. The *inverse descent set* of a permutation π, denoted $\mathrm{iDes}(\pi)$, is the set of descents of π^{-1}. In other words, it is the set of entries a for which $a+1$ occurs earlier in the permutation.

The inverse descent set is preserved under ϕ since, if a occurs after $a+1$, then it is added at a later step in the Foata bijection, and the two can then never be in the same block at a step after that. Thus they can never switch places relative to each other.

The inverse descent set arises in the decomposition of certain symmetric functions into Gessel's quasisymmetric functions (see e.g. [17]). As a more immediate application of studying the relative orderings of entries in the word, we note that there is a *standardization* map on words that is compatible with Foata's bijection.

Definition 2.3.9. Given a word $w_1, \ldots, w_n$, we define the ***standardization*** of w to be the unique permutation π for which $\pi_i < \pi_j$ if and only if either $w_i < w_j$ or $w_i = w_j$ and $i < j$.

For instance, the standardization of 21132 is 31254. Notice that the blocks in the steps of the Foata bijection are the same for both permutations, and indeed this is true in general.

2.4 Partitions and tableaux

In this section we recall some combinatorial basics involving partitions and Young tableaux. A ***partition*** of a positive integer n is a sequence $(\lambda_1, \ldots, \lambda_k)$ of positive integers, called the ***parts*** of the partition, satisfying

$$\lambda_1 \geq \cdots \geq \lambda_k \text{ and } \sum_{i=1}^{k} \lambda_i = n.$$

We write $\ell(\lambda)$ to denote the number of parts of λ. We also write $|\lambda| = \sum \lambda_i$ to denote the ***size*** of λ.

The ***Young diagram*** of the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is the set of pairs (i, j) with $1 \leq j+1 \leq k$ and $1 \leq i+1 \leq \lambda_{j+1}$. Such a pair is called a *cell* of the diagram. We draw[1] the Young diagram as the partial grid of unit squares whose lower left corners are at the cells (i, j) of λ. For instance, the Young diagram of the partition $(5, 2, 2, 1)$ is shown below:

[1]Young diagrams as described here are often referred to as being drawn in "French notation", in which the rows are listed from bottom to top. The "English notation" lists the rows from top to bottom.

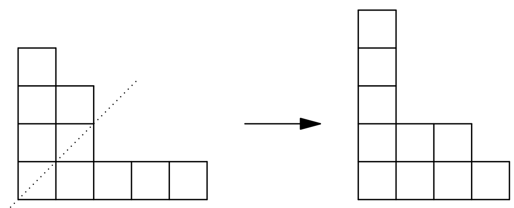
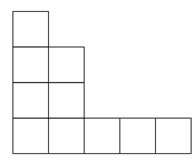


Figure 2.1: A partition λ and its transpose λ^*.



We often refer to a partition λ and its Young diagram interchangeably. For instance we may refer to the cells or the squares of λ. We refer to the ***rows*** and ***columns*** of a partition according to its diagram.

The ***dominance*** partial ordering on partitions is given by $\lambda \geq \mu$ if and only if $\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$ for all i.

The ***conjugate*** or ***transpose*** of a partition λ is the partition λ^* formed by reflecting λ about the diagonal $y = x$. That is, (j, i) is a cell of λ^* if and only if (i, j) is a cell of λ. For instance, if $\lambda = (5, 2, 2, 1)$ then $\lambda^* = (4, 3, 1, 1, 1)$.

The ***arm*** (resp. ***leg***) of a cell x in λ are respectively the sets of squares to the right of (resp. above) x in its row (resp. column). We write $a(x)$ and $l(x)$ for the sizes of the arm and leg of a cell x, and define the ***hook*** of x to be the union of the arm, leg, and $\{x\}$. A partition is a ***hook shape*** if for every cell (i, j) we have $ij = 0$.

We occasionally work with certain subsets of the cells of a Young diagram of a partition, so we define a general ***diagram*** to be any finite collection of cells (i, j) with i, j nonnegative integers.

A ***filling*** of a diagram D is a map $\sigma : D \to \mathbb{Z}_+$ (which we sometimes write $\sigma : \mu \to \mathbb{Z}_+$), and we represent a filling by writing $\sigma(x)$ inside the square of x for each cell x. We call $\sigma(x)$ the ***entry*** written in cell x. The ***content*** of a filling σ is the tuple

$$\text{content}(\sigma) = (|\sigma^{-1}(1)|, |\sigma^{-1}(2)|, \ldots, |\sigma^{-1}(n)|)$$

where n is the largest entry in σ. The ***reading word*** of a filling is the word formed by concatenating the rows from top to bottom (as one would read a book). The total ordering of the entries determined by left-to-right order in the reading word is called the ***reading order***.

A filling is a ***semistandard young tableau***, or SSYT, if the entries are weakly increasing left-to-right in each row and strictly increasing bottom-to-top in each column, that is, if

$\sigma(i,j) \le \sigma(i+1,j)$ and $\sigma(i,j) < \sigma(i,j+1)$ for all i and j. A semistandard Young tableau of shape λ is ***standard*** if its content is $(1,1,1,\ldots,1)$ for some number of 1's.

The tableau shown below is a semistandard Young tableau with content $(2,2,3,1,0,1)$.

4	6		
2	3	3	
1	1	2	3

We will often be working with the set of all fillings of a fixed content and shape, as defined below.

Definition 2.4.1. We use the notation $\mathcal{F}_\mu^\alpha$ for the set

$$\mathcal{F}_\mu^\alpha = \{\sigma : \mu \to \mathbb{Z}_+ | \mathrm{content}(\sigma) = \alpha\}.$$

2.5 The statistics inv, maj, and cocharge

In [17], the statistics inv and maj were extended to Young diagram fillings, and we recall the definitions here.

Definition 2.5.1. Given a filling σ of a partition μ, let $w^{(1)}, \ldots, w^{(\mu_1)}$ be the words formed by the successive columns of σ, read from top to bottom. Then

$$\mathrm{maj}(\sigma) = \sum_s \mathrm{maj}(w^{(s)}).$$

Example 2.5.2. The major index of the filling in Figure 2.2 is 7, since the first column has major index 6, the second has major index 0, and the third column, 1.

Remark 2.5.3. The major index restricts to the usual major index on words in the case that the partition is a single column.

2		
2		
1	1	
4	3	**6**
3	5	3

Figure 2.2: A filling of a Young diagram. Descents are shown in boldface, and attacking pairs are connected with gray lines.

For the statistic inv, we start with the definition provided in [17]. A ***descent*** is an entry which is strictly greater than the entry just below it.

Definition 2.5.4. An ***attacking pair*** in a filling σ of a Young diagram is a pair of entries u and v with $u > v$ satisfying one of the following conditions:

1. u and v are in the same row, with u to the left of v, or

2. u is in the row above v and strictly to its right.

Definition 2.5.5. The statistic inv on fillings is defined as

$$\operatorname{inv}(\sigma) = |\operatorname{Attack}(\sigma)| - \sum_{d \in \operatorname{Des}(\sigma)} a(d),$$

where $\operatorname{Attack}(\sigma)$ is the set of attacking pairs in σ, $\operatorname{Des}(\sigma)$ is the set of descents, and $a(d)$ is the length of the arm of the descent d.

Example 2.5.6. In Figure 2.2, there are 4 attacking pairs, and the arms of the descents have lengths 0, 2, and 0. Thus $\operatorname{inv}(\sigma) = 4 - 2 = 2$ in this case.

Remark 2.5.7. The inv statistic restricts to the usual inv on words in the case that the partition is a single row.

For our purposes, we will also need the following cleaner definition of the inv statistic. This more closely resembles the inv statistic on a permutation.

Definition 2.5.8. Let σ be any filling of a Young diagram with letters from a totally ordered alphabet A, allowing repeated letters. A ***relative inversion*** of a filling σ of a Young diagram is a pair of entries u and v in the same row, with u to the left of v, such that if b is the entry directly below u, one of the following conditions is satisfied:

- $u < v$ and b is between u and v in size, in particular $u \leq b < v$.

- $u > v$ and b is not between u and v in size, in particular either $b < v < u$ or $v < u \leq b$,

If u and v are on the bottom row, we treat b as any value less than $\min(u, v)$, usually 0 in the case $A = \mathbb{Z}_+$.

Remark 2.5.9. The conditions above for u and v in a triple (u, v, b) to form a relative inversion are equivalent to the statement that the ordering of the sizes of u, b, v orients the triple counterclockwise: either $b < v < u$, $v < u \leq b$, or $u \leq b < v$.

Example 2.5.10. In Figure 2.2, there are 2 relative inversions: $(5, 3)$ in the bottom row, and $(3, 6)$ in the second row.

In fact, the number of relative inversions in a filling σ is always equal to $\mathrm{inv}(\sigma)$. In [17], the authors introduce the related notion of an *inversion triple*. Relative inversions are simply the inversion triples that contribute 1 to $\mathrm{inv}(\sigma)$. The description in terms of relative inversions allows us to think of the inv as being computed row by row (just as maj is computed column by column).

For completeness, we include here a proof that $\mathrm{inv}(\sigma)$ is equal to the number of relative inversions of σ.

Proposition 2.5.11. *The quantity* $\mathrm{inv}(\sigma)$ *is equal to the number of relative inversions of* σ*.*

Proof. Recall that $\mathrm{inv}(\sigma)$ is the total number of attacking pairs minus the arms of the descents. Each descent of the form $u > b$ where b is the entry directly below u contributes -1 towards $\mathrm{inv}(\sigma)$ for each v to the right of u in the same row. Call such pairs (u, v) *descent-arm pairs*. Each attacking pair contributes $+1$ towards $\mathrm{inv}(\sigma)$.

Define a *good triple* to be a triple of entries (u, v, b) where u is directly above and adjacent to b and v is to the right of u in its row, where we also allow b to be directly below the entire tableau with a value of 0. Then each descent-arm pair or attacking pair is a member of a unique good triple, and contributes -1 or $+1$, respectively, to $\mathrm{inv}(\sigma)$. Therefore, $\mathrm{inv}(\sigma)$ is the sum of the contributions of all such pairs in each such triple.

A simple case analysis shows that each good triple contributes a total of 1 if it is a relative inversion and 0 otherwise. Thus $\mathrm{inv}(\sigma)$ is the total number of relative inversions. $\square$

We now introduce the ***cocharge*** statistic on words, a variant of the *charge* statistic first defined by Lascoux and Schützenberger in [28]. To define it, we first recall the definition of Knuth equivalence.

Definition 2.5.12. Given a word $w = w_1 \cdots w_n$ of positive integers, a *Knuth move* consists of either:

- A transposition of the form $xyz \to xzy$ where x, y, z are consecutive letters and $y < x \le z$ or $z < x \le y$
- A transposition of the form $xyz \to yxz$ where x, y, z are consecutive letters and $x \le z < y$ or $y \le z < x$.

Two words $w, \widetilde{w}$ are said to be *Knuth equivalent*, written $w \sim \widetilde{w}$, if one can be reached from the other via a sequence of Knuth moves. Knuth equivalence is an equivalence relation on words.

The cocharge can be defined in terms of Knuth equivalence classes as follows.

Definition 2.5.13. Given a word $w = w_1, \cdots, w_n$ with partition content μ, the *cocharge* of w, denoted $\mathrm{cc}(w)$ is the unique statistic satisfying the following properties:

1. It is constant on Knuth equivalence classes, that is, if w is Knuth equivalent to $\widetilde{w}$ then $\mathrm{cc}\, w = \mathrm{cc}\, \widetilde{w}$.

2. If $w = w_1 w_2 \cdots w_n$ and $w \neq 1$, let $\mathrm{cyc}(w) = w_2 w_3 \cdots w_n w_1$ be the word formed by moving the first letter to the end. Then

$$\mathrm{cc}(\mathrm{cyc}(w)) = \mathrm{cc}(w) - 1.$$

3. If the letters of w are weakly increasing then $\mathrm{cc}(w) = 0$.

There is also an algorithmic way of computing cocharge.

Definition 2.5.14. Let w be a word with partition content μ, so that it has μ_1 1's, μ_2 2's, and so on. Let $w^{(1)}$ be the subword formed by scanning w from right to left until finding the first 1, then continuing to scan until finding a 2, and so on, wrapping around cyclically if need be. Let $w^{(2)}$ be the subword formed by removing $w^{(1)}$ from w and performing the same process on the remaining word, and in general define $w^{(i)}$ similarly for $i = 1, \ldots, \mu_1$.

It turns out that

$$\mathrm{cc}(w) = \sum_i \mathrm{cc}(w^{(i)}),$$

(see, e.g., [17]) and one can compute the cocharge of a word $w^{(i)}$ having distinct entries $1, \ldots, k$ by the following process.

1. Set a counter to be 0, and label the 1 in the word with this counter, i.e. give it a subscript of 0.

2. If the 2 in the word is to the left of the 1, increment the counter by 1, and otherwise do not change the counter. Label the 2 with the new value of the counter.

3. Continue this process on each successive integer up to k, incrementing the counter if it is to the left of the previous letter.

4. When all entries are labeled, the sum of the subscripts is the cocharge.

There is a direct link between the major index of fillings σ having $\mathrm{inv}(\sigma) = 0$ and the cocharge of words. The connection lies in the ***cocharge word*** construction.

Definition 2.5.15. The ***cocharge word*** of a filling $\sigma : \mu \to \mathbb{Z}_+$ is the word $\mathrm{cw}(\sigma) = i_1 i_2 \cdots i_n$ consisting of the row indices of the cells $u_k = (i_k, j_k)$, where $u_1, u_2, \ldots, u_n$ is the ordering of the cells of μ such that $\sigma(u_1) \geq \sigma(u_2) \geq \cdots \geq \sigma(u_n)$, and for each constant segment $\sigma(u_j) = \cdots = \sigma(u_k)$, the cells $u_j, \cdots, u_k$ are in reverse reading order.

It turns out that for any filling σ such that $\mathrm{inv}(\sigma) = 0$, we have $\mathrm{maj}(\sigma) = \mathrm{cc}(\mathrm{cw}(\sigma))$. (See [17] for the proof.) Notice also that the content of the cocharge word is μ, and the following proposition shows that if we are given the cocharge word along with the content α of σ then we can recover σ uniquely.

Definition 2.5.16. We say that a sequence of numbers $a_1, \ldots, a_n$ is in ***cyclic order*** if there exists an index $i \in [n]$ for which

$$a_{i+1} \leq a_{i+2} \leq \cdots \leq a_n \leq a_1 \leq a_2 \leq \cdots \leq a_i.$$

The above definition is used throughout, as well as in the proof of the following proposition.

Proposition 2.5.17. *Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition. Given a tuple of multisets $(A_1, \ldots, A_k)$ of positive integers where $|A_i| = \mu_i$ for all i, there is a unique filling σ of μ with* $\mathrm{inv}(\sigma) = 0$ *whose i-th row contains precisely the numbers in A_i for all i.*

Proof. Since $\mathrm{inv}(\sigma) = 0$, the bottom row has the elements of A_1 in increasing order from left to right. We now induct on the rows. Suppose row i is filled in with entries $b_1, \ldots, b_r$ left to right. The leftmost entry a_1 of row $i + 1$ must be the smallest element of A_{i+1} that comes after b_1 in cyclic order. Then the next entry a_2 must be the smallest element of $A_{i+1} \setminus \{a_1\}$, that comes after b_2 in cyclic order, and so on. This uniquely determines row $i + 1$. $\square$

It follows that the cocharge word map is an isomorphism

$$\mathrm{cw} : (\mathcal{F}^{\alpha}_{\mu}|_{\mathrm{inv}=0}; \mathrm{maj}) \to (W^{\mu}_{n}; \mathrm{cc})$$

for any α such that $\sum \alpha_i = |\mu| = n$.

Remark 2.5.18. Throughout, we will use the phrase "rearrange the entries in each row in the unique way such that $\mathrm{inv}(\sigma) = 0$" to refer to the unique filling given by Proposition 2.5.17.

2.6 Symmetric functions

Let S_∞ be the "infinite symmetric group" generated by adjacent transpositions $(i, i+1)$ on the positive integers, and consider its action by permuting the variables of the formal power series ring $K[[x_1, x_2, x_3, \ldots]]$ where K is any field of characteristic 0. This restricts to an action on the homogeneous series of (total) degree d for each d. Then the ring of ***symmetric functions*** over K, is the ring generated by the homogeneous invariants of each degree:

$$\Lambda_K(x_1, x_2, \ldots) = \bigoplus_d K[[x_1, x_2, x_3, \ldots]]_d^{S_\infty}$$

The multiplication is naturally inherited from $K[[x_1, x_2, x_3, \ldots]]$. We often simply write X to denote the infinite set of variables $x_1, x_2, \ldots$, so that $\Lambda_K(X)$ is the ring of symmetric functions in these variables.

There are several natural bases for the ring of symmetric functions, all indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$. We refer the reader to [30] or [35] for thorough introductions to the theory of symmetric functions. For convenience we recall the basic definitions below, which we will use throughout.

Monomial symmetric functions:

$$m_\lambda = \sum_{i_1<\cdots<i_k} x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$$

Homogeneous symmetric functions: $h_\lambda = h_{\lambda_1} \cdot h_{\lambda_2} \cdot \cdots \cdot h_{\lambda_k}$ where

$$h_d = \sum_{i_1\le\cdots\le i_d} x_{i_1} \cdots x_{i_d}$$

Elementary symmetric functions: $e_\lambda = e_{\lambda_1} \cdot e_{\lambda_2} \cdot \cdots \cdot e_{\lambda_k}$ where

$$e_d = \sum_{i_1<\cdots<i_d} x_{i_1} \cdots x_{i_d}$$

Power sum symmetric functions: $p_\lambda = p_{\lambda_1} \cdot p_{\lambda_2} \cdot \cdots \cdot p_{\lambda_k}$ where

$$p_d = x_1^d + x_2^d + x_3^d + \cdots$$

Schur functions:

$$s_\lambda = \sum_{T\in \mathrm{SSYT}(\lambda)} x_1^{\alpha_1(T)} x_2^{\alpha_2}(T) \cdots$$

where $\mathrm{SSYT}(\lambda)$ is the set of all semistandard Young tableaux of shape λ, and $\alpha_i(\sigma)$ is the number of i's in T.

The ***Hall inner product*** is an inner product on $\Lambda_k(X)$ defined by setting

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu},$$

and extending by linearity. This inner product has the properties that $\langle h_\mu, m_\lambda \rangle = \delta_{\lambda\mu}$ and $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$, where $z_\lambda = \prod_i i^{m_i} m_i!$ where m_i is the number of times i occurs in λ.

It is well-known (see [30] or [33]) that the irreducible representations of the symmetric group S_n are also indexed by the partitions λ of size n, and we write V^λ to denote the irreducible S_n-module (sometimes called the *Specht module*) arising in this way. Let $S_n^\vee$ denote the set of all representations of S_n.

There is a natural correspondence between representations of the symmetric group S_n and a certain class of symmetric functions, that captures other aspects of the representation theory as well. The correspondence is given by the ***Frobenius map*** $\mathrm{Frob} : S_n^\vee \to \Lambda_{\mathbb{Q}}(X)$, given by

$$\mathrm{Frob}(V) = \frac{1}{n!} \sum_{\pi\in S_n} \chi_V(\pi) p_\lambda$$

where χ_V is the character of the representation V. Under this map we have $\mathrm{Frob}(V^\lambda) = s_\lambda$ for the irreducible representations V^λ, along with the rules

$$\mathrm{Frob}(V \oplus W) = \mathrm{Frob}(V) + \mathrm{Frob}(W) \tag{2.1}$$
$$\mathrm{Frob}(V \otimes W) = \mathrm{Frob}(V) \cdot \mathrm{Frob}(W) \tag{2.2}$$

Since any representation decomposes uniquely as a direct sum of irreducible representations, equation (2.1) implies that the range of Frob consists precisely of the symmetric functions that are positive integer sums of Schur functions s_λ. We call such a symmetric function ***Schur positive***.

Throughout, we will be working with generalizations of the Frobenius map that capture the structure of a graded or bi-graded S_n-module, as follows. For a graded S_n-module $V = \bigoplus_d V_d$, we define $\mathrm{Frob}_q(V) = \sum_d q^d \,\mathrm{Frob}(V_d)$, and for a bigraded S_n-module $V = \bigoplus_{i,j} V_{i,j}$, we define

$$\mathrm{Frob}_{q,t}(V) = \sum_{i,j} q^i t^j \,\mathrm{Frob}(V_{i,j}).$$

The map $\mathrm{Frob}_{q,t}$ (respectively Frob_q) gives a morphism from the ring of $\mathbb{Z}$-linear combinations of S_n-modules, with the additive and multiplicative structure of $\oplus$ and $\otimes$, to the ring $\Lambda_{\mathbb{Q}(q,t)}(X)$ (respectively $\Lambda_{\mathbb{Q}(q)}(X)$.) We say that a symmetric function $f \in \Lambda_{\mathbb{Q}(q,t)}(X)$ is ***Schur positive*** if it is a linear combination of Schur functions s_λ with coefficients in $\mathbb{Z}_+[q,t]$, and likewise for $\Lambda_{\mathbb{Q}(q)}(X)$. Note that a symmetric function is Schur positive if and only if it is the Frobenius image of a (bi-)graded S_n-module.

Several natural examples of bigraded S_n-modules, which arise naturally in the theory of Macdonald polymonials, begin with the ***diagonal action*** of S_n on $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. In the diagonal action, a permutation π sends a function $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$ to

$$f(x_{\pi_1}, \ldots, x_{\pi_n}, y_{\pi_1}, \ldots, y_{\pi_n}),$$

simultaneously permuting the two sets of variables. The quotient of this polynomial ring by the ideal S_+ of positive-degree invariants is called the ring of *diagonal harmonics*, written

$$\mathrm{DH}_n = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/S_+.$$

This quotient ring inherits an S_n-action from the diagonal action, and has been the subject of much recent study in algebra and combinatorics (see [16]).

Finally, we will require the notation of ***plethystic substitution***, an operation on symmetric functions defined as follows. Let $A = A(a_1, a_2, \ldots) \in \mathbb{Z}[[a_1, a_2, \ldots]]$ be a formal sum of monomials with integer coefficients in the variables a_i. For a power sum symmetric function $p_k = x_1^k + x_2^k + x_3^k + \cdots$, the plethystic substitution of A into p_k is the expression

$$p_k[A] = A(a_1^k, a_2^k, \ldots).$$

For a general symmetric function f, express $f = f(p_1, p_2, \ldots)$ as a polynomial in the power sum symmetric functions p_k. Then

$$f[A] = f(p_1[A], p_2[A], p_3[A], \ldots).$$

Plethystic substitution can be used to define a Hopf algebra structure on the ring of symmetric functions [7] with the power sum symmetric functions as the primitive elements. Certain plethystic substitutions, in particular the substitutions $f[(1-t)X]$ and $f[\frac{X}{1-t}]$ where $X = x_1 + x_2 + \cdots$ have important representation theoretic meanings (see [20], Proposition 3.3.1).

2.7 Hall-Littlewood polynomials

The ***Hall-Littlewood polynomials*** are t-analogs of certain classical bases of the space of symmetric functions. The original definition of the Hall-Littlewood polynomials is as the collection of polynomials $P_\lambda(X;t)$ in n variables $X = \{x_1, \ldots, x_n\}$ given by

$$P_\lambda(x_1, \ldots, x_n; t) = \frac{1}{\prod_i (m_i)_t!} \sum_{\pi \in S_n} \pi \left(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j} \right), \tag{2.3}$$

where $k = \ell(\lambda)$, m_i is the number of i's that occur in λ, and, for any positive integer m,

$$(m)_t! = (1)(1+t)(1+t+t^2)\cdots(1+t+t^2+\cdots+t^{m-1})$$

is the ***t-factorial***. One can show that the Hall-Littlewood polynomials are restriction-compatible:

$$P(x_1, \ldots, x_{n-1}, 0; t) = P(x_1, \ldots, x_{n-1}; t),$$

and we can therefore extend the Hall-Littlewood polynomials to the limiting symmetric functions $P(X;t)$ in an infinite collection of variables $X = \{x_1, x_2, \ldots\}$. (For this result and the basic facts about Hall-Littlewood polynomials that follow, see Macdonald's book [30], Chapter III.)

The Hall-Littlewood polynomials have the following specializations:

$$P_\lambda(X;0) = s_\lambda, \qquad P_\lambda(X;1) = m_\lambda, \qquad P_{(1^r)}(X;t) = e_r.$$

The polynomials P_λ form an orthogonal basis of $\Lambda_{\mathbb{Q}(t)}$ with respect to a symmetric t-analog of the Hall inner product given by

$$\langle f, g \rangle_t = \left\langle f, g\left[\frac{X}{1-t}\right] \right\rangle.$$

Indeed, if we define $b_\lambda(t) = \prod (m_i)_t!(1-t)^{m_i}$ and set $Q_\lambda(X;t) = b_\lambda(t) P_\lambda(X;t)$, then

$$\langle P_\lambda(X;t), Q_\mu(X;t) \rangle_t = \delta_{\lambda\mu}.$$

Since the P_λ's form a basis, we can ask for the transition matrix between the P_λ's and s_λ's. It turns out that

$$s_\lambda = \sum K_{\lambda\mu}(t) P_\mu(X;t)$$

where $K_{\lambda\mu}(t)$ is a polynomial in t with positive integer coefficients, giving a t-analog of the Kostka numbers $K_{\lambda\mu}$ that arise from the specialization $t = 1$. In fact, there is a statistic on semistandard Young tableaux called ***charge***, for which

$$K_{\lambda\mu}(t) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} t^{\mathrm{charge}(T)},$$

so in fact it is the t-series of a certain combinatorial statistic on the set $\mathrm{SSYT}(\lambda, \mu)$ of semistandard Young tableaux of shape λ and content μ.

Definition 2.7.1. The ***charge*** of a word w with partition content μ is the quantity

$$\mathrm{charge}(w) = n(\mu) - \mathrm{cc}(w)$$

where

$$n(\mu) = \sum (i-1)\mu_i$$

is the maximum value of the cocharge on W_n^μ. The ***charge*** of a semistandard Young tableau T is defined to be the charge of its reading word.

It is not difficult to transform this equation into a statement about Schur positivity of certain "transformed" Hall-Littlewood polynomials. In particular, define $H_\mu(X;t) = Q_\mu(\left[\frac{X}{1-t}\right];t)$. Then

$$\begin{aligned} \langle H_\mu(X;t), s_\lambda\rangle &= \langle s_\lambda, H_\mu(X;t)\rangle \\ &= \langle s_\lambda, Q_\mu(X;t)\rangle_t \\ &= \langle Q_\mu(X;t), \sum_\nu K_{\lambda\nu}(t) P_\nu(X;t)\rangle \\ &= K_{\lambda\mu}(t). \end{aligned}$$

Thus $H_\mu(X;t) = \sum_\lambda K_{\lambda\mu}(t)s_\lambda$, and so the polynomials H_μ are Schur positive. In fact, they arise naturally as the Frobenius image of a graded S_n-module when we make the further transformation to reverse the grading, by defining

$$\widetilde{H}_\mu(X;t) = t^{n(\mu)} H_\mu(X;t^{-1}).$$

(Note that $n(\mu)$ is the maximum value of the charge statistic on $\mathrm{SSYT}(\lambda, \mu)$.) We likewise define

$$\widetilde{K}_{\lambda\mu}(t) = t^{n(\mu)} K_{\lambda\mu}(t),$$

and define the ***cocharge*** of a semistandard Young tableau T to be $\mathrm{cc}(T) = n(\mu) - \mathrm{charge}(T)$. We call these polynomials the (transformed) ***t-Kostka polynomials***.

The symmetric functions $\widetilde{H}_\mu(X;t)$ are the Frobenius image of the ***Garsia-Procesi modules*** R_μ. These graded S_n-modules are the cohomology rings of the fibers of the Springer resolution (see [9] and [37]), and a concrete algebraic description was given in [13]. In particular, Garsia and Procesi consider the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ and define the ideal

$$I_\mu = (e_r(S) \mid k \geq r > k - d_k(\mu), |S| = k, S \subset \{x_1, \ldots, x_n\})$$

where

$$d_k(\mu) = \mu_1^* + \cdots + \mu_k^*$$

is the sum of the first k columns of μ, and where $e_r(S)$ is the "partial" elementary symmetric function in the variables in the set S. Then if

$$R_\mu = \mathbb{Q}[x_1, \ldots, x_n]/I_\mu,$$

the action of S_n on the variables gives rise to an action on R_μ and we have

$$\mathrm{Frob}_t(R_\mu) = \widetilde{H}_\mu(X; t).$$

In [13], the authors recursively define a set of monomials $\mathcal{B}(\mu)$, and show they form a basis of R_μ. To state this recursion we require two more definitions, which follow the notation in [13].

Definition 2.7.2. Given a partition μ, define $\mu^{(i)}$ to be the partition formed by removing the corner square from the column a_i containing the last square in the i-th row μ_i.

Definition 2.7.3. Given a set of monomials $\mathcal{C}$ and a monomial m, we write $m \cdot \mathcal{C}$ to denote the set of all monomials of the form $m \cdot x$ where $x \in \mathcal{C}$.

The following recursion defines the sets $\mathcal{B}(\mu)$.

Definition 2.7.4. The sets $\mathcal{B}(\mu)$ are defined by $\mathcal{B}((1)) = \{1\}$ and the recursion

$$\mathcal{B}(\mu) = \bigsqcup_{i-1}^{\mu_1^*} x_n^{i-1} \cdot \mathcal{B}(\mu^{(i)}).$$

We refer to these sets as the ***Garsia-Procesi module bases***.

2.8 Macdonald polynomials and a symmetry problem

The ***Macdonald polynomials*** are a two-parameter analog of certain bases of symmetric functions that specialize to the Hall-Littlewood polynomials. They were originally defined as the eigenvectors $P_\lambda(X; q, t)$ of certain operators on the ring $\Lambda_{\mathbb{Q}(q,t)}(X)$. They are constructed in such a way that they are the unique collection of functions satisfying certain orthogonality and triangularity conditions with respect to the q, t-Hall inner product defined by

$$\langle f, g \rangle_{q,t} = \left\langle f, g \left[\frac{1-q}{1-t} X \right] \right\rangle.$$

Definition 2.8.1. The polynomials $P_\lambda(X; q, t) \in \Lambda_{\mathbb{Q}(q,t)}(X)$ are the unique collection of polynomials satisfying:

- $P_\lambda = \sum_{\mu \le \lambda} a_{\lambda\mu}(q, t) m_\mu$ where $a_{\lambda\lambda} = 1$, and
- $\langle P_\lambda, P_\mu \rangle = 0$ if $\lambda \neq \mu$.

While the Macdonald polynomials do not have a known formula analogous to Equation (2.3), one can show that they have the following specializations:

$$P_\lambda(X; t, t) = s_\lambda, \qquad P_\lambda(X; 0, t) = P_\lambda(X; t), \qquad P_\lambda(X; 1, t) = e_{\lambda^*}, \qquad P_{(1^r)}(X; q, t) = e_r.$$

Macdonald defined the "integral form" of P_λ, written J_λ, to be a q, t-scalar multiple of P_λ and conjectured that its coefficients with respect to certain other bases are *polynomials* in q and t, rather than simply being rational functions. In particular, define

$$J_\lambda(X; q, t) = c_\lambda(q, t) P_\lambda(X; q, t)$$

where

$$c_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)} t^{l(s)+1}).$$

(Recall that $a(s)$ and $l(s)$ are the lengths of the *arm* and *leg* of the square s respectively.) Let $K_{\lambda\mu}(q, t)$ be the coefficients of J_μ in terms of the basis $s_\lambda[X/(1-t)]$, so that we have

$$J_\mu(X; q, t) = \sum_\lambda K_{\lambda\mu}(q, t) s_\lambda \left[\frac{X}{1-t} \right].$$

The ***Macdonald positivity conjecture***, in its original form, is that the coefficients $K_{\lambda\mu}(q, t)$ are elements of $\mathbb{Z}_+[q, t]$. As in the Hall-Littlewood case, it is convenient to make a transformation so that this conjecture can be expressed as a problem of Schur positivity. In particular, let

$$H_\mu(X; q, t) = J_\mu[\frac{X}{1-t}; q, t]$$

and let

$$\widetilde{H}_\mu = t^{n(\mu)} H_\mu(X; q, t^{-1}).$$

Also define the ***q, t-Kostka polynomials*** as $\widetilde{K}_{\lambda\mu}(q, t) = t^{n(\mu)} K_{\lambda\mu}(q, t^{-1})$. Then

$$\widetilde{H}_\mu(X; q, t) = \sum \widetilde{K}_{\lambda\mu}(q, t) s_\lambda,$$

and the Macdonald positivity conjecture can be rephrased as the claim that $\widetilde{K}_{\lambda\mu}(q, t)$ are polynomials with positive integer coefficients (i.e., that the H_μ's are Schur positive). The transformed Macdonald polynomials $\widetilde{H}_\mu(X; q, t)$ are uniquely determined by the somewhat more symmetric orthogonality relations

- $\widetilde{H}_\mu[(1-q)X; q, t] \in \mathbb{Q}(q,t)\{s_\lambda : \lambda \geq \mu\}$
- $\widetilde{H}_\mu[(1-t)X; q, t] \in \mathbb{Q}(q,t)\{s_\lambda : \lambda \geq \mu^*\}$
- $\widetilde{H}_\mu[1; q, t] = 1$.

The Macdonald positivity conjecture was first proved by Haiman [22], who showed that the polynomials $\widetilde{H}_\mu(X; q, t)$ are the Frobenius characteristics of certain bi-graded S_n-modules called the ***Garsia-Haiman modules***. For a partition μ, choose any ordering $(a_1, b_1), (a_2, b_2), \ldots$ of its cells, and define

$$\Delta_\mu = \det(x_i^{a_j} y_i^{b_j}).$$

Notice that Δ_μ is, up to sign, independent of the choice of ordering of the cells. Then, define

$$R_\mu = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/J_\mu$$

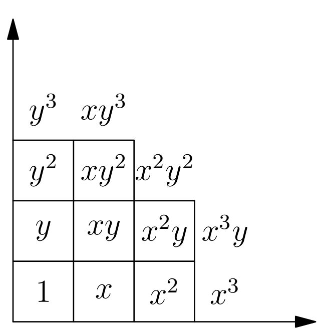
where J_μ is the ideal of all polynomials $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$ whose associated differential operator $\partial f = f(\partial x_i, \partial y_j)$ kills Δ_μ. Then R_μ inherits the structure of a bigraded S_n-module from the diagonal action on $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

As mentioned in the introduction, it was conjectured in [12] that $\mathrm{Frob}_{q,t}(R_\mu) = \widetilde{H}_\mu(X; q, t)$, and the authors reduced the problem to showing that R_μ has dimension $n!$ as a $\mathbb{C}$-vector space. This became known as the $n!$ ***conjecture***. In order to prove the $n!$ conjecture, Haiman established a link between the symmetric functions $\widetilde{H}_\mu(X; q, t)$ and the geometry of the ***Hilbert scheme*** of n points in the plane. [22] The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ is a moduli space which, as a set, consists of all ideals $I \subset \mathbb{C}[x, y]$ for which $\dim_{\mathbb{C}} \mathbb{C}[x, y]/I = n$.

A "generic" element of the Hilbert scheme,

$$I = \bigcap (x - x_i, y - y_i),$$

corresponds to a set of n distinct points $(x_i, y_i) \in \mathbb{C}^2$. Haiman considered the fixed points of the action of the torus $(\mathbb{C}^*)^2$ on $\mathrm{Hilb}_n(\mathbb{C}^2)$ (inherited from the natural multiplication action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^2$). These fixed points are precisely the ***monomial ideals*** I_μ generated by the monomials $x^i y^j$ such that (i, j) is not a square of μ.



There is a further construction [20] known as the ***isospectral Hilbert scheme***, written X_n. This can be thought of as the moduli space of ideals $I \in \mathrm{Hilb}_n(\mathbb{C}^2)$ together with an ordering of the n points of the zero locus of I, counted with multiplicity. In particular, there is a natural map $\rho : X_n \to \mathrm{Hilb}_n(\mathbb{C}^2)$ such that the following diagram commutes:

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "\rho"'] & \mathbb{C}^{2n} \arrow[d] \\ \mathrm{Hilb}_n(\mathbb{C}^2) \arrow[r] & \mathbb{C}^{2n}/S_n \end{tikzcd}$$

(In the diagram, the horizontal arrows take a point of the Hilbert scheme or isospectral Hilbert scheme to its corresponding list of points or set of points respectively.) Finally, let $\mathcal{H}_\mu = (\rho_* \mathcal{O}_X)_{I_\mu}$. By showing that the isospectral Hilbert scheme X_n is Cohen-Macaulay and Gorenstein, Haiman proved that

$$\mathcal{H}_\mu \cong R_\mu = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/J_\mu,$$

and that this space has dimension $n!$. [22]

There is no known generalization of the cocharge formula for the coefficients of the form

$$\widetilde{K}_{\lambda\mu}(q,t) = \sum_T q^{s(T)} t^{r(T)},$$

where T ranges over an appropriate set of Young tableaux and r and s are combinatorial statistics. This remains an important open problem in the theory of Macdonald polynomials.

Despite this mystery, a different combinatorial formula for the transformed Macdonald polynomials $\widetilde{H}_\mu$ has been found, and appeared in the literature in [17] in 2004. The authors prove that

$$\widetilde{H}_\mu(x; q, t) = \sum_\sigma q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma, \tag{2.4}$$

where the sum ranges over all fillings σ of the diagram of μ with positive integers, and x^σ is the monomial $x_1^{m_1} x_2^{m_2} \cdots$ where m_i is the number of times the letter i occurs in σ. Here inv and maj are the statistics on fillings defined in Section 2.5.

Since this combinatorial formula for $\widetilde{H}_\mu(x; q, t)$ is an expansion in terms of monomials rather than Schur functions, it does not give an immediate answer to the Macdonald positivity conjecture. Indeed, it perhaps raises more questions than it answers. For one, there is a well-known q, t-symmetry relation for the transformed Macdonald polynomials $\widetilde{H}_\mu(x; q, t)$, namely

$$\widetilde{H}_\mu(x; q, t) = \widetilde{H}_{\mu^*}(x; t, q).$$

This is obvious from the triangularity conditions that define $\widetilde{H}_\mu$, and is also clear from Haiman's geometric interpretation. [22] When combined with the combinatorial formula, however, we obtain a remarkable generating function identity:

$$\sum_{\sigma:\mu\to\mathbb{Z}_+} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^{\sigma} = \sum_{\rho:\mu^*\to\mathbb{Z}_+} q^{\mathrm{maj}(\rho)} t^{\mathrm{inv}(\rho)} x^{\rho}. \tag{2.5}$$

Setting $t = 1$ and $\mu = (n)$ and taking the coefficient of $x_1 \cdots x_n$ on both sides, this reduces to

$$\sum_{w\in S_n} q^{\mathrm{inv}(w)} = \sum_{w\in S_n} q^{\mathrm{maj}(w)}.$$

This is precisely the equation describing the equidistribution of the Mahonian statistics inv and maj on permutations, which we discussed in Sections 2.2 and 2.3

In light of this, it is natural to ask if there is an elementary combinatorial proof of (2.5), in the sense of the problem stated in the introduction. We now restate this problem below using the notation introduced in Section 2.1.

Problem 1.0.1. Find a natural isomorphism of weighted sets

$$\varphi : (\mathcal{F}; \mathrm{inv}, \mathrm{maj}) \to (\mathcal{F}; \mathrm{maj}, \mathrm{inv})$$

which interchanges inv and maj and sends a partition shape to its conjugate. That is, for any a, b, μ, α, the map φ restricts to a bijection

$$\varphi : \mathcal{F}^{\alpha}_{\mu}|_{\mathrm{inv}=a,\mathrm{maj}=b} \to \mathcal{F}^{\alpha}_{\mu^*}|_{\mathrm{inv}=b,\mathrm{maj}=a}.$$

Remark 2.8.2. In [17], the authors give a combinatorial proof of the fact that the polynomials $\widetilde{H}_\mu$ are symmetric in the variables x_i. We will make use of this fact repeatedly, rearranging the entries of α as needed. In other words, to solve Problem 1.0.1, it suffices to find a map φ that restricts to bijections $\mathcal{F}^{\alpha}_{\mu}|_{\mathrm{inv}=a,\mathrm{maj}=b} \to \mathcal{F}^{r(\alpha)}_{\mu^*}|_{\mathrm{inv}=b,\mathrm{maj}=a}$ where r is some bijective map that rearranges the entries of α.

Chapter 3

The inv statistic for Hall-Littlewood polynomials

We now turn to the specialization of Equation (2.5) at $q = 0$. The symmetry relation becomes

$$\widetilde{H}_\mu(x;0,t) = \widetilde{H}_{\mu^*}(x;t,0),$$

which is a symmetry relation between the transformed *Hall-Littlewood polynomials* $\widetilde{H}_\mu(x;t) := \widetilde{H}_\mu(x;0,t)$. In this case the equation reduces to becomes

$$\sum_{\substack{\sigma:\mu\to\mathbb{Z}_+\\ \mathrm{inv}(\sigma)=0}} t^{\mathrm{maj}(\sigma)} x^\sigma = \sum_{\substack{\rho:\mu^*\to\mathbb{Z}_+\\ \mathrm{maj}(\rho)=0}} t^{\mathrm{inv}(\rho)} x^\rho. \tag{3.1}$$

Combinatorially, we would like to find natural morphisms

$$\varphi : (\mathcal{F}^\alpha_\mu|_{\mathrm{inv}=0};\mathrm{maj}) \to (\mathcal{F}^{r(\alpha)}_{\mu^*}|_{\mathrm{maj}=0};\mathrm{inv})$$

of weighted sets. For the bijection $r(\alpha)$, we will use the *reverse* map of Definition 3.2.2 below.

As noted in Section 2.5, the left hand side above can alternatively be described in terms of the cocharge statistic on words. In this chapter we consider the right hand side, and study fillings ρ having $\mathrm{maj}(\rho) = 0$, equipped with the inv statistic.

3.1 Inversion words and diagrams

In analogy with the cocharge word defined in [17], for fillings ρ having $\mathrm{maj}(\rho) = 0$, we can form an associated *inversion word* and describe a statistic on the inversion word that measures $\mathrm{inv}(\rho)$ in the case that $\mathrm{maj}(\rho) = 0$.

Definition 3.1.1. Let ρ be a filling of shape μ having $\mathrm{maj} = 0$. We define the ***inversion word*** of ρ as follows. Starting with the smallest value that appears in the filling, write the column numbers of the entries with that value as they appear in reading order, and then proceed with the second smallest entry and so on.

For instance, the filling:

$$\begin{array}{|c|c|c|c|}\cline{1-1} 1 \\ \cline{1-3} 2 & 2 & 4 \\ \hline 4 & 3 & 4 & 1 \\ \hline 5 & 3 & 5 & 2 \\ \hline\end{array}$$

has inversion word 141242231313.

In order to compute $\mathrm{inv}(\sigma)$ given only its inversion word, we will use a visual representation of the inversion word, which we call a *diagram*.

Definition 3.1.2. Fix a linearly ordered finite multiset A, with elements $a_1 \le a_2 \le \cdots \le a_n$. The ***diagram*** a function $f : A \to \mathbb{Z}_+$ is the plot of the function with respect to the ordering on A. We say that the diagram has ***shape*** μ if $|f^{-1}(i)| = \mu_i$ for each i.

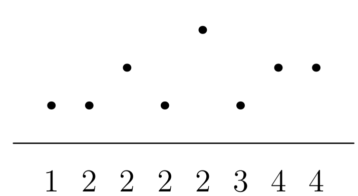
The diagrams we will be using, defined below, are essentially the plot of the inversion word, considered as a function on a multiset.

Definition 3.1.3. Let ρ be a filling of μ^* having $\mathrm{maj}(\rho) = 0$, and let w be the inversion word of ρ. Let A be the multiset consisting of the entries of ρ, ordered from least to greatest and in reading order in the case of a tie. Let $f : A \to \mathbb{Z}_+$ be the function given by $f(a_i) = w_i$. We define $\mathrm{InvPlot}(\rho)$ to be the diagram of the function f, whose plot has μ_j dots in the j-th row.

Notice that the InvPlot of a filling of shape μ^* has shape μ, the conjugate shape. For instance, the tableau

$$\begin{array}{|c|c|c|}\cline{1-1} 1 \\ \cline{1-2} 2 & 2 \\ \cline{1-2} 2 & 4 \\ \hline 3 & 4 & 2 \\ \hline\end{array}$$

has maj $= 0$, and its inversion word is 11213122. Its plot is as follows.



To compute the number of inversions, we define the ***inversion labeling*** of a diagram to be the result of labeling each row of dots μ_i in the diagram with the numbers $1, 2, \ldots, \mu_i$ from right to left:

$$\begin{array}{cccccccc}
 & & & & 1 & & & \\
 & & 3 & & & & 2 & 1 \\
4 & 3 & & 2 & & 1 & & \\
\hline
1 & 2 & 2 & 2 & 2 & 3 & 4 & 4
\end{array}$$

Finally, an ***inversion*** in the diagram of a function $f : A \to \mathbb{Z}_+$, labeled as above, is a pair of entries $a < b$ in the ordered multiset A for which either:

I. The dots above a and b have the same label and $f(a) > f(b)$, or

II. The dot in position a is labeled i and the dot in position b is labeled $i + 1$, and $f(b) > f(a)$.

So there are 3 inversions in the diagram above, two of type I and one of type II:

$$\begin{array}{cccccccc}
 & & & & 1 & & & \\
 & & 3 & & & & 2 & 1 \\
4 & 3 & & 2 & & 1 & & \\
\hline
1 & 2 & 2 & 2 & 2 & 3 & 4 & 4
\end{array}$$

For fillings σ with $\mathrm{maj}(\sigma) = 0$, there are no descents, and so the number of inversions in $\mathrm{InvPlot}(\sigma)$ is equal to $\mathrm{inv}(\sigma)$. In particular, type I and II inversions correspond to attacking pairs in the same row or on adjacent rows, respectively.

Remark 3.1.4. The type I and II inversions also correspond to the two types of inversions used to define the dinv statistic on parking functions. Indeed, this was the original motivation for the full definition of the inv statistic. [15]

We now classify the types of diagrams that arise as the InvPlot of a filling.

Definition 3.1.5. A consecutive subsequence is in ***inversion-friendly order*** if, when each row is labeled from right to left as above, all dots of label $i+1$ in the subsequence occur before the dots of label i for all i, and the dots of any given label appear in increasing order from bottom to top.

An example of an inversion-friendly subsequence is shown below.

$$\begin{array}{cccccccc}
 & & & & & & & 2 \\
 & 5 & & & 4 & & & \\
 & & & 4 & & 3 & 2 & \\
5 & & 4 & & & & &
\end{array}$$

It is easy to check that, in the plot of any filling ρ having $\mathrm{maj}(\rho) = 0$, every subsequence above a fixed letter of the alphabet A is in inversion-friendly order. We claim that the converse is true as well, namely, that every diagram having all such subsequences in inversion-friendly order corresponds to a unique Young diagram filling ρ having $\mathrm{maj}(\rho) = 0$.

Definition 3.1.6. A diagram is of ***inversion word type*** if every subsequence determined by a fixed letter of A is in inversion-friendly order.

We let $\mathrm{ID}_{\mu,A}$ the set of all diagrams of shape μ of inversion word type over A. We equip $\mathrm{ID}_{\mu,A}$ with its inv statistic to make it into a weighted set.

Proposition 3.1.7. *Let μ be a partition of n, and let A be a multiset of n positive integers with content α. The map* InvPlot *is an isomorphism of weighted sets*

$$\mathrm{InvPlot} : (\mathcal{F}^{\alpha}_{\mu^*}|_{\mathrm{maj}=0}; \mathrm{inv}) \to (\mathrm{ID}_{\mu,A}; \mathrm{inv}).$$

Proof. As noted above, this is a map of sets that preserves the inv statistic since there are no descents. To show it is bijective, we construct its inverse.

Let D be an arbitrary diagram in $\mathrm{ID}_{\mu,A}$, and let $f : A \to \mathbb{Z}_+$ be the corresponding map. For any $a \in A$ let $\ell(a)$ be the label on the dot at height $f(a)$. Then let ρ be the filling of shape μ^* in which $a \in A$ is placed in the square in column $f(a)$ from the left, and height $\ell(a)$ from the bottom. By the definition of InvPlot, we have that $\mathrm{InvPlot}(\rho) = D$, and furthermore if $D = \mathrm{InvPlot}(\sigma)$ then $\rho = \sigma$. Thus the map sending D to ρ is the inverse of InvPlot. □

We will show in Section 3.4 that the inversion-friendly diagrams are in weight-preserving bijection with a certain generalization of Carlitz codes, thereby generalizing the map invcode of the Carlitz bijection.

There is another natural question that arises from this line of inquiry. One can show the Schur-positivity of the Hall-Littlewood polynomials directly from the $t = 0$ specialization of the combinatorial formula for Macdonald polynomials, as done in [17] via the cocharge word construction. It would be useful to do the same from the $q = 0$ side, via the inversion word construction. This may allow us to obtain a different statistic on semistandard Young tableaux that also describes the Kostka polynomials, in the following sense.

Problem 3.1.8. *Construct a map f from $\mathrm{ID}_{\mu,A}$ to the set S of pairs (P, Q) of semistandard Young tableaux of the same shape such that Q has entries A, and a natural statistic $\widetilde{\mathrm{inv}}$ on semistandard Young tableaux such that if $f(W) = (P, Q)$ then $\mathrm{inv}(W) = \widetilde{\mathrm{inv}}(P)$.*

In the $t = 0$ specialization where the statistics are maj and cocharge, the map f is simply the RSK correspondence (see [17]). However, the inv statistic on inversion words is not invariant under Knuth equivalence, and so the RSK correspondence would not suffice for Problem 3.1.8.

A solution to this problem would give rise to a Schur expansion for the Hall-Littlwood polynomials of the form

$$\widetilde{H}(X;t) = \sum_{\lambda} \left(\sum_{T \in \mathrm{SSYT}(\lambda,\mu)} t^{\widetilde{\mathrm{inv}}(T)} \right) s_\lambda,$$

giving us an alternative expansion of the Kostka polynomials as $\widetilde{K}_{\lambda\mu}(t) = \sum_{T\in\mathrm{SSYT}(\lambda,\mu)} t^{\widetilde{\mathrm{inv}}(T)}$. It would therefore be of interest to try to find a statistic $\widetilde{\mathrm{inv}}$ that satisfies the conjecture and which is easier or faster to compute than the rather unwieldy cocharge statistic.

3.2 The Carlitz bijection on words

Notice that the Carlitz bijection gives rise to a bijection ϕ satisfying Problem 1.0.1 for one-column shapes $\mu = (1, 1, \ldots, 1)$ having content $\alpha = (1, 1, \ldots, 1)$. Indeed, $\mathrm{inv}(\sigma) = 0$ for any filling σ of a one-column shape μ, and $\mathrm{maj}(\rho) = 0$ for any filling ρ of its one-row conjugate μ^*. Since $\mathrm{maj}(\sigma)$ and $\mathrm{inv}(\rho)$ in this case are the same as maj and inv of their reading words, this determines a bijection for distinct entries ($\alpha = (1, 1, \ldots, 1)$.)

In order to generalize Carlitz codes to the general Hall-Littlewood case, we now generalize the Carlitz bijection to words, i.e. fillings with *any* content α for one-column shapes μ.

Definition 3.2.1. Let $A = (a_1^{\alpha_1}, a_2^{\alpha_2}, \ldots, a_k^{\alpha_k})$ be any finite multiset of size n, with an ordering "$<$" such that $a_1 < a_2 < \cdots < a_k$, and let μ be a partition of n. We say that a word c of length n is ***A-weakly increasing*** if every subword of the form

$$c_{\alpha_1+\cdots+\alpha_i}, c_{\alpha_1+\cdots\alpha_i+1}, \ldots c_{\alpha_1+\cdots+\alpha_i+\alpha_{i+1}-1}$$

is weakly increasing.

For instance, if $A = \{1, 1, 2, 3, 3, 3, 4, 4\}$, ordered by magnitude, then the word 23711213 is A-weakly increasing, since the subwords 23, 7, 112, and 13, corresponding to each letter of A, are weakly increasing.

We also will make use of Macdonald symmetry in the variables x_i by defining a weight-preserving bijection on alphabets.

Definition 3.2.2. The ***reverse*** of the content $\alpha = (\alpha_1, \ldots, \alpha_M)$ is the tuple

$$r(\alpha) = (\alpha_M, \alpha_{M-1}, \ldots, \alpha_1).$$

In terms of alphabets, let A be a finite multiset of positive integers with maximum element M. The ***content*** of A is α if α_i is the multiplicity of i in A. The ***complement*** of A, denoted $\overline{A}$, is the multiset consisting of the elements $M + 1 - a$ for all $a \in A$. Notice that the content of $\overline{A}$ is $r(\alpha)$.

If A is an ordered alphabet, then its complement inherits this ordering: if $a < b$ in A then $\overline{a} < \overline{b}$ in $\overline{A}$.

For instance, the complement of the multiset

$$\{1, 2, 2, 2, 2, 3, 4, 4\}$$

is $\{4, 3, 3, 3, 3, 2, 1, 1\}$, and correspondingly, $r(1, 4, 1, 2) = (2, 1, 4, 1)$.

We generalize Carlitz's codes as follows.

Definition 3.2.3. Let $C_{(1^n),A}$ denote the subset of C_n consisting of all Carlitz codes of length n which are A-weakly increasing. This subset inherits the Σ statistic from C_n.

We now can define bijections

$$\mathrm{invcode} : (\mathcal{F}^{\alpha}_{(1^n)}; \mathrm{inv}) \to (C_{(1^n),A}; \Sigma)$$

and

$$\mathrm{majcode} : (\mathcal{F}^{r(\alpha)}_{(n)}; \mathrm{maj}) \to (C_{(1^n),A}; \Sigma).$$

Definition 3.2.4. Let w be a word consisting of the letters in the ordered alphabet $A = a_1 \leq \cdots \leq a_n$ (corresponding to a filling of a horizontal shape), with ties among the letters broken in the order they appear in w. The ***inversion code*** of w is the code $\mathrm{invcode}(w) = c_1 \cdots c_n$ where c_i is the number of inversions having a_i as the smaller entry of the inversion.

For example, the inversion code of the filling

3	2	4	1	3	2

is 313010, since the 1 is the smaller entry of 3 inversions, the first 2 is the smaller entry of 1 inversion, the second 2 is the smaller entry of 3 inversions, and so on.

Proposition 3.2.5. *The map* invcode *is an isomorphism of weighted sets*

$$\mathrm{invcode} : \mathcal{F}^{\alpha}_{(1^n)} \to C_{(1^n),A}.$$

The above proposition will be implied by Proposition 3.4.3, and so we omit the proof.

To define the map majcode, we first require a standardization rule for fillings of columns.

Definition 3.2.6. Let σ be any filling of a column of height n with positive integers. We define the **standardization labeling** on repeated entries as follows.

1. Let i be a letter that occurs k times in σ. Remove any entries larger than i to form a smaller column σ'.

2. Find the bottommost i that is either on the very bottom of σ' or has entries a and b above and below it with $a > b$. Assign this i a label of k and remove it. Repeat this process, labeling the next i by $k - 1$ and so on, until there are no i's left that satisfy this condition.

3. Finally, remove and label any remaining i's in order from top to bottom, decreasing the label by one each time.

We define Standardize(σ) as the unique column filling using labels $1, 2, \ldots, n$ that respects the ordering of the entries of σ and breaks ties according to the standardization labeling.

Proposition 3.2.7. *For any column filling σ with alphabet A, let $\rho = \mathrm{Standardize}(\sigma)$. Then ρ and σ have the same major index, and* majcode(ρ) *is A-weakly increasing.*

The key to the proof is the following technical lemma. Define a *consecutive block* of n's in a filling to be a maximal consecutive run of entries in a column which are all filled with the letter n.

Lemma 3.2.8. *Given a filling of a one-column shape $\mu = (1^r)$ having largest entry n, there is a unique way of ordering the n's in the filling, say $n_1, \ldots, n_{\alpha_n}$, such that the following two conditions are satisfied.*

1. *Any consecutive block of n's in the column appears in the sequence in order from bottom to top, and*

2. *If we remove $n_1, \ldots, n_{\alpha_n}$ in that order, and let d_i be the amount that the major index of the column decreases at the i-th step, then the sequence $d_1, d_2, \ldots, d_{\alpha_n}$ is weakly increasing.*

Proof. We first show (1) that there is a *unique* choice of entry labeled n at each step which minimizes d and is at the bottom of a consecutive block, and then that (2) the resulting sequence d_i is weakly increasing. For any entry x, we define $\psi_x(\sigma)$ to be the column formed by removing the entry x from σ.

To prove (1), consider the bottommost entries of each consecutive block of n's. We wish to show that no two of these n's have the same value of $d = \mathrm{maj}(\sigma) - \mathrm{maj}(\psi_n(\sigma))$ upon removal. So, suppose there is an n in the i-th square from the top and an n in the j-th square from the top, each at the bottom of their blocks, and call them n_i and n_j to distinguish them. Assume for contradiction that removing either of the n's results in a decrease by d of the major index.

Suppose an entry n has an entry a above it and b below. In $\psi_n(\sigma)$, a and b are adjacent, and they can either form a descent or not. If they do, then $d = \mathrm{maj}(\sigma) - \mathrm{maj}(\psi(\sigma))$ is equal to the number of descents below and including that n, and if they do not, then d is equal to the sum of the number of descents *strictly* below the n *plus* the position of the n from the top. We consider several cases based on the two possibilities for each of n_i and n_j.

If either n_i or n_j is at the very bottom of the filling, then removing that entry results in $d = 0$, and the other does not, so we may assume neither of n_i or n_j is in the bottom row.

Case 1: Each of n_i and n_j forms a new descent upon removal, in $\psi_{n_i}(\sigma)$ and $\psi_{n_j}(\sigma)$. Assume without loss of generality that $i < j$, and let t be the number of descents weakly below position j (meaning its position from the top is greater than or equal to j) and s the number of descents weakly below position i. Then since the n_i is at the bottom of its block, it is a descent, so $s > t$. Since s and t are the values of d for the removal of the two n's, we have a contradiction.

Case 2: Neither n_i nor n_j, upon removal, forms a new descent. In this case, assume without loss of generality that $i < j$ and let t be the number of descents *strictly* below position j. Let r be the number of descents strictly between rows i and j. Since the n's are at the bottom of their blocks, the two n's are descents as well, so the values of d upon removing the n's are $i + r + t + 1$ and $j + t$. By our assumption, these are equal, and so we have

$$\begin{aligned} i + r + 1 + t &= j + t \\ j - i - 1 &= r \end{aligned}$$

But $j - i - 1$ counts the number of squares strictly between positions i and j. Since r is the number of squares in this set which are descents, this means that every square between i and j must be a descent. But the square in position j has the highest possible label n, so the square just before it (above it) cannot be a descent. Hence we have a contradiction.

Case 3: One of the two n's, say the one in position i, forms a new descent upon removal, and the other does not. Then in this case defining t as the number of descents *strictly* below position j and s the number of descents *weakly* below position i, the two values of d are $j+t$ and s. So $j + t = s$ by our assumption, and so $j = s - t$, which implies $s - t > 0$, or $s > t$. Thus, necessarily $i < j$.

Now, $s - t$ is the number of descents between positions i and j, inclusive. Since $i \geq 1$ there are at most j such squares, and the one preceding j cannot be a descent since there is an n in the j-th position. Thus this quantity $s - t$ is strictly less than j, but we showed before that $j = s - t$, a contradiction. This completes the proof of claim (1).

For claim (2), consider any two consecutive d values in this process, say d_1 and d_2 for simplicity, that correspond to the largest value n. Let n_1 and n_2 be the corresponding copies of n. We wish to show that $d_1 \leq d_2$.

First, notice that if n_1 and n_2 were in the same consecutive block before removal, we have $d_1 = d_2$ unless n_2 is a block of length 1 in $\psi(\sigma)$, in which case $d_2 \geq d_1$.

So we may assume that n_1 and n_2 were in different consecutive blocks before removal. In this case the removal of n_1 may only change the value of d on removing n_2 by at most one, namely by either shifting it back by one position if n_1 is above n_2 in the column, or by removing one descent from below n_2, if n_1 is below n_2. Thus $d_2 = \mathrm{maj}(\psi_{n_1}(\sigma)) - \mathrm{maj}(\psi_{n_2}(\psi_{n_1}(\sigma)))$ is at most one less than $\mathrm{maj}(\sigma) - \mathrm{maj}(\psi_{n_2}(\sigma))$. Since n_1 was chosen so as to minimize d_1, and we showed in our proof of (1) that the choice is unique, this implies that $d_2 + 1 > d_1$. Thus $d_2 \geq d_1$, as desired.

This completes the proof of (2). □

Proposition 3.2.7 now follows from the proof of the above lemma.

We now can define the map majcode on words, that is, for one-column fillings.

Definition 3.2.9. Let σ be any filling of a column shape $\mu = (1^r)$. We define $\mathrm{majcode}(\sigma) = \mathrm{majcode}(\mathrm{Standardize}(\sigma))$, where majcode of a standard filling is defined to be the majcode of its reading word (which is a permutation).

Example 3.2.10. Let σ be the one-column filling whose reading word is 6434666251664, the standardization labeling on the 6's is shown by the subscripts:

$$6_2\,4\,3\,4\,6_3\,6_4\,6_5\,2\,5\,1\,6_1\,6_6\,4$$

Since this one-column shape has size 13, the filling Standardize(σ) will have the 6's relabeled as the numbers from 8 to 13 according to the subscripts above:

$$\mathbf{9}\,4\,3\,4\,\mathbf{10}\,\mathbf{11}\,\mathbf{12}\,2\,5\,1\,\mathbf{8}\,\mathbf{13}\,4$$

We then remove the $13, 12, \ldots, 8$ in order. This results in a sequence of difference values $1, 3, 3, 3, 5, 7$, which is weakly increasing.

We are left with a column with reading word 4342514, in which there is only one 5, so Standardize changes that to a 7. We remove this to obtain a difference of 1 in the major index. We are left with 434214, in which the 4's are standardized as follows:

$$4_1 3 4_2 2 1 4_3 \to 435216.$$

Removing these in order from 6 down to 1 decreases the major index by $0, 2, 3, 2, 1, 0$, respectively. Therefore,

$$\mathrm{majcode}(\sigma) = 1, 3, 3, 3, 5, 7, 1, 0, 2, 3, 2, 1, 0.$$

Note that this sequence is $\{6, 6, 6, 6, 6, 6, 5, 4, 4, 4, 3, 2, 1\}$-weakly increasing.

Proposition 3.2.11. *The map* majcode *is a weighted set isomorphism* $\mathcal{F}_{(1^n)}^{r(\alpha)} \to C_{(1^n),A}$ *for any alphabet A with content α, and any one-column partition shape* (1^n).

Proof. Carlitz's work shows that majcode is an isomorphism in the case that $\alpha = (1, 1, \ldots, 1)$, i.e. A has one of each letter from 1 to n. In the case of repeated entries, we note that majcode is still injective. Indeed, given a code corresponding to a filling, there is a unique place to insert the next number at each step - by applying the Standardize map, using Carlitz's bijection, and then un-standardizing in the unique way so that the order of entries is preserved and the resulting alphabet is A.

Now, notice that by our definition of majcode and Lemma 3.2.8, the codes we get are A-weakly increasing. We claim that they are also Carlitz codes: at the i-th step, there are $n - i + 1$ letters remaining, and the difference d_i is either the position of the letter we're removing plus the number of descents strictly below it, or the number of descents weakly below it. Therefore, the maximum value of d_i is $n - i + 1$, and so $d_1 d_2 \cdots d_n$ is a Carlitz code and is A-weakly increasing. It follows that majcode is an injective morphism of weighted sets $\mathcal{F}_{(1^n)}^{r(\alpha)}|_{\mathrm{inv}=0} \to C_{(1^n),A}$.

Finally, notice that the two sets have the same cardinality: each has cardinality $\binom{n}{\alpha}$ where α is the content of the alphabet A. It follows that majcode is bijective, as desired. $\square$

3.3 Generalized Carlitz codes

In the context of Hall-Littlewood symmetry, we can think of the Carlitz bijection as a solution to the case in which $\mu = (1^n)$ is a straight shape with one column, filled with distinct entries. Thus, we wish to generalize the notion of a Carlitz code to fillings of arbitrary shapes having inv or maj equal to 0, using arbitrary alphabets.

Our generalization is motivated by the monomial basis of the Garsia-Procesi modules in [13], which are closely connected to the cocharge (maj) statistic. We define a generalized Carlitz code as follows.

Definition 3.3.1. A word having letters in $\{0, 1, 2, \ldots\}$ is ***Yamanouchi*** (also often called *lattice* or *ballot*) if every suffix contains at least as many i's as $i + 1$'s for all $i \geq 0$.

A word w has ***content*** $\alpha = (\alpha_1, \ldots, \alpha_k)$ if exactly α_i of the entries of w are equal to $i - 1$ for each i. We also sometimes say it has content A where A is the multiset of letters of w.

Finally, a word $w = w_1 \cdots w_n$ is μ-***sub-Yamanouchi***, or μ-***Carlitz***, if there exists a Yamanouchi word $v = v_1 \cdots v_n$ of content μ such that $w_i < v_i$ for all i.

Example 3.3.2. The sub-Yamanouchi words for shape $\mu = (1, 1, 1, \ldots, 1)$ are precisely the classical Carlitz codes.

We will see that the μ-sub-Yamanouchi words are the correct analog of Carlitz codes in the case that our Young diagram fillings have distinct entries. However, in general we require the following more precise definition.

Definition 3.3.3. We define $C_{\mu,A}$ to be the collection of all μ-sub-Yamanouchi codes which are A-weakly increasing (see Definition 3.2.1). We call such codes ***generalized Carlitz codes***, and we equip this collection with the statistic $\Sigma : C_{\mu,A} \to \mathbb{Z}$ by $\Sigma(c) = \sum c_i$, forming a weighted set $(C_{\mu,A}; \Sigma)$.

We now introduce the concept of the *monomial* of a code. The next three definitions are compatible with the notation in [13].

Definition 3.3.4. Fix variables $x_1, x_2, \ldots$. For any finite code c of length n, define its ***monomial*** to be

$$x^c = x_n^{c_1} x_{n-1}^{c_2} \cdots x_1^{c_n}.$$

Also let $\mathcal{C}_A(\mu)$ be the set of all monomials x^c of μ-sub-Yamanouchi words c that are A-weakly increasing.

In [13], the authors define similar sets of monomials $\mathcal{B}(\mu)$, which form bases of the modules R_μ that arise naturally in the study of the Hall-Littlewood polynomials. We will see that in the case $A = \{1, 2, \ldots, n\}$, we have $\mathcal{C}_A(\mu) = \mathcal{B}(\mu)$, by showing that the sets $\mathcal{C}_A(\mu)$ satisfy a generalized version of the recursion in [13]. To state this recursion we require two more definitions, which follow the notation in [13].

Definition 3.3.5. Given a partition μ, define $\mu^{(i)}$ to be the partition formed by removing the corner square from the column a_i containing the last square in the i-th row μ_i.

Definition 3.3.6. Given a set of monomials $\mathcal{C}$ and a monomial m, we write $m \cdot \mathcal{C}$ to denote the set of all monomials of the form $m \cdot x$ where $x \in \mathcal{C}$.

The following recursion defines the sets $\mathcal{B}(\mu)$.

Definition 3.3.7. The sets $\mathcal{B}(\mu)$ are defined by $\mathcal{B}((1)) = \{1\}$ and the recursion

$$\mathcal{B}(\mu) = \bigsqcup_{i-1}^{\mu_1^*} x_n^{i-1} \cdot \mathcal{B}(\mu^{(i)}).$$

We refer to these sets as the Garsia-Procesi module bases.

We require one new definition in order to state our general recursion in the next proposition.

Definition 3.3.8. Let $A = \{a_1, a_2, \ldots, a_n\}$ with $a_1 \le a_2 \le \cdots \le a_n$ be a multiset of positive integers, and let λ be a partition of $n-1$. We define $\mathcal{C}_A^{(t)}(\lambda)$ to be the set of all monomials x^d of λ-sub-Yamanouchi words $d_1, \ldots, d_{n-1}$ that are $A \setminus \{a_1\}$-weakly increasing and if $a_1 = a_2$ then $d_1 \ge t$.

Proposition 3.3.9 (General Recursion)**.** *For any partition μ of n and any multiset of positive integers $A = \{a_1, a_2, \ldots, a_n\}$ with $a_1 \le a_2 \le \cdots \le a_n$, we have*

$$\mathcal{C}_A(\mu) = \bigsqcup_{i=1}^{\mu_1^*} x_n^{i-1} \cdot \mathcal{C}_A^{(i-1)}(\mu^{(i)}).$$

Before proving this proposition, we explore some of its basic consequences. Notice that in the case $A = \{1, 2, \ldots, n\}$, since there are no repeated entries, Proposition 3.3.9 reduces to

$$\mathcal{C}_{[n]}(\mu) = \bigsqcup_{i-1}^{\mu_1^*} x_n^{i-1} \cdot \mathcal{C}(\mu^{(i)}).$$

Since this is the same as the recursion given for the sets $\mathcal{B}(\mu)$ described in the previous section, and $\mathcal{C}_{\{1\}}((1)) = \{x_1\} = \mathcal{B}((1))$, we have the following corollary.

Corollary 3.3.10. *If $A = \{1, 2, \ldots, n\}$, we have $\mathcal{C}_A(\mu) = \mathcal{B}(\mu)$.*

As noted in [13], we can now also enumerate the sets $\mathcal{C}_A(\mu)$ in the case $A = \{1, 2, \ldots, n\}$. For, in this case the simplified recursion gives

$$|\mathcal{C}_A(\mu)| = \sum_i |\mathcal{C}_A(\mu^{(i)})|$$

with $|\mathcal{C}_{\{1\}}((1))| = 1$. But the multinomial coefficients $\binom{n}{\mu}$ satisfy $\binom{1}{1} = 1$ and the same recursion:

$$\binom{n}{\mu} = \sum_i \binom{n}{\mu^i}.$$

Corollary 3.3.11. *If $A = \{1, 2, \dots, n\}$, we have*

$$|\mathcal{C}_A(\mu)| = \binom{n}{\mu}.$$

We now prove Proposition 3.3.9.

Proof. The sets forming the union on the right hand side are disjoint because the i-th set consists only of monomials having x_n^{i-1} as their power of x_n. We now show inclusion both ways.

($\subseteq$) Let $x^c \in \mathcal{C}_A(\mu)$ where $c = c_1, \dots, c_n$ is a μ-sub-Yamanouchi word which is A-weakly increasing. Let $i = c_1 + 1$, so that $c_1 = i - 1$. Also let $c' = c_2, \dots, c_n$. Notice that if $a_1 = a_2$ then $c_2 \geq i - 1$, and c' is $A \setminus \{a_1\}$-weakly increasing. Thus, to show $x^c \in x^{i-1}\mathcal{C}_A^{(i-1)}(\mu^{(i)})$, we just need to show that c' is $\mu^{(i)}$-sub-Yamanouchi.

Since c is μ-sub-Yamanouchi, there exists a Yamanouchi word d having μ_i entries equal to $i - 1$ for each i, for which $x^c | x^d$. Let t be the highest index such that $\mu_{t+1} = \mu_i$. Then $\mu^{(i)} = (\mu_1, \mu_2, \dots, \mu_t - 1, \cdots, \mu_k)$. So, we wish to show that we can form a new μ-Yamanouchi word b from d so that we still have $x^c | x^b$ but $b_1 = t$. This way c' will be $\mu^{(i)}$-sub-Yamanouchi, with respect to $b' = b_2, \dots, b_n$.

We have $\mu_{t+2} < \mu_{t+1}$ by our assumption defining t, so there are strictly more t's than $t + 1$'s in d. Notice that this means we can move the *leftmost* t in d any number of spots to the left without changing the fact that the word is Yamanouchi.

Also notice that $d_1 \geq c_1 = i - 1$. But since there are exactly as many $i - 1$'s as i's, $i + 1$'s, and so on up to t in d, we must in fact have $d_1 \geq t$, for otherwise the suffix $d_2, \dots, d_n$ would not satisfy the Yamanouchi property. So $d_1 \geq t$.

Now, let d_r be the leftmost t in d. We form a subword of d as follows. Let d_1 be the first letter of our subword. Then let d_{p_1} be the leftmost letter between d_1 and d_r with $t \leq d_{p_1} \leq d_r$, if it exists. Then let d_{p_2} be the first letter between d_{p_1} and d_r for which $t \leq d_{p_2} \leq d_{p_1}$, and so on until we reach a point at which no such letter exists. We now have a subsequence of letters $d_1, d_{p_1}, d_{p_2}, \dots, d_{p_k}, d_r = t$ where d_r is the leftmost t in d. We define b to be the word formed from d by cyclically shifting this subsequence, replacing d_{p_i} with $d_{p_{i-1}}$ for all $i > 1$, replacing d_{p_1} with d_1, and replacing d_1 with d_{p_k}.

For instance, if $\mu = (4, 3, 3, 2, 2)$, $i - 1 = 1$, then $t = 2$, and we might have

$$c = 120412130010100$$

with

$$d = \mathbf{43}04\mathbf{2}2130021100.$$

Then the subword of d consists of those letters in boldface above, and we cyclically shift the boldface letters to the right in their positions to form

$$b = \mathbf{24}04\mathbf{3}2130021100,$$

which is still μ-Yamanouchi and still dominates c in the sense that $x^c | x^b$.

To verify that in general $x^c | x^b$, notice that $c_1 = i - 1 \le t$, and since the other letters in the subword decrease to the right, we have $b_i \ge d_i$ for all $i > 1$. Thus each $b_i \ge c_i$ for all i, and so $x^c | x^b$.

To show that b is still Yamanouchi, notice that to form b from d, we have moved the leftmost t all the way to the left (which, we noted above, preserves the Yamanouchi property) and moved each d_{p_j} to the right without crossing over any element having value $d_{p_j} - 1$ (for otherwise our sequence d_{p_j} would have an extra element, a contradiction.) Thus we have not changed the property of there being at least as many $d_{p_j} - 1$'s as d_{p_j}'s in each suffix, and we have not changed the property that there are at least as many d_{p_j}'s as $d_{p_j} + 1$'s in each suffix, because we moved these elements to the right. The other Yamanouchi conditions remain unchanged, since we are only moving the letters d_{p_j}. Thus b is Yamanouchi as well.

($\supseteq$) For the other inclusion, let $c = c_1, \ldots, c_n$ be a word such that $x^c \in x^{i-1} \cdot \mathcal{C}_A^{(i)}(\mu^{(i)})$. Then $c' = c_2, \ldots, c_n$ is $\mu^{(i)}$-sub-Yamanouchi, so there exists a word $d' = d_2, \ldots, d_n$ which is Yamanouchi of content $\mu^{(i)}$ such that $x^{c'} | x^{d'}$. Let $d_1 = t$ where t is the highest index such that $\mu_{t+1} = \mu_i$. Then $d = d_1, \ldots, d_n$ is Yamanouchi of shape μ by the definition of $\mu^{(i)}$, and since $c_1 = i - 1$, we have $c_1 \le t = d_1$. Thus $x^c | x^d$. Finally, note that if $a_1 = a_2$ in A, then $c_2 \ge i - 1$ by the definition of $\mathcal{C}^{(i)}$. Thus c is A-weakly increasing. It follows that $x^c \in \mathcal{C}_A(\mu)$. $\square$

3.4 Inversion Codes

We can now generalize the inversion code of a permutation to arbitrary fillings ρ with $\mathrm{maj}(\rho) = 0$.

Definition 3.4.1. Let ρ be a filling of μ^* having $\mathrm{maj}(\rho) = 0$. Order its entries by size with ties broken in reading order to form a totally ordered alphabet $A = \{a_1, \ldots, a_n\}$. Then its ***inversion code***, denoted $\mathrm{invcode}(\rho)$, is the sequence $c_1 \cdots c_n$ whose i-th entry c_i is the number of attacking pairs having a_i as its smaller entry.

Example 3.4.2. Consider the following tableau.

1		
2	2	
2	4	
3	4	2

There are three attacking pairs in this diagram: the 2 in the bottom row is attacked by the 3 and 4 in its row, and the 3 is attacked by the 4 in the second row. When we order the entries in reading order and record the number of larger numbers that attack it, we get the following table.

Entries	1	2	2	2	2	3	4	4
Code	0	0	0	0	2	1	0	0

Therefore, the inversion code of the filling above is 00002100.

Theorem 3.4.3. *The inversion code of any filling $\rho \in \mathcal{F}^{\alpha}_{\mu^*}$ is α-weakly increasing and μ-sub-Yamanouchi. Moreover, the map*

$$\mathrm{invcode} : \mathcal{F}^{\alpha}_{\mu^*}\,|_{\mathrm{maj}=0} \to C_{\mu,A}$$

is an isomorphism of weighted sets.

To prove Theorem 3.4.3, we use the notation of inversion words and diagrams introduced in Section 3.1. We wish to show that the inversion-friendly diagrams are in weight-preserving bijection with generalized Carlitz codes.

Definition 3.4.4. The *inversion code* of a diagram w, denoted $\mathrm{invcode}(w)$, is the sequence $\{c_i\}$ whose i-th entry c_i is the number of inversion pairs of the form $(w(i), b)$.

Example 3.4.5. The inversion code of the following diagram is 00002100.

$$\begin{array}{cccccccc}
 & & & & 1 & & & \\
 & & 3 & & & & 2 & 1 \\
4 & 3 & & 2 & & 1 & & \\
\hline
1 & 2 & 2 & 2 & 2 & 3 & 4 & 4
\end{array}$$

Using Proposition 3.1.7, we can also define the inversion code of a filling $\rho \in \mathcal{F}^{\alpha}_{\mu}|_{\mathrm{maj}=0}$ to be

$$\mathrm{invcode}(\rho) := \mathrm{invcode}(\mathrm{InvPlot}(\rho)).$$

It is easy to see that this matches the definition of inversion code in Section 3.4. We can therefore rephrase Theorem 3.4.3 as follows.

Theorem 3.4.6 (Theorem 3.4.3, rephrased)**.** *The inversion code of any diagram in $\mathrm{ID}_{\mu,A}$ is α-weakly increasing and μ-sub-Yamanouchi. Moreover, the map*

$$\mathrm{invcode} : \mathrm{ID}_{\mu,A} \to C_{\mu,A}$$

is an isomorphism of weighted sets.

We break the proof into several lemmas.

Lemma 3.4.7. *The map* invcode *is a well-defined morphism from* $\mathrm{ID}_{\mu,A} \to C_{\mu,A}$ *for all* μ *and* A.

Proof. Let $w : A \to \mathbb{Z}_+$ be a diagram in $\mathrm{ID}_{\mu,A}$, and let $c = \mathrm{invcode}(w)$.

We first show that c is μ-sub-Yamanouchi. Let $i > 0$ and consider the subset of dots labeled i in the inversion labeling of w, say $w(r_1), \ldots, w(r_t)$ from left to right. We claim that $w(r_{t-j})$ is the left element of at most j inversions for each $j = 0, \ldots, t-1$. Indeed, $w(r_{t-j})$ is to the left of exactly j dots labeled i; those dots in a lower row form the Type I inversions with $w(r_{t-j})$. For Type II, the dots labeled $i+1$ in a higher row must have an i to the right of them, so correspond to one of the dots labeled i in a higher row and to the right of $w(r_{t-j})$. Thus $w(r_{t-j})$ is the left element of at most j inversions, and so $c_{r_{t-j}} \leq j$.

It follows that $c_{r_1}, \ldots, c_{r_t}$ is an ordinary Carlitz code. Therefore, c can be decomposed into several Carlitz codes, one for each label, of lengths $\mu_1^*, \mu_2^*, \ldots$. Let d_i be the resulting upper bound on c_i for each i. Then d is a union of the sequences

$$\mu_i^*, \mu_i^* - 1, \ldots, 2, 1, 0$$

for each i, arranged so that each of these sequences retains its order. Thus d is a Yamanouchi code, since every entry d_i can be matched with a unique entry having value $d_i - 1$ to its right, namely the next entry in the corresponding subsequence. Note also that d is Yamanouchi of shape μ, since there are μ_1 zeroes, μ_2 ones, etc in d. Since c is bounded above component-wise by d, we have that c is μ-sub-Yamanouchi.

We now show that c is A-weakly increasing. It suffices to show that for any two consecutive dots $w(t), w(t+1)$ of w that are in inversion-friendly order, we have $c_t \leq c_{t+1}$. Suppose the dot $w(t)$ is labeled i in the inversion labeling, and $w(t+1)$ is labled j. Then by assumption, since they are in inversion-friendly order, we have either $i = j$ with the j in a higher row than i, or $j < i$. The i is the left element of c_t inversions and the j is the left element of c_{t+1} inversions.

First suppose $i = j$ and the j is in a higher row than the i, that is, $w(t+1) > w(t)$. If b is an index to the right of the i such that $(w(t), w(b))$ is an inversion, then there are three possibilities: First, $w(b)$ could be labeled i and be below $w(t)$, in which case $(w(t+1), w(b))$ is also an inversion. Second, $w(b)$ could be labeled $i+1$ and be above $w(t)$ but below $w(t+1)$, in which case there is a dot labeled i in row $w(b)$ to the right of b, forming an inversion with $w(t+1)$. And third, $w(b)$ could be labeled $i+1$ and be above row $w(t+1)$, in which case $(w(t+1), w(b))$ is also an inversion. Thus there is at least one inversion with $w(t)$ as its left element for every inversion with $w(t+1)$ as the left element, and so $c_t \leq c_{t+1}$ in this case.

Similarly, if $j < i$, then any dot labeled i or $i+1$ has a dot labeled j and a dot labeled $j+1$ to its right, and so $c_t \leq c_{t+1}$ in this case as well.

It follows that invcode is a well-defined map. □

Lemma 3.4.8. *The map* invcode *is injective.*

Proof of Theorem 3.4.3. We will show that given a code c, we can form an inversion-friendly diagram by placing dots above $c_1, c_2, \ldots, c_n$ from left to right. We claim that there is a unique height that is compatible with c at each step.

With the empty word as a trivial base case, we proceed inductively. Suppose we have already placed the first $t-1$ dots from the left. There may be several possible dot heights available for the t-th dot, depending on the shape μ and which dot heights have already been chosen. We claim that each possible height would result in a different value of the code number c_t. To show this, let $h_1 < h_2$ be two possible heights of the t-th dot. Since the first $t-1$ dots have been chosen and we know the shape of the diagram, the labels i and j of a dot at height h_1 or h_2 respectively are uniquely determined. We also note that the inversion code number c_t is uniquely determined by the choice of the t-th dot (given the first $t-1$ dots), since any row of length $\mu_r \geq i$ that did not have a dot labeled i among the first t values must necessarily have one afterwards, and so the set of label values in each row to the right of the t-th entry is determined.

So, let r be the inversion code number c_t that would result from the dot at height h_1 labeled i, and s the code number for h_2 labeled j. We wish to show that $s \neq r$, and we consider the cases $j \leq i$ and $j > i$ separately.

If $j \leq i$, let k be the number of dots labeled i that would be below and to the right of the $w(t)$ if $w(t) = h_1$ (labeled i). Then $r - k$ would be the number of $i+1$'s above and to the right of it. Each of the k rows having the i's also have j's weakly to the right of them because $j \leq i$, and each of the $r-k$ rows with the $i+1$'s have both a $j+1$ and a j to the right. Thus if $w(t) = h_2$ (labeled j) instead, the j would have at least r inversions, and so $s \geq r$. But if $w(t) = h_2$, then this j also forms an inversion with the j in row h_1, giving an extra inversion. Thus $s > r$, and so $s \neq r$ in this case.

If $j > i$, consider the s dots labeled j or $j+1$ that would form an inversion with $w(t)$ if $w(t) = h_2$. Then each of these rows would also contain an i or $i+1$ that would form an inversion with the i at height h_1, in addition to the row h_2 itself, showing that $r > s$. Thus $s \neq r$, as desired. $\square$

We have that $|C_{\mu,A}| = |\mathcal{C}_A(\mu)|$ by our definition of $\mathcal{C}$. Furthermore, when $A = \{1, 2, \ldots, n\}$ we have $|\,\mathrm{ID}_{\mu,A}\,| = \binom{n}{\mu}$ because we are simply counting the number of unrestricted diagrams having μ_1 dots in the first row, μ_2 in the second row, and so on. We can now conclude bijectivity in this case.

Corollary 3.3.11. The map invcode is bijective in the case $A = \{1, 2, \ldots, n\}$.

We are now ready to prove Theorem 3.4.3.

Proof. We already have shown (Corollary 3.3.11) that invcode is a bijective map $\mathrm{ID}_{\mu,[n]} \to C_{\mu,[n]}$. Notice that for any other alphabet $A = \{a_1, \ldots, a_n\}$, we have $\mathrm{ID}_{\mu,A} \subset \mathrm{ID}_{\mu,[n]}$ and $C_{\mu,A} \subset C_{\mu,[n]}$. We also know that the map

$$\mathrm{invcode} : \mathrm{ID}_{\mu,[n]} \to C_{\mu,[n]}$$

restricts to an injective map $\mathrm{invcode} : \mathrm{ID}_{\mu,A} \to C_{\mu,A}$ by Lemmas 3.4.7 and 3.4.8. It remains to show that it is surjective onto $C_{\mu,A}$.

Let $c \in C_{\mu,A} \subset C_{\mu,[n]}$. Then c is A-weakly increasing on constant letters of A. Let $d = \mathrm{invcode}^{-1}(c) \in \mathrm{ID}_{\mu,[n]}$. We wish to show that d is of inversion word type with respect to A, so that $d \in \mathrm{ID}_{\mu,A}$, that is, if $r < s$ and $a_r = a_s$ in A then $(d(a_r), d(a_s))$ is not an inversion. Suppose $(d(a_r), d(a_s))$ is an inversion. Then either $d(a_r)$ and $d(a_s)$ are both dots labeled i with $d(a_s) < d(a_r)$, or $d(a_r)$ is labeled i and $d(a_s)$ labeled $i+1$ with $d(a_s) > d(a_r)$.

In the first case, if $(d(a_s), d(a_t))$ is another inversion involving a_s, then either $d(a_t)$ is lower than $d(a_s)$ (and hence lower than $d(a_r)$) and labeled i, or it is above it and labeled $i+1$. If the former then $(d(a_r), d(a_t))$ is an inversion, and if the latter, either there is an i in the same row forming an inversion with $d(a_r)$, or the $i+1$ is above $d(a_r)$, forming an inversion with it. Thus $d(a_r)$ is the left element of at least as many inversions as $d(a_s)$, plus one for the inversion $(d(a_r), d(a_s))$. Thus $c_r > c_s$.

In the second case, if $(d(a_s), d(a_t))$ is another inversion, then $d(a_t)$ is either lower (but possibly above $d(a_r)$) and labeled $i+1$, or higher and labeled $i+2$. In the former case either $d(a_t)$ itself forms an inversion with $d(a_r)$ or the i in its row does. In the latter case the $i+1$ in its row forms an inversion with $d(a_r)$. Since $(d(a_r), d(a_s))$ is an inversion as well, we again have $c_r > c_s$. But this contradicts the fact that c is A-weakly increasing.

Hence invcode is surjective, and thus bijective, from $\mathrm{ID}_{\mu,A}$ to $C_{\mu,A}$. Clearly the map preserves the statistics: the sum of all the entries of the inversion code of a diagram is the total number of inversions of the diagram, so invcode sends inv to Σ. Therefore,

$$\mathrm{invcode} : \mathrm{ID}_{\mu,A} \to C_{\mu,A}$$

is an isomorphism of weighted sets. $\square$

Chapter 4

Major Index Codes in the Hall-Littlewood Case

In Section 3.4, we found an isomorphism of weighted sets, the map invcode, from $\mathcal{F}^{\alpha}_{\mu^*}|_{\mathrm{maj}=0}; \mathrm{inv})$ to $(C_{\mu,A}; \Sigma)$. To complete the proof of q, t-symmetry in the Hall-Littlewood case, it now suffices to find a weighted set isomorphism

$$\mathrm{majcode} : (\mathcal{F}^{\alpha}_{\mu}|_{\mathrm{inv}=0}; \mathrm{maj}) \to C_{\mu,A}$$

where α is the content of the alphabet A.

Recall the recursion for the μ-sub-Yamanouchi codes of content A from Proposition 3.3.9:

$$\mathcal{C}_A(\mu) = \bigsqcup_{i=1}^{\mu_1^*} x_n^{i-1} \cdot \mathcal{C}_A^{(i-1)}(\mu^{(i)}).$$

Using this recursion, one possible strategy for constructing majcode is by showing combinatorially that $\mathcal{F}^{\alpha}_{\mu}|_{\mathrm{inv}=0}$ satisfies a similar recursion.

In this section, we present some partial progress towards finding the map majcode. All of our work is based on the following four-step approach to the problem.

Step 1. Consider the content (1^n) corresponding to fillings with distinct entries, and find an explicit weighted set isomorphism

$$\psi : (\mathcal{F}^{(1^n)}_{\mu}|_{\mathrm{inv}=0}; \mathrm{maj}) \to \bigsqcup_{d} (\mathcal{F}^{(1^{n-1})}_{\mu^{(d+1)}}|_{\mathrm{inv}=0}; \mathrm{maj} + d).$$

That is, ψ should send an inversion-free filling T of μ to an inversion-free filling $\psi(T)$ of $\mu^{(d+1)}$ for some d, such that

$$\mathrm{maj}(\psi(T)) = \mathrm{maj}(T) - d.$$

Step 2. Define the majcode of a filling T having content (1^n) to be $d_1 d_2 \dots d_n$ where

$$d_k = \mathrm{maj}(\psi^k(T)) - \mathrm{maj}(\psi^{k-1}(T)).$$

Step 3. Check the base case of a single square, and conclude that because the recursion is satisfied, majcode is an isomorphism of weighted sets

$$(\mathcal{F}_\mu^{(1^n)}|_{\mathrm{inv}=0}, \mathrm{maj}) \to (C_{\mu,[n]}, \Sigma),$$

where $C_{\mu,[n]}$ are the generalized Carlitz codes of shape μ and content $[n]$.

Step 4. Show that there is a standardization map

$$\text{Standardize} : \mathcal{F}_\mu^{\alpha}|_{\mathrm{inv}=0} \to \mathcal{F}_\mu^{1^n}|_{\mathrm{inv}=0}$$

that respects maj, such that the composition majcode $\circ$ Standardize is a bijection to $C_{\mu,A}$ where A is the alphabet with content α. That is, show that after standardizing, we get a major index code which is A-weakly increasing, and none of these codes are mapped to twice.

4.1 Killpatrick's Method for Standard Fillings

For Step 1 in our strategy, in which $A = \{1, 2, \dots, n\}$ is an alphabet with no repeated letters, such a map can easily be extracted from the work of Killpatrick [26]. In this paper, the author gives a combinatorial proof of a recursion for a generating function involving the charge statistic ch (recall that $\mathrm{ch}(\mu) = n(\mu) - \mathrm{cc}(\mu)$ where $n(\mu) = \sum_i (i-1) \cdot \mu_i$.) Killpatrick defines W_μ to be the set of words of content μ, and lets $r_{i,\mu} = |\{j > i : \mu_j = \mu_i\}|$. The recursion is stated as:

$$\sum_{w \in W_\mu} q^{\mathrm{ch}(w)} = \sum_i q^{r_{i,\mu}} \sum_{w \in W_{\mu^{(i)}}} q^{\mathrm{ch}(w)}.$$

If we substitute $q \to 1/q$ and multiply both sides by $q^{n(\mu)}$, this becomes

$$\sum_{w \in W_\mu} q^{\mathrm{cc}(w)} = \sum_i q^{i-1} \sum_{w \in W_{\mu^{(i)}}} q^{\mathrm{cc}(w)},$$

which is equivalent to the recursion we stated in step 1 above. Killpatrick's work, when translated into the language of fillings via the cocharge word construction, gives a map ψ for step 1. We now give a translated description of the map ψ.

Definition 4.1.1. The ***crank*** of a filling σ having $\mathrm{inv}(\sigma) = 0$ and alphabet $\{1, \dots, n\}$ is the filling formed by (a) decreasing each entry $i \geq 2$ by 1 and replacing the entry 1 with n, and (b) rearranging the entries within each row in the unique way such that $\mathrm{inv} = 0$, as in Proposition 2.5.17.

Definition 4.1.2. A ***crank orbit*** is the set of all fillings obtained by repeatedly applying the crank to a filling.

Note that the crank orbits have sizes $|\mu|/d$ where d is some common divisor of the parts of μ. Furthermore, the crank orbits partition the fillings of μ into disjoint subsets. (See Figure 4.1.)

Let T be a filling with $\mathrm{inv}(T) = 0$. We now give a five-step algorithm (A-E below) for computing $\psi(T)$, and throughout we use the example

$$T = \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}.$$

A. List the crank orbits and mark the "special" fillings. List the entries of each crank orbit for μ in order by starting with a filling and repeatedly applying the crank. We define a ***special filling*** of such an orbit to be a filling for which the largest entry $n = |\mu|$ occurs in the bottom row of the tableau.

B. Assign difference values plus one to the special fillings. For each special filling $\sigma^\star$, define

$$\mathrm{diff}(\sigma^\star) = \mathrm{maj}(\mathrm{crank}^{-1}(\sigma^\star))) - \mathrm{maj}(\sigma^\star)$$

Assign to each special filling the number

$$\ell(\sigma^\star) = \mathrm{diff}(\sigma^\star) + 1.$$

The work in [26] shows that the values $\ell(\sigma^\star)$, ranging over all special fillings in a given orbit of size $|\mu|/d$, will be the sizes of the column lengths of the partition μ/d formed by dividing each of the rows of μ by d.

C. Assign labelings. Starting with any special filling $\sigma^\star$ in a given orbit, label it z_1 and proceed to label each entry in the orbit according to the following algorithm. Continue labeling entries by z_1 in order until we have either labeled $\ell(\sigma_1^\star)$ of them, or until we encounter another special filling $\sigma_2^\star$. In either case, change our label to z_2 and start labeling entries with z_2 starting from $\sigma_2^\star$ in the same manner. If we finish labeling $\ell(\sigma_2^\star)$ fillings with a_2 and have not finished labeling with z_1's, return to z_1 until it is finished or we reach the next star, which we label z_3, and so on. (See Figure 4.1.) It turns out that we will end up with the same partition of the orbit given by the labels z_i no matter which special filling that we start at. [26]

D. Sort the labeled entries into columns. For the crank orbit of T, sort all of the entries labeled z_1, all those labeled z_2, and so on each into their own column with the special entry at the bottom and the rest above it in the order they appear in the orbit. This forms a set of columns which, if arranged in decreasing order of height, forms the partition shape μ/d.

	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&4\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 4&1\\\hline 2&3\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 3&4\\\hline 1&2\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&3\\\hline 1&4\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 3&1\\\hline 2&4\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&4\\\hline 1&3\\\hline\end{array}$
special?	$\star$			$\star$	$\star$	
maj	0	1	2	1	1	2
$\ell(\sigma^\star)$	2			2	2	
label	z_1	z_1	z_2	z_2	z_1	z_1

Figure 4.1: The crank orbits for $\mu = (2,2)$, along with their special entries, maj and ℓ values, and labels.

For example, for the first orbit in Figure 4.1, we have two columns:

$$\begin{array}{cc} \begin{array}{|c|c|}\hline 4&1\\\hline 2&3\\\hline\end{array} & \begin{array}{|c|c|}\hline 3&4\\\hline 1&2\\\hline\end{array} \\ \\ \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} & \begin{array}{|c|c|}\hline 2&3\\\hline 1&4\\\hline\end{array} \end{array}$$

E. Bumping from the bottom. Find the location of T in one of the columns produced in Step 4. Let i be the number of fillings below T in its column. Let $T^\star$ be the special filling in the bottom row of the column of T. Let cell $c = (\mathrm{row}, \mathrm{col})$ be the corner of $T^\star$ that is removed to form shape $\mu^{(i)}$, and let $T^\star_\rightarrow$ be the filling formed by deleting columns $1, \ldots, \mathrm{col}-1$ of $T^\star$. So, in the running example, T is in the upper right corner of the diagram from Step 4, and so $c = 2$, $i = 1$, $T^\star = \begin{array}{|c|c|}\hline 2&3\\\hline 1&4\\\hline\end{array}$ and $T^\star_\rightarrow = \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}$.

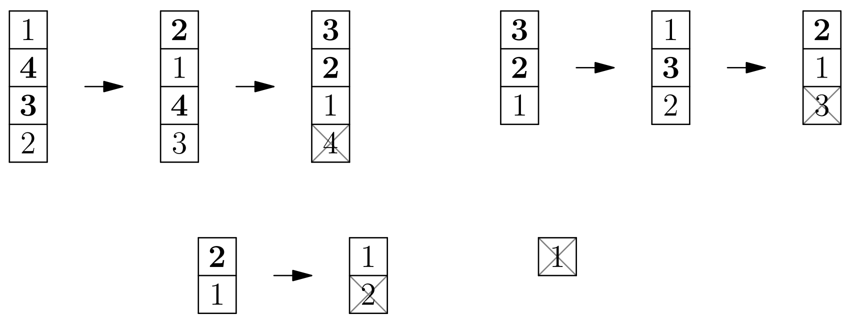
Define the ***bumping sequence*** to be the sequence of entries $b_{\mathrm{row}}, b_{\mathrm{row}-1}, \ldots, b_1$ where b_{row} is the entry in square c, and for all j, b_j is the entry in row j of $T^\star_\rightarrow$ which is the largest entry less than b_{j+1} (or the very largest entry, if b_{j+1} is less than all entries in row j). It turns out that we always have $b_1 = n$.

Finally, remove $b_1 = n$ and bump down $b_1, \ldots, b_{\mathrm{row}}$ one row each. Re-order the rows so that there are no inversions, and the resulting tableau is $\psi(T)$. In the running example we have $\psi(T) = \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\\hline 1&3\\\hline\end{array}$. Killpatrick's proof shows that $\mathrm{maj}(\psi(T)) = \mathrm{maj}(T) - i$. It follows that ψ gives rise to a map majcode that completes the proof of symmetry in this case.

Theorem 4.1.3. *In the case $\alpha = (1^n)$ of fillings with distinct entries, we have that $\varphi =$* $\mathrm{majcode}^{-1} \circ \mathrm{invcode}$ *is an isomorphism of weighted sets*

$$\varphi : \mathcal{F}_\mu^{(1^n)}|_{\mathrm{maj}=0} \to \mathcal{F}_{\mu^*}^{(1^n)}|_{\mathrm{inv}=0}.$$

However, Killpatrick's map majcode$'$ does not satisfy the requirements of Step 4. To illustrate this, we consider the case in which $\mu = (1^n)$ is a straight column shape. In this case, Killpatrick's bijection majcode$'$ can be rephrased as follows. Given a filling w of a straight column shape such as the one with reading word 1432 in the diagram below, check to see if the bottommost entry is n. If not, cyclically increase each entry by 1 modulo n. (Here we're using the inverse of the crank for convenience.) Continue applying the inverse crank until the bottommost entry is n, and let c_1 be the number of times we applied the inverse crank. (In the figure, $c_1 = 2$.)



Once the bottommost entry is n, we remove the bottom box, and repeat step 1 on the new tableau. The resulting number of inverse cranks used is recorded as c_2. (In the figure, $c_2 = 2$.) We continue until there are no boxes left, and set $\mathrm{majcode}'(w) = c_1 c_2 \cdots c_n$. (In the figure, $\mathrm{majcode}'(w) = 2210$.)

Now, suppose we had a standardization map Standardize as in Step 4. Consider the one-column tableaux having entries from the alphabet $\{2, 2, 1, 1, 1, 1\}$ and major index 4. There are three such tableaux:

1	1	**2**
1	1	1
1	2	**2**
2	**2**	1
1	1	1
2	1	1

There are also three (1^6)-sub-Yamanouchi codes that are $\{2, 2, 1, 1, 1, 1\}$-weakly increasing and sum to 4, namely:

$$040000$$
$$130000$$
$$220000$$

It follows that Standardize maps these three tableaux to the three standardized fillings whose codes majcode′ are 040000, 130000, and 220000, respectively. But these three tableaux are:

$$\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline \mathbf{5}\\\hline 1\\\hline 6\\\hline\end{array}\qquad\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline \mathbf{6}\\\hline 1\\\hline 5\\\hline\end{array}\qquad\begin{array}{|c|}\hline 2\\\hline 3\\\hline 5\\\hline \mathbf{6}\\\hline 1\\\hline 4\\\hline\end{array}$$

Therefore, the map Standardize cannot preserve the relative ordering of the entries, or even the positions of the descents. This makes it unlikely that a simple rule for such a standardization map exists. However, it is possible that there exists a more complicated combinatorial rule for such a map, and we leave this as an open question for future investigation.

Problem 4.1.4. *Construct a natural map* Standardize *that satisfies the conditions of Step* 4 *for Killpatrick's map* majcode$'$.

Since standardization is not immediate, we now return to Carlitz's bijection and generalize majcode to arbitrary inversion-free fillings of certain infinite families of shapes.

4.2 Cocharge Contribution and Structural Results

We now introduce some new definitions and technical lemmata which will be used throughout the proofs in this chapter.

Definition 4.2.1. The ***cocharge contribution*** $\mathrm{cc}_{(i,j)}(\sigma)$ of an entry $\sigma(i,j)$ of a filling σ is the number of descents that occur weakly below the entry (i,j) in its column, j.

It is easy to see that the cocharge contributions add up to the major index.

Proposition 4.2.2. *Let $\sigma : \mu \to \mathbb{Z}_+$ be any filling. Then* $\mathrm{maj}(\sigma)$ *is equal to the sum of the cocharge contributions of the entries of σ, i.e.*

$$\mathrm{maj}(\sigma) = \sum_{(i,j)\in\mu} \mathrm{cc}_{(i,j)}(\sigma).$$

We omit the proof, and refer the reader to the example in Figure 4.2.

Definition 4.2.3. Let w be any sequence consisting of k 0's and k 1's, and let $a_1, a_2, \ldots, a_k$ be any ordering of the 0's. We define the ***crossing number*** of w with respect to this ordering as follows. Starting with a_1, let b_1 be the first 1 to the right of a_1 in the sequence, possibly wrapping around cyclically if there are no 1's to the right of a_1. Then let b_2 be the first 1 cyclically to the right of a_2 other than b_1, and so on. Then the *crossing number* is the number of indices i for which b_i is to the left of a_i.

$$\begin{array}{|c|c|c|}\hline 3^2 & \mathbf{6}^2 & 7^1 \\ \hline \mathbf{10}^2 & \mathbf{5}^1 & \mathbf{11}^1 \\ \hline \mathbf{9}^1 & 1^0 & 2^0 \\ \hline 4^0 & 8^0 & 12^0 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 3^2 & \mathbf{6}^2 & 7^1 \\ \hline \mathbf{10}^2 & \mathbf{5}^1 & \mathbf{11}^1 \\ \hline \mathbf{9}^1 & 1^0 & 2^0 \\ \hline 4^0 & 8^0 \\ \cline{1-2} \end{array} \longrightarrow \begin{array}{|c|c|c|}\cline{1-2} 3^2 & \mathbf{6}^2 \\ \hline \mathbf{10}^2 & \mathbf{5}^1 & 7^1 \\ \hline \mathbf{9}^1 & 1^0 & \mathbf{11}^1 \\ \hline 4^0 & 8^0 & 2^0 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|}\cline{1-2} 3^2 & \mathbf{6}^2 \\ \hline \mathbf{10}^2 & 5^1 & \mathbf{7}^1 \\ \hline \mathbf{9}^1 & \mathbf{11}^1 & 1^0 \\ \hline 2^0 & 4^0 & 8^0 \\ \hline \end{array}$$

Figure 4.2: The cocharge contribution of the entries in each tableau is shown as a superscript. Notice that the sum of the cocharge contributions of a tableau is equal to its major index. In addition, the three-step process of Proposition 4.3.1 does not change the major index.

Example 4.2.4. If we order the 0's from left to right, the word 10110010 has crossing number 2.

Lemma 4.2.5. *Let w be any sequence consisting of k 0's and k 1's. Then its crossing number is independent of the choice of ordering of the 0's.*

Proof. Say that a word is 0-***dominated*** if every prefix has at least as many 0's as 1's. First, we note that there exists a cyclic shift of w which is 0-dominated. Indeed, consider the partial sums of the $(-1)^{w_i}$'s in the sequence, so that any 0 contributes $+1$ and any 1 contributes -1. The total sum is 0, and we can shift to start at the index of the minimal partial sum; the partial sums will now all be positive.

Now, we show by induction that any 0-dominated sequence has crossing number $m = 0$. It is clearly true for $k = 1$, since the only 0-dominated sequence is 01 in this case.

Suppose the claim holds for any 0-dominated sequence of $k-1$ 0's and $k-1$ 1's and let s be an 0-dominated sequence with k 0's. Choose an arbitrary 0 to be a_1, and denote it $\hat{0}$. Then since s is 0-dominated, the last term in s is a 1 and so $\hat{0}$ will be paired with a 1, denoted $\hat{1}$, to the right of it. Remove both $\hat{0}$ and $\hat{1}$ from s to form a sequence s' having $k-1$ 0's and $k-1$ 1's.

We claim that s' is 0-dominated. Note that all prefixes of s' that end to the left of $\hat{0}$ are unchanged, and hence still have at least as many 0's as 1's. Any prefix P' that ends between $\hat{0}$ and $\hat{1}$ is the result of removing $\hat{0}$ from a corresponding prefix P of s, which had at least as many 0's as 1's. If there were an equal number of 0's as 1's in P, then its last term is a 1. This means that $\hat{1}$ was not the first 1 to the right of the 0, a contradiction. So P has strictly more 0's than 1's, and so $P' = P \setminus \{\hat{0}\}$ has at least as many 0's as 1's. Finally, any prefix which ends to the right of $\hat{1}$ has one less 0 and one less 1 than the corresponding initial subsequence of s, and so it also has at least as many 0's as 1's. It follows that s' is 0-dominated.

By the inductive hypothesis, no matter how we order the remaining 0's, there are no crossing pairs. Since the choice of a_1 was arbitrary, the crossing number is 0 for any ordering of the 0's.

Returning to the main proof, let $w = w_1 w_2 \cdots w_{2k}$ and let i be such that the cyclic shift $w' = w_i w_{i+1} \cdots w_{2k} w_1 w_2 \cdots w_{i-1}$ is 0-dominated. Then every pairing in w' has the 0 to the left of the 1, and so the crossing number of w is the number of pairings in which the 0 is among $w_i \cdots w_{2k}$ and the 1 is among $w_1 \cdots w_{i-1}$. Hence, the crossing number is equal to the difference between the number of 1's and 0's among $w_1 w_2 \cdots w_{i-1}$. This is independent of the choice of order of the 0's, and the proof is complete. □

In the rest of the paper, if a row r is above a row s in a filling, we say that we ***rearrange*** r ***with respect to*** s if we place the entries of r in the unique ordering for which there are no inversions in row r, given that s is below it.

Lemma 4.2.6. *Let σ be a filling of the two-row shape (k, k) with $\mathrm{inv}(\sigma) = 0$. Let σ'_π be formed by rearranging the bottom row via the permutation π, and rearranging the top row with respect to the new bottom row. Then $\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma'_\pi)$.*

Proof. Let w be cocharge word of the diagram. No matter what the permutation of rows, the cocharge word will remain unchanged, a sequence of k 1's and k 2's. But the permutation of the bottom row determines a permutation of the 1's, and the subsequent ordering of the top row is determined by the process of selecting the first remaining 2 cyclically to the right of the 1 at each step. It forms a descent if and only if that 2 is to the left of the 1, i.e. if it contributes to the crossing number. So the number of descents is equal to the crossing number of the cocharge word (thinking of the 1's as 0's and the 2's as 1's), and by Lemma 4.2.5 the proof is complete. □

We now have the tools to prove the next technical lemma.

Lemma 4.2.7. *Let $a_1, \ldots, a_{w-1}$ be any positive integers, and suppose $b_1, \ldots, b_w$ are positive integers such that the partial tableau*

$$\begin{array}{ccccc} b_1 & b_2 & \cdots & b_{w-1} & b_w \\ a_1 & a_2 & \cdots & a_{w-1} & \end{array}$$

has no inversions among the b_i's. Then if we rearrange $a_1, \ldots, a_{w-1}$ in any way and then rearrange the b's in the unique way that guarantees no inversions among the b's, then the entry b_w is still in the last position. Furthermore, the total number of descents among $b_1, \ldots, b_w$ is unchanged after this operation.

Proof. Consider the cyclic ordering of $a_1, \ldots, a_{w-1}, b_1, \ldots, b_w$. Since there are no inversions among the b's, we have that a_i, b_i, b_w are in cyclic order for each i, possibly with $b_i = b_w$ or $a_i = b_w$.

Let $b_w, t_1, \ldots, t_{2w-2}$ be the ordering of these letters that is in cyclic order, with ties broken in such a way that b_w, a_i, b_i occur in that order in the sequence for each i. Then if we replace the a_i's with 0's and the b_i's with 1's, the suffix $t_1, \ldots, t_{2w-2}$ has crossing number 0 since each a_i is paired with b_i to its right.

It follows from Lemma 4.2.5 that, if we rearrange the a_i's, the crossing number is still 0 and so b_w still corresponds to the 1 at the beginning of the sequence. It follows that b_w is still in the last position in the new filling. Finally, by considering only the first $w-1$ columns, we can apply Lemma 4.2.6 to see that the total number of descents among $b_1, \ldots, b_{w-1}$ remains unchanged. □

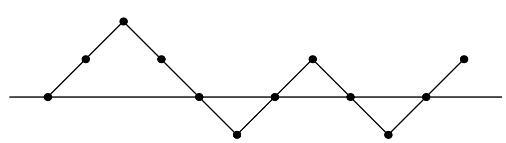
The following lemma is a sort of inverse to Lemma 4.2.7.

Lemma 4.2.8. *Given two collections of letters $b_1, \ldots, b_{w-1}$ and $a_1, \ldots, a_w$, there is a unique element a_i among $a_1, \ldots, a_w$ such that, in any two-row tableau with $a_1, \ldots, \hat{a}_i, \ldots, a_w$ as the entries in the bottom row and $b_1, \ldots, b_{w-1}, a_i$ as the entries in the top, with no inversions in the top row, the entry a_i occurs in the rightmost position in the top row.*

Proof. As usual, let us think of the a_i's as 0's and the b_i's as 1's in a cocharge word, arranged according to the magnitudes of the a_i's and b_i's. Then we have a sequence of w 0's and $w-1$ 1's, and we wish to show that there is a unique 0 that, when we change it to a 1, is not paired with any 0 when computing the crossing number. By Lemma 4.2.7, there is a unique such 1 in any word of $w-1$ 0's and w 1's.

So, by Lemma 4.2.7, it suffices to find a 0 in the original tableau such that upon removal, the remaining sequence starting with the entry to its right is 0-dominated. For instance, in the sequence 001110100, which has 5 zeros and 4 ones, if we remove the second-to-last zero and cyclically shift the letters so that the new sequence starts with the 0 to its right, we get the sequence 00011101, which is 0-dominated.

To show that there is a unique such 0, consider the up-down walk starting at 0 in which we move up one step for each 0 in the sequence and down one step for each 1. Then we end at height 1, since there is one more 0 than 1 in the sequence. For instance, the sequence 001110100 corresponds to the up-down walk:



Consider the last visit to the minimum height of this walk. If the minimum height is 0 then we simply remove the last 0 in the sequence and we are done. If the minimum height is less than 0, then there are at least two up-steps (0's) following it since it is the last visit to the min. The first of these up-steps corresponds to a 0 which we claim is our desired entry. Indeed, if we remove this 0, the walk starting at the next step and cycling around the end of the word is a positive walk, corresponding to a 0-dominated sequence.

It is easy to see that if we do the same with any of the other 0-steps, the resulting walk will not be positive and so the corresponding sequence will not be 0-dominated. This completes the proof. □

Notice that in a two-row shape with the bottom row ordered least to greatest and no inversions in the second row, the descents must be "left-justified": they must occur in columns $1, \ldots, k$ for some k. For, if $b_r > a_r$ is a descent and $b_{r-1} \le a_{r-1}$ is not, then $b_r > a_{r-1}$ by transitivity and we have $b_{r-1} \le a_{r-1} < b_r$, forming an inversion. Moreover, after the descents the b_i's are weakly increasing: $b_i \le b_j$ for $k < i < j$ - this follows directly from the fact that none of these b_i's are descents. The descents $b_1, \ldots, b_k$ are also weakly increasing; otherwise we would have an inversion. We will use this fact throughout.

Lemma 4.2.9. *Let $a_1 \le \cdots \le a_{w-1}$ and let $b_1, b_2, \ldots, b_w$ be numbers such that the partial tableau*

$$\begin{array}{ccccc} b_1 & b_2 & \cdots & b_{w-1} & b_w \\ a_1 & a_2 & \cdots & a_{w-1} & \end{array}$$

has no inversions in the second row. Then if we bump b_w down one row so that

$$a_1 \le a_2 \le \cdots \le a_t \le b_w < a_{t+1} \le \cdots \le a_{w-1}$$

is the bottom row, and leave $b_1, \ldots, b_{w-1}$ unchanged, then the new tableau still has no inversions, and the descents in the second row remain the same (and left-justified).

Proof. Let k be the number of descents among the b's. If $k = 0$, there are no descents, and we must have $b_w \le a_1$ so as not to have inversions. In this case, b_w drops down into the first position in the bottom row, and there are still no descents and no inversions since $b_1 \le b_2 \le \ldots \le b_{w-1}$ in this case.

If $k \ge 1$, then $b_k > a_k$ is the last descent. Since b_k and b_w do not form an inversion in the original tableau, we must either have $a_k < b_k \le b_w$ or $b_w \le a_k < b_k$. We consider these cases separately.

Case 1: Suppose $a_k < b_k \le b_w$. Then $t > k$, i.e. b_w drops to a position to the right of the last descent, after which point we have $b_i \le a_i$ for all such i. Thus, for instance, $b_{t+1} < a_{t+1}$, and since b_w and b_{t+1} did not originally form a descent, we must have $b_{t+1} \le b_w \le a_{t+1}$. This means that $b_{t+1} \le b_w$, so b_{t+1} still does not form a descent in the new tableau. Then, similarly we have $b_{t+2} \le b_w$, and so $b_{t+2} \le a_{t+1}$, and so on. Thus the descents have stayed the same in the new tableau.

Furthermore, since $b_i < b_w$ for all $i \ge t+1$ in this case, we have $b_i < b_w < a_i$ for all $i \ge t+1$, and since the b_i's after position k are weakly increasing, none of these form inversions. Since $b_1, \ldots, b_t$ are above the same letters $a_1, \ldots, a_t$ as before and are in the same positions relative to the other b_i's, they cannot be the left elements of inversions either.

Case 2: Suppose now that $b_w \le a_k < b_k$. If $b_w = a_k$ then in fact it drops to the right of a_k and it is the same as the previous case. So we can assume that $b_w < a_k < b_k$.

Then $t \le k$, i.e. b_w drops to a position underneath a descent of the original tableau shape. Since $b_w \le a_{t+1}$ and $a_{t+1} < b_{t+1}$ is a descent, we have $b_w < b_{t+1}$ and so b_{t+1} is still a descent in the new tableau. Similarly b_i is still a descent for all $i \le k$. To check that b_{k+1} is still *not* a descent, assume it is: that $a_k < b_{k+1}$. Then $b_w \le a_k < b_{k+1}$, and so $b_w \le a_{k+1} \le b_{k+1}$ since the original filling had no inversions. If $a_{k+1} < b_{k+1}$, we get a contradiction, so $a_{k+1} = b_{k+1}$. But then $b_w = a_{k+1}$, contradicting the fact that $b_w < a_{k+1}$. Thus there is not an inversion in the $(k+1)$st position. Hence the descents stay the same in this case as well.

Furthermore, consider b_i and b_j with $i < j < w$: if i is among $1, \ldots, t$ then b_i and b_j do not form an inversion since b_i is still above a_i. If i and j are both among $t+1, \ldots, k$, then they do not form an inversion, since b_i and b_j are both descents and $b_i < b_j$. If i is among $t+1, \ldots, k$ and $j > k$, note that $b_j < b_w$ since it is in the run of non-descents of the b's. Hence $b_j < a_i$ by transitivity, and so $b_j < a_i < b_i$ since b_i is a descent. This implies that b_i and b_j do not form an inversion. Finally, if $i > k$ and $j > i$, we are once again in the run of non-descents at the end, which is weakly increasing, and hence there are no inversions since none are descents. We conclude that the b_i's have no inversions among them in this case either. $\square$

Lemma 4.2.10. *Let $a_1, \ldots, a_{w-1}$, $b_1, \ldots, b_w$, and c_w be numbers such that the partial filling*

$$\begin{array}{ccccc} & & & & c_w \\ b_1 & b_2 & \cdots & b_{w-1} & b_w \\ a_1 & a_2 & \cdots & a_{w-1} & \end{array}$$

has no inversions in the second row. Then there exists an ordering $t_1, \ldots, t_w$ of $a_1, \ldots, a_{w-1}, b_w$ such that if $s_1, \ldots, s_w$ is the unique ordering of $b_1, \ldots, b_{w-1}, c_w$ for which the partial filling

$$\begin{array}{cccc} s_1 & s_2 & \cdots & s_w \\ t_1 & t_2 & \cdots & t_w \end{array}$$

has no inversions in the second row, then the entry c_w is directly above b_w in the new filling.

Proof. Let T' be the two-row filling consisting of the s's and t's as in the statement of the lemma. Let x be the cocharge word of T', with the bottom row indexed by 0 and the top by 1. Then x consists of 0's and 1's, and as in Lemma 4.2.5, the number of descents in T' is the crossing number of this word. So b_w is one of the 0's in this word, and c_w is one of the 1's, and we wish to show that there is some ordering of the 0's in which b_w is paired with c_w.

Assume to the contrary that b_w cannot be paired with c_w no matter how we order the 0's. Choose a cyclic shift $\widetilde{x}$ of x whose crossing number is 0, as we did in Lemma 4.2.5. If b_w is to the left of c_w in $\widetilde{x}$, then since it can't be paired with c_w there must be an index k between that of b_w and c_w at which the prefix of the first k letters is 0-dominated. For, if there were more 0's than 1's at every step up to c_w then we can pair off the other 0's starting from the left until c_w is the first 1 to the right of b_w. This means we can choose a different cyclic ordering, starting at the $k+1$st letter, for which the crossing number is also 0. In this cyclic shift, c_w is to the left of b_w. So we have reduced to the case that c_w is to the left of b_w.

In this case, c_w is one of the 1's, and b_w is one of the 0's, e.g. in the 0-dominated sequence 001011, we might have c_w be the third entry and b_w the fourth. Before we dropped down the b_w and c_w, we had a tableau whose cocharge word looked like this word except with the 0 of b_w replaced by a 1, and the 1 of c_w replaced by a 2 (in the example, this would give us the word 002111.) Remove the 2 from this word. In the resulting word of 0's and 1's, since we have bumped up a 0 to a 1 but removed one of the 1's before it, every prefix is 0-dominated except the entire word, which has one more 1 than it has 0's. Thus the very last 1 is the only entry which is not paired. But b_w is, by assumption, the entry which is unpaired in the original ordering. This is a contradiction, since b_w was a 0 in the bumped-down word and hence could not have been in the last position.

It follows that there must exist an ordering of the 0's in which b_w is paired with c_w. This completes the proof. $\square$

4.3 Bumping from the Bottom

The Carlitz bijection on words, defined in Section 3.2, gives a map majcode for arbitrary fillings of one-column shapes μ. We now present a strategy towards generalizing to all shapes μ, and show that rectangles behave similarly to one-column shapes.

Our primary tool is the following technical result. This proposition generalizes the fact that if we remove the largest entry n from the bottom of a one-column shape, we get a major index code entry $d = 0$.

Proposition 4.3.1. *Suppose $\sigma : \mu \to \mathbb{Z}_+$ is a filling for which $\mathrm{inv}(\sigma) = 0$ and the largest entry n appears in the bottom row. Let $\sigma_\downarrow : \mu^{(1)} \to \mathbb{Z}_+$ be the filling obtained by:*

1. *Removing the rightmost n from the bottom row of σ, which must be in the rightmost column since $\mathrm{inv}(\sigma) = 0$,*
2. *Shifting each of the remaining entries in the rightmost column down one row,*
3. *Rearranging the entries in each row in the unique way so that $\mathrm{inv}(\sigma_\downarrow) = 0$.*

Then the major index does not change:

$$\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma_\downarrow).$$

To prove this, we require two further lemmas. In all of what follows, we let $\sigma : \mu \to \mathbb{Z}_+$ be a filling with $\mathrm{inv}(\sigma) = 0$ whose largest entry appears in the bottom row, and let $\sigma_\downarrow : \mu^{(1)} \to \mathbb{Z}_+$ be constructed from σ as above.

Lemma 4.3.2. *Suppose $\mathrm{inv}(\sigma) = 0$. Let $i \geq 1$ be an index such that $\mu_{i+1} = \mu_1$, i.e. the $(i+1)$st row of μ is as long as the bottom row. Then we have*

$$\mathrm{cc}_{(i+1,\mu_1)}(\sigma) + \sum_{1 \leq j \leq \mu_1 - 1} \mathrm{cc}_{(i,j)}(\sigma) = \sum_{1 \leq j \leq \mu_1} \mathrm{cc}_{(i,j)}(\sigma_\downarrow).$$

Proof. We induct on i. For the base case, $i = 1$, the left hand side is the total cocharge contribution of the entries $(1,1), (1,2), \ldots, (1, \mu_1 - 1)$ and the entry $(2, \mu_1)$. The square $(1, \mu_1)$ is filled with the largest number n, by our assumption that n appears in the bottom row and the fact that $\mathrm{inv}(\sigma) = 0$. Thus the entry in $(2, \mu_1)$ cannot be a descent, and so the cocharge contribution of all of these entries are 0. Thus the left hand side is 0. The right hand side is also 0, since it is the sum of the cocharge contributions from the bottom row of $\sigma_\downarrow$.

For the induction, let $i > 1$ and suppose the claim is true for $i-1$. Then the induction hypothesis states that

$$s := \mathrm{cc}_{(i,\mu_1)}(\sigma) + \sum_{1\le j\le \mu_1-1} \mathrm{cc}_{(i-1,j)}(\sigma) = \sum_{1\le j\le\mu_1} \mathrm{cc}_{(i-1,j)}(\sigma_\downarrow).$$

Then if there are k descents among the entries $(i+1,\mu_1)$ and $(i,1),\ldots,(i,\mu_1-1)$ of σ, then their total cocharge contribution is equal to $s+k$, since they are the entries strictly above those that contribute to the left hand side of the equation above.

So, to show that

$$\mathrm{cc}_{(i+1,\mu_1)}(\sigma) + \sum_{1\le j\le \mu_1-1} \mathrm{cc}_{(i,j)}(\sigma) = \sum_{1\le j\le\mu_1} \mathrm{cc}_{(i,j)}(\sigma_\downarrow),$$

it suffices to show that the total cocharge contribution of the i-th row of $\sigma_\downarrow$ is also $s+k$. By the induction hypothesis it is equivalent to show that there are k descents among the entries in the i-th row of $\sigma_\downarrow$.

Now, let $w = \mu_1$ be the width of the tableau, and let $a_1,\ldots,a_{w-1}$ be the first $w-1$ entries in row $i-1$ of σ. Let $b_1,\ldots,b_w$ be the elements of row i, and let c_w be the entry in square $(i+1,w)$, above b_w.

$$\begin{array}{ccccc} & & & & c_w \\ b_1 & b_2 & \cdots & b_{w-1} & b_w \\ a_1 & a_2 & \cdots & a_{w-1} & \end{array}$$

Consider the $2\times w$ tableau T' with bottom row elements $a_1,\ldots,a_{w-1},b_w$ and top row elements $b_1,\ldots,b_{w-1},c_w$. By Lemma 4.2.10, there is a way of rearranging the bottom row of T' such that if we rearrange the top row respectively, then c_w lies above b_w. This suffices, for now the remaining columns will form a tableau with no inversions in the second row, with $a_1,\ldots,a_{w-1}$ and $b_1,\ldots,b_{w-1}$ as the entries of the rows. By Lemma 4.2.6 this has the same number of descents independent of the ordering of the a_i's, and c_w will be a descent or not depending on whether it was a descent before. Thus there are still k descents in the i-th row. □

Lemma 4.3.2 shows that the cocharge contribution is conserved for rows i for which $\mu_{i+1} = \mu_1$. The next lemma will show that the cocharge contribution is unchanged for higher rows as well. Again, here σ is a filling having its largest entry n occurring in the bottom row.

Lemma 4.3.3. *Suppose* $\mathrm{inv}(\sigma) = 0$, *and the rightmost (wth) column of μ has height $\mu^*_w = h$. Then in $\sigma_\downarrow$, row h consists of the first $w-1$ letters of row h of σ in the same order, and their cocharge contributions are the same as they were in σ.*

It follows from this lemma that all higher rows are unchanged as well, and combining this with Lemma 4.3.2, it will follow that $\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma_\downarrow)$.

Proof. We induct on h, the height of the rightmost column. For $h = 1$ and $h = 2$, we are done by previous lemmata (see Lemma 4.2.9). So, suppose $h \geq 3$ and the claim holds for all smaller h.

Performing the operation of Proposition 4.3.1, suppose we have bumped down all but the topmost entry (in row h) of the rightmost column and rearranged each row with respect to the previous. Let rows $h-2$, $h-1$, and h have contents:

$$\begin{array}{ccccc} d_1 & d_2 & \cdots & d_{w-1} & d_w \\ c_1 & c_2 & \cdots & c_{w-1} & \\ x_1 & x_2 & \cdots & x_{w-1} & x_w \end{array}$$

Notice that, by the induction hypothesis, the entries $c_1, \ldots, c_{w-1}$ are the same as they were in σ before bumping down c_w and have the same cocharge contributions as they did before. Thus the row of d's as shown is currently the same as row h of σ. So, we wish to show that upon bumping d_w down and rearranging all rows so that the filling has no inversions, the entries in row h are still $d_1, d_2, \ldots, d_{w-1}$ in that order, and that these entries have the same cocharge contributions as they did before.

We first show that the entries $d_1, \ldots, d_{w-1}$ do not change their positions upon bumping d_w down to row $h-1$ (and rearranging so that there are still no inversions.) We proceed by strong induction on the width w. For the base case, $w = 2$, we have that d_1 is the only entry left in the top row, and therefore cannot change its position.

Now, assume that the claim is true for all widths less than w. If d_w bumps down and inserts in a row t above x_t, then the numbers $c_1, \ldots, c_{t-1}$ are still above $x_1, \ldots, x_{t-1}$ respectively since they are still first in cyclic order after each. Likewise the entries $d_1, \ldots, d_{t-1}$ remain the same in this case. Thus we may delete the first $t-1$ columns and reduce to a smaller case, in which the claim holds by the induction hypothesis. This allows us to assume that when d_w bumps down, it is in the first column, above x_1, and so the tableau looks like:

$$\begin{array}{ccccc} d_* & d_* & \cdots & d_* & \\ d_w & c_* & \cdots & c_* & c_* \\ x_1 & x_2 & \cdots & x_{w-1} & x_w \end{array}$$

where the $*$'s are an appropriate permutation of the indices for $d_1, \ldots, d_{w-1}$ and $c_1, \ldots, c_{w-1}$.

We now show that $d_1, \ldots, d_r$ remain in their respective positions for all $r \geq 1$, by induction on r. (So, we are doing a triple induction on the height, the width of the tableau, and the index of the d's). For the base case, we wish to show that d_1 is the entry above d_w in the new tableau. We have, from the fact that $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma_\downarrow) = 0$, that the following triples are in cyclic order for any k such that $2 < k < w$:

1. (x_1, d_w, c_1), with possible equalities $x_1 = c_1$, $d_w = c_1$

2. (x_1, c_1, c_k), with possible equalities $x_1 = c_k$, $c_1 = c_k$
3. (c_k, d_k, d_w), with possible equalities $c_k = d_w$, $d_k = d_w$
4. (c_1, d_1, d_k), with possible equalities $c_1 = d_k$, $d_1 = d_k$

Combining (1) and (2) above, we have that (x_1, d_w, c_1, c_k) are in cyclic order, and so in particular (d_w, c_1, c_k) is in cyclic order. Combining this with (3) above, we have (d_k, d_w, c_1, c_k) are in cyclic order, and in particular so are (d_k, d_w, c_1). Using this and (4), we have (d_k, d_w, c_1, d_1) are in cyclic order, and in particular either $c_1 \neq d_1$ or $c_1 = d_1 = d_k$, and so (d_w, d_1, d_k) are in cyclic order with either $d_w \neq d_1$ or $d_w = d_k = d_1$; this implies that d_1 and d_k will not form an inversion if d_1 is placed above the d_w. Thus d_1 does indeed stay in the leftmost column.

For the induction step, suppose $d_1, \ldots, d_{r-1}$ are in columns $1, \ldots, r-1$ respectively in $\sigma_\downarrow$. We wish to show that d_r must be in the rth position. To do so, first notice that since c_i is first in cyclic order after x_i among $c_i, c_{i+1}, \ldots, c_{w-1}$ for each i, we have that for each i, the element that appears above x_i after bumping d_w down is among $c_1, \ldots, c_i$.

Suppose c_k is above x_k in the new tableau for some $k \leq r-1$. Then d_k is in this column as well by the induction hypothesis, and so removing this entire column will not affect the relative ordering of the remaining entries. But now d_r is the $(r-1)$st of the d's in question, and therefore must be in the $(r-1)$st position by the induction hypothesis, and so must be in the rth position in the full tableau (prior to removing the kth column).

Otherwise, if c_k is never above x_k for any $k \leq r-1$, we have that c_1 must appear above x_2, since it can only be c_1 or c_2 but is not c_2 by assumption. Then, c_2 must appear above x_3, and so on, up to c_{r-2} appearing above x_{r-1}. If c_r appears above x_r, then d_r must be above that since we knew from the previous tableau that it is first in cyclic order after c_r among $d_r, \ldots, d_{w-1}$. So the only case that remains is where c_{r-1} appears in column r, above x_r. The diagram is as follows:

$$\begin{array}{ccccccc} d_1 & d_2 & d_3 & \cdots & d_{r-1} & d_* & \cdots \\ d_w & c_1 & c_2 & \cdots & c_{r-2} & c_{r-1} & \cdots \\ x_1 & x_2 & x_3 & \cdots & x_{r-1} & x_r & \cdots \end{array}$$

We wish to show that the entry d_* above is d_r. First, we claim that $(d_w, c_1, c_2, \ldots, c_r)$ are in cyclic order. For, we have (x_1, c_1, c_2) and (x_1, d_w, c_1) are in cyclic order, so (x_1, d_w, c_1, c_2) are. Since (x_2, c_1, c_2) and (x_2, c_2, c_3) are in cyclic order, we have that (x_2, c_1, c_2, c_3) are in cyclic order. Since (x_1, c_1, c_3) are in cyclic order as well, we can combine this with the last two observations to deduce that

$$(x_1, d_w, c_1, c_2, c_3)$$

are in cyclic order. Now, we can use the triples (x_3, c_3, c_4), (x_3, c_2, c_3), and (x_2, c_2, c_4) to deduce that $(x_2, c_1, c_2, c_3, c_4)$ are in cyclic order as well. But since (x_1, c_1, c_4) are in cyclic order, this means that

$$(x_1, d_w, c_1, c_2, c_3, c_4)$$

are in cyclic order as well, and so on. At each step, to add c_k to the list we only need consider rows up to that of x_{k-1}. Hence, the process continues up to $k = r$.

Finally, notice that since we are only concerned with relative cyclic order of the entries to determine their positions, we may cyclically increase all the entries modulo the highest entry in such a way that $d_w \le c_1 \le c_2 \le \cdots \le c_r$ in actual size. Furthermore, since we are currently only concerned with the position of d_r, which is determined by its relative ordering with d_i for $i > r$ and with c_{r-1}, we may assume that $c_r \le c_{r+1} \le c_{r+2} \le \ldots \le c_n$ are increasing as well; it will make no difference as to the value of d_*. But then the top two rows behave exactly as in the two-row case of Lemma 4.2.9. We know that d_r occurs in the rth column from this lemma, and the induction is complete.

We have shown that $d_1, \ldots, d_{w-1}$ retain their ordering, and it remains to show that they retain their cocharge contributions. If any c_k lies above x_k, and hence d_k above it, the column has not changed and so d_k does indeed retain its cocharge contribution. So, as before, we may remove such columns and reduce to the case in which the entries are:

$$\begin{array}{cccccc} d_1 & d_2 & d_3 & \cdots & d_{w-1} & \\ d_w & c_1 & c_2 & \cdots & c_{w-2} & c_{w-1} \\ x_1 & x_2 & x_3 & \cdots & x_{w-1} & x_w \end{array}$$

For the first column, we have that (x_1, d_w, c_1) are in cyclic order since d_w and c_1 do not form an inversion. Moreover, either $x_1 \ne d_w$ or $x_1 = d_w = c_1$, in which case we may assume that d_w is in fact located in the second column instead, and reduce to a smaller case. So we may assume $x_1 \ne d_w$. In addition, (c_1, d_1, d_w) are in cyclic order, with $c_1 \ne d_1$ unless $c_1 = d_1 = d_w$, and if $d_1 = d_w$ then we must have $d_1 = d_2 = \cdots = d_w$ so that d_1 does not form an inversion with any element in the new tableau. We now consider three cases based on the actual ordering of x_1, d_w, c_1 (which are in cyclic order):

Case 1: Suppose $x_1 < d_w \le c_1$. Then since (c_1, d_1, d_w) are in cyclic order, either d_1 is greater than both c_1 and d_w or less than or equal to both. Since both c_1 and d_w are descents when over x_1, the cocharge contribution of d_1 is unchanged in this case.

Case 2: Suppose $d_w \le c_1 \le x_1$. Then in this case neither c_1 nor d_w is a descent when in the first column, and the same analysis as in Case 1 shows that d_1 has the same cocharge contribution in either case.

Case 3: Suppose $c_1 \le x_1 < d_w$. Then $c_1 \le d_1 \le d_w$. If d_1 is strictly greater than c_1, it forms a descent with c_1 and not with d_w. But note that d_w is a descent when in the first column, and c_1 is not, so the total number of descents weakly beneath d_1 balances out and is equal in either case. If $d_1 = c_1$, then $d_1 = d_2 = \cdots = d_w$, which is impossible since then $c_1 = d_w$. So the cocharge contribution of d_1 is the same in this case as well.

This completes the proof that d_1 retains the same cocharge contribution. We now show the same holds for an arbitrary column i.

In the i-th column, we have d_i above c_{i-1} above x_i. Note that (c_i, d_i, d_w) and (d_w, c_{i-1}, c_i) are in cyclic order (the latter by the above argument which showed that $d_w, c_1, c_2, \ldots, c_{w-1}$ are in cyclic order given that the c_i's are arranged as above), so (c_i, d_i, d_w, c_{i-1}) are in cyclic order. In particular (c_i, d_i, c_{i-1}) are in cyclic order. Moreover, if $c_i = d_i$ then $d_i = d_w = c_i$. Since d_w, c_{i-1}, c_i are in cyclic order we must have $c_i = d_i = c_{i-1}$ in this situation.

We also have that (x_i, c_{i-1}, c_i) are in cyclic order, and by a similar argument as above we can assume $x_i \neq c_{i-1}$. So either $x_i < c_{i-1} \leq c_i$, $c_i \leq x_i < c_{i-1}$, or $c_{i-1} \leq c_i \leq x_i$. The exact same casework as above for these three possibilities then shows that d_i retains its cocharge contribution. $\square$

Proposition 4.3.1 now follows immediately from Lemmas 4.3.2 and 4.3.3 and Proposition 4.2.2.

Lemma 4.2.8 allows us to recover σ from a tableau $\sigma_\downarrow$ whose second-longest row μ_k is one square shorter than its longest rows (μ_1 through μ_{k-1}). We simply raise the appropriate entry a_i from row μ_{k-1} to row μ_k, then do the same from row μ_{k-2} to μ_{k-1}, and so on, and finally insert a number n in the bottom row, where n is larger than all of the other entries in $\sigma_\downarrow$.

Example 4.3.4. Applying the process $\sigma \to \sigma_\downarrow$ in the tableau below, the major indexes of the starting tableau and the ending tableau are both 10.

$$
\begin{array}{|c|c|c|}
\hline 3 & \mathbf{6} & 7 \\
\hline \mathbf{10} & \mathbf{5} & \mathbf{11} \\
\hline \mathbf{9} & 1 & 2 \\
\hline 4 & 8 & 12 \\
\hline
\end{array}
\rightarrow
\begin{array}{|c|c|c|}
\hline 3 & \mathbf{6} & 7 \\
\hline \mathbf{10} & \mathbf{5} & \mathbf{11} \\
\hline \mathbf{9} & 1 & 2 \\
\hline 4 & 8 & \multicolumn{1}{c}{} \\
\cline{1-2}
\end{array}
\rightarrow
\begin{array}{|c|c|c|}
\cline{1-2} 3 & \mathbf{6} & \multicolumn{1}{c}{} \\
\hline \mathbf{10} & \mathbf{5} & 7 \\
\hline \mathbf{9} & 1 & \mathbf{11} \\
\hline 4 & 8 & 2 \\
\hline
\end{array}
\rightarrow
\begin{array}{|c|c|c|}
\cline{1-2} 3 & \mathbf{6} & \multicolumn{1}{c}{} \\
\hline \mathbf{10} & 5 & \mathbf{7} \\
\hline \mathbf{9} & \mathbf{11} & 1 \\
\hline 2 & 4 & 8 \\
\hline
\end{array}
$$

4.4 Reducing Rectangles to Columns

Using Proposition 4.3.1, we can provide a new combinatorial proof of the recurrence of Garsia and Procesi for all rectangular shapes $\mu = (a, a, a, \ldots, a)$. This also will provide the first letter of majcode for rectangular shapes.

Theorem 4.4.1. *Let $A = \{1, 2, \ldots, n\}$ be the alphabet with content $\alpha = (1^n)$, and let $\mu = (a, a, a, \ldots, a)$ be a rectangle shape of size n. Then there is a weighted set isomorphism*

$$\psi : (\mathcal{F}_\mu^{(1^n)}|_{\mathrm{inv}=0}; \mathrm{maj}) \to \bigsqcup_{d=0}^{\mu_1^*-1} (\mathcal{F}_{\mu^{(d+1)}}^{(1^{n-1})}|_{\mathrm{inv}=0}; \mathrm{maj} + d)$$

defined combinatorially by the following process.

1. *Given a filling $\sigma : \mu \to \mathbb{Z}_+$ with distinct entries $1, \ldots, n$ and $\mathrm{inv}(\sigma) = 0$, let i be the row containing the entry n. Split the filling just beneath row i to get two fillings σ_{top} and σ_{bot} where σ_{bot} consists of rows $1, \ldots, i-1$ of σ and σ_{top} consists of rows i and above.*

2. *Rearrange the entries of the rows of* σ_{top} *in the unique way that forms a filling* $\widetilde{\sigma_{top}}$ *for which* $\mathrm{inv}(\widetilde{\sigma_{top}}) = 0$.

3. *Apply the procedure of Proposition 4.3.1 to* $\widetilde{\sigma_{top}}$*, that is, removing the* n *from the bottom row and bumping each entry in the last column down one row. Let the resulting tableau be called* τ.

4. *Place* τ *on top of* σ_{bot} *and rearrange all rows to form a tableau* ρ *having* $\mathrm{inv}(\rho) = 0$. *Then we define* $\psi(\sigma) = \rho$.

Moreover, if $\mathrm{maj}(\sigma) - \mathrm{maj}(\psi(\sigma)) = d$*, then* $0 \le d < \mu_1^*$ *and we assign* $\psi(\sigma)$ *to the dth set in the disjoint union.*

Remark 4.4.2. Theorem 4.4.1 gives a new combinatorial proof of the recursion

$$\sum_{\substack{\sigma:\mu\to\mathbb{Z}_+\\ \mathrm{inv}(\sigma)=0}} q^{\mathrm{maj}(\sigma)} = \sum_{d} q^{d-1} \sum_{\substack{\rho:\mu^{(d)}\to\mathbb{Z}_+\\ \mathrm{inv}(\rho)=0}} q^{\mathrm{maj}(\rho)}$$

of Garsia and Procesi for rectangular shapes μ.

The map ψ of Theorem 4.4.1 is illustrated by the example below.

$$\begin{array}{|c|c|c|}\hline 6&8&9\\\hline 3&4&11\\\hline 15&1&5\\\hline 13&14&2\\\hline 7&10&12\\\hline\end{array}
\;\longrightarrow\;
\begin{array}{c}\begin{array}{|c|c|c|}\hline 6&8&9\\\hline 3&11&4\\\hline 1&5&\cancel{15}\\\hline\end{array}\\ \begin{array}{|c|c|c|}\hline 13&14&2\\\hline 7&10&12\\\hline\end{array}\end{array}
\;\longrightarrow\;
\begin{array}{c}\begin{array}{|c|c|}\hline 6&8\\\hline 3&11\\\hline 1&5\\\hline\end{array}\;\begin{array}{|c|}\hline 9\\\hline 4\\\hline\end{array}\\ \begin{array}{|c|c|c|}\hline 13&14&2\\\hline 7&10&12\\\hline\end{array}\end{array}
\;\longrightarrow\;
\begin{array}{|c|c|c|}\hline 6&8&\\\hline 3&9&11\\\hline 1&4&5\\\hline 13&14&2\\\hline 7&10&12\\\hline\end{array}$$

We now prove Theorem 4.4.1.

Proof of Theorem 4.4.1. It is clear that ψ is a morphism of weighted sets, preserving the statistics, so we only need to show that ψ is a bijection. To do so, we construct an inverse map $\phi = \psi^{-1}$ that takes a pair (ρ, d) and returns an appropriate filling $\sigma : \mu \to \mathbb{Z}_+$, where $\rho : \mu^{(d-1)} \to \mathbb{Z}_+$ is a filling with no inversions using the letters $1, \ldots, n-1$, and d is a number with $0 \le d \le \mu_1^* - 1$. For simplicity let $h = \mu_1^*$ be the height of μ.

Let (ρ, d) be such a pair. Consider the fillings $\sigma_1, \sigma_2, \ldots, \sigma_h$ formed as follows. Let σ_h be the tableau obtained by inserting the number n into the top row of ρ and rearranging the entries of the top row so that $\mathrm{inv}(\sigma_h) = 0$. Let σ_{h-1} be the tableau formed from ρ by first moving the unique element of the $(h-1)$st row given by Lemma 4.2.8 to the top row, and then inserting n into the $(h-1)$st row and rearranging all rows so that there are no inversions again. Then, let σ_{h-1} be formed from ρ by first moving the same element, call it a_{h-1}, up to the top row, then using Lemma 4.2.8 again to move an element a_{h-2} from row $h-2$ to row $h-1$, and finally inserting n in row $h-2$ and rearranging the rows again so that there are no inversions. Continuing in this manner, we define each of $\sigma_1, \ldots, \sigma_h$ likewise, and it is easy to see that $\psi(\sigma_i) = \rho$ for all i, by using Lemma 4.2.7 repeatedly.

Now, we wish to show that the numbers $d_i = \mathrm{maj}(\sigma_i) - \mathrm{maj}(\rho)$ for $i = 1, \ldots, h$ form a permutation of $0, \ldots, h-1$. Let $a_1, \ldots, a_{h-1}$ be the elements of rows $1, \ldots, h-1$ that were moved up by 1 in each of the steps as described above. By Proposition 4.3.1, the filling σ_1, whose rightmost column has entries $a_{h-1}, a_{h-2}, \ldots, a_1, n$ from top to bottom, has the same major index as ρ. So $d_1 = 0$, and $\mathrm{maj}(\sigma_1) = \mathrm{maj}(\rho)$. We will now compare all other σ_i's to σ_1 rather than to ρ.

We claim that the difference in the major index from σ_1 to σ_i is the same as the difference obtained when moving n up to row i (and shifting all lower entries down by one) in the one-column filling having reading word $a_{h-1}, a_{h-2}, \ldots, a_1, n$. Then, by Carlitz's original bijection, we will be done, since each possible height gives a distinct difference value d between 0 and $h-1$.

To proceed, consider the total number of descents in each row. In σ_i, the entry n is in the i-th row. Let τ consist of the top $h-i$ rows of this filling, arranged so that $\mathrm{inv}(\tau) = 0$. Then the top $h-i-1$ rows (row 2 to $h-i$ of τ) are the same as in σ_1, with the same descents. Thus if we rearrange *every* row with respect to the one beneath, including rows $i-1$ and below to form σ_i, each row also has the same number of descents as it does in σ_1 by Lemma 4.2.6.

We now show the same is true for row $i+1$. In τ, we have a_i above n, and the remaining entries in that row are above the same set of entries they were in σ_1. So the number of descents in row $i+1$ goes down by 1 from σ_1 to σ_i if $a_i > a_{i-1}$, and otherwise it remains the same.

For rows i and below, we use Lemma 4.2.10. For any row t from 2 to i, the entries of row $t-1$ can be rearranged so that if row t is arranged on top of it with no inversions, the entry a_t lies in the space above a_{t-1} (or n lies above a_{i-1} in the case $t = i$.) The remaining entries in the top row of this two-row arrangement are then above the same set of entries they were in σ_1, with no inversions between them, and by Lemma 4.2.6 they have the same number of descents among them. So, the descents have only changed by what the comparison of each a_t with a_{t-1} (or n with a_{i-1}) contributes.

Therefore, the number of descents in a given row of σ_i, relative to σ_1, can either increase by 1, stay the same, or decrease by 1, according to whether it does in the one-column shape filled by $a_h, \ldots, a_1, n$ when we move n up to height i.

Now, for rectangular shapes, if p_t is the total number of descents in row t, it is easy to see that the total cocharge contribution (major index) of the filling is the sum of the partial sums

$$p_1 + (p_1 + p_2) + (p_1 + p_2 + p_3) + \cdots + (p_1 + \cdots + p_h).$$

Since the values of p_t in σ_i differ by 0 or ± 1 from the corresponding values of σ_1, it follows that the difference d_i is the sum of the partial sums of these differences. But this is the same as the difference in the one-column case we are comparing to. This completes the proof. $\square$

We now state some important corollaries that follow from Theorem 4.4.1.

Corollary 4.4.3. *Let $h = \mu_1^*$ be the height of the rectangle shape μ. Since μ is a rectangle, the shape $\mu^{(d+1)}$ is independent of $d \in \{0, \ldots, h-1\}$, so let $\mu_\downarrow$ be this shape. Let $\rho \in \mathcal{F}_{\mu_\downarrow}^{(1^{n-1})}$, so that there is a copy ρ_d of ρ in $\mathcal{F}_{\mu^{(d+1)}}^{(1^{n-1})}|_{\mathrm{inv}=0}$ for all $d = 0, \ldots, h-1$. Let $\sigma_d = \psi^{-1}(\rho_d)$ for each d. Then for each $i = 1, \ldots, h$, the largest entry n occurs in the i-th row in exactly one of $\sigma_0, \ldots, \sigma_{h-1}$.*

The next theorem, which also follows directly from the proof of Theorem 4.4.1, suggests that the standardization map for rectangle shapes can be inherited from the standardization map for single-column shapes described above.

Theorem 4.4.4 (Reducing rectangles to columns)**.** *For $\sigma \in \mathcal{F}_\mu^{(1^n)}|_{\mathrm{inv}=0}$ with μ a rectangle, the value of $d = \mathrm{maj}(\sigma) - \mathrm{maj}(\psi(\sigma))$ can be determined as follows. Let σ_1 be the unique element of $\mathcal{F}_\mu^{(1^n)}|_{\mathrm{inv}=0}$ for which n is in the bottom row and $\psi(\sigma_1) = \psi(\sigma)$, so that $\sigma_{1\downarrow} = \psi(\sigma_1) = \psi(\sigma)$. Let $a_{h-1}, \ldots, a_1, n$ be the entries of the rightmost column of σ_1 from top to bottom. Then d is the same as the difference in the major index obtained from inserting n into the i-th position in the one-column shape with reading word $a_{h-1}, \ldots, a_1$.*

This theorem is so crucial to the proofs of the results in the next section that it is helpful to give the sequence of a_i's its own name. We call it the *bumping sequence* of σ.

Definition 4.4.5. Let σ be a filling of a rectangle shape μ having height h, with distinct entries $1, 2, \ldots, n$. The ***bumping sequence*** of σ is the collection of entries $a_1, a_2, \ldots, a_{h-1}$ defined as in Theorem 4.4.4 above. If n is in the i-th row of σ, then $a_1, \ldots, a_{i-1}$ are in rows 1 through $i-1$ respectively, and $a_i, \ldots, a_{h-1}$ are in rows $i+1$ through h.

We can also say something about the position of these a_i's given the position of the largest entry.

Proposition 4.4.6. *Let μ be a rectangle shape of height h, and let $\sigma \in \mathcal{F}_\mu^{(1^n)}$ with its largest entry n in row i. Then if $a_1, \ldots, a_{h-1}$ is the bumping sequence of σ, then $a_{i+2}, \ldots, a_{h-1}$ all occur in columns weakly to the right of the n, and each a_j is weakly to the right of a_{j-1} for $j \geq i+3$.*

Proof. Let $c_1, \ldots, c_r, n, c_{r+1}, \ldots, c_{m-1}$ be the entries in row i from left to right. Consider the reordering of row i given by $c_1, \ldots, c_{m-1}, n$ and order row $i+1$ with respect to this ordering. Let the numbers in the new ordering in row $i+1$ be $b_1, \ldots, b_{m-1}, a_i$. Then a_i is the same as the value of a_i from Theorem 4.4.4 by Lemma 4.2.7; that is, a_i would lie above n if we ordered $c_1, \ldots, c_{m-1}$ by size as well.

Now, since $c_1, \ldots, c_r$ are the first r entries in both orderings of row i, it follows that $b_1, \ldots, b_r$ must be the first r entries in both corresponding orderings of row $i+1$. Thus a_i, not being equal to any of $b_1, \ldots, b_r$, must be weakly to the right of the column that n is in.

The same argument can be used to show that a_{i+1} is weakly to the right of a_i as well, and so on. This completes the proof. $\square$

4.5 Three Row Shapes and Fat Hooks

We now provide a complete bijection majcode in the case that $\mu = (\mu_1, \mu_2, \mu_3)$ is a partition with at most three rows.

We start with the definition of majcode for two-row shapes, which we will use as part of the algorithm for three rows.

Lemma 4.5.1. *Let $\mu = (\mu_1, \mu_2)$ be any two-row shape of size n. Then there is a weighted set isomorphism*

$$\psi : (\mathcal{F}_{\mu}^{(1^n)}|_{\mathrm{inv}=0}; \mathrm{maj}) \to \bigsqcup_{d=0}^{1} (\mathcal{F}_{\mu^{(d+1)}}^{(1^{n-1})}|_{\mathrm{inv}=0}; \mathrm{maj} + d)$$

defined combinatorially by the following process. Given an element σ of $\mathcal{F}_{\mu}^{(1^n)}|_{\mathrm{inv}=0}$, that is, a filling of the two-row shape μ having no inversions, consider its largest entry n.

1. *If the n is in the bottom row, define $\psi(\sigma) = \sigma_{\downarrow}$ as in Proposition 4.3.1.*
2. *If the n is in the second row, remove it and re-order the remaining entries in the top row so that there are no inversions. Let $\psi(\sigma)$ be the resulting filling.*

Proof. We first show that ψ is a morphism of weighted sets. If the n we remove is in the bottom row, then by Proposition 4.3.1, the new filling $\sigma_{\downarrow} = \psi(\sigma)$ is in $\mathcal{F}_{\mu^{(1)}}^{(1^{n-1})}|_{\mathrm{inv}=0}$ and has the same major index as σ. This means that $\sigma_{\downarrow}$ is in the $d = 0$ component of the disjoint union

$$\bigsqcup_{d=0}^{1} (\mathcal{F}_{\mu^{(d+1)}}^{(1^{n-1})}|_{\mathrm{inv}=0}; \mathrm{maj} + d),$$

and the statistic is preserved in this case.

Otherwise, if the n is in the second (top) row, then $\sigma' = \psi(\sigma)$ is in $\mathcal{F}_{\mu^{(2)}}^{(1^{n-1})}|_{\mathrm{inv}=0}$. We wish to show that the difference in major index, $d = \mathrm{maj}(\sigma) - \mathrm{maj}(\sigma')$, is 1 in this case. Indeed, notice that the bottom row remains unchanged after removing the n, and so the difference in major index will be the same as if we ignore the extra $\mu_1 - \mu_2$ numbers at the end of the bottom row and consider just the rectangle that includes the second row instead. By Theorem 4.4.1, it follows that $d = 1$. Therefore ψ is a morphism of weighted sets.

To show that ψ is bijective, we construct an inverse map ϕ. First, let $\sigma' \in \mathcal{F}_{\mu^{(1)}}^{(1^{n-1})}|_{\mathrm{inv}=0}$. Then we can insert n into the bottom row, and if μ is a rectangle also bump up one of the entries of the bottom row according to Lemma 4.2.8. This creates a filling σ of shape μ having the same major index as σ'. We define $\phi(\sigma') = \sigma$, which defines an inverse map for ψ on the restriction of ψ to $\psi^{-1}\left(\mathcal{F}_{\mu^{(1)}}^{(1^{n-1})}|_{\mathrm{inv}=0}\right)$.

Now let σ' be a filling of shape $\mu^{(2)}$. The shape $\mu^{(2)}$ has a longer first row than second row, so we can insert n into the second row and rearrange the row entries to obtain an inversion-free filling σ of shape μ and content α. We define $\phi(\sigma') = \sigma$, and by Theorem 4.4.1

applied to the two-row rectangle inside μ of width equal to the top row of μ, the major index increases by 1 from σ' to σ. Thus ϕ is an inverse to ψ on $\mathcal{F}_{\mu^{(1)}}^{(1^{n-1})}|_{\text{inv}=0}$, and ψ is bijective. $\square$

We now complete the entire bijection for two rows by defining a standardization map for two-row fillings.

Definition 4.5.2. For a two-row shape $\mu = (\mu_1, \mu_2)$, we define the map

$$\text{Standardize} : \mathcal{F}_\mu^\alpha|_{\text{inv}=0} \to \mathcal{F}_\mu^{1^n}|_{\text{inv}=0}$$

as follows. Given a filling $\sigma \in \mathcal{F}_\mu^\alpha|_{\text{inv}=0}$, define Standardize($\sigma$) to be the filling of μ with content (1^n) that respects the ordering of the entries of σ by size, with ties broken by reading order.

Example 4.5.3. The standardization map for two rows is illustrated below.

$$\begin{array}{|c|c|c|c|c|}\hline 3 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 3 & 3 & 3 \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|}\hline 5 & 1 & 2 & 6 & 7 \\ \hline 3 & 4 & 8 & 9 & 10 \\ \hline \end{array}$$

We can now define majcode for two-row shapes.

Definition 4.5.4. Let $\mu = (\mu_1, \mu_2)$ be a two-row shape of size n. Given a filling σ of μ, let $\overline{\sigma} = \text{Standardize}(\sigma)$. Then we define $\text{majcode}(\sigma) = d_1 d_2 \cdots d_n$ where

$$d_i = \text{maj}(\psi^{i-1}(\overline{\sigma})) - \text{maj}(\psi^i(\overline{\sigma})),$$

and where ψ is the map defined in Lemma 4.5.1.

Remark 4.5.5. Notice that, given a filling σ of μ having arbitrary content, we have

$$\text{majcode}(\sigma) = \text{majcode}(\text{Standardize}(\sigma)).$$

Theorem 4.5.6. *The map* majcode *defined on two-row shapes* $\mu = (\mu_1, \mu_2)$ *is an isomorphism of weighted sets*

$$\mathcal{F}_\mu^\alpha|_{\text{inv}=0} \to C_{\mu,A}$$

for each alphabet A *and corresponding content* α*.*

Proof. Putting together the recursions of Lemma 4.5.1 and Lemma 3.3.9, we have that for the content (1^n) corresponding to alphabet $[n]$, the map majcode is a weighted set isomorphism

$$\mathcal{F}_\mu^{(1^n)}|_{\text{inv}=0} \to C_{\mu,[n]}.$$

Now, let A be any alphabet with content α. Let σ be a filling of μ with content α. Then we know $\text{majcode}(\sigma) = \text{majcode}(\text{Standardize}(\sigma))$, so $\text{majcode}(\sigma) \in C_{\mu,[n]}$. In other words, majcode(σ) is μ-sub-Yamanouchi. In addition, since Standardize is an injective map (there is clearly only one way to un-standardize a standard filling to obtain a filling with a given alphabet), the map majcode, being a composition of Standardize and the majcode for standard fillings, is injective as well on fillings with content α.

We now wish to show that $\mathrm{majcode}(\sigma) = d_1, \ldots, d_n$ is A-weakly increasing, implying that majcode is an injective morphism of weighted sets to $C_{\mu,A}$. To check this, let $\widetilde{\sigma} = \mathrm{Standardize}(\sigma)$. Then any repeated letter from σ will become a collection of squares that have consecutive entries and are increasing in reading order in $\widetilde{\sigma}$. Neither of the two operations of the map ψ affects the reading order of such subcollections since consecutive integers a and $a+1$ cannot occur in reverse order in a filling with distinct entries and no inversions. So, it suffices to show that if the largest entry m of σ occurs i times, then $d_1 \le \cdots \le d_i$.

In $\widetilde{\sigma}$, the m's of σ become the numbers $n-i+1, n-i+2, \ldots, n$, and occur in reading order. Thus we remove any of these that occur in the bottom row first, and for those we have $d_t = 0$. We continue removing these from the bottom row until there are none left in the bottom row. Then the remaining d_t's up to d_i will equal 1. Therefore, $d_1 \le d_2 \le \cdots \le d_i$, as required.

Finally, it now suffices to show that $\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0}$ as the same size as $C_{\mu,A}$, since then the injective map majcode is in fact a bijection. Note that the fillings σ of μ with content α and $\mathrm{inv}(\sigma) = 0$ are in one-to-one correspondence with the partitions of the alphabet A of content α into blocks (disjoint sub-multisets) of size $\mu_1, \mu_2, \ldots$, by considering the contents of each row. This is the same as the number of ways of filling the conjugate shape μ^* with the reverse alphabet in such a way that $\mathrm{maj} = 0$, by considering the contents of each column. It follows that $\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0}$ has the same size as $\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0}$, which in turn has the same size as $C_{\mu,A}$ by Theorem 3.4.3. □

Corollary 4.5.7. *For any two-row shape μ and content α, the map* $\mathrm{invcode}^{-1} \circ \mathrm{majcode}$ *is an isomorphism of weighted sets from $\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0} \to \mathcal{F}_{\mu^*}^{r(\alpha)}|_{\mathrm{maj}=0}$. This gives a combinatorial proof of the identity*

$$\widetilde{H}_\mu(x; 0, t) = \widetilde{H}_{\mu^*}(x; t, 0)$$

for two-row shapes.

Example 4.5.8. In Figure 4.3, the map majcode is applied to a two-row filling σ. The figure shows that $\mathrm{majcode}(\sigma) = 100010$. If we apply $\mathrm{invcode}^{-1}$ to this code using the reversed alphabet, we obtain the filling ρ below:

2	1
2	2
4	3

Notice that $\mathrm{maj}(\sigma) = \mathrm{inv}(\rho) = 2$.

We now have the tools to extend our map ψ to three-row shapes.

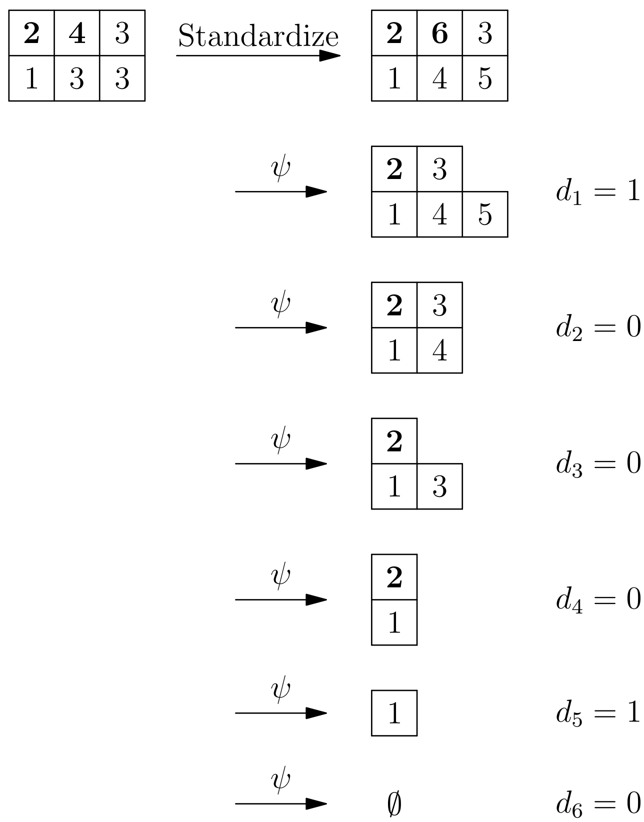


Figure 4.3: The map majcode for two-row shapes.

Definition 4.5.9. Let σ be any filling of a three-row shape $\mu = (\mu_1, \mu_2, \mu_3)$, and let σ' be the $3 \times \mu_3$ rectangle contained in σ. Let n be the largest entry in σ. Choosing one of these n's, say n_i, we define $\psi_{n_i}(\sigma)$ by the following process.

1. If n_i is to the right of σ', remove the n as in the two-row algorithm to form $\psi_{n_i}(\sigma)$.

2. If n_i is in the bottom row and in σ', then σ is a rectangle and we let $\psi_{n_i}(\sigma) = \sigma_{\downarrow}$.

3. If n_i is in the second row and in σ', let a_2 be the top entry of the bumping sequence of σ'. Let b be the entry in square $(\mu_2 + 1, 2)$ if it exists, and let $b = n + 1$ otherwise. If $b \geq a_2$, then remove n_i and bump down a_2 to the second row, and if $b < a_2$, simply remove n_i. Rearrange the modified rows so that there are no inversions, and let $\psi_{n_i}(\sigma)$ be the resulting filling.

4. If n_i is in the top row and in σ', let a_1, a_2 be the bumping sequence of the $3 \times \mu_3$ rectangle in σ. If $a_2 > a_1$ or $\mu_2 = \mu_3$, then remove n_i from σ. Otherwise, if $a_2 \leq a_1$, remove n and bump a_2 up to the top row. Rearrange the modified rows so that there are no inversions, and let $\psi_{n_i}(\sigma)$ to be the resulting filling.

Lemma 4.5.10. *Let $\mu = (\mu_1, \mu_2, \mu_3)$ be any three-row shape of size n. Then the map $\psi = \psi_n$ defined above is a morphism of weighted sets when restricted to fillings having distinct entries. That is, in the case of distinct entries there is a unique choice of n, and*

$$\psi : (\mathcal{F}_{\mu}^{(1^n)}|_{\mathrm{inv}=0}; \mathrm{maj}) \to \bigsqcup_{d=0}^{2} (\mathcal{F}_{\mu^{(d+1)}}^{(1^{n-1})}|_{\mathrm{inv}=0}; \mathrm{maj} + d)$$

is a morphism of weighted sets.

Proof. We wish to show that ψ is a morphism of weighted sets, i.e. that it preserves the statistics on the objects. If the n is in the bottom row, then $\psi(n)$ is in the $d = 0$ component of the disjoint union and the maj is preserved, by Proposition 4.3.1. If n is in the second row and to the right of column μ_3, then by Lemma 4.5.1 the difference in maj upon removing it is 1 and we obtain a filling in the $d = 1$ component of the disjoint union.

This leaves us with two possibilities: n is in the second row and weakly to the left of column μ_3, or n is in the top (third) row. In either case, if $\mu_2 = \mu_3$ then the mapping is the same as that in Theorem 4.4.1, and we get a map to either the $d = 1$ or $d = 2$ component of the disjoint union. So we may assume $\mu_2 \neq \mu_3$.

Case 1. Suppose that n is in the second row. We have two subcases to consider: $b < a_2$ and $b \geq a_2$.

If $b < a_2$, $\psi(\sigma)$ is formed by removing the n and rearranging so that there are no inversions. Note that any entry i to the right of n in row 2 is less than the entry directly south of n. Furthermore, such entries i are not descents and are increasing from left to right. Thus these entries simply slide to the left one space each to form $\psi(\sigma)$ after removing the n. So b is the only new entry to be weakly to the left of column μ_3 in $\psi(\sigma)$. Since b is not a descent, the effect on the major index is the same as if we simply replaced n by b in σ'. Consider any arrangement of the second row of σ' in which n is at the end, and arrange the top row relative to this ordering. Then a_2 is at the end of this top row by its definition, and so replacing n by b will make a_2 a descent and thereby increase the total cocharge contribution by 1. By Lemma 4.2.6 this is the same as the increase in the cocharge contribution from σ to $\psi(\sigma)$. Hence $\psi(\sigma)$ lies in the $d = 1$ component of the disjoint union.

If $b \geq a_2$, we claim that if a_1 is the entry in the bottom row of the bumping sequence, then $b < a_1$. If a_1 is to the right of the column that n is in then the claim clearly holds. Otherwise, let $a_1, d_1, d_2, \ldots, d_i$ be the consecutive entries in the bottom row starting from a_1 and ending at the entry d_i beneath the n, and let $c_1, \ldots, c_i$ be the entries in the second row from the entry above a_1 to the entry just before the n. The c_j's are all descents, and the c_j's and d_j's are both increasing sequences. Since there are no inversions in the second row, we have $b < d_i$. Since removing the n and bumping up a_1 results in the a_1 at the end of the second row by definition, upon doing this the d_i's all slide to the left one space, and the c_i's must also remain in position and remain descents by Proposition 4.3.1. In particular, this means that $d_i < c_i$, and so $b < c_i$ as well. But then since there are no inversions it follows that $b < d_{i-1}$, which is less than c_{i-1}, and so on. Continuing, we find that $b < a_1$ as claimed.

Since $b \geq a_2$ by assumption, it follows that $a_2 < a_1$ and so removing the n and bumping down a_2 in the rectangle results in a difference in major index of 2 by Theorem 4.4.4. Note also that if we perform this bumping in the entire filling σ, the entry a_2 ends up to the left of column $\mu_3 + 1$ since $a_2 \leq b$ and hence it is to the left of b in the second row. Thus the entries to the right of the rectangle are preserved, and $\mathrm{maj}(\psi(\sigma)) = \mathrm{maj}(\sigma) - 2$. It follows that $\psi(\sigma)$ lies in the $d = 2$ component of the disjoint union.

Case 2. Suppose n is in the top row. If $a_2 > a_1$, then removing n results in the major index decreasing by $d = 2$, and so $\psi(\sigma)$ is in the $d = 2$ component of the disjoint union. Otherwise, $a_2 \leq a_1$. Since $\mu_2 \neq \mu_3$, we remove the n and bump a_2 up to the top row.

Since $a_2 \leq a_1$, by Theorem 4.4.4 we find that simply removing the n results in a decrease by 1 in the major index. Since the top row has had a descent removed (by the proof of Theorem 4.4.1), it follows that the empty space created in the top row was not above a descent, for otherwise the major index would decrease by 2. Thus in particular b is not a descent.

It follows that if $\widetilde{\sigma}$ is formed by bumping up a_2 and inserting n in the second row, then the n, being the last descent in the second row, will appear among the first μ_3 columns of $\widetilde{\sigma}$. In addition, since $a_2 \leq a_1$ this results in an *increase* in major index by 1 from σ to $\widetilde{\sigma}$, by Corollary 4.4.3.

We now wish to show that $b \geq a_2$; if so, we claim removing n from $\widetilde{\sigma}$ will result in a decrease by 2 in the major index, and will also result in the tableau $\psi(\sigma)$, thereby showing that $\mathrm{maj}(\psi(\sigma)) = \mathrm{maj}(\sigma) - 1$ and so $\psi(\sigma)$ is in the $d = 1$ component. To see that the major index decreases by 2 on removing n, note that by Proposition 4.4.4, the effect of removing the n is the same as replacing n by b in the one-column shape with entries a_2, n, a_1. If $b \geq a_2$ then we have that $b < a_1$ by the same argument as in Case 1 above, and so the major index decreases by 2. Thus it suffices to show $b \geq a_2$.

If a_2 is not a descent of σ, this is clear, so suppose a_2 is a descent of σ in the second row. Let c be the entry directly below a_2, and assume for contradiction that $b < a_2$. Then $b < c$, and furthermore the first non-descent in row 2, say e, is less than c. Note that by our above argument we know that e lies within the rectangle σ'.

Now, we restrict our attention to σ' and bump a_2 and a_1 up one row each, and consider the ordering of the bottom row in which we place c in the column one to the left of the column that e was contained in and shift the remaining entries to the left to fill the row. Rearranging the new second row with respect to the first, we consider the position of a_1 relative to c. If a_1 is to the left of the c we have a contradiction since a_1 must land in column μ_3 by Lemma 4.2.7 and the definition of bumping sequence. Therefore the entries in the second row to the left of c are unchanged. Since $a_1 \geq a_2 \geq c$, and all remaining entries in the second row are either a_1 or are less than c, we have that a_1 must be on top of the c in the second row. This is again a contradiction, since this implies that a_1 does not land in column μ_3. It follows that $b \geq a_2$, as desired.

This completes the proof that ψ is a well-defined morphism of weighted sets. $\square$

We have shown that ψ is a morphism, and we now show it is bijective.

Lemma 4.5.11. *The map ψ of Lemma 4.5.10 is an isomorphism.*

Proof. We know from the lemma above that ψ is a morphism; it suffices to show that it is bijective. First notice that the cardinality of $\left(\mathcal{F}_{\mu}^{(1^n)}\right)\Big|_{\mathrm{inv}=0}$ is

$$\binom{n}{\mu},$$

and the cardinality of $\left(\bigsqcup_{d=0}^{2}\mathcal{F}_{\mu^{(d+1)}}^{(1^{n-1})}\right)\Big|_{\mathrm{inv}=0}$ is

$$\binom{n-1}{\mu_1-1,\mu_2,\mu_3}+\binom{n-1}{\mu_1,\mu_2-1,\mu_3}+\binom{n-1}{\mu_1,\mu_2,\mu_3-1}.$$

Thus the cardinalities of the two sets are equal, and so it suffices to show that ψ is surjective.

To do so, choose an element ζ of the codomain. Then ζ can lie in any one of the three components of the disjoint union $\left(\bigsqcup_{d=0}^{2}\mathcal{F}_{\mu^{(d+1)}}^{(1^{n-1})}\right)|_{\mathrm{inv}=0}$, and we consider these three cases separately.

Case 1: Suppose ζ lies in the $d=0$ component. Then we can insert n in the bottom row so as to reverse the map of Proposition 4.3.1, and we obtain an element σ of $\left(\mathcal{F}_{\mu}^{(1^n)}\right)|_{\mathrm{inv}=0}$ which maps to ζ under ψ.

Case 2: Now, suppose ζ lies in the $d=1$ component. If $\mu_2=\mu_3$ then $\mu^{(1)}=(\mu_1,\mu_2,\mu_3-1)$ and so we can find a filling σ of μ that maps to ζ by Proposition 4.4.1. Otherwise, the shape of ζ is (μ_1,μ_2-1,μ_3) and we wish to find a filling σ of shape μ for which $\psi(\sigma)=\zeta$. Let ρ be the filling of μ formed by inserting n into the second row and rearranging entries so that there are no inversions. Notice that if the n lies to the right of column μ_3 then $\psi(\rho)=\zeta$ and we are done.

So, suppose n lies in the $3\times\mu_3$ rectangle in ρ. Let a_1 and a_2 be the bumping sequence of this rectangle. Since n is the rightmost descent in the second row of ρ, inserting it did not change the cocharge contribution of the portion to the right of column μ_3; there were no descents there in σ and there are none in ρ. Let b be the entry in column μ_3+1, row 2 of ρ. If $b<a_2$, then $\psi(\rho)=\zeta$ and we are done.

Otherwise, if $b\geq a_2$, then by the argument in Lemma 4.5.10 we know that $\mathrm{maj}(\rho)-\mathrm{maj}(\zeta)=2$. We have that $\tau:=\psi(\rho)$ is the filling formed by removing the n and bumping a_2 down to the second row, and that $\mathrm{maj}(\rho)-\mathrm{maj}(\tau)=2$. Hence $\mathrm{maj}(\tau)=\mathrm{maj}(\zeta)$. Since $b\geq a_2$, a_2 lies to the left of b in τ and hence is weakly to the left of column μ_3. So, let σ be the tableau formed by inserting n in the top row of τ. Now σ has shape μ, and can be formed directly from ρ by shifting the position of n among a_1 and a_2 as in Theorem 4.4.4.

It follows that $\mathrm{maj}(\sigma)-\mathrm{maj}(\rho)=\pm1$, and so $\mathrm{maj}(\sigma)-\mathrm{maj}(\tau)$ is equal to 1 or 3. It is not 3 because τ is formed from σ by removing an n from the top row, which changes the major index by at most 2 by Theorem 4.4.1. It follows that $\mathrm{maj}(\sigma)-\mathrm{maj}(\tau)=1$, and therefore $a_2\leq a_1$ by Theorem 4.4.4. Thus $\psi(\sigma)=\zeta$ by the definition of ψ.

Case 3: Suppose ζ is in the $d = 2$ component. If $\mu_2 = \mu_3$ then we simply insert n into ζ in either row 2 or 3 according to Theorem 4.4.4 to obtain a tableau σ with $\psi(\sigma) = \zeta$.

Otherwise, if $\mu_2 \neq \mu_3$, ζ has shape $(\mu_1, \mu_2, \mu_3 - 1)$. Let ρ be the tableau of shape μ formed by inserting n in the top row of ζ. Let a_1 and a_2 be the entries in row 1 and 2 corresponding to this n in the $3 \times \mu_3$ rectangle contained in ρ. Then if $a_2 > a_1$, $\psi(\rho) = \zeta$ and we're done.

If instead $a_2 \leq a_1$, then removing n from ρ decreases its major index by 1. Since the number of descents in the top row goes down by exactly 1 by Lemma 4.5.1, we can conclude that the entry in row 2, column μ_3 is a non-descent; otherwise removing n from ρ would decrease the major index by 2. So, let σ be the filling formed by removing n from ρ, bumping a_2 to the top row, and inserting n in the second row. Since there are non-descents in the rectangle we have that n lies in the rectangle in σ as well.

Finally, again by the argument used for Lemma 4.5.10 we have that $a_2 \leq b$ where b is the entry in row 2, column $\mu_3 + 1$ in σ. Thus $\psi(\sigma) = \zeta$ as desired. □

We can now complete the three-row case by defining its standardization map for fillings with repeated entries. This definition is designed to force the majcode sequences to be A-weakly increasing.

Definition 4.5.12. Given a filling σ of μ, define Standardize(σ) as follows. First, for any letter i that occurs with multiplicity in σ, label the i's with subscripts in reading order to distinguish them. If we bump one of them up or down one row, choose the one to bump from the row in question that preserves their reading order.

Let n be the largest entry that occurs in σ. For each such n_t compute $d_t = \mathrm{maj}(\sigma) - \mathrm{maj}(\psi_{n_i}(\sigma))$, and let $d = \min_t(\{d_t\})$. Let n_r be the last n in reading order for which $d_r = d$. Form the filling $\psi_{n_r}(\sigma)$, and repeat the process on the new filling. Once there are no n's left to remove, similarly remove the $n-1$'s, and so on until the empty tableau is reached.

Now, consider the order in which we removed the entries of σ and change the corresponding entries to $N, N-1, \ldots, 1$ in that order, where $N = |\mu|$. The resulting tableau is Standardize(σ).

We can now define majcode for three-row shapes.

Definition 4.5.13. Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a three-row shape of size n. Given a filling σ of μ, let $\overline{\sigma} = \mathrm{Standardize}(\sigma)$. Then we define $\mathrm{majcode}(\sigma) = d_1 d_2 \cdots d_n$ where

$$d_i = \mathrm{maj}(\psi^{i-1}(\overline{\sigma})) - \mathrm{maj}(\psi^i(\overline{\sigma})),$$

and where ψ is the map defined in Lemma 4.5.10.

Remark 4.5.14. Notice that, given a filling σ of μ having arbitrary content, we have

$$\mathrm{majcode}(\sigma) = \mathrm{majcode}(\mathrm{Standardize}(\sigma)).$$

Theorem 4.5.15. *The map* majcode *defined on three-row shapes* $\mu = (\mu_1, \mu_2)$ *is an isomorphism of weighted sets*

$$\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0} \to C_{\mu,A}$$

for each alphabet A *and corresponding content* α.

To prove Theorem 4.5.15, we first state a structure lemma about three-row shapes with no inversions.

Lemma 4.5.16. *If the consecutive entries* $b_1, \dots, b_n$ *in some row of a filling with no inversions are directly above a weakly increasing block of squares* $c_1 \le \cdots \le c_n$ *in the row below, then there exists a* k *for which* $b_1, \dots, b_k$ *are descents and* $b_{k+1}, \dots, b_n$ *are not descents. Moreover* $b_1 \le \cdots \le b_k$ *and* $b_{k-1} \le \cdots \le b_n$ *are both increasing blocks of squares.*

Proof. This is clear by the definition of inversions. □

In particular, the second row has one (possibly empty) block of descents and one (possibly empty) block of non-descents. The third row has up to two blocks of descents, one for each of the blocks in the second row, and so on.

We also need to show that the cardinalities of the sets are equal in the case of repeated entries.

Lemma 4.5.17. *We have*

$$\left|\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0}\right| = |C_{\mu,A}|$$

for any alphabet A *with content* α *and any shape* μ.

Proof. Given an alphabet A, the cocharge word of any filling using the letters in A has the property that it is weakly increasing on any run of a repeated letter, where we list the elements of A from largest to smallest. Furthermore, the cocharge word has content μ. It is not hard to see that a word is the cocharge word of a filling in $\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0}$ if and only if it has content μ and is weakly increasing over repeated letters of A, listed from greatest to least.

Recall that the fillings in $\mathcal{F}_{\mu^*}^{r(\alpha)}|_{\mathrm{maj}=0}$ can be represented by their *inversion word*, and a word is an inversion word for such a filling if and only if it has content μ and every subsequence corresponding to a repeated letter of the reversed alphabet is in inversion-friendly order. By swapping the inversion-friendly order for weakly increasing order above each repeated letter, we have a bijection between inversion words and cocharge words, and hence a bijection (of sets, not of weighted sets) from $\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0}$ to $\mathcal{F}_{\mu^*}^{r(\alpha)}|_{\mathrm{maj}=0}$. By Theorem 3.4.3, we have that

$$\left|\mathcal{F}_{\mu^*}^{r(\alpha)}|_{\mathrm{maj}=0}\right| = |C_{\mu,A}|,$$

and so the cardinality of $\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0}$ is equal to $|C_{\mu,A}|$ as well. □

We can now prove Theorem 4.5.15.

Proof. By Lemmas 4.5.10, 4.5.11, and 3.3.9, we have that for the content (1^n) corresponding to alphabet $[n]$, the map majcode is a weighted set isomorphism

$$\mathcal{F}_{\mu}^{(1^n)}|_{\mathrm{inv}=0} \to C_{\mu,[n]}.$$

Now, let A be any alphabet with content α. Let σ be a filling of μ with content α. Then we know majcode(σ) = majcode(Standardize(σ)), so majcode$(\sigma) \in C_{\mu,[n]}$. In other words, majcode(σ) is μ-sub-Yamanouchi. In addition, since Standardize is an injective map (there is clearly a unique way to un-standardize a standard filling to obtain a filling with a given alphabet), the map majcode, being a composition of Standardize and the majcode for standard fillings, is injective as well on fillings with content α.

We now wish to show that majcode$(\sigma) = d_1, \ldots, d_n$ is A-weakly increasing, implying that majcode is an injective morphism of weighted sets to $C_{\mu,A}$. By Lemma 4.5.17 this will imply that it is an isomorphism of weighted sets. It suffices to show this for the largest letter m of A by the definition of standardization. Suppose m occurs i times. We wish to show that $d_j \le d_{j+1}$ for all $j \le i-1$. So choose $j \le i-1$.

Suppose $d_j = 0$. Then by the definition of Standardize, we have that the m we removed from $\psi^{j-1}(\sigma)$ was in the bottom row. If there are still m's in the bottom row of $\psi^j(\sigma)$ then $d_{j+1} = 0$ as well. Otherwise $d_{j+1} > 0$, so $d_j \le d_{j+1}$ in this case.

Suppose $d_j = 1$. Then the m we removed from $\psi^{j-1}(\sigma)$ was in either the first or second row and there were no m's in the bottom row. By the definition of ψ, there are therefore no m's in the bottom row of $\psi(\sigma)$ either, and so $d_{j+1} \ge 1 = d_j$.

Finally, suppose $d_j = 2$. Let m_j be the m we remove from $\psi^{j-1}(\sigma)$ to obtain $d_j = 2$. As in the previous case we have $d_{j+1} \ge 1$, and we wish to show $d_{j+1} \ne 1$. Let m_{j+1} be the corresponding m. Since d_j is minimal for $\psi^{j-1}(\sigma)$, there are no m's in $\psi^{j-1}(\sigma)$ which we can treat as the largest entry and remove according to ψ to form $d_j = 1$. Therefore if we removed m_{j+1} before m_j we would also have a difference of 2 in the major index.

We consider three subcases separately for the locations of m_j and m_{j+1}: they can either both be in the second row, m_j can be in the second row with m_{j+1} in the third (top) row, or they can both be in the top row. No other possibilities exist because they must occur in reverse reading order, and cannot be in the bottom row since $d_j = 2$.

Subcase 1: Suppose both m_j and m_{j+1} are in the second row. Then m_{j+1} and m_j are at the end of the block of descents in that order, and weakly to the left of column μ_3. Let b be the entry in row 2, column $\mu_3 + 1$. Let a_2 be the entry in the third row in the bumping sequence of m_j, and let a_2' be the entry in the bumping sequence of m_{j+1} in $\psi^j(\sigma)$. Since $d_j = 2$, we have $a_2 \le b$ and $b < m$, and so $a_2 \ne m$. Therefore no new m's are dropped down. In other words, m_{j+1} is indeed the m that will be removed upon applying ψ the second time.

We now need to check that m_{j+1} remains to the left of column μ_3 after applying ψ. Indeed, by Proposition 4.4.4, we have that the number of descents in row 2 goes down by one, and the number of descents in the top row remains the same, upon applying ψ to $\psi^{j-1}(\sigma)$. Since there are no m's in the bottom row, m_{j+1} is the rightmost descent in the second row of $\psi^j(\sigma)$, and the descent we lost was m_j, so m_{j+1} remains in its column.

We now just need to show that $a_2' \le b'$, where b' is the entry in row 2, column μ_3 after applying ϕ. Either $b' = b$, $b' = a_2$, or b' is the entry b_0 that is bumped out from the first $\mu_3 - 2$ columns when we drop down a_2.

Consider any ordering of the first μ_3 entries of the second row of $\psi^{j-1}(\sigma)$ such that the two m's (m_{j+1} and m_j) are at the end in that order, and also place b_0 in the third-to-last position. Now, rearrange the entries above these so that there are no inversions. We know that a_2 is at the end of the top row, above m_j, by its definition. Let a be the entry above m_{j+1} and let c be the entry to the left of that (if such a column exists.)

If $b' = a_2$, then the first $\mu_3 - 2$ entries of the second row are unchanged on applying ψ. In our new ordering above, this means $a_2' = a$, and since a and a_2 occur above the two m's in our new ordering, we have $a \le a_2$. It follows that $a_2' \le b'$.

If $b' \ne a_2$, then b' is either b or b_0. To find a_2' in our new ordering, bumping down the a_2 can be thought of as replacing the b_0 with a_2 and rearranging the top row again so that there are no inversions. The first $\mu_3 - 3$ entries remain in the same positions, and either c or a lies above m_{j+1} based on which comes later in cyclic order after a_2. So either $a_2' = a$ or $a_2' = c$.

We now have to show that whether a_2' is a or c, it is less than both b and b_0. Notice that $a_2 \le b_0$: Since m_{j+1} stays in its place, either a_2 replaces a larger entry among the descents in the second row, which in turn bumps out a larger entry b_0 among the non-descents, or it replaces a non-descent itself and displaces a larger non-descent b_0 to its right. So if $a_2' = a$, then since $a \le a_2$ we have $a_2' \le a_2 \le b_0$ and also $a_2' \le a_2 \le b$ since $a_2 \le b$.

Finally, if $a_2' = c$, then a_2, a, c are in cyclic order. If $c \le a_2$ we are done by the above argument. Otherwise $a_2 < a \le c$ or $a_2 = a = c$, in which case $a_2' = a$ and we are done by the previous case. So $a_2 < a \le c$, but we already know $a \le a_2$, so we have a contradiction. It follows that $a_2' \le b'$ as desired.

Subcase 2: Suppose m_j and m_{j+1} are in the top row. Then by Lemma 4.5.16 and since there are no m's in the second row by the definition of Standardize, the m's are either in the first or second block of descents in the third row. If either of them is in the second block, it is clear that removing m_j results in $d_j = 1$, not 2, a contradiction. So they are both in the first block, themselves above descents in the second row, with m_{j+1} and m_j adjacent and in that order.

Now, removing m_j will cause the block of non-descents to its right to slide to the left one space (since they are necessarily less than the entry beneath m_j). If the *second* block of non-descents in the third row is nonempty, one of these will replace the last entry above the descents in the second row, since all of these are still less than the entry below m_j and the least among the entries to the right will replace it. In that case the number of descents to the right of m_j is unchanged, and so $d_j = 1$, a contradiction. Thus there are no non-descents in the second block, i.e. above the non-descents in row 2.

Because of this, removing m_j simply causes all the entries to its right to slide to the left one space, and the first descent to its right becomes a non-descent. The same then happens when we remove m_{j+1} by the same argument. It follows that $d_{j+1} = 2$ in this case.

Subcase 3: Suppose m_j is in the second row and m_{j+1} in the top. Then the m_{j+1} is to the left of m_j, in the first block of descents in the third row, since otherwise we would have a difference of 1 on removing m_{j+1}. Moreover, as in the previous case, the top row has no non-descents above the non-descents in row 2.

So, let $a_1, \ldots, a_r$ be the entries in row 3 that lie weakly to the right of m_j's column. Then a_1 is not a descent and each of $a_2, \ldots, a_r$ are descents. Let $m_j, b_2, \ldots, b_r$ be the entries below them. If we rearrange these in the second row in the increasing order $b_2, \ldots, b_r, m_j$, and then rearrange the a_i's above them as $a_{\sigma(1)}, \ldots, a_{\sigma(r)}$ so that there are no inversions, there are still $r-1$ descents among the $a_{\sigma(i)}$'s by Lemma 4.2.6. These descents must be $a_{\sigma(1)}, \ldots, a_{\sigma(r-1)}$ by Lemma 4.5.16, and the last entry $a_{\sigma(r)}$ above the m_j is the entry in m_j's bumping sequence.

Now, to form $\psi^j(\sigma)$, we remove m_j and drop down $a_{\sigma(r)}$. Notice that the entries in the top row to the left of where m_j was are unchanged: consider the $3 \times \mu_3$ rectangle and bump the m_j down to the bottom row according to Theorem 4.4.1. Then bump it out according to Proposition 4.3.1, which leaves us with the same top row as that of $\psi^j(\sigma)$. The entire top row save for the last entry is unchanged upon applying Proposition 4.3.1, and so having the m_j inserted into the second row instead can only change the entries to the right of it in the row above. Thus the entries to its left in the top row are unchanged, and have the same cocharge contribution as well.

Finally, in the columns weakly to the right of the column that m_j was in, the entries in the top row are $a_{\sigma(1)}, \ldots, a_{\sigma(r-1)}$ in some order. We claim that the entries in the second row are formed by replacing at most one of $b_2, \ldots, b_r$ by a smaller entry, which is either $a_{\sigma(r)}$ or something bumped to the right by $a_{\sigma(r)}$ if $a_{\sigma(r)}$ lands in a column to the left of the b_i's. Indeed, the only way it would be a larger entry replacing them is if a descent replaced m_j, but in this case we would have $d_j = 1$ since the number of descents in the second row would be the same, and the number of descents in the top row would decrease by only 1.

Therefore, the entries $a_{\sigma(1)}, \ldots, a_{\sigma(r-1)}$ are all descents in the top row, and so removing m_{j+1} still results in a difference $d_{j+1} = 2$. In particular, the descents formed by m_{j+1} and one of the a_i's are removed, since the a's all slide one position to the left, and did not form new descents upon removing the m_{j+1} before the m_j.

This completes the proof. □

Corollary 4.5.18. *For any three-row shape μ and content α, the map* $\mathrm{invcode}^{-1} \circ \mathrm{majcode}$ *is an isomorphism of weighted sets from $\mathcal{F}_\mu^\alpha|_{\mathrm{inv}=0} \to \mathcal{F}_{\mu^*}^{r(\alpha)}|_{\mathrm{maj}=0}$. This gives a combinatorial proof of the identity*

$$\widetilde{H}_\mu(x; 0, t) = \widetilde{H}_{\mu^*}(x; t, 0)$$

for three-row shapes.

Example 4.5.19. We demonstrate all of the above maps on the filling σ below, with its repeated entries labeled with subscripts in reading order to distinguish them.

$$\begin{array}{|c|c|cc}
\cline{1-2}
\mathbf{8}_1 & 1 & \multicolumn{2}{c}{} \\
\hline
\mathbf{5} & \mathbf{8}_2 & 2_1 & \multicolumn{1}{c|}{2_2} \\
\hline
3 & 4 & 6 & \multicolumn{1}{c|}{8_3} \\
\hline
\end{array}$$

We will standardize and compute majcode simultaneously. To decide which of the 8's to remove first, we look at which would give the smallest first majcode. This is clearly the 8_3 in the bottom row, so we remove it and bump down the 2.

$$\begin{array}{|c|c|c|c}
\cline{1-2}
\mathbf{8}_1 & 1 & \multicolumn{2}{c}{} \\
\cline{1-3}
\mathbf{5} & \mathbf{8}_2 & 2_1 & \multicolumn{1}{c}{} \\
\hline
2_2 & 3 & 4 & \multicolumn{1}{c|}{6} \\
\hline
\end{array}$$

To decide which of the remaining 8's to remove next, note that they both would decrease maj by 2, and so we remove the one that comes last in reading order, namely 8_2. Since $1 < 2$ we bump down the 1.

$$\begin{array}{|c|c|c|c}
\cline{1-1}
\mathbf{8} & \multicolumn{3}{c}{} \\
\cline{1-3}
\mathbf{5} & 1 & 2_1 & \multicolumn{1}{c}{} \\
\hline
2_2 & 3 & 4 & \multicolumn{1}{c|}{6} \\
\hline
\end{array}$$

Finally, when we remove the last 8, the maj decreases by 2, so we do not have to lift the 5 up to the third row.

$$\begin{array}{|c|c|c|c}
\cline{1-3}
\mathbf{5} & 1 & 2_1 & \multicolumn{1}{c}{} \\
\hline
2_2 & 3 & 4 & \multicolumn{1}{c|}{6} \\
\hline
\end{array}$$

We can now use the two-row algorithm to complete the process, and we find $\mathrm{majcode}(\sigma) = 0220100000$. The corresponding inversion diagram for the reverse alphabet $\{1, 1, 1, 3, 4, 5, 6, 7, 7, 8\}$ is shown below.

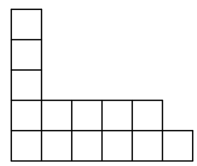
$$\begin{array}{cccccccccc}
 & 2 & 1 & & & & & & & \\
 & & & 4 & 3 & & & 2 & & 1 \\
4 & & & & & 3 & 2 & & 1 & \\
\hline
1 & 1 & 1 & 3 & 4 & 5 & 6 & 7 & 7 & 8
\end{array}$$

Finally, we can reconstruct from this the filling $\rho = \mathrm{invcode}^{-1}(0220100000)$ below.

$$\begin{array}{|c|c|c}
\cline{1-2}
1 & 3 & \multicolumn{1}{c}{} \\
\cline{1-2}
5 & 4 & \multicolumn{1}{c}{} \\
\hline
6 & 7 & \multicolumn{1}{c|}{1} \\
\hline
7 & 8 & \multicolumn{1}{c|}{1} \\
\hline
\end{array}$$

Note that $\mathrm{inv}(\rho) = \mathrm{maj}(\sigma) = 5$, and $\mathrm{inv}(\sigma) = \mathrm{maj}(\rho) = 0$.

Remark 4.5.20. The map above essentially uses the fact that a three-row shape is the union of a rectangle and a two-row shape. For *any* shape that is the union of a rectangle and two rows, a similar map can be used to remove the first n, and so for "fat hook" shapes consisting of two rows plus a long column, a similar algorithm also produces a valid majcode map.



In general, however, the resulting shape on removing the first n is no longer the union of a rectangle and a two-row shape, and we cannot use an induction hypothesis.

However, we believe that this method may generalize to all shapes, as follows.

Problem 4.5.21. *Extend the map ψ for three-row shapes to all shapes inductively, as follows. First extend it to shapes which are the union of a three-row shape and a single column, then use this to extend it to shapes which are the union of a three-row shape and a rectangle shape, using Theorem 4.4.4. Then iterate this new map on any four-row shape, so* majcode *can then be defined on four-row shapes, and so on.*

4.6 Application to Cocharge

Proposition 4.3.1 reveals an interesting property of the cocharge statistic on words, defined in Section 2.5.

As mentioned in Section 2.5, for any filling $\sigma \in \mathcal{F}|_{\mathrm{inv}=0}$ we have $\mathrm{maj}(\sigma) = \mathrm{cc}(\mathrm{cw}(\sigma))$. Therefore, we can translate some of our results regarding such fillings to properties of words and the‘ir cocharge. We first require the following fact.

Lemma 4.6.1. *If $\sigma \in \mathcal{F}|_{\mathrm{inv}=0}$ and $w = \mathrm{cw}(\sigma)$, the words $w^{(i)}$ correspond to the columns of σ, in the sense that the letters in the subword $w^{(i)}$ are in positions corresponding to the entries in column i in σ.*

Proof. If $w = \mathrm{cw}(\sigma)$ and σ has alphabet $A = a_1 \geq \ldots \geq a_n$, the letters a_i for which the corresponding letter w_i equals r are the entries in row r. The smallest - that is rightmost - a_i, say a_{i_0}, for which $w_i = 1$ is the leftmost entry of the bottom row, i.e. the bottom entry of the first column. The second entry of the first column is then the first a_i in cyclic order after a_{i_0} for which $w_i = 2$. This corresponds to the 2 in the subword $w^{(1)}$, and similarly the letters in $w^{(1)}$ correspond to the entries in the first column.

A similar argument shows that the second column corresponds to $w^{(2)}$, and so on. $\square$

In particular, Proposition 4.3.1 states that if the largest entry of a filling $\sigma \in \mathcal{F}_\mu^{(1^n)}|_{\mathrm{inv}=0}$ is in the bottom row, then we can remove it, bump down any entries in its (rightmost) column, and rearrange the rows to get a filling with no inversions. By Lemma 4.6.1, this translates to the following result in terms of words.

Theorem 4.6.2. *Let $w = w_1 \cdots w_n$ be a word with partition content μ for which $w_1 = 1$. Let $w^{(1)}, \cdots, w^{(\mu_1)}$ be its decomposition into subwords as in Definition 2.5.14. Then $w_1 \in w^{(\mu_1)}$, and if w' is the word formed by removing w_1 from w and also decreasing each letter that is in $w^{(\mu_1)}$ by one, then*

$$\mathrm{cc}(w) = \mathrm{cc}(w').$$

This theorem fills a gap in our understanding of cocharge, as it gives a recursive way of dealing with words that start with 1. These are the only words that do not satisfy the relation $\mathrm{cc}(\mathrm{cyc}(w)) = \mathrm{cc}(w) - 1$ of Definition 2.5.13.

Example 4.6.3. Consider the word 15221432313. It has three 1's, three 2's, and three 3's, but only one 4 and 5, so to find the word $w^{(\mu_1)} = w^{(3)}$ we can ignore the 4 and 5. The words $w^{(1)}$, $w^{(2)}$, and $w^{(3)}$, ignoring the 4 and 5, are the subwords listed below:

w	1	5	2	2	1	4	3	2	3	1	3
$w^{(1)}$							3	2		1	
$w^{(2)}$				2	1						3
$w^{(3)}$	1		2						3		

and so the word w' is formed by removing the leading 1 and decreasing the 2 and 3 from $w^{(3)}$. Thus

$$w' = 5121432213.$$

We also find that $\mathrm{cc}(w) = \mathrm{cc}(w') = 12$.

Chapter 5

Two further results on Macdonald q, t-symmetry

5.1 Specialization at $t = 1$

In this section, we give a combinatorial proof of the specialization of Problem 1.0.1 at $t = 1$, namely $\widetilde{H}_\mu(x; q, 1) = \widetilde{H}_{\mu^*}(x; 1, q)$.

By the combinatorial formula in [17], it suffices to prove that, for any content α,

$$\sum_{\substack{\sigma:\mu\to\mathbb{Z}_+\\|\sigma|=\alpha}} q^{\mathrm{maj}(\sigma)} = \sum_{\substack{\rho:\mu^*\to\mathbb{Z}_+\\|\rho|=\alpha}} q^{\mathrm{inv}(\rho)}. \tag{5.1}$$

To prove this, we can extend either the Foata or the Carlitz bijection. We will work with the Carlitz bijection for consistency, but the bijection can easily be built in the same manner using the Foata bijection. Let

$$f = \mathrm{invcode}^{-1} \circ \mathrm{majcode}$$

be the Carlitz bijection on permutations of a given ordered alphabet with n distinct entries. We first prove Equation 5.1 in the case that $\alpha = (1, 1, \ldots, 1)$.

Proposition 5.1.1. *For any fixed partition λ, we have $\sum_\sigma q^{\mathrm{maj}(\sigma)} = \sum_\rho q^{\mathrm{inv}(\rho)}$ where the first sum ranges over all fillings $\sigma : \lambda \to \mathbb{Z}_+$ of λ with distinct entries, and the second ranges over all fillings $\rho : \lambda^* \to \mathbb{Z}_+$ of the conjugate partition λ^* with distinct entries.*

Proof. We extend the bijection f as follows.

Given a filling σ of λ, let $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ be the words formed by reading each of the columns of λ from top to bottom. Let $w^{(i)} = f(v^{(i)})$ for each i, so that $\mathrm{maj}(v^{(i)}) = \mathrm{inv}(w^{(i)})$. Notice that $\mathrm{maj}(\lambda) = \sum_{i=1}^k \mathrm{maj}(v^{(i)})$. We aim to construct a filling ρ of λ^* such that $\mathrm{inv}(\rho) = \sum_{i=1}^k \mathrm{inv}(w^{(i)})$.

Let the bottom row of ρ be $w^{(1)}$. To construct the second row, let $t_1 = w_1^{(1)}$ be the corner letter. Let $x_1, x_2, \ldots, x_r$ be the unique ordering of the letters of $w^{(2)}$ for which the sequence $t_1, x_1, x_2, \ldots, x_r$ is in cyclic order. Notice that if x_i is placed in the square above t_1, it would be part of exactly i relative inversions to the right of it, since $x_1, \ldots, x_{i-1}$ would form inversions with it and the others would not.

Now, in $w^{(2)}$, let i_k be the number of inversions whose left element is the kth letter of $w^{(2)}$. Then write x_{i_1} in the square above t_1 in order to preserve the number of inversions the first letter is a part of. Then for the square above $t_2 = w_2^{(1)}$, similarly order the remaining x's besides x_{i_1} in cyclic order after t_2, and write down in this square the unique such x_{i_2} for which it is the left element of exactly i_2 inversions in its row. Continue this process for each $k \leq r$ to form the second row of the tableau.

Continue this process on each subsequent row, using the words $w^{(3)}, w^{(4)}, \ldots$, to form a tableau ρ. We define $f(\sigma) = \rho$, and it is easy to see that this construction process is reversible (strip off the top row and rearrange according to inversion numbers, then strip off the second, and so on.) Thus we have extended the Carlitz bijection to tableaux of content $\alpha = (1, 1, \ldots, 1)$, proving the result in this case. □

Using this proposition, we prove two technical lemmata about the q-series involved. Define $\mathrm{inv}_w(R)$ to be the number of relative inversions in a row R given a filling w of the row directly beneath it.

Lemma 5.1.2. *Let R be the $(i+1)$st row in a partition diagram λ for some $i \geq 1$. Let $w = w_1, \ldots, w_{\lambda_i}$ be a fixed filling of the i-th row, underneath R. Let $a_1, \ldots, a_{\lambda_{i+1}}$ be any λ_{i+1} distinct positive integers. Then*

$$\sum q^{\mathrm{inv}_w(R)} = (\lambda_i)_q!$$

where the sum ranges over all fillings of the row R with the integers $a_1, \ldots, a_{\lambda_{i+1}}$ in some order.

Proof. We know that

$$\sum_{r \in S_{\lambda_{i+1}} \cdot (a)} q^{\mathrm{inv}(r)} = (n)_q!.$$

We use a similar process to that in Proposition 5.1.1 to construct a bijection ϕ from the set of permutations r of $a_1, \ldots, a_{\lambda_{i+1}}$ to itself such that $\mathrm{inv}_w(\phi(r)) = \mathrm{inv}(r)$.

Namely, let $r = r_1, \ldots, r_{\lambda_{i+1}}$ be a permutation of $a_1, \ldots, a_{\lambda_{i+1}}$ and let i_k be the number of inversions that r_k is a part of in r for each k. Let $x_0, \ldots, x_{\lambda_{i+1}}$ be the ordering of the letters of r for which $w_1, x_0, \ldots, x_{\lambda_{i+1}}$ is in cyclic order. Let the first letter of $\phi(r)$ be x_{i_1}, remove x_{i_1} from the sequence, and repeat the process to form the entire row from the letters of r. Let $\phi(r)$ be this row.

The map ϕ can be reversed by using the all-0's word for w and using the same process as above to recover r from $\phi(r)$. Thus ϕ is bijective. Moreover $\mathrm{inv}_w(\phi(r)) = \mathrm{inv}(r)$ by construction. This completes the proof. □

Lemma 5.1.3. *Let r be the $(i+1)$st row in a partition diagram λ for some $i \geq 1$. Let $w = w_1, \ldots, w_{\lambda_i}$ be a fixed filling of the row directly underneath r. Let $a_1, \ldots, a_{\lambda_{i+1}}$ be positive integers, with multiplicities $m_1, \ldots, m_k$. Then*

$$\sum q^{\mathrm{inv}_w(r)} = \binom{\lambda_{i+1}}{m_1, \ldots, m_k}_q = \frac{(\lambda_{i+1})_q!}{(m_1)_q! \cdots (m_k)_q!}$$

where the sum ranges over all distinct fillings of the row r with the integers $a_1, \ldots, a_{\lambda_{i+1}}$ in some order.

Proof. Multiplying both sides of the relation by $(m_1)_q! \cdots (m_k)_q!$, we wish to show that

$$(m_1)_q! \cdots (m_k)_q! \sum q^{\mathrm{inv}_w(r)} = (\lambda_{i+1})_q!.$$

This follows immediately by interpreting $(\lambda_{i+1})_q!$ and each $(m_i)_q!$ as in Lemma 5.1.2, and assigning all possible orderings to the repeated elements and counting the total number of relative inversions in each case. □

We are now ready to prove Equation 5.1.

Theorem 5.1.4. *We have*

$$\sum_{\substack{\sigma:\mu \to \mathbb{Z}_+ \\ |\sigma| = \alpha}} q^{\mathrm{maj}(\sigma)} = \sum_{\substack{\rho:\mu^* \to \mathbb{Z}_+ \\ |\rho| = \alpha}} q^{\mathrm{inv}(\rho)}.$$

Proof. We break down each sum according to the contents of the columns of μ and the rows of μ^*, respectively. For a given multiset of contents of the columns, where the entries in the i-th column have multiplicities $m_1^{(i)}, \ldots, m_{k_i}^{(i)}$, we have that

$$\sum_{\sigma} q^{\mathrm{maj}(\sigma)} = \prod_i \binom{\mu_i'}{m_1^{(i)}, \ldots, m_{k_i}^{(i)}}_q,$$

where the sum ranges over all fillings σ with the given column entries. By Lemma 5.1.3, we have that the corresponding sum over fillings ρ with the given contents in the rows of μ^* is the same:

$$\sum_{\rho} q^{\mathrm{inv}(\rho)} = \prod_i \binom{\mu_i'}{m_1^{(i)}, \ldots, m_{k_i}^{(i)}}_q.$$

Summing over all possible choices of the entries from α for each column of μ, the result follows. □

5.2 Hook Shapes

We now demonstrate a bijective proof of Problem 1.0.1 in the case that μ is a ***hook shape***, that is, $\mu = (m, 1, 1, 1, \ldots, 1)$ for some m. There is a known combinatorial formula for the q, t-Kostka poloynomials in the case of hook shapes μ given by Stembridge [36], but it does not involve the inv and maj statistics.

The symmetry of inv and maj was demonstrated for fillings of hook shapes having *distinct* entries in [8], and makes use of the Foata bijection. In this section, we instead use the Carlitz bijection to prove the result, which will hold for arbitrary fillings by the results in Section 3.2.

Lemma 5.2.1. *We have the following two facts about one-column and one-row shapes respectively.*

- *Given a filling σ of a one-column shape, suppose $A = a_1 \geq \cdots \geq a_n$ is the alphabet of its entries written in the standardization order as in Proposition 3.2.7, from greatest to least. Then if a_i is the bottommost entry in σ, then the first* 0 *in* majcode(σ) *is in position i from the left.*

- *Given a filling ρ of a one-row shape, suppose $A = a_1 \leq \cdots \leq a_n$ is the alphabet of its entries written in order with ties broken in reading order. Then if a_i is the leftmost entry in σ, then the first* 0 *in* invcode(σ) *is in position i from the left.*

Proof. For the filling σ of a one-column shape, recall that we define majcode by removing the entries one at a time from greatest to least in standardization order. The only time the difference in major index is 0 is when the entry is on the bottom, and so the first time this occurs is when we remove the bottommost entry a_i from the filling (i.e. at the i-th step).

For the filling ρ of a one-row shape, note that the leftmost entry a_i always has an inversion code number of 0. Moreover, if any entry b to its right also has an inversion code number of 0, then $b \geq a_i$ for otherwise it would be the smaller entry of an inversion (with a_i itself). It follows that a_i is the smallest entry whose inversion code number is 0. $\square$

We now define a map from fillings of hook shapes to *pairs* of partial codes that we call *hook codes*.

Definition 5.2.2. Let σ be a filling of a hook shape μ. We define the ***hook codes*** of σ to be the pair of codes consisting of the invcode of its bottom row and the majcode of its leftmost column, along with the data of which entries occur in the row and which occur in the column.

Notice that, by the standardization orderings on the row and column of μ as defined in Section 3.2, if the corner square in μ is one of the repeated letters a of the filling, then it is considered the largest a in its column and the smallest a in its row.

Thus we can define a standardization ordering on fillings of hook shapes: we order the letters from smallest to largest, with the following tie-breaking rules.

- If two copies of the letter a appear in the left column, the tie is broken as in Section 3.2.
- If they appear in the bottom row, then the leftmost a comes first.
- If one appears in the column and the other in the row, the one in the column comes first.

This enables us to represent hook codes visually, as shown in the following example.

Example 5.2.3. Consider the filling σ of a hook shape shown below. The 2 in the corner is considered to be greater than the 2 above it and less than the 2 to its right. To represent the hook code of σ, we write the entries of the filling in the standardization ordering, and write the invcode and (the reverse of) majcode of the bottom row and left column respectively underneath the corresponding letters.

$$\begin{ytableau} 4 \\ 2 \\ 1 \\ 2 & 5 & 3 & 2 & 5 \end{ytableau}$$

	1	2	2	2	3	4	5	5
invcode			0	2	1		0	0
majcode	0	1	0			2		

Notice that the majcode is written *backwards*, because the entries are in increasing order.

We now characterize the pairs of codes that correspond to fillings of hook shapes.

Lemma 5.2.4. *Let μ be a hook shape of height h and width l with $h+l-1=n$, and let $A=\{a_1 \leq \cdots \leq a_n\}$ be an ordered multiset. A pair of partial codes (X,Y) of lengths l and h respectively is a hook code of some filling σ of μ if and only if the four conditions below are satisfied.*

1. *The leftmost 0 of X matches the rightmost 0 of Y.*
2. *The two codes do not overlap in any other position, and every position is part of at least one of the two codes.*
3. *The code X is an element of C_l and is A-weakly increasing, where we restrict A to the l letters corresponding to the positions of the entries of X.*
4. *The code Y, when read backwards, is an element of C_h and is A-weakly increasing, where we restrict A to the h letters corresponding to the positions of the entries of Y.*

Proof. First we show that the hook code of any filling σ of μ satisfies the four conditions. Condition 1 follows immediately from Lemma 5.2.1, because the major index code is written in reverse order. Condition 2 is clear since every entry is in either the row or the column and only the corner square is in both. Conditions 3 and 4 follow immediately from the definition of hook codes.

Now, suppose we have a pair of codes satisfying Conditions 1–4. Then there is a unique way to form a row and a column of entries based on their elements, since they are both valid Carlitz codes and are A-weakly increasing by Conditions 3 and 4. Because of Condition 1 and Lemma 5.2.1, the leftmost entry of the row is the same as the bottommost entry of the column, and so we can put them together to form a filling σ of a hook shape. Because of Condition 2, the hook shape μ has the appropriate size and shape, and we are done. $\square$

Using Lemma 5.2.4, we can now define our bijection.

Definition 5.2.5. For any hook shape μ and content α, let $\phi : \mathcal{F}_\mu^\alpha \to \mathcal{F}_{\mu^*}^{r(\alpha)}$ be the map defined by interchanging the pair of hook codes of a given filling and writing them backwards, and also reversing its alphabet.

Example 5.2.6. Starting with the tableau in Example 5.2.3, if we reverse the alphabet, interchange invcode and majcode, and write the codes in backwards order, then we obtain the filling and pair of codes below. It follows that the filling in Example 5.2.3 maps to the filling below under ϕ.

$$\begin{array}{|c|c|c|c|}
\hline
4 \\
\cline{1-1}
3 \\
\cline{1-1}
1 \\
\cline{1-1}
1 \\
\hline
4 & 5 & 2 & 4 \\
\hline
\end{array}$$

	1	1	2	3	4	4	4	5
invcode			2			0	1	0
majcode	0	0		1	2	0		

Theorem 5.2.7. *We have that*

$$\operatorname{maj}(\phi(\sigma)) = \operatorname{inv}(\sigma)$$

and

$$\operatorname{inv}(\phi(\sigma)) = \operatorname{maj}(\sigma)$$

for any filling σ of a given hook shape μ. Moreover, ϕ is a bijection from $\mathcal{F}_\mu^\alpha$ to $\mathcal{F}_{\mu^*}^{r(\alpha)}$ for any content α.

Proof. Clearly ϕ interchanges inv and maj, since it interchanges the invcode and majcode of the filling. To show it is a well-defined map into fillings of the conjugate shape, note that reversing and interchanging the codes and reversing the alphabet results in a pair of codes that satisfy conditions 1-4 of Lemma 5.2.4.

Finally, ϕ is a bijection - in fact, it is an involution - because the operations of reversing the alphabet, interchanging the pair of codes, and writing the codes in the reverse order are all involutions. $\square$

Corollary 5.2.8. *The map ϕ above satisfies the conditions of the Problem 1.0.1, proving combinatorially that*

$$\widetilde{H}_\mu(x;q,t) = \widetilde{H}_{\mu^*}(x;t,q)$$

when μ is a hook shape.

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