Title
On the k-Schur Positivity of k-Bandwidth LLT Polynomials

Permalink
https://escholarship.org/uc/item/26x8z6dz

Author
Miller, Christopher Roman

Publication Date
2019

Peer reviewed|Thesis/dissertation

On the k-Schur Positivity of k-Bandwidth LLT Polynomials

by

Christopher R. Miller

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Mark Haiman, Chair
Professor Mary Gaillard
Professor Denis Auroux

Spring 2019

On the k-Schur Positivity of k-Bandwidth LLT Polynomials

Abstract

On the k-Schur Positivity of k-Bandwidth LLT Polynomials

by

Christopher R. Miller

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Mark Haiman, Chair

Both LLT polynomials and k-Schur functions were derived from the study of Macdonald polynomials, and have proved to be fruitful areas of study. A well-known conjecture due to Haglund and Haiman states that k-bandwidth LLT polynomials expand positively into k-Schur functions. This is trivial in the case $k = 1$ and has been recently proved for $k = 2$. In this work, we present a proof for the case $k = 3$. In doing so, we introduce a new method for establishing linear relations among LLT polynomials.

Contents

Acknowledgments

This work is the result of years of meetings with my advisor, Mark Haiman. So first and foremost, I would like to acknowledge Mark for his unending enthusiasm, patience, and mentorship. Essentially every conjecture I have worked on in Berkeley has been sourced by him. Without his guidance, I would have never known that these problems existed, let alone that I could solve them.

I am also thankful for the mentorship I received from mathematicians around the world: Maria Gillespie, Mateusz Michałek, Jarek Buczyński, Kaie Kubjas, Alexander Engström, Lauren Williams, Anne Schilling, Sylvie Corteel. These are the people who have worked with me, published with me, advised me and invited me to seminars. They make the big world of mathematics a friendlier place.

I have benefitted from discussions with many graduate students over the years, far too many to name. But I am especially thankful for Jeremy Meza, who has long been my closest collaborator. He has been there to hear my very best ideas, and my very worst. Wherever he is, he brings optimism, energy and passion to his work.

I am further indebted to my roommates and officemates: Rockford Sison, Wish Sison, Noble Macfarlane, Maxim Wimberley, Eric Hallman, Alexander Carney, Delanie Lowe, Nicholas Ryder, Christian Chavez, Elizabeth Ferme, Hero Ashman, Alexander Bertoloni Meli, Lisa Nguyen. These are the people who have made the worst days of mathematical malaise not just tolerable, but beautiful.

Lastly, I must acknowledge my family, without whom I would be lost: Betty, Dave, Mike, Fanny, Natalie, Rob, Renee, Adriana and Pablo.

Chapter 1

Introduction and Background

1.1 Introduction

An open problem in the study k-Schur functions is to prove that k-bandwidth LLT polynomials expand positively into k-Schur functions. This conjecture is sometimes known as the Haglund-Haiman conjecture. Since Macdonald polynomials indexed by k-bounded partitions are known to expand positively into k-bandwidth LLT polynomials [1], a proof of the Haglund-Hamain conjecture is sufficient to prove the k-Schur positivity of Macdonald polynomials. This conjecture has been verified combinatorially in the case that $k = 1$ [2] and $k = 2$ [3]. Here we present a combinatorial proof in the case that $k = 3$.

After introducing the relevant background, this work has three parts. In the first part, we provide a new method (Theorem 2.2.1, Lemma 2.3.1) for proving identities among LLT polynomials. A common sort of decomposition will be

$$\mathcal{G}_{\vec{\lambda}}(X; q) = q^m \mathcal{G}_{\vec{\mu}}(X; q) + q^n \mathcal{G}_{\vec{\nu}}(X; q) \tag{1.1}$$

where $\vec{\mu}$ and $\vec{\nu}$ are derived from $\vec{\lambda}$, and m, n are integers. Each relation in fact describes infinitely many equalities of LLT polynomials, and for this reason we call them LLT equivalence relations. We verify one LLT equivalence relation for every ordered pair of 3-bandwidth skew partitions, and present them in appendix A.

In the second part, we use these identities of LLT polynomials to prove that 3-bandwidth LLT polynomials can be written as $\mathbb{N}[q, q^{-1}]$-linear combination of LLT polynomials for which there is no possible decomposition (Theorem 3.1.9). For this reason, we call these special LLT polynomials 3-indecomposables. This part relies heavily on the equations derived in the previous section, and is essentially a collection of proofs by induction.

Finally, the third part of this work show that these 3-indecomposables are 3-Schur functions (Lemma 3.3.1). This proof come in two parts. First, we use a theorem of Haiman and Grojnowski to show that these 3-indecomposable LLT polynomials are generalized Hall-Littlewood polynomials (Lemma 3.2.1). Then a theorem of Lam and Morse realizes these particular generalized Hall-Littlewood polynomials as 3-Schur functions (Lemma 3.3.1).

Together, this proves that 3-bandwidth LLT polynomials are $\mathbb{N}[q, q^{-1}]$-linear combinations of 3-Schur functions. Since 3-Schur functions have a monic term, this is sufficient to prove k-Schur positivity.

1.2 Partitions and Tableaux

A partition of $n \in \mathbb{N}$ is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\sum_{i=1}^{\infty} \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1}$ for all i. We will write $\lambda \vdash n$ to denote a partition of n. Since a partition necessarily has only a finite number of nonzero entries, we often use the notation $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where λ_ℓ is the last nonzero term. The number $\ell = \ell(\lambda)$ is sometimes called the length of the partition. We will interchangeably use the cell-wise notation, which defines a partition λ as a set $\{(i, j) \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$. As the name suggests, elements of these sets are called cells.

If λ and μ are a pair of partitions such that $\lambda_i \geq \mu_i$ for all i, then we define the skew partition λ/μ as the set difference $\lambda \smallsetminus \mu = \{(i, j) \mid 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i\}$. Given a cell $x = (i, j) \in \lambda$, the *content* of the cell is an integer $c(x) = j - i$. Cells with the same content are said to fall on the same *content line*.

A Young tableau of shape λ is a function $T : \lambda/\mu \to \mathbb{N}$ labeling the cells that is increasing along columns, and weakly increasing along rows, i.e. $T(i, j) < T(i + 1, j)$ and $T(i, j) \leq T(i, j + 1)$ for all i, j. This is known as the *semistandard condition*, and we will also refer to these tableaux as *semistandard tableaux*. The *weight* of a semistandard tableau $w(T)$ is a sequence $(w_1(T), w_2(T), \ldots)$ such that $w_i(T)$ is the multiplicity of i in T. If $w(T) = (1, 1, \ldots, 1, 0 \ldots)$ then T is said to be a *standard tableau*. Semistandard tabelaux of shape λ and weight μ comprise a set $\text{SSYT}(\lambda, \mu)$, and such standard tableaux are denoted $\text{SYT}(\lambda, \mu)$. We will often make reference to the sets $\text{SSYT}(\lambda) = \cup_\mu \text{SSYT}(\lambda, \mu)$ and $\text{SYT}(\lambda) = \cup_\mu \text{SYT}(\lambda, \mu)$.

Given a semistandard tableau T, the monomial x^T is defined to be $x_1^{w_1(T)} x_2^{w_2(T)} \cdots$, and the Schur function $s_\lambda(X) = s_\lambda(x_1, x_2, \ldots)$ by

$$s_\lambda(X) = \sum_{T \in \text{SSYT}(\lambda)} x^T \tag{1.2}$$

Allowing Λ to denote the ring of symmetric functions in the alphabet $X = x_1 + x_2 + \cdots$, it is a classical theorem that $s_\lambda(X)$ is a symmetric function [4].

1.3 d-cores and Ribbon Tableaux

A d-*ribbon* is a connected skew partition that contains no 2×2 box, and has d many cells. Given a cell $x \in \lambda$, the *hook* of $x = (a, b)$ counts the number of cells above x or to to its right, including x. In notation, $\text{hook}(x) = \text{hook}(a, b) = |\{(i, b) \in \lambda \mid i \geq a\}| + |\{(a, j) \in \lambda \mid j > b\}|$.

A partition λ is said to be a *d-core* if there does not exist μ such that λ/μ is a d-ribbon. Equivalently, λ is a d-core if there does not exist $x \in \lambda$ such that d divides hook(x) [5].
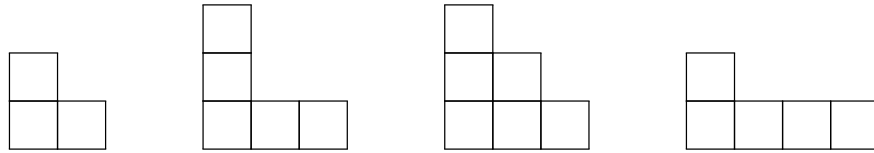
Figure 1.1: Some examples of 4-cores

Given a partition λ, we can consider a sequence $\{\lambda^{(i)}\}_{i=1}$ of partitions produced by repeatedly removing d-ribbons. In other words, $\lambda^{(1)} = \lambda$ and $\lambda^{(i)}/\lambda^{(i+1)}$ is well defined and a d-ribbon for all i. Since $\lambda^{(i+1)}$ has strictly fewer cells than $\lambda^{(i)}$, this process terminates with a partition from which no d-ribbon can be removed; by definition a d-core. So we can rewrite the sequence as $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(D)}$, where $\lambda^{(D)}$ is a d-core. It is a fundamental theorem that $\lambda^{(D)}$ is independent of the sequence of ribbons removed [5]. This justifies defining a function $\mathrm{core_d}(\lambda) = \lambda^{(D)}$ which returns the unique d-core that results from removing d-ribbons from λ. In Figure 1.2 we see an example of stripping away ribbons to produce a d-core, with $\lambda = (6, 6, 6, 5)$ and $\mathrm{core_d}(\lambda) = (2, 1)$.
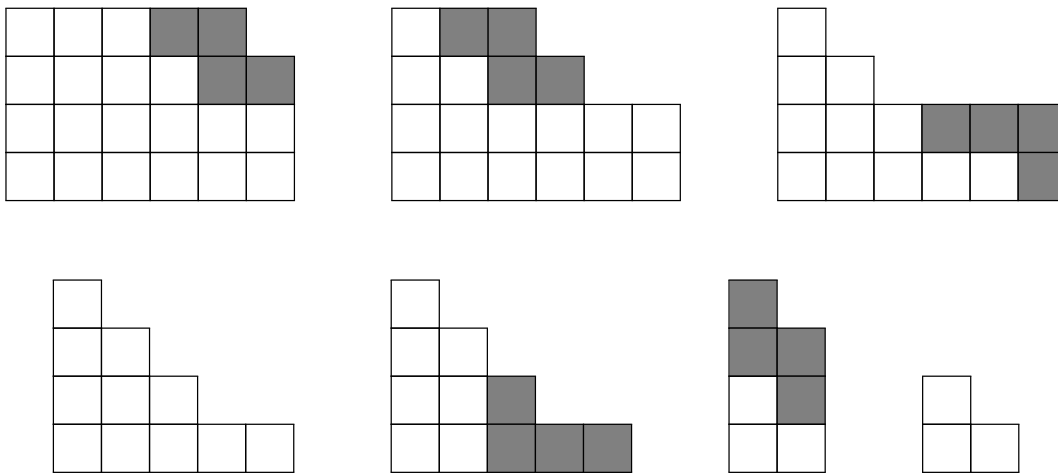
Figure 1.2: A sequence or partitions produced by removing ribbons.

Suppose that λ and μ are a pair of partitions such that there exists a sequence $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(P)})$ where $\lambda^{(i)}/\lambda^{(i+1)}$ is a d-ribbon for all i and $\lambda^{(P)} = \mu$. Then λ/μ is said to be *d-tileable*. It is a fact that λ/μ is d-tileable if and only if $\mathrm{core_d}(\lambda) = \mathrm{core_d}(\mu)$.

The *head* of a ribbon is the cell with greatest content (i.e. the bottom-right) and the *tail* is the cell with least content (i.e. top-left). If a skew shape λ/μ is tileable by d-ribbons such that the head of each ribbon is directly above no other cell, then λ/μ is said to be a *horizontal d-ribbon strip*. A *semistandard d-ribbon tableau of shape λ/μ and weight ν* is

a function $T : \lambda/\mu \to \mathbb{N}$ such that the cells labeled i form a horizontal d-ribbon strip for all i, and T is nondecreasing along columns and rows. The *weight* of a d-ribbon tableau is a sequence $w(T) = (w_1(T), w_2(T), \ldots)$ such that $w_i(T) = |\{x \in \lambda/\mu \mid T(x) = i\}|/d$. We denote the set of all d-ribbon tableaux of shape λ/μ and weight ν by $\mathrm{Tab}_d(\lambda/\mu, \nu)$, and the set of all d ribbon tableaux of shape λ/μ by $\mathrm{Tab}_d(\lambda/\mu)$. We note that when $d = 1$, the definition of d-ribbon tableau is the same as for a semistandard Young Tableau.
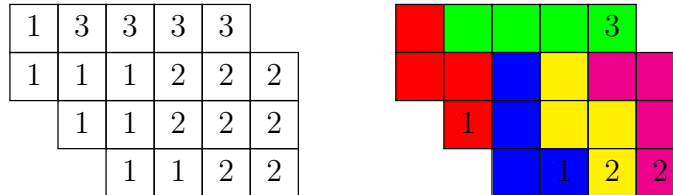

Figure 1.3: A ribbon tableau drawn two different ways

Figure 1.3 is an example of the above definitions. It is a 4-ribbon tableau of shape λ/μ where $\lambda = (6, 6, 6, 5)$ and $\mu = (2, 1)$. We know immediately that λ/μ is 4-tileable, because $\mathrm{core}_4(\lambda) = \mathrm{core}_4(\mu) = (2, 1)$, as was shown in Figure 1.2. Figure 1.3 displays a ribbon tableau of weight $(2, 2, 1)$.

It is a fact that every horizontal d-ribbon tableau can be tiled by d-ribbons in exactly one way that respects the horizontal strip condition [6, Definition 4]. Thus a semistandard d-ribbon tableau comes naturally tiled by d-ribbons. We will abuse notation and write $R \in T$ to denote a ribbon within a semistandard d-ribbon tableau.

Given a d-tileable skew partition λ/μ and a cell $x \in \lambda/\mu$, we define the adjusted content $\tilde{c}_d(x)$ to be the unique integer such that $c(x) = \tilde{c}_d(x) \cdot d + r$ where $1 \le r \le d$. When it is unambiguous, we will write $\tilde{c}(x)$ in place of $\tilde{c}_d(x)$.

1.4   Multiskew Partitions and d-Quotients

A *multiskew partition* is a finite sequence of skew partitions. The *length* of a multiskew partition is the number of entries. Given two multiskew partitions with the same length $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ and $\vec{\mu} = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(d)})$ whose entries are all straight shapes such that $\mu^{(i)} \subseteq \lambda^{(i)}$, we will sometimes write $\vec{\lambda}/\vec{\mu}$ to denote the sequence of skew partitions $(\lambda^{(1)}/\mu^{(1)}, \lambda^{(2)}/\mu^{(2)}, \ldots, \lambda^{(d)}/\mu^{(d)})$. We will always use superscripts to index the entries of a multiskew partition to distinguish them from the parts of a partition. We will interchange-ably treat $\vec{\lambda}$ as a sequence of skew partitions, and as a disjoint union of cells $\vec{\lambda} = \sqcup_i \lambda^{(i)}$

A *semistandard multiskew tableau of shape $\vec{\lambda}$ and weight ν* is a function $\vec{T} : \sqcup_i \lambda^{(i)} \to \mathbb{N}$ such that the restriction to each skew partition $T_{\lambda^{(i)}} : \lambda^{(i)} \to \mathbb{N}$ is a semistandard Young tableau. The weight of a a multiskew tableau is the sum of the weights of each restriction: $w(\vec{T}) = \sum_i w(T_{\lambda^{(i)}})$. We will denote the set of multiskew tableau of shape $\vec{\lambda}$ and weight μ by $\mathrm{SSYT}(\vec{\lambda}, \mu)$, and the set of all multiskew tableau with fixed shape by $\mathrm{SSYT}(\vec{\lambda})$.

There is an important relationship between d-tileable partitions and multiskew partitions of length d. Further, this relationship descends to the level of tableaux, and asserts a correspondence between semistandard d-ribbon tableaux and semistandard multiskew tableau of a fixed shape and weight. We will call this map quot_d, and describe it now.

The map quot_d was first given by Stanton and White [7] as a bijection $\text{quot}_d : \text{Tab}_d(\lambda/\mu) \to \text{SSYT}(\vec{\beta})$ such that for every weight ν, the restriction $\text{quot}_d |_\nu : \text{Tab}_d(\lambda/\mu, \nu) \to \text{SSYT}(\vec{\beta}, \nu)$ is bijective.

We will define quot_d on the level of tableaux, where it is more intuitive. Let $S \in \text{Tab}_d(\lambda/\mu, \nu)$ be a d-ribbon tableau. Then to each ribbon $R \in S$, we can assign the content of its head $c(h_R)$, which decomposes as $c(h_R) = \tilde{c}(h_R) \cdot d + i_R$ where $1 \leq i_R \leq d$. So for each ribbon $R \in S$, we collect the following tuple of information: $(i_R, \tilde{c}(h_R), S(h_R))$. We simultaneously produce $\vec{\beta}$ and $T = \text{quot}_d(S) \in \text{SSYT}(\vec{\beta}, \nu)$. The pair $(\vec{\beta}, T)$ is defined by the following properties.

- $\beta^{(i)}$ has one cell for each ribbon $R \in T$ such that $i_R = i$.

- For each such ribbon $R \in T$, the corresponding cell $x \in \beta$ satisfies $c(x) = \tilde{c}(h_R)$ and $T(x) = S(h_R)$.

This defines a tableau as a collection of cells with assigned content and weight. If λ/μ is connected, then $\beta^{(i)}$ is unique for all i, up to diagonal translation within the plane. It is a theorem that the multiskew partition $\vec{\beta}$ depends only on λ/μ, and not on the particular ribbon tableau S.

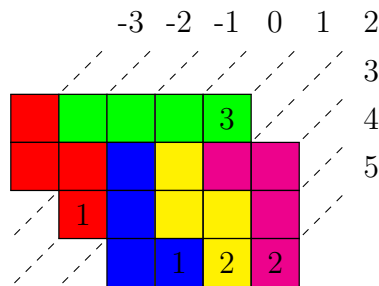We end this section with an example of the map quot_d seen in Figure 1.4.

Figure 1.4: A ribbon tableau drawn with content.

Here $\lambda = (6, 6, 6, 5)$ and $\mu = (2, 1)$, We will see that $\vec{\beta} = ((2,2)/(1,1), \emptyset, (1)/(0), (2,2)/(2))$. The weight ν is $(2, 2, 1)$. Since the d-ribbon tableau T has 5 ribbons, there are 5 tuples to consider:

$$(i_R, \tilde{c}(h_R), T(h_R)) = (1, 0, 3), (1, 1, 2), (3, 0, 1), (4, -1, 1), (4, 0, 2)$$

We can read off the following:

- $\beta^{(1)}$ has two cells x_1, x_2 such that $c(x_1) = 0$, $T(x_1) = 3$, $c(x_2) = 1$, and $T(x_2) = 2$.

- $\beta^{(2)}$ has no cells.

- $\beta^{(3)}$ has one cell x_3 such that $c(x_3) = 0$ and weight $T(x_3) = 1$.

- $\beta^{(4)}$ has two cells x_4, x_5 such that $c(x_4) = -1$, $T(x_4) = 1$, $c(x_5) = 0$, and $T(x_5) = 2$.

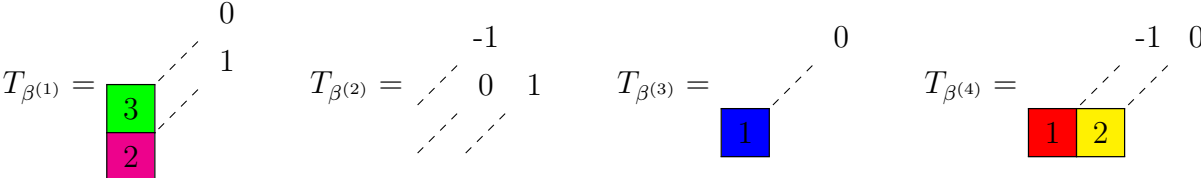There is a unique multiskew tableau that meets these conditions, it is displayed in Figure 1.5.

Figure 1.5: An example of $\text{quot}_\text{d}(\lambda/\mu) = \vec{\beta}$.

We often prefer to write multiskew tableaux and or partitions within a single plane, as in Figure 1.6. This notation will be especially useful when we define new variant LLT polynomials in the next section.
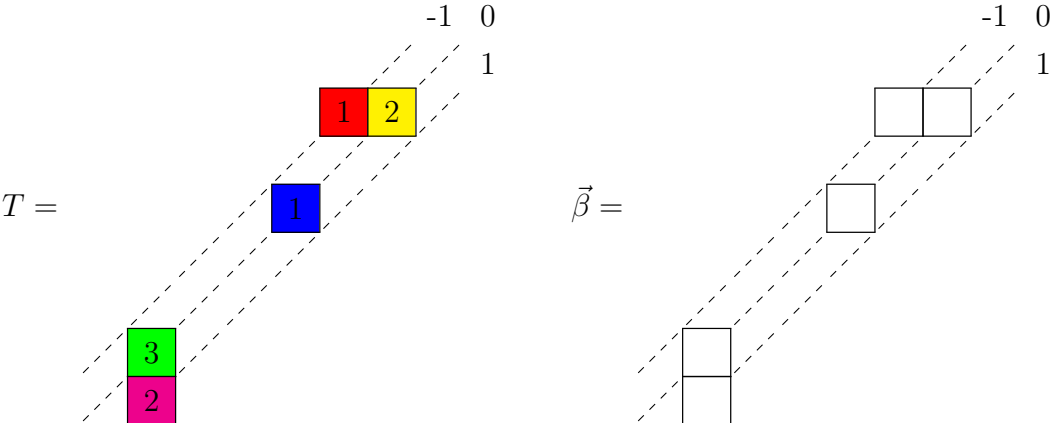
Figure 1.6: An alternative way to draw T and $\vec{\beta}$ in a single plane.

1.5 LLT Polynomials

Lascoux, Leclerc and Thibon introduced what are now called LLT polynomials in 1997 [8], where they were defined in terms of ribbon tableaux. LLT polynomials were famously used to expand Macdonald polynomials [1], and are themselves known to be Schur positive [9] [10]. Every LLT polynomial is a q-analog of a product of Schur functions.

We will introduce two definitions of LLT polynomials, and show they are essentially equivalent. The first definition is given by a sum over d-ribbon tableaux, the second by a sum over multiskew tableaux.

We define the *spin* of a ribbon R to be $S(R) = \frac{\text{height}(R)-1}{2}$, where height$(R)$ is the number of rows occupied by the ribbon. We can extend this definition to a d-ribbon tableau $T \in \text{Tab}_d(\lambda/\mu)$ by $S(T) = \sum_{R \in T} S(R)$. For example, the ribbon tableau in Figure 1.3 has spin 4.

Using spin, we can define the *combinatorial LLT polynomials* [10]:

$$G_{\lambda/\mu}^{(d)}(X; u) = \sum_{T \in \text{Tab}_d(\lambda/\mu)} u^{2 \cdot S(T)} x^T \tag{1.3}$$

In fact these are not quite the spin symmetric functions defined by Lascoux, Leclerc and Thibon, who have the same formula but substitute $q = u^2$ [8].

There is another way to describe LLT polynomials in terms of multiskew partitions. Let $\vec{\beta} = (\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(d)})$ be a multiskew partition, and let $T \in \text{SSYT}(\vec{\beta})$. Then we count *inversions* by the number of pairs of cells $(x, y) \in \beta^{(i)} \times \beta^{(j)}$ where $T(x) > T(y)$ and either

- $c(x) = c(y) - 1$ and $i > j$ or

- $c(x) = c(y)$ and $i < j$.

We say that the values $(T(x), T(y))$ form an *inversion*, and that the cells x, y form an *attacking pair*. We denote the number of inversions in a multiskew tableau T by inv(T). In Figure, 1.6 the multiskew tableau T has inv$(T) = 2$. Inversion counting can be reformulated by considering the *content reading word*. The content reading word of a multiskew tableau T is formed by writing down the entries of T by reading to the northeast along content lines, leaving entries for blanks. For example Figure 1.6 has content reading word $w = (\_, \_, \_, 1, 3, \_, 1, 2, 2, \_, \_, \_)$. Since T is a 4-tuple, we count inversions by looking for inversions in w such that the indices of the entries differ by less than 4. In w, this leaves us with the pairs $(3, 1)$ and $(3, 2)$, which are the same pairs that appear in the standard formulation.

We use inv to define another type of LLT polynomials. Haiman and Grojnowski call these the *new variant LLT polynomials* [10]:

$$\mathcal{G}_{\vec{\beta}}(X; q) = \sum_{T \in \text{SSYT}(\vec{\beta})} q^{\text{inv}(T)} x^T. \tag{1.4}$$

It is worth noting that setting $q = 1$ results in a product of Schur functions: $\mathcal{G}_{\vec{\beta}}(X; 1) = \prod_{i=1}^{d} s_{\lambda^{(i)}}$. This is the easiest way to see that LLT polynomials are a q-analog of a product of Schur functions.

The combinatorial LLT polynomials and the new variant LLT polynomials are related by the map quot$_d : \text{Tab}_d(\lambda/\mu) \to \text{SSYT}(\vec{\beta})$ defined in Section 1.4. On the level of tableaux, quot$_d$ satisfies the following.

$$S^*(\lambda/\mu) - S(T) = \text{inv}(\text{quot}_{\text{d}}(T)) - \text{inv}_*(\vec{\beta}) \tag{1.5}$$

where $S^*(\lambda/\mu) = \max\{S(T) \mid T \in \text{Tab}_d(\lambda/\mu)\}$ and $\text{inv}_*(\vec{\beta}) = \min\{\text{inv}(T) \mid T \in \text{SSYT}(\vec{\beta})\}$ [11]. We derive that

$$
\begin{aligned}
G_{\lambda/\mu}^{(d)}(X;u) &= \sum_{T \in \text{Tab}_d(\lambda/\mu)} u^{2 \cdot S(T)} x^T \\
&= \sum_{T \in \text{Tab}_d(\lambda/\mu)} q^{S(T)} x^T \\
&= \sum_{T \in \text{Tab}_d(\lambda/\mu)} q^{S^*(\lambda/\mu) - \text{inv}(\text{quot}_{\text{d}}(T)) + \text{inv}_*(\vec{\beta})} x^T \\
&= q^c \sum_{T \in \text{Tab}_d(\lambda/\mu)} q^{-\text{inv}(\text{quot}_{\text{d}}(T))} x^T \\
&= q^c \sum_{T \in \text{SSYT}(\vec{\beta})} q^{-\text{inv}(T)} x^T \\
&= q^c \mathcal{G}_{\vec{\beta}}(X;q^{-1})
\end{aligned}
\tag{1.6}
$$

where $c = S^*(\lambda/\mu) + \text{inv}_*(\vec{\beta})$. Rearranging, we see that

$$\mathcal{G}_{\vec{\beta}}(X;q) = u^e G_{\lambda/\mu}^{(d)}(X;u^{-1}) \tag{1.7}$$

where $q = u^2$ and $e = 2 \cdot S^*(\lambda/\mu) + 2 \cdot \text{inv}_*(\text{quot}_{\text{d}}(\lambda/\mu))$ depends only on the shape λ/μ. There is an alternative formulation of the inversion statistic due to Schilling, Shimozono, and White. Their statistic inv' satisfies $\text{inv}'(\text{quot}_{\text{d}}(T)) = S^*(T) - S(T)$, which can be useful in some settings [12]. Nonetheless, we will use inv as defined in this thesis originally.

We end this section with an example. Consider the d-tileable skew partition in λ/μ in Figure 1.3 and the multiskew partition $\vec{\beta}$ in Figure 1.6. We calculate

- $G_{\lambda/\mu}^{(d)}(X;u) = u^2 s_{(2,1,1,1)} + u^4 s_{(2,2,1)} + (u^6 + u^4) s_{(3,1,1)} + u^6 s_{(3,2)} + u^8 s_{(4,1)}$

- $\mathcal{G}_{\vec{\beta}}(X;q) = q^5 s_{(2,1,1,1)} + q^4 s_{(2,2,1)} + (q^4 + q^3) s_{(3,1,1)} + q^3 s_{(3,2)} + q^2 s_{(4,1)}$

and note that $u^{12} G_{\lambda/\mu}^{(d)}(X;u^{-1}) = \mathcal{G}_{\vec{\beta}}(X;q)$.

1.6 k-Schur functions

In 2000, the study of q,t-Kostka polynomials led Lapointe, Lascoux, and Morse to the conjecture that a filtration of the space Λ_t of graded symmetric functions can be used to derive a refinement of Macdonald Positivity [13]. This filtration is given by $\Lambda_t^{(1)} \subset \Lambda_t^{(2)} \subset \cdots \subset \Lambda_t$,

where $\Lambda_t^{(k)}$ is the span of $\{H_\lambda(X;t)\}_{\lambda \in \mathcal{P}_k}$, $H_\lambda(X;t)$ is a Hall-Littlewood polynomial, and $\mathcal{P}_k = \{\lambda | \lambda_1 \le k\}$, the set of *k-bounded partitions*. The space $\Lambda_t^{(k)}$ can also be identified as the span of $\{s_\lambda[\frac{x}{1-q}]\}_{\lambda \in \mathcal{P}_k}$

For each space $\Lambda_t^{(k)}$ they conjectured that there is a basis $\{s_\lambda^{(k)}(X;t)\}_{\lambda_1 \le k}$ satisfying the following.

1. $H_\lambda(X;q;t) = \sum\limits_{\mu \in \mathcal{P}_k} K_{\mu\lambda}^{(k)}(q;t)s_\mu^{(k)}(X;t)$ where $K_{\mu\lambda}^{(k)}(q;t) \in \mathbb{N}[q;t]$ and $\lambda \in \mathcal{P}_k$

2. $s_\lambda^{(k)}(X;t) = \sum\limits_{\mu \in \mathcal{P}_{k+1}} b_{\lambda,\mu}^{(k,k+1)}(t)s_\mu^{(k+1)}(X;t)$ where $b_{\lambda,\mu}^{(k,k+1)} \in \mathbb{N}[t]$ and $\lambda \in \mathcal{P}_k$

The first statement claims that Macdonald polynomials can be expanded positively in these bases. The second holds that each basis vector $s_\lambda^{(k)}(X;t)$ expands positively in the subsequent basis $\{s_\lambda^{(k+1)}\}_{\lambda \in \mathcal{P}_{k+1}}$. These conjectured symmetric functions $s_\lambda^{(k)}(X;t)$ are called *k*-Schur functions.

At current, there are four conjecturally equivalent definitions of *k*-Schur functions. Each has its own merits, and we recommend that the interested reader look to [5] for more information. For all definitions, it is known that $s_\lambda^{(k)}(X;t) = s_\lambda(X)$ when $k > |\lambda|$. If the *k*-branching rule holds (Property 2), then iterative application shows that all *k*-Schur functions are Schur positive, with coefficients in $\mathbb{N}[t]$. Further, if Macdonald polynomials are *k*-Schur positive (Property 1), then we can conclude Macdonald positivity. It is unsurprising, then, that *k*-Schur functions are a very active area of study.

We will use a symmetric operator definition, as in [13][14]. Given a *k*-bounded partition λ, the *k*-Schur functions $s_\lambda^{(k)}(X;t)$ can be defined recursively as follows

$$s_\lambda^{(k)}(X;t) = T_{\lambda_1} B_{\lambda_1} s_{(\lambda_2,\lambda_3,\dots)}^{(k)}(X;t) \text{ where } s_{()}^{(k)}(X;t) = 1. \tag{1.8}$$

We must define the operators T_n and B_n to make sense of this. The operator B_n appears in the construction of the Hall-Littlewood symmetric functions, and is called a *vertex operator*. It is defined by

$$B_n = \sum_{i=0} s_{i+n}[X]s_i[X(t-1)]^\perp \tag{1.9}$$

where $f^\perp$ is the adjoint of f with respect to the Hall inner product. This definition generalizes to partitions by the following:

$$B_\lambda = \prod_{1 \le i < j \le \ell(\lambda)} (1 - te_{ij})B_{\lambda_1}B_{\lambda_2} \cdots B_{\lambda_\ell} \tag{1.10}$$

where $e_{ij}B_{\lambda_1} \cdots B_{\lambda_\ell} = B_{\lambda_1} \cdots B_{\lambda_i+1} \cdots B_{\lambda_j-1} \cdots B_{\lambda_\ell}$.

In order to define the operator T_n, we must first make sense of what is called the *k-split basis*. Given a *k*-bounded partition λ, we define the *k*-split of λ to be the sequence of
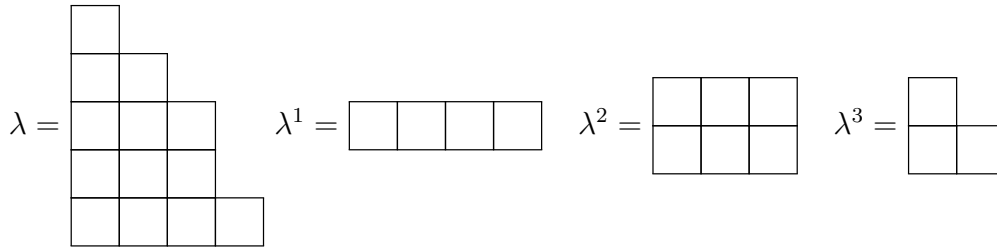
Figure 1.7: A 4 bounded partition and its 4-split

partitions $\lambda^{\to k} = (\lambda^1, \lambda^2, \ldots \lambda^r)$, where the entries concatenate to form λ, and every entry other than λ^r has largest hook equal to k. See Figure 1.7 for an example.

Given a k-bounded partition λ and its k-split $\lambda^{\to k} = (\lambda^1, \lambda^2, \ldots \lambda^r)$, we can define the *k-split polynomials.*

$$S_\lambda^k = B_{\lambda^1} S_{(\lambda^2, \ldots, \lambda^r)}^k \text{ with } S_{()}^k = 1 \tag{1.11}$$

The set $\{S_\lambda\}_{\lambda_1 \leq k}$ is provably a basis for $\Lambda_t^{(k)}$, and is usually called the k-split basis. Due to this fact, it is sufficient to define the operator T_n on the k-split basis:

$$T_n S_\lambda^k = \begin{cases} S_\lambda^k & \text{if } \lambda_1 = n \\ 0 & \text{if } \lambda_1 \neq n \end{cases} \tag{1.12}$$

We end this section by computing $s_{3,3,2,1}^{(4)}(X;t)$ as a demonstration.

$$
\begin{aligned}
s_{()}^{(4)}(X;t) &= 1 \\
s_{(1)}^{(4)}(X;t) &= T_1 B_1 s_{()}^{(4)}(X;t) \\
&= s_{(1)} \\
s_{(2,1)}^{(4)}(X;t) &= T_2 B_2 s_{(1)}^{(4)}(X;t) \\
&= s_{(2,1)} \\
s_{(3,2,1)}^{(4)}(X;t) &= T_3 B_3 s_{(2,1)}^{(4)}(X;t) \\
&= s_{(3,2,1)} + t s_{(4,2)} \\
s_{(3,3,2,1)}^{(4)}(X;t) &= T_3 B_3 s_{(3,2,1)}^{(4)}(X;t) \\
&= s_{(3,3,2,1)} + t s_{(4,3,1,1)} + t s_{(4,3,2)} \\
&\quad + t^2 s_{(4,4,1)} + t^2 s_{(5,3,1)} + t^3 s_{(5,4)}
\end{aligned}
\tag{1.13}
$$

1.7 Haglund-Haiman Conjecture

In his 2006 ICM talk, Haiman announced a conjecture stating that certain LLT polynomials are k-Schur positive. This conjecture can be stated in two ways, depending on the definition

one uses for LLT polynomials. We will state the conjecture in both settings, and show that they are equivalent.

Conjecture 1.7.1. *Let λ/μ be a skew partition. If there exist $d, k, r \in \mathbb{N}$ such the content of every cell in λ/μ is contained in the interval $[r, r + dk - 1]$, then*

$$u^{-2h} \cdot G^{(d)}_{\lambda/\mu}(X; u) = \sum_{\lambda} K_{\lambda}(q) s^{(k)}_{\lambda}(X; q) \tag{1.14}$$

where $u^2 = q$, $K_{\lambda} \in \mathbb{N}[q]$ and $h = S_*(\lambda/\mu) = \min\{s(T) \mid T \in Tab_d(\lambda/\mu)\}$.

In other words, if the contents of a d-ribbon partition are in an interval of diagonal width dk, then the corresponding combinatorial LLT polynomial is k-Schur positive. The term u^{-2h} appears only to ensure that the coefficients $u^{-2h} \cdot G^{(k)}_{\lambda/\mu}(X; u)\langle x_{\lambda} \rangle$ are polynomials in q, and in particular have no fractional exponents. We will refer to this as the Haglund-Haiman conjecture.

We are interested in an equivalent statement about new variant LLT polynomials.

Conjecture 1.7.2. *Let $\vec{\beta}$ be a multiskew partition. If there exists $k, r \in \mathbb{N}$ such that $c(x) \in [r, k + r - 1]$ for all $x \in \vec{\beta}$ then*

$$\omega \mathcal{G}_{\vec{\beta}}(X; q) = \sum_{\lambda} K_{\lambda}(q) s^{(k)}_{\lambda}(X; q) \tag{1.15}$$

where $K_{\lambda}(q) \in \mathbb{N}[q]$ and ω is the involution on Λ such that $\omega s_{\lambda} = s_{\lambda'}$.

In other words, if there are only k many consecutive contents occupied by the cells of $\vec{\beta}$, then the corresponding new variant LLT polynomial is k-Schur positive, after being acted on by ω. Multiskew partitions satisfying this restriction on content are said to be k-*bandwidth*, as in Figure 1.8. In the remainder of this section, we will show that Conjecture 1.7.1 and Conjecture 1.7.2 are equivalent.
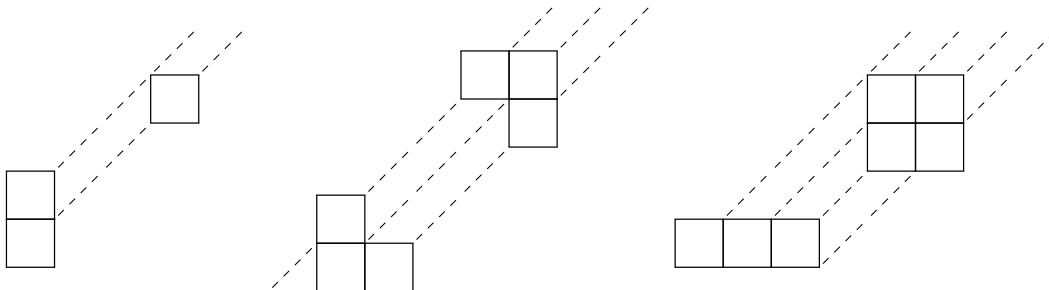

Figure 1.8: 2-bandwidth, 3-bandwidth and 4-bandwidth multiskew partitions.

Lemma 1.7.3. *The Haglund-Haiman conjecture is equivalent to the following: If $\vec{\beta}$ is a k-bandwidth multiskew partition, then there exists $m \in \mathbb{N}$ such that*

$$q^m \cdot \mathcal{G}_{\vec{\beta}}(X; q^{-1}) = \sum_\lambda K_\lambda(q) s_\lambda^{(k)}(q) \tag{1.16}$$

where $K_\lambda(q) \in \mathbb{N}[q]$.

Proof. The equivalence of these two conjectures can be derived form the map quot_d described in a previous section. We first check that the restrictions on content are preserved by quot_d. Let λ/μ be a d-tileable partition such that the content of every cell falls in the range $[r, r + dk - 1 + (d-1)]$. Since $G_{\lambda/\mu}^{(k)}(X; u)$ is independent of the position of λ/μ within the plane, we can translate λ/μ to have content entirely within the interval $[1, dk + (d-1)]$. The map quot_d only considers the contents of the heads of ribbons in tilings of λ/μ, which have content in the range $[1, dk]$. Then the adjusted contents $\tilde{c}(x)$ fall within the interval $[0, k-1]$ for all $x \in \lambda/\mu$. We conclude that d-tileable skew partitions in the content interval $[1, dk + (d-1)]$ are in bijection with multiskew partitions in the interval $[0, k-1]$.

Now we must conclude that Equation 1.7.3 holds. From a Equation 1.7, we see that for any h

$$u^{-2h} \cdot G_{\lambda/\mu}^{(d)}(X; u) = u^{-2h} u^{-e} \mathcal{G}_{\vec{\beta}}(X; q^{-1}) = q^{\frac{-e-2h}{2}} \mathcal{G}_{\vec{\beta}}(X; q^{-1}) \tag{1.17}$$

where $e = e(\lambda/\mu)$ and $\vec{\beta} = \mathrm{quot}_d(\lambda/\mu)$. Thus $\mathcal{G}_{\vec{\beta}}(X; q^{-1})$ and $G_{\lambda/\mu}^{(d)}(X; u)$ are the same up to a power of u. The degree of both sides is minimized when $h = S_*(\lambda/\mu)$.

We see that if $u^{-2S_*(\lambda/\mu)} \cdot G_{\lambda/\mu}^{(d)}(X; u)$ is k-Schur positive, then there exists $m \in \mathbb{Z}$ such that $q^m \cdot \mathcal{G}_{\vec{\beta}}(X; q^{-1})$ is k-Schur positive. To see the converse, suppose there exists m such that $q^m \cdot \mathcal{G}_{\vec{\beta}}(X; q^{-1})$ is k-Schur positive. We may assume that m is minimal, meaning that $q^m \cdot \mathcal{G}_{\vec{\beta}}(X; q^{-1}) = u^{-2S_*(\lambda/\mu)} \cdot G_{\lambda/\mu}^{(d)}(X; u)$. Thus $u^{-2S_*(\lambda/\mu)} \cdot G_{\lambda/\mu}^{(d)}(X; u)$ is k-Schur positive. $\square$

In the next lemma, we must make sense of how the automorphism ω acts on LLT polynomials indexed by multiskew partitions. Recall that for any partitions λ, we have $\omega s_\lambda = s_{\lambda'}$, where λ' is the transpose of λ.

Lemma 1.7.4. *Let $\vec{\beta}$ be a multiskew partition. Then $\omega \mathcal{G}_{\vec{\beta}}(X; q) = q^I \mathcal{G}_{\omega_0 \vec{\beta}'}(X; q^{-1})$, where β' denotes the entry-wise transpose of β, ω_0 is the permutation whose action is to reverse the order of all the entries in $\vec{\beta}$ and $I = I(\vec{\beta})$ is the number of pairs of cells in $\vec{\beta}$ that form an attacking pair.*

Proof. Haglund, Haiman, Loehr and Ulyanov [11] tell us how ω acts on LLT polynomials that are indexed by d-tileable parititons:

$$\omega G_{\lambda/\mu}^{(d)}(X; u) = u^{(d-1)|\lambda/\mu|/d} G_{\lambda'/\mu'}^{(d)}(X; u^{-1}) \tag{1.18}$$

Further, we have that

$$\mathcal{G}_{\vec{\beta}}(X; q) = u^e G^{(k)}_{\lambda/\mu}(X; u^{-1}) \tag{1.19}$$

where $\vec{\beta} = \text{quot}_\text{d}(\lambda/\mu)$ and $e = e(\lambda/\mu) = 2 \cdot S^*(\lambda/\mu) + 2 \cdot \text{inv}_*(\vec{\beta})$. So,

$$
\begin{aligned}
\omega \mathcal{G}_{\vec{\beta}}(X; q^{-1}) &= \omega u^{-e(\lambda/\mu)} \cdot G^{(d)}_{\lambda/\mu}(X; u) \\
&= u^{-e(\lambda/\mu)} u^{(d-1)|\lambda/\mu|/d} \cdot G^{(d)}_{\lambda'/\mu'}(X; u^{-1}) \\
&= u^{-e(\lambda/\mu)} u^{(d-1)|\lambda/\mu|/d} u^{-e(\lambda'/\mu')} \cdot u^{e(\lambda'/\mu')} G^{(d)}_{\lambda'/\mu'}(X; u^{-1}) \\
&= u^{-e(\lambda/\mu)} u^{(d-1)|\lambda/\mu|/d} u^{-e(\lambda'/\mu')} \cdot \mathcal{G}_{\omega_0 \vec{\beta}'}(X; q) \\
&= u^{-[e(\lambda/\mu)+e(\lambda'/\mu')]} u^{(d-1)|\lambda/\mu|/d} \cdot \mathcal{G}_{\omega_0 \vec{\beta}'}(X; q)
\end{aligned}
\tag{1.20}
$$

We have used the fact that $\text{quot}_\text{d}(\lambda'/\mu') = \omega_0 \vec{\beta}'$, which we will prove later in this section. For now, we seek to simplify the term $e(\lambda/\mu) + e(\lambda'/\mu')$. Let $T \in \text{Tab}_d(\lambda/\mu)$ be a d-ribbon tableau such that $S(T) = S^*(\lambda/\mu)$. Without loss of generality, we may assume that T is standard. Then the transpose T' is also a standard ribbon tableau but for the transposed shape λ'/μ'. It is clear that the spin $S(T')$ can be compute directly from T by interchanging the height and *width* or the ribbons. More formally, define the width of a ribbon $\text{width}(R)$ to be the number of columns it occupies, so that $S(T') = \sum_{R \in T} \frac{\text{width}(R)-1}{2}$.

Since the sum of the height and width of a ribbon is $d+1$, it is clear that $S(U) + S(U')$ is constant for all $U \in \text{Tab}_d(\lambda/\mu)$. Since T maximizes S on $\text{Tab}_d(\lambda/\mu)$, we have that $S(T')$ minimizes S on $\text{Tab}_d(\lambda'/\mu')$. Thus $S(T') = S_*(\lambda'/\mu')$.

$$
\begin{aligned}
e(\lambda/\mu) + e(\lambda'/\mu') &= 2 \cdot S^*(\lambda/\mu) + 2 \cdot \text{inv}_*(\text{quot}_\text{d}(\lambda/\mu)) + 2 \cdot S^*(\lambda'/\mu') + 2 \cdot \text{inv}_*(\text{quot}_\text{d}(\lambda'/\mu')) \\
&= 2 \cdot S(T) + 2 \cdot \text{inv}(\text{quot}_\text{d}(T)) + 2 \cdot S(T') + 2 \cdot \text{inv}(\text{quot}_\text{d}(T')) \\
&= 2 \cdot S(T) + 2 \cdot S(T') + 2 \cdot \text{inv}(\text{quot}_\text{d}(T)) + 2 \cdot \text{inv}(\text{quot}_\text{d}(T')) \\
&= \sum_{R \in T} [\text{height}(R) - 1] + \sum_{R \in T} [\text{width}(R) - 1] \\
&\qquad + 2 \cdot \text{inv}(\text{quot}_\text{d}(T)) + 2 \cdot \text{inv}(\text{quot}_\text{d}(T')) \\
&= \sum_{R \in T} (\text{height}(R) + \text{width}(r) - 2) + 2 \cdot \text{inv}(\text{quot}_\text{d}(T)) + 2 \cdot \text{inv}(\text{quot}_\text{d}(T')) \\
&= \sum_{R \in T} (d - 1) + 2 \cdot \text{inv}(\text{quot}_\text{d}(T)) + 2 \cdot \text{inv}(\text{quot}_\text{d}(T')) \\
&= (|\lambda/\mu|/d)(d - 1) + [2 \cdot \text{inv}(\text{quot}_\text{d}(T)) + 2 \cdot \text{inv}(\text{quot}_\text{d}(T'))]
\end{aligned}
\tag{1.21}
$$

Now we need only simplify the term $2 \cdot \text{inv}(\text{quot}_\text{d}(T)) + 2 \cdot \text{inv}(\text{quot}_\text{d}(T'))$. The map d-quotient: $\text{Tab}_d(\lambda/\mu) \to \vec{\beta}$ takes a ribbon whose head has content $p \cdot d + r$ and sends it to a cell in the rth position with content p.

We would like to see what happens when we apply $\mathrm{quot_d}$ to the transpose of a ribbon tableau. Recall that $\mathrm{quot_d}$ assigns a ribbon whose head has content $p \cdot d + r$ to a cell in the rth position with content p. In the previous section, we have forced the residues to fall in the interval $[1, d]$. Here, the proofs are much cleaner if we instead keep the residues in the interval $[0, d-1]$ and zero-index our multiskew partitions. We note that these two different notions of $\mathrm{quot_d}$ result in different multiskew partitions, but their corresponding LLT polynomials are identical. For more detail, see Definition 3.1.1 and Proposition 3.1.2

Let R be a ribbon in T whose head has content $pd + r$ with $r \in [0, d-1]$. Then the same ribbon in T' has a tail with content $-pd - r$, and a head with content $-pd - r + (d-1) = (-p) \cdot d + (d - r - 1)$. We see that, instead of being sent to the cell with content p in position r, the transposed ribbon is sent to the cell with content $-p$ in position $(d-1) - r$. Thus, $\mathrm{quot_d}(T')$ can be computed from $\mathrm{quot_d}(T)$ by reversing the order of the entries and taking transposes of each. In other words, $\mathrm{quot_d}(T') = \omega_0 \, \mathrm{quot_d}(T)'$, where $\omega_0 \in S_n$ is the longest word. Since T was standard, there is no issue maintaining the semistandard condition. Since $T' \in \mathrm{Tab}_d(\lambda'/\mu')$ and $\mathrm{quot_d}$ is defined on skew partitions, we also have that $\mathrm{quot_d}(\lambda'/\mu') = \omega_0\beta'$, thus completing the proof of Equation 1.20.

If we let x, y be a pair of cells in $\vec{\beta}$ and U any standard multiskew tableau of shape $\vec{\beta}$, then $U(x) > U(y)$ or $U(y) > U(x)$. In any case, the same inequality holds for the same pair of cells in $w_0 U'$. So if x, y form an attacking pair, then either U or $w_0 U'$ will yield an inversion from this pair. Letting $U = \mathrm{quot_d}(T)$, we have that $\mathrm{inv}(\mathrm{quot_d}(T)) + \mathrm{inv}(\mathrm{quot_d}(T'))$ counts the number of attacking pairs in $\vec{\beta}$, which we will call $I = I(\vec{\beta})$.

We compute

$$
\begin{aligned}
e(\lambda/\mu) + e(\lambda'/\mu') &= (|\lambda/\mu|/d)(d-1) + 2 \cdot \mathrm{inv}(\mathrm{quot_d}(T)) + 2 \cdot \mathrm{inv}(\mathrm{quot_d}(T')) \\
&= (|\lambda/\mu|/d)(d-1) + 2 \cdot I(\vec{\beta})
\end{aligned}
\tag{1.22}
$$

so that

$$
\begin{aligned}
\omega \mathcal{G}_{\vec{\beta}}(X; q^{-1}) &= u^{-[e(\lambda/\mu)+e(\lambda'/\mu')]} u^{(d-1)|\lambda/\mu|/d} \mathcal{G}_{\omega_0\vec{\beta}'}(X; q) \\
&= u^{-[(|\lambda/\mu|/d)(d-1)+2\cdot I]} u^{(d-1)|\lambda/\mu|/d} \mathcal{G}_{\omega_0\vec{\beta}'}(X; q) \\
&= u^{-2I} \mathcal{G}_{\omega_0\vec{\beta}'}(X; q) \\
&= q^{-I} \mathcal{G}_{\omega_0\vec{\beta}'}(X; q)
\end{aligned}
\tag{1.23}
$$

Substituting $q \mapsto q^{-1}$ establishes the lemma. □

We end this section with a proof that Conjecture 1.7.1 and Conjecture 1.7.2 are equivalent.

Proof. From Lemma 1.7.3, we see that the statement of the Haglund-Haiman conjecture is that for any k-bandwidth multiskew partition $\vec{\beta}$ there exists m such that $q^m \cdot \mathcal{G}_{\vec{\beta}}(X; q^{-1})$ is

k-Schur positive. Recall that the term q^m only exists to ensure that all of the coefficients $q^m \cdot \mathcal{G}_{\vec{\beta}}(X; q^{-1})\langle x_\lambda \rangle$ are polynomials in q. From Lemma 1.7.4, we have

$$q^I \cdot \mathcal{G}_{\vec{\beta}}(X; q^{-1}) = \omega \mathcal{G}_{\omega_0 \vec{\beta}'}(X; q) \tag{1.24}$$

It is clear that $q^I \cdot \mathcal{G}_{\vec{\beta}}(X; q^{-1})\langle x_\lambda \rangle$ is a polynomial in q for all λ. So we have resolved the Haglund-Haiman conjecture to showing that $\omega \mathcal{G}_{\omega_0 \vec{\beta}'}(X; q)$ is k-Schur positive for k-bandwidth multiskew parititons β. Since that the map $\vec{\beta} \mapsto \omega_0 \vec{\beta}'$ preserves the k-bandwidth property and is an involution on the set of multiskew partitions, we see that the Haglund-Haiman conjecture is equivalent to proving the k-Schur positivity of $\omega \mathcal{G}_{\beta}(X; q)$ for any k-bandwidth partition β. $\qquad \square$

1.8 Generalized Hall-Littlewood Polynomials

Our connection between k-Schur functions and LLT polynomials comes from generalized Hall-Littlewood polynomials. A theorem of Haiman and Grojnowski proves that in some cases, LLT polynomials actually are generalized Hall-Littlewood polynomials [10]. As we will see in Proposition 1.8.2 and Lemma 3.3.1, some generalized Hall-Littlewood polynomials are are provably k-Schur positive.

A *generalized Hall-Littlewood polynomial* $P_{\vec{\lambda}}$ is indexed by a tuple of straight shape partitions $\vec{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(d)})$. Given $\vec{\lambda}$, we define these polynomials as

$$P_{\vec{\lambda}}(X; q) = B_{\lambda^{(1)}} \cdots B_{\lambda^{(d)}}(1) \tag{1.25}$$

where $B_\lambda^{(i)}$ is a vertex operator as in Equation 1.10. If $\lambda^{(i)}$ is a single part for all i and $\vec{\lambda}$ concatenates to make a single partition, then $P_{\vec{\lambda}} = H_\lambda(X; t)$, a Hall-Littlewood polynomial. Lapointe and Morse prove an important property of k-Schur functions that will help us to relate them to generalized Hall-Littlewood polynomials. We will refer to it as the *k-rectangle property*.

Theorem 1.8.1 (*k*-rectangle property). *[6, Theorem 26] If μ, ν, λ are partitions where $\lambda = (\mu, \nu)$ and $\mu_{\ell(\mu)} > r \geq \nu_1$, then*

$$B_{r^{k+1-r}} s_\lambda^{(k)}(X; t) = t^{|\mu|-\ell(\mu)r} s_{(\mu, r^{k+1-r}, \nu)}^{(k)} \tag{1.26}$$

This theorem allows us to build up k-Schur functions by acting on the left with operators B_λ, so long as λ is a rectangle with greatest hook equal to k.

As an example of the efficacy of this theorem, consider the following proposition.

Proposition 1.8.2. *Fix $k \in \mathbb{N}$. If $\vec{\lambda}$ is such that every entry is a k-rectangle and the concatenation of these partitions is also a partition λ, then*

$$P_{\vec{\lambda}}(X; q) = s_\lambda^{(k)}(X; q) \tag{1.27}$$

Proof. Since $\vec{\lambda}$ concatenates to form a single partition, it must be the case that $\lambda^{(i)}$ forms a sequence of rectangles of nonincreasing width. In particular we can write our sequence of operators as $B_{\lambda^{(1)}} \cdots B_{\lambda^{(d)}}(1) = B_{r_1^{k+1-r_1}} \cdots B_{r_d^{k+1-r_d}}(1)$ where r_i is a weakly decreasing sequence.

Since the rectangles become larger in the order of application, we have $B_{r^{k+1-r}} s_\nu^{(k)}(X; t) = s_{(r^{k+1-r}, \nu)}^{(k)}$. This follows from the fact that $\mu = \emptyset$ in the notation used in the previous theorem.

So the application of these operators is quite nice, and results in a single k-Schur function. $\qquad \square$

A corollary of 1.8.1 is that all k-Schur functions can be built out a finite set of k-Schur functions whose indexing partition does not contain any k-rectangle. This is called a *k-irreducible*. This is done by applying a sequence of operators indexed by k-rectangles to said k-irreducibles. It is shown in [6] that for each k, there are exactly $k!$ such k-irreducibles.

We would like to relate this theory to the theory of LLT polynomials. This connection is made in a theorem from Haiman and Grojnowski, originally conjectured by Shimozono and Weyman [15].

Theorem 1.8.3. *[10, Theorem 7.15] Let $\lambda = (m_1^{r_1}, \ldots, m_k^{r_k})$ where $m_1 \geq \cdots \geq m_k$. Let $\vec{\beta}/\vec{\gamma}$ be a multiskew partition such that $\beta^{(i)}/\gamma^{(i)}$ is a translation of $m_i^{r_i}$, the southeast corners of the rectangles $\beta^{(i)}/\gamma^{(i)}$ have weakly decreasing content, and the southwest corners have weakly increasing content. Then there is an integer m such that*

$$P_\lambda(X; q^{-1}) = q^m \mathcal{G}_{\vec{\beta}/\vec{\gamma}}(X; q) \tag{1.28}$$

Chapter 2

LLT-equivalence

2.1 What is LLT-equivalence?

In this section we explore the ways in which LLT polynomials with different indexing multiskew partitions can be related to one another. Our motivation comes from the recent work of Lee [3] who introduced so called 'local linearity relations' for LLT polynomials indexed by partitions consisting of single cells.

Definition 2.1.1. *Let $\vec{\lambda}$ and $\vec{\lambda}'$ be multi-skew partitions. We will say they are LLT-equivalent if $\mathcal{G}_{(\vec{\lambda},\vec{\mu})}(X;q) = \mathcal{G}_{(\vec{\lambda}',\vec{\mu})}(X;q)$ for all multi-skew partitions $\vec{\mu}$.*

LLT-equivalence is a powerful property, because it allows us to identify complicated LLT polynomials with potentially simpler ones. We first demonstrate an example of this definition in Figure 2.1. There are two multiskew partitions, each drawn with content lines. It does not matter what the precise values of the content lines are, except that they must be the same for both multiskew partitions. In some cases, we will assume without loss of generality that the content lines correspond to contents $0, 1$ and 2.
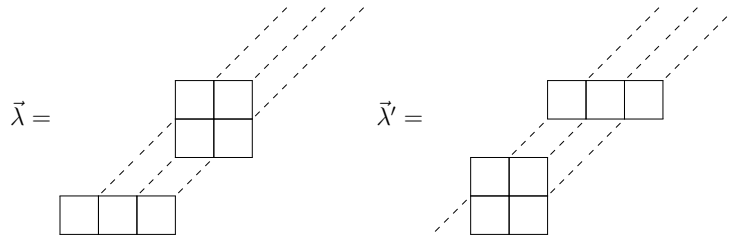
Figure 2.1: LLT-equivalent multiskew partitions

The multi-skew partitions in Figure 2.1 are in fact LLT-equivalent. So $\mathcal{G}_{\vec{\lambda}}(X;q) = \mathcal{G}_{\vec{\lambda}'}(X;q)$. Further, we can append any multiskew partition $\vec{\mu}$, as in Figure 2.2, and the result will satisy $\mathcal{G}_{(\vec{\lambda},\vec{\mu})}(X;q) = \mathcal{G}_{(\vec{\lambda}',\vec{\mu})}(X;q)$.
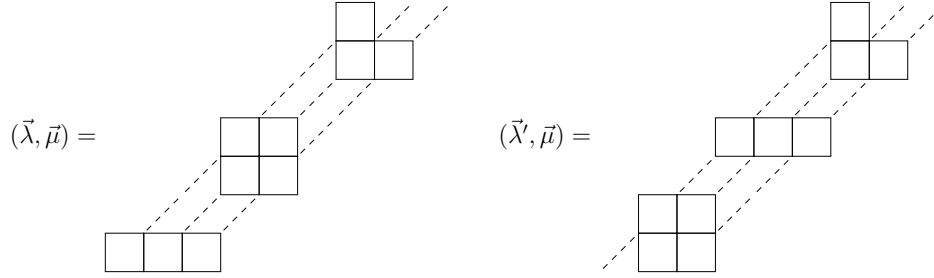
Figure 2.2: A pair satisfying $\mathcal{G}_{(\vec{\lambda},\vec{\mu})}(X;q) = \mathcal{G}_{(\vec{\lambda}',\vec{\mu})}(X;q)$

While a pair of LLT-equivalent multiskew partitions necessarily yield the same LLT polynomial, the converse does not hold. To see this, we produce mutliskew partitions $\vec{\lambda}$, $\vec{\lambda}'$ and $\vec{\mu}$ such that $\mathcal{G}_{\vec{\lambda}}(X;q) = \mathcal{G}_{\vec{\lambda}'}(X;q)$ but $\mathcal{G}_{(\vec{\lambda},\vec{\mu})}(X;q) \neq \mathcal{G}_{(\vec{\lambda}',\vec{\mu})}(X;q)$. Figures 2.3 and 2.4 demonstrate such an example, and prove that LLT-equivalence is stronger than simple equality of LLT polynomials.
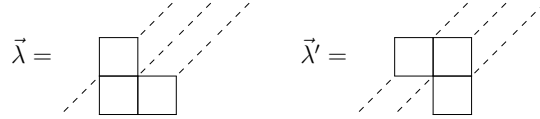
Figure 2.3: A pair satisfying $\mathcal{G}_{\vec{\lambda}}(X;q) = \mathcal{G}_{\vec{\lambda}'}(X;q)$
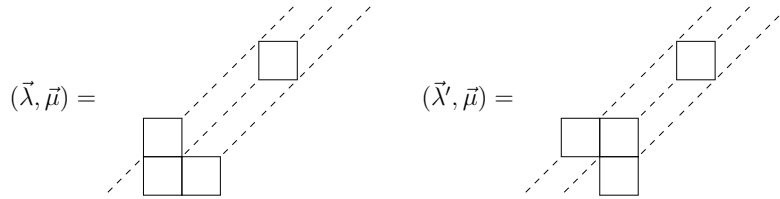
Figure 2.4: Here, $\mathcal{G}_{(\vec{\lambda},\vec{\mu})}(X;q) \neq \mathcal{G}_{(\vec{\lambda}',\vec{\mu})}(X;q)$

In Figure 2.3, we have that $\mathcal{G}_{\vec{\lambda}}(X;q) = s_{(2,1)}(X)$ by definition, and $\mathcal{G}_{\vec{\lambda}'}(X;q) = s_{(2,1)}(X)$ due to the Littlewood-Richardson rule. In Figure 2.4, $\mathrm{inv}_*(\vec{\lambda},\vec{\mu}) = 0$, while $\mathrm{inv}_*(\vec{\lambda}',\vec{\mu}) = 1$. So it's easy to see that $\mathcal{G}_{(\vec{\lambda},\vec{\mu})}(X;q) \neq \mathcal{G}_{(\vec{\lambda}',\vec{\mu})}(X;q)$. Thus $\vec{\lambda}$ and $\vec{\lambda}'$ are not LLT-equivalent.

We extend the above definition to LLT-equivalence of sums of LLT polynomials.

Definition 2.1.2. *Let $\sum_i a_i(q)\vec{\lambda}_{(i)}$ and $\sum_j b_j(q)\vec{\nu}_{(j)}$ be $\mathbb{N}[q]$-linear combinations of multiskew partitions. They are said to be LLT-equivalent if*

$$\sum_i a_i(q)\mathcal{G}_{(\vec{\lambda}_{(i)},\vec{\mu})}(X;q) = \sum_j b_j(q)\mathcal{G}_{(\vec{\nu}_{(j)},\vec{\mu})}(X;q) \tag{2.1}$$

for every multiskew partition μ.

A typical use of this type of statement is to decompose one LLT polynomial as an $\mathbb{N}[q]$-linear combination of other LLT polynomials. In Figures 2.5 and 2.6, we display one such example from the literature, due to Lee [3].
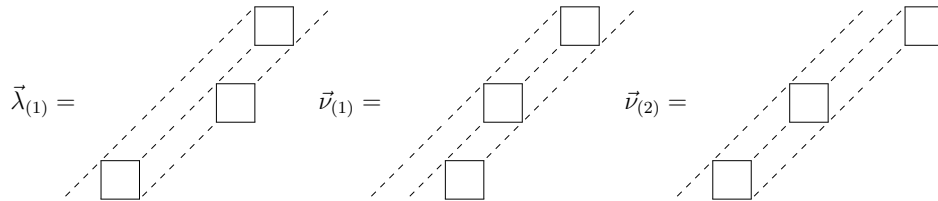
$$\vec{\lambda}_{(1)} = \qquad \vec{\nu}_{(1)} = \qquad \vec{\nu}_{(2)} =$$

Figure 2.5: Multiskew Partitions for Local Linearity Relations

In Figure 2.5, we define three multiskew partitions $\vec{\lambda}_{(1)}, \vec{\nu}_{(1)}$ and $\vec{\nu}_{(2)}$. These satisfy the equation

$$(q+1)\mathcal{G}_{\vec{\lambda}_{(1)}}(X;q) = \mathcal{G}_{\vec{\nu}_{(1)}}(X;q) + q\,\mathcal{G}_{\vec{\nu}_{(2)}}(X;q) \tag{2.2}$$

and further, they satisfy $(q+1)\mathcal{G}_{(\vec{\lambda}_{(1)},\vec{\mu})}(X;q) = \mathcal{G}_{(\vec{\nu}_{(1)},\vec{\mu})}(X;q) + q\,\mathcal{G}_{(\vec{\nu}_{(2)},\vec{\mu})}(X;q)$ for any multiskew partition μ. So we can say that $(q+1)\vec{\lambda}_{(1)}$ and $\vec{\nu}_{(1)} + q\,\vec{\nu}_{(2)}$ are LLT-equivalent. We will prove this fact at the end of this chapter, since Lee's proof only applies to the case that $\vec{\mu}$ is a sequence of single-celled partitions. We will often abuse notation and write that $\mathbb{N}[q]$-combinations of multiskew partitions are equal to denote that they are LLT-equivalent. See Figure 2.6 for example.

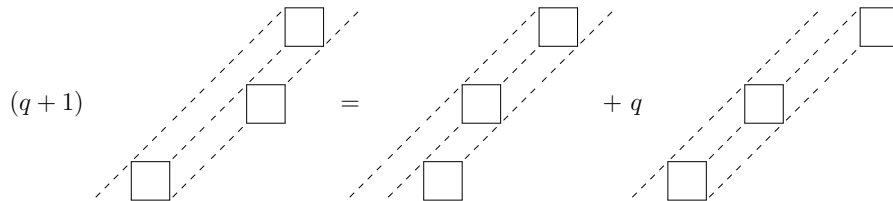$$(q+1) \qquad = \qquad + q$$

Figure 2.6: Local Linearity Relations

Identifying LLT-equivalent multiskew partitions is useful because they allow us to rewrite complicated LLT polynomials as a combination of simpler terms. This will help us to deduce the Schur and k-Schur expansions of a large class of LLT polynomials from the expansions of a small subset.

We present in the next section a sufficient condition to prove LLT-equivalence.

2.2 LLT-equivalence techniques

A sufficient condition for the LLT-equivalence of a pair of multiskew parititons is to identify a weight, inversion, and content-preserving bijection between their corresponding semistandard

multiskew tableaux. In notation, we would prove that there exists a bijection $f : \mathrm{SSYT}(\vec{\lambda}) \to \mathrm{SSYT}(\vec{\lambda'})$ satisfying the following properties. We use bag notation $\wr X \backslash$ to denote a multiset.

1. $w(T) = w(f(T))$

2. For every content c_0, $\wr T(x) \mid x \in \vec{\lambda}$ and $c(x) = c_0 \backslash = \wr T'(x) \mid x \in \vec{\lambda'}$ and $c(x) = c_0 \backslash$

3. $\mathrm{inv}(T) = \mathrm{inv}(f(T))$

Property 2 is the most technical of these. Consider the set of all cells in $\vec{\lambda}$ on the diagonal with content c_0. Together, the weights of these cells form a multiset. Property 2 is equivalent to saying that for any c_0, the cells of content c_0 in $\vec{\lambda}$ and $\vec{\lambda'}$ yield the same multiset of weights. One can think of a bijection having this property as preserving the weight along diagonals. In fact, Property 2 implies Property 1 for this reason.

We will prove that this definition is sufficient to determine LLT-equivalence. We will first need the fact that inv decomposes into components. Given a concatenated pair of multiskew partitions $\vec{\beta} = (\vec{\lambda}, \vec{\mu})$ and a multiskew tableau $T \in \mathrm{SSYT}(\vec{\beta}) = \mathrm{SSYT}(\vec{\lambda}) \times \mathrm{SSYT}(\vec{\mu})$, consider the restrictions $T_{\vec{\lambda}} \in \mathrm{SSYT}(\vec{\lambda})$ and $T_{\vec{\mu}} \in \mathrm{SSYT}(\vec{\mu})$. Then $\mathrm{inv}(T) = \mathrm{inv}(T_{\vec{\lambda}}) + \mathrm{inv}(T_{\vec{\mu}}) + \widetilde{\mathrm{inv}}(T_{\vec{\lambda}}, T_{\vec{\mu}})$, where $\widetilde{\mathrm{inv}}(T_{\vec{\lambda}}, T_{\vec{\mu}})$ counts the number of inversions between pairs of cells in $\vec{\lambda} \times \vec{\lambda'}$.

We see Figure 2.7 for a visual, where we have $\vec{\lambda} = ((3), (2))$ and $\vec{\mu} = ((3)/(1))$. We compute inversions: $\mathrm{inv}(T) = 4$, $\mathrm{inv}(T_{\vec{\lambda}}) = 2$, $\mathrm{inv}(T_{\vec{\mu}}) = 0$. That leaves $\widetilde{\mathrm{inv}}(T_{\vec{\lambda}}, T_{\vec{\mu}}) = 2$, which corresponds to the two pairs $(6, 4)$ and $(4, 2)$.
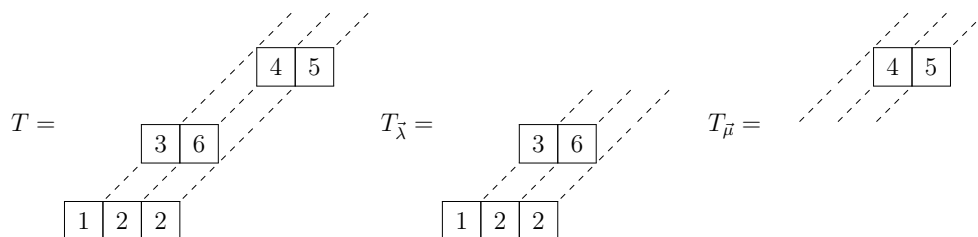

Figure 2.7: Some multiskew tableaux

If $f : \mathrm{SSYT}(\vec{\lambda}) \to \mathrm{SSYT}(\vec{\lambda'})$ is weight, content and inversion preserving bijection and $T \in \mathrm{SSYT}(\vec{\lambda}) \times \mathrm{SSYT}(\vec{\mu})$, then $\widetilde{\mathrm{inv}}(T_{\vec{\lambda}}, T_{\vec{\mu}}) = \widetilde{\mathrm{inv}}(f(T_{\vec{\lambda}}), T_{\vec{\mu}})$. This is due to the fact that $\widetilde{\mathrm{inv}}(T_{\vec{\lambda}}, T_{\vec{\mu}})$ can be computed from the multisets $\wr (T_{\vec{\lambda}}(x), c(x)) \mid x \in \vec{\lambda} \backslash$ and $\wr (T_{\vec{\mu}}(x), c(x)) \mid x \in \vec{\mu} \backslash$, which are preserved by f due to Property 2. For some intuition, consider the inversions $(6, 4)$ and $(4, 2)$ that make up $\widetilde{\mathrm{inv}}(T_{\vec{\lambda}}, T_{\vec{\mu}})$ in 2.7. These pairs would still contribute to $\widetilde{\mathrm{inv}}(T_{\vec{\lambda}}, T_{\vec{\mu}})$ if the cells were translated to other parts of their corresponding multiskew partition with the same content. In general, we are taking advantage of the fact that all of the cells in $\vec{\lambda}$ come from skew partitions with smaller index than the cells in $\vec{\mu}$ do. This means that whether or not a pair forms an inversion depends only on their contents and weights, both of which are preserved by f.

Let $\vec{\lambda}, \vec{\lambda}'$. Suppose that there exists a weight, content and inversion-preserving bijection between $\mathrm{SSYT}(\vec{\lambda})$ and $\mathrm{SSYT}(\vec{\lambda}')$.

$$\begin{aligned}
\mathcal{G}_{(\vec{\lambda},\vec{\mu})}(X;q) &= \sum_{T \in \mathrm{SSYT}(\vec{\lambda},\vec{\mu})} x^T q^{\mathrm{inv}(T)} \\
&= \sum_{T_{\vec{\lambda}} \in \mathrm{SSYT}(\vec{\lambda})} \sum_{T_{\vec{\mu}} \in \mathrm{SSYT}(\vec{\mu})} x^{T_{\vec{\lambda}}} x^{T_{\vec{\mu}}} q^{\mathrm{inv}(T_{\vec{\lambda}}) + \mathrm{inv}(T_{\vec{\mu}}) + \widetilde{\mathrm{inv}}(T_{\vec{\lambda}}, T_{\vec{\mu}})} \\
&= \sum_{T_{\vec{\lambda}} \in \mathrm{SSYT}(\vec{\lambda})} \sum_{T_{\vec{\mu}} \in \mathrm{SSYT}(\vec{\mu})} x^{f(T_{\vec{\lambda}})} x^{T_{\vec{\mu}}} q^{\mathrm{inv}(f(T_{\vec{\lambda}})) + \mathrm{inv}(T_{\vec{\mu}}) + \widetilde{\mathrm{inv}}(f(T_{\vec{\lambda}}), T_{\vec{\mu}})} \\
&= \sum_{T_{\vec{\lambda}'} \in \mathrm{SSYT}(\vec{\lambda}')} \sum_{T_{\vec{\mu}} \in \mathrm{SSYT}(\vec{\mu})} x^{T_{\vec{\lambda}'}} x^{T_{\vec{\mu}}} q^{\mathrm{inv}(f(T_{\vec{\lambda}'})) + \mathrm{inv}(T_{\vec{\mu}}) + \widetilde{\mathrm{inv}}(f(T_{\vec{\lambda}'}), T_{\vec{\mu}})} \\
&= \sum_{T \in \mathrm{SSYT}(\vec{\lambda}',\vec{\mu})} x^T q^{\mathrm{inv}(T)} \\
&= \mathcal{G}_{(\vec{\lambda}',\vec{\mu})}(X;q)
\end{aligned} \tag{2.3}$$

Thus we have concluded that $\vec{\lambda}$ and $\vec{\lambda}'$ are LLT-equivalent. This offers a path towards proving that large collections of multiskew partitions are LLT-equivalent. Perhaps it is possible to produce weight, content and inversion-preserving bijections in some systematic way. This is the direction that Lee followed in [3], where he established the local linearity relations.

Unfortunately it is generally very hard to prove that these bijections exists between multiskew partitions, though there are some easy cases. In Figure 2.8 we have plotted a pair of multiskew partitions that are provably LLT-equivalent.
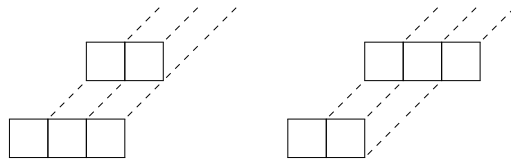

Figure 2.8: A pair of partitions that are provably LLT-equivalent

We prove LLT-equivalence by establishing an appropriate bijection between their semi-standard Young tableaux. The bijection f is displayed in Figure 2.9, where it is assumed that $a \geq b \geq c$ and $d \geq e$. Since there are still many possible relations among the entries, the bijection breaks down into three cases.

1. $a \geq d$

2. $a < d$ and $b \geq e$

   3. $a < d$ and $b < e$

It is obvious from Figure 2.9 that f is content and weight-preserving. It is less obvious that f is inversion-preserving, or even a bijection. Both of these facts follow from a more general analysis of multiskew partitions indexed by horizontal strips, which we defer to the appendix. For now, consider the tableau T that satisfies $(a, b, c, d, e) = (3, 2, 2, 5, 2)$. If we want to compute $f(T)$, then we are in Case 2. Before applying the bijection, we see that T has only one inversion: $(5, 3)$. After applying the bijection, we compute that $f(T)$ still has only one inversion: $(5, 2)$. So the number of inversions is preserved, and this will be the case for all tableaux.
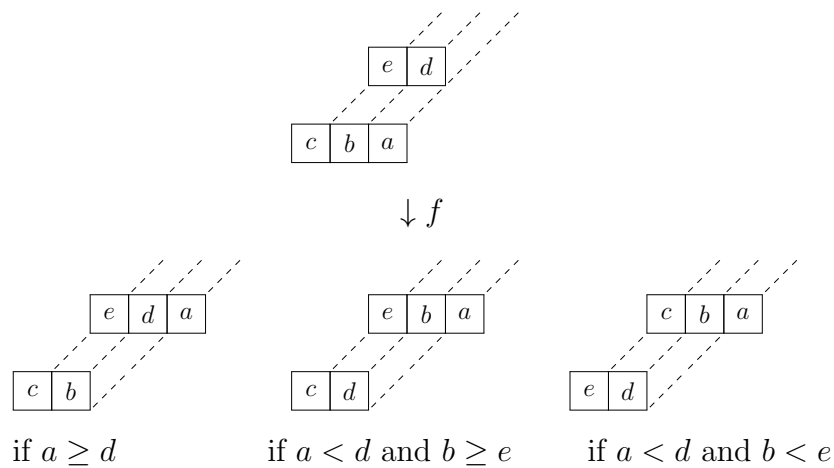
Figure 2.9: A weight, content, and inversion-preserving bijection.

We are often interested in relationships among LLT polynomials that are more complicated than simple equality. We can establish LLT-equivalence relations by producing bijections that are not inversion-preserving, but are up to a constant. If we replace Property 3 with the statement $\mathrm{inv}(T) = \mathrm{inv}(f(T)) + C$ for some fixed $C \in \mathbb{Z}$, then we can show that $q^C \vec{\lambda}$ and $\vec{\lambda}'$ are LLT-equivalent. The proof of this fact is essentially identical to the previous case, and follows from the more general proof of Theorem 2.2.1 at the end of this section. In Figure 2.10, we have a pair of multiskew partitions that yield the LLT-equivalence $q\vec{\lambda} = \vec{\lambda}'$. They are simple enough that we can produce a weight and content preserving bijection between their semistandard young tableaux, and also show that it preserves inversions up to a constant.
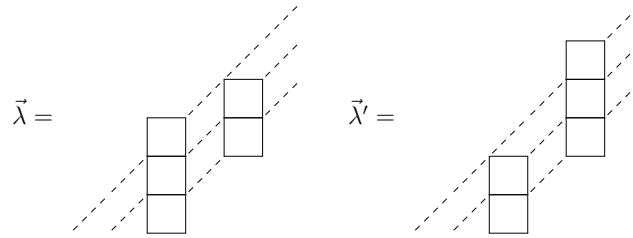
Figure 2.10: A pair of multiskew partitions that are provably LLT-equivalent up to a power of q

We display the corresponding bijection g in Figure 2.11. The reader may prove that this function is indeed a bijection, and that it increases the number of inversions by 1 for any input. We will prove this as part of a more general analysis in the appendix. The bijection splits into three cases, depending on the relative values of the entries in the columns. We can describe an arbitrary tableau by allowing $a > b > c$ to be the entries in the first column and $d > e$ to be the entries in the second. Then the bijection breaks into three cases:

1. $a > d$

2. $a \le d$ and $b > e$

3. $a \le d$ and $b \le e$

For example, if T is a tableau satisfying $(a, b, c, d, e) = (3, 2, 1, 5, 4)$, then applying g puts us in Case 3. There is one inversion in T: $(5, 1)$. We compute that there are two inversions in $g(T)$: $(4, 1)$ and $(5, 2)$. So we see that the number of inversion has increased by exactly one, which will always be the case. Since this bijection clearly preserves content and weight, g establishes that the multiskew partitions are LLT-equivalent up to a power of q.

In full generality, we would like to have a method for proving that pairs of $\mathbb{N}[q]$-linear combinations of multiskew partitions are LLT-equivalent. This will also be achievable by an appropriate weight, content and inversion-preserving bijection. We again use bag notation $\wr X \wr$ to denote multisets.

Theorem 2.2.1. *Let $\sum\limits_{i} q^{r_i} \vec{\lambda}_{(i)}$ and $\sum\limits_{j} q^{s_j} \vec{\nu}_{(j)}$ be $\mathbb{N}[q]$-linear combinations of multiskew partitions. Then they are LLT-equivalent if there exists a bijection $f : \bigsqcup\limits_{i} SSYT(\vec{\lambda}_{(i)}) \to \bigsqcup\limits_{j} SSYT(\vec{\nu}_{(j)})$ that has the following properties.*

1. *If $(T', j) = f(T, i)$, then $w(T) = w(T')$.*

2. *If $(T', j) = f(T, i)$, then for any content c_0, $\wr T(x) \mid x \in \operatorname{sh}(T)$ and $c(x) = c_0 \wr = \wr T'(x) \mid x \in \operatorname{sh}(T')$ and $c(x) = c_0 \wr$.*

3. *If $(T', j) = f(T, i)$, then $\operatorname{inv}(T') + s_j = \operatorname{inv}(T) + r_i$.*
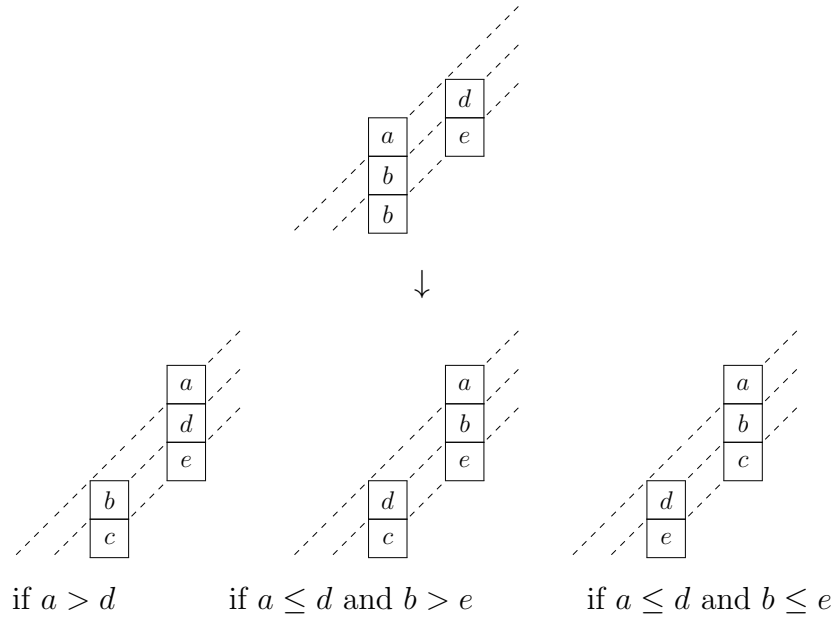
Figure 2.11: A weight and content-preserving bijection that increases the number of inversions by 1.

Here we have used the notation (T, i) to represent the tableau $T \in \text{SSYT}(\vec{\lambda}_{(i)})$ and to distinguish it from other copies of T that appear in the domain. In other words, $\bigsqcup_i \{T \mid T \in \text{SSYT}(\vec{\lambda}_{(i)})\} = \bigcup_i \{(T, i) \mid T \in \text{SSYT}(\vec{\lambda}_{(i)})\}$. We will refer to functions that meet these conditions as inversion-preserving, even though they only preserve the inversions up to a constant.

Proof. Given f as described, we define $D_{ij} = \text{SSYT}(\vec{\lambda}_{(i)}) \cap f^{-1}(\text{SSYT}(\vec{\nu}_{(j)}))$ and $f_{ij} = f|_{D_{ij}}$ for all i, j. This collection of functions should be thought of as the restriction of the bijection f to the components of its domain and codomain. Observe that for $T \in D_{ij}$, $f(T) = f_{ij}(T)$ and $\text{inv}(T) = \text{inv}(f(T)) - r_i + s_j$. Let $\vec{\mu}$ by an arbitrary multiskew partition. We wish to show that $\sum_i q^{r_i} \mathcal{G}_{(\vec{\lambda}_{(i)}, \vec{\mu})}(X; q) = \sum_j q^{s_j} \mathcal{G}_{(\vec{\nu}_{(j)}, \vec{\mu})}(X; q)$.

$$
\begin{aligned}
\sum_i q^{r_i} \mathcal{G}_{(\vec{\lambda}_{(i)}, \vec{\mu})}(X; q) &= \sum_i q^{r_i} \sum_{T \in \mathrm{SSYT}(\vec{\lambda}_{(i)}, \vec{\mu})} x^T q^{\mathrm{inv}(T)} \\
&= \sum_i q^{r_i} \sum_{T_{\vec{\lambda}} \in \mathrm{SSYT}(\vec{\lambda}_{(i)})} \sum_{T_{\vec{\mu}} \in \mathrm{SSYT}(\vec{\mu})} x^{T_{\vec{\lambda}}} x^{T_{\vec{\mu}}} q^{\mathrm{inv}(T_{\vec{\lambda}}) + \mathrm{inv}(T_{\vec{\mu}}) + \widetilde{\mathrm{inv}}(T_{\vec{\lambda}}, T_{\vec{\mu}})} \\
&= \sum_i q^{r_i} \sum_j \sum_{T_{\vec{\lambda}} \in D_{ij}} \sum_{T_{\vec{\mu}} \in \mathrm{SSYT}(\vec{\mu})} x^{T_{\vec{\lambda}}} x^{T_{\vec{\mu}}} q^{\mathrm{inv}(T_{\vec{\lambda}}) + \mathrm{inv}(T_{\vec{\mu}}) + \widetilde{\mathrm{inv}}(T_{\vec{\lambda}}, T_{\vec{\mu}})} \\
&= \sum_i q^{r_i} \sum_j \sum_{T_{\vec{\lambda}} \in D_{ij}} \sum_{T_{\vec{\mu}} \in \mathrm{SSYT}(\vec{\mu})} x^{f_{ij}(T_{\vec{\lambda}})} x^{T_{\vec{\mu}}} q^{\mathrm{inv}(f_{ij}(T_{\vec{\lambda}})) + s_j - r_i + \mathrm{inv}(T_{\vec{\mu}}) + \widetilde{\mathrm{inv}}(f_{ij}(T_{\vec{\lambda}}), T_{\vec{\mu}})} \\
&= \sum_i \sum_j \sum_{T_{\vec{\lambda}} \in D_{ij}} \sum_{T_{\vec{\mu}} \in \mathrm{SSYT}(\vec{\mu})} x^{f_{ij}(T_{\vec{\lambda}})} x^{T_{\vec{\mu}}} q^{\mathrm{inv}(f_{ij}(T_{\vec{\lambda}})) + s_j + \mathrm{inv}(T_{\vec{\mu}}) + \widetilde{\mathrm{inv}}(f_{ij}(T_{\vec{\lambda}}), T_{\vec{\mu}})} \\
&= \sum_j \sum_i \sum_{T'_{\vec{\lambda}} \in f(D_{ij})} \sum_{T_{\vec{\mu}} \in \mathrm{SSYT}(\vec{\mu})} x^{T'_{\vec{\lambda}}} x^{T_{\vec{\mu}}} q^{\mathrm{inv}(T'_{\vec{\lambda}}) + s_j + \mathrm{inv}(T_{\vec{\mu}}) + \widetilde{\mathrm{inv}}(T'_{\vec{\lambda}}, T_{\vec{\mu}})} \\
&= \sum_j \sum_{T'_{\vec{\lambda}} \in \mathrm{SSYT}(\vec{\nu}_{(j)})} \sum_{T_{\vec{\mu}} \in \mathrm{SSYT}(\vec{\mu})} x^{T'_{\vec{\lambda}}} x^{T_{\vec{\mu}}} q^{\mathrm{inv}(T'_{\vec{\lambda}}) + s_j + \mathrm{inv}(T_{\vec{\mu}}) + \widetilde{\mathrm{inv}}(T'_{\vec{\lambda}}, T_{\vec{\mu}})} \\
&= \sum_j \sum_{T' \in \mathrm{SSYT}(\vec{\nu}_{(j)}, \vec{\mu})} x^{T'} q^{\mathrm{inv}(T')} \\
&= \sum_j q^{s_j} \mathcal{G}_{(\vec{\nu}_{(j)}, \vec{\mu})}(X; q)
\end{aligned}
$$

$$(2.4)$$

Thus we have proved that the LLT polynomials are identical for any concatenated sequence $\vec{\mu}$, and that $\sum_i q^{r_i} \vec{\lambda}_{(i)}$ and $\sum_j q^{s_j} \vec{\nu}_{(j)}$ are LLT-equivalent. $\qquad\square$

Since the sums $\sum_i q^{r_i} \vec{\lambda}_{(i)}$ and $\sum_j q^{s_j} \vec{\nu}_{(j)}$ in the previous theorem can have repeated mul-tiskew parititons, this type of weight, content and inversion-preserving bijection can be used to show that any $\mathbb{N}[q]$-linear combination of multiskew partitions are LLT-equivalent. In particular, one could produce an explicit bijection that proves the LLT-equivalence of the local linearity relations in Figure 2.2. Lee's original proof [3][Theorem 3.4] uses alternative methods, and only holds for multiskew partitions where every partition is a single cell. We would like to have a bijection as in 2.2.1, which would prove LLT-equivalence entirely.

2.3 LLT-equivalence by computational means

In this section we will lay out a computational method for verifying LLT-equivalence of $\mathbb{N}[q]$-linear combinations of multiskew partitions. In particular, we will prove the existence

of weight, content, and inversion-preserving bijections between the appropriate collections of semistandard Young tableaux.

Lemma 2.3.1. *Let $\sum_i q^{r_i}\vec{\lambda}_{(i)}$ and $\sum_j q^{s_j}\vec{\nu}_{(j)}$ be $\mathbb{N}[q]$-linear combinations of multiskew partitions. If there exist content and inversion-preserving bijections $f_\alpha : \bigsqcup_i SSYT(\vec{\lambda}_{(i)}, \alpha) \rightarrow \bigsqcup_j SSYT(\vec{\nu}_{(j)}, \alpha)$ for all left-justified compositions α, then $\sum_i q^{r_i}\vec{\lambda}_{(i)}$ and $\sum_j q^{s_j}\vec{\nu}_{(j)}$ are LLT-equivalent.*

We will say that a sequence $\{a_i\}$ is left-justified if there exists k such that $a_1, a_2, \ldots a_k$ are nonzero but $a_{k+1}, a_{k+2}, \ldots$ are zero.

Proof. Suppose $\{f_\alpha\}$ is as described. We wish to construct a function $f : \bigsqcup_i \text{SSYT}(\vec{\lambda}_{(i)}) \rightarrow \bigsqcup_j \text{SSYT}(\vec{\nu}_{(j)})$ that is weight, content, and inversion preserving. We will define f on an arbitrary tableau $(T, I) \in \bigsqcup_i \text{SSYT}(\vec{\lambda}_{(i)})$. To make this proof more readable, we will write T instead of (T, i) when there is no ambiguity. The weight of T is an arbitrary composition $w(T) = (w_1(T), w_2(T), \ldots)$. Let $\{a_i\}_{i=1}^k$ be the sequence of indices such that $w_{a_i}(T) \neq 0$. Since $\{a_i\}$ is an increasing sequence, this defines an invertible function $g_{w(T)} : \mathbb{N} \rightarrow \mathbb{N}$ by

$$g_{w(T)}(i) = \begin{cases} a_i & \text{if } \leq k \\ 0 & \text{otherwise} \end{cases}$$

Recalling that a tableau is properly defined as a function, we produce a new tableau: $g_{w(T)}^{-1} \circ T$. We think of this tableau as being essentially the same as T, but with all entries rescaled to be as small as possible. See Figure 2.12 for an example.
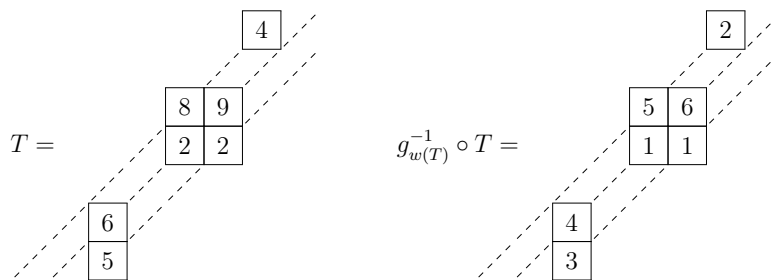

Figure 2.12: A tableau T and its rescaled image under $g_{w(T)}^{-1}$

This function $g_{w(T)}^{-1} \circ T$ tableau has two important properties.

- The weight of $g_{w(T)}^{-1} \circ T$ is left-justified.

- The relative order of the entries is preserved. In particular, $T(x) \geq T(y)$ if and only if $g_{w(T)}^{-1} \circ T(x) \geq g_{w(T)}^{-1} \circ T(y)$.

Since $w(g_{w(T)}^{-1} \circ T)$ is a left-justified sequence, there exists a weight, inversion, and content preserving bijection $f_\alpha : \bigsqcup_i \mathrm{SSYT}(\vec{\lambda}_{(i)}, \alpha) \to \bigsqcup_j \mathrm{SSYT}(\vec{\nu}_{(j)}, \alpha)$, where $\alpha = w(g_{w(T)}^{-1} \circ T)$.

We define the function $f : \bigsqcup_i \mathrm{SSYT}(\vec{\lambda}_{(i)}) \to \bigsqcup_j \mathrm{SSYT}(\vec{\nu}_{(j)})$ by

$$f(T) = g_{w(T)} \circ f_\alpha \circ g_{w(T)}^{-1} \circ T \tag{2.5}$$

Where $\alpha = w(g_{w(T)}^{-1} \circ T)$. We must prove that f is weight, content, and inversion-preserving.

We first see that f is weight preserving. If T has weight $w(T) = (w_1(T), w_2(T), \ldots)$, then $g_{w(T)}^{-1} \circ T$ has weight $(w_{a_1}(T), w_{a_2}(T), \ldots, w_{a_k}(T), 0, \ldots)$, where $\{w_{a_i}\}_{i=1}^k$ are the nonzero entries of $w(T)$. By definition, $f_\alpha \circ g_{w(T)}^{-1} \circ T$ has the same weight as $g_{w(T)}^{-1} \circ T$. Since $g_{w(T)}$ and $g_{w(T)}^{-1}$ have the inverse effect on weight, we see that $g_{w(T)} \circ f_\alpha \circ g_{w(T)}^{-1} \circ T$ has weight $(w_1(T), w_2(T), \ldots)$.

To see that f is content-preserving, we must show that $\wr(T(x), c(x))|x \in \mathrm{sh}(T)\wr = \wr(f(T)(x), c(x))|x \in \mathrm{sh}(f(T))\wr$ for any tableau T. Equivalently, we fix any content p and show that

$$\wr T(x)|x \in \mathrm{sh}(T) \text{ and } c(x) = p\wr \ = \ \wr f(T)(x)|x \in \mathrm{sh}(f(T)) \text{ and } c(x) = p\wr. \tag{2.6}$$

We first note that $\wr g_{w(T)}^{-1} \circ T(x)|x \in \mathrm{sh}(T) \text{ and } c(x) = p\wr = \wr f_\alpha \circ g_{w(T)}^{-1} \circ T(x)|x \in \mathrm{sh}(T) \text{ and } c(x) = p\wr$, because f_α is content-preserving by definition. Since these sets are the same, so are their images under $g_{w(T)}$. Thus equation 2.6 holds.

We must see that f is inversion preserving. Since $g_{w(T)}$ preserves the relative order of the entries of T, it is sufficient to see the effect of f_α on the inversion count. Let $(T, i) \in \mathrm{SSYT}(\vec{\lambda}_{(i)}) \cap f^{-1}(\mathrm{SSYT}(\vec{\nu}_{(j)}))$. Then $\mathrm{inv}(f_\alpha \circ g_{w(T)}^{-1} \circ T) = \mathrm{inv}(g_{w(T)}^{-1} \circ T) + r_i - s_j$. So, $\mathrm{inv}(f(T)) = \mathrm{inv}(T) + r_i - s_j$, and we conclude that f is inversion-preserving.

$\square$

If α is a composition of n, then $\mathrm{SSYT}(\vec{\lambda}, \alpha)$ is nonempty if and only if n is also the size of λ. So $\bigsqcup_i \mathrm{SSYT}(\vec{\lambda}_{(i)})$ is nonempty if and only if one of the multiskew partitions $\vec{\lambda}_{(1)}, \vec{\lambda}_{(2)}, \ldots, \vec{\lambda}_{(l)}$ has a size n. For any n, there are only finitely many left-justified compositions, thus there are only finitely many left-justified compositions α such that $\bigsqcup_i \mathrm{SSYT}(\vec{\lambda}_{(i)}, \alpha)$ is nonempty.

So this theorem allows us to systematically produce weight, content and inversion-preserving bijections. Thus we can prove that combinations of multiskew partitions are LLT-equivalent. We proceed with an example by proving Lee's local linearity relations, as in Figure 2.6.

Lee's theorem is that $(q + 1)\vec{\lambda}_{(1)} = \vec{\lambda}_{(2)} + q\vec{\lambda}_{(3)}$. Each multiskew partition has size 3, so we are only interested in compositions of 3. Thus we need only find content and inversion-preserving bijections

$$f_\alpha : \mathrm{SSYT}(\vec{\lambda}_{(1)}, \alpha) \sqcup \mathrm{SSYT}(\vec{\lambda}_{(1)}, \alpha) \to \mathrm{SSYT}(\vec{\lambda}_{(2)}, \alpha) \sqcup \mathrm{SSYT}(\vec{\lambda}_{(3)}, \alpha) \qquad (2.7)$$

for $\alpha \in \{(1, 1, 1), (2, 1, 0), (1, 2, 0), (3, 0, 0)\}$. Since there are only finitely many such multiskew tableaux, we can confirm that a bijection exists via computer search, which it does. Since Lee's theorem was original formulated in the setting of unicellular partitions, this computation in fact results in a slightly stronger statement allowing for arbitrary skew partitions to be appended.

This method of establishing LLT-equivalence relations will be especially useful for us in the next section, where we establish relations among all pairs of 3-bandwidth skew partitions.

We end this section with a related conjecture.

Conjecture 2.3.2. *Let* $\sum_i q^{r_i} \vec{\lambda}_{(i)}$ *and* $\sum_j q^{s_j} \vec{\nu}_{(j)}$ *be* $\mathbb{N}[q]$*-linear combinations of multiskew partitions, and let* u_c *be a partition that this a single with content* $c \in \mathbb{Z}$. *Then* $\sum_i q^{r_i} \vec{\lambda}_{(i)}$ *and* $\sum_j q^{s_j} \vec{\nu}_{(j)}$ *are LLT-equivalent if and only if for all* $c \in \mathbb{Z}$, $\sum_i q^{r_i} \mathcal{G}_{(\vec{\lambda}_{(i)}, u_c)}(X; q) = \sum_j q^{r_j} \mathcal{G}_{(\vec{\nu}_{(j)}, u_c)}(X; q)$.

$$(2.8)$$

The guiding principle behind this conjecture is that LLT polynomials can be distinguished by identifying the content lines on which they differ. In particular, if this conjecture holds and there are two multiskew partitions that are not LLT-equivalent, then there should exist a single-celled partition that, when appended, produces differing LLT polynomials. See for example Figure 2.3 and Figure 2.4.

We note that if the content c is outside the interval of contents occupied by a multiskew partition $\vec{\lambda}$, then $\mathcal{G}_{(\vec{\lambda}, u_c)}(X; q) = \mathcal{G}_{(\vec{\lambda})}(X; q) s_{(1)}(X; q)$. Thus all but finitely many values of c will result in the same LLT polynomial, and it is unnecessary to confirm equality for all c.

If verified, this conjecture will yield a much simpler method for determining the LLT equivalence of multiskew partitions.

Chapter 3

k-Schur Positivity for k=3

In this section we seek to establish the k-Schur positivity of k-bandwidth LLT polynomials for $k = 3$. This will come in three steps:

1. Using LLT-equivalence, we will determine that all LLT polynomials are $\mathbb{N}[q, q^{-1}]$-linear combinations of LLT polynomials whose indices are drawn from a special set. These special LLT polynomials cannot be expanded into simpler terms, so we will call them 3-indecomposable, or indecomposable when there is no ambiguity.

2. Using a result of Haiman and Grojnowski, we will show that these 3-indecomposable LLT polynomials are in fact generalized Hall-Littlewood polynomials after applying ω.

3. We will show that these generalized Hall-Littlewood polynomials are in fact 3-Schur functions.

Let's see this in a bit more detail:

$$\mathcal{G}_{\vec{\lambda}}(X; q) = \sum_i q^{r_i} \mathcal{G}_{\vec{\mu}_{(i)}}(X; q) \text{ (expand into indecomposables)}$$

$$\omega\mathcal{G}_{\vec{\lambda}}(X; q) = \sum_i q^{r_i} \omega\mathcal{G}_{\vec{\mu}_{(i)}}(X; q)$$

$$= \sum_i q^{r_i} q^{s_i} P_{\vec{\nu}_i}(X; q) \text{ (indecomposables are generalized Hall-Littlewood polynomials)}$$

$$= \sum_i q^{r_i+s_i} s^{(3)}_{\vec{\nu}_i}(X; q) \text{ (these generalized Hall-Littlewood polynomials are 3-Schur functions)}$$

$$(3.1)$$

This confirms that $\omega\mathcal{G}_{\vec{\lambda}}(X; q)$ can be written as an $\mathbb{N}[q, q^{-1}]$-linear combination of 3-Schur functions. Since every k-Schur function has a monic term and all of the coefficients of $\omega\mathcal{G}_{\vec{\lambda}}(X; q)$ are polynomials in q, this is sufficient to determine that all coefficients $q^{r_i+s_i}$ in the above expansion are positive powers. Thus $\omega\mathcal{G}_{\vec{\lambda}}(X; q)$ is 3-Schur positive.

We will also refer to the the indexing multiskew partitions of 3-indecomposable LLT polynomials as 3-indecomposable. There are a few choices that can be made for the collection of 3-indecomposable multiskew partitions, but we will fix here an unambiguous definition.

Definition 3.0.1. *A 3-bandwidth multiskew partition $\vec{\lambda}$ is said to be indecomposable if it can be written as $(\vec{\lambda}_{init}, \vec{\lambda}_{hor}, \vec{\lambda}_{square}, \vec{\lambda}_{vert})$, where*

- *$\vec{\lambda}_{hor}$ is a sequence of 3-rectangles that are connected horizontal strips*

- *$\vec{\lambda}_{square}$ is a sequence of 3-rectangles that are 2x2 squares.*

- *$\vec{\lambda}_{vert}$ is a sequence of 3-rectangles that are connected vertical strips*

and $\vec{\lambda}_{init}$ is any of the following:

- *empty.*

- *a single skew partition containing one cell having content 0.*

- *a horizontal domino occupying the content interval $[0, 1]$ and then a vertical domino occupying the same content interval.*

Here a domino is any connected skew partition of size two, and we assume that all ccells have content in the interval $[0, 2]$

Intuitively, a multiskew partition is 3-indecomposable if it consists of a sequence of three rectangles preceded by a multiskew partition $\vec{\lambda}_{\text{init}}$. In Figure 3.1 we see all of the possible multiskew partitions $\vec{\lambda}_{\text{init}}$.
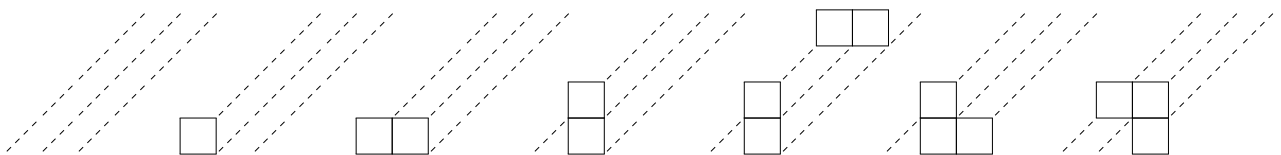
Figure 3.1: All multiskew vectors $\vec{\lambda}_{\text{init}}$.

In the first section, we will show that all 3-bandwidth LLT polynomials can be expanded as $\mathbb{N}[q, q^{-1}]$-linear combination of indecomposables.

3.1 Three-bandwidth LLT polynomials expand into indecomposables

This section relies heavily on the methods established in Section 2.3. We will establish a number of LLT-equivalence relations for 3-bandwidth multiskew partitions.

First, we must explain a nice property of LLT polynomials that we will call cycling.

Definition 3.1.1. *If $\vec{\lambda} = (\lambda^{(1)}/\mu^{(1)}, \lambda^{(2)}/\mu^{(2)}, \ldots, \lambda^{(d)}/\mu^{(d)})$, then we define the cycling operator by*

$$\text{cycle}(\vec{\lambda}) = (\kappa(\lambda^{(d)}/\mu^{(d)}), \lambda^{(1)}/\mu^{(1)}, \ldots, \lambda^{(d-1)}/\mu^{(d-1)}) \tag{3.2}$$

where $\kappa(\lambda^{(d)}/\mu^{(d)})$ is a skew partition with the same shape as $\lambda^{(d)}/\mu^{(d)}$ but all of the contents are increased by 1. Formally, $\kappa(\lambda^{(d)}/\mu^{(d)}) = (\lambda_1^{(d)}+1, \lambda_2^{(d)}+1, \ldots, \lambda_{\ell(\lambda^{(d)})}^{(d)}+1, \lambda_{\ell(\lambda^{(d)})+1}^{(d)}, \ldots)/(\mu_1^{(d)}+1, \mu_2^{(d)}+1, \ldots, \mu_{\ell(\lambda^{(d)})}^{(d)}+1, \mu_{\ell(\lambda^{(d)})+1}^{(d)}, \ldots).$

Intuitively, cycling increments each partition to the next entry. Since the last partition has no subsequent entry, it instead increments to the front, but also the content of every cell increases. See Figure 3.2 Note that this process is invertible, allowing that we identify multiskew partitions that are diagonal translations of one-another. Following the Stanton-White correspondence, one can see that this action corresponds precisely to increasing the content of every cell in a d-tileable partition by 1. This is left as an exercise to the reader, and is an alternative way to prove the following proposition.
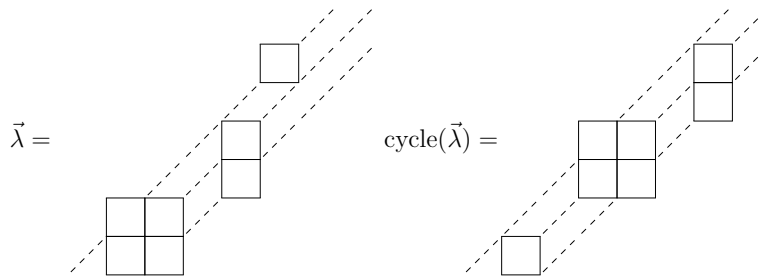
Figure 3.2: An example of cycling

Proposition 3.1.2. *If $\vec{\lambda}$ is a multiskew partition, then $\mathcal{G}_{\vec{\lambda}}(X; q) = \mathcal{G}_{\text{cycle}(\vec{\lambda})}(X; q)$.*

Proof. Observe that cycling descends to a bijection on semistandard tableaux by rotating the entries the same way it rotates the cells. Additionally, this map preserves the content

reading word, so that $\mathrm{inv}(T) = \mathrm{inv}(\mathrm{cycle}(T))$ for any tableau T. Thus

$$
\begin{aligned}
\mathcal{G}_{\vec{\lambda}}(X;q) &= \sum_{T \in \mathrm{SSYT}(\vec{\lambda})} q^{\mathrm{inv}(T)} x^T \\
&= \sum_{T \in \mathrm{SSYT}(\vec{\lambda})} q^{\mathrm{inv}(\mathrm{cycle}(T))} x^{\mathrm{cycle}(T)} \\
&= \sum_{T' \in \mathrm{cycle}(\mathrm{SSYT}(\vec{\lambda}))} q^{\mathrm{inv}(T')} x^{T'} \\
&= \sum_{T' \in \mathrm{SSYT}(\mathrm{cycle}(\vec{\lambda}))} q^{\mathrm{inv}(T')} x^{T'} \\
&= \mathcal{G}_{\mathrm{cycle}(\vec{\lambda})}(X;q)
\end{aligned}
\tag{3.3}
$$

$\square$

We can use this fact to derive a consequence of LLT-equivalence.

Proposition 3.1.3. *Let $\sum_i a_i(q)\vec{\lambda}_{(i)}$ and $\sum_j b_j(q)\vec{\nu}_{(j)}$ be LLT-equivalent $\mathbb{N}[q]$-linear combinations of multiskew partitions. If $\vec{\gamma}$ and $\vec{\mu}$ are multiskew partitions, then*

$$
\sum_i a_i(q)\mathcal{G}_{(\vec{\gamma}, \vec{\lambda}_{(i)}, \vec{\mu})}(X;q) = \sum_j b_j(q)\mathcal{G}_{(\vec{\gamma}, \vec{\nu}_{(j)}, \vec{\mu})}(X;q). \tag{3.4}
$$

Proof. If $\vec{\gamma} = (\gamma^{(1)}/\tau^{(1)}, \gamma^{(2)}/\tau^{(2)}, \ldots, \gamma^{(g)}/\tau^{(g)})$, then we can compute

$$
\mathrm{cycle}^{-g}(\vec{\gamma}, \vec{\lambda}_{(i)}, \vec{\mu}) = (\vec{\lambda}_{(i)}, \vec{\mu}, \kappa^{-1}(\gamma^{(1)}/\tau^{(1)}), \kappa^{-1}(\gamma^{(2)}/\tau^{(2)}), \ldots, \kappa^{-1}(\gamma^{(g)}/\tau^{(g)})) \tag{3.5}
$$

where $\kappa^{-1}(\gamma/\tau) = (\gamma_1, \gamma_1, \gamma_2, \gamma_3, \ldots,)/(\gamma_1, \tau_1, \tau_2, \tau_3, \ldots)$. This is effectively an inverse to the map κ in Definition 3.1.2, since we identify skew partitions that are the same shape and have cell-wise the same content. Note that κ^{-1} decreases the content of each cell by at most one.

If we use the shorthand $\kappa^{-1}(\vec{\gamma}) = (\kappa^{-1}(\gamma^{(1)}/\tau^{(1)}), \kappa^{-1}(\gamma^{(2)}/\tau^{(2)}), \ldots, \kappa^{-1}(\gamma^{(g)}/\tau^{(g)}))$, then LLT-equivalence tells us that

$$
\sum_i a_i(q)\mathcal{G}_{(\vec{\lambda}_{(i)}, \vec{\mu}, \kappa^{-1}(\vec{\gamma}))} = \sum_j a_i(q)\mathcal{G}_{(\vec{\nu}_{(j)}, \vec{\mu}, \kappa^{-1}(\vec{\gamma}))} \tag{3.6}
$$

since $\vec{\lambda}_{(i)}$ and $\vec{\nu}_{(j)}$ are the leftmost multiskew partitions in each summand. Putting this together yields the desired equation.

$$\sum_i a_i(q)\mathcal{G}_{(\vec{\gamma},\vec{\lambda}_{(i)},\vec{\mu})}(X;q) = \sum_i a_i(q)\mathcal{G}_{\text{cycle}^{-k}(\vec{\gamma},\vec{\lambda}_{(i)},\vec{\mu})}(X;q)$$

$$= \sum_i a_i(q)\mathcal{G}_{(\vec{\lambda}_{(i)},\vec{\mu},\kappa^{-1}(\vec{\gamma}))}(X;q)$$

$$= \sum_j a_i(q)\mathcal{G}_{(\vec{\nu}_{(j)},\vec{\mu},\kappa^{-1}(\vec{\gamma}))}(X;q) \qquad (3.7)$$

$$= \sum_j b_j(q)\mathcal{G}_{\text{cycle}^{-k}(\vec{\gamma},\vec{\nu}_{(j)},\vec{\mu})}(X;q)$$

$$= \sum_j b_j(q)\mathcal{G}_{(\vec{\gamma},\vec{\nu}_{(j)},\vec{\mu})}(X;q)$$

$$\square$$

The point of this theorem is largely to simplify notation, because now we don't need to cycle our multiskew partitions before using LLT-equivalence. The intuition behind the theorem is that LLT-equivalence is actually a statement about arbitrary sequences of skew partitions appended before or after the multiskew partitions of interest.

The method outlined in the previous section can be used to validate LLT-equivalence relations. We find that these equations often take on the form

$$(\lambda^{(1)}, \lambda^{(2)}) = q^m(\lambda^{(2)}, \lambda^{(1)}) \qquad (3.8)$$

where m is some integer. We will say that such pairs of skew partitions $\lambda^{(1)}, \lambda^{(2)}$ are commuting. These pairs are of interest to us, because they allow us to rearrange the entries of multiskew partitions while keeping essentially the same LLT polynomial. We display all the commuting relations in the table in appendix A, which is read as follows: If $\lambda^{(1)}$ and $\lambda^{(2)}$ satisfy equation 3.8, then m can be found in row $\lambda^{(1)}$ and column $\lambda^{(2)}$ in the table in appendix A.
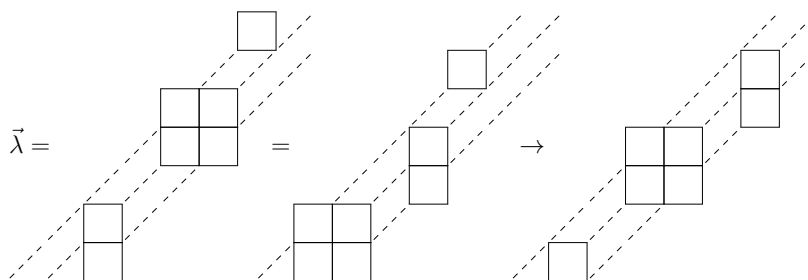
Figure 3.3: An LLT-equivalence relation and cycling

In Figure 3.3 we have an example of commuting partitions. If $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$, then $\lambda^{(1)}$ and $\lambda^{(2)}$ are commuting. We look at row $\lambda^{(1)}$ and column $\lambda^{(2)}$ in the table in appendix

A to find the LLT-equivalence relation $(\lambda^{(1)}, \lambda^{(2)}) = q(\lambda^{(2)}, \lambda^{(1)})$. Every entry in the table in appendix A corresponds to such an equation. But there are also many empty cells in the table, which correspond to pairs that do not commute. In each of these cases, there is a decomposition of the corresponding LLT polynomial as a sum of several LLT polynomials. These are written out in appendix A as well.

In Figure 3.3, we also introduce some notation. Here an equality $(=)$ refers to LLT-equivalence, and an arrow $(\rightarrow)$ refers to either the operator cycle or the operator cycle^{-1}. Since LLT-equivalence and cycling both preserve LLT polynomials up to a power of q, this notation really demonstrates to us how LLT polynomials decompose. We will ignore everywhere powers of q, because in our proofs we are only trying to establish the existence of $\mathbb{N}[q, q^{-1}]$-linear combinations, we do not care about what exactly these coefficients are. We will also simplify our proofs by making the arbitrary choice that all of our skew partitions have content in the interval $[0, 2]$ unless otherwise stated. We will use these relations to decompose any LLT polynomial into a sum of polynomials whose multiskew partitions pairwise commute.

Lemma 3.1.4. *Let $\vec{\lambda}$ be any 3-bandwidth multiskew partition. Then $\mathcal{G}_{\vec{\lambda}}$ can be written as a $\mathbb{N}[q, q^{-1}]$-linear combination of LLT polynomials indexed by multiskew partitions $\vec{\nu}$ such that for every i, j, $\nu^{(i)}$ and $\nu^{(j)}$ commute.*

Proof. Recall that a 3-rectangle is a rectangular partition occupying three content lines. We proceed by strong induction on the number n of indices i such that $\lambda^{(i)}$ is not a 3-rectangle.

If $n = 0$, then $\vec{\lambda}$ consists entirely of 3-rectangles. Note from the table in appendix A that 3-rectangles commute with every skew partition. Thus all of the entries pairwise commute.

If $n = 1$, then $\vec{\lambda}$ must have exactly 1 entry which is not a 3-rectangle. Note from table A that this skew partition necessarily commutes with every other skew partition, because all the rest are 3-rectangles, which commute with every skew partition.

We assume the lemma for the case that there are n skew partitions that are not 3-rectangles.

Suppose there are $n + 1$ many skew partitions that do not commute with at least one other skew partition. Let $\{a_i\}_{i=0}^{d-n-1}$ be the set such that $\lambda^{(a_i)}$ commutes with all the entries of $\vec{\lambda}$. Let $\{b_i\}_{i=0}^{n+1}$ be the complementary indices. By moving those commuting partitions to the right, we see that there exists $m \in \mathbb{Z}$ such that $\mathcal{G}_{\vec{\lambda}} = q^m \cdot \mathcal{G}_{(\lambda^{(b_1)}, \dots, \lambda^{(b_{n+1})}, \lambda^{(a_1)}, \dots \lambda^{(a_{d-n-1})})}$.

By definition, $\lambda^{(b_1)}$ does not commute with some entry of $\{\lambda^{(b_i)}\}_{i=1}^{n+1}$. Let b_k be the smallest such index. Then $\lambda^{(b_1)}$ commutes with $\lambda^{(b_j)}$ for $j \in [1, k-1]$. Thus there exists $m \in \mathbb{Z}$ such that $\mathcal{G}_{\vec{\lambda}} = q^m \cdot \mathcal{G}_{(\lambda^{(b_2)}, \dots, \lambda^{(b_1)}, \lambda^{(b_k)}, \dots \lambda^{(b_{n+1})}, \lambda^{(a_1)}, \dots \lambda^{(a_{d-n-1})})}$. In particular, there are two adjacent terms $\lambda^{(b_1)}, \lambda^{(b_k)}$. Thus we see it is sufficient to consider the case where there is an adjacent pair of skew partitions that do not commute.

Reindexing and relabeling, suppose there are $n + 1$ many skew partitions in $\vec{\lambda}$ that are not 3-rectangles. Without loss of generality we can assume that there is an adjacent pair with indices k and $k + 1$ that do not commute with each other. Neither can be a 3-rectangle, as 3-rectangles commute with all skew partitions. We see appendix A for an LLT-equivalent

decomposition of this pair, and note that all pairs decompose into a sum of two terms. There are two cases, which depend on both the pair of skew partitions, and the two terms in the sum.

Case 1. Both terms in the sum have at most one entry that is not a 3-rectangle. In this case, we have a decomposition that replaces a multiskew partition containing $n + 1$ entries that are not 3-rectangles with two terms, each containing either $n - 1$ or n entries that are not 3-rectangles. This case is complete by induction.

Case 2a. One partition in the pair is a skew hook shape, and that other is a straight hook shape. This occurs in just two situations, see for example Figure 3.4.
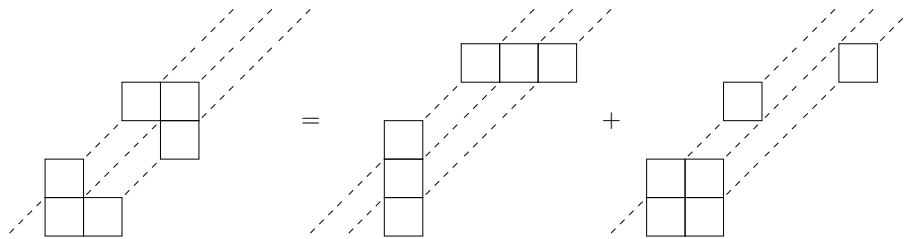
Figure 3.4: A decomposition of two hook shapes.

Case 2b. One partition in the pair is a skew hook shape or straight hook shape, and the other is a single cell with content 1. This occurs in just four situations, see for example Figure 3.5.
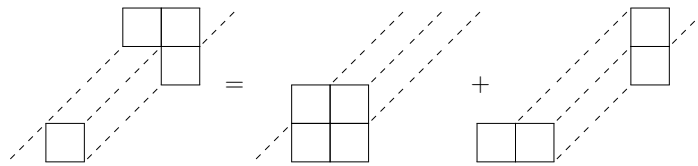
Figure 3.5: A decomposition of a single cell and a hook shape.

In either Case 2a or Case 2b there are two terms in the decomposition. One of these terms yields $n - 1$ many entries that are not 3-rectangles, so we can ignore it. The other term is of interest: it yields $n + 1$ entries that are not 3-rectangles, which offers no improvement. But we can proceed as above to permute the entries of this multiskew partition so that two non-commuting skew partitions are adjacent, and repeat this inductive step. Note that the processes in Case 2a and Case 2b remove either one or two hook shapes. After repeatedly rearranging and decomposing, either all skew partitions will commute or there will be no more hook shapes and we will be in Case 1, resulting in a decrease in the number of entries that are not 3-rectangles and the completion of this proof.

□

We note that the decomposition described here uses only the LLT-equivalence relations in Appendix A. This fact can be used to strengthen the statement of Lemma 3.1.4, but we will not use this. Instead we note that the number of cells in each content line must be constant across all multiskew partitions considered in such a decomposition. While this is in fact true of all LLT-equivalence relations, perhaps the quickest way to see this is to observe that all of the relations in Appendix A have this property.

Lemma 3.1.5. *Let $\vec{\lambda}$ be any sequence of 3-bandwidth skew partitions. Then $\mathcal{G}_{\vec{\lambda}}$ can be written as an $\mathbb{N}[q, q^{-1}]$-linear combination of LLT polynomials indexed by multiskew partitions such that all entries pairwise commute, and each skew partition occupies the content interval $[2], [1, 2]$, or $[0, 2]$.*

Proof. Following Lemma 3.1.4, we may assume without loss of generality that the entries of $\vec{\lambda}$ pairwise commute. Then the partitions may be reorganized such $\vec{\lambda} = (\vec{\lambda}_{[2]}, \vec{\lambda}_{[1,2]}, \vec{\lambda}_{[0,2]}, \vec{\lambda}_{[0,1]}, \vec{\lambda}_{[0]}, \vec{\lambda}_{[1]})$, where

- $\vec{\lambda}_{[2]}$ consists of partitions occupying the content interval content $[2]$

- $\vec{\lambda}_{[1,2]}$ consists of partitions occupying the content interval content $[1, 2]$

- $\vec{\lambda}_{[0,2]}$ consists of partitions occupying the content interval content $[0, 2]$

- $\vec{\lambda}_{[0,1]}$ consists of partitions occupying the content interval content $[0, 1]$

- $\vec{\lambda}_{[0]}$ consists of partitions occupying the content interval content $[0]$

- $\vec{\lambda}_{[1]}$ consists of partitions occupying the content interval content $[1]$

Since this is every possible content interval, this form is general, and we will call it commuting form. In Figure 3.6 we see two examples of multiskew partitions in commuting form.

So we assume that $\vec{\lambda}$ is in commuting form. We proceed by cycling. Recall that cycling a multiskew partiton preserves the corresponding LLT polynomial, and increases the content of all of the cells of the last skew partition, while repositioning it to be the leading skew partition. We apply cycle a many times, where is the number of entries of $\vec{\lambda}$ that are also in $\vec{\lambda}_{[0,1]}, \vec{\lambda}_{[0]}$ or $\vec{\lambda}_{[1]}$. Then cycle$^a(\vec{\lambda})$ is of the following form.

1. Partitions occupying contents $[1, 2]$

2. Partitions occupying content $[1]$

3. Partitions occupying content $[2]$

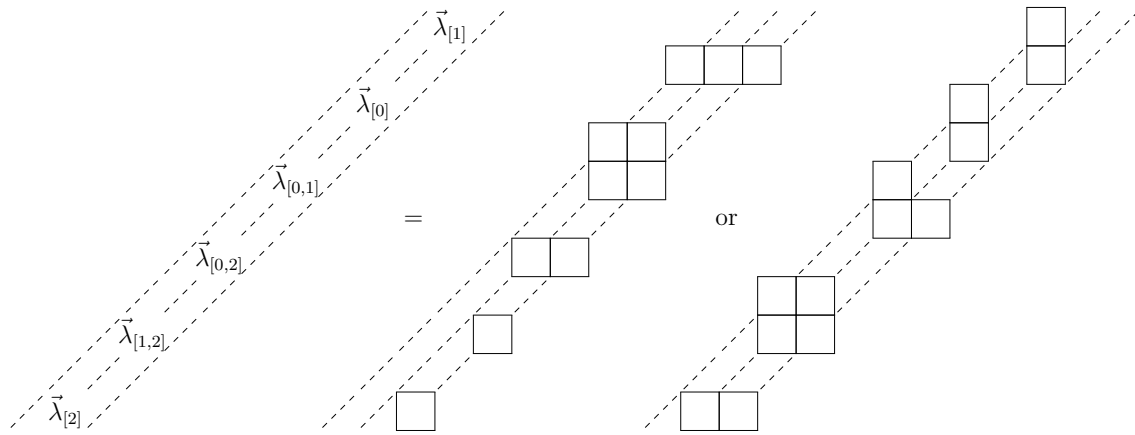4. Partitions occupying contents $[1, 2]$

Figure 3.6: Mutliskew partitions in commuting form

5. Partitions occupying contents $[0, 2]$

It is uncertain whether or not the entries of $\text{cycle}^a(\vec{\lambda})$ actually pairwise commute. So we proceed as in Lemma 3.1.4 and write $\text{cycle}^a(\vec{\lambda})$ as a sum of terms where the multiskew partitions pairwise commute. Picking any one, we are back to the start of this proof. Call this new multiskew partition $\vec{\gamma}$. Since the expansion from Lemma 3.1.4 preserves the number of boxes on each content line, the overall content of $\vec{\gamma}$ is greater than the overall content of $\vec{\lambda}$. Formally, we let $c(\vec{\lambda})$ be the sum of the contents of all cells in all entries of $\vec{\lambda}$. Then it is clear that $c(\vec{\gamma}) = c(\text{cycle}^a(\vec{\lambda})) \geq c(\vec{\lambda}) + a$, since cycling increases the overall content by at least one.

Thus we have a multi-step process that applies to any multiskew partition $\vec{\lambda}$ such that the entries of $\vec{\lambda}$ pairwise commute:

1. Permute the entries of $\vec{\lambda}$ so that it's in commuting form.

2. Let a be the number of entries in $\vec{\lambda}_{[0,1]}, \vec{\lambda}_{[0]}$ or $\vec{\lambda}_{[1]}$, and compute $\text{cycle}^a(\vec{\lambda})$.

3. Expand $\text{cycle}^a(\vec{\lambda})$ into multiskew partitions whose entries pairwise commute.

4. Consider any one of the resulting multiskew partitions.

Each iteration of this process changes the class of relevant multiskew partitions to a class of ones with larger overall content. Since all of our multiskew partitions are assumed to be on the content lines $0, 1$ and 2, the overall content of any multiskew partition is bounded above by $2n$, where n is the number of cells in the multiskew partition. So this process must terminate. Moreover, repeated application of this process limits to a class of multiskew partitions where all the skew partitions pairwise commute, but $a = 0$ in step 2. This means that $\vec{\lambda}_{[0,1]}, \vec{\lambda}_{[0]}$ and $\vec{\lambda}_{[1]}$ are all empty sequences. Thus the terminal class of multiskew partitions are sequences of partitions that can be written as $(\vec{\lambda}_{[2]}, \vec{\lambda}_{[1,2]}, \vec{\lambda}_{[0,2]})$, where

- $\vec{\lambda}_{[2]}$ consists of partitions occupying the content interval content $[2]$.

- $\vec{\lambda}_{[1,2]}$ consists of partitions occupying the content interval content $[1, 2]$.

- $\vec{\lambda}_{[0,2]}$ consists of partitions occupying the content interval content $[0, 2]$.

This matches the form in the statement of this lemma, so we have completed the proof.
$\square$

We will call the from in the statement of Lemma 3.1.5 expanding form.

Lemma 3.1.6. *Let $\vec{\lambda}$ be any sequence of 3-bandwidth skew partitions. Then $\mathcal{G}_{\vec{\lambda}}(X; q)$ can be written as an $\mathbb{N}[q, q^{-1}]$-linear combination of LLT polynomials indexed by multiskew partitions that satisfy either of the following:*

- *A sequence of 3-rectangles and hook shapes such that all skew partitions pairwise commute.*

- *A sequence of single cells with content 2, dominoes occupying the content interval $[1, 2]$, and 3-rectangles such that all skew partitions pairwise commute.*

Proof. Following Lemma 3.1.4 and Lemma 3.1.5, it is sufficient to prove this lemma for an arbitrary multiskew partition $\vec{\lambda}$ in expanding form.

If $\vec{\lambda}$ contains no entries that are hook shapes, then it has the desired form and we are done. Otherwise, $\vec{\lambda}$ contains some hook shapes. Since all entries of $\vec{\lambda}$ pairwise commute, we can assume that all hook shapes appear at the end of the sequence. There are only two 3-bandwidth hook shape skew partitions. Up to translation they are $\gamma_l = (2, 1)$ and $\gamma_r = (2, 2)/(1)$. Since γ_l and γ_r do not commute (see Appendix A), at most one of them appears in $\vec{\lambda}$.

We will continue by considering multiple cases. In each we will decompose $\vec{\lambda}$ into several terms that have strictly fewer entries that are hook shapes while still being in expanding form. Since $\vec{\lambda}$ is in expanding form, all of its entries pairwise commute. So without loss of generality we can assume that $\vec{\lambda} = (\vec{\lambda}_{[2]}, \vec{\lambda}_{[1,2]}, \vec{\lambda}_{\text{rect}}, \vec{\lambda}_{\text{hook}})$, where

- $\vec{\lambda}_{[2]}$ consists of partitions occupying the content interval content $[2]$.

- $\vec{\lambda}_{[1,2]}$ consists of partitions occupying the content interval content $[1, 2]$.

- $\vec{\lambda}_{\text{rect}}$ consists of partitions that are 3-rectangles in the content interval content $[0, 2]$.

- $\vec{\lambda}_{\text{hook}}$ consists of translations of either γ_l or γ_r with content in the interval $[0, 2]$.

Case 1: $\vec{\lambda}_{[2]}$ is nonempty, and $\vec{\lambda}_{\text{hook}}$ contains one or more hook shapes γ_l. Then the last entry of $\vec{\lambda}$ is γ_l and the first entry is a single cell with content 2. We proceed as in Figure 3.7.
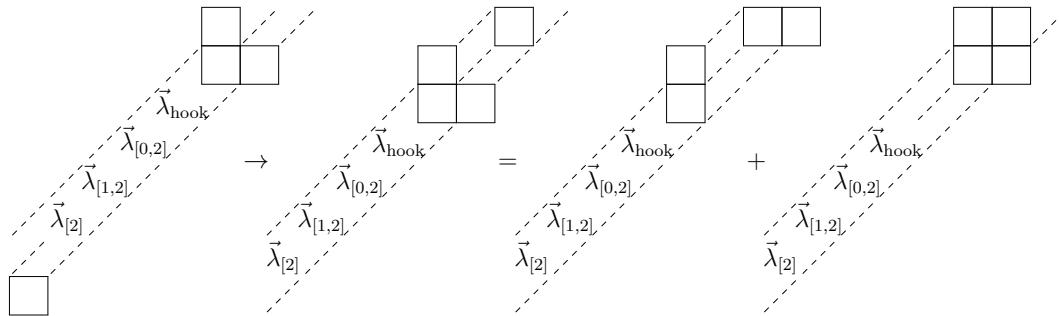

Figure 3.7: Case 1

Figure 3.7 yields two multiskew partitions. The entries of the second multiskew partition all pairwise commute, and they can be permuted to be in expanding form. The first multiskew partition requires some more work, which we display in Figure 3.8.
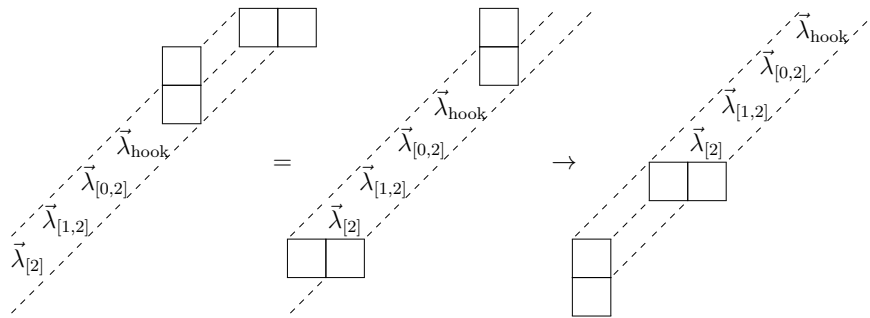
Figure 3.8: Case 1 continued

The procedure in Figure 3.8 results in a multiskew partition that we will call $\vec{\lambda}'$. The next step is to consider further cases:

Case 1a: The subsequence $\vec{\lambda}_{\text{hook}}$ in $\vec{\lambda}'$ is empty. In this case we are done, as all entries of $\vec{\lambda}'$ pairwise commute and it is in expanding form.

Case 1b: The subsequence $\vec{\lambda}_{\text{hook}}$ in $\vec{\lambda}'$ is nonempty. Since $\vec{\lambda}'$ contains a vertical domino with content in the interval $[1, 2]$ and a hook shape γ_l, not all of its entries pairwise commute. So we must proceed with one more expansion, which we display in Figure 3.9. The result of this decomposition is two terms in which all the entries pairwise commute, so they can be permuted to be in expanding form.
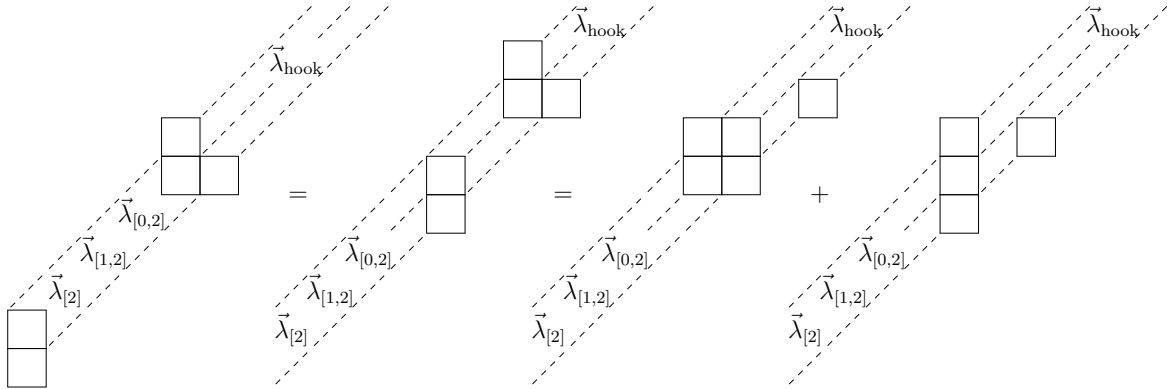
Figure 3.9: Case 1b

The procedure in Case 1 replaces a multiskew partition in expanding form with several multiskew partitions in expanding form, except the new ones have strictly fewer hook shapes than the original. Thus we can apply this process iteratively until each multiskew partition either has no hook shapes, or has no single cells with content 1. This leads us to the next case.

Case 2: $\vec{\lambda}_{[2]}$ is empty, $\vec{\lambda}_{[1,2]}$ contains one or more horizontal dominos, and $\vec{\lambda}_{\text{hook}}$ contains one or more hook shapes γ_l. Without loss of generality, the last entry of $\vec{\lambda}$ is a γ_l and the first entry is a horizontal domino with content in the interval $[1, 2]$. Then we proceed as in Figure 3.10.
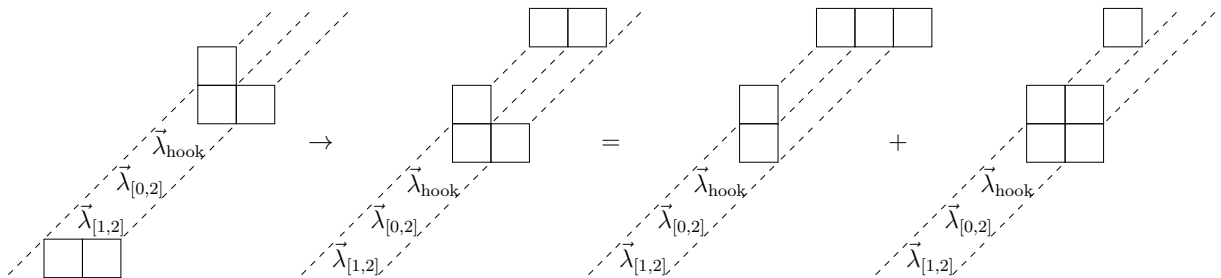
Figure 3.10: Case 2

There are two terms in the decomposition in Figure 3.10, and we must write each in terms of multiskew partitions in expanding form. The first term can be resolved to Case 1a or Case 1b, as in Figure 3.11.
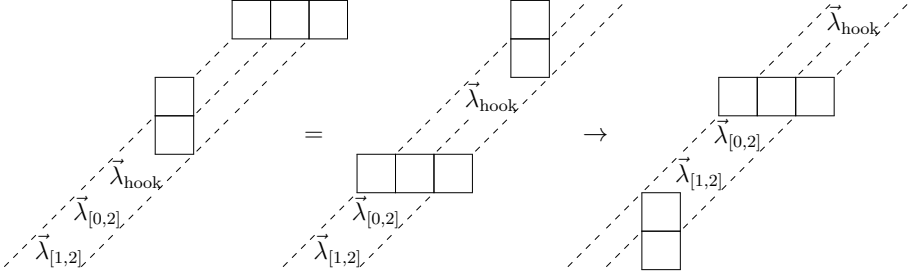
Figure 3.11: Case 2, term 1

The second term in the decomposition in Figure 3.10, requires more work, as seen in Figure 3.12.
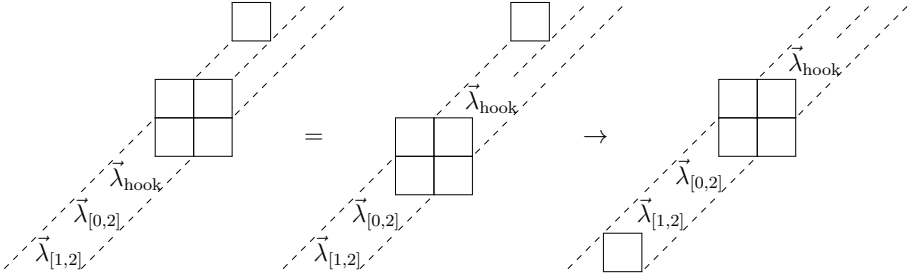
Figure 3.12: Case 2, term 2

We call the last multiskew partition in Figure 3.12 $\vec{\lambda}''$. Then there are two cases, depending on how many hook shapes are in $\vec{\lambda}''$

Case 2a: The subsequence $\vec{\lambda}_{\text{hook}}$ in $\vec{\lambda}''$ is empty. Then we proceed as in Figure 3.13 to put the multiskew partition into expanding form.
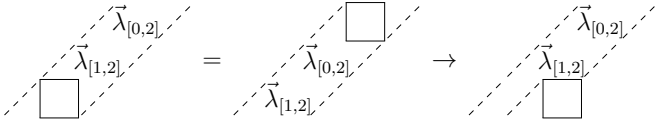
Figure 3.13: Case 2a

Case 2b: The subsequence $\vec{\lambda}_{\text{hook}}$ in $\vec{\lambda}''$ is nonempty. Then we proceed as in Figure 3.14.
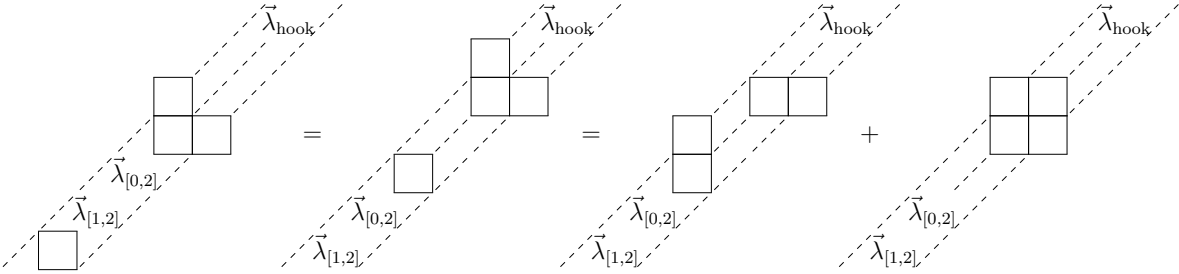
Figure 3.14: Case 2b

The expansion in Figure 3.14 results in two multiskew partitions. The second is already in expanding form, but the first needs to be touched up. We put this multiskew partition into the form of Case 1b by following Figure 3.15.
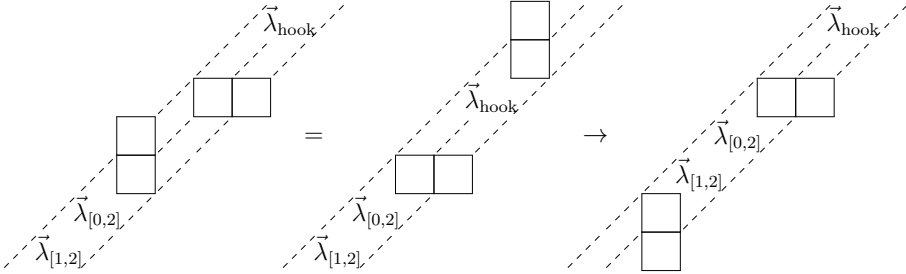
Figure 3.15: Case 2b continued

Both Case 1 and Case 2 expand a given multiskew partition in terms of others that have strictly fewer hook shapes. By repeatedly applying each case, we are eventually left with muliskew partitions in expanding form that satisfy $\vec{\lambda}_{\text{hook}} = \emptyset$ or $\vec{\lambda}_{[2]} = \vec{\lambda}_{[1,2]} = \emptyset$. Thus every multiskew partition can be expanded into of multiskew partitions of this type.

There are two additional cases that are left as exercises to the reader, since they are essentially the same as the proofs presented here.

Case 3: $\vec{\lambda}_{[2]}$ is nonempty, and $\vec{\lambda}_{\text{hook}}$ contains one or more hook shapes γ_r. See Figure 3.16

Case 4: $\vec{\lambda}_{[2]}$ is empty, $\vec{\lambda}_{[1,2]}$ contains one or more vertical dominos, and $\vec{\lambda}_{\text{hook}}$ contains one or more hook shapes γ_r. See Figure 3.16.
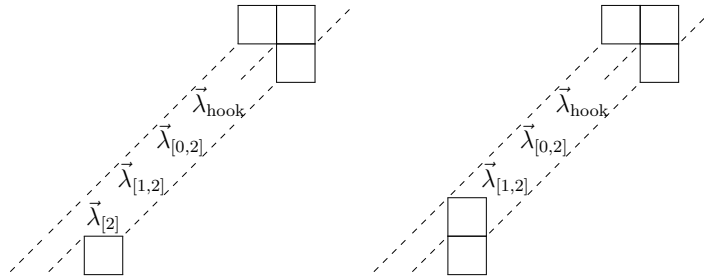
Figure 3.16: Case 3 (left) and Case 4 (right)

$\square$

The result of this lemma is that we need only consider multiskew partitions in expanding form of two types: those with hook shapes, and those without. We will deal with each case separately in the next two lemmas, and show that they can be simplified yet further.

Lemma 3.1.7. *Let $\vec{\lambda}$ be a 3-bandwidth multiskew partition in expanding form that contains no hook shapes. Then $\mathcal{G}_{\vec{\lambda}}(X;q)$ can be written as a $\mathbb{N}[q,q^{-1}]$-linear combination of LLT polynomials indexed by partitions satisfying any of the following:*

- *A sequence of 3-rectangles followed by at most one hook shape.*

- *A single cell with content 2 or a domino with content in the interval $[1,2]$ followed by a sequence of 3-rectangles.*

- *A horizontal domino with content in the interval $[1,2]$ followed by a vertical domino with content in the interval $[1,2]$ followed by a sequence of 3-rectangles.*

Proof. Given $\vec{\lambda}$ as described, we first note that all entries of $\vec{\lambda}$ pairwise commute. Up to a power of q, we can rearrange the terms and still obtain an LLT-equivalent multiskew partition. Without loss of generality, we assume $\vec{\lambda} = (\vec{\lambda}_{\text{cell}}, \vec{\lambda}_{\text{hor}}, \vec{\lambda}_{\text{vert}}, \vec{\lambda}_{\text{rect}})$, where

- $\vec{\lambda}_{\text{cell}}$ is a sequence of single cells with content 2

- $\vec{\lambda}_{\text{hor}}$ is a sequence of horizontal dominoes with content in the interval $[1,2]$

- $\vec{\lambda}_{\text{vert}}$ is a sequence of vertical dominoes with content in the interval $[1,2]$

- $\vec{\lambda}_{\text{rect}}$ is a sequence of 3-rectangles with content in the interval $[0,2]$

We will say that a multiskew partition of this form is in rectangular expanding form. See Figure 3.17 for an example.
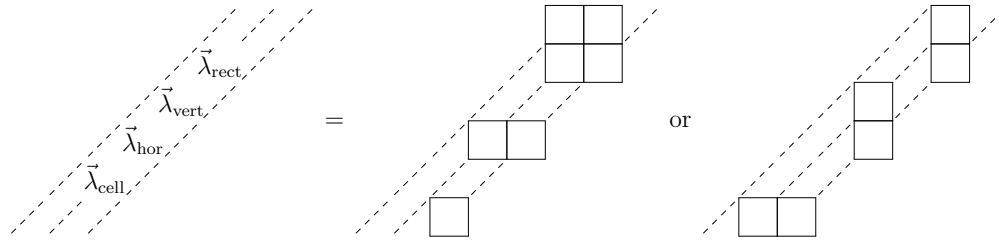
Figure 3.17: Rectangular expanding form

We will decompose multiskew partitions in rectangular expanding form into other multiskew partitions in rectangular expanding form, while reducing the number of entries that are not 3-rectangless. As in the previous lemmas, we will follow an algorithm that has several cases.

Case 1: $\vec{\lambda}_{\text{cell}}$ consists of more than one cell. Then we proceed as in Figure 3.18. We note that this process reduces the number of entries that are not 3-rectangles by one. Further, all entries in both multiskew partition pairwise commute, so the entries can be permuted into expanding form.
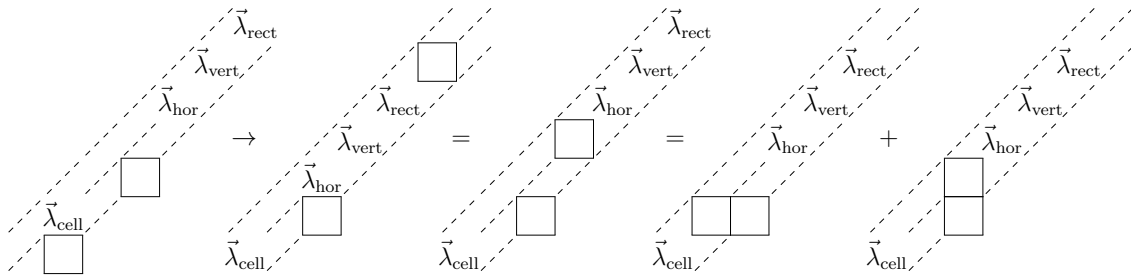
Figure 3.18: Case 1

Case 2: $\vec{\lambda}_{\text{cell}}$ consists at most one cell, while $\vec{\lambda}_{\text{hor}}$ consists of more than one domino. Then we proceed as in Figure 3.19.
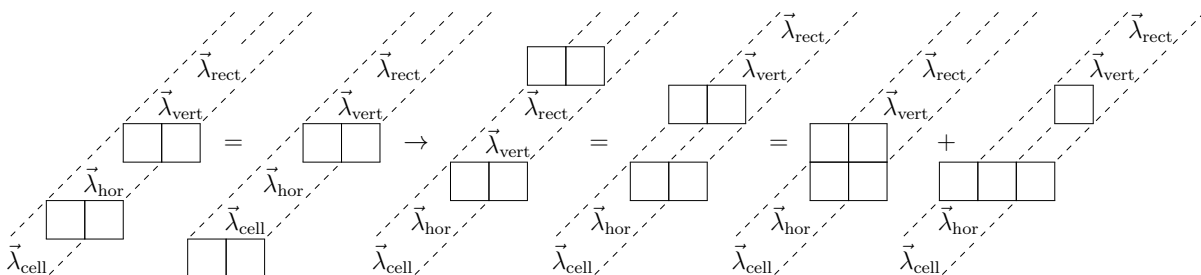
Figure 3.19: Case 2

In Figure 3.19, we are left with two multiskew partitions. When compared to the original multiskew partition $\vec{\lambda}$, both have strictly fewer entries that are not 3-rectangles. In the first

multiskew partition, all of the entries pairwise commute, and we can permute the entries until it is in rectangular expanding form. The second multiskew partition can also be put into rectangular expanding form, which we display in Figure 3.20.
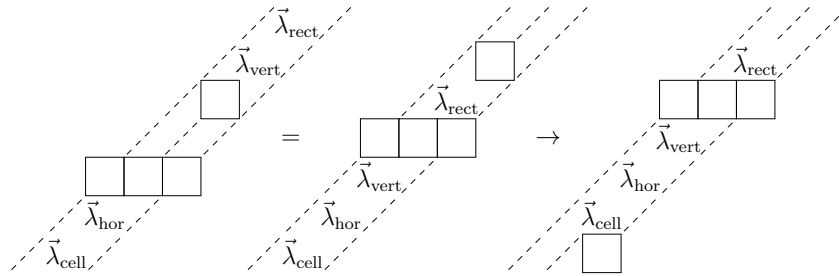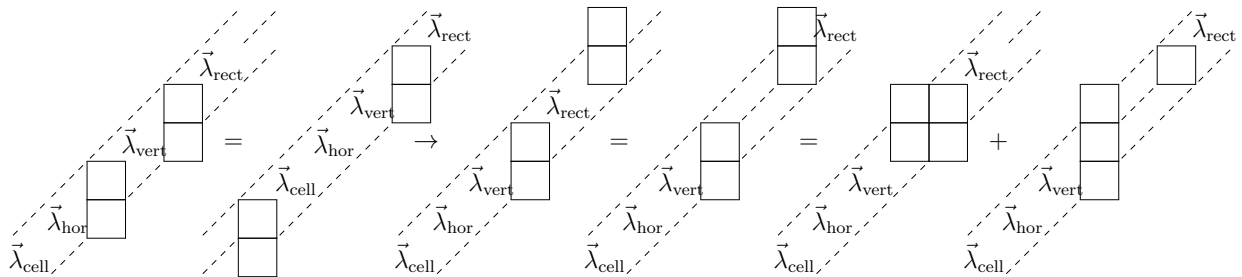
Figure 3.20: Case 2 continued

Case 3: $\vec{\lambda}_{\text{cell}}$ and $\vec{\lambda}_{\text{hor}}$ both consist of at most one entry, while $\vec{\lambda}_{\text{vert}}$ consists of more than one domino. Then we proceed as in Figure 3.21.

Figure 3.21: Case 3

In Figure 3.21, we are again left with two multiskew partitions. When compared to the original multiskew partition $\vec{\lambda}$, both have strictly fewer entries that are not 3-rectangles. The first multiskew partition is already in rectangular expanding form. The second multiskew partition can also be put into rectangular expanding form, which we see in Figure 3.22.
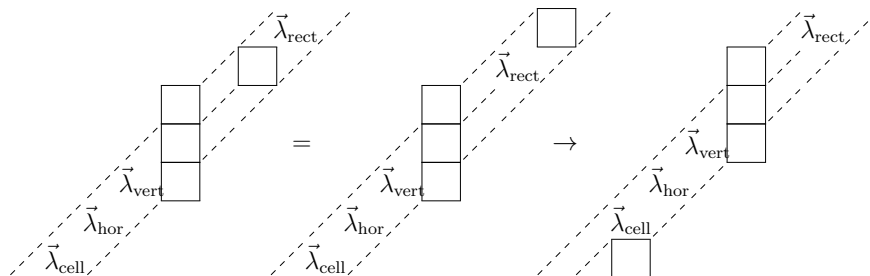
Figure 3.22: Case 3 continued

We have described this algorithm over the three cases. This algorithm expands an LLT polynomial into several LLT polynomials whose indexing multiskew partitions all have strictly fewer entries that are not 3-rectangles, and it eventually terminates when there are one or fewer entries in each of $\vec{\lambda}_{\text{cell}}$, $\vec{\lambda}_{\text{hor}}$ and $\vec{\lambda}_{\text{vert}}$. Note that the resulting multiskew partitions are also in rectangular expanding form.

There are a few cases, depending on which of $\vec{\lambda}_{\text{cell}}$, $\vec{\lambda}_{\text{hor}}$ and $\vec{\lambda}_{\text{vert}}$ are nonempty.

Case A: If $\vec{\lambda}_{\text{cell}}$ is empty, then there is at most one horizontal domino, at most one vertical domino, and any number of 3-rectangles in the multiskew partition. This is one of the desired forms, so we are done.

Case B: If $\vec{\lambda}_{\text{cell}}$ has one entry, but $\vec{\lambda}_{\text{hor}}$ and $\vec{\lambda}_{\text{vert}}$ are both empty. Then the multiskew partition consists of just a single cell followed by 3-rectangles, which is also of the desired form.

Case C: If $\vec{\lambda}_{\text{cell}}$ and $\vec{\lambda}_{\text{hor}}$ have one entry, but $\vec{\lambda}_{\text{vert}}$ is empty. Then we proceed as in Figure 3.23. The entries of the resulting multiskew partitions pairwise commute, and they can be permuted to be in the desired form.
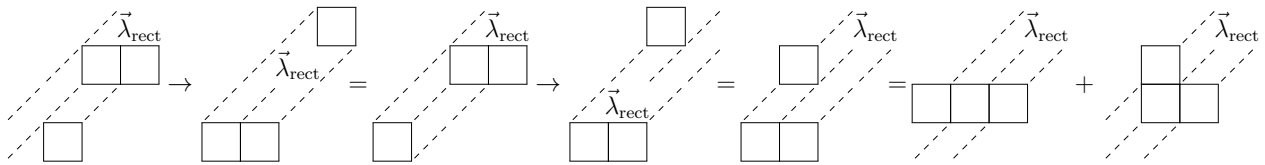

Figure 3.23: Case C

Case D: If $\vec{\lambda}_{\text{cell}}$ and $\vec{\lambda}_{\text{vert}}$ have one entry, but $\vec{\lambda}_{\text{hor}}$ is empty. Then we proceed as in Figure 3.24. The entries of the resulting multiskew partitions pairwise commute, and they can be permuted to be in the desired form.
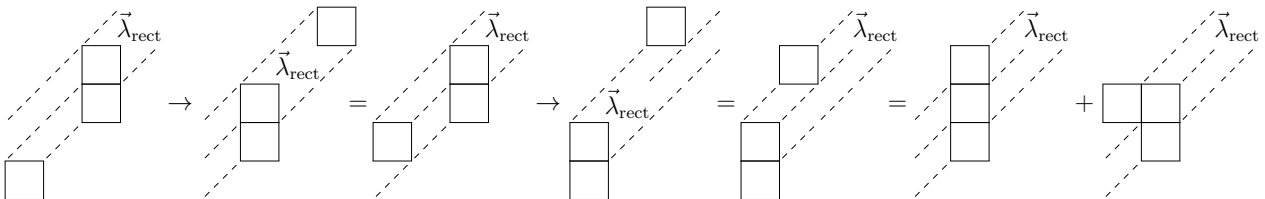

Figure 3.24: Case D

Case E: If $\vec{\lambda}_{\text{cell}}$, $\vec{\lambda}_{\text{vert}}$ and $\vec{\lambda}_{\text{hor}}$ all have one entry. Then we proceed as in Figure 3.25, Figure 3.26 and Figure 3.27.
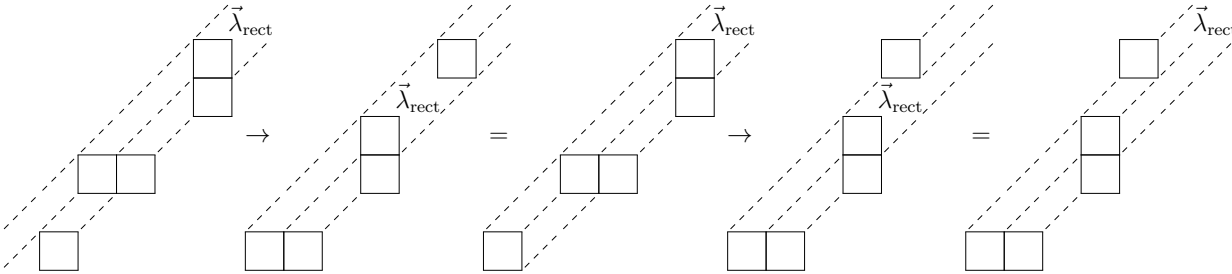
Figure 3.25: Case E

Figure 3.25 results in a single multiskew partition. We expand this multiskew partition in Figure 3.26.
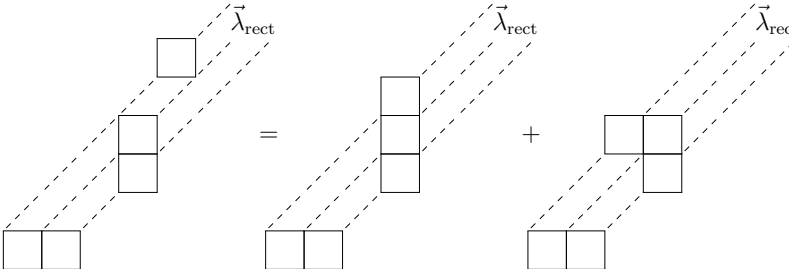
Figure 3.26: Case E continued

Figure 3.26 results in two multiskew partitions. In the first, all entries pairwise commute, and we can permute them to put the multiskew partition into the desired form. The second multiskew partition requires more finesse, which we display in Figure 3.27.
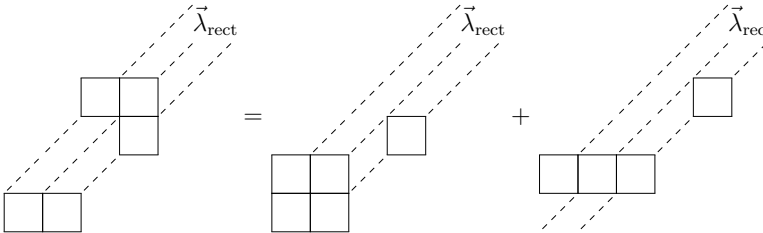
Figure 3.27: Case E continued

Figure 3.27 results in two multiskew partitions. In both, all entries pairwise commute, and we can permute them to be in the desired form.

Thus we have expanded $\vec{\lambda}$ into mutliskew partitions of the desired form. Since $\vec{\lambda}$ was an arbitrary multiskew partition in rectangular expanding form, this completes the proof.

□

Lemma 3.1.8. *Let $\vec{\lambda}$ be a 3-bandwidth multiskew partition in expanding form that consists entirely of 3-rectangles and hook shapes. Then $\mathcal{G}_{\vec{\lambda}}(X; q)$ can be written as an $\mathbb{N}[q, q^{-1}]$-linear combination of LLT polynomials whose indexing multiskew partitions are any of the following:*

- *A sequence of 3-rectangles followed by at most one hook shape.*

- *A single cell with content 2 or a domino with content in the interval $[1, 2]$ followed by a sequence of 3-rectangles.*

- *A horizontal domino with content in the interval $[1, 2]$ followed by a vertical domino with content in the interval $[1, 2]$ followed by a sequence of 3-rectangles.*

Proof. We will assume in this proof that the hook shapes in $\vec{\lambda}$ are all translations $\gamma_l = (2, 1)$. The work is essentially the same in the case of $\gamma_r = (2, 2)/(1, 0)$.

Since $\vec{\lambda}$ is in expanding form, all of the skew partitions pairwise commute. So without loss of generality, we can assume that $\vec{\lambda}$ can be written as $(\vec{\lambda}_{\text{rect}}, \vec{\lambda}_{\text{hook}})$, where

- $\vec{\lambda}_{\text{rect}}$ is a sequence of 3-rectangles with content in the interval $[0, 2]$

- $\vec{\lambda}_{\text{hook}}$ is a sequence of hook shapes with content in the interval $[0, 2]$.

If $\vec{\lambda}_{\text{hook}}$ has only one entry, then $\vec{\lambda}$ is of the desired form, and we are done. Otherwise, $\vec{\lambda}_{\text{hook}}$ can be broken down into two components of equal or nearly equal size, which we will call $\vec{\gamma}_1$ and $\vec{\gamma}_2$. If there are n hook shapes in $\vec{\lambda}_{\text{hook}}$, then $\vec{\gamma}_1$ has $\lceil \frac{n}{2} \rceil$ entries and $\vec{\gamma}_2$ has $\lfloor \frac{n}{2} \rfloor$ many entries, all of which are also hook shapes. In particular, $\vec{\gamma}_1$ has at most one more element than $\vec{\gamma}_2$. We proceed by permuting the entries of $\vec{\lambda}$ and applying $\text{cycle}^{-\lfloor \frac{n}{2} \rfloor}$ as in Figure 3.28. Here $\vec{\gamma}_1, \vec{\gamma}_2$ and $\vec{\lambda}_{\text{rect}}$ are each represented on a single content line, but actually span the adjacent content lines as well. This observation is important, because this proof will require us to use four content lines.
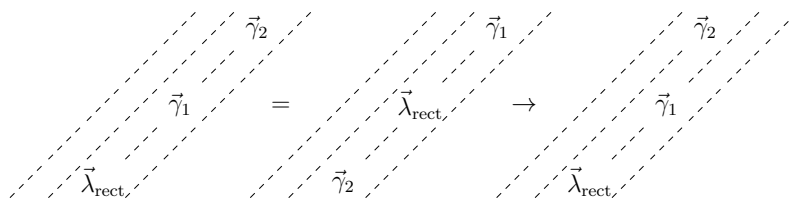

Figure 3.28: Positioning hook shapes

We see in Appendix A that there are LLT-equivalence relations among hook shapes that are translations of the same skew partition, but with differing content. This is precisely the situation presented in Figure 3.28. We proceed with these LLT-equivalence relations in Figure 3.29 by considering one hook shape from $\vec{\gamma}_1$ and one hook shape from $\vec{\gamma}_2$.
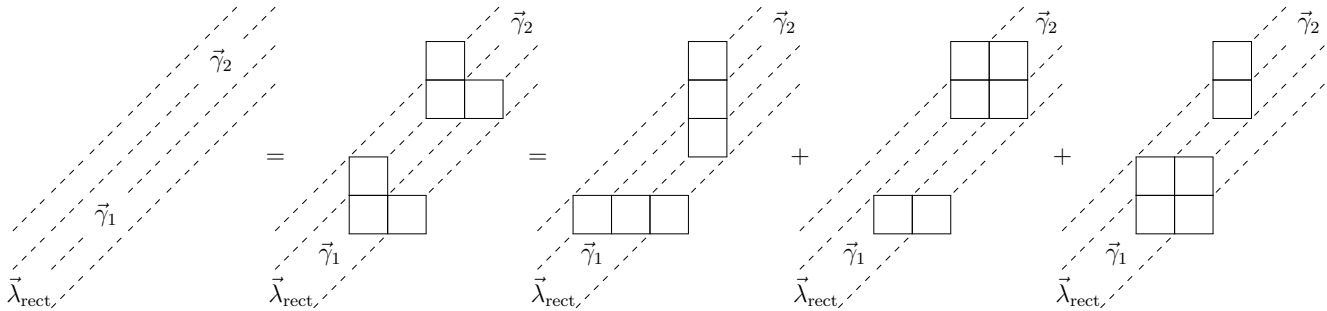
Figure 3.29: A decomposition of two hook shapes

The decomposition in Figure 3.29 results in three terms. We note that each term has exactly two fewer hook shapes than the original multiskew partition. For this reason, we will iteratively apply this decomposition until our multiskew partitions have either zero or one hook shape. At each step, we will show that these multiskew partitions can be organized as the following sequence:

- $\vec{\lambda}_{\text{hor}}$, a sequence of horizontal dominoes with content in the interval $[1, 2]$.

- $\vec{\lambda}_{\text{rect}}$, a sequence of 3-rectangles with content in the interval $[0, 2]$.

- $\vec{\gamma}_1$, a sequence of hook shapes with content in the interval $[0, 2]$.

- $\vec{\gamma}_2$, a sequence of hook shapes with content in the interval $[-1, 1]$.

- $\vec{\lambda}_{\text{vert}}$, a sequence of vertical dominoes with content in the interval $[-1, 0]$.

Since there are three terms, we need three figures. These can be seen in Figure 3.30, Figure 3.31 and Figure 3.32.
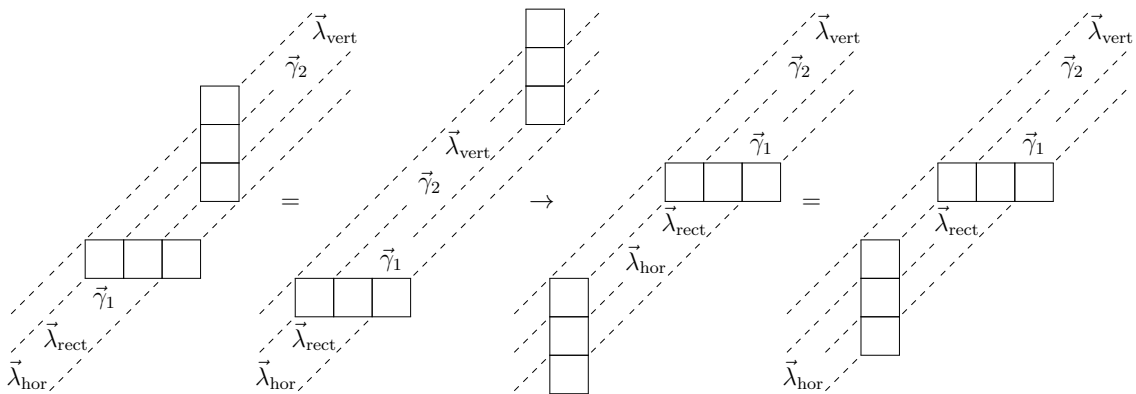

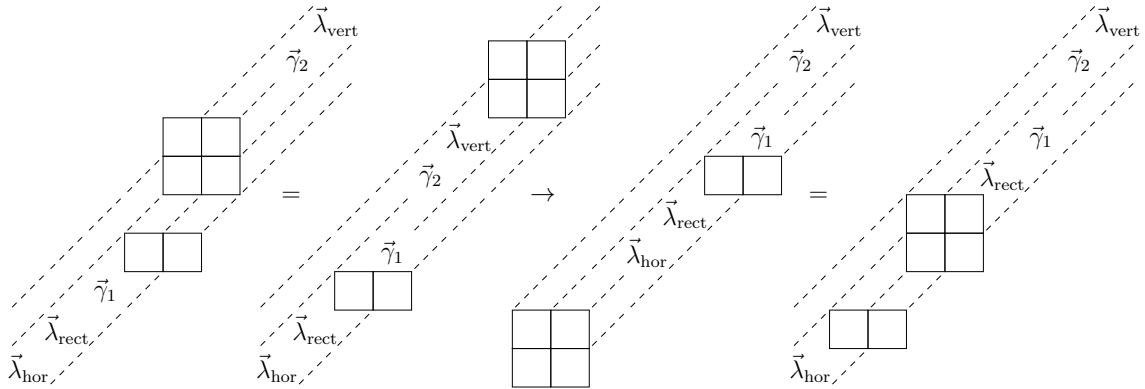Figure 3.30: The first term in the decomposition of two hook shapes.

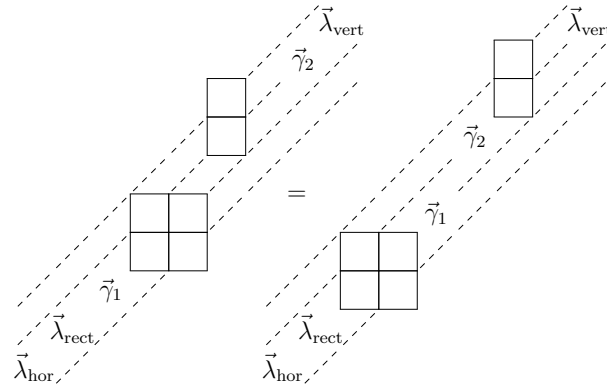Figure 3.31: The second term in the decomposition of two hook shapes.

Figure 3.32: The third term in the decomposition of two hook shapes.

This process terminates when $\vec{\gamma}_2$ is empty. At this point, $\vec{\gamma}_1$ has either one or zero entries. Regardless, we can cycle any such multiskew partition as in Figure 3.33. For emphasis, we use superscripts to denote the content interval of the multiskew partitions.
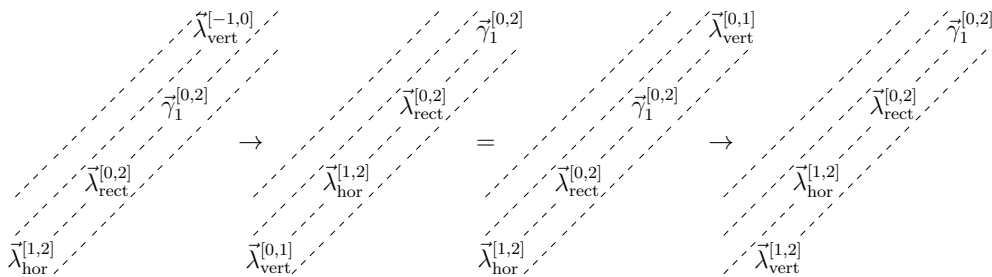
Figure 3.33: Straightening out the multiskew partitions

Formally, Figure 3.33 describes some multiskew partitions as the following sequence:

- $\vec{\lambda}_{\text{vert}}$, a sequence of vertical dominoes with content in the interval $[1, 2]$.

- $\vec{\lambda}_{\text{hor}}$, a sequence of horizontal dominoes with content in the interval $[1, 2]$.

- $\vec{\lambda}_{\text{rect}}$, a sequence of 3-rectangles with content in the interval $[0, 2]$.

- $\vec{\gamma}_1$, a sequence consisting of one or zero hook shapes with content in the interval $[0, 2]$.

We call any such multiskew partition $\vec{\lambda}'$. There are a few cases, depending on which of $\vec{\lambda}_{\text{hor}}$, $\vec{\lambda}_{\text{vert}}$ and $\vec{\gamma}_1$ are empty. In most cases, we will show that the $\vec{\lambda}'$ can be decomposed into terms that are all in rectangular expanding form, as defined in Lemma 3.1.7. Then they must expand into terms of the desired form by the statement of this same lemma.

Case 1: $\vec{\gamma}_1$ is empty. So this multiskew partition consists of no hook shapes. Then $\vec{\lambda}' = (\vec{\lambda}_{\text{vert}}, \vec{\lambda}_{\text{hor}}, \vec{\lambda}_{\text{rect}})$ is in expanding form and contains no hook shapes. So the entries can be permuted to put it into rectangular expanding form.

Case 2: $\vec{\gamma}_1$ is nonempty and $\vec{\lambda}_{\text{vert}}$ is also nonempty. Then we proceed as in Figure 3.9. The decomposition yields two terms, neither contain any hook shapes. In both, all entries pairwise commute and the entries can be permuted to be in rectangular expanding form.

Case 3: $\vec{\gamma}_1$ and $\vec{\lambda}_{\text{hor}}$ are nonempty, but $\vec{\lambda}_{\text{vert}}$ is empty. Then we proceed as in Figure 3.10, which results in two terms. The first term can be put into rectangular expanding form via the process in Figure 3.11. The second term can also be put into this form via the processes in Figure 3.12 and Figure 3.13.

Case 4: $\vec{\gamma}_1$ is nonempty, but $\vec{\lambda}_{\text{hor}}$ are $\vec{\lambda}_{\text{vert}}$ empty. Then $\vec{\lambda}'$ is a sequence of $3-$rectangles followed by a single hook shape. Thus, $\vec{\lambda}'$ is in the desired form.

$\square$

This leads us to the main theorem of this section.

Theorem 3.1.9. *Let $\vec{\lambda}$ be any 3-bandwidth multiskew partition. Then $\mathcal{G}_{\vec{\lambda}}(X; q)$ can be written as a $\mathbb{N}[q, q^{-1}]$-linear combination of LLT polynomials indexed by multiskew partitions that are all 3-indecomposable.*

Proof. Let $\vec{\lambda}$ be any 3-bandwidth multiskew partition. Then by Lemma 3.1.6, $\mathcal{G}_{\vec{\lambda}}(X; q)$ can be written as an $\mathbb{N}[q, q^{-1}]$-linear combination of LLT polynomials indexed by multiskew partitions that satisfy either of the following:

- A sequence of 3-rectangles and hook shapes such that all skew partitions pairwise commute.

- A sequence of single cells with content 2, dominoes occupying the content interval $[1, 2]$, and 3-rectangles such that all skew partitions pairwise commute.

Let $\vec{\mu}$ be any such multiskew partition. If $\vec{\mu}$ meets the first condition, then we can apply Lemma 3.1.7. If $\vec{\mu}$ meets the second condition, then we can apply Lemma 3.1.8. The result is that $\mathcal{G}_{\vec{\mu}}(X; q)$ can be written as an $\mathbb{N}[q, q^{-1}]$-linear combination of LLT polynomials indexed by multiskew partition that satisfy any of the following conditions:

1. A sequence of 3-rectangles.

2. A sequence of 3-rectangles followed by at most one hook shape.

3. A single cell with content 2 followed by a sequence of 3-rectangles.

4. A domino with content in the interval $[1, 2]$ followed by a sequence of 3-rectangles.

5. A horizontal domino with content in the interval $[1, 2]$ followed by a vertical domino with content in the interval $[1, 2]$ followed by a sequence of 3-rectangles.

Let $\vec{\nu}$ be any such multiskew partition. It is clear that the entries of $\vec{\nu}$ all pairwise commute. If $\vec{\nu}$ meets condition 1, then the entries can be permuted to yield an indecomposable $\vec{\lambda}$ where $\vec{\lambda}_{\text{init}}$ is the empty partition.

Let $\vec{\nu}$ be any such multiskew partition. It is clear that the entries of $\vec{\nu}$ all pairwise commute. If $\vec{\nu}$ meets condition 2, then the entries can be permuted to yield an indecomposable $\vec{\lambda}$ where $\vec{\lambda}_{\text{init}}$ is a hook shape.

If $\vec{\nu}$ meets condition 3, then we can proceed as in Figure **??** to yield an indecomposable $\vec{\lambda}$ where $\vec{\lambda}_{\text{init}}$ is a single cell with content 0.
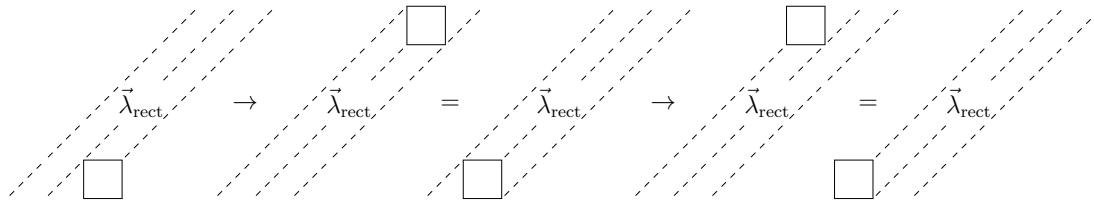

Figure 3.34: Cycling a single cell

If $\vec{\nu}$ meets condition 4, then $\text{cycle}^{-1}(\vec{\nu})$ consists of 3-rectangles and a single domino with content in the interval $[0, 1]$. Since all entries pairwise commute, we can permute the entries to yield an indecomposable $\vec{\lambda}$ where $\vec{\lambda}_{\text{init}}$ is a domino with content in the interval $[0, 1]$.

If $\vec{\nu}$ meets condition 5, then $\text{cycle}^{-2}(\vec{\nu})$ consists of 3-rectangles, a horizontal domino with content in the interval $[0, 1]$, and a vertical domino with content in the interval $[0, 1]$. Since all of these entries pairwise commute, we can permute the entries to yield an indecomposable $\vec{\lambda}$ where $\vec{\lambda}_{\text{init}}$ is a horizontal domino with content in the interval $[0, 1]$ followed by a vertical domino with content in the interval $[0, 1]$.

This completes the proof.

□

3.2 LLT polynomials indexed by indecomposables are generalized Hall-Littlewood polynomials

Using a theorem of Grojnowski and Haiman, we prove that 3-indecomposable LLT polynomials can be identified as generalized Hall-Littlewood polynomials after certain transformations.

Theorem 3.2.1. *Let $\vec{\lambda}$ be a 3-indecomposable multiskew partition. Then there exists a sequence of partitions $(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(d)})$ and integer m such that $q^m \omega \mathcal{G}_{\vec{\lambda}}(X; q) = P_{(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(d)})}(X; q)$, a generalized Hall-Littlewood polynomial.*

Proof. By definition, $\vec{\lambda}$ can be written as $(\vec{\lambda}_{\text{init}}, \vec{\lambda}_{\text{hor}}, \vec{\lambda}_{\text{square}}, \vec{\lambda}_{\text{vert}})$. In Lemma 1.7.4, we proved that $\omega \mathcal{G}_{\vec{\lambda}}(X; q) = q^m \mathcal{G}_{\omega_0 \vec{\lambda}'}(X, q^{-1})$ for some integer m, where ω_0 is the permutation the reverses the entries of $\vec{\lambda}'$. We observe that

$$
\begin{aligned}
\omega_0 \vec{\lambda}' &= \omega_0 (\vec{\lambda}_{\text{init}}, \vec{\lambda}_{\text{hor}}, \vec{\lambda}_{\text{square}}, \vec{\lambda}_{\text{vert}})' \\
&= \omega_0 (\vec{\lambda}'_{\text{init}}, \vec{\lambda}'_{\text{hor}}, \vec{\lambda}'_{\text{square}}, \vec{\lambda}'_{\text{vert}}) \\
&= (\vec{\lambda}'_{\text{vert}}, \vec{\lambda}'_{\text{square}}, \vec{\lambda}'_{\text{hor}}, \omega_0 \vec{\lambda}'_{\text{init}})
\end{aligned}
\tag{3.9}
$$

The transpose of a 3-rectangle is another 3-rectangle. If we assume that all contents fall within the interval $[0, 2]$, then $\vec{\lambda}'_{\text{init}}$ consists of either zero, one or two skew partitions, each containing a cell with content 2. Additionally, $\vec{\lambda}'_{\text{vert}}$ is a sequence of connected horizontal strips, $\vec{\lambda}'_{\text{square}}$ is a sequence of 2x2 squares, $\vec{\lambda}'_{\text{hor}}$ is a sequence of connected vertical strips, and $\omega_0 \vec{\lambda}'_{\text{init}}$ is any of the following:

- empty.

- a single skew partition containing a cell having content 2.

- a horizontal domino occupying the content interval $[1, 2]$ and then a vertical domino occupying the same content interval.

In each case, we will show that $\mathcal{G}_{\omega_0 \vec{\lambda}'}(X, q^{-1})$ is a generalized Hall-Littlewood polynomial. We proceed as in [10], by considering $\omega_0 \vec{\lambda}'$ as a sequence of multiskew partitions $\vec{\beta}/\vec{\gamma}$. In this setting, $\vec{\beta}$ can be thought of as a sequence of partitions of length $r_1, r_2, \ldots, r_d$. If we let n be the sum of these values, then we pick $L = GL_{r_1} \times GL_{r_2} \times \cdots \times GL_{r_d}$ to be a Levi subgroup of GL_n. We let W be the Weyl group of G and W_J be the Weyl group of L.

We define $X_+(L) = \{\lambda \in X | \langle \alpha_i, \lambda \rangle \geq 0 \, \forall i \in J\}$ to be the cone of dominant weights for the Levi L. It is clear that $\vec{\beta}, \vec{\gamma} \in X_+(L)$. We also define $X_{++}(L) = \{\lambda \in X | \langle \alpha_i, \lambda \rangle > 0 \, \forall i \in J\}$ to be the set of regular and dominant weights for the Levi subgroup $L \subseteq G$. Lastly, we define $\rho_L = (\rho_{r_1}, \rho_{r_2}, \ldots, \rho_{r_d})$ where $\rho_r = (0, -1, -2, \ldots, 1 - r)$.

We say that a weight $\lambda \in X_{++}(L)$ is L-quasi-dominant [10] if

$$X_{++}(L) \cap (\lambda + Q_+) \cap \mathrm{Conv}(W\lambda) = \{\lambda\} \tag{3.10}$$

where Q_+ is the positive root lattice, and $\mathrm{Conv}(W\lambda)$ is the convex hull of all permutations of λ.

We wish to show that $\mathcal{G}_{\omega_0\vec{\lambda}'}(X, q^{-1})$ is a generalized Hall-Littlewood polynomial. Haiman and Grojnowski prove that it is sufficient to check that $\vec{\beta} + \rho_L$ and $-\omega_0^J(\vec{\gamma} + \rho_L)$ are L-quasi-dominant [10], where ω_0^J is the longest word in W_J. We will now show that these weights are L-quasi-dominant.

Since we assume that $\omega_0\vec{\lambda}'$ is within the content interval $[0, 2]$, we can write down explicitly what the entries are. Among the 3-rectangles, we have

- The horizontal strip 3-rectangles are $(3)/(0)$

- The square 3-rectangles are $(3, 3)/(1, 1)$

- The vertical strip 3-rectangles are $(3, 3, 3)/(2, 2, 2)$

Additionally, there are the entries of $\omega_0\vec{\lambda}'_{\mathrm{init}}$, which are drawn from the following:

- The vertical domino is $(3, 3)/(2, 2)$.

- The horizontal domino is $(3)/(1)$.

- The single cell is $(3)/(2)$.

- The straight hook shape γ_l is $(3, 2)/(1, 1)$

- The skew hook shape γ_r is $(3, 3)/(2, 1)$.

Given this information, there are only a few options for $\vec{\beta}$ and $\vec{\gamma}$. We will deal with each case separately, and show that $\vec{\beta} + \rho_L$ and $-\omega_j^J(\vec{\gamma} + \rho_L)$ are L-quasi-dominant.

Case 1: $\omega_0\vec{\lambda}'_{\mathrm{init}}$ is empty. In this case,

$$\begin{aligned}
\vec{\beta} &= ((3), \ldots, (3), (3, 3), \ldots (3, 3), (3, 3, 3), \ldots, (3, 3, 3)) \\
\vec{\beta} + \rho_L &= ((3), \ldots, (3), (3, 2), \ldots (3, 2), (3, 2, 1), \ldots, (3, 2, 1))
\end{aligned} \tag{3.11}$$

Let $x \in X_{++}(L) \cap ((\beta + \rho_L) + Q_+) \cap \mathrm{Conv}(W(\beta + \rho_L))$, then $x \geq (\beta + \rho_L)$ in dominance order. Assume for contradiction that $x > (\beta + \rho_L)$. Then there is some first entry where $x > (\beta + \rho_L)$. It must be at one of the positions $\{0, r_1, r_1 + r_2, \ldots, r_1 + \cdots + r_k\}$ to maintain the strictly dominant property, because these indices point to the first entry of each tuple. Since all of these entries are equal to 3, this means that x has an entry greater than 3. By convexity, $x \notin \mathrm{Conv}(W(\beta + \rho_L))$. Thus we have a contradiction and conclude that $x = \vec{\beta} + \rho_L$, so that $(\beta + \rho_L)$ is L-quasi-dominant.

We continue by considering $\vec{\gamma}$. Here,

$$\vec{\gamma} = ((0), \dots, (0), (1,1), \dots (1,1), (2,2,2), \dots, (2,2,2))$$
$$-\omega_0^J(\vec{\gamma} + \rho_L) = ((0), \dots, (0), (0,-1), \dots (0,-1), (0,-1,-2), \dots, (0,-1,-2)) \tag{3.12}$$

The proof that $-\omega_0^J(\vec{\gamma} + \rho_L)$ is L-quasi-dominant is identical to the proof for $\vec{\beta}$.
Case 2: $\omega_0 \vec{\lambda}'_{\text{init}}$ consists of one vertical domino. In this case,

$$\vec{\beta} = ((3), \dots, (3), (3,3), \dots (3,3), (3,3,3), \dots, (3,3,3), (3,3))$$
$$\vec{\beta} + \rho_L = ((3), \dots, (3), (3,2), \dots (3,2), (3,2,1), \dots, (3,2,1), (3,2)) \tag{3.13}$$

The proof that $\vec{\beta} + \rho_L$ is L-quasi-dominant is the same as the proof in Case 1.

$$\vec{\gamma} = ((0), \dots, (0), (1,1), \dots (1,1), (2,2,2), \dots, (2,2,2), (2,2))$$
$$-\omega_0^J(\vec{\gamma} + \rho_L) = ((0), \dots, (0), (0,-1), \dots (0,-1), (0,-1,-2), \dots, (0,-1,-2), (-1,-2)) \tag{3.14}$$

Let $x \in X_{++}(L) \cap ((-\omega_0^J(\vec{\gamma} + \rho_L) + Q_+) \cap \text{Conv}(W(-\omega_0^J(\vec{\gamma} + \rho_L)))$, then $x \geq -\omega_0^J(\vec{\gamma} + \rho_L)$ in dominance order. Assume for contradiction that $x > -\omega_0^J(\vec{\gamma} + \rho_L)$. Then there is some first entry where $x > -\omega_0^J(\vec{\gamma} + \rho_L)$. It must be at one of the positions $\{0, r_1, r_1 + r_2, \dots, r_1 + \dots + r_k\}$. If the first position where $x > -\omega_0^J(\vec{\gamma} + \rho_L)$ is in the set $\{0, r_1, r_1 + r_2, \dots, r_1 + \dots + r_{k-1}\}$, then x has an entry greater than 0. By convexity, $x \notin \text{Conv}(W(-\omega_0^J(\vec{\gamma} + \rho_L))$ yielding a contradiction. If the first position where $x > -\omega_0^J(\vec{\gamma} + \rho_L)$ is $r_1 + \dots + r_k$, then x and $-\omega_0^J(\vec{\gamma} + \rho_L)$ differ only in the final tuple. Since $x \in (-\omega_0^J(\vec{\gamma} + \rho_L) + Q_+)$ and the final tuple has 2 entries, this means that the final entry of x is smaller than the final entry of $-\omega_0^J(\vec{\gamma} + \rho_L)$. So x contains an entry that is smaller than -2, again contradicting the fact that $x \in \text{Conv}(W(-\omega_0^J(\vec{\gamma} + \rho_L)))$. Thus $x = -\omega_0^J(\vec{\gamma} + \rho_L)$, so that $-\omega_0^J(\vec{\gamma} + \rho_L)$ is L-quasi-dominant.
Case 3: $\omega_0 \vec{\lambda}'_{\text{init}}$ consists of one horizontal domino. In this case,

$$\vec{\beta} = ((3), \dots, (3), (3,3), \dots (3,3), (3,3,3), \dots, (3,3,3), (3))$$
$$\vec{\beta} + \rho_L = ((3), \dots, (3), (3,2), \dots (3,2), (3,2,1), \dots, (3,2,1), (3)) \tag{3.15}$$

The proof that $\vec{\beta} + \rho_L$ is L-quasi-dominant is the same as the proof in Case 1.

$$\vec{\gamma} = ((0), \dots, (0), (1,1), \dots (1,1), (2,2,2), \dots, (2,2,2), (1))$$
$$-\omega_0^J(\vec{\gamma} + \rho_L) = ((0), \dots, (0), (0,-1), \dots (0,-1), (0,-1,-2), \dots, (0,-1,-2), (-1)) \tag{3.16}$$

Let $x \in X_{++}(L) \cap ((-\omega_0^J(\vec{\gamma} + \rho_L) + Q_+) \cap \text{Conv}(W(-\omega_0^J(\vec{\gamma} + \rho_L)))$, then $x \geq -\omega_0^J(\vec{\gamma} + \rho_L)$ in dominance order. Assume for contradiction that $x > -\omega_0^J(\vec{\gamma} + \rho_L)$. Then there is some first entry where $x > -\omega_0^J(\vec{\gamma} + \rho_L)$. It must be at one of the positions $\{0, r_1, r_1 + r_2, \dots, r_1 + \dots + r_k\}$. Like in the previous cases, if the first position where $x > -\omega_0^J(\vec{\gamma} + \rho_L)$ is in the set $\{0, r_1, r_1 + r_2, \dots, r_1 + \dots + r_{k-1}\}$, then convexity is contradicted. The first entry where $x > \omega_0^J(\vec{\gamma} + \rho_L)$ also cannot be $r_1 + \dots + r_k$, because this is the final entry, and $x \in ((-\omega_0^J(\vec{\gamma} + \rho_L) + Q_+)$. Thus we conclude that $-\omega_0^J(\vec{\gamma} + \rho_L)$ is L-quasi-dominant.

Case 4: $\omega_0 \vec{\lambda}'_{\text{init}}$ consists of one single cell. In this case, $\vec{\beta}$ is identical to Case 3, so the proof that it is L-quasi-dominant is also identical to the proof in Case 1.

$$\vec{\gamma} = ((0), \dots, (0), (1,1), \dots (1,1), (2,2,2), \dots, (2,2,2), (2))$$
$$-\omega_0^J(\vec{\gamma} + \rho_L) = ((0), \dots, (0), (0,-1), \dots (0,-1), (0,-1,-2), \dots, (0,-1,-2), (-2))$$
$$\tag{3.17}$$

The proof that $-\omega_0^J(\vec{\gamma} + \rho_L)$ is L-quasi-dominant is the same as the proof in Case 3.

Case 5: $\omega_0 \vec{\lambda}'_{\text{init}}$ consists of one hook shape γ_l. In this case

$$\vec{\beta} = ((3), \dots, (3), (3,3), \dots (3,3), (3,3,3), \dots, (3,3,3), (3,2))$$
$$\vec{\beta} + \rho_L = ((3), \dots, (3), (3,2), \dots (3,2), (3,2,1), \dots, (3,2,1), (3,1))$$
$$\tag{3.18}$$

Let $x \in X_{++}(L) \cap ((\beta + \rho_L) + Q_+) \cap \text{Conv}(W(\beta + \rho_L))$, then $x \geq (\beta + \rho_L)$ in dominance order. Assume for contradiction that $x > (\beta + \rho_L)$. Then there is some first entry where $x > (\beta + \rho_L)$. As in previous cases, this position can only fall in the last tuple. The last tuple has only two entries. If the first entry where $x > -\omega_0^J(\beta + \rho_L)$ is $r_1 + \dots + r_k$, then x contains an entry greater than 3, contradicting convexity like before. The first entry where $x > \beta + \rho_L$ cannot be $r_1 + \dots + r_k + 1$, because then $x \notin ((\beta + \rho_L) + Q_+)$. Thus we conclude that $(\beta + \rho_L)$ is L-quasi-dominant.

$$\vec{\gamma} = ((0), \dots, (0), (1,1), \dots (1,1), (2,2,2), \dots, (2,2,2), (1,1))$$
$$-\omega_0^J(\vec{\gamma} + \rho_L) = ((0), \dots, (0), (0,-1), \dots (0,-1), (0,-1,-2), \dots, (0,-1,-2), (0,-1))$$
$$\tag{3.19}$$

The the proof that $-\omega_0^J(\vec{\gamma} + \rho_L)$ is L-quasi-dominant is identical to the proof for $\vec{\beta} + \rho_L$ in Case 1.

Case 6: $\omega_0 \vec{\lambda}'_{\text{init}}$ consists of one hook shape γ_r. In this case,

$$\vec{\beta} = ((3), \dots, (3), (3,3), \dots (3,3), (3,3,3), \dots, (3,3,3), (3,3))$$
$$\vec{\beta} + \rho_L = ((3), \dots, (3), (3,2), \dots (3,2), (3,2,1), \dots, (3,2,1), (3,2))$$
$$\tag{3.20}$$

The proof that $\vec{\beta} + \rho_L$ is L-quasi-dominant is the same as the proof in Case 1.

$$\vec{\gamma} = ((0), \ldots, (0), (1,1), \ldots (1,1), (2,2,2), \ldots, (2,2,2), (2,1))$$
$$-\omega_0^J(\vec{\gamma} + \rho_L) = ((0), \ldots, (0), (0,-1), \ldots (0,-1), (0,-1,-2), \ldots, (0,-1,-2), (0,-2))$$
$$(3.21)$$

The proof that $-\omega_0^J(\vec{\gamma} + \rho_L)$ is L-quasi-dominant is the same as the proof for $\beta + \rho_L$ in Case 5.

Case 7: $\omega_0\vec{\lambda}'_{\text{init}}$ consists of one horizontal domino and one vertical domino. In this case,

$$\vec{\beta} = ((3), \ldots, (3), (3,3), \ldots (3,3), (3,3,3), \ldots, (3,3,3), (3), (3,3))$$
$$\vec{\beta} + \rho_L = ((3), \ldots, (3), (3,2), \ldots (3,2), (3,2,1), \ldots, (3,2,1), (3), (3,2))$$
$$(3.22)$$

The proof that $\vec{\beta} + \rho_L$ is L-quasi-dominant is the same as the proof in Case 1.

$$\vec{\gamma} = ((0), \ldots, (0), (1,1), \ldots (1,1), (2,2,2), \ldots, (2,2,2), (1), (2,2))$$
$$-\omega_0^J(\vec{\gamma} + \rho_L) = ((0), \ldots, (0), (0,-1), \ldots (0,-1), (0,-1,-2), \ldots, (0,-1,-2), (-1), (-1,-2))$$
$$(3.23)$$

Let $x \in X_{++}(L) \cap ((-\omega_0^J(\vec{\gamma} + \rho_L) + Q_+) \cap \text{Conv}(W(-\omega_0^J(\vec{\gamma} + \rho_L)))$, then $x \geq -\omega_0^J(\vec{\gamma} + \rho_L)$ in dominance order. Assume for contradiction that $x > -\omega_0^J(\vec{\gamma} + \rho_L)$. Then there is some first entry where $x > -\omega_0^J(\vec{\gamma} + \rho_L)$. It must be at one of the positions $\{0, r_1, r_1 + r_2, \ldots, r_1 + \cdots + r_k\}$. If the first position where $x > -\omega_0^J(\vec{\gamma} + \rho_L)$ is in the set $\{0, r_1, r_1 + r_2, \ldots, r_1 + \cdots + r_{k-2}\}$, then x has an entry greater than 0. By convexity, $x \notin \text{Conv}(W(-\omega_0^J(\vec{\gamma} + \rho_L)))$ yielding a contradiction.

So, x and $-\omega_0^J(\vec{\gamma} + \rho_L)$ can only differ in the last two tuples: $((-1), (-1, -2))$. The final entry of x cannot be smaller than the final entry of $-\omega_0^J(\vec{\gamma} + \rho_L)$, because that would require x to have an entry smaller than -2, contradicting the convexity requirement. So we are looking for a triple $(a, b, c) \in ((-1, -1, -2) + Q_+)$ such that $(a, b, c) > (-1, -1, -2)$ in dominance order, $b > c$ and $c \geq -2$. This is clearly impossible, so we conclude that no such x exists, and $-\omega_0^J(\vec{\gamma} + \rho_L)$ is L-quasi-dominant.

We have concluded that in each case $\vec{\beta} + \rho_L$ and $-\omega_0^J(\vec{\gamma} + \rho_L)$ are L-quasi-dominant. So from [10, Theorem 7.15] we see that

$$\text{Ind}_{L,q^{-1}}^G(\chi_{\vec{\beta}}(L) \otimes \chi_{\vec{\gamma}}^*(L)) = q^m \mathcal{G}_{\vec{\beta}/\vec{\gamma}}(X; q)) \tag{3.24}$$

Further, if $(\chi_{\vec{\beta}}(L) \otimes \chi_{\vec{\gamma}}^*(L))$ is irreducible then there exists $\mu \in X_+(L)$ with

$$\chi_\mu(L) = \chi_{\vec{\beta}}(L) \otimes \chi_{\vec{\gamma}}^*(L) \tag{3.25}$$

and we conclude that there exists m such that

$$P_\mu(X; q^{-1}) = q^m \mathcal{G}_{\vec{\beta}/\vec{\gamma}}(X; q). \tag{3.26}$$

In any case, we would like to find a sequence μ such 3.25 holds. It is a well-known fact that $\chi_{\vec{\gamma}}(L)^* = \chi_{-\omega_0^J \vec{\gamma}}(L)$. From the list of cases above, we see that if $\omega_0 \lambda'_{\text{init}} \neq \gamma_r$, then $\vec{\gamma}$ is constant in every tuple. Thus $\omega_0^J \vec{\gamma} = \vec{\gamma}$ and $\chi_{\vec{\gamma}}(L)^* = \chi_{-\vec{\gamma}}(L)$.

Further, since $\chi_{-\vec{\gamma}}(L)$ is also constant on every tuple,

$$
\begin{aligned}
\vec{\beta} &= ((3), \ldots, (3), (3,3), \ldots (3,3), (3,3,3), \ldots, (3,3,3), \vec{\beta}_{\text{init}}) \\
\vec{\gamma} &= ((0), \ldots, (0), (1,1), \ldots (1,1), (2,2,2), \ldots, (2,2,2), \vec{\gamma}_{\text{init}}) \\
\chi_\mu &= \chi_{\vec{\beta}} \otimes \chi_{\vec{\gamma}}^* \\
\mu &= \vec{\beta} - \vec{\gamma} \\
&= ((3), \ldots, (3), (2,2), \ldots (2,2), (1,1,1), \ldots, (1,1,1), \vec{\beta}_{\text{init}} - \vec{\gamma}_{\text{init}})
\end{aligned}
\tag{3.27}
$$

In keeping with the cases above, here are the corresponding values of μ.

1. $((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots, (1,1,1))$

2. $((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots, (1,1,1), (1,1))$

3. $((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots, (1,1,1), (2))$

4. $((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots, (1,1,1), (1))$

5. $((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots, (1,1,1), (2,1))$

The last case involves results involves the skew shape γ_r, so we will have to be a bit more careful when doing computations. For readability, we will condense sequences like $(3), \ldots, (3)$ to simply (3). This does not change the proof. In this case,

$$
\begin{aligned}
\vec{\beta} &= ((3), (3,3), (3,3,3), (3,3)) \\
\vec{\gamma} &= ((0), (1,1), (2,2,2), (2,1)) \\
-\omega_0^J \vec{\gamma} &= ((0), (-1,-1), (-2,-2,-2), (-1,-2)) \\
&= ((0), (-1,-1), (-2,-2,-2), (-2,-2)) + ((0), (0,0), (0,0,0), (1,0))
\end{aligned}
\tag{3.28}
$$

We see that

$$
\begin{aligned}
\chi_{\vec{\beta}} \otimes \chi_{\vec{\gamma}}^* &= \chi_{((3),(3,3),(3,3,3),(3,3))} \otimes \chi_{((0),(-1,-1),(-2,-2,-2),(-2,-2))} \otimes \chi_{((0),(0,0),(0,0,0),(1,0))} \\
&= \chi_{((3),(2,2),(1,1,1),(1,1))} \otimes \chi_{((0),(0,0),(0,0,0),(1,0))} \\
&= \chi_{((3),(2,2),(1,1,1),(2,1))}
\end{aligned}
\tag{3.29}
$$

The last equality follows from a Pieri rule for $\mathfrak{gl}_2$.

Thus we have proved that $\vec{\beta} + \rho_L$ and $-\omega_0^J(\vec{\gamma} + \rho_L)$ are both L-quasidominant, and that there exists a sequence of straight shapes μ such that $\chi_\mu(L) = \chi_{\vec{\beta}}(L) \otimes \chi_{\vec{\gamma}}(L)^*$. For all such triples $(\vec{\beta}, \vec{\gamma}, \mu)$, we conclude that there exists m such that

$$P_\mu(X; q) = q^m \mathcal{G}_{\vec{\beta}/\vec{\gamma}}(X; q^{-1}). \tag{3.30}$$

Thus we conclude that all LLT polynomials can be decomposed into generalized Hall-Littlewood polynomials.

$\square$

3.3 LLT polynomials indexed by indecomposables are 3-Schur functions

In the previous section, we concluded that 3-indecomposable LLT polynomials are generalized Hall-Littlewood polynomials, up to a power of q. The converse is certainly not always true, and there is a relatively small class of generalized Hall-Littlewood polynomials that are LLT polynomials. In the following theorem, we will show that all such generalized Hall-Littlewood polynomials are 3-Schur functions.

Lemma 3.3.1. *If $\vec{\lambda}$ is as in Lemma 3.2.1, then there exists m such that $q^m \omega \mathcal{G}_{\vec{\lambda}}(X; q)$ is a 3-Schur function.*

Proof. From Lemma 3.2.1, we see that it is equivalent to consider the LLT polynomial $\mathcal{G}_{\vec{\beta}/\vec{\gamma}}(X; q^{-1})$, where $\vec{\beta}/\vec{\gamma} = \omega_0 \vec{\lambda}'$.

In every case except one, all entries of $\vec{\beta}/\vec{\gamma}$ are translations of straight shape partitions. In these cases, we have that there exists an integer m such that

$$P_\mu(X; q) = q^m \mathcal{G}_{\vec{\beta}/\vec{\gamma}}(X; q) \tag{3.31}$$

where μ is a sequence of straight shape partitions such that the ith partition is a translation of the ith skew partition in $\vec{\beta}/\vec{\gamma}$. There are only a few options for the form of μ:

- $\mu = ((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots (1,1,1))$

- $\mu = ((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots (1,1,1), (1))$

- $\mu = ((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots (1,1,1), (1,1))$

- $\mu = ((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots (1,1,1), (2))$

- $\mu = ((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots (1,1,1), (2,1))$

- $\mu = ((3), \ldots, (3), (2,2), \ldots, (2,2), (1,1,1), \ldots (1,1,1), (2), (1,1))$

All of these sequences have the same general form, which is a sequence of 3-rectangles followed by one or two entries that are not 3-rectangles. If we label the sequence of partitions that are not 3-rectangles as μ_{irred}, then the corresponding generalized Hall-Littlewood polynomial is

$$P_\mu(X;q) = B_{(3)} \cdots B_{(3)} B_{(2,2)} \cdots B_{(2,2)} B_{(1,1,1)} \cdots B_{(1,1,1)} P_{\mu_{\text{irred}}}(X;q) \qquad (3.32)$$

From Theorem 1.8.1 in Section 1.8 we see that it is sufficient to determine that $P_{\mu_{\text{irred}}}(X;q)$ is a 3-Schur function, because any raising operator B_λ where λ is a 3-rectangle transforms a 3-Schur function into another 3-Schur function, up to a power of q.

Using SAGE, we check that $P_{\mu_{\text{irred}}}$ is a 3-Schur function for each μ_{irred}.

If $\mu_{\text{irred}} = \emptyset, (1), (2), (1,1)$, or $(2,1)$, then $\mu_{\text{irred}}^{\to 3} = \mu_{\text{irred}}$, and $P_{\mu_{\text{irred}}}$ is a single 3-split polynomial, a Schur function and a 3-Schur function.

So we have

$$P_\emptyset = 1 = s_\emptyset^{(3)} \qquad (3.33)$$

$$P_{(1)} = B_{(1)}(1) = s_{(1)} = s_{(1)}^{(3)} \qquad (3.34)$$

$$P_{(2)} = B_{(2)}(1) = s_{(2)} = s_{(2)}^{(3)} \qquad (3.35)$$

$$P_{(1,1)} = B_{(1,1)}(1) = s_{(1,1)} = s_{(1,1)}^{(3)} \qquad (3.36)$$

$$P_{(2,1)} = B_{(2,1)}(1) = s_{(2,1)} = s_{(2,1)}^{(3)} \qquad (3.37)$$

$$P_{((2),(1,1))} = B_{(2)}B_{(1,1)}(1) = s_{(2,1,1)} + ts_{(3,1)} = s_{(2,1,1)}^{(3)} \qquad (3.38)$$

$\square$

We note that this theorem has a close relationship with the formulation of k-Schur functions given by Morse and Lapointe in [14]. In particular, they outline how we can produce any k-Schur function by combining raising operators corresponding to k-rectangles with a finite collection of partitions that are said to be k-irreducible. This theory is seen in our work, where there are exactly 6 multiskew partitions that μ_{irred} can be, and an arbitrarily large sequence of 3-rectangles. For this reason, we can produce any 3-Schur function as a 3-bandwidth LLT polynomial.
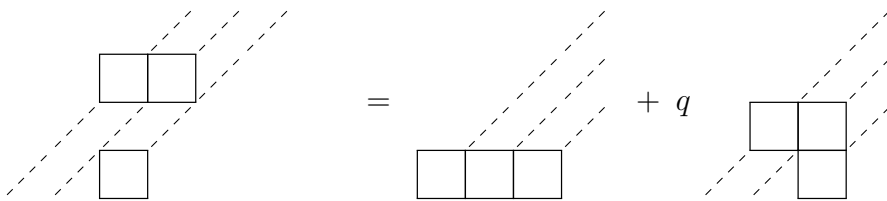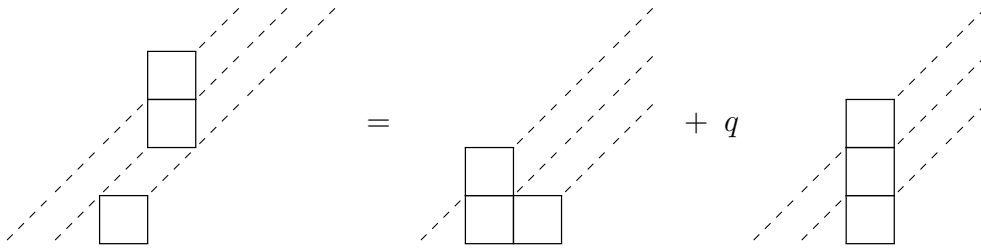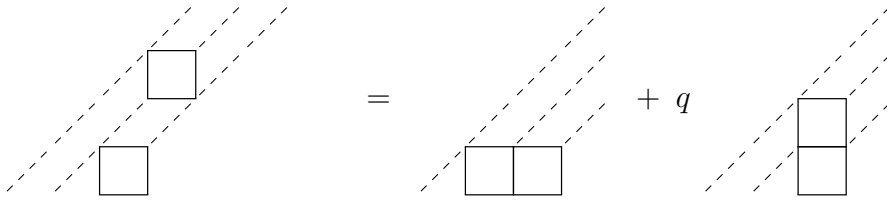
Bibliography

[1] Jim Haglund, Mark Haiman, and Nick Loehr. "A combinatorial formula for Macdonald polynomials". In: *Journal of the American Mathematical Society* 18.3 (2005), pp. 735–761.

[2] Sami Assaf and Sara Billey. "Affine dual equivalence and k-Schur functions". In: *arXiv preprint arXiv:1201.2128* (2012).

[3] Seung Jin Lee. "Linear relations on LLT polynomials and their k-Schur positivity for k= 2". In: *arXiv preprint arXiv:1807.03951* (2018).

[4] Richard P. Stanley. *Enumerative Combinatorics: Volume 2*. 2nd. New York, NY, USA: Cambridge University Press, 2011. ISBN: 1107602629, 9781107602625.

[5] T. Lam et al. *k-Schur Functions and Affine Schubert Calculus*. Fields Institute Monographs. Springer New York, 2014. ISBN: 9781493906826. URL: https://books.google.com/books?id=q%5C_W%5C_AwAAQBAJ.

[6] Francois Descouens. "A generating algorithm for ribbon tableaux and spin polynomials". In: *arXiv preprint math/0611824* (2006).

[7] Dennis Stanton and Dennis E. White. "A Schensted Algorithm for Rim Hook Tableaux." In: *J. Comb. Theory, Ser. A* 40 (Nov. 1985), pp. 211–247. DOI: 10.1016/0097-3165(85)90088-3.

[8] Alain Lascoux, Bernard Leclerc, and Jean-Yves Thibon. "Ribbon tableaux, Hall–Littlewood functions, quantum affine algebras, and unipotent varieties". In: *Journal of Mathematical Physics* 38.2 (1997), pp. 1041–1068.

[9] Bernard Leclerc and Jean-Yves Thibon. "Littlewood–Richardson Coefficients and Kazhdan–Lusztig Polynomials". In: *Combinatorial Methods in Representation Theory*. Tokyo, Japan: Mathematical Society of Japan, 2000, pp. 155–220. DOI: 10.2969/aspm/02810155. URL: https://doi.org/10.2969/aspm/02810155.

[10] I Grojnowski and M Haiman. "Affine Hecke algebras and positivity of LLT and Macdonald polynomials". In: (Feb. 2019).

[11] J Haglund et al. "A Combinatorial Formula for the Character of the Diagonal Coinvariants". In: *Duke Mathematical Journal* 126 (Nov. 2003). DOI: 10.1215/S0012-7094-04-12621-1.

[12]   Anne Schilling, Mark Shimozono, and D E. White. "Branching formula for q-Littlewood–Richardson coefficients". In: *Advances in Applied Mathematics - ADVAN APPL MATH* 30 (Feb. 2003), pp. 258–272. DOI: `10.1016/S0196-8858(02)00535-3`.

[13]   Luc Lapointe, Alain Lascoux, Jennifer Morse, et al. "Tableau atoms and a new Macdonald positivity conjecture". In: *Duke Mathematical Journal* 116.1 (2003), pp. 103–146.

[14]   Luc Lapointe and Jennifer Morse. "Schur function identities, their t-analogs, and k-Schur irreducibility". In: *Advances in Mathematics* 180.1 (2003), pp. 222–247.

[15]   Mark Shimozono and Jerzy Weyman. "Graded Characters of Modules Supported in the Closure of a Nilpotent Conjugacy Class". In: *Eur. J. Comb.* 21.2 (Feb. 2000), pp. 257–288. ISSN: 0195-6698. DOI: `10.1006/eujc.1999.0344`. URL: `http://dx.doi.org/10.1006/eujc.1999.0344`.
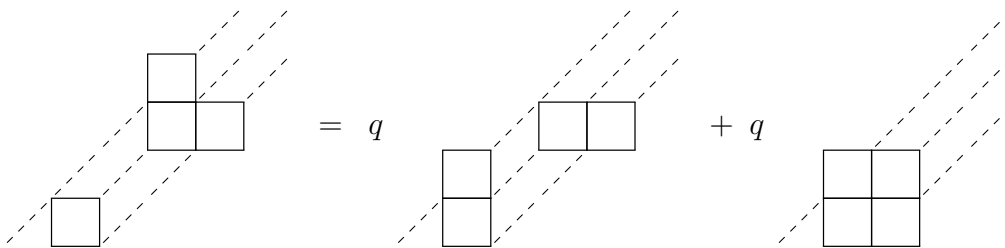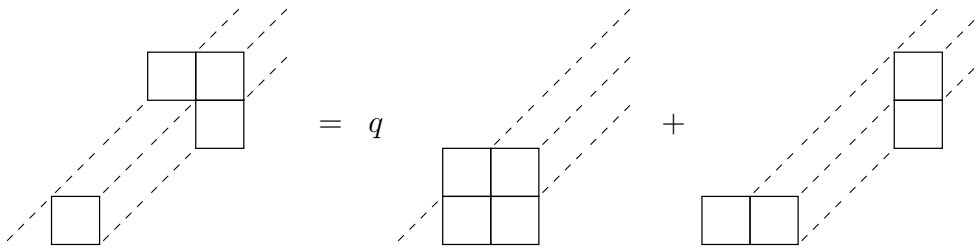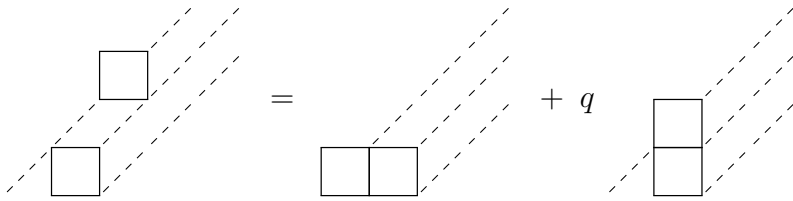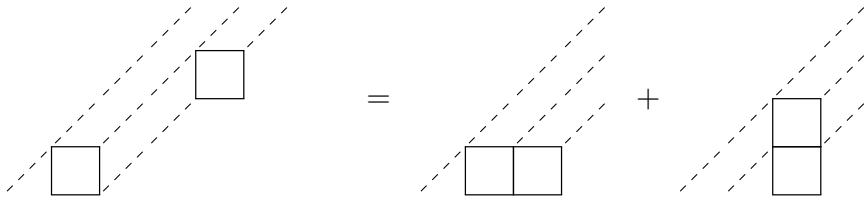
Appendix A

Figures

This appendix contains LLT-equivalence statements, each of which was verified by a computer search. For each pair of skew partitions, we compute an LLT-equivalent $\mathbb{N}[q]$-linear combination of multiskew partitions. We denote LLT-equivalence with an equals sign (=). Because the LLT polynomial of a multiskew partition is invariant under translation within the plane, we can assume that our 3-bandwidth partitions fall within the content interval $[0, 2]$.

The table here displays commuting pairs, which are skew partitions that satisfy $(\lambda_1, \lambda_2) = q^m(\lambda_2, \lambda_1)$ for some integer m. In this case, m can be found in the row labeled by λ_1 and the column labeled by λ_2. We have filled the skew partitions with their content, to make it clear where they fall within the plane.

The gaps within the table correspond to pairs that do not commute. For each of these, we produce an LLT-equivalent combination of multiskew partitions. Each page corresponds to a row of the table. So the first page contains LLT-equivalence relations where the first skew partition is a single cell with content two.

Finally, the last page contains two LLT-equivalence relations where the mulstiskew partitions cannot fit on three content lines. In this case, we think of content interval being $[-1, 2]$.

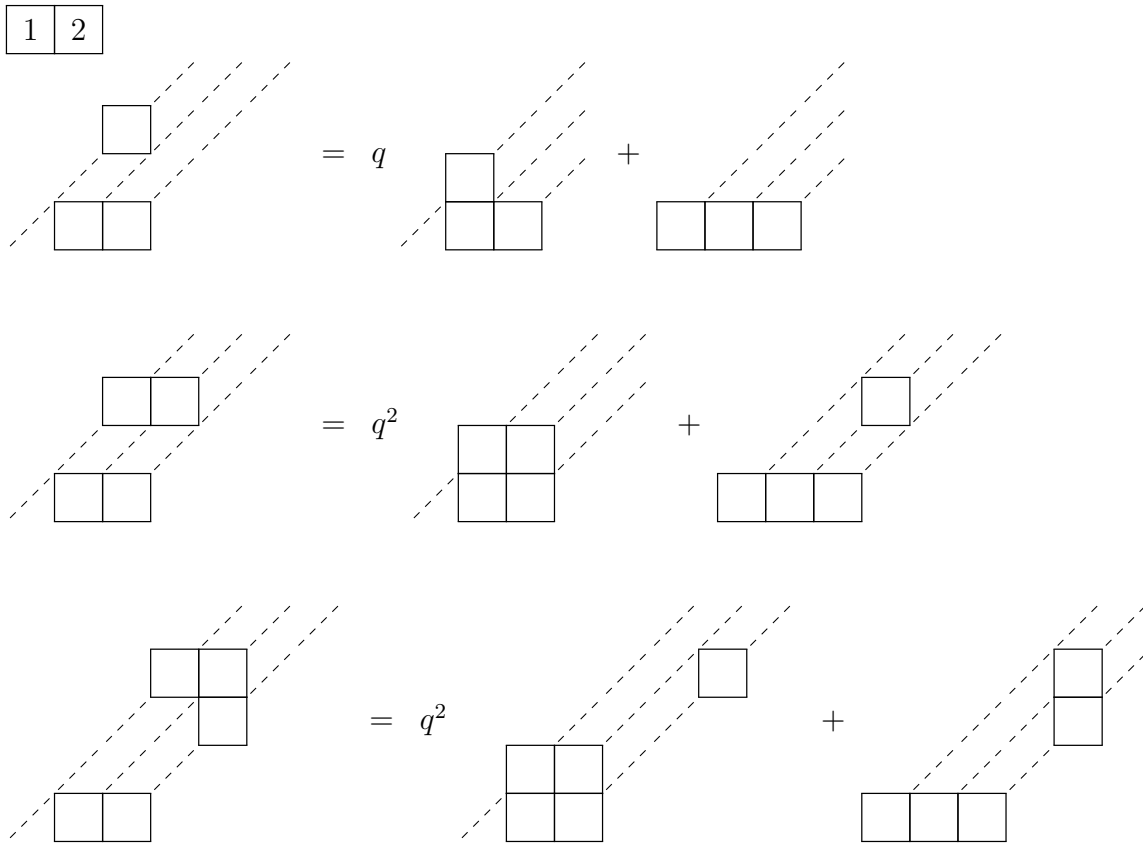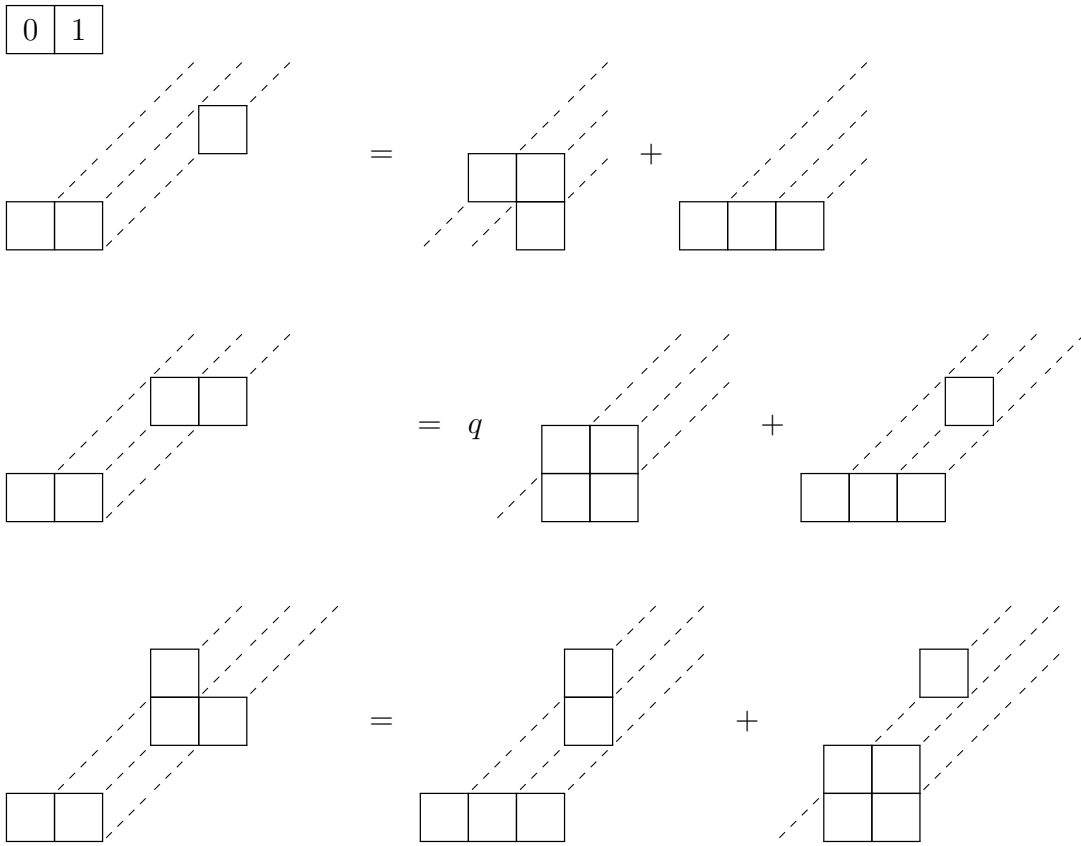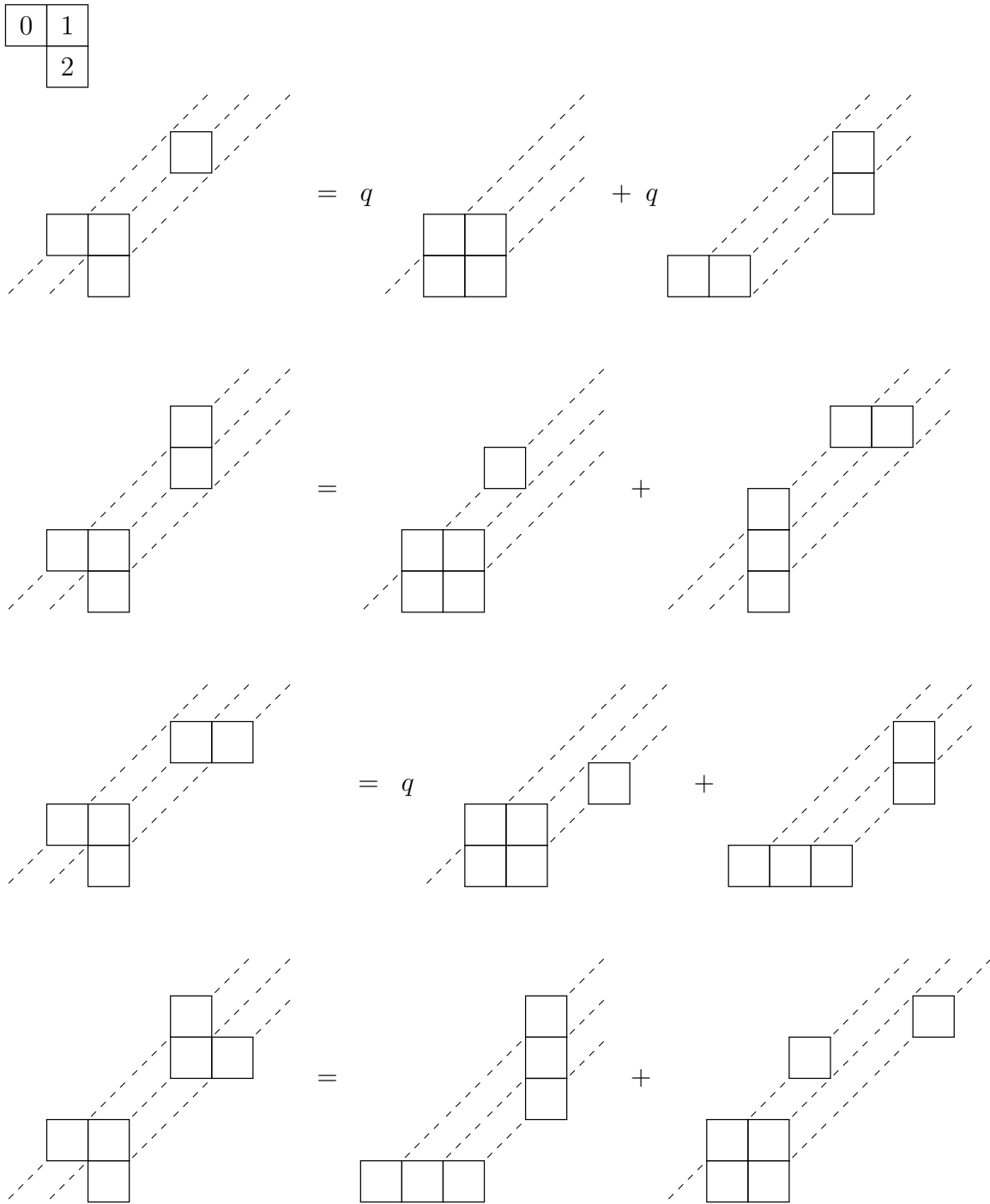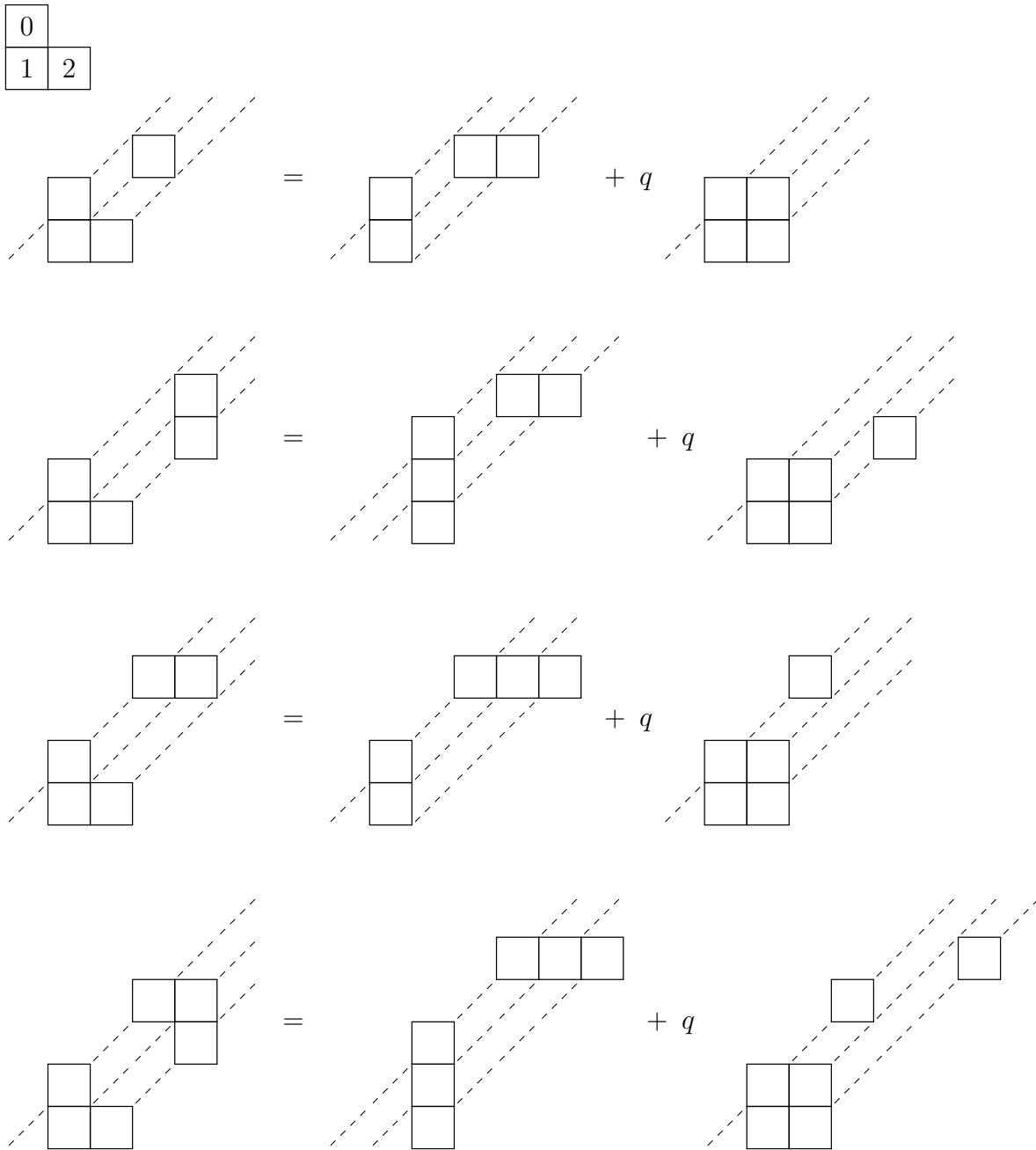|	`2`	`1`	`0`	`1;2`	`0;1`	`1 2`	`0 1`	`0 1;·2`	`0;1 2`	`0;1;2`	`0 1;1 2`	`0 1 2`
`2`	0		0	1		0		1	0	1	1	0
`1`		0		-1	1	0	0			0	0	0
`0`	0		0		-1		0	0	-1	-1	-1	0
`1;2`	-1	1		0		0	1	0		1	1	0
`0;1`		-1	1		0	-1	0		0	-1	-1	0
`1 2`	0	0		0	1	0			1	1	1	0
`0 1`		0	0	-1	0		0	-1		-1	-1	0
`0 1;·2`	-1		0	0			1	0		0	0	0
`0;1 2`	0		1		0	-1			0	0	0	0
`0;1;2`	-1	0	1	-1	1	-1	1	0	0	0	0	0
`0 1;1 2`	-1	0	1	-1	1	-1	1	0	0	0	0	0
`0 1 2`	0	0	0	0	0	0	0	0	0	0	0	0

2

1

0

Additional LLT-equivalance relations