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Stability of Simultaneous Input and State Estimation Algorithms

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Stability of Simultaneous Input and State Estimation Algorithms

A Thesis submitted in partial satisfaction of the requirements for the degree Master of Science

in

Engineering Sciences (Mechanical Engineering)

by

Mohammed Alyaseen

Committee in charge:

Professor Robert Bitmead, Chair
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The Thesis of Mohammed Alyaseen is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

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I would also like to thank Mohammad Ali Abooshahab and Morten Hovd from the Norwegian University of Science and Technology (NTNU) for first sharing the problem, providing insightful help in the investigation, and aiding in the preparation of results for publication.

Results in Sections 3, 4, and 6 are provisionally accepted to be published in Automatica journal in a paper co-authored with Mohammad Ali Abooshahab, Morten Hovd, and Robert Bitmead. The author of this thesis was a primary investigator of those results and is the primary author of those sections as they appear in this thesis. Also, Figures 1 and 2 are similar to a figure appearing in our paper in Automatica with minor differences.

ABSTRACT OF THE THESIS

Stability of Simultaneous Input and State Estimation Algorithms

by

Mohammed Alyaseen

Master of Science in Engineering Sciences (Mechanical Engineering)

University of California San Diego, 2021

Professor Robert Bitmead, Chair

Algorithms for simultaneous input and state estimation (SISE algorithms) that are optimal in the minimum-variance unbiased sense have been in development for decades. The stability of such algorithms is not guaranteed. For time invariant systems, this thesis derives necessary and sufficient stability conditions. In the square case, where the number of inputs equals the number of outputs, the exact positions of the algorithm's poles are established. In the non-square case, the derived stability condition is formulated as a detectability condition. Those necessary and sufficient conditions are generalized to sufficient only conditions for the time varying case. Lastly, the effect of delay in the system under consideration on the SISE algorithm is explored. A general method, inspired by fixed lag smoothing, is proposed to handle cases where a general delay is present.

1 Introduction

Consider a problem of a rocket modeled as a dynamical system with thrust as input and the rocket's trajectory as output. Now consider the unfavorable circumstance where somehow there is a malfunction in the thrust. It is desirable in this situation to have a reasonable detection of that unknown malfunction in the thrust of which no direct measurement is available and where the only available information is measures of the rocket trajectory such as position, velocity, etc. This hypothetical problem is an instance of a more general problem. The general problem asks if, for a dynamical system actuated by some unknown input signal, there is a way to reasonably estimate the system's states and unknown input signals given noisy measurements of the system's output. It is necessary to add to the problem description that no information about the statistical structure of the unknown signal is assumed since any unrealistic assumption about the unknown input signals will affect the performance of the desired estimator. This excludes the possibility of treating the problem with state extension approaches. An interest in this estimation problem is at least as old as J D Glover's 1969 paper [1]. Motivated by problems where the input signals are completely unknown like in the rocket example (which is taken from him), he developed an algorithm that is supposed to estimate the unknown input signal of discrete linear time varying (D-LTV) systems. Through the course of the decades following his work and until recently, the range of applications of this problem only widened. To emphasize this wide range and show the amount of attention this problem has caught, some areas of applications are described below.

One early set of applications is the estimation of states of geophysical systems where the input is an environmental phenomenon that can only be detected in a finite number of places. An example of such an application is found in [2] where the mean areal precipitation is estimated at some time given measurements of precipitation only at finite number of locations. Another area of application is detection of isolated impulse inputs [3, 4] that have unknown magnitude and act on the system at an unknown time. This area includes problems like failure detection to which the rocket example above belongs. Another application that appeared recently in the literature is the estimation of states of power systems where some elements, like power consumption on the consumers end, are largely unknown [5]. In fact, problems encountered in this last application are the motive behind the theoretical development of the material presented in this thesis. It can be seen from this short but broad list of applications presented thus far that in some cases such as failure detection, it is desirable primarily to estimate the unknown input without much emphasis on state estimation, while in other cases such as that of power systems, the desired quantities to be estimated are the states of the system. This, however, is a matter of application while in theory, both problems are instances of the broader problem of simultaneous input and states estimation (SISE).

The theoretical development of algorithms to solve the described problem was as rich as its applications. It may be that the absence of any information about the unknown input opened many possible ways to deal with it. Many algorithms where proposed to solve this problem and many methods were used to arrive at such algorithms. For example, Kitanidis [2] developed an algorithm to estimate the state only of a linear system with no feedthrough term. Hsieh [6] extended his algorithm to estimate the unknown input as well. An extension to systems where a full column rank direct feedthrough matrix is present was done by others like Darouach, Zasadzinski, and Boutayeb [7] as well as Gillijns and de Moor [8]. Moreover, Yong, Zhu, and Frazzoli [9] produced an algorithm that makes no assumption about the feedthrough matrix. In all these works, the emphasis was on an algorithm that would produce the minimum variance unbiased estimate of the state and the unknown inputs i.e. MVU algorithms. It should not be taken that the minimum variance unbiased approach was the only way by which such algorithms were produced. For instance, Li [10] took a Bayesian approach, where the unbiasedness requirement is dropped, and developed a filter that he proved to be a minimum-mean-square-error estimator. Bitmead, Abooshahab, and Hovd [11] took another approach in which the unknown input is modelled as a white noise with unbounded variance and a SISE algorithm is derived as a limiting case of the Kalman Filter. Fang, de Callafon, and Cortes [12] extended the algorithm even beyond the linear framework and proposed a method of solving the SISE problem in the non-linear context.

Another line of research, which was arguably less active than the first, has been finding stability conditions for those algorithms. This line includes the paper of Darouach and Zasdzinski [13] where necessary and sufficient stability conditions are deduced for the SISE algorithm of an LTI system. Yong and co-authors [9] recovered Darouach's conditions while deriving sufficient only stability conditions for the LTV case. This variety of algorithms present in literature that deal with the problem in different cases or even in the same case but with different approaches should not be thought of as totally distinct algorithms. In almost every attempt at a solution of that problem, the property of optimality in some sense was sought. This provided a basic connection, often explicitly stated, between relatively new and older approaches to deal with this problem. Even in [10] where the approach taken is Bayesian and where the unbiasedness of the estimator is not sought, the algorithm developed there is proved to be identical to that of [2] which is built as a classical minimum variance unbiased point estimator.

1.1 Contributions & organization

As shown in the above literature review, there is a large set of works that studies a minimum variance unbiased algorithmic solution for the problem of estimating states and unknown inputs of a system. This thesis belongs to this set. However, there is no attempt to derive a new algorithm that estimates states and unknown inputs, since there are already more than enough. The main contribution of this work is to attempt to bring some clarity and unity to this picture by establishing precisely the connection of these algorithms to system inversion and optimal estimation by deriving necessary and sufficient conditions for stability starting with the time-invariant (LTI) case. What makes the necessary and sufficient stability conditions derived here for LTI systems different from those of Darouach [13] and Yong et. al [9] is that we arrive at our conditions without any appeal to the optimality of the SISE algorithm. We also use the same analysis method to recover the sufficient only stability conditions for linear time-varying (LTV) systems. Further, a new method is proposed to solve the SISE problem when an indefinite delay is present in the system under consideration. The presence of delay violates the assumptions for many SISE algorithms such as [14] and [8]. The method proposed here reduces delayed systems to forms on which classical SISE algorithms can be applied.

Section 2 presents the SISE problem for a linear time-invariant system. Section 3 studies the zero direct feedthrough case and the corresponding SISE algorithm of [14] and shows that, in the square case where the number of measurements equals the number of disturbance channels, the input estimator is a one-step delayed inverse of the system and the state estimator is a plant simulation. Stability depends on the transmission zeros of the former system. These necessary and sufficient stability conditions then are extended to the non-square case with more measurements. This involves the Riccati difference equation and a detectability condition. Section 4 expands this analysis to the full column-rank direct feedthrough case. Section 6 contains the extension of stability results to time-varying systems via the Riccati equation. Section 5 brings unity to the distinct algorithms developed for distinct cases by showing that those different cases are due primarily to different delay structures of the system under study. It also reduces the general delay problem to the problem analyzed in Section 4. Finally, Section 7 waves some difficulties raised in [11] in the notion of model-free unbiased estimation by interpreting the SISE algorithm as a point estimator.

2 Problem Formulation

As noted earlier in section 1.1, for the purpose of clarity we will begin by performing our analysis on the LTI case and then extend results to LTV systems. Thus, consider a system modeled by the following state space realization.

$$x_{t+1} = Ax_t + Gd_t + w_t \tag{1}$$

$$y_t = Cx_t + Hd_t + v_t, \tag{2}$$

where $x_t \in \mathbb{R}^n$ is the state, $d_t \in \mathbb{R}^m$ is the unknown input, and $y_t \in \mathbb{R}^p$ is the system's output. The initial state x_0 is a random variable with a known covariance matrix P_0. The signals w_t and v_t are zero mean independent white noises that are also independent of x_0. The covariances of w_t and v_t are $Q \geqslant 0$ and $R > 0$, respectively. We will call the term Hd_t the feedthrough term, and H the feedthrough matrix. Note that, following [14] and [8], the system considered has no known input. This is not a limiting assumption as the algorithm developed is easily extensible to the case with known input as will be shown below. The conditions of whiteness and independence of w_t and v_t are not limiting either as known from Kalman filtering theory (Chapter 5.5 of Anderson and Moore [15]). Denote the measurements signal $\mathbf{Y}^t \triangleq \{y_t, \ldots, y_1, y_0\}$. The aim of SISE is to produce from $\mathbf{Y}^t$, a recursive filtered state estimate, $\hat{x}_{t|t}$, and filtered and/or smoothed MVU estimates, $\hat{d}_{t|t+1}$ or $\hat{d}_{t|t}$, depending on the properties of the system matrices. We make the following assumption.

Assumption 1. *System (1-2) has $[A, C]$ observable, $[A, Q^{1/2}]$ stabilizable, $P_0 \geq 0$, $Q \geq 0$, and $R > 0$.*

As mentioned in Section 1.1, the SISE algorithm that will be analyzed in this thesis is built as an unbiased-minimum-variance estimator. However, recall from the literature review in Section 1 that there are many formulations of the minimum variance unbiased (MVU) SISE algorithm that are not totally distinct. For that, it should be stated clearly what formulation will be followed in this thesis and what is the motivation behind its choice. This demands a little more detailed review of the history of development of the MVU formulation of SISE. The tradition began with Kitanidis [2] who formulated an algorithm that estimates the state only, for a system with zero feedthrough term. Darouach et. al [13] introduced a new approach for the state filter and established stability conditions for it. Hsieh [6] extended Kitanidis' algorithm to estimate the input as well for systems with no feedthrough but without attempting to prove the optimality of the input estimate in the sense of minimum variance. In a central paper [14], Gillijns and de Moor united all results of the three previous authors by developing an algorithm that simultaneously estimates the state and the unknown input. They proved the optimality of their algorithm in the minimum variance sense. They brought unity to previous works by proving that Kitanidis' [2] algorithm implicitly performs input estimation that is exactly like the one presented in Hseih [6] and that the latter's input estimate is optimal in the sense of minimum variance. In a following paper [8], the same authors did similar work but with the case where the system has a full rank feedthrough term. However, They did not discuss the stability of their algorithms in either of their papers. Young, Zhu, and Frazzoli [9] on the other hand established stability results and an extension of Gillijns and De Moor's SISE to the case of a system with a non-full rank feedthrough matrix. Given the central role played by Gillijns and De Moor's two papers [14, 8] in uniting old algorithms and providing a solid ground for new treatments such as that of Yong et al.

[9], we chose in this thesis to analyze the algorithms presented in [14] and [8] for both cases of zero and full rank feedthrough term respectively.

2.1 Inclusion of a known input signal

It is seen from the problem formulation above that the system that will be considered has no known input signal. This assumption enhances the simplicity and conciseness of the material. However, as will be shown in what follows, this assumption is by no means limiting. Consider the following system with both known and unknown input signals (u_t and d_t respectively)

$$\bar{x}_{t+1} = A\bar{x}_t + Bu_t + Gd_t + w_t \tag{3}$$

$$\bar{y}_t = C\bar{x}_t + Du_t + Hd_t + v_t. \tag{4}$$

Thanks to linearity, we can split the state of this system into two components, one expresses the response to the known input and nothing more, while the other expresses the response to initial conditions and all other disturbances. Specifically, define the following deterministic system

$$\begin{aligned} x'_{t+1} &= Ax'_t + Bu_t \\ y'_t &= Cx'_t + Du_t \\ x'_0 &= 0. \end{aligned}$$

Note that, given the original system matrices and the known input signal u_t, the state x'_t and output y'_t of this system can be exactly determined for any finite time t. Now we define

$$\begin{aligned} x_{t+1} &\triangleq \bar{x}_{t+1} - x'_{t+1} \\ &= A\bar{x}_t + Bu_t + Gd_t + w_t - Ax'_t - Bu_t \\ &= A(\bar{x}_t - x'_t) + Gd_t + w_t \\ &= Ax_t + Gd_t + w_t. \end{aligned} \tag{5}$$

We similarly define the output

$$\begin{aligned} y_t &= \bar{y}_t - y'_t \\ &= C\bar{x}_t + Du_t + Hd_t + v_t - Cx'_t - Du_t \\ &= C(\bar{x}_t - x'_t) + Hd_t + v_t \\ &= Cx_t + Hd_t + v_t. \end{aligned} \tag{6}$$

The system represented by (5) and (6) is of the same form as the system (1-2) which is used in the problem formulation. Thus, any SISE algorithms designed for (1-2) can be readily applied to the system with known input signals (3-4). This is done by supplying $y_t = \bar{y}_t - y'_t$ as the SISE algorithm input, where $\bar{y}_t$ is measured and y'_t is calculated. Then the state estimate for (3-4) is given as $\hat{\bar{x}}_{t|t} = \hat{x}_{t|t} + x'_t$, where $\hat{x}_{t|t}$ is the state estimate provided by the SISE algorithm and x'_t is calculated.

3 Zero direct feedthrough

For $H = 0$ in (2), SISE from [14] is shown as Algorithm 1.

Algorithm 1: SISE for $H = 0$

$$X_t = AP_{t-1}A^T + Q, \quad (7)$$

$$K_t = X_t C^T (CX_t C^T + R)^{-1}, \quad (8)$$

$$M_t = [G^T C^T (CX_t C^T + R)^{-1} CG]^{-1} \times G^T C^T (CX_t C^T + R)^{-1}, \quad (9)$$

$$P_t = (I - K_t C)\left[(I - GM_t C)X_t \times (I - GM_t C)^T + GM_t R M_t^T G^T\right] + K_t R M_t^T G^T, \quad (10)$$

$$\hat{d}_{t-1|t} = M_t(y_t - CA\hat{x}_{t-1|t-1}), \quad (11)$$

$$\hat{x}_{t|t} = A\hat{x}_{t-1|t-1} + G\hat{d}_{t-1|t} + K_t \times (y_t - CA\hat{x}_{t-1|t-1} - CG\hat{d}_{t-1|t}). \quad (12)$$

$$\text{cov}(x_t|\mathbf{Y}^t) = P_t, \quad (13)$$

under the following structural condition.

Assumption 2.

$$\textit{rank}\, CG = m. \quad (14)$$

An immediate observation is that SISE contains no specific information related to a model for the unmeasured disturbance d_t. Indeed, it is frequently claimed that signal $\{d_t : t = 0, 1, \ldots\}$ possesses no model whatsoever. In fact, it will be argued that the unknown input signal d_t at time t is treated as a mere parameter of the model not a stochastic signal itself. Although this is not explicitly stated in some works in which the SISE algorithm is developed such as [14], failure to appreciate this point would bring some misunderstandings related to the unbiasedness of SISE as will be discussed in Section 7. Evidently, Assumption 2 requires $p \geq m$ and $\operatorname{rank} C \geq \operatorname{rank} G = m$. Firstly, we treat the square case, $p = m$, where the number of measurements equals the dimension of the disturbance input. Then we shall derive more general results.

3.1 Stability analysis for the square case without feedthrough

From Assumption 2 when $p = m$, CG is invertible. Since, from (9), $M_t CG = I$ or $M_t = (CG)^{-1}$, we have

$$\hat{d}_{t-1|t} = (CG)^{-1}(y_t - CA\hat{x}_{t-1|t-1}), \tag{15}$$

$$0 = y_t - CA\hat{x}_{t-1|t-1} - CG\hat{d}_{t-1|t}, \tag{16}$$

$$\hat{x}_{t|t} = A\hat{x}_{t-1|t-1} + G\hat{d}_{t-1|t}, \tag{17}$$

$$= [I - G(CG)^{-1}C]A\hat{x}_{t-1|t-1} + G(CG)^{-1}y_t. \tag{18}$$

This estimation algorithm:

- is time-invariant;
- does not depend on Q or R, the noise variances;
- is independent from the covariance calculations;
- has zero $\hat{x}_{t|t}$ innovations (16), (12).

Algorithm 1 reduces to (15-18).

$$\hat{x}_{t|t} = [I - G(CG)^{-1}C]A\hat{x}_{t-1|t-1} + G(CG)^{-1}y_t,$$
$$\hat{d}_{t-1|t} = -(CG)^{-1}CA\hat{x}_{t-1|t-1} + (CG)^{-1}y_t.$$

Note that, using the Woodbury matrix identity twice, we may rewrite the SISE y_t-to-$\hat{d}_{t-1|t}$ transfer function as

$$\begin{aligned}
&(CG)^{-1} - (CG)^{-1}CA(zI - A + G(CG)^{-1}CA)^{-1}G(CG)^{-1} \\
&\quad = \left[CG + CA(zI - A)^{-1}G\right]^{-1}, \\
&\quad = \left[C\left(I + A(zI - A)^{-1}\right)G\right]^{-1}, \\
&\quad = \left[C\left(I - z^{-1}A\right)^{-1}G\right]^{-1}, \\
&\quad = \left[zC(zI - A)^{-1}G\right]^{-1}. \qquad (19)
\end{aligned}$$

The filtered state estimate error satisfies

$$\begin{aligned}
\tilde{x}_{t|t} &\triangleq x_t - \hat{x}_{t|t}, \\
&= [I - G(CG)^{-1}C]A\tilde{x}_{t-1|t-1} \\
&\quad + [I - G(CG)^{-1}C]w_{t-1} - G(CG)^{-1}v_t.
\end{aligned}$$

The stability of SISE and the boundedness of the covariance of $\tilde{x}_{t|t}$, depends on the eigenvalues of $[I - G(CG)^{-1}C]A$.

Theorem 1. *For system (1-2) subject to $p = m$ and Assumptions 1 and 2, the eigenvalues of the SISE estimator system matrix, $[I - G(CG)^{-1}C]\,A$, lie at the transmission zeros of the square transfer function $zC(zI - A)^{-1}G$. Accordingly, the SISE estimator is asymptotically stable if and only if these transmission zeros all lie inside the unit circle.*

The proof of this theorem follows immediately from (19). We see that, in the square case, the poles of SISE are located precisely at the transmission zeros of a one-step delayed version of the d_t-to-y_t transfer function of the model (1-2). SISE therefore is performing system inversion to recover $\hat{d}_{t-1|t}$ from $\mathbf{Y}^t$. The dependent recursion (17) for $\hat{x}_{t|t}$ is merely a simulation of the state equation (1) driven by $\hat{d}_{t-1|t}$. Effectively all the information in $\mathbf{Y}^t$ is used in generating the disturbance estimate, leaving simulation (17) to generate the state estimate.

3.2 System transformation for the non-square case

Due to Assumption 2 the only non-square case permissible is the one with number of outputs p greater than the number of unknown inputs m. Also, the fact that the solution of the square case was significantly facilitated by recognizing the invertibility of the matrix CG gives the insight that transforming the non-square case to a system where an invertibility of this sort is used will simplify the problem. Keeping in mind Assumption 2, take the singular value decomposition of the full column rank $p \times m$ matrix CG.

$$\begin{aligned} \mathtt{svd}(CG) &= U\Sigma V^T, \\ &= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \bar{\Sigma}_{m\times m} \\ 0_{p-m\times m} \end{bmatrix} V^T \end{aligned}$$

where $\bar{\Sigma}$ is an $m \times m$ diagonal matrix of non-zero singular values. Transform the original output signal y_t from (2),

$$\begin{aligned}\bar{y}_t &= U^T y_t = U^T C x_t + U^T v_t \\ &= \bar{C} x_t + \bar{v}_t \\ &= \begin{bmatrix} U_1^T C \\ \bar{U}_2^T C \end{bmatrix} x_t + \begin{bmatrix} U_1^T v_t \\ \bar{U}_2^T v_t \end{bmatrix} \\ &= \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} x_t + \begin{bmatrix} \bar{v}_{1,t} \\ \bar{v}_{2,t} \end{bmatrix}. \end{aligned} \tag{20}$$

By construction,

$$\begin{aligned} \bar{C}G = U^T C G &= \Sigma V^T \\ \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} G &= \begin{bmatrix} \bar{C}_1 G \\ 0 \end{bmatrix}, \end{aligned} \tag{21}$$

where $\det(\bar{C}_1 G) \neq 0$. It should be noted that the covariance matrix of $\bar{v}_t$ will change to $\bar{R} = U^T R U$, where R is the covariance matrix of the untransformed measurement noise of the original system, v_t. This does not affect assumption of positive definiteness of measurement noise covariance.[1]

3.3 Stability analysis for the non-square case without feedthrough

Since it is shown in the previous section that any system (1-2) on which Assumptions 1 and 2 hold can be transformed to a system (1-20), we will analyze the stability of the

[1] A simple proof is that for any $v = Uv' \neq 0 \in \mathbb{R}^k$ and full column rank U, we have: $R \in \mathbb{R}^{k \times k} > 0 \rightarrow (\forall x \in \mathbb{R}^k \neq 0)(x^T R x > 0) \rightarrow v^T R v > 0 \rightarrow v'^T (U^T R U) v' > 0$.

transformed system. Theorem 2 which is the central theorem of this subsection is mentioned first. The rest of the section consists of a proof and an interpretation of this theorem.

Theorem 2. *Subject to assumptions 1 & 2, the SISE algorithm of system (1-20) as presented in Algorithm 1 is stable if and only if* $\{A(I - G(\bar{C}_1 G)^{-1}\bar{C}_1), \bar{C}_2\}$ *is detectable.*

As mentioned in the introduction, the method used to derive the stability conditions for the SISE algorithm is reducing its state estimator's covariance matrix limit as time increases indefinitely to a solution of an algebraic Riccati equation (ARE) that is dependent on the system matrices only. To do so, note that $\hat{d}_{t-1|t}$ can be eliminated from the state estimate $\hat{x}_{t|t}$ by substituting (11) in (12). Doing so reduces the state estimation component of Algorithm 1 to

$$\hat{x}_{t|t} = (I - K_t\bar{C})(I - GM_t\bar{C})A\hat{x}_{t-1|t-1} + \left[GM_t + K_t(I - \bar{C}GM_t)\right] y_t. \tag{22}$$

Now, denote

$$\begin{aligned} \mathcal{H}_t &\triangleq GM_t - K_t\bar{C}GM_t + K_t \\ I - \mathcal{H}_t\bar{C} &= I - GM_t\bar{C} + K_t\bar{C}GM_tC - K_t\bar{C} \\ &= (I - K_t\bar{C})(I - GM_t\bar{C}). \end{aligned} \tag{23}$$

The closed loop equation of the SISE estimate of the state (22) can now be written as

$$\hat{x}_{t|t} = (I - \mathcal{H}_t\bar{C})A\hat{x}_{t-1} + \mathcal{H}_t\bar{y}_t.$$

Define

$$
\begin{aligned}
\tilde{x}_t &\triangleq x_t - \hat{x}_{t|t} \\
&= Ax_{t-1} + Gd_{t-1} + w_{t-1} - (I - \mathcal{H}_t\bar{C})A\hat{x}_{t-1|t-1} - \mathcal{H}_t\bar{y}_t \\
&= Ax_{t-1} + Gd_{t-1} + w_{t-1} - (I - \mathcal{H}_t\bar{C})A\hat{x}_{t-1|t-1} \\
&\quad - \mathcal{H}_t(\bar{C}Ax_{t-1} + \bar{C}Gd_{t-1} + \bar{C}w_{t-1} + \bar{v}_t) \\
&= (I - \mathcal{H}_t\bar{C})A\tilde{x}_{t-1} + (I - \mathcal{H}_t\bar{C})Gd_{t-1} + (I - \mathcal{H}_tC)w_{t-1} - \mathcal{H}_t\bar{v}_t.
\end{aligned}
$$

Now, from (9), $M_t\bar{C}G = I$. Hence, by (23),

$$
\begin{aligned}
(I - \mathcal{H}_t\bar{C})G &= G - GM_t\bar{C}G - K_t\bar{C}GM_t\bar{C}G + K_t\bar{C}G \\
&= 0.
\end{aligned}
$$

So, the unknown input d_{t-1} is eliminated which reduces the innovations $\tilde{x}_t$ to

$$
\tilde{x}_t = (I - \mathcal{H}_t\bar{C})A\tilde{x}_{t-1} + (I - \mathcal{H}_t\bar{C})w_{t-1} - \mathcal{H}_tv_t. \tag{24}
$$

Now by (13),

$$
\begin{aligned}
P_t &= \mathrm{cov}(x_t|\bar{\mathbf{Y}}_{\mathbf{t}}) \\
&= \mathrm{E}[\tilde{x}_t\tilde{x}_t^T]
\end{aligned}
$$

Note that $\mathrm{E}[\tilde{x}_{t-1}w_{t-1}^T]$, $\mathrm{E}[\tilde{x}_{t-1}\bar{v}_t^T]$, and $\mathrm{E}[w_{t-1}\bar{v}_t^T]$ are zero matrices (the first two from model and algorithm and the last from the noise independence assumption). So, a recursive equation for P_t equivalent to (13) can be derived directly from (24), namely,

$$P_t = (I - \mathcal{H}_t \bar{C})(AP_{t-1}A^T + Q)(I - \mathcal{H}_t \bar{C})^T + \mathcal{H}_t \bar{R} \mathcal{H}_t^T. \tag{25}$$

Using (7),

$$X_{t+1} = A(I - \mathcal{H}_t \bar{C})X_t(I - \mathcal{H}_t \bar{C})^T A^T + A\mathcal{H}_t \bar{R} \mathcal{H}_t^T A^T + Q. \tag{26}$$

Note that (26) is in the form of a recursive Lyapunov equation. However, the matrix $\mathcal{H}_t$ is dependent on X_t so it is not directly possible to find stability criteria. To deal with this problem, (26) is reduced to a Riccati difference equation which explicitly depends only on the system matrices and the matrix X_t. This reduction is made possible by the system transformation discussed in Section 3.2. The following lemma establishes this reduction.

Lemma 1. *Splitting the transformed covariance matrix $\bar{R}$ to*

$$\bar{R} = \begin{bmatrix} \bar{R}_1 & \bar{R}_2 \\ \bar{R}_2^T & \bar{R}_3 \end{bmatrix},$$

the recursion (26) *is equivalent to the Riccati Difference Equation (RDE):*

$$\begin{aligned} X_{t+1} = {} & \bar{A}X_t\bar{A}^T - (\bar{A}X_t\bar{C}_2^T - AG(\bar{C}_1G)^{-1}R_2)(\bar{C}_2X_t\bar{C}_2^T + \bar{R}_3)^{-1} \\ & \times (\bar{A}X_t\bar{C}_2^T - AG(\bar{C}_1G)^{-1}\bar{R}_2)^T + \bar{Q}, \end{aligned} \tag{27}$$

where,

$$\bar{A} = A(I - G(\bar{C}_1G)^{-1}\bar{C}_1), \quad \bar{Q} = AG(\bar{C}_1G)^{-1}R_1(G(\bar{C}_1G)^{-1})^T A^T + Q. \tag{28}$$

Proof. keeping (21) in mind, define $Z \triangleq C_1 G$, so det$Z \neq 0$. Denote

$$\begin{aligned}(\bar{C}X_t\bar{C}^T + \bar{R})^{-1} &\triangleq Y \\ &= \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}.\end{aligned} \tag{29}$$

Now,

$$\mathcal{H}_t = GM_t - K_t\bar{C}GM_t + K_t,$$

where

$$\begin{aligned}M_t &= (G^T\bar{C}^TYCG)^{-1}G^T\bar{C}^TY, \\ K_t &= X_t\bar{C}^TY.\end{aligned}$$

So,

$$\begin{aligned}\mathcal{H}_t = G &\left(\begin{bmatrix} Z^T & 0 \end{bmatrix} Y \begin{bmatrix} Z \\ 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} Z^T & 0 \end{bmatrix} Y - X_t \begin{bmatrix} \bar{C}_1^T & \bar{C}_2^T \end{bmatrix} Y \begin{bmatrix} Z \\ 0 \end{bmatrix} \\ &\times \left(\begin{bmatrix} Z^T & 0 \end{bmatrix} Y \begin{bmatrix} Z \\ 0 \end{bmatrix}\right)^{-1} \begin{bmatrix} Z^T & 0 \end{bmatrix} Y + X_t \begin{bmatrix} \bar{C}_1^T & \bar{C}_2^T \end{bmatrix} Y \\ &= \begin{bmatrix} GZ^{-1} & GZ^{-1}Y_1^{-1}Y_2 \end{bmatrix} + X_t \begin{bmatrix} 0 & \bar{C}_2(Y_3 - Y_2^TY_1^{-1}Y_2) \end{bmatrix}\end{aligned} \tag{30}$$

But from (29),

$$(\bar{C}X_t\bar{C}^T + \bar{R}) = \begin{bmatrix} \bar{C}_1 X_t \bar{C}_1^T + \bar{R}_1 & \bar{C}_1 X_t \bar{C}_2^T + \bar{R}_2 \\ \bar{C}_2 X_t \bar{C}_1^T + \bar{R}_2^T & \bar{C}_2 X_t \bar{C}_2^T + \bar{R}_3 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1}.$$

By the formula for inverting 2×2 block matrices,

$$\begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y_1^{-1} + Y_1^{-1}Y_2(Y_3 - Y_2^T Y_1^{-1} Y_2)^{-1} Y_2^T Y_1^{-1} & -Y_1^{-1}Y_2(Y_3 - Y_2^T Y_1^{-1} Y_2)^{-1} \\ -(Y_3 - Y_2^T Y_1^{-1} Y_2)^{-1} Y_2^T Y_1^{-1} & (Y_3 - Y_2^T Y_1^{-1} Y_2)^{-1}. \end{bmatrix}$$

From the right column matrix blocks,

$$\begin{aligned} \bar{Y} \triangleq (Y_3 - Y_2^T Y_1^{-1} Y_2) &= (\bar{C}_2 X_t \bar{C}_2^T + \bar{R}_3)^{-1}, \qquad (31) \\ Y_1^{-1} Y_2 &= -(\bar{C}_1 X_t \bar{C}_2^T + \bar{R}_2)\bar{Y}. \end{aligned}$$

Substituting (31) in (30) gives

$$\begin{aligned} \mathcal{H}_t &= \begin{bmatrix} GZ^{-1} & -GZ^{-1}(\bar{C}_1 X_t \bar{C}_2^2 + \bar{R}_2)\bar{Y} \end{bmatrix} + \begin{bmatrix} 0 & X_t \bar{C}_2^T \bar{Y} \end{bmatrix}, \\ &= \begin{bmatrix} GZ^{-1} & ((I - GZ^{-1}\bar{C}_1)X_t \bar{C}_2^T - GZ^{-1}\bar{R}_2)\bar{Y} \end{bmatrix}. \\ \mathcal{H}_t &= \begin{bmatrix} GZ^{-1} & \mathcal{H}\bar{Y} \end{bmatrix}, \qquad (32) \end{aligned}$$

with,

$$\mathcal{H} \triangleq ((I - GZ^{-1}\bar{C}_1)X_t \bar{C}_2^T - GZ^{-1}\bar{R}_2). \qquad (33)$$

Substitution of (32) in (26) gives

$$
\begin{aligned}
X_{t+1} &= A((I-\mathcal{H}_t\bar{C})X_t(I-\mathcal{H}_t\bar{C})^T+\mathcal{H}_t\bar{R}\mathcal{H}_t^T)A^T+Q, \\
&= A\left((I-GZ^{-1}\bar{C}_1-\mathcal{H}\bar{Y}\bar{C}_2)X_t(I-GZ^{-1}\bar{C}_1-\mathcal{H}\bar{Y}\bar{C}_2)^T+\right. \\
&\quad \left.\mathcal{H}_t\begin{bmatrix}\bar{R}_1 & \bar{R}_2 \\ \bar{R}_2^T & \bar{R}_3\end{bmatrix}\mathcal{H}_t^T\right)A^T+Q, \\
&= A((I-GZ^{-1}\bar{C}_1)X_t(I-GZ^{-1}\bar{C}_1)^T, \\
&\quad -\mathcal{H}\bar{Y}\bar{C}_2X_t(I-GZ^{-1}\bar{C}_1)^T-(I-GZ^{-1}\bar{C}_1)X\bar{C}_2^T\bar{Y}\mathcal{H}^T+\mathcal{H}\bar{Y}\bar{C}_2X\bar{C}_2^T\bar{Y}\mathcal{H}^T, \\
&\quad +GZ^{-1}\bar{R}_1(GZ^{-1})^T+\mathcal{H}\bar{Y}\bar{R}_2^T(GZ^{-1})^T+GZ^{-1}\bar{R}_2\bar{Y}\mathcal{H}^T\mathcal{H}\bar{Y}\bar{R}_3)\bar{Y}\mathcal{H}^T)A^T+Q, \\
&= A((I-GZ^{-1}\bar{C}_1)X_t(I-GZ^{-1}\bar{C}_1)^T- \\
&\quad \mathcal{H}\bar{Y}[\bar{C}_2X_t(I-GZ^{-1}\bar{C}_1)^T-\bar{R}_2^TZ^{-T}G^T]- \\
&\quad [(I-GZ^{-1}\bar{C}_1)X_t\bar{C}_2^T-GZ^{-1}\bar{R}_2]\bar{Y}\mathcal{H}^T \\
&\quad +\mathcal{H}\bar{Y}(\bar{C}_2X_t\bar{C}_2^T+\bar{R}_3)\bar{Y}\mathcal{H}^T+GZ^{-1}\bar{R}_1(GZ^{-1})^T)A^T+Q.
\end{aligned}
$$

The last equation can be simplified by introducing $\bar{Y}$ and $\mathcal{H}$ from (31) and (33), respectively.

$$
\begin{aligned}
X_{t+1} &= A((I-GZ^{-1}\bar{C}_1)X_t(I-GZ^{-1}\bar{C}_1)^T-\mathcal{H}\bar{Y}\mathcal{H}^T-\mathcal{H}\bar{Y}\mathcal{H}^T+\mathcal{H}\bar{Y}\bar{Y}^{-1}\bar{Y}\mathcal{H}^T \\
&\quad +GZ^{-1}\bar{R}_1(GZ^{-1})^T)A^T+Q, \\
&= A((I-GZ^{-1}\bar{C}_1)X_t(I-GZ^{-1}\bar{C}_1)^T-\mathcal{H}\bar{Y}\mathcal{H}^T+GZ^{-1}\bar{R}_1(GZ^{-1})^T)A^T+Q.
\end{aligned}
$$

Substituting the definitions (31) and (33) gives

$$
\begin{aligned}
X_{t+1} &= \bar{A}X_t\bar{A}^T-(\bar{A}X_t\bar{C}_2^T-AG(\bar{C}_1G)^{-1}\bar{R}_2)(\bar{C}_2X_t\bar{C}_2^T+\bar{R}_3)^{-1}\times \\
&\quad (\bar{A}X_t\bar{C}_2^T-AG(\bar{C}_1G)^{-1}\bar{R}_2)^T+\bar{Q},
\end{aligned}
$$

where,

$$\bar{A} = A(I - G(\bar{C}_1 G)^{-1}\bar{C}_1), \quad \bar{Q} = AG(\bar{C}_1 G)^{-1}\bar{R}_1(G(\bar{C}_1 G)^{-1})^T A^T + Q.$$

This ends the proof of the lemma. □

After the recursive covariance equation for the SISE algorithm is reduced to the Riccati difference equation (27), we apply the results of convergence of RDE's to (27) to establish the stability conditions for the SISE algorithm. The stability conditions are those conditions that ensure that the following two propositions hold.

1. The discrete-time algebraic Riccati equation (DARE)

$$\begin{aligned} X = \bar{A}X\bar{A}^T - (\bar{A}X\bar{C}_2^T - AG(\bar{C}_1 G)^{-1}\bar{R}_2)(\bar{C}_2 X\bar{C}_2^T + \bar{R}_3)^{-1} \times \\ (\bar{A}X\bar{C}_2^T - AG(\bar{C}_1 G)^{-1}\bar{R}_2)^T + \bar{Q}, \end{aligned} \tag{34}$$

   that corresponds to the RDE (27), has a unique positive definite stabilizing solution $\bar{X}$.

2. The matrix X_t of (27) approaches the stabilizing solution $\bar{X}$ of (34) as $t \to \infty$.

The conditions for both of these propositions are established in the following proof of Theorem 2.

Proof of theorem 2

Theorem 2 states that the SISE algorithm as presented in Algorithm 1 of a system (1-20), subject to Assumptions 1 & 2, is stable if and only if $\{A(I - G(\bar{C}_1 G)^{-1}\bar{C}_1), \bar{C}_2\}$ is detectable.

Proof. In this proof, we make use of Theorem E.5.1 and Theorem 14.7.2 from Kaileth, Sayed, and Hassibi [16]. Both theorems are reproduced in the Appendix. Appealing to Theorem E.5.1 in , the DARE (34) has a unique positive semi-definite stabilizing solution $\bar{X}$ if and only if

1. $\{\bar{A}^s, \bar{Q}^{s/2}\}$ is controllable on the unit circle and
2. $\{\bar{A}, \bar{C}_2\}$ is detectable,

   where,

$$\bar{Q}^s \triangleq \bar{Q} - AG(\bar{C}_1 G)^{-1} \bar{R}_2 \bar{R}_3^{-1} \bar{R}_2^T (\bar{C}_1 G)^{-T} G^T A^T,$$
$$\bar{A}^s \triangleq \bar{A} - AG(\bar{C}_1 G)^{-1} \bar{R}_2 \bar{R}_3^{-1} \bar{C}_2.$$

Also, according to Theorem 14.7.2 in [16], the matrix X_t of (27) approaches the stabilizing solution $\bar{X}$ of (34) as $t \to \infty$ if Condition 2 above holds and

3. $\{\bar{A}^s, \bar{Q}^{s/2}\}$ in stabilizable, and
4. $X_0 > -X^a$,

   where X^a is a certain positive definite matrix. (For details, see the Appendix.)

But the conditions 1, 3, and 4 are implied by Assumption 1 as will be shown below. Hence, the SISE algorithm (Algorithm 1) is stable if and only if Condition 2 holds, namely, if and only if $\{\bar{A}, \bar{C}_2\} = \{A(I - G(\bar{C}_1 G)^{-1}\bar{C}_1), \bar{C}_2\}$ is detectable.

By Assumption 1, the covariance Q satisfies $Q \geq 0$, which by the definition of X_t in (7) implies Condition 4. Also, by Assumption 1, $\{A, Q^{1/2}\}$ is stabilizable. This implies Condition 3 as the following reasoning suggests. If $\{A, Q^{1/2}\}$ is stabilizable then there exists a matrix N_1 such that the eigenvalues of $A+Q^{1/2}N_1$ can all be placed in the unit circle. Note that

$$\begin{aligned}
\mathcal{Q}^s &= \bar{Q} - AG(\bar{C}_1G)^{-1}\bar{R}_2\bar{R}_3^{-1}\bar{R}_2^T(\bar{C}_1G)^{-T}G^TA^T, \\
&= Q + AG(\bar{C}_1G)^{-1}\bar{R}_1(G(\bar{C}_1G)^{-1})^TA^T - AG(\bar{C}_1G)^{-1}\bar{R}_2\bar{R}_3^{-1}\bar{R}_2^T(\bar{C}_1G)^{-T}G^TA^T, \\
&= Q + AG(\bar{C}_1G)^{-1}(\bar{R}_1 - \bar{R}_2\bar{R}_3^{-1}\bar{R}_2^T)(\bar{C}_1G)^{-T}G^TA^T.
\end{aligned}$$

Take

$$\mathcal{Q}^{s/2} = \begin{bmatrix} Q^{1/2} & AG(\bar{C}_1G)^{-1}(\bar{R}_1 - \bar{R}_2\bar{R}_3^{-1}\bar{R}_2^T)^{1/2} \end{bmatrix}.$$

Denote

$$N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix},$$

with the above definition of N_1 and the choice of

$$N_2 = (\bar{R}_1 - \bar{R}_2\bar{R}_3^{-1}\bar{R}_2^T)^{-1/2}(\bar{C}_1 + \bar{R}_2\bar{R}_3^{-1}\bar{C}_2).$$

Note that $\left(\bar{R}_1 - \bar{R}_2\bar{R}_3^{-1}\bar{R}_2^T\right)$ is always invertible since it is a Schur complement of the leading block in the positive definite matrix $\bar{R}$. This choice of N yields

$$\begin{aligned}
A^s + Q^{s/2}N &= \bar{A} - AG(\bar{C}_1G)^{-1}\bar{R}_2\bar{R}_3^{-1}\bar{C}_2 \\
&\quad + \begin{bmatrix} Q^{1/2} & AG(\bar{C}_1G)^{-1}(\bar{R}_1 - \bar{R}_2\bar{R}_3^{-1}\bar{R}_2^T)^{1/2} \end{bmatrix} \\
&\quad \times \begin{bmatrix} N_1 \\ (\bar{R}_1 - \bar{R}_2\bar{R}_3^{-1}\bar{R}_2^T)^{-1/2}(\bar{C}_1 + \bar{R}_2\bar{R}_3^{-1}\bar{C}_2) \end{bmatrix}, \\
&= \bar{A} - AG(\bar{C}_1G)^{-1}\bar{R}_2\bar{R}_3^{-1}\bar{C}_2 + Q^{1/2}N_1 \\
&\quad + AG(\bar{C}_1G)^{-1}(\bar{R}_1 - \bar{R}_2\bar{R}_3^{-1}\bar{R}_2^T)^{1/2}(\bar{R}_1 - \bar{R}_2\bar{R}_3^{-1}\bar{R}_2^T)^{-1/2}(\bar{C}_1 + \bar{R}_2\bar{R}_3^{-1}\bar{C}_2), \\
&= A - AG(\bar{C}_1G)^{-1}\bar{C}_1 - AG(\bar{C}_1G)^{-1}\bar{R}_2\bar{R}_3^{-1}\bar{C}_2 + Q^{1/2}N_1 + AG(\bar{C}_1G)^{-1}\bar{C}_1 \\
&\quad + AG(\bar{C}_1G)^{-1}\bar{R}_2\bar{R}_3^{-1}\bar{C}_2, \\
&= A + Q^{1/2}N_1,
\end{aligned}$$

whose eigenvalues all lie within the unit circle by the definition of N_1. Hence, $\{F^s, Q^{s/2}\}$ is stabilizable. Finally, note that Condition 3 implies Condition 1, since all eigenvalues on the unit circle can be placed within the unit circle by the above choice of N. This concludes the proof. $\square$

We already know from Theorem 1 that the eigenvalues of $A(I - G(\bar{C}_1G)^{-1}\bar{C}_1)$ are stable if and only if $z\bar{C}_1(zI - A)^{-1}G$ is minimum-phase. We see that, when $p > m$, the surfeit of measurements beyond those strictly needed to produce $\hat{d}_{t-1|t}$ are brought to bear on estimating x_t. The stability of SISE depends on either the square system $(A, \bar{C}_1)$ yielding stability via Theorem 1, i.e. via stable transmission zeros, or there being sufficient information in the additional measurements to stabilize the estimator.

3.4 Further Reduction

In this subsection, a transformation alternative to that of Section 3.2 is introduced to further simplify the derivation of Theorem 2. The transformation is a variation of the technique in [9]. Consider again the singular value decomposition of the $p \times m$ matrix CG.

$$
\begin{aligned}
\texttt{svd}(CG) &= U\Sigma V^T, \\
&= \begin{bmatrix} U_m & U_{p-m} \end{bmatrix} \begin{bmatrix} \bar{\Sigma} \\ 0 \end{bmatrix} V^T.
\end{aligned}
$$

Define the $p \times p$ non-singular transformation

$$
\mathcal{T} = \begin{bmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{bmatrix} \triangleq \begin{bmatrix} I_m & -U_m^T R U_{p-m} (U_{p-m}^T R U_{p-m})^{-1} \\ 0 & I_{p-m} \end{bmatrix} \begin{bmatrix} U_m^T \\ U_{p-m}^T \end{bmatrix}, \tag{35}
$$

where R is the covariance matrix of the measurement noise of the original system. Now, transform the original output signal y_t,

$$
\begin{aligned}
\bar{y}_t = \mathcal{T} y_t &= \begin{bmatrix} \mathcal{T}_1 C \\ \mathcal{T}_2 C \end{bmatrix} x_t + \begin{bmatrix} \mathcal{T}_1 v_t \\ \mathcal{T}_2 v_t \end{bmatrix} \\
&= \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} x_t + \begin{bmatrix} \bar{v}_{1,t} \\ \bar{v}_{2,t} \end{bmatrix}.
\end{aligned} \tag{36}
$$

As in (21), This transformation yields

$$
\begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} G = \begin{bmatrix} \bar{C}_1 G \\ 0 \end{bmatrix},
$$

where $\det(\bar{C}_1 G) \neq 0$. What is different in this transformation though is that the covariance of the transformed noise, $\bar{v}_t$, is made block diagonal.

$$\begin{aligned}\bar{R} = \text{cov}\begin{bmatrix}\bar{v}_{1,t}\\ \bar{v}_{2,t}\end{bmatrix} &= \text{cov}(\mathcal{T}v_t) = \mathcal{T}\text{cov}(v_t)\mathcal{T}^T = \mathcal{T}R\mathcal{T}^T,\\ &= \begin{bmatrix}\mathcal{T}_1\\ \mathcal{T}_2\end{bmatrix} R \begin{bmatrix}\mathcal{T}_1^T & \mathcal{T}_2^T\end{bmatrix},\\ &= \begin{bmatrix}\mathcal{T}_1 R\mathcal{T}_1^T & \mathcal{T}_1 R\mathcal{T}_2^T\\ \mathcal{T}_2 R\mathcal{T}_1^T & \mathcal{T}_2 R\mathcal{T}_2^T\end{bmatrix}.\end{aligned}$$

However,

$$\begin{aligned}\mathcal{T}_1 R\mathcal{T}_2^T &= (U_m^T - U_m^T R U_{p-m}(U_{p-m}^T R U_{p-m})^{-1} U_{p-m}^T) R U_{p-m}^T\\ &= 0_{m\times p-m}.\end{aligned}$$

Hence,

$$\bar{R} = \begin{bmatrix}\bar{R}_1 & 0\\ 0 & \bar{R}_2\end{bmatrix}$$

This transformation significantly simplifies (27) and thus the derivations of Lemma 1 and Theorem 2. The simplified version of (27) will be used in Section 6 to extend stability results to time varying systems. More, a similar transformation will be used for stability results of the case with direct feedthrough.

Results in Section 3 are provisionally accepted to be published in Automatica journal in a paper co-authored with Mohammad Ali Abooshahab, Morten Hovd, and Robert Bitmead. The author of this thesis was a primary investigator of those results and is the primary author of those sections as they appear in this thesis.

4 Full-rank direct feedthrough

When $H \neq 0$ in (2), SISE alters. Gillijns and De Moor [8] provide a SISE algorithm, subject to the following assumption.

Assumption 3. *rank* $H = m$.

Subject to this assumption, the SISE algorithm for time-invariant system (1-2) is given as Algorithm 2.

Algorithm 2: SISE for rank$H = m$

$$\hat{x}_{t|t-1} = A\hat{x}_{t-1|t-1} + G\hat{d}_{t-1|t-1}, \tag{37}$$

$$P^x_{t|t-1} = \begin{bmatrix} A & G \end{bmatrix} \begin{bmatrix} P^x_{t-1|t-1} & P^{xd}_{t-1|t-1} \\ P^{dx}_{t-1|t-1} & P^d_{t-1|t-1} \end{bmatrix} \begin{bmatrix} A^T \\ G^T \end{bmatrix} + Q,$$

$$\tilde{R}_t = CP^x_{t|t-1}C^T + R, \tag{38}$$

$$M_t = (H^T\tilde{R}_t^{-1}H)^{-1}H^T\tilde{R}_t^{-1}, \tag{39}$$

$$\hat{d}_{t|t} = M_t(y_t - C\hat{x}_{t|t-1}), \tag{40}$$

$$P^d_{t|t} = (H^T\tilde{R}_t^{-1}H)^{-1},$$

$$K_t = P^x_{k|k-1}C^T\tilde{R}_t^{-1}, \tag{41}$$

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t(y_t - C\hat{x}_{t|t-1} - H\hat{d}_{t|t}), \tag{42}$$

$$P^x_{t|t} = P^x_{t|t-1} - K_t(\tilde{R}_t - HP^dH^T)K_t^T,$$

$$P^{xd}_{t|t} = \left(P^{dx}_{t|t}\right)^T = -K_tHP^d_{t|t}.$$

In this section, an analysis analogous to that of Section 3 will be followed. It is worth noting from (39) that $M_tH = I$. This implies that the SISE algorithm that gives a filtered estimate $\hat{d}_{t|t}$ rather than a smoothed estimate $\hat{d}_{t|t+1}$ is difined if and only if rank $H = m$, i.e. the matrix H has full column rank. This is where Assumption 3 is exploited. This condition is significant for the derivations in the next subsection. When rank $H < m$, [9] provides ULISE, a carefully developed SISE algorithm which uses the singular value decomposition as in subsection 4.2 but more widely to handle the more complicated interaction between filtered and smoothed estimates for d_t. Another method by which the case of rank $H < m$ is reduced to the case of rank $H = m$ will also be derived in Section 5.

4.1 Stability analysis for the square case with feedthrough

Noting that Algorithm 2 is defined if and only if rank $H = m$, the feedthrough term H in the square case is an $m \times m$ full rank matrix and hence, invertible. By (39),

$$\begin{aligned} M_t &= (H^T \tilde{R}_t^{-1} H)^{-1} H^T \tilde{R}_t^{-1} \\ &= H^{-1} \tilde{R}_t H^{-T} H^T \tilde{R}_t^{-1} = H^{-1}. \end{aligned}$$

Substitution in (40) yields

$$\hat{d}_t = H^{-1}(y_t - C\hat{x}_{t|t-1}) \tag{43}$$

$$0 = y_t - C\hat{x}_{t|t-1} - H\hat{d}_t. \tag{44}$$

It is clear from (44) and (42) that in the square case, all the measurements of time t are used to estimate d_t and the filtered value of the state estimate $\hat{x}_{t|t}$ is the same as the previous predicted estimate $\hat{x}_{t|t-1}$. The input estimate $\hat{d}_{t|t}$ is then used to predict the value of the state at time $t + 1$ by mere simulation. To describe the dynamics of Algorithm 2 in the square case, the closed-loop formula of the predicted state estimate is obtained by eliminating $\hat{d}_{t|t}$ as follows.

$$\begin{aligned} \hat{x}_{t|t} &= \hat{x}_{t|t-1} \\ \hat{x}_{t+1|t} &= A\hat{x}_{t|t-1} + G\hat{d}_{t|t} \\ &= A\hat{x}_{t|t-1} + GH^{-1}(y_t - C\hat{x}_{t|t-1}) \\ &= (A - GH^{-1}C)\hat{x}_{t|t-1} + GH^{-1}y_t. \end{aligned} \tag{45}$$

Theorem 3. *For system (1-2) subject to $p = m$ and Assumptions 1 and 3, the eigenvalues of* *the SISE estimator system matrix,* $A - GH^{-1}C$*, lie at the transmission zeros of the square* d_t*-to-*y_t *transfer function* $H + C(zI - A)^{-1}G$*. Accordingly, the SISE estimator is asymptotically stable if and only if these transmission zeros all lie inside the unit circle.*

Proof. applying the woodbury identity to the d_t-to-y_t square transfer function,

$$\begin{aligned}&\left[H + C(zI - A)^{-1}G\right]^{-1}\\&\qquad = H^{-1} - H^{-1}C(zI - A + GH^{-1}C)^{-1}GH^{-1}.\end{aligned}$$

Which is exactly the y_t-to-$\hat{d}_{t|t}$ transfer function as can be seen from (45) and (43). The poles of this transfer function lie at the transmission zeros of the original system's d_t-to-y_t transfer function. □

As in the zero feedthrough case, the square SISE simplifies to (45) and (43) which means again that Algorithm 2 becomes time invariant, independent from noise variances and from estimates' covariance calculations, and as pointed out earlier has zero filtering innovation. It is also possible as in the zero feedthrough case to calculate the exact positions of the poles of the SISE in this case as Theorem 3 proves. However, there is an important difference between the square cases with full-rank feedthrough and zero feedthrough. When the feedthrough matrix is full-rank, SISE (Algorithm 2) inverts the system (1-2); while when the feedthrough is the zero matrix, SISE (Algorithm 1) inverts a delayed version of the system (1-2). That is roughly because in the latter case, the input d_t does not affect y_t, it only affects y_{t+1}. This will be examined in detail in Section 5.

4.2 System transformation for the non-square case

As in the zero feedthrough case, the simplicity and neatness of the result of the square case result from the fact that the feedthrough matrix H should be invertible. This gives the indication that some sort of this invertibility of the feedthrough matrix for the non-square case will simplify the solution. This insight goes hand in hand with the condition rank $H = m$ which is necessary for derivation of the algorithm. Take the singular value decomposition of matrix H

$$\begin{aligned}\texttt{svd}(H) &= U\Sigma V^T \\ &= \begin{bmatrix} U_m & U_{p-m} \end{bmatrix} \begin{bmatrix} \bar{\Sigma} \\ 0 \end{bmatrix} V^T.\end{aligned}$$

Define the non-singular transformation

$$\mathcal{T} = \begin{bmatrix} \mathcal{T}_1 \\ \mathcal{T}_2 \end{bmatrix} \triangleq \begin{bmatrix} I_m & -U_m^T R U_{p-m}(U_{p-m}^T R U_{p-m})^{-1} \\ 0 & I_{p-m} \end{bmatrix} \begin{bmatrix} U_m^T \\ U_{p-m}^T \end{bmatrix}. \tag{46}$$

Now, transform the original output signal y_t,

$$\begin{aligned}\bar{y}_t = \mathcal{T} y_t &= \bar{C} x_t + \bar{H} d_t + \bar{v}_t \\ &= \begin{bmatrix} \mathcal{T}_1 C \\ \mathcal{T}_2 C \end{bmatrix} x_t + \begin{bmatrix} \mathcal{T}_1 H \\ \mathcal{T}_2 H \end{bmatrix} d_t + \begin{bmatrix} \mathcal{T}_1 v_t \\ \mathcal{T}_2 v_t \end{bmatrix}.\end{aligned}$$

By a reasoning similar to that of Section 3.4, note that

$$\bar{y}_t = \begin{bmatrix} \bar{C}_1 \\ \bar{C}_2 \end{bmatrix} x_t + \begin{bmatrix} \bar{H}_1 \\ 0 \end{bmatrix} d_t + \begin{bmatrix} \bar{v}_{1,t} \\ \bar{v}_{2,t} \end{bmatrix}, \tag{47}$$

where $\bar{H}_1 = \mathcal{T}_1 \bar{H}$, which is non-singular by construction. Following the reasoning in Section 3.4, note that the noise covariance of the original system, R, is altered without effect on positive definiteness by the transformation as follows.

$$\bar{R} = \text{cov} \begin{bmatrix} \bar{v}_{1,t} \\ \bar{v}_{2,t} \end{bmatrix} = \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & \bar{R}_2 \end{bmatrix}. \tag{48}$$

The above transformation (47 - 48) can be performed whenever Assumption 3 holds. It is for this transformed system that the stability results are derived in the next section.

4.3 Stability analysis for the non-square case with feedthrough

The central theorem of this section for which the rest of the section is dedicated is the following.

Theorem 4. *Subject to Assumptions 1 & 3, the SISE algorithm of system (1-47) as presented in Algorithm 2 is stable if and only if* $\{A - G\bar{H}_1^{-1}\bar{C}_1, \bar{C}_2\}$ *is detectable.*

As in Section 4.3, a lemma analogous to Lemma 1 is required before the proof of Theorem 4 can be presented. Again, the method used to derive the stability conditions for the SISE algorithm is reducing its state estimator's covariance recursive equation to a Riccati difference equation. First, eliminate the input estimate $\hat{d}_{t|t}$ from the prior state estimate $\hat{x}_{t+1|t}$

by substituting (40) in (42) and then both in (37). The resulting estimator is

$$\hat{x}_{t+1|t} = (A - AK_t\bar{C} + AK_t\bar{H}M_t\bar{C} - GM_t\bar{C})\hat{x}_{t|t-1} + (AK_t - AK_t\bar{H}M_t + GM_t)\bar{y}_t. \tag{49}$$

Defining

$$\mathcal{L}_t \triangleq AK_t - AK_t\bar{H}M_t + GM_t$$

allows (49) to be written as

$$\hat{x}_{t+1|t} = (A - \mathcal{L}_t\bar{C})\hat{x}_{t|t-1} + \mathcal{L}_t\bar{y}_t. \tag{50}$$

Define

$$\begin{aligned}
\tilde{x}_{t+1} &\triangleq x_{t+1} - \hat{x}_{t+1|t}, \\
&= Ax_t + Gd_t + w_t - (A - \mathcal{L}_t\bar{C})\hat{x}_{t|t-1} - \mathcal{L}_t(\bar{C}x_t + \bar{H}d_t + \bar{v}_t), \\
&= (A - \mathcal{L}_t\bar{C})\tilde{x}_t + (G - \mathcal{L}_t\bar{H})d_t + w_t - \mathcal{L}\bar{v}_t.
\end{aligned}$$

Noting that $M_t\bar{H} = I$,

$$G - \mathcal{L}_t\bar{H} = G - AK_t\bar{H} + AK_t\bar{H}M_t\bar{H} - GM_t\bar{H} = 0$$

Hence,

$$\tilde{x}_{t+1} = (A - \mathcal{L}_t\bar{C})\tilde{x}_t + w_t - \mathcal{L}\bar{v}_t$$

Define $P_{t+1} \triangleq P^x_{t+1|t} = \text{cov}(\hat{x}_{t+1|t}) = E[\tilde{x}_{t+1}\tilde{x}^T_{t+1}]$.

$$\begin{aligned} P_{t+1} &= E\left[\tilde{x}_{t+1}\tilde{x}^T_{t+1}\right] \\ &= (A - \mathcal{L}_t\bar{C})P_t(A - \mathcal{L}_t\bar{C})^T + Q + \mathcal{L}_t\bar{R}\mathcal{L}_t^T \end{aligned} \tag{51}$$

The following lemma is the counterpart of Lemma 1 but for the nonzero feedthrough case. Similarly, it will be exploited to prove Theorem 4.

Lemma 2. *For the system (1-47), the following RDE is equivalent to* (51)*.*

$$P_{t+1} = \bar{A}P_t\bar{A} - \bar{A}P_t\bar{C}_2^T(\bar{C}_tP_t\bar{C}_t^T + \bar{R}_2)^{-1}\bar{C}_2P_t\bar{A}^T + \bar{Q}, \tag{52}$$

where

$$\bar{A} = (A - G\bar{H}_1^{-1}\bar{C}_1), \quad \bar{Q} = Q + G\bar{H}_1^{-1}\bar{R}_1\bar{H}_1^{-T}G^T$$

Proof. Split $\tilde{R}_t^{-1}$ of (38) into

$$\tilde{R}_t^{-1} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}. \tag{53}$$

It should be noted that by Assumption 1, $R > 0$ which implies $\tilde{R}_t > 0$. Hence, Y_1 and its schur complement $(Y_3 - Y_2^TY_1^{-1}Y_2)$ are non-singular. Now M_t and K_t of (39) and (41) can be written as

$$
\begin{aligned}
M_t &= \left(\begin{bmatrix} \bar{H}_1^T & 0 \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \begin{bmatrix} \bar{H}_1 \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \bar{H}_1^T & 0 \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}, \\
&= \begin{bmatrix} \bar{H}_1^{-1} & \bar{H}_1^{-1} Y_1^{-1} Y_2 \end{bmatrix}, \\
K_t &= P_t \begin{bmatrix} \bar{C}_1^T & \bar{C}_2^T \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\mathcal{L}_t &= AK_t + AK_t \bar{H} M_t - GM_t \qquad (54) \\
&= AP_t \begin{bmatrix} \bar{C}_1^T & \bar{C}_2^T \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} + AP_t \begin{bmatrix} \bar{C}_1^T & \bar{C}_2^T \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \begin{bmatrix} \bar{H}_1 \\ 0 \end{bmatrix} \begin{bmatrix} \bar{H}_1^{-1} & \bar{H}_1^{-1} Y_1^{-1} Y_2 \end{bmatrix} \\
&\quad - G \begin{bmatrix} \bar{H}_1^{-1} & \bar{H}_1^{-1} Y_1^{-1} Y_2 \end{bmatrix} \\
&= \begin{bmatrix} G\bar{H}_1^{-1} & AP_t \bar{C}_2^T \left(Y_3 - Y_2^T Y_1^{-1} Y_2 \right) + G \bar{H}_1^{-1} Y_1^{-1} Y_2 \end{bmatrix} \qquad (55)
\end{aligned}
$$

From (53),

$$
\tilde{R}_t = (\bar{C} P_t \bar{C}^T + \bar{R}) = \begin{bmatrix} \bar{C}_1 P_t \bar{C}_1^T + \bar{R}_1 & \bar{C}_1 P_t \bar{C}_2^T \\ \bar{C}_2 P_t \bar{C}_1^T & \bar{C}_2 P_t \bar{C}_2^T + \bar{R}_2 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1}.
$$

By the formula of partitioned matrix inversion,

$$
\begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y_1^{-1} + Y_1^{-1} Y_2 (Y_3 - Y_2^T Y_1^{-1} Y_2)^{-1} Y_2^T Y_1^{-1} & -Y_1^{-1} Y_2 (Y_3 - Y_2^T Y_1^{-1} Y_2)^{-1} \\ -(Y_3 - Y_2^T Y_1^{-1} Y_2)^{-1} Y_2^T Y_1^{-1} & (Y_3 - Y_2^T Y_1^{-1} Y_2)^{-1} \end{bmatrix}.
$$

Thus,

$$\bar{Y} \triangleq (Y_3 - Y_2^T Y_1^{-1} Y_2) = (\bar{C}_2 P_t \bar{C}_2^T + \bar{R}_2)^{-1}, \tag{56}$$

$$Y_1^{-1} Y_2 = -\bar{C}_1 P_t \bar{C}_2^T \bar{Y}.$$

Substituting for $\bar{Y}$ and $Y_1^{-1} Y_2$ into (55) yields

$$\mathcal{L}_t = \begin{bmatrix} G\bar{H}_1^{-1} & \bar{A} P_t \bar{C}_2^T \bar{Y} \end{bmatrix}, \tag{57}$$

where $\bar{A} = A - G\bar{H}_1^{-1} C_1$. To conclude, substituting for $\mathcal{L}_t$ from (57) into (51) gives

$$\begin{aligned} P_{t+1} &= (A - \mathcal{L}_t \bar{C}) P_t (A - \mathcal{L}_t \bar{C})^T + Q + \mathcal{L}_t \bar{R} \mathcal{L}_t^T \\ &= (A - G\bar{H}_1^{-1} \bar{C}_1 - \bar{A} P_t \bar{C}_2^T \bar{Y} \bar{C}_2) P_t (A - G\bar{H}_1^{-1} \bar{C}_1 - \bar{A} P_t \bar{C}_2^T \bar{Y} \bar{C}_2)^T \\ &\quad + G\bar{H}_1^{-1} \bar{R}_1 \bar{H}_1^{-T} G^T + \bar{A} P_t \bar{C}_2^T \bar{Y} \bar{R}_2 \bar{Y} \bar{C}_2 P_t \bar{A}^T + Q \\ &= \bar{A} P_t \bar{A}^T - \bar{A} P_t \bar{C}_2^T \bar{Y} \bar{C}_2 P_t \bar{A}^T - \bar{A} P_t \bar{C}_2^T \bar{Y} \bar{C}_2 P_t^T \bar{A}^T + \bar{A} P_t \bar{C}_2^T \bar{Y} \bar{C}_2 P_t \bar{C}_2^T \bar{Y} \bar{C}_2 P_t^T \bar{A}^T \\ &\quad + G\bar{H}_1^{-1} R_1 \bar{H}_1^{-T} G^T + \bar{A} P_t \bar{C}_2^T \bar{Y} \bar{R}_2 \bar{Y} \bar{C}_2 P_t \bar{A}^T + Q \\ &= \bar{A} P_t \bar{A}^T - 2\bar{A} P_t \bar{C}_2^T \bar{Y} \bar{C}_2 P_t^T \bar{A}^T + \bar{A} P_t \bar{C}_2^T \bar{Y} (\bar{C}_2 P_t \bar{C}_2^T + \bar{R}_2) \bar{Y} \bar{C}_2 P_t \bar{A}^T \\ &\quad + G\bar{H}_1^{-1} \bar{R}_1 \bar{H}_1^{-T} G^T + Q. \end{aligned}$$

By (56), $(\bar{C}_2 P_t \bar{C}_2^T + \bar{R}_2) = \bar{Y}^{-1}$. So,

$$\begin{aligned} P_{t+1} &= \bar{A} P_t \bar{A}^T - \bar{A} P_t \bar{C}_2^T \bar{Y} \bar{C}_2 P_t^T \bar{A}^T + G\bar{H}_1^{-1} \bar{R}_1 \bar{H}_1^{-T} G^T + Q \\ &= \bar{A} P_t \bar{A}^T - \bar{A} P_t \bar{C}_2^T (\bar{C}_2 P_t \bar{C}_2^T + \bar{R}_2)^{-1} \bar{C}_2 P_t^T \bar{A}^T + \bar{Q} \end{aligned}$$

This concludes the proof. □

As evident from the discussion presented so far, the analysis of the case where rank $H = m$ follows closely the case where $H = 0$. The reader would have already conjectured that the proof of Theorem 4 parallels that of Theorem 2. Indeed, this is true. However, the proof of Theorem 4 will be fully established below to show how the different transformation simplifies the proof as it may have been already seen that it simplified the proof of Lemma 2. Theorems E.5.1 and 14.7.1 [16], reproduced in the Appendix, will be utilized as done in the proof of Theorem 2.

Proof of Theorem 4

Theorem 4 states that the SISE algorithm as presented in Algorithm 2 of a system (1 - 47) subject to Assumptions 1 and 3 is stable if and only if $\{A - G\bar{H}_1^{-1}\bar{C}_1, \bar{C}_2\}$ is detectable.

Proof. Appealing to Theorem E.5.1 in [16], the DARE

$$P = \bar{A}P\bar{A}^T - \bar{A}P\bar{C}_2^T(\bar{C}_2P\bar{C}_2^T + \bar{R}_2)^{-1}\bar{C}_2P^T\bar{A}^T + \bar{Q} \tag{58}$$

that corresponds to the RDE (52) has a unique positive semi-definite stabilizing solution $\bar{P}$ if and only if

1. $\{\bar{A}, \bar{Q}^{1/2}\}$ is controllable on the unit circle and
2. $\{\bar{A}, \bar{C}_2\}$ is detectable.

Also, according to Theorem 14.7.2 in [16], the matrix P_t of (52) approaches the stabilizing solution $\bar{P}$ of (58) as $t \to \infty$ if Condition 2 above holds and

3. $\{\bar{A}, \bar{Q}^{1/2}\}$ is stabilizable, and

4. $P_0 > -P^a$,

   where P^a is a certain positive definite matrix. (For details, see the Appendix.)

But the conditions 1, 3, and 4 are implied by Assumption 1 as will be shown below. Hence, the SISE algorithm (Algorithm 2) is stable if and only if Condition 2 holds, namely, if and only if $\{\bar{A}, \bar{C}_2\} = \{A - G\bar{H}_1^{-1}\bar{C}_1, \bar{C}_2\}$ is detectable.

By Assumption 1, $P_0 \geq 0$, which implies Condition 4. Also, by Assumption 1, $\{A, Q^{1/2}\}$ is stabilizable. This implies Condition 3 as the following reasoning suggests. If $\{A, Q^{1/2}\}$ is stabilizable then there exists a matrix N_1 such that the eigenvalues of $A + Q^{1/2}N_1$ can all be placed in the unit circle. Note that

$$\bar{Q} = Q + G\bar{H}_1^{-1}R_1\bar{H}_1^{-T}G^T.$$

Take

$$\bar{Q}^{1/2} = \begin{bmatrix} Q^{1/2} & G\bar{H}^{-1}\bar{R}_1^{1/2} \end{bmatrix}.$$

Now denote

$$N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix},$$

with the previous choice of N_1 and the choice of

$$N_2 = \bar{R}_1^{-1/2}\bar{C}_1.$$

This choice of N yields

$$\begin{aligned}\bar{A}+\bar{Q}^{1/2}N &= A-G\bar{H}_1^{-1}\bar{C}_1+\begin{bmatrix}Q^{1/2} & G\bar{H}_1^{-1}\bar{R}_1^{1/2}\end{bmatrix}\begin{bmatrix}N_1\\ \bar{R}_1^{-1/2}\bar{C}_1\end{bmatrix},\\ &= A-G\bar{H}_1^{-1}C_1+Q^{1/2}N_1+G\bar{H}_1^{-1}\bar{R}_1^{1/2}\bar{R}_1^{-1/2}\bar{C}_1,\\ &= A+Q^{1/2}N_1,\end{aligned}$$

whose eigenvalues all lie within the unit circle by the definition of N_1. Hence, $\{\bar{A},\bar{Q}^{1/2}\}$ is stabilizable. Finally, note that Condition 3 implies Condition 1, since all eigenvalues on the unit circle can be placed within the unit circle by the above choice of N. This concludes the proof. □

The relative simplicity of (52) compared to (27) is evident. This is due to the transformation which produces block diagonal measurement noise covariance. This transformation made the derivations of both Lemma 2 and Theorem 4 simpler.

Results in Section 4 are provisionally accepted to be published in Automatica journal in a paper co-authored with Mohammad Ali Abooshahab, Morten Hovd, and Robert Bitmead. The author of this thesis was a primary investigator of those results and is the primary author of those sections as they appear in this thesis.

5 SISE and delay

So far, both formulations of the SISE problem we studied can be summarized in Figure 1. The case where $\ell=1$ is that with $H=0$, while the case where $\ell=0$ is that with rank$H=m$. Recall also that in Section 2, it was stated that the aim of the SISE algorithm is to produce from $\mathbf{Y}^t \triangleq \{y_t, \ldots, y_1, y_0\}$, a recursive filtered state estimate, $\hat{x}_{t|t}$, and filtered and/or smoothed MVU estimates, $\hat{d}_{t|t+1}$ or $\hat{d}_{t|t}$, depending on the properties of the system matrices.

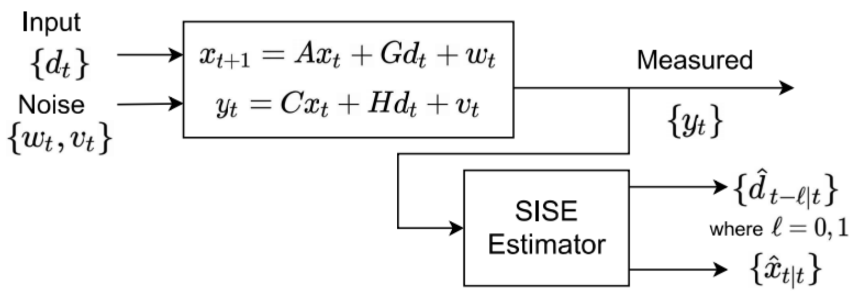

Figure 1: SISE Problem Formulation

Thus, it is now clear that the filtered and smoothed estimates, $\hat{d}_{t|t}$ and $\hat{d}_{t|t+1}$, correspond to the cases of full rank feedthrough matrix and zero feedthrough matrix respectively. What has been done is a derivation and analysis of stability criteria for both cases subject to Assumtions 2 or 3. In this section, a connection between both cases and the meaningfulness of those assumptions will be established through a discussion of the relation of SISE to the delay structure of the system. Denote the d_t-to-y_t transfer function of system (1-2) by $Z(z)$. In terms of system matrices,

$$Z(z) = H + C(zI - A)^{-1}G. \tag{59}$$

Using the Neumann series representation (geometric series generalization), this can also be written as

$$\begin{aligned} Z(z) &= H + z^{-1}C(I - z^{-1}A)^{-1}G \\ &= H + z^{-1}CG + z^{-2}CAG + \cdots + z^{-(n+1)}CA^nG + \ldots . \end{aligned} \tag{60}$$

Thus, using the inverse z-transform, the time response y_t at time t of system (1-2) to any sequence of input signals $\{d_t\}$ can by written as

$$y_t = Hd_t + CGd_{t-1} + CAGd_{t-2} + \cdots + CA^nGd_{t-(n+1)} + \ldots. \tag{61}$$

This shows clearly the relevance of Assumptions 2 and 3. When $H = 0$, the construction of an unbiased estimator of d_t as a function of $\mathbf{Y}^t$ is impossible, since $\mathbf{Y}^t$ is independent of d_t. However, there is hope for constructing an unbiased estimator of d_{t-1} as a function of $\mathbf{Y}^t$, given an unbiased estimate of x_{t-1}. But here also, (61) shows that the problem is indeterminate unless the matrix CG is has full column rank, that is rank$(CG) = m$, which is exactly what Assumption 2 ensures. On the other hand, when $H \neq 0$, $\mathbf{Y}^t$ becomes dependant on d_t. But the construction of an unbiased estimator of d_t as a function of $\mathbf{Y}^t$, given an unbiased estimate of x_t, becomes an indeterminate problem unless rank $H = m$, which is exactly what Assumption 3 ensures. This explanation shows that some of the different formulations of SISE present in the literature are due primarily to differences in the system's delay structure which is fully captured by properties of the unit impulse response matrices $H, CG, CAG, CA^2G, \ldots$, also known as the Markov parameters of the system (1-2).

Now consider the following question. What if $H \neq 0$ but rank$H < m$. The singular value decomposition of H_t thus becomes

$$H_t = \begin{bmatrix} U_{1,t} & U_{2,t} \end{bmatrix} \begin{bmatrix} \Sigma_t & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1,t}^T \\ V_{2,t}^T \end{bmatrix} \tag{62}$$

This does not fit with the assumptions of any of the two limiting cases studied above. From (61), it is clear that seeking an unbiased estimator of d_t that is dependant on $\mathbf{Y}^t$ only poses an indeterminate problem. However, as (61) also shows if shifted one step forward, y_{t+1} is also dependant on d_t. This situation raises the question about what conditions should hold for H and CG such that an unbiased estimator of d_t can be constructed as a function of $\mathbf{Y}^{t+1}$. This problem is well studied by Yong, Zhu, and Frazzoli in [17], where the two techniques of filtering and smoothing described above are carefully unified to obtain estimates of d_t using both y_t and y_{t+1}. The algorithm resulting from this unifying approach is thus called *Unified Linear Input & State Estimator* (ULISE). The method used in ULISE depends on splitting the unknown input d_t into two additive components $d_{1,t}$ and $d_{2,t}$ such that $d_t = d_{1,t} + d_{2,t}$. This decomposition is made cleverly in a matter that allows $d_{1,t}$ to be unbiasedly estimated from y_t and $d_{2,t}$ to be unbiasedly estimated from y_{t+1}. Thus, an estimate $\hat{d}_{t|t+1}$ is obtained allowing a prior estimate of x_{t+1}, and thus goes the filter. It turns out that the condition that guarantees the possibility of constructing the above described estimator of d_t is that

$$\text{rank}(C_{2,t}G_{2,t-1}) + \text{rank}(H_t) = m, \tag{63}$$

where $C_{2,t}$ is time varying equivalent of that given in transformation (47), and $G_{2,t-1} = G_{t-1}V_{2,t-1}$ with $V_{2,t-1}$ from (62). All this leads naturally to the question, what if (63) is not fulfilled? Is there a formulation of the SISE problem with a general delay structure. Figure 2 shows the formulation of this problem graphically. Note, however, that when $\ell > 1$, the state estimator can only be a smoother. That is, if an unbiased estimate of d_t can only be obtained after the application of d_t to the system by $\ell > 1$ time steps, then an unbiased estimate of the state x_{t+1} is only possible at time $t+\ell$ since any unbiased estimator of x_{t+1} should be a function of the estimate of d_t. A thorough discussion of this general delay problem is presented by the authors of [17] in another paper [18], where d_t is split into as much components as necessary for obtaining unbiased estimates of each component from consecutive measurements. There method, however, is complex and the complexity increases as the delay increases. Thus, I propose a simpler method here that is inspired by fixed-lag smoothing. The simplicity of the method that will be proposed here springs from the fact that it reduces the general delay case to the case with full rank feedthrough matrix. This simplicity, though, is not without its expenses that will be discussed after the method is explained.

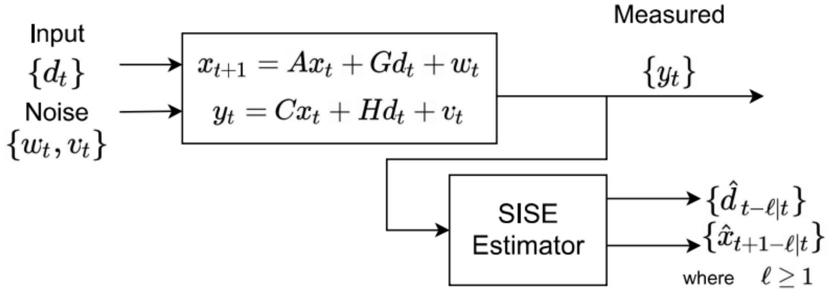

Figure 2: SISE Problem Formulation with General Delay

5.1 Reduction of systems with a general delay to a system with no delay

The consecutive output signals of (1-2) can be written as

$$
\begin{aligned}
y_t &= Cx_t + Hd_t + v_t \\
y_{t+1} &= CAx_t + Hd_{t+1} + CGd_t + Cw_t + v_{t+1} \\
y_{t+2} &= CA^2x_t + Hd_{t+2} + CGd_{t+1} + CAGd_t + Cw_{t+1} + CAw_t + v_{t+2} \\
&\vdots \\
y_{t+k} &= CA^kx_t + CA^{k-1}Gd_t + CA^{k-2}Gd_{t+1} + \cdots + Hd_{t+k} \\
&\quad CA^{k-1}w_t + CA^{k-2}w_{t+1} + \cdots + Cw_{t+k-1} + v_{t+k}.
\end{aligned}
$$

In matrix form,

$$\begin{bmatrix} y_t \\ y_{t+1} \\ y_{t+2} \\ \vdots \\ y_{t+k} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^k \end{bmatrix} x_t + \begin{bmatrix} H & 0 & 0 & 0 & \dots & 0 \\ CG & H & 0 & 0 & \dots & 0 \\ CAG & CG & H & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ CA^{k-1}G & CA^{k-2}G & & \dots & \dots & H \end{bmatrix} \begin{bmatrix} d_t \\ d_{t+1} \\ d_{t+2} \\ \vdots \\ d_{t+k} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ C & 0 & 0 & 0 & \dots & 0 \\ CA & C & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ CA^{k-1} & CA^{k-2} & & \dots & \dots & C \end{bmatrix} \begin{bmatrix} w_t \\ w_{t+1} \\ \vdots \\ w_{t+k-1} \end{bmatrix} + \begin{bmatrix} v_t \\ v_{t+1} \\ v_{t+2} \\ \vdots \\ v_{t+k} \end{bmatrix}. \tag{64}$$

The number of block rows is $k+1$ which is not yet determined. Part of the problem is to find what is the minimum sufficient k such that d_t can be unbiasedly estimated. Next we define a transformation to be applied on (64) that enables the estimation of d_t given $\mathbf{Y}^{t+k}$. To do this we first define the matrices $T_1, \dots, T_{k+1}$ according to Algorithm 3. Note, however, that to define those matrices the system (1-2) should comply with Assumption 4.

Assumption 4. *$p \geq m$, that is the number of outputs of (1-2) is greater than or equal to the number of inputs.*

Algorithm 3: Define $T_1, T_2, \dots, T_{k+1}$

Initialization:

$\text{svd}(H) = \begin{bmatrix} U_{1,1} & U_{1,2} \end{bmatrix} \begin{bmatrix} \Sigma_{1,r_1 \times r_1} & 0 \\ 0_{(p-r_1)\times r_1} & 0 \end{bmatrix} V_1, \quad \bar{R}_1 = R,$

$\bar{U}_{1,1} = U_{1,1}^T - U_{1,1}^T \bar{R}_1 U_{1,2} (U_{1,2}^T \bar{R}_1 U_{1,2})^{-1} U_{1,2}^T$

$T_1 = \bar{U}_{1,1} \in \mathbb{R}^{r_1 \times p},$

for $i = 2$ **to** k **do:**

$$\text{svd}(U_{i-1,2}^T U_{i-2,2}^T \dots U_{1,2}^T C A^{i-2} G) = \begin{bmatrix} U_{i,1} & U_{i,2} \end{bmatrix} \begin{bmatrix} \Sigma_{i,r_i \times r_i} & 0 \\ 0 & 0 \end{bmatrix} V_i, \tag{65}$$

where $U_{i,1} \in \mathbb{R}^{(p-r_1-\dots-r_{i-1})\times r_i}, \quad U_{i,2} \in \mathbb{R}^{(p-r_1-\dots-r_{i-1})\times(p-r_1-\dots-r_i)}$

$$\bar{R}_i = U_{i-1,2}^T \bar{R}_{i-1} U_{i-1,2} \tag{66}$$

$$\bar{U}_{i,1} = U_{i,1}^T - U_{i,1}^T \bar{R}_i U_{i,2} (U_{i,2}^T \bar{R}_i U_{i,2})^{-1} U_{i,2}^T \in \mathbb{R}^{r_i \times p} \tag{67}$$

$$T_i = \bar{U}_{i,1} U_{i-1,2}^T U_{i-2,2}^T \dots U_{1,2}^T \in \mathbb{R}^{r_i \times p} \tag{68}$$

Finalization:

$T_{k+1} = U_{k,2}^T U_{k-1,2}^T \dots U_{1,2}^T \in \mathbb{R}^{(p-r_1-\dots-r_k)\times p}$

Define the matrix

$$T \triangleq \begin{bmatrix} T_1^T & T_2^T & \dots & T_{k+1}^T \end{bmatrix}^T.$$

Note that the number of rows of T is $r_1 + r_2 + \dots + r_k + (p - r_1 - \dots - r_k) = p$. Hence $T \in \mathbb{R}^{p\times p}$. This seemingly complicated definition of T gives the elegant properties presented in the following lemma.

Lemma 3. *The following propositions hold for the matrix T.*

1. *T is non-singular,*
2.
   - $T_2 H = T_3 H = \dots = T_{k+1} H = 0,$
   - $T_i C A^j G = 0,\ \forall\, j \in \{0, 1, \dots, k-2\}$ *and* $i \in \{j+3, j+4, \dots, k+1\}$
3. *TRT^T is a block diagonal matix with blocks $(T_1 R T_1^T, T_2 R T_2^T, \dots, T_{k+1} R T_{k+1}^T)$ in this order.*

Proof. Each proposition is proved separately,

1. For any $0 \leq i \leq k$,

$$\begin{bmatrix} \bar{U}_{i,1}^T \\ U_{i,2}^T \end{bmatrix} = \begin{bmatrix} I_{r_i} & -U_{i,1}^T \bar{R}_i U_{i,2} (U_{i,2}^T \bar{R}_i U_{i,2})^{-1} U_{i,2}^T \\ 0 & I_{p-r_1-\cdots-r_i} \end{bmatrix} \begin{bmatrix} U_{i,1}^T \\ U_{i,2}^T \end{bmatrix}.$$

Thus, $\begin{bmatrix} \bar{U}_{i,1} & U_{i,2} \end{bmatrix}^T$ is non-singular. The matrix T can be written in the following form

$$T = \begin{bmatrix} I & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \bar{U}_{k,1} \\ 0 & 0 & \dots & U_{k,2} \end{bmatrix} \dots \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \bar{U}_{3,1} \\ 0 & 0 & U_{3,2} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \bar{U}_{2,1} \\ 0 & U_{2,2} \end{bmatrix} \begin{bmatrix} \bar{U}_{1,1} \\ U_{1,2} \end{bmatrix}.$$

Thus, T is non-singular too.

2. • From the initialization of Algorithm 3, $U_{1,2}^T H = 0$. Thus, $T_2 H = T_3 H = \cdots = T_{k+1} H = 0$.

   • For any j and i, if $0 \leq j \leq k-2$ and $j+3 \leq i \leq k$ then $T_i = \bar{T}_i U_{j+2,2}^T U_{j+1,2}^T \dots U_{1,2}^T$, with a suitable $\bar{T}_i$ obtained easily from (68). But by the definition of $U_{j+2,2}$ in (65),

$$T_i = \bar{T}_i U_{j+2,2}^T (U_{j+1,2}^T \dots U_{1,2}^T C A^j G) = 0.$$

   A similar argument can be cast for the proposition that $T_{k+1} C A^{k-2} G = 0$.

3. Note that

$$TRT^T = \begin{bmatrix} T_1RT_1^T & T_1RT_2^T & \dots & T_1RT_{k+1}^T \\ T_2RT_1^T & T_2RT_2^T & \dots & T_2RT_{k+1}^T \\ \vdots & & \ddots & \vdots \\ T_{k+1}RT_1^T & & \dots & T_{k+1}RT_{k+1}^T \end{bmatrix}$$

But when $i < j$,

$$T_iRT_j^T = \bar{U}_{i,1}U_{i-1,2}^TU_{i-2,2}^T \dots U_{1,2}^TR\left(\bar{T}_jU_{i,2}^TU_{i-1,2}^T \dots U_{1,2}^T\right)^T$$

with a suitable $\bar{T}_j$ obtained from (68). Thus, using (66),

$$\begin{aligned} T_iRT_j^T &= \bar{U}_{i,1}U_{i-1,2}^TU_{i-2,2}^T \dots U_{1,2}^TRU_{1,2} \dots U_{i-1,2}U_{i,2}\bar{T}_j^T \\ &= \bar{U}_{i,1}\bar{R}_iU_{i,2}\bar{T}_j^T \end{aligned}$$

By (67),

$$\begin{aligned} T_iRT_j^T &= \bar{U}_{i,1}\bar{R}_iU_{i,2}\bar{T}_j^T \\ &= \left(U_{i,1}^T - U_{i,1}^T\bar{R}_iU_{i,2}\left(U_{i,2}^T\bar{R}_iU_{i,2}\right)^{-1}U_{i,2}^T\right)\bar{R}_iU_{i,2}\bar{T}_j^T \\ &= 0 \end{aligned}$$

Thus, the upper triangular blocks part of TRT^T are all zeros. And since TRT^T is symmetric, the lower triangular blocks are also zeros. This proves Proposition 3.

□

We now transform (64) by the matrix $\mathcal{T} \triangleq \text{diag}\{T, T, T \dots, T\} \in \mathbb{R}^{(k+1)p\times(k+1)p}$.

Proposition 1 in Lemma 3 ensures that $\mathcal{T}$ is a valid transformation. This transformation yields

$$\begin{bmatrix} Ty_t \\ Ty_{t+1} \\ Ty_{t+2} \\ \vdots \\ Ty_{t+k} \end{bmatrix} = \begin{bmatrix} TC \\ TCA \\ TCA^2 \\ \vdots \\ TCA^k \end{bmatrix} x_t + \begin{bmatrix} TH & 0 & 0 & 0 & \dots & 0 \\ TCG & TH & 0 & 0 & \dots & 0 \\ TCAG & TCG & TH & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ TCA^{k-1}G & TCA^{k-2}G & & \dots & \dots & TH \end{bmatrix} \begin{bmatrix} d_t \\ d_{t+1} \\ d_{t+2} \\ \vdots \\ d_{t+k} \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ TC & 0 & 0 & 0 & \dots & 0 \\ TCA & TC & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ TCA^{k-1} & TCA^{k-2} & & \dots & \dots & TC \end{bmatrix} \begin{bmatrix} w_t \\ w_{t+1} \\ \vdots \\ w_{t+k-1} \end{bmatrix} + \begin{bmatrix} Tv_t \\ Tv_{t+1} \\ Tv_{t+2} \\ \vdots \\ Tv_{t+k} \end{bmatrix}. \tag{69}$$

Exploiting Proposition 2 in Lemma 3,

$$TH = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_k \\ T_{k+1} \end{bmatrix} H = \begin{bmatrix} T_1H \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad TCG = \begin{bmatrix} T_1CG \\ T_2CG \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, TCA^{k-2}G = \begin{bmatrix} T_1CA^{k-2}G \\ \vdots \\ \vdots \\ T_kCA^{k-2}G \\ 0 \end{bmatrix}.$$

Thus, if from (69) we extract the $k+1$ row blocks that depend only on d_t and have zero blocks multiplied by $d_{t+1}, d_{t+2}, \ldots$ etc., we get

$$\begin{bmatrix} T_1 y_t \\ T_2 y_{t+1} \\ T_3 y_{t+2} \\ \vdots \\ T_{k+1} y_{t+k} \end{bmatrix} = \begin{bmatrix} T_1 C \\ T_2 CA \\ T_3 CA^2 \\ \vdots \\ T_{k+1} CA^k \end{bmatrix} x_t + \begin{bmatrix} T_1 H \\ T_2 CG \\ T_3 CAG \\ \vdots \\ T_{k+1} CA^{k-1} G \end{bmatrix} d_t + \mathcal{V}_t, \tag{70}$$

where

$$\mathcal{V}_t = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ T_2 C & 0 & 0 & \ldots & 0 \\ T_3 CA & T_3 C & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ T_{k+1} CA^{k-1} & T_{k+1} CA^{k-2} & \ldots & \ldots & T_{k+1} C \end{bmatrix} \begin{bmatrix} w_t \\ w_{t+1} \\ \vdots \\ w_{t+k-1} \end{bmatrix} + \begin{bmatrix} T_1 v_t \\ T_2 v_{t+1} \\ T_3 v_{t+2} \\ \vdots \\ T_{k+1} v_{t+k} \end{bmatrix}. \tag{71}$$

It is thus possible to define a new system

$$x_{t+1} = Ax_t + Gd_t + w_t \tag{72}$$

$$\mathcal{Y}_t = \mathcal{C} x_t + \mathcal{H} d_t + \mathcal{V}_t, \tag{73}$$

where $\mathcal{C}$ and $\mathcal{H}$ are the matrix coefficients of x_t and d_t in (70), respectively. System (72-73) has the same form as system (1-2) with the exception that the noise signal $\mathcal{V}_t$ is colored and correlated with the process noise signals $w_t, w_{t+1}, \ldots, w_{t+k-1}$. As known from Kalman filtering theory, these difficulties can be overcome by finding a an equivalent system with white and uncorrelated noise signals (Chapter 5.5 of Anderson and Moore [15]). It should be noted that Proposition 3 in Lemma 3 ensures that the part of $\mathcal{V}_t$ that depends on $v_t, v_{t+1}, \ldots, v_{t+k}$

is white. That is because

$$E\left(\begin{bmatrix} T_1 v_t \\ T_2 v_{t+1} \\ \vdots \\ T_{k+1} v_{t+k} \end{bmatrix} \begin{bmatrix} T_1 v_{t+1} \\ T_2 v_{t+2} \\ \vdots \\ T_{k+1} v_{t+k+1} \end{bmatrix}^T\right) = 0.$$

This simplifies the problem of $\mathcal{V}_t$ being colored. The transformation of system (72-73) into an equivalent system with white and uncorrelated noise will not be done for the general case in this thesis. Assuming this is done, SISE of Algorithm 2 can be applied directly on (72-73) to estimate d_t and x_{t+1} given $y_t, y_{t+1}, \ldots, y_{t+k}$ if rank $\mathcal{H} = m$. Thus the number of consecutive measurements needed for unbiasedly estimating d_t is the number, $k+1$, that lets rank $\mathcal{H} = m$. But, one should not try adding row blocks to $\mathcal{H}$ until eternity, since there is an upper bound for rank $\mathcal{H}$ that can be a priori calculated. This is formulated in the following lemma.

Lemma 4. *rank* $\mathcal{H} \leq$ *rank* $\begin{bmatrix} H^T & (CG)^T & (CAG)^T & \ldots & (CA^{n-1}G)^T \end{bmatrix}$*, where n is the dimension of the state x_t.*

Proof.

$$\text{rank}\mathcal{H} = \text{rank}\begin{bmatrix} T_1 H \\ T_2 CG \\ \vdots \\ T_{k+1} CA^{k-1} G \end{bmatrix},$$

$$= \text{rank}\left(\begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_{k+1} \end{bmatrix}\begin{bmatrix} H \\ CG \\ \vdots \\ CA^{k-1} G \end{bmatrix}\right),$$

$$\leq \min\left(\text{rank}\begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_{k+1} \end{bmatrix}, \text{rank}\begin{bmatrix} H \\ CG \\ \vdots \\ CA^{k-1} G \end{bmatrix}\right).$$

Keeping in mind Assumption 4 and Proposition 1 of Lemma 3,

$$\text{rank}\begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_{k+1} \end{bmatrix} = \left(\sum_{i=1}^{k+1} \text{rank} T_i,\right) = p \geq m.$$

Thus,

$$\operatorname{rank}\mathcal{H} \leq \min\left(p, \operatorname{rank}\begin{bmatrix} H \\ CG \\ \vdots \\ CA^{k-1}G \end{bmatrix}_{(k+1)p\times m}\right)$$
$$= \operatorname{rank}\begin{bmatrix} H \\ CG \\ \vdots \\ CA^{k-1}G \end{bmatrix}_{(k+1)p\times m}.$$

For $k \leq n$, the lemma clearly holds. But for $k > n$, Cayley-Hamilton Theroem comes to our rescue allowing us to prove that

$$\operatorname{rank}\begin{bmatrix} H^T & (CG)^T & \dots & (CA^{k-1}G)^T \end{bmatrix}^T = \operatorname{rank}\begin{bmatrix} H^T & (CG)^T & \dots & (CA^{n-1}G)^T \end{bmatrix}^T$$

however large is k. This proves Lemma 4. $\square$

Lemma 3 says that if there can not be found a $k \leq n$ for which $\operatorname{rank}\mathcal{H} = m$, then such a k does not exist. This provides an effective necessary and sufficient condition for any Discrete LTI system with a finite number of states. Note also that there is no need to re-derive stability conditions for the general delay case since it is reduced to the maximum rank feedthrough case to which stability conditions are found in Section 4.

As can be seen so far, an explicit solution for SISE with no conditions on the feedthrough matrix is found by reducing it to the case with full rank feedthrough matrix. The only steps that need to be taken are running Algorithm 3 until $\operatorname{rank}\mathcal{H} = m$, reformulating (72-73) to an equivalent system with white uncorrelated noise, and applying Algorithm 2 to get the desired estimates. As mentioned earlier, the general delay SISE problem was carefully solved by Yong et al. [18] but in another complicated way that depends on decomposing the d_t into k components each of which is estimated by an output signal at consecutive times. Aside from the problem of noise correlation and color, the method presented here is simpler. In fact, they do not explicitly formulate an algorithm for the general delay case but show how to recursively do so and give an example for systems with delay latency $\ell = 2$. However, there method is superior in giving estimates of components of d_t as soon as they are available. In contrast, the method presented here gives the estimate of the whole signal d_t in one late shot. In terms of the estimation of the state, x_{t+1}, both algorithms give a smoothed value $\hat{x}_{t+1|t+k}$ after the whole estimate of d_t is available.

6 Extension of stability conditions to time-varying systems

Developments so far have been limited to the time-invariant case and have availed themselves of concepts of transmission zeros and stable invertibility, each of which is problematic to extend to time-varying systems. However, since alternative results have been phrased for the time-varying case, we consider this extension now, relying on examination of SISE recursions via Riccati difference equations in the proofs of Lemmas 1 and 2. Note, however, that the transformation of Section 3.4, where the measurement noise covariance becomes block diagonal, is used here for the case of $H = 0$.

Appealing to [14, 8] for the time-varying SISE algorithms in the case zero direct feedthrough and with application of the transformation (36) rather than (21), Riccati equation

(27) becomes

$$X_{t+1} = \bar{A}_t X_t \bar{A}_t^T - \bar{A}_t X_t \bar{C}_{2,t}^T (\bar{C}_{2,t} X_t \bar{C}_{2,t}^T + \bar{R}_{2,t})^{-1} \times (\bar{A}_t X_t C_{2,t}^T)^T + \bar{Q}_t,$$

where,

$$\begin{aligned}
\bar{A}_t &= A_t(I - G_{t-1}(\bar{C}_{1,t} G_{t-1})^{-1} \bar{C}_{1,t}), \\
\bar{Q}_t &= A_t G_{t-1}(\bar{C}_{1,t} G_{t-1})^{-1} \bar{R}_{1,t} (G_{t-1}(\bar{C}_{1,t} G_{t-1})^{-1})^T A_t^T \\
&\quad + Q_t,
\end{aligned}$$

and, in the case of full-rank feedthrough, (52) becomes

$$P_{t+1} = \hat{A}_t P_t \hat{A}_t^T - (\hat{A}_t P_t \bar{C}_{2,t}^T)(\bar{C}_{2,t} P_t \bar{C}_{2,t} + \bar{R}_{2,t})^{-1} \times (\hat{A}_t P_t \bar{C}_{2,t}^T)^T + \hat{Q}_t,$$

where,

$$\hat{A}_t = A_t - G_t \bar{H}_{1,t}^{-1} \bar{C}_{1,t}, \quad \hat{Q}_t = Q_t + G_t \bar{H}_{1,t}^{-1} \bar{R}_{1,t} \bar{H}_{1,t}^{-T} G_t^T.$$

with now time-varying quantities $\{A_t, G_t, \ldots, \}$. We may appeal to standard sufficient results, e.g. Theorem 5.3 in [19], on the exponential stability of the Kalman filter subject to uniform reachability and detectability. Subject to the uniform satisfaction of time-varying equivalents of Assumptions 1, 2 and/or 3 as appropriate, this extends our stability conditions to the uniformly time-varying case. However, these stability conditions are now sufficient only.

Results in Section 6 are provisionally accepted to be published in Automatica journal in a paper co-authored with Mohammad Ali Abooshahab, Morten Hovd, and Robert Bitmead. The author of this thesis was a primary investigator of those results and is the primary author of those sections as they appear in this thesis.

7 A note on the meaningfulness of unbiasedness in the SISE context

In their paper *A Kalman-filtering derivation of simultaneous input and state estimation* [11], Bitmead and co-authors criticize the attempt to estimate the unknown input signal without attributing any model to it. They say,

> "*It is usually attributed to John von Neumann or to Stanislaw Ulam that the study of non-equilibrium thermodynamics in Physics is akin to the study of non-elephants in Zoology. By the same token, the study of model-free estimation is an unhelpful even meaningless description in this domain.*"

Unsatisfied by the model-free description of the SISE problem, they re-derived the SISE algorithm of [14] as a Kalman filter by explicitly modelling the unknown input sequence $\{d_t\}$ as a white noise with a covariance approaching infinity. They hold that, unless the signal d_t is given a statistical model, the property of *unbiasedness* of the estimator is not used in its probabilistically standard sense. In this thesis, I used the term *unbiased* to describe the SISE estimator without adopting the views of Bitmead et. al. And thus, in what follows, I will defend the traditional account of SISE by appealing to the standard statistical concepts of point estimators and point estimates.

Definitions of a point estimators, point estimates, and bias

In the following definitions, I follow the classic text *Introduction to Mathematical Statistics* of Hogg and Craig [20]. Let a random variable Y have a probability density function (pdf) that is of known functional form that depends upon an unknown parameter θ. Thus, to each value of θ there corresponds a pdf of Y that we denote by $f_Y(y;\theta)$. It should be noted here that θ is considered only an unknown **constant** parameter not a random variable. Now, let $Y_1, Y_2, \ldots, Y_n$ denote random samples of the random variable Y. A point estimator $\hat{\Theta}$ of the parameter θ is defined as a function $\hat{\Theta} = g(Y_1, Y_2, \ldots, Y_n)$. The aim in the construction of the function $g(Y_1, Y_2, \ldots, Y_n)$ is that when the observed experimental values $y_1, y_2, \ldots, y_n$ of the random samples $Y_1, Y_2, \ldots, Y_n$ are substituted in g, it yields a "good" estimate of θ. We denote the estimate by $\hat{\theta} = g(y_1, y_2, \ldots, y_n)$. Note that the estimator $\hat{\Theta}$ is a random variable since it is a function of the random samples $Y_1, Y_2, \ldots$ while the estimate $\hat{\theta}$ is only a numerical value like θ. One of the quality measures of estimators that give a sense to the word "good" as used above is unbiasedness. According to this formulation, the bias of a point estimator is defined as

$$\text{bais} \triangleq E[\hat{\Theta}] - \theta.$$

Thus, an unbiased point estimator $\hat{\Theta}$ is an estimator with bias $= 0$.

Input estimation as unbiased point estimation

Consider the system (1-2)

$$x_{t+1} = Ax_t + Gd_t + w_t$$
$$y_t = Cx_t + Hd_t + v_t.$$

with w_t and v_t being white, uncorrelated, and zero-mean noises. We consider the sequence $\{d_t \in \mathbb{R}^m\}$ a sequence of **determinate** but unknown parameters. The state at time t, x_t, is a random variable since it is a function of the random variable x_0 and the sequence of random variables $\{w_t\}$. Thus the measurement at time t, y_t, is a random variable too, parametrized by the **constant** d_t. Assume that a function $\hat{X}_t(Y_t)$ is given as an unbiased point estimator of $E[x_t]$ such that $E[\hat{X}_t(Y_t)] = E[x_t]$. Define a function $\hat{D}_t(Y_t)$ as

$$\begin{aligned} \hat{D}_t(Y_t) &\triangleq M_t(y_t - C\hat{X}_t(Y_t)) \\ &= M_t(Cx_t + Hd_t + v_t - C\hat{X}_t(Y_t)), \end{aligned} \tag{74}$$

Where M_t is an arbitrary matrix of suitable dimensions. Evidently, $\hat{D}_t(Y_t)$ is a random variable with an expected value of

$$E[\hat{D}_t(Y_t)] = M_t(CE[x_t] + HE[d_t] + E[v_t] - CE[\hat{X}_t(Y_t)]).$$

But since d_t is a constant and $E[\hat{X}_t(Y_t)] = E[x_t]$ by definition, we get

$$E[\hat{D}_t(Y_t)] = M_t H d_t.$$

Thus, $\hat{D}_t(Y_t)$ is an **unbiased point estimator** of the **parameter** d_t if and only if $M_t H = I$ which exactly agrees with SISE formulation of [8] that was reproduced as Algorithm 2.

Moreover, if $M_t H = I$, a function $\hat{X}_{t+1}(Y_{t+1})$ can be defined as

$$\hat{X}_{t+1}(Y_{t+1}) = A\hat{X}_t(Y_t) + G\hat{D}_t(Y_t),$$

which has an expected value of

$$\begin{aligned} E[\hat{X}_{t+1}(Y_{t+1})] &= AE[\hat{X}_t(Y_t)] + GE[\hat{D}_t(Y_t)] \\ &= AE[x_t] + Gd_t \\ &= E[x_{t+1}]. \end{aligned}$$

Thus, $\hat{X}_{t+1}(Y_{t+1})$ is an unbiased estimator of $E[x_{t+1}]$. This allows us to define $D_{t+1}(Y_{t+1})$ as an unbiased estimator of d_{t+1} similar to what was done for d_t but with the condition $M_{t+1}H = I$. This can go on indefinitely. Thus, a strictly statistical ground has been established to confirm the meaningfulness of the claim that SISE is an unbiased estimator of the signal d_t without committing to any statistical model for it. SISE does exactly that and adds to it a modification of the estimators $\hat{X}_t(Y_t)$ and $\hat{D}_t(Y_t)$ to minimize the variance. The same reasoning applied above can be easily extended to the case of zero feedthrough.

8 Appendix

Theorems E.5.1 and 14.7.2 from [16] are alluded to in the proofs of Theorems 2 and 4. Thus, Theorems E.5.1 and 14.7.2 from [16] will be reproduced here for completeness.

Theorem (E.5.1 from [16])**.** *Consider the discrete-time algebraic Riccati equation (DARE)*

$$P = FPF^* + GQG^* - (FPH^* + GS)(\mathcal{R} + HPH^*)^{-1}(FPH^* + GS)^*. \tag{75}$$

Then the following two statements are equivalent.

(i) $\{F, H\}$ *is detectable and* $\{F^s, G\mathcal{Q}^{s/2}\}$ *is controllable on the unit circle, where* $F^s \triangleq F - GS\mathcal{R}^{-1}H$ *and* $Q^s \triangleq \mathcal{Q} - S\mathcal{R}^{-1}S^*$.

(ii) The DARE has a stabilizing P, i.e., one for which the matrix $F - K_pH$ *is stable, where* $K_p = (FPH^* + GS)(\mathcal{R} + HPH^*)^{-1}$.

Moreover, any such stabilizing solution is unique and positive semi-definite.

Theorem (14.7.2 from [16])**.** *Consider the Riccati recursion*

$$P_{i+1} = FP_iF^* + G\mathcal{Q}G^* - (FP_iH^* + GS)(\mathcal{R} + HP_iH^*)^{-1}(FP_iH^* + GS)^*,$$

where $\{F, H\}$ *is detectable and* $\{F^s, GQ^{s/2}\}$ *is stabilizable (*F^s *and* Q^s *are defined as in Theorem E.5.1 above). Suppose, moreover, that the initial condition* P_0 *is a Hermitian matrix such that*

$$I + (P^a)^{*/2}P_0(P^a)^{1/2} > 0,$$

where $P^a = (P^a)^{1/2}(P^a)^{*/2}$ *is a certain positive definite matrix (Check [16] for details). Then* P_i *converges to the unique stabilizing solution,* P*, of the DARE corresponding to the above RDE.*

References

[1] J.D. Glover. The linear estimation of completely unknown signals. *IEEE Transactions on Automatic Control*, AC-14(6):766–769, 1969.

[2] P.K. Kitanidis. Unbiased minimum-variance linear state estimation. *Automatica*, 23(6):775–778, 1987.

[3] Alan S. Willsky and Harold L. Jones. A generalized likelihood ratio approach to the detection and estimation of jumps in linear systems. *IEEE Transactions on Automatic Control*, 1976.

[4] P. Sanyal and C.N. Shen. Bayes' decision rule for rapid detection and adaptive estimation scheme with space applications. *IEEE Transactions on Automatic Control*, AC-19(3):228–231, 1974.

[5] Morten Hovd Mohammad Ali Abooshahab and Robert R. Bitmead. Monitoring disturbances and states in partially known power systems.

[6] C-S. Hsieh. Robust two-stage Kalman filters for systems with unknown inputs. *IEEE Transactions on Automatic Control*, 45(12):2374–2378, December 2000.

[7] M. Zasadzinski M. Darouach and M. Boutayeb. Extension of minimum variance estimation for systems with unknown inputs. *Automatica*, 2003.

[8] S. Gillijns and B. De Moor. Unbiased minimum-variance input and state estimation for linear discrete-time sysems with direct feedthrough. *Automatica*, 43:934–937, 2007.

[9] Sze Zhong Yong, Minghui Zhu, and Emilio Frazzoli. A unified filter for simultaneous input and state estiamtin of linear discrete-time stochastic systems. *Automatica*, 63:321–329, 2016.

[10] Baibing Li. State estimation with partially observed inputs: A unified kalman filtering approach. *Automatica*, 2013.

[11] R.R. Bitmead, M. Hovd, and M.A. Abooshahab. A Kalman-filtering derivation of simultaneous input and state estimation. *Automatica*, 108:108478, 2019.

[12] Huazhen Fang, Raymond A. de Callafon, and Jorge Cortés. Simultaneous input and state estimation for nonlinear systems with applications to flow field estimation. *Automatica*, 49(9):2805–2812, 2013.

[13] M. Darouach and M. Zasdzinski. Unbiased minimum variance estimation for systems with unknown exogenous inputs. *Automatica*, 1997.

[14] S. Gillijns and B. De Moor. Unbiased minimum-variance input and state estimation for linear discrete-time sysems. *Automatica*, 43:111–116, 2007.

[15] Brian DO Anderson and John B Moore. *Optimal filtering*. Courier Corporation, 2012.

[16] A. Sayed T. Kailath and B. Hassibi. *Linear Estimation*. Prentice-hall, Upper Saddle River, NJ, USA, 2000.

[17] Sze Zheng Yong, Minghui Zhu, and Emilio Frazzoli. A unified filter for simultaneous input and state estimation of linear discrete-time stochastic systems. *Automatica*, 63:321–329, 2016.

[18] S. Z. Yong, M. Zhu, and E. Frazzoli. Simultaneous input and state estimation with a delay. In *2015 54th IEEE Conference on Decision and Control (CDC)*, pages 468–475, 2015.

[19] B.D.O. Anderson and J.B. Moore. Detectability and stabilizability of time-varying discrete-time linear systems. *SIAM J. Control Optimization*, 19(1):20–32, 1981.

[20] R.V. Hogg and A.T. Craig. *Introduction to Mathematical Statistics*. Macmillan Publishing, 1978.