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UNIVERSITY OF CALIFORNIA, SAN DIEGO

**Extensions of the Reciprocity Method in Consecutive Pattern Avoidance
in Permutations**

A Dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Quang Tran Bach

Committee in charge:

Professor Jeffrey Remmel, Chair
Professor Ronald Graham
Professor Ramamohan Paturi
Professor Brendon Rhoades
Professor Jacques Verstraete

2017

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The Dissertation of Quang Tran Bach is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2017

DEDICATION

To my little brother, my trusty friend and loyal companion.

EPIGRAPH

*Once you eliminate the impossible, whatever remains, no matter how improbable,
must be the truth.*

—Sir Arthur Conan Doyle, *Sherlock Holmes*

TABLE OF CONTENTS

	Signature Page	iii
	Dedication	iv
	Epigraph	v
	Table of Contents	vi
	List of Figures	viii
	List of Tables	x
	Acknowledgements	xi
	Vita	xv
	Abstract of the Dissertation	xvi
Chapter 1	Introduction	1
	1.1 A quick history of permutation patterns	1
	1.2 Symmetric functions and brick tabloids	7
	1.3 The main goal of this thesis	16
Chapter 2	The reciprocity method	22
	2.1 Pattern matching in the cycle structure of permutations	23
	2.2 The reciprocity method	28
	2.3 Results of the reciprocity method	35
	2.3.1 The case $\Gamma = \Gamma_{k_1, k_2}$	35
	2.3.2 Adding an identity permutation to Γ_{k_1, k_2}	44
	2.3.3 The cases $\{1324, 123\}$ and $\{1324 \dots p, 123 \dots p-1\}$ for $p \geq 5$	56
Chapter 3	The case of multiple descents	74
	3.1 A new involution	75
	3.2 Results of the new involution	82
	3.2.1 The case $\Gamma = \{14253, 15243\}$	84
	3.2.2 The case $\Gamma = \{142536\}$	91
	3.2.3 The proof of Theorem 12	106
	3.2.4 The remaining cases of $\tau = 152634, \tau = 152436, \tau =$ 162435 , and $\tau = 142635$	115

Chapter 4	Refinements of the c-Wilf equivalent relation	131
	4.1 The proof of Theorem 18	138
	4.2 The proof of Theorem 19	151
Chapter 5	Generating Function for Initial and Final Descents	156
	5.1 The proof of Theorem 22	158
	5.2 Generating function for the number of initial and final descents	165
Bibliography	175

LIST OF FIGURES

Figure 1.1:	The six $(1^2, 2^2)$ -brick tabloids of shape (6).	7
Figure 1.2:	The weights of the six $(1^2, 2^2)$ -brick tabloids of shape (6).	11
Figure 1.3:	The $(1^2, 2^2)$ -brick tabloids of shape (6) under multiple weights.	13
Figure 2.1:	The construction of a filled-labeled-brick tabloid.	30
Figure 2.2:	$I_\Gamma(O)$ for O in Figure 2.1.	32
Figure 2.3:	Patterns for two consecutive brick of size 2 in a fixed point of I_Γ	62
Figure 2.4:	The bijection ϕ	63
Figure 2.5:	The bijection $\theta_{n,k}$	64
Figure 2.6:	The bijection $\theta_{n,k}^{-1}$	65
Figure 2.7:	The flip of Dyck path.	66
Figure 2.8:	The bijection $\beta_{n,k}$	67
Figure 2.9:	The bijection $\beta_{n,k}^{-1}$	68
Figure 2.10:	A fixed point with Γ_p -matches starting at c_i for $i = 1, \dots, m - 1$	71
Figure 3.1:	An example of the involution J_Γ	77
Figure 3.2:	A fixed point of $J_{\{15342\}}$	82
Figure 3.3:	The possible choice for d in Subcase 2a.	88
Figure 3.4:	Subcase 2b.	90
Figure 3.5:	A 142536-match as a 2-line array.	93
Figure 3.6:	Fixed points that start with series of τ -matches.	95
Figure 3.7:	Fixed points that start with series of τ -matches in Case 2.1.	96
Figure 3.8:	The Hasse diagram of \bar{G}_{6k+4} for $k = 0, 1, 2$	98
Figure 3.9:	The Hasse diagram of D_n	98
Figure 3.10:	Partitioning the Hasse Diagram of \bar{G}_{6k+4}	99
Figure 3.11:	The matrix M_7	100
Figure 3.12:	Fixed points that start with series of τ -matches in Case 2.2.	100
Figure 3.13:	i starts brick b_{2k+3}	102
Figure 3.14:	The Hasse diagram of \bar{G}_{6k+2} for $k = 0, 1, 2$	104
Figure 3.15:	Partitioning the Hasse Diagram of \bar{G}_{6k+2}	105
Figure 3.16:	The matrix P_7	106
Figure 3.17:	The Hasse diagram associated with τ_a	106
Figure 3.18:	The Hasse diagram of overlapping τ_a -mathces.	107
Figure 3.19:	Subcases (2.A) and (2.B).	110
Figure 3.20:	The ordering of $\{\sigma_2, \sigma_4, \dots, \sigma_{4k}\}$ for $k = 2$ (left) and for general k (right).	118
Figure 3.21:	The ordering of $\{\sigma_1, \sigma_2, \dots, \sigma_{10}\}$ in S_2	120
Figure 3.22:	The ordering of $\{\sigma_1, \sigma_2, \dots, \sigma_{4k+2}\}$ in S_k (top) and its simplified structure (bottom).	122
Figure 3.23:	The generalized diagram for $L(k, n)$	122
Figure 3.24:	The ordering of the first $4k + 2$ cells of O for $\tau = 142635$	127

Figure 3.25: The Hasse diagram for $L(a_1, \dots, a_{n+1}; b_1, \dots, b_n; c_1, \dots, c_n)$	128
Figure 4.1: The construction of a filled, labeled brick tabloid.	141
Figure 5.1: An example of the object created from equation (5.1).	160
Figure 5.2: The image of the filled, labeled brick tabloid from Figure 5.1. . .	162
Figure 5.3: A fixed point of the involution I	163
Figure 5.4: A "regular" fixed point of the involution I'	170

LIST OF TABLES

Table 2.1:	The polynomials $U_{\Gamma_{2,2},n}(-y)$ for $\Gamma_{2,2} = \{1324, 1423\}$	42
Table 2.2:	The polynomials $U_{\Gamma_{6,4},n}(-y)$	44
Table 2.3:	The polynomials $U_{\Gamma_{2,2,2},2k}(-y)$ for $\Gamma_{2,2,2} = \{1324, 1423, 123\}$	49
Table 2.4:	The polynomials $U_{\Gamma_{2,2,2},2k+1}(-y)$ for $\Gamma_{2,2,2} = \{1324, 1423, 123\}$	49
Table 2.5:	The polynomials $U_{\Gamma_{2,2,3},3k}(-y)$ for $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$	51
Table 2.6:	The polynomials $U_{\Gamma_{2,2,3},3k+1}(-y)$ for $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$	52
Table 2.7:	The polynomials $U_{\Gamma_{2,2,3},3k+2}(-y)$ for $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$	52
Table 2.8:	The polynomials $U_{\Gamma,n}(-y)$ for $\Gamma = \{1324, 123\}$	60
Table 3.1:	The polynomials $U_{\Gamma,n}(-y)$ for $\Gamma = \{14253, 15243\}$	90
Table 3.2:	The polynomials $MN_{\Gamma,n}(x, y)$ for $\Gamma = \{14253, 15243\}$	91
Table 3.3:	The polynomials $U_{\tau,n}(y)$ for $\tau = 142536$	107
Table 3.4:	The first eight values of S_k	123
Table 3.5:	The first eight values of L_k	129
Table 4.1:	The c-Wilf equivalent classes of length 5.	150

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my family for their unconditional love and support. I have been extremely fortunate in my life to have such an amazing and unique family. My parents have sacrificed a lot in order for me to have a better life and education. To this day, I still remember my Dad Bạch Công Sơn making trips of 60 kilometers one way between home and work almost every day of the week. Dad has been spending more than 30 years of his life riding on an old scooter amidst the wind, sand, and dust of the polluted rural areas in Vietnam and under the harsh sunlight and downpour rain of a tropical country just to be with the family and to take me to school in the morning. My Mom Trần Thị Thiệu and my Aunt Trần Thị Thoa have been working non-stop from 5:00 a.m. to midnight everyday in order to provide me a perfect childhood with the best education. To my family in the United States, I would like to thank my Uncle Trần Gia Tuấn for his tremendous amount of help and counsel with my everyday problems as I am trying to adapt and integrate to the new life in a foreign country. In fact, Uncle Tuấn has been a father figure for me in my father's absence. I would like to thank my Grandpa Trần Gia Tá, my Aunts Trần Tina, Tôn Nữ Lan Hương, and Trần Kim Trúc for treating me as their own son, being constantly concerned about my well-being, and for putting food on the table and a roof over my head over the past ten years. I owe a tremendous amount of gratitude to all the members of my family as they have played an important role in the development of my identity and shaping the individual that I am today.

I would like to thank my thesis advisor, Professor Jeffrey Rimmel, who has supported me throughout my graduate studies with his patience, knowledge, and wisdom. It has been an utmost honor to study under the guidance of a brilliant

mathematician and dedicated educator like Professor Remmel. He has ignited in me the joy, passion, and enthusiasm for the study of enumerative combinatorics and permutation patterns. Professor Remmel has also contributed greatly to my rewarding experience at the University of California, San Diego (UCSD) by engaging me with numerous intellectual ideas and research projects, funding my trips to various meetings and conferences, referring me to many employment opportunities to develop my curriculum vitae, providing tips and feedbacks for my teaching skills, and always demanding a high quality of work in all of our papers. Most importantly, over my tenure at UCSD, Professor Remmel has always been by my side helping me through the arduous journey of graduate school. This dissertation would not have been possible without the dedicated guidance and unyielding support of Professor Remmel over the past four years.

I also thank Professor Adriano Garsia, Dr. Kenneth Barrese, and the members of the thesis committee: Professors Ronald Graham, Ramamohan Paturi, Brendon Rhoades, and Jacques Verstraete, for their time and interest in my works, as well as their valuable comments and constructive feedback on my thesis.

I would like to thank Dr. Gabriele Wienhausen, Ms. Susan Rinaldi, and the Supplemental Instruction staff members from the Teaching and Learning Commons who have provided support for the classes I taught at UCSD. Dr. Wienhausen and Ms. Rinaldi introduced me to the Supplemental Instruction model, an academic support program which aims at maximizing students' involvement with the course materials and bridging the gap between the instructor and students. My time working under their supervision and mentorship has provided me with a lot of valuable experience, vastly altered my perspective on teaching and learning mathematics, and trained me

to be a better educator. It is my sincere hope that their experimental Triton Prep program will be successful and will help many incoming UCSD students flourish in their academic careers.

I would like to thank a special lady in my life, Trần Trịnh Ngọc Trâm, who has always been there cheering me up and listening to my “boring and technical” math talks. Trâm makes me a more caring person for knowing her.

I thank my good friend Allison Richey for her time and effort in proofreading many of my essays and papers, including portions of this dissertation thesis. I truly appreciate our camaraderie and cherish the time spent with Allison exploring the city of San Diego and sampling the various ethnic cuisine the city has to offer. My time at UCSD was also enriched and made enjoyable due to the companionship of my friendly and cheerful fellow graduate students (several of whom are now doctors and professors): Shaunak Das, Chris Deotte, Miles Jones, Janine LoBue-Tiefenbruck, Ran Pan, Roshil Paudyal, Dun Qiu, Luvreet Sangha, Sittipong Thamrongpaibroj, and Nan Zou. It has been a great pleasure to discuss various ideas for projects, papers, and homework problems with each and everyone of them. I also thank Mr. Wilson Cheung for helping me maintain my course websites and providing me with tech support. In addition, I gratefully acknowledge the financial support of the Mathematics department which made my Ph.D. work possible.

Chapter 2 is based on the paper *Generating functions for descents over permutations which avoid sets of consecutive patterns*, published in the Australasian Journal of Combinatorics in 2016. Bach, Quang; Remmel, Jeffrey. The dissertation author was the primary investigator of this paper.

Part of Chapter 3 is based on an unpublished paper by Remmel and the

dissertation author. It has been submitted for publication to the Discrete Mathematics journal and the preprint is available at <https://arxiv.org/abs/1702.08125>. The last part of this chapter consists of new materials in which the dissertation author is the main researcher.

The materials of Chapter 4 is based on the paper *Descent c -Wilf Equivalence*, published in the Discrete Mathematics and Theoretical Computer Science journal in 2017. Bach, Quang; Remmel, Jeffrey. The dissertation author was the primary investigator of the material.

Lastly, Chapter 5 is currently being prepared for submission for publication of the material. Bach, Quang; Remmel, Jeffrey. The dissertation author was the primary investigator of the results presented in this thesis.

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PUBLICATIONS

Q.T. Bach and J.B. Remmel, Generating functions for descents over permutations which avoid sets of consecutive patterns, *Australas. J. Combin.*, **64(1)** (2016), 194-231.

Q.T. Bach, R. Paudyal, and J.B. Remmel, A Fibonacci analogue of Stirling numbers, submitted to *Discrete Math*, <http://arxiv.org/abs/1510.04310v2> (2016).

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Q.T. Bach and J.B. Remmel, Generating functions for permutations which avoid consecutive patterns with multiple descents, submitted to *Discrete Math*, <https://arxiv.org/abs/1702.08125> (2017).

Q.T. Bach, R. Paudyal, and J.B. Remmel, Q -analogues of the Fibo-Stirling numbers, to appear in the *Journal of Integer Sequences*, <https://arxiv.org/abs/1701.07515> (2017).

ABSTRACT OF THE DISSERTATION

**Extensions of the Reciprocity Method in Consecutive Pattern Avoidance
in Permutations**

by

Quang Tran Bach

Doctor of Philosophy in Mathematics

University of California, San Diego, 2017

Professor Jeffrey Remmel, Chair

Let S_n denote the symmetric group. For any $\sigma \in S_n$, we let $\text{des}(\sigma)$ denote the number of descents of σ , $\text{inv}(\sigma)$ denote the number of inversions of σ , and $\text{LRmin}(\sigma)$ denote the number of left-to-right minima of σ . Jones and Remmel developed the *Reciprocity Method* to study the generating functions of the form

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} x^{\text{LRmin}(\sigma)} y^{1+\text{des}(\sigma)}$$

where $\mathcal{NM}_n(\tau)$ is the set of permutations σ in the symmetric group S_n which have

no consecutive τ -matches and τ is a permutation that starts with 1 and has exactly one descent.

In this thesis, we extend the reciprocity method to study the generating functions of the form

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\Gamma)} x^{\text{LRmin}(\sigma)} y^{1+\text{des}(\sigma)}$$

in the case where Γ is a set of permutations such that, for all $\tau \in \Gamma$, τ starts with 1 but we do not put any conditions on the number of descents in τ . In addition, we can also obtain the q -analog for the reciprocity method and compute the generating functions of the form

$$\text{INM}_\Gamma(t, q, z) = 1 + \sum_{n \geq 0} \frac{t^n}{[n]_q!} \text{INM}_{\Gamma, n}(q, z)$$

where $\text{INM}_{\Gamma, n}(q, z) = \sum_{\sigma \in \mathcal{NM}_n(\Gamma)} z^{\text{des}(\sigma)+1} q^{\text{inv}(\sigma)}$. Our results from this extension then lead us to define natural refinements for the c -Wilf equivalence relation. That is, if $\mathbf{stat}_1, \dots, \mathbf{stat}_k$ are permutations statistics, we say that two sets of permutations Γ and Δ are $(\mathbf{stat}_1, \dots, \mathbf{stat}_k)$ - c -Wilf equivalent if for all $n \geq 1$,

$$\sum_{\sigma \in \mathcal{NM}_n(\Gamma)} \prod_{i=1}^k x_i^{\mathbf{stat}_i(\sigma)} = \sum_{\sigma \in \mathcal{NM}_n(\Delta)} \prod_{i=1}^k x_i^{\mathbf{stat}_i(\sigma)}.$$

This enables us to give many examples of pairs of permutations α and β in S_j which are $\text{des-}c$ -Wilf equivalent, (des, inv) - c -Wilf equivalent, and $(\text{des}, \text{inv}, \text{LRmin})$ - c -Wilf equivalent.

Chapter 1

Introduction

1.1 A quick history of permutation patterns

We first start with the basic definitions and terminologies for permutations and permutation patterns.

We let S_n denote the group all permutations of length n . That is, S_n is the set of all one-to-one maps $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ under composition. We let $S_\infty = \cup_{n \geq 0} S_n$. Given $\sigma \in S_n$, we shall write $\sigma = \sigma_1 \dots \sigma_n$ where $\sigma_i = \sigma(i)$. This way of writing permutations is often referred to as *one-line notation*.

If $\sigma = \sigma_1 \dots \sigma_n \in S_n$, then we let $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $des(\sigma) = |Des(\sigma)|$ denote the *descent set* and *the number of descents of σ* , respectively. We let $inv(\sigma) = |\{(i, j) : 1 \leq i < j \leq n \ \& \ \sigma_i > \sigma_j\}|$ denote *the number of inversions of σ* and $coinv(\sigma) = |\{(i, j) : 1 \leq i < j \leq n \ \& \ \sigma_i < \sigma_j\}|$ denote *the number of coinversions of σ* . We define the *reverse of σ* , σ^r , to be the permutation $\sigma^r = \sigma_n \sigma_{n-1} \dots \sigma_2 \sigma_1$ and the *complement of σ* , σ^c , to be the permutation $\sigma^c = \sigma_1^c \sigma_2^c \dots \sigma_n^c$ where $\sigma_i^c = n + 1 - \sigma_i$ for each $1 \leq i \leq n$. We say that σ_j is a *left-to-right minima* of σ if $\sigma_i > \sigma_j$ for all $i < j$.

For example, the left-to-right minima of $\sigma = 938471625$ are 9, 3 and 1.

Given a sequence $\tau = \tau_1 \cdots \tau_n$ of distinct positive integers, we define the *reduction* of τ , $\text{red}(\tau)$, to be the permutation of S_n that results by replacing the i -th smallest element of τ by i for each i . For example $\text{red}(53962) = 32541$.

We define the usual p, q -analogues of n , $n!$, and $\binom{n}{k}$ as

$$\begin{aligned} [n]_{p,q} &= p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q}, \\ [n]_{p,q}! &= [1]_q [2]_q \cdots [n]_q, \text{ and} \\ \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!}. \end{aligned}$$

We shall use the standard conventions that $[0]_{p,q} = 0$ and $[0]_{p,q}! = 1$. Setting $p = 1$ in $[n]_{p,q}$, $[n]_{p,q}!$, and $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ yields $[n]_q$, $[n]_q!$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q$, respectively.

Let $\tau = \tau_1 \dots \tau_j \in S_j$ and $\sigma = \sigma_1 \dots \sigma_n \in S_n$. Then we say that

1. τ *occurs* in σ if there exists $1 \leq i_1 < \cdots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_j}) = \tau$,
2. there is a τ -*match starting in position* i in σ if $\text{red}(\sigma_i \sigma_{i+1} \dots \sigma_{i+j-1}) = \tau$, and
3. σ *avoids* τ if there is no occurrence of τ in σ .

We note that a τ -match in σ is often referred to as a consecutive occurrence of τ in σ . There are many variations of the notions a pattern occurring in a permutation σ including barred patterns, vincular patterns, bivincular patterns, partially ordered patterns, and mesh patterns. These types of patterns are described in Kitaev's book [29] which gives a broad introduction to the study of permutations patterns and its applications. However, we shall not study such variations in this thesis. .

We let $\mathcal{S}_n(\tau)$ denote the set of permutations of S_n which avoid τ and $\mathcal{NM}_n(\tau)$

denote the set of permutations of S_n which have no τ -matches. Let $S_n(\tau) = |\mathcal{S}_n(\tau)|$ and $NM_n(\tau) = |\mathcal{NM}_n(\tau)|$. If α and β are elements of S_j , then we say that α is *Wilf-equivalent to β* if $S_n(\alpha) = S_n(\beta)$ for all $n \geq 1$ and we say that α is *consecutive-Wilf-equivalent (c-Wilf-equivalent) to β* if $NM_n(\alpha) = NM_n(\beta)$ for all n . For any permutations τ and σ , we let $\tau\text{-mch}(\sigma)$ denote *the number of τ -matches of σ* .

These definitions are easily extended to sets of permutations. That is, if $\Gamma \subseteq S_j$, then we say that

1. Γ *occurs in σ* if there exists $1 \leq i_1 < \dots < i_j \leq n$ such that $\text{red}(\sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_j}) \in \Gamma$,
2. there is a Γ -*match starting in position i* in σ if $\text{red}(\sigma_i\sigma_{i+1}\dots\sigma_{i_j-1}) \in \Gamma$, and
3. σ *avoids Γ* if there is no occurrence of Γ in σ .

We let $\mathcal{S}_n(\Gamma)$ denote the set of permutations of S_n which avoid Γ and $\mathcal{NM}_n(\Gamma)$ denote the set of permutations of S_n which have no Γ -matches. We let $S_n(\Gamma) = |\mathcal{S}_n(\Gamma)|$ and $NM_n(\Gamma) = |\mathcal{NM}_n(\Gamma)|$. If Γ and Δ are subsets of S_j , then we say that Γ is *Wilf-equivalent to Δ* if $S_n(\Gamma) = S_n(\Delta)$ for all n and we say that Γ is *c-Wilf-equivalent to Δ* if $NM_n(\Gamma) = NM_n(\Delta)$ for all n . For any permutation σ and set of permutations Γ , we let $\Gamma\text{-mch}(\sigma)$ denote the number of Γ -matches of σ .

It is easy to see that the Wilf equivalence classes and c-Wilf equivalence classes are closed under the operations of reverse and complement. It immediately follows that there are at most two Wilf-equivalence classes in S_3 , namely $\{123, 321\}$ and $\{132, 213, 231, 312\}$. One of the first major results in the subject is due to Knuth in 1969, which says that the number of 321-avoiding permutations is equal to that of 132-avoiding permutations. Thus, in fact, all permutations in S_3 are Wilf equivalent. Moreover, for all $\tau \in S_3$, $S_n(\tau) = C_n$ where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number.

There are three Wilf-equivalence classes in S_4 .

- The first Wilf-equivalence class in S_4 is

$\{\mathbf{1234}, 1243, 1432, 2134, 2143, 2341, 3142, 3214, 3412, 4123, 4312, 4321\}$. Bóna [10]

gave an exact formula for $S_n(1342)$.

- The second Wilf-equivalence class in S_4 is

$\{\mathbf{1342}, 1423, 2314, 2413, 3142, 2431, 3124, 3241, 4132, 4213\}$. Gessel [21] in 1990

gave an explicit formula of $S_n(\tau)$ for any τ in this class.

- The third Wilf-equivalence class in S_4 is $\{\mathbf{1324}, 4231\}$. It is still an open problem to find an explicit formula for $S_n(1324)$ or find a generating function for $S_n(1324)$. There is a recursive formula given by Marinov and Radoičić [34] in 2003. The upper and lower bounds for the growth of this class are provided by Bóna [11] and Bevan [9] in 2015.

In addition, there are numerous results that involve Wilf equivalent classes for sets of two or more patterns of various lengths.

There are also refinements to the Wilf-equivalence and c-Wilf equivalence relation. For any permutation statistic **stat** on permutations and any pair of permutations α and β in S_j , we say that α is **stat**-Wilf equivalent to β if for all $n \geq 1$

$$\sum_{\sigma \in \mathcal{S}_n(\alpha)} x^{\mathbf{stat}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(\beta)} x^{\mathbf{stat}(\sigma)}. \quad (1.1)$$

More generally, if $\mathbf{stat}_1, \dots, \mathbf{stat}_k$ are permutations statistics, then we say that α and

β are $(\mathbf{stat}_1, \dots, \mathbf{stat}_k)$ -Wilf equivalent if for all $n \geq 1$,

$$\sum_{\sigma \in \mathcal{S}_n(\alpha)} \prod_{i=1}^k x_i^{\mathbf{stat}_i(\sigma)} = \sum_{\sigma \in \mathcal{S}_n(\beta)} \prod_{i=1}^k x_i^{\mathbf{stat}_i(\sigma)}. \quad (1.2)$$

Replacing $\mathcal{S}_n(\tau)$ by $\mathcal{NM}_n(\tau)$ in equations (1.1) and (1.2) above gives us analogous refinements for the c-Wilf equivalent relation.

The study of patterns in permutations and words has quite a long history which can be dated back to Euler in 1749 and later MacMahon in the 1880s. In 1749, Leonhard Euler introduced polynomials of the form

$$\sum_{k=0}^{n-1} (k+1)^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}$$

where $A_n(t) = \sum_{\sigma \in S_n} x^{\text{des}(\sigma)}$ is the Eulerian polynomial, the generating function of the the number of descents over the symmetric group S_n . In the 1880s, MacMahon gave generating functions for the distribution of inversions in permutations and words. In modern day terminology, these results correspond to occurrences or consecutive occurrences of the pattern 21.

The origin of the modern day study of patterns in words can be traced back to papers by Rotem, Rogers, and Knuth in 1970s and has been an active area of research since then. It started with an exercise proposed by Knuth in the first volume of his book "The Art of Computer Programming" [32]. In this particular exercise, Knuth asked his reader to show that the number of *stack-sortable* permutations of length n is given by $\frac{1}{n+1} \binom{2n}{n}$, the n -th Catalan number. Here, a *stack* is a last-in first-out linear sorting device that allows two operations *push* and *pop*. The input of the algorithm is a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ of length n . In the first step of

the algorithm, we *push* σ_1 into the stack. Next, we compare σ_1 with the left-most remaining element in the input, namely σ_2 . If $\sigma_1 < \sigma_2$, we *pop* σ_1 out of the stack, set σ_1 as the first element of the output, and *push* σ_2 into the stack. Otherwise, we simply *push* σ_2 into the stack on top of σ_1 . Subsequently, in each stage of the algorithm, we compare the left-most remaining element in the input with the top element in the stack. The process ends when all the elements have been placed into the output. If the output is another n -permutation $s(\sigma) = \sigma'_1 \sigma'_2 \cdots \sigma'_n$ such that $\sigma'_1 < \sigma'_2 < \cdots < \sigma'_n$ then we say σ is *stack-sortable*. It is well-known that the number of stack-sortable permutation of length n equals to the number of permutations of the same length that avoid the pattern 231. Therefore, this exercise provides the first explicit application of permutation patterns in computer science.

The notion of patterns in permutations and words has also proved to be a useful language in a variety of seemingly unrelated problems including the theory of Kazhdan-Lusztig polynomials, singularities of Schubert varieties, Chebyshev polynomials, rook polynomials for Ferrers boards, and various other sorting algorithms and sortable permutations. In addition, the study of patterns in permutations and words also arises in computational biology and theoretical physics. Many tools have been developed to study a variety of problems such as how to count the number of permutations and words that avoid a given pattern or collection of patterns or how to find the generating function for the number of occurrences of a pattern or collection of patterns. There also are two recent books in this area, one by Kitaev [29] which studies patterns in permutations and another by Heubach and Mansour [23] which studies patterns in words. There is also an annual conference dedicated purely to the study of patterns in permutations and words called "Permutation Patterns," which was organized for the

first time at the University of Otago in Dunedin, New Zealand, in 2003.

1.2 Symmetric functions and brick tabloids

In this section, we give the necessary background on symmetric functions that will be used throughout this thesis.

A *partition of n* is a sequence of positive integers $\lambda = (\lambda_1, \dots, \lambda_s)$ such that $0 < \lambda_1 \leq \dots \leq \lambda_s$ and $n = \lambda_1 + \dots + \lambda_s$. We shall write $\lambda \vdash n$ to denote that λ is a partition of n and we let $\ell(\lambda)$ denote the number of parts of λ . When a partition of n involves repeated parts, we shall often use exponents in the partition notation to indicate these repeated parts. For example, we will write $(1^2, 4^5)$ for the partition $(1, 1, 4, 4, 4, 4, 4)$.

If $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of n , then a λ -brick tabloid of shape n is a filling of a rectangle consisting of n cells with bricks of sizes $\lambda_1, \dots, \lambda_k$ in such a way that no two bricks overlap. For example, Figure 1.1 shows the six $(1^2, 2^2)$ -brick tabloids of shape (6) .

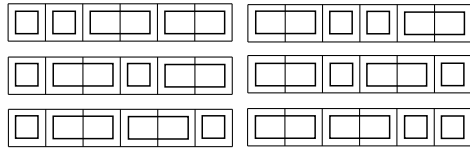


Figure 1.1: The six $(1^2, 2^2)$ -brick tabloids of shape (6) .

Let $\mathcal{B}_{\lambda,n}$ denote the set of λ -brick tabloids of shape (n) and let $B_{\lambda,n}$ be the number of λ -brick tabloids of shape (n) . If $B \in \mathcal{B}_{\lambda,n}$, we will write $B = (b_1, \dots, b_{\ell(\lambda)})$ if the lengths of the bricks in B , reading from left to right, are $b_1, \dots, b_{\ell(\lambda)}$. For example, the brick tabloid in the top right position in Figure 1.1 is denoted as $(1, 2, 2, 1)$.

Let Λ denote the ring of symmetric functions in infinitely many variables

x_1, x_2, \dots and let Λ_n be the vector space of symmetric functions of degree n . We shall define the n^{th} elementary and homogeneous symmetric functions e_n and h_n through their respective generating functions.

Let $E(t)$ denote the generating function for the sequence e_0, e_1, e_2, \dots . We define e_n by

$$E(t) = \sum_{n=0}^{\infty} e_n t^n = \prod_{i=1}^{\infty} (1 + x_i t) = (1 + x_1 t)(1 + x_2 t) \cdots .$$

For example, if $0 = x_4 = x_5 = \dots$ then $E(t)$ becomes

$$\begin{aligned} & (1 + x_1 t)(1 + x_2 t)(1 + x_3 t) \\ &= 1 + (x_1 + x_2 + x_3)t^1 + (x_1 x_2 + x_2 x_3 + x_3 x_1)t^2 + x_1 x_2 x_3 t^3, \end{aligned}$$

which means that the first few elementary symmetric functions in three variables are $e_0 = 1$, $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1 x_2 + x_2 x_3 + x_3 x_1$, and $e_3 = x_1 x_2 x_3$.

The n^{th} homogeneous symmetric function h_n is defined in a similar manner to e_n . The generating function for h_n is defined to be

$$H(t) = \sum_{n=0}^{\infty} h_n t^n = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}.$$

For example, if $0 = x_4 = x_5 = \dots$ then $H(t)$ becomes

$$\begin{aligned} & \left(\frac{1}{1 - x_1 t} \right) \left(\frac{1}{1 - x_2 t} \right) \left(\frac{1}{1 - x_3 t} \right) \\ &= (1 + x_1 t + x_1^2 t^2 + \cdots)(1 + x_2 t + x_2^2 t^2 + \cdots)(1 + x_3 t + x_3^2 t^2 + \cdots) \\ &= 1 + (x_1 + x_2 + x_3)t^1 + (x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_3 x_1)t^2 + \cdots, \end{aligned}$$

so the first few homogeneous symmetric functions in three variables are $h_0 = 1$, $h_1 = x_1 + x_2 + x_3$, and $x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1$.

Lastly, the n^{th} power symmetric function p_n is defined to be

$$p_n(x_1, x_2, x_3, \dots) = x_1^n + x_2^n + x_3^n + \dots .$$

It follows directly from their definition that

$$H(t) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} = \prod_{i=1}^{\infty} \frac{1}{1 + x_i(-t)} = \frac{1}{E(-t)}.$$

For any partition $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$, $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$. It is well known that $\{e_\lambda : \lambda \vdash n\}$, $\{h_\lambda : \lambda \vdash n\}$, and $\{p_\lambda : \lambda \vdash n\}$ are bases for Λ_n , for all n . So the functions e_0, e_1, \dots form an algebraically independent set of generators for Λ , and hence, every element in Λ can be uniquely expressed as a polynomial in the functions e_1, e_1, \dots, e_N for some N . This means that a ring homomorphism θ on Λ can be defined by simply specifying $\theta(e_n)$ for all n . This is the basic idea for the *homomorphism method* which was initiated by the work of Brenti. In [12], Brenti defined a ring homomorphism θ by setting

$$\theta(e_n) = \frac{(-1)^{n-1}(x-1)^{n-1}}{n!}$$

and used it to obtain the following well-known generating function

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} = \frac{x-1}{x - e^{(x-1)z}}. \quad (1.3)$$

However, Brenti did not use any results on the combinatorics of the transition matrices

between bases of symmetric function. It was Remmel and his students who combined homomorphisms and such combinatorics. Further details on these results can be found in the book [37].

The most important foundation for Remmel's development of the homomorphism method is perhaps the following identity, which was proved by Egecioğlu and Remmel in [17]

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda. \quad (1.4)$$

This interpretation of h_n in terms of e_n will aid us in describing the behavior of the homomorphism θ when applied to the homogeneous symmetric functions, which in turn will allow us to find generating functions for permutation statistics.

In addition, in the book "Counting with Symmetric Functions" by Mendes and Remmel [37], the authors also explored other relationships between different symmetric functions. For example, it is well known that for $n \geq 1$,

$$\sum_{i=0}^{n-1} (-1)^i e_i p_{n-i} = (-1)^{n-1} n e_n.$$

Now by expanding $E(-z) \sum_{n=1}^{\infty} p_n z^n$ and applying this result, we obtain

$$E(-z) \sum_{n=1}^{\infty} p_n z^n = \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} (-1)^i e_i p_{n-1} \right) = \sum_{n=1}^{\infty} (-1)^{n-1} n e_n z^n.$$

Thus,

$$\sum_{n=1}^{\infty} p_n z^n = \frac{\sum_{n=1}^{\infty} (-1)^{n-1} n e_n z^n}{E(-z)}. \quad (1.5)$$

Small modifications to the brick tabloids can help us describe the interpretation

of p_n in terms of e_n . Let ν be a function on the set of non-negative numbers. For each $B = (b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda, n}$, we define the weight $\omega_\nu(B)$ of B to be $\omega_\nu(B) = \nu(b_1)$ and let

$$\omega_\nu(\mathcal{B}_{\lambda, n}) = \sum_{B \in \mathcal{B}_{\lambda, n}} \omega_\nu(B).$$

For example, let $n = 8$ and $\lambda = (1^2, 3^2)$. Then the weights of the six brick tabloids in $\mathcal{B}_{\lambda, (n)}$ are given in Figure 1.2 so that $\omega_\nu(\mathcal{B}_{(1^2, 3^2), 8}) = 3\nu(1) + 3\nu(3)$.

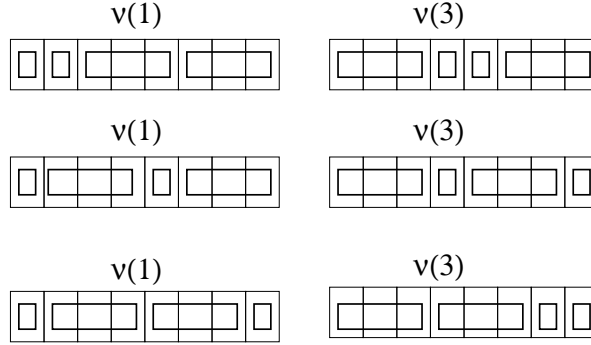


Figure 1.2: The weights of the six $(1^2, 2^2)$ -brick tabloids of shape (6) .

In [33] and [36], the authors defined a new symmetric function $p_{n, \nu}$ for each n by setting

$$p_{n, \nu} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \omega_\nu(\mathcal{B}_{\lambda, (n)}) e_\lambda. \quad (1.6)$$

Therefore, by (1.5),

$$\sum_{n \geq 1} p_{n, \nu} t^n = \frac{\sum_{n \geq 1} (-1)^{n-1} \nu(n) e_n t^n}{E(-t)} = \frac{\sum_{n \geq 1} (-1)^{n-1} \nu(n) e_n t^n}{\sum_{n \geq 0} (-1)^n e_n t^n}. \quad (1.7)$$

Furthermore, we observe that when $\nu(n) = 1$, the above equation becomes

$$1 + \sum_{n \geq 1} p_{n,\nu} t^n = 1 + \frac{\sum_{n \geq 1} (-1)^{n-1} e_n t^n}{\sum_{n \geq 0} (-1)^n e_n t^n} = \frac{1}{E(-t)} = 1 + \sum_{n \geq 1} h_n t^n.$$

It follows from (1.5) that when $\nu(n) = n$ for all $n \geq 1$, $p_{n,\nu}$ equals the power symmetric function p_n .

Another natural extension to the method is given by Mendes, Remmel, and Riehl in [38] where the authors extended the above result to the case of multiple weight functions. Now suppose that we are given r weight functions α_i , for $1 \leq i \leq r$, each is defined on the set of non-negative numbers. For each $B \in \mathcal{B}_{\lambda,n}$, where $\ell(\lambda) \geq r$, we define

$$\omega_{\alpha_1, \dots, \alpha_r}(B) = \alpha_1(b_1) \dots \alpha_r(b_r) \quad \text{for } \ell(\lambda) \geq r.$$

We also define

$$\omega_{\alpha_1, \dots, \alpha_r}(\mathcal{B}_{\lambda,n}) = \sum_{B \in \mathcal{B}_{\lambda,n}} \omega_{\alpha_1, \dots, \alpha_r}(B).$$

For example, if $B = (b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}$ and we let $w_{\alpha_1, \alpha_2}(B) = \alpha_1(b_1)\alpha_2(b_{\ell(\lambda)})$ when $\ell(\lambda) \geq 2$, then the weights of the six brick tabloids of shape $n = 8$ and type $\lambda = (1^2, 3^2)$ are given in Figure 1.3 below. Here,

$$\omega_{\alpha_1, \alpha_2}(\mathcal{B}_{(1^2, 3^2), 8}) = \alpha_1(1)\alpha_2(1) + \alpha_1(3)\alpha_2(3) + 2\alpha_1(1)\alpha_2(3) + 2\alpha_1(3)\alpha_2(1).$$

In [38], Mendes, Remmel, and Riehl showed that

$$\begin{aligned}
 p_{n;\alpha_1,\dots,\alpha_r} &= \sum_{\lambda \vdash n, \ell(\lambda) \geq r} (-1)^{n-\ell(\lambda)} \omega_{\alpha_1,\dots,\alpha_r}(\mathcal{B}_{\lambda,n}) e_\lambda \\
 &= \sum_{\substack{T=(b_1,\dots,b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n} \\ \ell(\lambda) \geq r, b_i \geq 1}} (-1)^{n-\ell(\lambda)} \omega_{\alpha_1,\dots,\alpha_r}(T) \prod_{i=1}^{\ell(\lambda)} e_{b_i}.
 \end{aligned}$$

So that

$$\sum_{n \geq r} p_{n;\alpha_1,\dots,\alpha_r} t^n = \frac{\prod_{i=1}^r (\sum_{n \geq 1} (-1)^{n-1} \alpha_i(n) e_n z^n)}{\sum_{n \geq 0} (-z)^n e_n} \tag{1.8}$$

This formula (1.8) provided an extension to the homomorphism method which allowed the authors of [38] to give several combinatorial proofs for the generating functions counting the numbers of descents in permutations with prescribed descent run lengths. In Chapter 5, we shall explore another application of this result in computing the generating functions counting the number of initial and final descents in permutations.

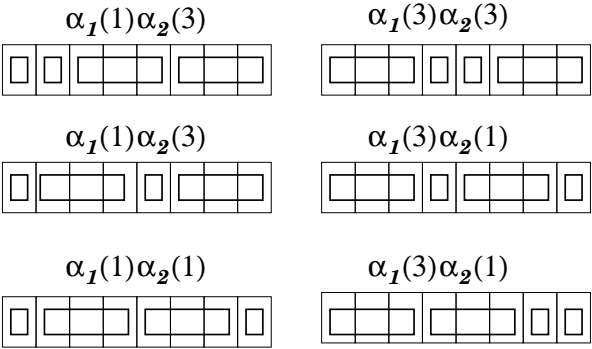


Figure 1.3: The $(1^2, 2^2)$ -brick tabloids of shape (6) under multiple weights.

There have been many other applications and extensions of Brenti’s homomorphism method over the recent years by Remmel and his co-authors. In [8], Beck and

Remmel defined a new homomorphism $\theta_{p,q}$ by

$$\theta_{p,q}(e_n) = \frac{(-1)^{n-1}(x-1)^{n-1}q^{\binom{n}{2}}}{[n]_{p,q}!}$$

where $[n]_{p,q}!$ is the p, q -analogue of n defined in the previous section. They then used this homomorphism to show that

$$[n]_{p,q}!\theta_{p,q}(h_n) = \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}.$$

This in turn allowed them to obtain the p, q -analogue of (1.3) as follows.

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} = \frac{x-1}{x - \mathbf{e}_{p,q}^{t(x-1)}} \quad (1.9)$$

where

$$\mathbf{e}_{p,q}^x = 1 + \sum_{n \geq 1} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} x^n.$$

One can also apply the homomorphism method to the case of alternating permutations. Here, we say that the permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is *alternating* if $\sigma_{i-1} > \sigma_i$ and $\sigma_i < \sigma_{i+1}$ for even i . In addition, an alternating permutation of even length is called *even alternating* while an alternating permutation of an odd length is called *odd alternating*. Now consider the following homomorphism. $\varphi(e_n) = \frac{(-1)^n}{n!} g(n)$ where $g(n) = 0$ if n is odd and $g(n) = (-1)^{n/2}$ if n is even. Then it can be shown that $(2n)!\varphi(h_{2n}) = A_{2n}$, where A_{2n} is the number of even alternating permutations of length $2n$. This leads to the following generating function

$$\sum_{n \geq 0} \frac{t^{2n}}{(2n)!} A_{2n} = \left(\sum_{n \geq 0} (-1)^n \frac{t^{2n}}{(2n)!} \right)^{-1} = \sec(t). \quad (1.10)$$

Similar argument can be used to show that $(2n - 1)! \varphi(p_{2n}) = A_{2n-1}$, where A_{2n-1} is the number of odd alternating permutations of length $2n - 1$. Therefore,

$$\begin{aligned}
\sum_{n \geq 1} \frac{t^{2n-1}}{(2n-1)!} A_{2n-1} &= \frac{1}{t} \varphi \left(\sum_{n \geq 1} p_{2n} t^{2n} \right) \\
&= \frac{1}{t} \frac{\sum_{n \geq 1} (-1)^{n-1} (2n) \varphi(e_{2n}) t^{2n}}{\sum_{n \geq 0} \varphi(e_{2n}) (-t)^{2n}} \\
&= \frac{\sum_{n \geq 1} (-1)^{n-1} \frac{t^{2n-1}}{(2n-1)!}}{\sum_{n \geq 0} (-1)^n \frac{t^{2n}}{(2n)!}} \\
&= \frac{\sin(t)}{\cos(t)} = \tan(t). \tag{1.11}
\end{aligned}$$

Combining (1.10) and (1.11) gives us the following well-known generating function for the number of alternating permutations

$$\sum_{n=0}^{\infty} A_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \#|\sigma \in S_n \text{ is alternating}| \frac{t^n}{n!} = \sec(t) + \tan(t).$$

In [46], Wagner extended this idea to compute the generating functions over the wreath product $C_k \wr S_n$. Specifically, Wagner considered the signed permutations in $C_k \wr S_n$ with signs in the set $Ck = \{\epsilon, \epsilon^2, \dots, \epsilon^k\}$ where $\epsilon = e^{2\pi i/k}$, and defined a partial ordering Ω on the signed letters such that $\epsilon^i 1 <_{\Omega} \epsilon^i 2 <_{\Omega} \dots <_{\Omega} \epsilon^i n$ for all $0 \leq i \leq k$ together with a partial ordering Γ such that $\epsilon^i a <_{\Gamma} \epsilon^j b$ if $a < b$ for all i, j . The number of $(C_k \wr S_n)$ -descents and $(C_k \wr S_n)$ -inversions of an element $\sigma \in C_k \wr S_n$ are given by

$$\begin{aligned}
\text{des}_k(\sigma) &= |\{i : 1 \leq i \leq n-1, \sigma_i >_{\Omega} \sigma_{i+1}\}| \quad \text{and} \\
\text{inv}_k(\sigma) &= |\{(i, j) : 1 \leq i < j \leq n, \sigma_i >_{\Gamma} \sigma_j\}|.
\end{aligned}$$

This enabled the author of [46] to obtain the generating function for

$$\sum_{\sigma \in C_k \S S_n} \epsilon(\sigma)^m x^{\text{des}_k(\sigma)} y^{\text{inv}_k(\sigma)}$$

and several related results. Langley and Remmel in [?] considered a sequence of permutations $\Sigma = (\sigma^{(1)}, \dots, \sigma^{(L)})$ in S_n and define the *common descents* of the set Σ to be

$$\text{Comdes}(\Sigma) = \left(\bigcap_{i=1}^L \text{Des}(\sigma^{(i)}) \right) \quad \text{and} \quad \text{comdes}(\Sigma) = |\text{Comdes}(\Sigma)|.$$

Applying this method, they obtained an analogue for the generating functions of the form

$$\sum_{n \geq 0} \frac{t^n}{[n]_{P,Q}!} \sum_{\Sigma = (\sigma^{(1)}, \dots, \sigma^{(L)}) \in S_n^L} x^{\text{comdes}(\Sigma)} Q^{\text{inv}(\Sigma)} P^{\text{coinv}(\Sigma)}$$

as an P, Q -analog of (1.9) where $Q = (q_1, \dots, q_L)$, $P = (p_1, \dots, p_L)$, and these statistics are defined as

$$Q^{\text{inv}(\Sigma)} = \prod_{i=1}^L q_i^{\text{inv}(\sigma^{(i)})} \quad \text{and} \quad P^{\text{coinv}(\Sigma)} = \prod_{i=1}^L p_i^{\text{coinv}(\sigma^{(i)})}.$$

1.3 The main goal of this thesis

The main focus of this thesis is to apply an extension of the basic homomorphism method to study the distribution of descents over $\mathcal{NM}_n(\Gamma)$, the set of permutations of length n which have no consecutive occurrences of Γ , where Γ is a set

of permutations. Specifically, we shall consider generating functions of the form

$$\text{NM}_\Gamma(t, x, y) = \sum_{n \geq 0} \frac{t^n}{n!} \text{NM}_{\Gamma, n}(x, y) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\Gamma)} x^{\text{LRmin}(\sigma)} y^{1+\text{des}(\sigma)}.$$

In the case where Γ consists of a single permutation τ , we shall simply write $\text{NM}_\tau(t, x, y)$ for $\text{NM}_\Gamma(t, x, y)$.

Jones and Remmel [24–27] developed what they called the *reciprocity method* to compute the generating function $\text{NM}_\tau(t, x, y)$ for certain families of permutations τ such that τ starts with 1 and $\text{des}(\tau) = 1$. The basic idea of their approach is as follows. If τ starts with 1, then the results in [25] allows us to write $\text{NM}_\tau(t, x, y)$ in the form

$$\text{NM}_\tau(t, x, y) = \left(\frac{1}{U_\tau(t, y)} \right)^x$$

where $U_\tau(t, y) = \sum_{n \geq 0} U_{\tau, n}(y) \frac{t^n}{n!}$.

Next one writes

$$U_\tau(t, y) = \frac{1}{1 + \sum_{n \geq 1} \text{NM}_{\tau, n}(1, y) \frac{t^n}{n!}}. \quad (1.12)$$

One can then use the homomorphism method to give a combinatorial interpretation of the right-hand side of (1.12) which can be used to find simple recursions for the coefficients $U_{\tau, n}(y)$. This homomorphism method was first introduced by Brenti [12] and later developed by Remmel and his students which is the subject of the book “Counting with Symmetric Functions” by Mendes and Remmel [37]. The so-call homomorphism method derives generating functions for various permutation statistics by applying a ring homomorphism defined on the ring of symmetric functions Λ in infinitely many

variables x_1, x_2, \dots to simple symmetric function identities such as $H(t) = 1/E(-t)$, where $H(t)$ and $E(t)$ are the generating functions for the homogeneous and elementary symmetric functions defined above.

In their case, Jones and Remmel defined a homomorphism θ_τ on Λ by setting

$$\theta_\tau(e_n) = \frac{(-1)^n}{n!} \text{NM}_{\tau,n}(1, y).$$

Then

$$\theta_\tau(E(-t)) = \sum_{n \geq 0} \text{NM}_{\tau,n}(1, y) \frac{t^n}{n!} = \frac{1}{U_\tau(t, y)}.$$

Hence

$$U_\tau(t, y) = \frac{1}{\theta_\tau(E(-t))} = \theta_\tau(H(t)),$$

which implies that

$$n! \theta_\tau(h_n) = U_{\tau,n}(y).$$

Thus, if we can compute $n! \theta_\tau(h_n)$ for all $n \geq 1$, then we can compute the polynomials $U_{\tau,n}(y)$ and the generating function $U_\tau(t, y)$, which in turn allows us to compute the generating function $\text{NM}_\tau(t, x, y)$. Jones and Remmel [26, 27] showed that one can interpret $n! \theta_\tau(h_n)$ as a certain signed sum of the weights of filled, labeled brick tabloids when τ starts with 1 and $\text{des}(\tau) = 1$. They then defined a weight-preserving, sign-reversing involution I on the set of such filled, labeled brick tabloids which allowed them to give a relatively simple combinatorial interpretation for $n! \theta_\tau(n_n)$. Consequently, they showed how such a combinatorial interpretation allowed them to prove that for certain families of such permutations τ , the polynomials $U_{\tau,n}(y)$ satisfy simple recursions.

In [3], Remmel and the dissertation author extended the reciprocity method to study the polynomials $U_{\Gamma,n}(y)$ where

$$U_{\Gamma}(t, y) = 1 + \sum_{n \geq 1} U_{\Gamma,n}(y) \frac{t^n}{n!} = \frac{1}{1 + \sum_{n \geq 1} \text{NM}_{\Gamma,n}(1, y) \frac{t^n}{n!}}$$

in the case where Γ is a set of permutations such that for all $\tau \in \Gamma$, τ starts with 1 and $\text{des}(\tau) \leq 1$. Specifically, we studied the case where

$$\Gamma_{k_1, k_2} = \{\sigma \in S_p : \sigma_1 = 1, \sigma_{k_1+1} = 2, \sigma_1 < \sigma_2 < \cdots < \sigma_{k_1}, \sigma_{k_1+1} < \sigma_{k_1+2} < \cdots < \sigma_p\}.$$

That is, Γ_{k_1, k_2} consists of all permutations σ of length p where 1 is in position 1, 2 is in position $k_1 + 1$, and σ consists of two increasing sequences, one starting at 1 and the other starting at 2. Interestingly, our extension for the reciprocity method also applies even when the permutations in Γ do not have the same length nor the same descent set. In addition, we also investigated a new phenomenon that arises when adding the identity permutation $12 \dots k$ to the family Γ_{k_1, k_2} . Let $\Gamma_{k_1, k_2, s} = \Gamma_{k_1, k_2} \cup \{1 \dots s(s+1)\}$ for some $s \geq \max(k_1, k_2)$. In certain cases, we were able to obtain explicit formulas for the polynomials $U_{\Gamma_{k_1, k_2, s}, n}(y)$ for certain values of k_1, k_2 , and s . For instance, if $\Gamma = \{1324, 123\}$, then we proved the following result for the polynomials $U_{\Gamma, n}(y)$'s. For all $n \geq 0$,

$$U_{\Gamma, 2n}(y) = \sum_{k=0}^n \frac{(2k+1) \binom{2n}{n-k}}{n+k+1} (-y)^{n+k+1} \text{ and } U_{\Gamma, 2n+1}(y) = \sum_{k=0}^n \frac{2(k+1) \binom{2n+1}{n-k}}{n+k+2} (-y)^{n+k}.$$

Another example where we could find an explicit formula is the case $\Gamma_{2,2,s} = \{1324, 1342, 123\}$ where we showed that $U_{\Gamma_{2,2,s}, 1}(y) = -y$, and for $n \geq 2$, the polyno-

mials $U_{\Gamma_{2,2,s,n}}(y)$'s satisfy the recursion

$$U_{\Gamma_{2,2,s,n}}(y) = -yU_{\Gamma_{2,2,s,n-1}}(y) - \sum_{k=0}^{s-2} ((n-k-1)yU_{\Gamma_{2,2,s,n-k-2}}(y) + (n-k-2)y^2U_{\Gamma_{2,2,s,n-k-3}}(y)).$$

Using these recursions, we proved that

$$U_{\Gamma_{2,2,2,2n}}(y) = \sum_{i=0}^n (2n-1) \downarrow\downarrow_{n-i} (-y)^{n+i} \text{ and } U_{\Gamma_{2,2,2,2n+1}}(y) = \sum_{i=0}^n (2n) \downarrow\downarrow_{n-i} (-y)^{n+1+i}$$

where for any x , $(x) \downarrow\downarrow_0 = 1$ and $(x) \downarrow\downarrow_k = x(x-2)(x-4) \cdots (x-2k-2)$ for $k \geq 1$.

Remmel and the dissertation author also further extended the reciprocity method to study the generating functions $NM_{\Gamma}(t, x, y)$ where all the permutations Γ start with 1 but there is no restriction on the number of descents in a permutations in Γ . While the basic concepts of the reciprocity method still hold, the involution defined by Jones and Remmel no longer works. Thus, we defined a new sign-reversing, weight-preserving mapping J_{Γ} and, under this new involution, we were able to compute the recursion for the polynomials $U_{\Gamma,n}(y)$ for the special cases where $\tau \in \Gamma$ such that $\text{des}(\tau) = j \geq 1$ and the bottom elements of these descents are $2, \dots, j+1$ when reading from left to right. In most of the cases here, the analysis of the fixed points of the involution J_{Γ} can be associated with counting the number of linear extensions for certain Hasse diagrams.

Finally, we can obtained the q -analog for the reciprocity method and computed

the generating functions of the form

$$\text{INM}_\Gamma(t, q, z) = 1 + \sum_{n \geq 0} \frac{t^n}{[n]_q!} \text{INM}_{\Gamma, n}(q, z)$$

where $\text{INM}_{\Gamma, n}(q, z) = \sum_{\sigma \in \mathcal{NM}_n(\Gamma)} z^{\text{des}(\sigma)+1} q^{\text{inv}(\sigma)}$. Our results from this extension led us to define natural refinements for the c -Wilf equivalence relation. We also gave many examples of pairs of permutations α and β such that α and β are (des, inv) - c -Wilf equivalent, $(\text{des}, \text{LRmin})$ - c -Wilf equivalent, and $(\text{des}, \text{inv}, \text{LRmin})$ - c -Wilf equivalent.

The remainder of this thesis is organized as follows. Chapter 2 starts with the results from Jones and Remmel on pattern matching in cycle structure which leads to the development of their *reciprocity method*. In the same chapter, we also describe Bach and Remmel's extension to the reciprocity method to the case where Γ is a family of permutations that start with 1 and have $\text{des}(\tau) \leq 1$ for all $\tau \in \Gamma$. In Chapter 3, we provide a new involution which will allow us to remove the restriction on the number of descents in the forbidden patterns. We also consider examples where the forbidden patterns $\tau = \tau_1 \cdots \tau_6$ with $\tau_1 = 1, \tau_3 = 2, \tau_5 = 3$ and $\text{des}(\tau) = 2$. In Chapter 4, we give the q -analogue to the reciprocity method and discuss several refinements for the c -Wilf equivalent relation. We also provide conditions on permutations α and β in S_j which will guarantee that α and β are des - c -Wilf equivalent, (des, inv) - c -Wilf equivalent, or $(\text{des}, \text{inv}, \text{LRmin})$ - c -Wilf equivalent. Lastly, in Chapter 5, we consider several other applications of Brenti's homomorphism method in finding the generating functions for the number of initial and final descents in permutations.

Chapter 2

The reciprocity method

In this chapter, we extend the reciprocity method of Jones and Remmel [26, 27] to study generating functions of the form $\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\Gamma)} x^{\text{LRmin}\sigma} y^{1+\text{des}(\sigma)}$ where Γ is a set of permutations which start with 1 and have at most one descent, $\mathcal{NM}_n(\Gamma)$ is the set of permutations σ in the symmetric group S_n which have no Γ -matches, $\text{des}(\sigma)$ is the number of descents of σ and $\text{LRmin}\sigma$ is the number of left-to-right minima of σ . We also briefly introduce the study by Jones and Remmel on pattern matching in cycle structure and use their result to show that this generating function is of the form $\left(\frac{1}{U_\Gamma(t, y)}\right)^x$ where $U_\Gamma(t, y) = \sum_{n \geq 0} U_{\Gamma, n}(y) \frac{t^n}{n!}$ and the coefficients $U_{\Gamma, n}(y)$ satisfy some simple recursions in the case where Γ equals $\{1324, 123\}$, $\{1324 \cdots p, 12 \cdots (p-1)\}$ and $p \geq 5$, or Γ is the set of permutations $\sigma = \sigma_1 \cdots \sigma_n$ of length $n = k_1 + k_2$ where $k_1, k_2 \geq 2$, $\sigma_1 = 1$, $\sigma_{k_1+1} = 2$, and $\text{des}(\sigma) = 1$.

2.1 Pattern matching in the cycle structure of permutations

Jones and Remmel [24] initiated the study of study pattern matching conditions in the cycle structure of a permutation. Suppose that $\tau = \tau_1 \cdots \tau_j$ is a permutation in S_j and σ is a permutation in S_n with k cycles C_1, \dots, C_k . Here, we shall always write cycles in the form $C_i = (c_{0,i} \dots, c_{p_i-1,i})$ where $c_{0,i}$ is the smallest element in C_i and p_i is the length of the cycle C_i . In the cycle structure of σ , we shall always arrange the cycles by increasing smallest elements, i.e., we arrange the k cycles of σ so that $c_{0,1} < \dots < c_{0,k}$. Then we say that σ has a *cycle τ -match* if there is an i such that $C_i = (c_{0,i} \dots, c_{p_i-1,i})$ where $p_i \geq j$ and an τ such that $\text{red}(c_{r,i}c_{r+1,i} \cdots c_{r+j-1,i}) = \tau$, where we take indices of the form $r+s \pmod{p_i}$. We denote the number of cycle τ -match in σ by $c\text{-}\tau\text{-mch}(\sigma)$. For example, if $\tau = 2\ 1\ 3$ and $\sigma = (1, 10, 9)(2, 3)(4, 8, 5, 7, 6)$ then $9\ 1\ 10$ is a cycle τ -match in the first cycle while $7\ 5\ 8$ and $6\ 4\ 7$ are cycle τ -matches in the third cycle. In addition, $c\text{-}\tau\text{-mch}(\sigma) = 3$.

Similarly, we say that τ *cycle occurs* in σ if there exists an i such that $C_i = (c_{0,i} \dots, c_{p_i-1,i})$ where $p_i \geq j$ and there is an r with $0 \leq r \leq p_i - 1$ with indices $0 \leq i_1 \leq \dots \leq i_{j-1} \leq p_i - 1$ such that $\text{red}(c_{r,i}c_{r+i_1,i} \cdots c_{r+i_{j-1},i}) = \tau$ where again, we take indices of the form $r+s \pmod{p_i}$. We say that σ *cycle avoids* τ if there is no cycle occurrence of τ in σ . For example, if $\tau = 1\ 2\ 3$ and $\sigma = (1, 10, 9)(2, 3)(4, 8, 5, 7, 6)$ then $4\ 5\ 7, 4\ 5\ 6$, and $5\ 6\ 8$ are cycle occurrences of τ in the third cycle.

We can extend of the notion of cycle matches and cycle occurrences to sets of permutations in the obvious fashion. That is, suppose that Γ is a set of permutations in S_j and σ is a permutation in S_n with k cycles $C_1 \dots C_k$. Then we say that σ has a *cycle*

Γ -*match* if there is an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \geq j$ and an τ such that $\text{red}(c_{r,i}c_{r+1,i} \cdots c_{r+j-1,i}) \in \Gamma$, where we take indices of the form $r + s \pmod{p_i}$. We say that Γ *cycle occurs* in σ if there exists an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \geq j$ and there is an r with $0 \leq r \leq p_i - 1$ with indices $0 \leq i_1 \leq \dots \leq i_{j-1} \leq p_i - 1$ such that $\text{red}(c_{r,i}c_{r+i_1,i} \cdots c_{r+i_{j-1},i}) \in \Gamma$ where again, we take indices of the form $r + s \pmod{p_i}$. We say that σ *cycle avoids* Γ if there is no occurrence of Γ in σ .

For $\Gamma \subset S_j$, we let $\mathcal{CS}_n(\Gamma)$ denote the set of permutations of S_n which cycle avoid Γ and $\mathcal{NCM}_n(\Gamma)$ denote the set of permutations of S_n which have no cycle Γ -matches. We let $CS_n(\Gamma) = |\mathcal{CS}_n(\Gamma)|$ and $NCM_n(\Gamma) = |\mathcal{NCM}_n(\Gamma)|$.

Given a cycle $C = (c_0, \dots, c_{p-1})$ where c_0 is the smallest element in the cycle, we let $\text{cdes}(C) = 1 + \text{des}(c_0 \cdots c_{p-1})$. Thus, $\text{cdes}(C)$ counts the number of descent pairs as we traverse once around the cycle because the extra factor of 1 counts the descent pair $c_{p-1} > c_0$. For example, if $C = (1, 5, 3, 7, 2)$ then $\text{cdes}(C) = 3$ which counts the descent pairs 53, 72, and 21 as we traverse once around C . By convention, if $C = (c_0)$ is one-cycle then $\text{cdes}(C) = 1$. If $\sigma \in S_n$ is a permutation with k cycles $C_1 \dots C_k$, then we define $\text{cdes}(\sigma) = \sum_{i=1}^k \text{cdes}(C_i)$. We let $\text{cyc}(\sigma)$ denote the number of cycles in σ .

In [24], Jones and Remmel studied the generating functions

$$CA_\Gamma(t, x, y) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{CS}_n(\Gamma)} x^{\text{cyc}(\sigma)} y^{\text{cdes}(\sigma)}, \text{ and}$$

$$NCM_\Gamma(t, x, y) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCM}_n(\Gamma)} x^{\text{cyc}(\sigma)} y^{\text{cdes}(\sigma)}.$$

Their approach was to use the theory of exponential structures to reduce the problem down to studying pattern matching in n -cycles. That is, let \mathcal{L}_m denote the set of

m -cycles in S_m . Suppose that R is a ring and for each $m \geq 1$, we have a weight function $W_m : \mathcal{L}_m \rightarrow R$. We let $W(L_m) = \sum_{C \in \mathcal{L}_m} W_m(C)$. Now suppose that $\sigma \in S_n$ with the cycle structure $\sigma = C_1 \cdots C_k$. For each i , if C_i is of size m then we let $W(C_i) = W_m(\text{red}(C_i))$. Lastly, we define the weight of σ , $W(\sigma)$ by

$$W(\sigma) = \prod_{i=1}^k W(C_i).$$

Let $\mathcal{C}_{n,k}$ denote the set of all permutations of S_n with k cycles, then it is shown in the book "Enumerative Combinatorics, vol. 2" by Stanley [43] that

$$1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^n x^k \sum_{\sigma \in \mathcal{C}_{n,k}} W(\sigma) = e^{x \sum_{m \geq 1} \frac{W(L_m)t^m}{m!}}. \quad (2.1)$$

Now let $\mathcal{CS}_{n,k}(\Gamma)$ denote the set of permutations $\sigma \in S_n$ with k cycles such that σ cycle avoids Γ and let $\mathcal{NCM}_{n,k}(\Gamma)$ denote the set of permutations $\sigma \in S_n$ with k cycles such that σ has no cycle Γ -matches. Similarly, let $\mathcal{L}_m^{ca}(\Gamma)$ be the set of m cycles $\gamma \in S_m$ such that γ cycle avoids Γ with $L_m^{ca}(\Gamma) = |\mathcal{L}_m^{ca}(\Gamma)|$ and let $\mathcal{L}_m^{ncm}(\Gamma)$ be the set of m cycles $\gamma \in S_m$ such that γ has no cycle Γ -match with $L_m^{ncm}(\Gamma) = |\mathcal{L}_m^{ncm}(\Gamma)|$. We are interested in the special cases of weight functions W_m^{ca} where $W_m^{ca}(C) = 1$ if C cycle avoids a set of permutations and $W_m^{ncm}(C) = 0$ otherwise, or $W_m^{ncm}(C) = 1$ if C has no cycle Γ -matches and $W_m^{ncm}(C) = 0$ otherwise. Then under these special weight functions, (2.1) becomes

$$\text{CA}_\Gamma(t, x, y) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^n x^k \sum_{\sigma \in \mathcal{CS}_{n,k}(\Gamma)} y^{\text{cdes}(\sigma)} = e^{x \sum_{m \geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{ca}(\Gamma)} y^{\text{cdes}(C)}}$$

and

$$\text{NCM}_\Gamma(t, x, y) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^n x^k \sum_{\sigma \in \mathcal{NCM}_{n,k}(\Gamma)} y^{\text{cdes}(\sigma)} = e^{x \sum_{m \geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{\text{ncm}}(\Gamma)} y^{\text{cdes}(C)}}.$$

In the case $\Gamma = \{\tau\}$, Jones and Remmel showed that if $\tau \in S_j$ is a permutation that starts with 1, then we can reduce the problem of finding $\text{NCM}_\tau(t, x, y)$ to the usual problem of finding the generating function of permutations that have no τ -matches. Their approach is as follows. Suppose we are given a permutation $\sigma \in S_n$ with k cycles $C_1 \cdots C_k$. Assume we have arranged the cycles so that the smallest element in each cycle is on the left and we arrange the cycles by decreasing smallest elements. We let $\bar{\sigma}$ be the permutation that arises from $C_1 \cdots C_k$ by erasing all the parenthesis and commas. For example, if $\sigma = (7, 10, 9, 11)(4, 8, 6)(1, 5, 3, 2)$ then $\bar{\sigma} = 7\ 10\ 9\ 11\ 4\ 8\ 6\ 1\ 5\ 3\ 2$. It is easy to see that the minimal elements of the cycles correspond to left-to-right minima in $\bar{\sigma}$. It is also easy to see that under the bijection $\sigma \rightarrow \bar{\sigma}$, $\text{cdes}(\sigma) = \text{des}(\bar{\sigma}) + 1$ since every left-to-right minima is part of a descent pair in $\bar{\sigma}$.

In [24], Jones and Remmel proved that if $\tau \in S_j$ and τ starts with 1, then for any $\sigma \in S_n$,

1. σ has k cycles if and only if $\bar{\sigma}$ has k left-to-right minima,
2. $\text{cdes}(\sigma) = \text{des}(\bar{\sigma}) + 1$, and
3. σ has no cycle- τ -match if and only if $\bar{\sigma}$ has no τ -match.

The consequence of this result is that we can automatically obtain refinements of generating functions for the number of permutations with no τ -matches when τ

starts with 1. Specifically,

$$\begin{aligned}
\text{NM}_\tau(t, x, y) &= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} x^{\text{LRmin}(\sigma)} y^{1+\text{des}(\sigma)} \\
&= 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{k=1}^n x^k \sum_{\bar{\sigma} \in \mathcal{NCM}_{n,k}(\tau)} y^{\text{cdes}(\bar{\sigma})} \\
&= e^{x \sum_{m \geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{\text{ncm}}(\tau)} y^{\text{cdes}(C)}} = \text{NCM}_\tau(t, x, y).
\end{aligned}$$

This implies

$$\text{NM}_\tau(t, 1, y) = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} y^{1+\text{des}(\sigma)} = e^{\sum_{m \geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{\text{ncm}}(\tau)} y^{\text{cdes}(C)}}.$$

So

$$\sum_{m \geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{\text{ncm}}(\tau)} y^{\text{cdes}(C)} = \ln(\text{NM}_\tau(t, 1, y))$$

which then gives

$$\text{NM}_\tau(t, x, y) = \text{NCM}_\tau(t, x, y) = e^{x \ln(\text{NM}_\tau(t, 1, y))} = (\text{NM}_\tau(t, 1, y))^x.$$

Hence, if we let

$$U_\Gamma(t, y) = \sum_{n=0}^{\infty} \frac{t^n}{n!} U_{\tau,n}(y) = \frac{1}{\text{NM}_\Gamma(t, 1, y)} = \frac{1}{1 + \sum_{n \geq 1} \frac{t^n}{n!} \text{NM}_{\Gamma,n}(1, y)}, \quad (2.2)$$

then it is the case that

$$\text{NM}_\tau(t, x, y) = \sum_{n \geq 0} \frac{t^n}{n!} \text{NM}_{\Gamma,n}(x, y) = \left(\frac{1}{U_\tau(t, y)} \right)^x.$$

Here, we can now exploit the identity $H(t) = 1/E(-t)$ between the generating functions for the homogeneous and elementary symmetric functions and Egecioglu and Remmel's formula in (1.4) to define a homomorphism in order to give a combinatorial interpretation of the right-hand side of (2.2) which can be used to find simple recursions for the coefficients $U_{\tau,n}(y)$. This homomorphism method shall be discussed in greater details in the upcoming section.

2.2 The reciprocity method

In this section, we shall introduce Bach and Remmel's extension to the reciprocity method to find a combinatorial interpretation for $U_{\Gamma,n}(y)$ in the case where Γ is a set of permutations which all start with 1 and have at most one descent. We can assume that Γ contains at most one identity permutation. That is, if $12 \cdots s$ and $12 \cdots t$ are in Γ for some $s < t$, then if we consecutively avoid $12 \cdots s$, we automatically consecutively avoid $12 \cdots t$. Thus $\mathcal{NM}_n(\Gamma) = \mathcal{NM}_n(\Gamma - \{12 \cdots t\})$ for all n . In the case where Γ contains only one permutation τ , we simply replace Γ by τ to obtain the original reciprocity method introduced by Jones and Remmel in [25–27].

We want give a combinatorial interpretation to

$$U_{\Gamma}(t, y) = \frac{1}{\text{NM}_{\Gamma}(t, 1, y)} = \frac{1}{1 + \sum_{n \geq 1} \frac{t^n}{n!} \text{NM}_{\Gamma,n}(1, y)},$$

where

$$\text{NM}_{\Gamma,n}(1, y) = \sum_{\sigma \in \mathcal{NM}_n(\Gamma)} y^{1+\text{des}(\sigma)}.$$

We define a homomorphism θ_{Γ} on the ring of symmetric functions Λ by setting

$\theta_\Gamma(e_0) = 1$ and, for $n \geq 1$,

$$\theta_\Gamma(e_n) = \frac{(-1)^n}{n!} \text{NM}_{\Gamma,n}(1, y).$$

It follows that

$$\begin{aligned} \theta_\Gamma(H(t)) &= \sum_{n \geq 0} \theta_\Gamma(h_n) t^n = \frac{1}{\theta_\tau(E(-t))} = \frac{1}{1 + \sum_{n \geq 1} (-t)^n \theta_\Gamma(e_n)} \\ &= \frac{1}{1 + \sum_{n \geq 1} \frac{t^n}{n!} \text{NM}_{\Gamma,n}(1, y)} = U_\Gamma(t, y). \end{aligned}$$

By (1.4), we have

$$\begin{aligned} n! \theta_\Gamma(h_n) &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \theta_\Gamma(e_\lambda) \\ &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \sum_{(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{b_i}}{b_i!} \text{NM}_{\Gamma, b_i}(1, y) \\ &= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \binom{n}{b_1, \dots, b_{\ell(\lambda)}} \prod_{i=1}^{\ell(\lambda)} \text{NM}_{\Gamma, b_i}(1, y). \quad (2.3) \end{aligned}$$

Next, we want to give a combinatorial interpretation to the right hand side of (2.3). We select a brick tabloid $B = (b_1, b_2, \dots, b_{\ell(\lambda)})$ of shape (n) filled with bricks whose sizes induce the partition λ . We interpret the multinomial coefficient $\binom{n}{b_1, \dots, b_{\ell(\lambda)}}$ as the number of ways to choose an ordered set partition $\mathcal{S} = (S_1, S_2, \dots, S_{\ell(\lambda)})$ of $\{1, 2, \dots, n\}$ such that $|S_i| = b_i$ for $i = 1, \dots, \ell(\lambda)$. For each brick b_i , we then fill the cells of b_i with numbers from S_i such that the entries in the brick reduce to a permutation $\sigma^{(i)} = \sigma_1 \cdots \sigma_{b_i}$ in $\mathcal{NM}_{b_i}(\Gamma)$. We label each descent of σ that occurs within each brick as well as the last cell of each brick by y . This accounts for the

factor $y^{\text{des}(\sigma^{(i)})+1}$ within each brick. Finally, we use the factor $(-1)^{\ell(\lambda)}$ to change the label of the last cell of each brick from y to $-y$. We will denote the filled labeled brick tabloid constructed in this way as $\langle B, \mathcal{S}, (\sigma^{(1)}, \dots, \sigma^{(\ell(\lambda))}) \rangle$.

For example, when $n = 17$, $\Gamma = \{1324, 1423, 12345\}$, and $B = (9, 3, 5, 2)$, consider the ordered set partition $\mathcal{S} = (S_1, S_2, S_3, S_4)$ of $\{1, 2, \dots, 17\}$, where

$$S_1 = \{2, 5, 6, 9, 11, 15, 16, 17, 19\}, S_2 = \{7, 8, 14\}, S_3 = \{1, 3, 10, 13, 18\}, S_4 = \{4, 12\},$$

and the permutations $\sigma^{(1)} = 1 \ 2 \ 4 \ 6 \ 5 \ 3 \ 7 \ 9 \ 8 \in \mathcal{NM}_9(\Gamma)$, $\sigma^{(2)} = 1 \ 3 \ 2 \in \mathcal{NM}_7(\Gamma)$, $\sigma^{(3)} = 5 \ 1 \ 2 \ 4 \ 3 \in \mathcal{NM}_5(\Gamma)$, and $\sigma^{(4)} = 2 \ 1 \in \mathcal{NM}_2(\Gamma)$. The construction of $\langle B, \mathcal{S}, (\sigma^{(1)}, \dots, \sigma^{(4)}) \rangle$ is then pictured in Figure 2.1.

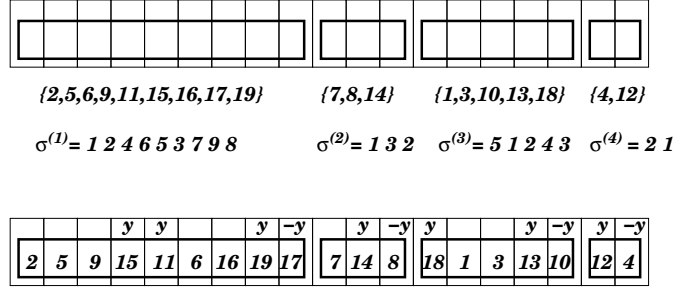


Figure 2.1: The construction of a filled-labeled-brick tabloid.

We can then recover the triple $\langle B, (S_1, \dots, S_{\ell(\lambda)}), (\sigma^{(1)}, \dots, \sigma^{(\ell(\lambda))}) \rangle$ from B and the permutation σ which is obtained by reading the entries in the cells from right to left. We let $\mathcal{O}_{\Gamma, n}$ denote the set of all filled labeled brick tabloids created this way. That is, $\mathcal{O}_{\Gamma, n}$ consists of all pairs $O = (B, \sigma)$ where

1. $B = (b_1, b_2, \dots, b_{\ell(\lambda)})$ is a brick tabloid of shape n ,
2. $\sigma = \sigma_1 \cdots \sigma_n$ is a permutation in S_n such that there is no Γ -match of σ which lies entirely in a single brick of B , and

3. if there is a cell c such that a brick b_i contains both cells c and $c+1$ and $\sigma_c > \sigma_{c+1}$, then cell c is labeled with a y and the last cell of any brick is labeled with $-y$.

We define the sign of each O to be $\text{sgn}(O) = (-1)^{\ell(\lambda)}$. The weight $W(O)$ of O is defined to be the product of all the labels y used in the brick. Thus, the weight of the filled labeled brick tabloid from Figure 2.1 above is $W(O) = y^{11}$. It follows that

$$n!\theta_\Gamma(h_n) = \sum_{O \in \mathcal{O}_{\Gamma,n}} \text{sgn}(O)W(O). \quad (2.4)$$

We next define a sign-reversing, weight-preserving involution $I : \mathcal{O}_{\Gamma,n} \rightarrow \mathcal{O}_{\Gamma,n}$. Given a filled labeled brick tabloid $(B, \sigma) \in \mathcal{O}_{\Gamma,n}$ where $B = (b_1, \dots, b_k)$, we read the cells of (B, σ) from left to right, looking for the first cell c for which either

- (i) cell c is labeled with a y , or
- (ii) cell c is at the end of brick b_i where $\sigma_c > \sigma_{c+1}$ and there is no Γ -match of σ that lies entirely in the cells of the bricks b_i and b_{i+1} .

In case (i), we define $I_\Gamma(B, \sigma)$ to be the filled labeled brick tabloid obtained from (B, σ) by breaking the brick b_j that contains cell c into two bricks b'_j and b''_j where b'_j contains the cells of b_j up to and including the cell c while b''_j contains the remaining cells of b_j . In addition, we change the label of cell c from y to $-y$. In case (ii), $I_\Gamma(B, \sigma)$ is obtained by combining the two bricks b_i and b_{i+1} into a single brick b and changing the label of cell c from $-y$ to y . If neither case occurs, then we let $I_\Gamma(B, \sigma) = (B, \sigma)$.

For instance, the image of the filled labeled brick tabloid from the Figure 2.1 under this involution is shown below in Figure 2.2.

			$-y$	y			y	$-y$		y			y	$-y$	y	$-y$		
2	5	9	15	11	6	16	19	17	7	14	8	18	1	3	13	10	12	4

Figure 2.2: $I_\Gamma(O)$ for O in Figure 2.1.

We claim that as long as each permutation in Γ has at most one descent, then I_Γ is an involution. Let (B, σ) be an element of $\mathcal{O}_{\gamma, n}$ which is not a fixed point of I . Suppose that $I(B, \sigma)$ is defined using case (i) where we split a brick b_j at cell c which is labeled with a y . In that case, we let a be the number in cell c and a' be the number in cell $c + 1$ which must also be in brick b_j . Since cell c is labeled with y , it must be the case that $a > a'$. Moreover, there can be no cell labeled y that occurs before cell c since otherwise we would not use cell c to define $I(B, \sigma)$. In this case, we must ensure that when we split b_j into b'_j and b''_j , we cannot combine the brick b_{j-1} with b'_j because the number in that last cell of b_{j-1} is greater than the number in the first cell of b'_j and there is no Γ -match in the cells of b_{j-1} and b'_j since in such a situation, $I_\Gamma(I_\Gamma(B, \sigma)) \neq (B, \sigma)$. However, since we always take an action on the leftmost cell possible when defining $I_\Gamma(B, \sigma)$, we know that we cannot combine b_{j-1} and b_j so that there must be a Γ -match in the cells of b_{j-1} and b_j . Moreover, if we could now combine bricks b_{j-1} and b'_j , then that Γ -match must have involved the number a' and the number in cell d which is the last cell in brick b_{j-1} . But that is impossible because then there would be two descents among the numbers between cell d and cell $c + 1$ which would violate our assumption that the elements of Γ have at most one descent. Thus whenever we apply case (i) to define $I_\Gamma(B, \sigma)$, the first action that we can take is to combine bricks b'_j and b''_j so that $I_\Gamma^2(B, \sigma) = (B, \sigma)$.

If we are in case (ii), then again we can assume that there are no cells labeled y that occur before cell c . When we combine brick b_i and b_{i+1} , then we will label cell c with a y . It is clear that combining the cells of b_i and b_{i+1} cannot help us combine

the resulting brick b with b_{j-1} since, if there were a Γ -match that prevented us from combining bricks b_{j-1} and b_j , then that same Γ -match will prevent us from combining b_{j-1} and b . Thus, the first place where we can apply the involution will again be cell c which is now labeled with a y so that $I_\Gamma^2(B, \sigma) = (B, \sigma)$.

It is clear that if $I_\Gamma(B, \sigma) \neq (B, \sigma)$, then

$$\text{sgn}(B, \sigma)W(B, \sigma) = -\text{sgn}(I_\Gamma(B, \sigma))W(I_\Gamma(B, \sigma)).$$

Thus it follows from (2.4) that

$$n!\theta_\Gamma(h_n) = \sum_{O \in \mathcal{O}_{\Gamma, n}} \text{sgn}(O)W(O) = \sum_{O \in \mathcal{O}_{\Gamma, n}, I_\Gamma(O)=O} \text{sgn}(O)W(O).$$

Hence if all permutations in Γ have at most one descent, then

$$U_{\Gamma, n}(y) = \sum_{O \in \mathcal{O}_{\Gamma, n}, I_\Gamma(O)=O} \text{sgn}(O)W(O). \quad (2.5)$$

Thus to compute $U_{\Gamma, n}(y)$, we must analyze the fixed points of I_Γ .

If (B, σ) where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$ is a fixed point of the involution I_Γ , then (B, σ) cannot have any cell labeled y which means that the elements of σ that lie within any brick b_j of B must be increasing. If it is the case that an identity permutation $12 \cdots (k+1)$ is in Γ , then no brick of B can have length greater than k . Next, consider any two consecutive bricks b_i and b_{i+1} in B . Let c be the last cell of b_i and $c+1$ be the first cell of b_{i+1} . Then either $\sigma_c < \sigma_{c+1}$ in which case we say there is an increase between bricks b_i and b_{i+1} , or $\sigma_c > \sigma_{c+1}$ in which case we say there is a decrease between bricks b_i and b_{i+1} . In the latter case, there must

be a Γ -match of σ that lies in the cells of b_i and b_{i+1} which must necessarily involve σ_c and σ_{c+1} . Finally, we claim that since all the permutations in Γ start with 1, the minimal elements within the bricks of B must increase from left to right. That is, consider two consecutive bricks b_i and b_{i+1} and let c_i and c_{i+1} be the first cells of b_i and b_{i+1} , respectively. Suppose that $\sigma_{c_i} > \sigma_{c_{i+1}}$. Let d_i be the last cell of b_i . Then clearly $\sigma_{c_{i+1}} < \sigma_{c_i} \leq \sigma_{d_i}$ so that there is a decrease between brick b_i and brick b_{i+1} and hence there must be a Γ -match of σ that lies in the cells of b_i and b_{i+1} that involves the elements of σ_{d_i} and $\sigma_{c_{i+1}}$. But this is impossible since our assumptions ensure that $\sigma_{c_{i+1}}$ is the smallest element that lies in the bricks b_i and b_{i+1} so that it can only play the role of 1 in any Γ -match. But since every element of Γ starts with 1, then any Γ -match that lies in b_i and b_{i+1} that involves $\sigma_{c_{i+1}}$ must lie entirely in brick b_{i+1} which contradicts the fact that (B, σ) was a fixed point of I_Γ .

We have the following lemma describing the fixed points of the involution I_Γ .

Lemma 1. *Let Γ be a set of permutations which all start with 1 and have at most one descent. Let $\mathbb{Q}(y)$ be the set of rational functions in the variable y over the rationals \mathbb{Q} and let $\theta_\Gamma : \Lambda \rightarrow \mathbb{Q}(y)$ be the ring homomorphism defined by setting $\theta_\Gamma(e_0) = 1$, and $\theta_\Gamma(e_n) = \frac{(-1)^n}{n!} NM_{\Gamma,n}(1, y)$ for $n \geq 1$. Then*

$$n! \theta_\Gamma(h_n) = \sum_{O \in \mathcal{O}_{\Gamma,n}, I_\Gamma(O)=O} \text{sgn}(O) W(O)$$

where $\mathcal{O}_{\Gamma,n}$ is the set of objects and I_Γ is the involution defined above. Moreover, $O = (B, \sigma) \in \mathcal{O}_{\Gamma,n}$ where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$ is a fixed point of I_Γ if and only if O satisfies the following four properties:

1. there are no cells labeled with y in O , i.e., the elements in each brick of O are

increasing,

2. the first elements in each brick of O form an increasing sequence, reading from left to right,
3. if b_i and b_{i+1} are two consecutive bricks in B , then either (a) there is increase between b_i and b_{i+1} , i.e., $\sigma_{\sum_{j=1}^i b_j} < \sigma_{1+\sum_{j=1}^i b_j}$, or (b) there is a decrease between b_i and b_{i+1} , i.e., $\sigma_{\sum_{j=1}^i b_j} > \sigma_{1+\sum_{j=1}^i b_j}$, and there is a Γ -match contained in the elements of the cells of b_i and b_{i+1} which must necessarily involve $\sigma_{\sum_{j=1}^i b_j}$ and $\sigma_{1+\sum_{j=1}^i b_j}$, and
4. if Γ contains an identity permutation $12 \cdots (k+1)$, then $b_i \leq k$ for all i .

Note that since $U_{\Gamma,n}(y) = n! \theta_{\Gamma}(h_n)$, Lemma 1 gives us a combinatorial interpretation of $U_{\Gamma,n}(y)$. Since the weight of any fixed point (B, σ) of I_{Γ} is $-y$ raised to the number of bricks in B , it follows that $U_{\Gamma,n}(-y)$ is always a polynomial with non-negative integer coefficients.

2.3 Results of the reciprocity method

Having described the reciprocity methods, we now consider several results that arise by setting specific values to the family Γ .

2.3.1 The case $\Gamma = \Gamma_{k_1, k_2}$

Let $k_1, k_2 \geq 2$ and $p = k_1 + k_2$. We consider the family of permutations $\Gamma = \Gamma_{k_1, k_2}$ in S_p where

$$\Gamma_{k_1, k_2} = \{\sigma \in S_p : \sigma_1 = 1, \sigma_{k_1+1} = 2, \sigma_1 < \sigma_2 < \cdots < \sigma_{k_1}, \sigma_{k_1+1} < \sigma_{k_1+2} < \cdots < \sigma_p\}.$$

We then have the following result.

Theorem 1. Let $\Gamma = \Gamma_{k_1, k_2}$ where $k_1, k_2 \geq 2$, $m = \min\{k_1, k_2\}$, and $M = \max\{k_1, k_2\}$.

Then

$$NM_\Gamma(t, x, y) = \left(\frac{1}{U_\Gamma(t, y)} \right)^x \text{ where } U_\Gamma(t, y) = 1 + \sum_{n \geq 1} U_{\Gamma, n}(y) \frac{t^n}{n!},$$

$$U_{\Gamma, 1}(y) = -y, \text{ and for } n \geq 2,$$

$$U_{\Gamma, n}(y) = (1 - y)U_{\Gamma, n-1}(y) - y \binom{n-2}{k_1-1} \left(U_{\Gamma, n-M}(y) + y \sum_{i=1}^{m-1} U_{\Gamma, n-M-i}(y) \right).$$

Proof. By (2.5), we must show that the coefficients

$$U_{\Gamma, n}(y) = \sum_{O \in \mathcal{O}_{\Gamma, n}, I_\Gamma(O) = O} \text{sgn}(O) W(O)$$

have the following properties:

1. $U_{\Gamma, 1}(y) = -y$, and
2. $U_{\Gamma, n}(y) = (1 - y)U_{\Gamma, n-1}(y) - y \binom{n-2}{k_1-1} \left(U_{\Gamma, n-M}(y) + y \sum_{i=1}^{m-1} U_{\Gamma, n-M-i}(y) \right)$, where $m = \min\{k_1, k_2\}$ and $M = \max\{k_1, k_2\}$, for $n > 1$,

We will divide the proof into two cases, one where $k_1 \geq k_2$ and the other where $k_1 < k_2$.

Case 1. $k_1 \geq k_2$.

Let (B, σ) be a fixed point of I_Γ where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$. We know that 1 is in the first cell of (B, σ) . We claim that 2 must be in cell 2 or cell $k_1 + 1$ of (B, σ) . To see this, suppose that 2 is in cell c where $c \neq 2$ and $c \neq k_1 + 1$. Since there is no descent within any brick, 2 must be the first cell of its brick. Moreover,

since the minimal elements of the bricks form an increasing sequence, reading from left to right, 2 must be in the first cell of the second brick b_2 . Thus, 1 is in the first cell of the first brick b_1 and 2 is in the first cell of the second brick b_2 . Since $c > 2$, there is a decrease between bricks b_1 and b_2 and, hence, there must be a Γ -match of σ contained cells of b_1 and b_2 which involves 2 and the last cell of b_1 . Since all the elements of Γ start with 1, this Γ -match must also involve 1 since only 1 can play the role of 1 in a Γ -match that involves 2 and the last cell of b_1 . But in all such Γ -matches, 2 must be in cell $k_1 + 1$. Since $c \neq k_1 + 1$, this means that there can be no Γ -match contained in the cells of b_1 and b_2 which contradicts the fact that (B, σ) is a fixed point of I_Γ .

Thus, we have two subcases.

Subcase 1.A. 2 is in cell 2 of (B, σ) .

In this case there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of (B, σ) or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of (B, σ) . In either case, we know that 1 is not part of a Γ -match in (B, σ) . So if we remove cell 1 from (B, σ) and subtract 1 from the elements in the remaining cells, we will obtain a fixed point O' of I_Γ in $\mathcal{O}_{\Gamma, n-1}$.

Moreover, we can create a fixed point $O = (B, \sigma) \in \mathcal{O}_n$ satisfying conditions (1), (2), (3) and (4) of Lemma 1 where $\sigma_2 = 2$ by starting with a fixed point $(B', \sigma') \in \mathcal{O}_{\Gamma, n-1}$ of I_Γ , where $B' = (b'_1, \dots, b'_r)$ and $\sigma' = \sigma'_1 \cdots \sigma'_{n-1}$, and then letting $\sigma = 1(\sigma'_1 + 1) \cdots (\sigma'_{n-1} + 1)$, and setting $B = (1, b'_1, \dots, b'_r)$ or setting $B = (1 + b'_1, \dots, b'_r)$.

It follows that fixed points in Subcase 1.A will contribute $(1 - y)U_{\Gamma, n-1}(y)$ to $U_{\Gamma, n}(y)$.

Subcase 1.B. 2 is in cell $k_1 + 1$ of (B, σ) .

Since there is no decrease within the bricks of (B, σ) and the first numbers of

the bricks are increasing, reading from left to right, it must be the case that 2 is in the first cell of b_2 . Thus b_1 has exactly k_1 cells. In addition, b_2 has at least k_2 cells since otherwise, there could be no Γ -match contained in the cells of b_1 and b_2 and we could combine the bricks b_1 and b_2 , which would mean that (B, σ) is not a fixed point of I_Γ . By our argument above, it must be the case that the Γ -match of σ contained in the cells of b_1 and b_2 must start in the first cell. We first choose $k_1 - 1$ numbers to fill in the remaining cells of b_1 . There are $\binom{n-2}{k_1-1}$ ways to do this. For each such choice, we let O' be the result by removing the first k_1 cells from (B, σ) and replacing the i^{th} largest remaining number by i for $i = 1, \dots, n - k_1$, then O' will be a fixed point in $\mathcal{O}_{\Gamma, n-k_1}$ whose first brick is of size greater than or equal to k_2 .

On the other hand, suppose that we start with $O' \in \mathcal{O}_{\Gamma, n-k_1}$ which is a fixed point of I_Γ and whose first brick is of size greater than or equal to k_2 . Then we can take any $k_1 - 1$ numbers $1 < a_1 < a_2 < \dots < a_{k_1-1} \leq n$ and add a new brick at the start which contains $1, a_1, \dots, a_{k_1-1}$ followed by O'' which is the result of replacing the numbers in O' by the numbers in $\{1, \dots, n\} - \{1, a_1, \dots, a_{k_1-1}\}$ maintaining the same relative order, then we will create a fixed point O of I_Γ of size n whose first brick is of size k_1 and whose second brick starts with 2.

Thus we need to count the number of fixed points in $\mathcal{O}_{\Gamma, n-k_1}$ whose first brick has size at least k_2 . Suppose that $V = (D, \tau)$ is a fixed point of $\mathcal{O}_{\Gamma, n-k_1}$ where $D = (d_1, \dots, d_k)$ and $\tau = \tau_1 \dots \tau_{n-k_1}$. Now if $d_1 = j < k_2$, then there cannot be a decrease between bricks d_1 and d_2 because otherwise there would have been a Γ -match starting at cell 1 contained in the bricks d_1 and d_2 which is impossible since all permutations in Γ have their only descent at position $k_1 > j$. This means that the first brick d_1 must be filled with $1, \dots, j$. That is, since the minimal elements of the bricks

are increasing reading from left to right, we must have that the first element of d_2 , namely τ_{j+1} , is less than all the elements to its right and we have shown that all the elements in the first brick are less than τ_{j+1} . It follows that $\tau_1 \cdots \tau_{j+1} = 12 \cdots j(j+1)$. Therefore, if we let V' be the result of removing the entire first brick of V and subtracting j from the remaining numbers, then V' is a fixed point in $\mathcal{O}_{\Gamma, n-k_1-j}$.

It follows that

$$U_{\Gamma, n-k_1}(y) - \sum_{j=1}^{k_2-1} (-y) U_{\Gamma, n-k_1-j}(y)$$

equals the sum over all fixed points of $I_{\Gamma, n-k_1}$ whose first brick has size at least k_2 . Hence the contribution of fixed points in Subcase 1.B to $U_{\Gamma, n}(y)$ is

$$(-y) \binom{n-2}{k_1-1} \left(U_{\Gamma, n-k_1}(y) + \sum_{j=1}^{k_2-1} y U_{\Gamma, n-k_1-j}(y) \right).$$

Combining the two cases, we see that for $n > 1$,

$$U_{\Gamma, n}(y) = (1-y) U_{\Gamma, n-1}(y) - y \binom{n-2}{k_1-1} \left(U_{\Gamma, n-k_1}(y) + y \sum_{j=1}^{k_2-1} U_{\Gamma, n-k_1-j}(y) \right). \quad (2.6)$$

Case 2. $k_1 < k_2$.

Let $O = (B, \sigma)$ be a fixed point of I_{Γ} where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$. We know that 1 is in the first cell of O . By the same argument as in Case I, we know that 2 must be in cell 2 or cell $k_1 + 1$ of O . We now consider two subcases depending on the position of 2 in O .

Subcase 2.A. 2 is in cell 2 of (B, σ) .

By the same argument that we used in Subcase 1.A of Case 1, we can conclude

that the fixed points of I_Γ in Subcase 2.A will contribute $(1-y)U_{\Gamma,n-1}(y)$ to $U_{\Gamma,n}(y)$.

Subcase 2.B. 2 is in cell $k_1 + 1$ of (B, σ) .

Since the minimal elements of the bricks are increasing, reading from left to right, it must be the case that 2 is in the first cell of b_2 . Thus, b_1 has exactly k_1 cells, b_2 has at least k_2 cells, and there is a Γ_{k_1,k_2} -match in the cells of b_1 and b_2 which must start at cell 1.

We first choose $k_1 - 1$ numbers to fill in the remaining cells of b_1 . There are $\binom{n-2}{k_1-1}$ ways to do this. For each of such choice, let $d_1 < \dots < d_{k_2-k_1-1}$ be the smallest $k_2 - k_1 - 1$ numbers in $\{1, 2, \dots, n\} - \{\sigma_1, \dots, \sigma_{k_1+1}\}$. We claim that it must be the case that $\sigma_{k_1+1+i} = d_i$ for $i = 1, \dots, k_2 - k_1 - 1$. If not, let j be the least i such that $\sigma_{k_1+1+i} \neq d_i$. Then d_i cannot be in brick b_2 so that it must be the first element in brick b_3 . But then there will be a decrease between bricks b_2 and b_3 which means that there must be a Γ_{k_1,k_2} -match contained in the cells of b_2 and b_3 . Note that there is only one descent in each permutation of Γ_{k_1,k_2} and this descent must occur at position k_1 . It follows that this Γ_{k_1,k_2} -match must start at the $(k_2 - k_1)^{th}$ cell of b_2 . But this is impossible since our assumption will ensure that $\sigma_{k_1+1+(k_2-k_1-1)} = \sigma_{k_2} > d_i$.

It then follows that if we let O' be the result by removing the first k_2 cells from O and adjusting the remaining numbers in the cells, then O' will be a fixed point in $\mathcal{O}_{\Gamma,n-k_2}$ that starts with at least k_1 cells in the first brick. Then we can argue exactly as we did in Subcase 1.B to show that the contribution of fixed points in Subcase 2.B to $U_{\Gamma,n}(y)$ is

$$-y \binom{n-2}{k_1-1} \left(U_{\Gamma,n-k_2}(y) + \sum_{j=1}^{k_1-1} y U_{\Gamma,n-k_2-j}(y) \right).$$

It follows that in Case 2,

$$U_{\Gamma_{k_1, k_2}, n}(y) = (1 - y)U_{\Gamma_{k_1, k_2}, n}(y) - y \binom{n-2}{k_1-1} \left(U_{\Gamma, n-k_2}(y) + \sum_{j=1}^{k_1-1} y U_{\Gamma, n-k_2-j}(y) \right) \quad (2.7)$$

for $n > 1$.

Comparing equations (2.6) and (2.7), it is easy to see that if $m = \min(k_1, k_2)$ and $M = \max(k_1, k_2)$, then

$$U_{\Gamma_{k_1, k_2}, n}(y) = (1 - y)U_{\Gamma_{k_1, k_2}, n-1}(y) - y \binom{n-2}{k_1-1} \left(U_{\Gamma, n-M}(y) + y \sum_{i=1}^{m-1} U_{\Gamma, n-M-i}(y) \right)$$

for all $n > 1$ which proves Theorem 1. □

When $k_1 = k_2 = 2$, Theorem 1 gives us the following corollary.

Corollary 2. *For $\Gamma = \{1324, 1423\}$, then*

$$NM_{\Gamma}(t, x, y) = \left(\frac{1}{U_{\Gamma}(t, y)} \right)^x \text{ where } U_{\Gamma}(t, y) = 1 + \sum_{n \geq 1} U_{\Gamma, n}(y) \frac{t^n}{n!},$$

$U_{\Gamma, 1}(y) = -y$, and for $n \geq 2$,

$$U_{\Gamma, n}(y) = (1 - y)U_{\Gamma, n-1}(y) - y(n-2) (U_{\Gamma, n-2}(y) + yU_{\Gamma, n-3}(y)).$$

Thus, by Corollary 2,

$$U_{\Gamma_{2,2}, n}(y) = (1 - y)U_{\Gamma_{2,2}, n-1}(y) - y(n-2) (U_{\Gamma_{2,2}, n-2}(y) + yU_{\Gamma_{2,2}, n-3}(y)).$$

In Table 2.1, we computed $U_{\Gamma_{2,2}, n}(y)$ for $n \leq 14$. We observe that the poly-

nomials $U_{\Gamma_{2,2},n}(-y)$ in Table 2.1 are all log-concave. Here, a polynomial $P(y) = a_0 + a_1y + \dots + a_ny^n$ is called *log-concave* if $a_{i-1}a_{i+1} < a_i^2$, for all $i = 2, \dots, n-1$, and it is called *unimodal* if there exists an index k such that $a_i \leq a_{i+1}$ for $1 \leq i \leq k-1$ and $a_i \geq a_{i+1}$ for $k \leq i \leq n-1$. We conjecture that the polynomials $U_{\Gamma_{2,2},n}(-y)$ are log-concave, and hence, unimodal for all n . We checked this holds for $n \leq 21$.

Table 2.1: The polynomials $U_{\Gamma_{2,2},n}(-y)$ for $\Gamma_{2,2} = \{1324, 1423\}$

n	$U_{\Gamma_{2,2},n}(-y)$
1	y
2	$y + y^2$
3	$y + 2y^2 + y^3$
4	$y + 5y^2 + 3y^3 + y^4$
5	$y + 9y^2 + 11y^3 + 4y^4 + y^5$
6	$y + 14y^2 + 36y^3 + 19y^4 + 5y^5 + y^6$
7	$y + 20y^2 + 90y^3 + 85y^4 + 29y^5 + 6y^6 + y^7$
8	$y + 27y^2 + 188y^3 + 337y^4 + 162y^5 + 41y^6 + 7y^7 + y^8$
9	$y + 35y^2 + 348y^3 + 1057y^4 + 842y^5 + 273y^6 + 55y^7 + 8y^8 + y^9$
10	$y + 44y^2 + 591y^3 + 2749y^4 + 3875y^5 + 1731y^6 + 424y^7 + 71y^8 + 9y^9 + y^{10}$
11	$y + 54y^2 + 941y^3 + 6229y^4 + 14445y^5 + 10151y^6 + 3154y^7 + 621y^8$ $+ 89y^9 + 10y^{10} + y^{11}$
12	$y + 65y^2 + 1425y^3 + 12730y^4 + 44684y^5 + 52776y^6 + 22195y^7 + 5285y^8$ $+ 870y^9 + 109y^{10} + 11y^{11} + y^{12}$
13	$y + 77y^2 + 2073y^3 + 24022y^4 + 119432y^5 + 226116y^6 + 144007y^7 + 43133y^8$ $+ 8322y^9 + 1177y^{10} + 131y^{11} + 12y^{12} + y^{13}$
14	$y + 90y^2 + 2918y^3 + 42547y^4 + 284922y^5 + 807008y^6 + 830095y^7 + 331668y^8$ $+ 77027y^9 + 12487y^{10} + 1548y^{11} + 155y^{12} + 13y^{13} + y^{14}$

One might hope to prove the unimodality of the polynomials $U_{\Gamma_{2,2},n}(-y)$ by using the recursion

$$U_{\Gamma_{2,2},n}(-y) = (1+y)U_{\Gamma_{2,2},n-1}(-y) + (n-2)yU_{\Gamma_{2,2},n-2}(-y) + (n-2)y^2U_{\Gamma_{2,2},n-3}(-y) \quad (2.8)$$

and showing that for large enough n , the polynomials on the right hand side of (2.8) are all unimodal polynomials whose maximum coefficients occur at the same power

of y . There are two problems with this idea. First, assuming that $U_{\Gamma_{2,2},n}(-y)$ is a unimodal polynomial whose maximum coefficient occurs at y^j , then we know that $(1+y)U_{\Gamma_{2,2},n}(-y)$ is a unimodal polynomial. However, it could be that the maximum coefficient of $(1+y)U_{\Gamma_{2,2},n}(-y)$ occurs at y^j or at y^{j+1} . That is, if $P(y)$ is a unimodal polynomial whose maximum coefficient occurs at y^k , then $(1+y)P(y)$ could have its maximum coefficient occur at either y^k or y^{k+1} . For example,

$$(1+y)(1+5y+2y^2) = 1+6y+7y^2+2y^3$$

while

$$(1+y)(2+5y+y^2) = 2+7y+6y^2+y^3.$$

Thus where the maximum coefficient of $(1+y)U_{\Gamma_{2,2},n}(-y)$ occurs depends on the relative values of the coefficients on either side of the maximum coefficient of $U_{\Gamma_{2,2},n}(-y)$. For $n \leq 20$, the maximum coefficient of $(1+y)U_{\Gamma_{2,2},n}(-y)$ occurs at the same power of y where the maximum coefficient of $U_{\Gamma_{2,2},n}(-y)$ occurs, but it is not obvious that this holds for all n .

Second, it is not clear where to conjecture the maximum coefficients in the polynomials occur. That is, one might think from the table that for $n \geq 6$, the maximum coefficient in $U_{\Gamma_{2,2},n}(-y)$ occurs at $y^{\lfloor n/2 \rfloor + 1}$, but this does not hold up. For example, the maximum coefficient $U_{\Gamma_{2,2},18}(-y)$ occurs at y^8 and the maximum coefficient $U_{\Gamma_{2,2},19}(-y)$ occurs at y^9 . Moreover, the maximum coefficient $U_{\Gamma_{2,2},26}(-y)$ occurs at y^{12} and the maximum coefficient $U_{\Gamma_{2,2},27}(-y)$ occurs at y^{12} . Thus it is not clear how to use the recursion (2.8) to even prove the unimodality of the polynomials $U_{\Gamma_{2,2},n}(-y)$ much less prove that such polynomials are log concave.

When k_1 is larger than k_2 , the polynomials $U_{\Gamma_{k_1, k_2, n}}(-y)$ are not always unimodal. For example, consider the case where $k_1 = 6$ and $k_2 = 4$. Mathematica once again allows us to compute $U_{\Gamma_{6, 4, n}}(-y)$ for $n = 10$ and 11. It is quite easy to see from Table 2.2 that neither polynomial is unimodal.

Table 2.2: The polynomials $U_{\Gamma_{6, 4, n}}(-y)$

n	$U_{\Gamma_{6, 4, n}}(-y)$
10	$y + 65y^2 + 36y^3 + 84y^4 + 126y^5 + 126y^6 + 84y^7 + 36y^8 + 9y^9 + y^{10}$
11	$y + 192y^2 + 227y^3 + 120y^4 + 210y^5 + 252y^6 + 210y^7 + 120y^8 + 45y^9 + 10y^{10} + y^{11}$

2.3.2 Adding an identity permutation to Γ_{k_1, k_2}

In this subsection, we want to consider the effect of adding an identity permutation to Γ_{k_1, k_2} . To simplify our analysis, we shall consider only the case where $k_1 = k_2$, but the same type of analysis can be carried out in general. Thus, assume that $s \geq k_1 = k_2 \geq 2$ and let $\Gamma_{k_1, k_1, s} = \Gamma_{k_1, k_1} \cup \{12 \cdots s(s+1)\}$. Then we know that

$$U_{\Gamma_{k_1, k_1, s, n}}(y) = \sum_{O \in \mathcal{O}_{\Gamma_{k_1, k_1, s, n}}, I_{\Gamma_{k_1, k_1, s}}(O) = O} \text{sgn}(O)W(O).$$

We want to classify the fixed points of $I_{\Gamma_{k_1, k_1, s}}$ by the size of the first brick. By Lemma 1, it must be the case that the size of the first brick is less than or equal to s . We let $U_{\Gamma_{k_1, k_1, s, n}}^{(r)}(y)$ denote the sum of $\text{sgn}(O)W(O)$ over all fixed points of $I_{\Gamma_{k_1, k_1, s}}$ whose first brick is of size r . Thus,

$$U_{\Gamma_{k_1, k_1, s, n}}(y) = \sum_{r=1}^s U_{\Gamma_{k_1, k_1, s, n}}^{(r)}(y). \quad (2.9)$$

Now let $O = (B, \sigma)$ be a fixed point of $I_{\Gamma_{k_1, k_1, s}}$ where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$.

By our arguments above, if $b_1 < k_1$, then the elements in the first brick of (B, σ) are

$1, \dots, b_1$ so that for $1 \leq r < k_1$,

$$U_{\Gamma_{k_1, k_1, s, n}}^{(r)}(y) = -y U_{\Gamma_{k_1, k_1, s, n-r}}(y). \quad (2.10)$$

Let

$$U_{\Gamma_{k_1, k_1, s, n}}^{(\geq k_1)}(y) = \sum_{r=k_1}^s U_{\Gamma_{k_1, k_1, s, n}}^{(r)}(y)$$

be the sum of $\text{sgn}(O)W(O)$ over all fixed points of $I_{\Gamma_{k_1, k_1, s}}$ whose first brick has size greater than or equal to k_1 . Clearly,

$$\begin{aligned} U_{\Gamma_{k_1, k_1, s, n}}(y) &= U_{\Gamma_{k_1, k_1, s, n}}^{(\geq k_1)}(y) + \sum_{r=1}^{k_1-1} U_{\Gamma_{k_1, k_1, s, n}}^{(r)}(y) \\ &= U_{\Gamma_{k_1, k_1, s, n}}^{(\geq k_1)}(y) + \sum_{r=1}^{k_1-1} (-y) U_{\Gamma_{k_1, k_1, s, n-r}}(y) \end{aligned}$$

so that

$$U_{\Gamma_{k_1, k_1, s, n}}^{(\geq k_1)}(y) = U_{\Gamma_{k_1, k_1, s, n}}(y) + \sum_{r=1}^{k_1-1} y U_{\Gamma_{k_1, k_1, s, n-r}}(y). \quad (2.11)$$

Now suppose that $r > k_1$. Then we claim that $\sigma_i = i$ for $i = 1, \dots, r - k_1 + 1$.

That is, we know that $\sigma_1 = 1$ so that if it is not the case that $\sigma_i = i$ for $i = 1, \dots, r - k_1 + 1$, there must be a least $i \leq r - k_1 + 1$ which is not in the first brick of (B, σ) . Since there are no descents of σ within bricks and the minimal elements of the bricks of (B, σ) are increasing, reading from left to right, it must be that i is the first element of brick b_2 and there is a decrease between bricks b_1 and b_2 . Thus there is a $\Gamma_{k_1, k_1, s}$ -match that lies in the cells of b_1 and b_2 and the only place that

such a match can start is at cell $r - k_1 + 1$. But this is impossible since we would have $\sigma_{r-k_1+1} > i$ which is incompatible with having a $\Gamma_{k_1, k_1, s}$ -match starting at cell $r - k_1 + 1$. It follows that we can remove the first $r - k_1$ elements from (B, σ) and reduce the remaining elements by $r - k_1$ to produce a fixed point of $I_{\Gamma_{k_1, k_1, s}}$ of size $n - (r - k_1)$ whose first brick has size k_1 . Vice versa, if we start with a fixed point (D, τ) of $I_{\Gamma_{k_1, k_1, s}}$ of size $n - (r - k_1)$ where $D = (d_1, \dots, d_k)$, $\tau = \tau_1 \cdots \tau_{n-(r-k_1)}$, and $d_1 = k_1$, then if we add $1, \dots, r - k_1$ to the first brick and raise the remaining numbers by $r - k_1$, we will produce a fixed point of $I_{\Gamma_{k_1, k_1, s}}$ whose first brick is of size r . It follows that for $k_1 < r \leq s$,

$$U_{\Gamma_{k_1, k_1, s}, n}^{(r)}(y) = U_{\Gamma_{k_1, k_1, s}, n-(r-k_1)}^{(k_1)}(y). \quad (2.12)$$

Thus

$$U_{\Gamma_{k_1, k_1, s}, n}^{(\geq k_1)}(y) = \sum_{p=0}^{s-k_1} U_{\Gamma_{k_1, k_1, s}, n-p}^{(k_1)}(y). \quad (2.13)$$

Finally consider $U_{\Gamma_{k_1, k_1, s}, n}^{(k_1)}(y)$. Let (B, σ) be a fixed point of $I_{\Gamma_{k_1, k_1, s}}$ where $B = (b_1, \dots, b_k)$, $b_1 = k_1$, and $\sigma = \sigma_1 \cdots \sigma_n$. We then have two cases.

Case 1. 2 is in brick b_1 .

In this case, we claim that the first brick must contain the elements $1, \dots, k_1$. That is, in such a situation 1 cannot be involved in a $\Gamma_{k_1, k_1, s}$ -match in σ which means that there is not enough room for a $\Gamma_{k_1, k_1, s}$ -match that involves any elements from the first brick. Thus as before, we can remove the first brick from (B, σ) and subtract k_1 from the remaining elements of σ to produce a fixed point (D, τ) of $I_{\Gamma_{k_1, k_1, s}}$ of size $n - k_1$. Such fixed points contribute $(-y)U_{\Gamma_{k_1, k_1, s}, n-k_1}(y)$ to $U_{\Gamma_{k_1, k_1, s}, n}^{(k_1)}(y)$.

Case 2. 2 is in brick b_2 .

In this case, we can argue as above that 2 be the first cell of the second brick b_2 and b_2 starts at cell $k_1 + 1$. Then we have $\binom{n-2}{k_1-1}$ ways to choose the remaining elements in the first brick and if we remove the first brick and adjust the remaining elements, we will produce a fixed point (D, τ) of $I_{\Gamma_{k_1, k_1, s}}$ of size $n - k_1$ whose first brick is of size greater than or equal to k_1 . Such fixed points contribute $(-y)\binom{n-2}{k_1-1}U_{\Gamma_{k_1, k_1, s}, n-k_1}^{(\geq k_1)}(y)$ to $U_{\Gamma_{k_1, k_1, s}, n}^{(k_1)}(y)$.

It follows that

$$\begin{aligned} U_{\Gamma_{k_1, k_1, s}, n}^{(k_1)}(y) &= -yU_{\Gamma_{k_1, k_1, s}, n-k_1}(y) - y\binom{n-2}{k_1-1}U_{\Gamma_{k_1, k_1, s}, n-k_1}^{(\geq k_1)}(y) \\ &= -yU_{\Gamma_{k_1, k_1, s}, n-k_1}(y) - \\ &\quad y\binom{n-2}{k_1-1}\left(U_{\Gamma_{k_1, k_1, s}, n-k_1}(y) + y\sum_{r=1}^{k_1-1}U_{\Gamma_{k_1, k_1, s}, n-k_1-r}(y)\right) \end{aligned} \quad (2.14)$$

Putting equations (2.9), (2.10), (2.11), (2.12), (2.13), and (2.14) together, we see that

$$\begin{aligned} &U_{\Gamma_{k_1, k_1, s}, n}(y) \\ &= -y\sum_{r=1}^{k_1-1}U_{\Gamma_{k_1, k_1, s}, n-r}(y) + \sum_{p=0}^{s-k_1}U_{\Gamma_{k_1, k_1, s}, n-p}^{(k_1)}(y) \\ &= -y\sum_{r=1}^{k_1-1}U_{\Gamma_{k_1, k_1, s}, n-r}(y) - y\sum_{p=0}^{s-k_1}U_{\Gamma_{k_1, k_1, s}, n-p-k_1}(y) \\ &\quad + \binom{n-p-2}{k_1-1}\left(U_{\Gamma_{k_1, k_1, s}, n-p-k_1}(y) + y\sum_{a=1}^{k_1-1}U_{\Gamma_{k_1, k_1, s}, n-p-k_1-a}(y)\right) \\ &= -y\sum_{r=1}^{k_1-1}U_{\Gamma_{k_1, k_1, s}, n-r}(y) - y\left(\sum_{p=0}^{s-k_1}\left(1 + \binom{n-p-2}{k_1-1}\right)U_{\Gamma_{k_1, k_1, s}, n-p-k_1}(y) \right. \\ &\quad \left. + y\binom{n-p-2}{k_1-1}\sum_{a=1}^{k_1-1}U_{\Gamma_{k_1, k_1, s}, n-p-k_1-a}(y)\right). \end{aligned}$$

Thus we have the following theorem.

Theorem 3. Let $\Gamma_{k_1, k_1, s} = \Gamma_{k_1, k_1} \cup \{12 \cdots s(s+1)\}$ where $s \geq k_1$. Then $U_{\Gamma_{k_1, k_1, s}, 1}(y) = -y$ and for $n \geq 2$,

$$\begin{aligned} U_{\Gamma_{k_1, k_1, s}, n}(y) &= -y \sum_{r=1}^{k_1-1} U_{\Gamma_{k_1, k_1, s}, n-r}(y) \\ &\quad - y \left(\sum_{p=0}^{s-k_1} \left(1 + \binom{n-p-2}{k_1-1} \right) U_{\Gamma_{k_1, k_1, s}, n-p-k_1}(y) \right. \\ &\quad \left. + y \binom{n-p-2}{k_1-1} \sum_{a=1}^{k_1-1} U_{\Gamma_{k_1, k_1, s}, n-p-k_1-a}(y) \right). \end{aligned}$$

For example, if $k_1 = 2$, then

$$\begin{aligned} U_{\Gamma_{2,2,s}, n}(y) &= -y U_{\Gamma_{2,2,s}, n-1}(y) \\ &\quad - y \left(\sum_{p=0}^{s-2} (n-p-1) U_{\Gamma_{2,2,s}, n-2-p}(y) + (n-p-2) y U_{\Gamma_{2,2,s}, n-3-p}(y) \right). \end{aligned}$$

We shall further explore two special cases, namely, $k_1 = k_2 = s = 2$ where the recursion becomes

$$U_{\Gamma_{2,2,2}, n}(y) = -y U_{\Gamma_{2,2,2}, n-1}(y) - y(n-1) U_{\Gamma_{2,2,2}, n-2}(y) - y^2(n-2) U_{\Gamma_{2,2,2}, n-3}(y) \quad (2.15)$$

for $n > 1$, and $k_1 = k_2 = 2, s = 3$ where the recursion becomes

$$\begin{aligned} U_{\Gamma_{2,2,3}, n}(y) &= -y U_{\Gamma_{2,2,3}, n-1}(y) - y(n-1) U_{\Gamma_{2,2,3}, n-2}(y) - y^2(n-2) U_{\Gamma_{2,2,3}, n-3}(y) - \\ &\quad y(n-2) U_{\Gamma_{2,2,3}, n-3}(y) - y^2(n-3) U_{\Gamma_{2,2,3}, n-4}(y). \end{aligned} \quad (2.16)$$

Tables 2.3 and 2.4 below give the polynomials $U_{\Gamma_{2,2,2}, n}(-y)$ for even and odd

values of n , respectively.

Table 2.3: The polynomials $U_{\Gamma_{2,2,2},2k}(-y)$ for $\Gamma_{2,2,2} = \{1324, 1423, 123\}$

k	n	$U_{\Gamma_{2,2,2},2k}(-y)$
1	2	$y + y^2$
2	4	$3y^2 + 3y^3 + y^4$
3	6	$15y^3 + 15y^4 + 5y^5 + y^6$
4	8	$105y^4 + 105y^5 + 35y^6 + 7y^7 + y^8$
5	10	$945y^5 + 945y^6 + 315y^7 + 63y^8 + 9y^9 + y^{10}$
6	12	$10395y^6 + 10395y^7 + 3465y^8 + 693y^9 + 99y^{10} + 11y^{11} + y^{12}$
7	14	$135135y^7 + 135135y^8 + 45045y^9 + 9009y^{10} + 1287y^{11} + 143y^{12}$ $+13y^{13} + y^{14}$

Table 2.4: The polynomials $U_{\Gamma_{2,2,2},2k+1}(-y)$ for $\Gamma_{2,2,2} = \{1324, 1423, 123\}$

k	n	$U_{\Gamma_{2,2,2},2k+1}(-y)$
1	3	$2y^2 + y^3$
2	5	$8y^3 + 4y^4 + y^5$
3	7	$48y^4 + 24y^5 + 6y^6 + y^7$
4	9	$384y^5 + 192y^6 + 48y^7 + 8y^8 + y^9$
5	11	$3840y^6 + 1920y^7 + 480y^8 + 80y^9 + 10y^{10} + y^{11}$
6	13	$46080y^7 + 23040y^8 + 5760y^9 + 960y^{10} + 120y^{11} + 12y^{12} + y^{13}$
7	15	$645120y^8 + 322560y^9 + 80640y^{10} + 13440y^{11} + 1680y^{12} + 168y^{13}$ $+14y^{14} + y^{15}$

These data lead us to conjecture the following explicit formulas:

$$U_{\Gamma_{2,2,2},2k}(-y) = \sum_{i=0}^k (2k-1) \downarrow\downarrow_{k-i} y^{k+i} \quad (2.17)$$

$$U_{\Gamma_{2,2,2},2k+1}(-y) = \sum_{i=0}^k (2k) \downarrow\downarrow_{k-i} y^{k+1+i} \quad (2.18)$$

where $(x) \downarrow\downarrow_0 = 1$ and $(x) \downarrow\downarrow_k = x(x-2)(x-4) \cdots (x-2k-2)$ for $k \geq 1$.

These formulas can be proved by induction. Note that it follows from (2.15)

that for $n > 1$,

$$U_{\Gamma_{2,2,2},n}(-y) = yU_{\Gamma_{2,2,2},n-1}(-y) + y(n-1)U_{\Gamma_{2,2,2},n-2}(-y) - y^2(n-2)U_{\Gamma_{2,2,2},n-3}(-y). \quad (2.19)$$

One can directly check these formulas for $n \leq 3$. For $n > 3$, let $U_{\Gamma_{2,2,2},n}(-y)|_{y^k}$ be the coefficient of y^k in $U_{\Gamma_{2,2,2},n}(-y)$. Equation (2.19) allows us to write the coefficient of y^{k+1+i} , for $0 \leq i \leq k$, in $U_{\Gamma_{2,2,2},2k+1}(-y)$ as

$$\begin{aligned} U_{\Gamma_{2,2,2},2k+1}(-y)|_{y^{k+1+i}} &= U_{\Gamma_{2,2,2},2k}(-y)|_{y^{k+i}} + (2k)U_{\Gamma_{2,2,2},2k-1}(-y)|_{y^{k+i}} \\ &\quad - (2k-1)U_{\Gamma_{2,2,2},2k-2}(-y)|_{y^{k+i-1}} \\ &= (2k-1) \Downarrow\Downarrow_{k-i} + (2k) \cdot (2k-2) \Downarrow\Downarrow_{k-i} \\ &\quad - (2k-1) \cdot (2k-3) \Downarrow\Downarrow_{k-i} \\ &= (2k) \Downarrow\Downarrow_{k-i}. \end{aligned}$$

For the even case when $n = 2k$, the coefficient of y^{k+i} , for $0 \leq i \leq k$, in $U_{\Gamma_{2,2,2},2k}(-y)$ is

$$\begin{aligned} U_{\Gamma_{2,2,2},2k}(-y)|_{y^{k+i}} &= U_{\Gamma_{2,2,2},2k-1}(-y)|_{y^{k+i-1}} + (2k-1)U_{\Gamma_{2,2,2},2k-2}(-y)|_{y^{k+i-1}} \\ &\quad - (2k-2)U_{\Gamma_{2,2,2},2k-3}(-y)|_{y^{k+i-2}} \\ &= (2k-2) \Downarrow\Downarrow_{k-i} + (2k-1) \cdot (2k-3) \Downarrow\Downarrow_{k-i} \\ &\quad - (2k-2) \cdot (2k-4) \Downarrow\Downarrow_{k-i} \\ &= (2k-1) \Downarrow\Downarrow_{k-i}. \end{aligned}$$

This proves equations (2.17) and (2.18).

Hence, we can give a closed formula for $NM_{\Gamma_{2,2,2}}(t, x, y)$. That is, we have the following theorem.

Theorem 4.

$$NM_{\Gamma_{2,2,2}}(t, x, y) = \left(\frac{1}{1 + \left(\sum_{n \geq 1} \frac{t^n}{n!} \sum_{i=0}^k (2k-1) \Downarrow_{k-i} y^{k+i} \right) + \left(\sum_{n \geq 0} \frac{t^n}{n!} \sum_{i=0}^k (2k) \Downarrow_{k-i} y^{k+1+i} \right)} \right)^x.$$

It follows from (2.16) that

$$U_{\Gamma_{2,2,3},n}(-y) = yU_{\Gamma_{2,2,3},n-1}(-y) + y(n-1)U_{\Gamma_{2,2,3},n-2}(-y) + y(n-2)U_{\Gamma_{2,2,3},n-3}(-y) - y^2(n-2)U_{\Gamma_{2,2,3},n-3}(-y) - y^2(n-3)U_{\Gamma_{2,2,3},n-4}(-y).$$

The three tables 2.5, 2.6, and 2.7 give the polynomials $U_{\Gamma_{2,2,3},n}(y)$ for $n = 3k$, $n = 3k + 1$, and $n = 3k + 2$, respectively.

Table 2.5: The polynomials $U_{\Gamma_{2,2,3},3k}(-y)$ for $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$

k	n	$U_{\Gamma_{2,2,3},3k}(-y)$
1	3	$y + 2y^2 + y^3$
2	6	$4y^2 + 33y^3 + 19y^4 + 5y^5 + y^6$
3	9	$28y^3 + 767y^4 + 781y^5 + 267y^6 + 55y^7 + 8y^8 + y^9$
4	12	$280y^4 + 20496y^5 + 44341y^6 + 20765y^7 + 5137y^8 + 861y^9 + 109y^{10} + 11y^{11} + y^{12}$
5	15	$3640y^5 + 598892y^6 + 2825491y^7 + 2072739y^8 + 641551y^9 + 125111y^{10} + 17755y^{11} + 1977y^{12} + 181y^{13} + 14y^{14} + y^{15}$

For any $s \geq 3$, it is easy to see that the lowest power of y that occurs in $U_{\Gamma_{2,2,s},n}(-y)$ corresponds to brick tabloids where we use the minimum number of bricks. Since the maximum size of brick in a fixed point of $I_{\Gamma_{2,2,s}}$ is s , we see that the

Table 2.6: The polynomials $U_{\Gamma_{2,2,3},3k+1}(-y)$ for $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$

k	n	$U_{\Gamma_{2,2,3},3k+1}(-y)$
1	4	$5y^2 + 3y^3 + y^4$
2	7	$67y^3 + 81y^4 + 29y^5 + 6y^6 + y^7$
3	10	$1166y^4 + 3321y^5 + 1645y^6 + 417y^7 + 71y^8 + 9y^9 + y^{10}$
4	13	$23746y^5 + 160647y^6 + 128771y^7 + 41055y^8 + 8137y^9 + 1167y^{10}$ $+131y^{11} + 12y^{12} + y^{13}$
5	16	$550844y^6 + 8107518y^7 + 12109429y^8 + 5170965y^9 + 1225973y^{10}$ $+200253y^{11} + 24889y^{12} + 2493y^{13} + 209y^{14} + 15y^{15} + y^{16}$

Table 2.7: The polynomials $U_{\Gamma_{2,2,3},3k+2}(-y)$ for $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$

k	n	$U_{\Gamma_{2,2,3},3k+2}(-y)$
1	5	$7y^2 + 11y^3 + 4y^4 + y^5$
2	8	$70y^3 + 297y^4 + 157y^5 + 41y^6 + 7y^7 + y^8$
3	11	$910y^4 + 10343y^5 + 9223y^6 + 3069y^7 + 613y^8 + 89y^9 + 10y^{10} + y^{11}$
4	14	$14560y^5 + 390564y^6 + 687109y^7 + 306413y^8 + 74137y^9 + 12261y^{10}$ $+1537y^{11} + 155y^{12} + 13y^{13} + y^{14}$

minimum number of bricks that we can use for a fixed point of $I_{\Gamma_{2,2,s}}$ of length sn is n while the minimum number of bricks that we can use for a fixed point of $I_{\Gamma_{2,2,s}}$ of length $sn + j$ for $1 \leq j \leq s - 1$ is $n + 1$. We can prove the following general theorem for the coefficients of the lowest power of y that appears in $U_{\Gamma_{2,2,s},n}(-y)$.

Theorem 5. For $n \geq 1$,

$$U_{\Gamma_{2,2,s},sn}(-y)|_{y^n} = \prod_{i=1}^n ((i-1)s + 1) \quad (2.20)$$

and

$$U_{\Gamma_{2,2,s},sn+s-1}(-y)|_{y^{n+1}} = \prod_{i=1}^n ((i+1)s + 1). \quad (2.21)$$

Proof. For (2.20), we first notice that any fixed point (B, σ) of $I_{\Gamma_{2,2,s}}$ that contributes to $U_{\Gamma_{2,2,s},sn}(-y)|_{y^n}$ must have only bricks of size s . Thus $B = (s, \dots, s)$. We shall

prove (2.20) by induction on n . Clearly, $U_{\Gamma_{2,2,s,s}}(-y)|_y = 1$. Now suppose (B, σ) is a fixed point of $I_{\Gamma_{2,2,s}}$ of size sn where $\sigma = \sigma_1 \cdots \sigma_{sn}$. By our arguments above, the first $s - 1$ elements of the first brick must be $1, 2, \dots, s - 1$, reading from left to right. The element in the next cell σ_s can be arbitrary. That is, if it is equal to s , then there will be an increase between the first two bricks and if $\sigma_s > s$, then it must be the case that $\sigma_{s+1} = s$ in which case there will be a $\Gamma_{2,2,s}$ -match that involves the last two cells of the first brick and the first two cells of the next brick. We can then remove the first brick and adjust the remaining numbers to produce a fixed point O' of $I_{\Gamma_{2,2,s}}$ of length $s(n - 1)$ in which every brick is of size s . It follows by induction that

$$\begin{aligned} U_{\Gamma_{2,2,s,sn}}(-y)|_{y^n} &= ((n - 1)s + 1)U_{\Gamma_{2,2,s,s(n-1)}}(-y)|_{y^{n-1}} \\ &= ((n - 1)s + 1) \prod_{i=1}^{n-1} ((i - 1)s + 1) \\ &= \prod_{i=1}^n ((i - 1)s + 1). \end{aligned}$$

Next consider $U_{\Gamma_{2,2,s,2s-1}}(-y)|_{y^2}$. In this case, either the first brick is of size $s - 1$ or the first brick is of size s . If the first brick is of size s , then we can argue as above that the first $s - 1$ elements of the first brick are $1, \dots, s - 1$, and we have s choices for the last element of the first brick. If the first brick is of size $s - 1$, then we can argue as above that the first $s - 2$ elements of the first brick are $1, \dots, s - 2$, and we have $s + 1$ choices for the last element of the first brick. Thus

$$U_{\Gamma_{2,2,s,2s-1}}(-y)|_{y^2} = 2s + 1.$$

Next consider $U_{\Gamma_{2,2,s,(ns+s-1)}}(-y)|_{y^{n+1}}$. In such a situation, any fixed point

(B, σ) of $I_{\Gamma_{2,2,s}}$ that can contribute to $U_{\Gamma_{2,2,s,(ns+s-1)}}(-y)|_{y^{n+1}}$ must have n bricks of size s and one brick of size $s - 1$. If the first brick is of size s , then we can argue as above that the first $s - 1$ elements of the first brick are $1, \dots, s - 1$, and we have sn choices for the last element of the first brick. Then we can remove this first brick and adjust the remaining numbers to produce a fixed point O' in $I_{\Gamma_{2,2,s}}$ of size $(n - 1)s + s - 1$ which has $n - 1$ bricks of size s and one brick of size $s - 1$. If the first brick is of size $s - 1$, then we can argue as above that the first $s - 2$ elements of the first brick are $1, \dots, s - 2$, and we have $sn + 1$ choices for the last element of the first brick. Then we can remove this first brick and adjust the remaining numbers to produce a fixed point O' in $I_{\Gamma_{2,2,s}}$ of size ns which has n bricks of size s

Thus if $n \geq 2$,

$$\begin{aligned}
U_{\Gamma_{2,2,s,(ns+s-1)}}(-y)|_{y^{n+1}} &= (sn + 1)U_{\Gamma_{2,2,s,ns}}(-y)|_{y^n} + (sn)U_{\Gamma_{2,2,s,(n-1)s+s-1}}(-y)|_{y^n} \\
&= (sn + 1) \prod_{i=1}^n ((i - 1)s + 1) + (sn) \prod_{i=1}^{n-1} ((i + 1)s + 1) \\
&= (s + 1) \prod_{i=1}^{n-1} ((i + 1)s + 1) + (sn) \prod_{i=1}^{n-1} ((i + 1)s + 1) \\
&= ((n + 1)s + 1) \prod_{i=1}^{n-1} ((i + 1)s + 1) \\
&= \prod_{i=1}^n ((i + 1)s + 1).
\end{aligned}$$

□

Unfortunately, we cannot extend this type of argument to find the coefficients $U_{\Gamma_{2,2,s,ns+k}}(-y)|_{y^{n+1}}$ where $1 \leq k \leq s - 2$. The problem is that we have more than one choice for the sizes of the bricks in such cases. For example, to compute $U_{\Gamma_{2,2,3,4}}(-y)|_{y^3}$,

the brick sizes could be some rearrangement of (3,1) or (2,2). One can use our recursions to compute $U_{\Gamma_{2,2,s},ns+k}(-y)|_{y^{n+1}}$ for small values of s . For example, we can find all the coefficients of the lowest power of $U_{\Gamma_{2,2,3},n}(-y)$. That is, we claim

$$(i) \quad U_{\Gamma_{2,2,3},3k}(-y)|_{y^k} = \prod_{i=1}^k (3(i-1) + 1),$$

$$(ii) \quad U_{\Gamma_{2,2,3},3k+2}(-y)|_{y^{k+1}} = \prod_{i=1}^k (3(i+1) + 1), \text{ and}$$

$$(iii) \quad \text{if } A_k = U_{\Gamma_{2,2,3},3k+1}(-y)|_{y^{k+1}} \text{ then } A_1 = 5 \text{ and } A_k = (3k-1)A_{k-1} + (3k) \prod_{i=1}^{k-1} (3i+4) \\ \text{for all } k \geq 2.$$

Clearly, (i) and (ii) follow from our previous theorem. To prove (iii), note that

$$\begin{aligned} A_k &= U_{\Gamma_{2,2,3},3k+1}(-y)|_{y^{k+1}} = U_{\Gamma_{2,2,3},3k}(-y)|_{y^k} + (3k)U_{\Gamma_{2,2,3},3k-1}(-y)|_{y^k} + (3k-1)U_{\Gamma_{2,2,3},3k-2}(-y)|_{y^k} \\ &\quad - (3k-1)U_{\Gamma_{2,2,3},3k-2}(-y)|_{y^{k-1}} - (3k-2)U_{\Gamma_{2,2,3},3k-3}(-y)|_{y^{k-1}} \\ &= \prod_{i=1}^k (3i-2) + (3k) \prod_{i=1}^{k-1} (3i+4) + (3k-1)U_{\Gamma_{2,2,3},3k-2}(-y)|_{y^k} \\ &\quad - (3k-2) \prod_{i=1}^{k-1} (3i-2) \\ &= (3k) \prod_{i=1}^{k-1} (3i+4) + (3k-1)U_{\Gamma_{2,2,3},3k-2}(-y)|_{y^k} \\ &= (3k-1)A_{k-1} + (3k) \prod_{i=1}^{k-1} (3i+4). \end{aligned}$$

This explains all the coefficients for the smallest power of y in the polynomials $U_{\Gamma_{2,2,3},n}(-y)$ for the family $\Gamma_{2,2,3} = \{1324, 1423, 1234\}$.

2.3.3 The cases $\{1324, 123\}$ and $\{1324 \dots p, 123 \dots p-1\}$ for $p \geq 5$

In addition, we can also show that the reciprocity method applies even in cases where Γ is a family that contains permutations of different lengths. This is illustrated through the two following theorems.

Theorem 6. *Let $\Gamma = \{1324, 123\}$. Then*

$$NM_{\Gamma}(t, x, y) = \left(\frac{1}{U_{\Gamma}(t, y)} \right)^x \text{ where } U_{\Gamma}(t, y) = 1 + \sum_{n \geq 1} U_{\Gamma, n}(y) \frac{t^n}{n!},$$

$$U_{\Gamma, 1}(y) = -y, \text{ and for } n \geq 2,$$

$$U_{\Gamma, n}(y) = -yU_{\Gamma, n-1}(y) - yU_{\Gamma, n-2}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^k C_{k-1} U_{\Gamma, n-2k}(y).$$

Theorem 7. *Let $\Gamma = \{1324 \dots p, 123 \dots p-1\}$ where $p \geq 5$. Then*

$$NM_{\Gamma}(t, x, y) = \left(\frac{1}{U_{\Gamma}(t, y)} \right)^x \text{ where } U_{\Gamma}(t, y) = 1 + \sum_{n \geq 1} U_{\Gamma, n}(y) \frac{t^n}{n!},$$

$$U_{\Gamma, 1}(y) = -y, \text{ and for } n \geq 2,$$

$$U_{\Gamma, n}(y) = \sum_{k=1}^{p-2} (-y) U_{\Gamma, n-k}(y) + \sum_{k=1}^{p-2} \sum_{m=2}^{\lfloor \frac{n-k}{p-2} \rfloor} (-y)^m U_{\Gamma, n-k-(m-1)(p-2)}(y).$$

In the case of Theorem 6, the polynomials $U_{\{1324, 123\}, n}(-y)$ are the polynomials in the sequences A039598 and A039599 in On-line Encyclopedia of Integer Sequences [42] up to a power of y . The polynomials in sequences A039598 and A039599 are related to the expansions of the powers of x in terms of the Chebyshev polynomials of the second kind. We shall give a bijection between our combinatorial interpretation

of $U_{\{1324,123\},2n}(-y)$ and one of the known combinatorial interpretations for A039599, and a bijection between our combinatorial interpretation of $U_{\{1324,123\},2n+1}(-y)$ and one of the known combinatorial interpretations for A039598. This will allow us to give closed expressions for the polynomials $U_{\{1324,123\},n}(y)$. That is, we will prove that for all $n \geq 0$,

$$\begin{aligned} U_{\{1324,123\},2n}(y) &= \sum_{k=0}^n \frac{(2k+1) \binom{2n}{n-k}}{n+k+1} (-y)^{n+k+1} \text{ and} \\ U_{\{1324,123\},2n+1}(y) &= \sum_{k=0}^n \frac{2(k+1) \binom{2n+1}{n-k}}{n+k+2} (-y)^{n+k}. \end{aligned}$$

Proof of Theorem 6

Let $\Gamma = \{1324, 123\}$. Let (B, σ) be a fixed point I_Γ where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$. By Lemma 2, we know that all the bricks b_i must be of size 1 or 2. Since the minimal elements in bricks of B must weakly increase, we see that 1 must be in cell 1 and 2 must be either in b_1 or it is in the first cell of b_2 . Thus we have three possibilities.

Case 1. 2 is in b_1 .

In this case, b_1 must be of size 2 and we can remove b_1 from (B, σ) and reduce the remaining numbers by 2 to get a fixed point of I_Γ of size $n - 2$. It then easily follows that the fixed points in Case 1 contribute $-yU_{\Gamma,n-2}(y)$ to $U_{\Gamma,n}(y)$.

Case 2. 2 is in b_2 and $b_1 = 1$.

In this case, it is easy to see that 1 cannot be involved in any Γ -match so that we can remove b_1 from (B, σ) and reduce the remaining numbers by 1 to get a fixed point of I_Γ of size $n - 1$. It follows that the fixed points in Case 2 contribute $-yU_{\Gamma,n-1}(y)$ to $U_{\Gamma,n}(y)$.

Case 3. 2 is in b_2 and $b_1 = 2$.

In this case, there is descent between bricks b_1 and b_2 so that there must be a 1324-match in σ contained in the cells of b_1 and b_2 . In particular, this means $b_2 = 2$ and there is 1324-match starting at 1 in σ . We then have two subcases.

Subcase 3.A. There is no 1324-match in (B, σ) starting at cell 3

We claim that $\{\sigma_1, \dots, \sigma_4\} = \{1, 2, 3, 4\}$. If not, let $d = \min(\{1, 2, 3, 4\} - \{\sigma_1, \dots, \sigma_4\})$. Then d must be in cell 5, the first cell of brick b_3 and there is a decrease between bricks b_2 and b_3 since $d \leq 4 < \sigma_4$. Thus, in order to avoid combining bricks b_2 and b_3 , we need a 1324-match among the cells of these two bricks. However, the only possible 1324-match among the cells of b_2 and b_3 would have to start at cell 3 where $\sigma_3 = 2$. This contradicts the assumption that there is no 1324-match in (B, σ) starting at cell 3. As a result, it must be the case that the first four numbers must occupy the first four cells of (B, σ) so we must have $\sigma_1 = 1, \sigma_2 = 3, \sigma_3 = 2, \sigma_4 = 4$, and $\sigma_5 = 5$. It then follows that if we let O' be the result by removing the first four cells from (B, σ) and then subtract 4 from the remaining entries in the cells, then O' will be a fixed point in $\mathcal{O}_{\Gamma, n-4}$. It then easily follows that the contribution of fixed points in subcase 3.A to $U_{\Gamma, n}(y)$ is $(-y)^2 U_{\Gamma, n-4}(y)$.

Subcase 3.B. There is a 1324-match in O starting at cell 3

In this case, there is decrease between bricks b_2 and b_3 . Hence, the 1324-match starting at cell 3 must be contained in the cells of b_2 and b_3 so that b_3 must be of size 2. In general, suppose that the bricks b_2, \dots, b_{k-1} all have exactly two cells and there are 1324-matches starting at cells $1, 3, \dots, 2k-3$ but there is no 1324-match starting at cell $2k-1$ in O .

Similar to Subcase 3.A, we will show that $\{\sigma_1, \dots, \sigma_{2k}\} = \{1, 2, \dots, 2k\}$.

That is, the first $2k$ numbers must occupy the first $2k$ cells in O . If not, let $d = \min(\{1, 2, \dots, 2k\} - \{\sigma_1, \dots, \sigma_{2k}\})$. Since the minimal elements of the bricks are weakly increasing, it must be the case that d is in the first cell of b_{k+1} . Next, the fact that there are 1324-matches starting in cells $1, 3, \dots, 2k - 1$ easily implies that σ_{2k} is the largest element in $\{\sigma_1, \dots, \sigma_{2k}\}$ which means that $\sigma_{2k} > d$. But then there is a decrease between bricks b_k and b_{k+1} which means that there must be a 1324-match contained in the cells of b_k and b_{k+1} . This implies that there is a 1324-match starting at cell $2k - 1$ which contradicts our assumption.

Thus, if we remove the first $2k$ cells of (B, σ) and subtract $2k$ from the remaining elements, we will obtain a fixed point O' in $\mathcal{O}_{\Gamma, n-2k}$. Therefore, each fixed point O in this case will contribute $(-y)^k U_{\Gamma, n-2k}(y)$ to $U_{\Gamma, n}(y)$. The final task is to count the number of permutations $\sigma_1 \cdots \sigma_{2k}$ of S_{2k} that has 1324-matches starting at positions $1, 3, \dots, 2k - 3$. In [26], Jones and Remmel gave a bijection between the set of such σ and the set of paths of length $2k - 2$. Hence, there are C_{k-1} such fixed points, where $C_n = \frac{1}{n-1} \binom{2n}{n}$ is the n^{th} Catalan number. It then easily follows that the contribution of the fixed points in Subcase 3.B to $U_{\Gamma, n}(y)$ is

$$\sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^k C_{k-1} U_{\Gamma, n-2k}(y).$$

Hence, we know that $U_{\Gamma, 1} = -y$ and for $n > 1$,

$$U_{\Gamma, n}(y) = -y U_{\Gamma, n-1}(y) - y U_{\Gamma, n-2}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^k C_{k-1} U_{\Gamma, n-2k}(y).$$

This proves Theorem 6.

We have computed the polynomials $U_{\{1324, 123\}, n}(-y)$ for small n which are

given in the Table 2.8 below.

Table 2.8: The polynomials $U_{\Gamma,n}(-y)$ for $\Gamma = \{1324, 123\}$

n	$U_{\{1324,123\},n}(-y)$
1	y
2	$y + y^2$
3	$2y^2 + y^3$
4	$2y^2 + 3y^3 + y^4$
5	$5y^3 + 4y^4 + y^5$
6	$5y^3 + 9y^4 + 5y^5 + y^6$
7	$14y^4 + 14y^5 + 6y^6 + y^7$
8	$14y^4 + 28y^5 + 20y^6 + 7y^7 + y^8$
9	$42y^5 + 48y^6 + 27y^7 + 8y^8 + y^9$
10	$42y^5 + 90y^6 + 75y^7 + 35y^8 + 9y^9 + y^{10}$

We observe that, up to a power of y , the odd rows are the triangle A039598 in the OEIS and the even rows are the triangle A039599 in the OEIS. These tables arise from expanding the powers of x in terms of the Chebyshev polynomials of the second kind. Since there are explicit formula for entries in these tables, we have the following theorem.

Theorem 8. *Let $\Gamma = \{1324, 123\}$. Then for all $n \geq 0$,*

$$U_{\Gamma,2n}(y) = \sum_{k=0}^n \frac{(2k+1) \binom{2n}{n-k}}{n+k+1} (-y)^{n+k+1} \quad (2.22)$$

and

$$U_{\Gamma,2n+1}(y) = \sum_{k=0}^n \frac{2(k+1) \binom{2n+1}{n-k}}{n+k+2} (-y)^{n+k} \quad (2.23)$$

Proof. First we consider the polynomials $U_{\Gamma,2n+1}(-y)$ which correspond to the entries in the table $T(n, k)$ for $0 \leq k \leq n$ of entry A039598 in the OEIS. $T(n, k)$ has an

explicit formula, namely,

$$T(n, k) = \frac{2(k+1) \binom{2n+1}{n-k}}{n+k+2}$$

for all $n \geq 0$ and $0 \leq k \leq n$. Let $\mathcal{T}(n, k)$ be set all of paths of length $2n+1$ consisting of either up steps $(1, 1)$ or down steps $(1, -1)$ that start at $(0,0)$ and end at $(2n+1, 2k+1)$ which stay above the x -axis. Then one of the combinatorial interpretations of the $T(n, k)$'s is that $T(n, k) = |\mathcal{T}(n, k)|$. Let $\mathcal{F}_{2n+1, 2k+1}$ be the set of all fixed points of I_Γ with $2k+1$ bricks of size 1 and $n-k$ bricks of size 2. We will construct a bijection $\theta_{n,k}$ from $\mathcal{F}_{2n+1, 2k+1}$ onto $\mathcal{T}(n, k)$. Note all $(B, \sigma) \in \mathcal{F}_{2n+1, 2k+1}$ have weight $(-y)^{n+k+1}$ so that the bijections $\theta_{n,k}$ will prove (2.23).

First we must examine the fixed points of I_Γ in greater detail. Note that since Γ contains the identity permutation 123, all the bricks in any fixed point of I_Γ must be of size 1 or size 2. Next, we consider the structure of the fixed points of I_Γ which have k bricks of size 1 and ℓ bricks of size 2. Suppose (B, σ) is such a fixed point where $B = (b_1, \dots, b_{k+\ell})$ and that the bricks of size 1 in B are b_{i_1}, \dots, b_{i_k} where $1 \leq i_1 < \dots < i_k \leq k + \ell$. For any s , there cannot be a decrease between brick $b_{i_{j-1}}$ and brick b_{i_j} in B since otherwise we could combine bricks $b_{i_{j-1}}$ and b_{i_j} , which would violate our assumption that (B, σ) is a fixed point of I_Γ . Next we claim that if there are s bricks of size 2 that come before brick b_{i_j} so that b_{i_j} covers cell $2s + j$ in (B, σ) , then $\sigma_{2s+j} = 2s + j$ and $\{\sigma_1, \dots, \sigma_{2s+j}\} = \{1, \dots, 2s + j\}$. To prove this claim, we proceed by induction. For the base case, suppose that b_{i_1} covers cell $2s + 1$ so that (B, σ) starts out with s bricks of size 2. If $s = 0$, there is nothing to prove. Next suppose that $s = 1$. Then we know that in all fixed points of I_Γ , 2 must be in cell 2 or cell 3. Since there is an increase between b_1 and b_2 , it must be the case that 1 and 2 lie in b_1 and since the minimal elements in the brick form a weakly increasing sequence,

it must be the case that b_2 is filled with 3. If $s \geq 2$, then for $1 \leq i < s$, either there is an increase between b_i and b_{i+1} in which case the elements in b_i and b_{i+1} must match the pattern 1234, or there is a decrease between b_i and b_{i+1} in which case the four elements must match the pattern 1324. This means that if for each brick of size 2, we place the second element of the brick on the top of the first element, then any two consecutive bricks will be one of the two forms pictured in Figure 2.3. Thus if we consider the $s \times 2$ array built from the first s bricks of size 2, we will obtain a column strict tableaux with distinct entries of shape (s, s) . In particular, it must be the case that the largest element in the array is the element which appears at the top of the last column. That element corresponds to the second cell of brick b_s . Since there is an increase between brick b_s and brick b_{s+1} it must mean that the element in brick b_{s+1} is larger than any of the elements that appear in bricks b_1, \dots, b_s . Thus $\sigma_i < \sigma_{2s+1}$ for $i \leq 2s$. Since the minimal elements in the bricks are increasing, it follows that $\sigma_{2s+1} < \sigma_j$ for all $j > 2s + 1$ so that it must be the case that $\sigma_{2s+1} = 2s + 1$ and $\{\sigma_1, \dots, \sigma_{2s+1}\} = \{1, \dots, 2s + 1\}$. Thus the base case of our induction holds.

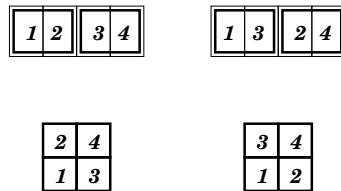


Figure 2.3: Patterns for two consecutive brick of size 2 in a fixed point of I_Γ .

We can repeat the same argument for i_j where $j > 1$. That is, by induction, we can assume that if there are r bricks of size 2 that precede brick $b_{i_{j-1}}$, then $\sigma_{2r+j-1} = 2r + j - 1$ and $\{\sigma_1, \dots, \sigma_{2r+j-1}\} = \{1, \dots, 2r + j - 1\}$. Hence if we remove these elements and subtract $2r + j - 1$ from the remaining elements in (B, σ) , we

would end up with a fixed point of I_γ . Thus we can repeat our argument for the base case to prove that if there are s bricks of size 2 between brick $b_{i_{j-1}}$ and b_{i_j} , then $\sigma_{2r+2s+j} = 2r + 2s + j$ and $\{\sigma_1, \dots, \sigma_{2r+2s+j}\} = \{1, \dots, 2r + 2s + j\}$.

Next we note that there is a well known bijection ϕ between standard tableaux of shape (n, n) and Dyck paths of length $2n$, see [43]. Here a Dyck path is path consisting of either up steps $(1, 1)$ or down steps $(1, -1)$ that starts at $(0, 0)$ and ends at $(2n, 0)$ which stays above the x -axis. Given a standard tableau T , $\phi(T)$ is the Dyck path whose i -th segment is an up step if i is in the first row and whose i -th segment is a down step if i is in the second row. This bijection is illustrated in Figure 2.4.

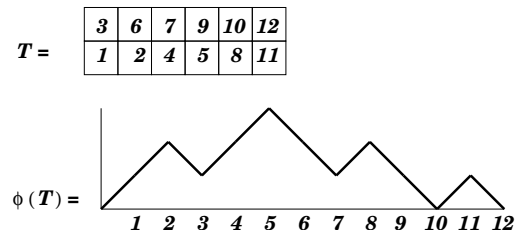


Figure 2.4: The bijection ϕ .

We can now easily describe our desired bijection $\theta_{n,k}$. Starting with a fixed point (B, σ) in $\mathcal{F}_{2n+1, 2k+1}$ where $B = (b_1, \dots, b_{n+k+1})$, we can rotate all the bricks of size 2 by -90 degrees and end up with an array consisting of bricks of size one and $2 \times r$ arrays corresponding to standard tableaux. For example, this step is pictured in the second row of Figure 2.5. By our remarks above, each $2 \times r$ array corresponds to standard tableaux of shape (r, r) where the entries lie in some consecutive sequence of elements from $\{1, \dots, 2n + 1\}$. Suppose that $b_{i_1}, \dots, b_{i_{2k+1}}$ are the bricks of size 1 in B where $i_1 < \dots < i_{2k+1}$. Let T_j be the standard tableau corresponding to the consecutive string of brick of size 2 immediately preceding brick b_{i_j} and P_i be the Dyck path $\phi(T_i)$. If there is no bricks of size 2 immediately preceding b_{i_j} , then P_j is just

the empty path. Finally let T_{2k+2} the standard tableau corresponding to the bricks of size 2 following $b_{i_{2k+1}}$ and P_{2k+2} be the Dyck path corresponding to $\phi(T_{2k+2})$ where again P_{2k+2} is the empty path if there are no bricks of size 2 following $b_{i_{2k+1}}$. Then

$$\theta_{n,k}(B, \sigma) = P_1(1, 1)P_2(1, 1) \dots P_{2k+1}(1, 1)P_{2k+2}.$$

For example, line 3 of Figure 2.5 illustrates this process. In fact, it easy to see that if i is in the bottom row of intermediate diagram for (B, σ) , then the i -th segment of $\theta_{n,k}(B, \sigma)$ is an up step and if i is in the top row of intermediate diagram for (B, σ) , then the i -th segment of $\theta_{n,k}(B, \sigma)$ is a down step.

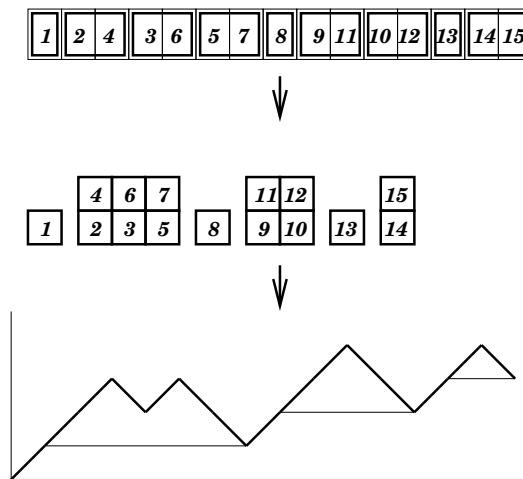


Figure 2.5: The bijection $\theta_{n,k}$.

The inverse of $\theta_{n,k}$ is also easy to describe. That is, given a path P in $\mathcal{T}(n, k)$, we let d_i be the step that corresponds to the last up step that ends at level i . Then P can be factored as

$$P_1 d_1 P_2 d_2 \dots P_{2k+1} d_{2k+1} P_{2k+2}$$

where each P_i is a path that corresponds to a Dyck path that starts at level $i - 1$ and ends at level $i - 1$ and stays above the line $x = i - 1$. Then for each i , $T_i = \phi^{-1}(P_i)$ is

a standard tableau. Using these tableaux and being cognizant of the restrictions on the initial segments of elements of $\mathcal{F}_{2n+1,2k+1}$ preceding bricks of size 1, one can easily reconstruct the 2 line intermediate array corresponding to $T_1d_1T_2d_2 \dots T_{2k+1}d_{2k+1}T_{2k+2}$. For example, this process is pictured on line 2 of Figure 2.6. Then we only have to rotate all the bricks of size corresponding to a bricks of height 2 by 90 degrees to obtain $\theta_{n,k}^{-1}(P)$. This step is pictured on line 3 of Figure 2.6.

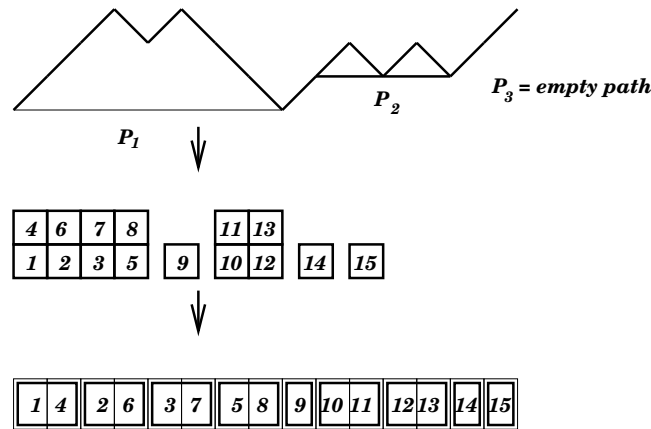


Figure 2.6: The bijection $\theta_{n,k}^{-1}$.

Next we consider the polynomials $U_{\Gamma,2n}(-y)$ which correspond to the entries in the table $R(n, k)$ for $0 \leq k \leq n$ of entry A039599 in the OEIS. $R(n, k)$ has an explicit formula, namely,

$$R(n, k) = \frac{(2k+1) \binom{2n}{n-k}}{n+k+1}$$

for all $n \geq 0$ and $0 \leq k \leq n$. Let $\mathcal{R}(n, k)$ be set all of paths of length $2n$ consisting of either up steps $(1, 1)$ or down steps $(1, -1)$ that start at $(0, 0)$ and end at $(2n, 0)$ that have k down steps that end on the line $x = 0$. Here there is no requirement that the paths stay above the x -axis. Then one of the combinatorial interpretations of the $R(n, k)$ s is that $R(n, k) = |\mathcal{R}(n, k)|$. Let $\mathcal{F}_{2n,2k}$ be the set of all fixed points of I_{Γ} with $2k$ bricks of size 1 and $n - k$ bricks of size 2. We will construct a bijection $\beta_{n,k}$ from

$\mathcal{F}_{2n,2k}$ onto $\mathcal{R}(n,k)$. Note all $(B, \sigma) \in \mathcal{F}_{2n,2k}$ weight $(-y)^{n+k}$ so that the bijections $\beta_{n,k}$ will prove (2.23).

We can now easily describe our desired bijection $\beta_{n,k}$. Starting with a fixed point (B, σ) in $\mathcal{F}_{2n,2k}$ where $B = (b_1, \dots, b_{n+k})$, we can rotate all the bricks of size 2 by -90 degrees and end up with an array consisting of bricks of size one and $2 \times r$ arrays corresponding to standard tableaux. For example, this step is pictured in the second row of Figure 2.8. By our remarks above, each $2 \times r$ array corresponds to standard tableaux of shape (r, r) where the entries lie in some consecutive sequence of elements from $\{1, \dots, 2n\}$. Suppose that $b_{i_1}, \dots, b_{i_{2k}}$ are the bricks of size 1 in B where $i_1 < \dots < i_{2k}$. Let T_s be the standard tableau corresponding to the bricks of size 2 immediately preceding brick b_{j_s} for $1 \leq s \leq 2n$ and let T_{2k+1} be the standard tableau corresponding to the bricks of size 2 following brick $b_{i_{2k}}$. For $i = 0, \dots, 2k+1$, let P_i be the Dyck path $\phi(T_i)$. In each case j where there are no such bricks of size 2, then P_j is just the empty path. For each such i , let \bar{P}_i denote the flip of P_i , i.e. the path that is obtained by flipping P_i about the x-axis. For example, the process of flipping a Dyck path is pictured in Figure 2.7.

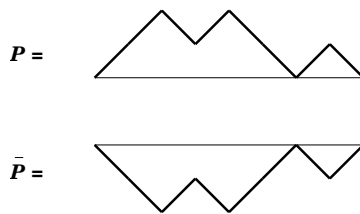


Figure 2.7: The flip of Dyck path.

Then

$$\beta_{n,k}(B, \sigma) = \bar{P}_1(1, 1)P_2(1, -1)\bar{P}_3(1, 1)P_4(1, -1) \dots \bar{P}_{2k-1}(1, 1)P_{2k}(1, -1)\bar{P}_{2k+1}.$$

That is, each pair $b_{i_{2j-1}}, b_{i_{2j}}$ will correspond to an up step starting at $x = 0$ followed by a Dyck path which starts at ends a line $x = 1$ followed by down step ending at $x = 0$. These segments are then connected by flips of Dyck path that stay below the x -axis. Thus $\beta_{n,k}(B, \sigma)$ will have exactly k down steps that end at $x = 0$. For example, line 3 of Figure 2.8 illustrates this process.

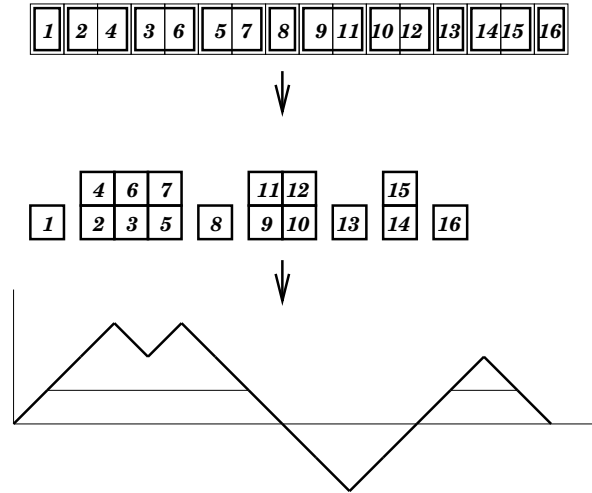


Figure 2.8: The bijection $\beta_{n,k}$.

The inverse of $\beta_{n,k}$ is also easy to describe. That is, given a path P in $\mathcal{R}(n, k)$, let f_1, \dots, f_k be the positions of the down steps that end at $x = 0$ and define e_1, \dots, e_k so that e_1 is the right most up step that starts at $x = 0$ and precedes f_1 and for $2 \leq i \leq k$, e_i is the right most up step that follows f_{i-1} and precedes f_i . It is then easy to see that the path Q_1 which precedes e_1 must be a path that starts at $(0,0)$ and ends at $(e_1 - 1, 0)$ and stays below the x -axis so that Q_1 is the flip of some Dyck path P_1 . Next, the path Q_2 between $(e_1, 1)$ and $(f_1 - 1, 1)$ must either be empty or is a path which stays above the line $x = 1$ and hence corresponds to the Dyck path P_2 . In general, the path Q_{2j-1} that starts at $(f_{j-1}, 0)$ and ends at $(e_j - 1, 0)$ must stay below the x -axis so that Q_{2j-1} is the flip of some Dyck path P_{2j-1} . Similarly, the

path Q_{2j} between $(e_j, 1)$ and $(f_j - 1, 1)$ must either be empty or is a path which stays above the line $x = 1$ and hence corresponds to the Dyck path P_{2j} . Finally, the path Q_{2k+1} which follows $(f_k, 0)$ is either empty or is a path that ends at $(2n, 0)$ and stays below the x -axis and, hence, corresponds to the flip of a Dyck path P_{2k+1} . In this way, we can recover the sequence of paths P_1, \dots, P_{2k+1} , which are either empty or Dyck paths, such that

$$P = \bar{P}_1(1, 1)P_2(1, -1)\bar{P}_3(1, 1)P_4(1, -1) \dots \bar{P}_{2k-1}(1, 1)P_{2k}(1, -1)\bar{P}_{2k+1}.$$

Then for each i , $T_i = \phi^{-1}(P_i)$ is either a standard tableau or the empty tableau. Using these tableaux and being cognizant of the restrictions on the initial segments of elements of $\mathcal{F}_{2n, 2k}$ preceding bricks of size one described above, one can easily reconstruct the 2 line intermediate arrays corresponding to $T_1 e_1 T_2 f_2 \dots T_{2k-1} e_{2k} T_{2k} f_{2k} T_{2k+1}$. For example, this process is pictured on line 2 of Figure 2.9. Then we only have to rotate all the bricks of size corresponding to a brick of height 2 by 90 degrees to obtain $\beta_{n,k}^{-1}(P)$. This step is pictured on line 3 of 2.9.

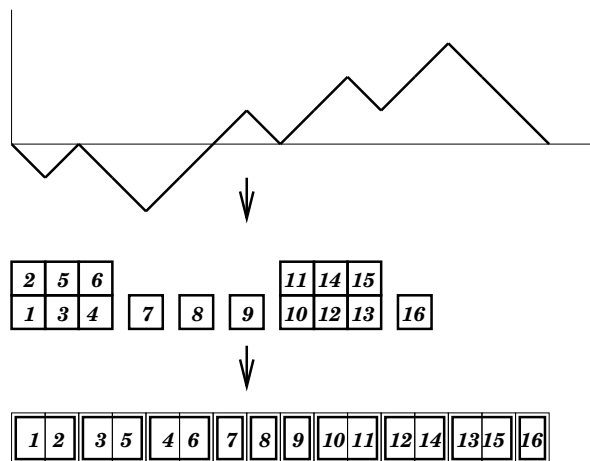


Figure 2.9: The bijection $\beta_{n,k}^{-1}$.

□

As a consequence of Theorem 8, we have a closed formula for $NM_{\{1323,123\}}(t, x, y)$.

Theorem 9.

$$NM_{\{1323,123\}}(t, x, y) = \left(\frac{1}{U_{\{1323,123\}}(t, y)} \right)^x \text{ where}$$

$$U_{\{1323,123\}}(t, y) = 1 + \sum_{n \geq 1} \frac{t^{2n}}{(2n)!} \left(\sum_{k=0}^n \frac{(2k+1) \binom{2n}{n-k}}{n+k+1} (-y)^{n+k} \right) \\ + \sum_{n \geq 0} \frac{t^{2n+1}}{(2n+1)!} \left(\sum_{k=0}^n \frac{2(k+1) \binom{2n+1}{n-k}}{n+k+2} (-y)^{n+k+1} \right).$$

The proof of Theorem 7.

Let $p \geq 5$ and $\Gamma_p = \{1324 \dots p, 123 \dots p-1\}$. It follows from Lemma 2 that any brick in a fixed point of I_{Γ_p} has size less than or equal to $p-2$.

Let (B, σ) be a fixed point of I_{Γ_p} where $B = (b_1, \dots, b_t)$ and $\sigma = \sigma_1 \dots \sigma_n$. Suppose that $b_1 = k$ where $1 \leq k \leq p-2$. If $b_1 = 1$, then $\sigma_1 = 1$ and we can remove brick b_1 from (B, σ) and subtract 1 from the remaining elements to obtain a fixed point O' of I_{Γ_p} of length $n-1$. It is easy to see that such fixed points contribute $-yU_{\Gamma_p, n-1}(y)$ to $U_{\Gamma_p}(y)$.

Next assume that $2 \leq k \leq p-2$. First we claim that $1, \dots, k-1$ must be in b_1 . That is, since the minimal elements in the bricks increase, reading from left to right, and the elements within each brick are increasing, it follows that the first element of brick b_2 is smaller than every element of σ to its right. Thus if there is an

increase between bricks b_1 and b_2 , it must be the case the elements in brick b_1 are the k smallest elements. If there is a decrease between bricks b_1 and b_2 , then there must be a $1324\dots p$ -match that lies in the cells of b_1 and b_2 which must start at position $k - 1$. Thus $\sigma_{k-1} < \sigma_{k+1}$ which means that $\sigma_1, \dots, \sigma_{k-1}$ must be the smallest $k - 1$ elements. We then have two cases depending on the position of k in σ .

Case 1. k is in the k^{th} cell of (B, σ) .

In this case, if we remove the entire brick b_1 from (B, σ) and subtract k from the numbers in the remaining cells, we will obtain a fixed point O' of $I_{\Gamma_p, n-k}$. It then easily follows that fixed points in Case 1 will contribute $-yU_{\Gamma_p, n-k}(y)$ to $U_{\Gamma_p, n}(y)$.

Case 2. k is in cell $k + 1$ of (B, σ) .

In this case, it is easy to see that k is in the first cell of the second brick in (B, σ) and there must be a $1324\dots p$ -match between the cells of the first two bricks. This match must start from cell $k - 1$ in O with the numbers $k - 1$ and k playing the roles of 1 and 2, respectively, in the match. This forces the brick b_2 to have exactly $p - 2$ cells. Thus we have two subcases.

Subcase 2.A. There is no $1324\dots p$ -match in (B, σ) starting at cell $k + p - 3$

In this case, we claim that $\{\sigma_1, \dots, \sigma_{k+p-2}\} = \{1, \dots, k + p - 2\}$. That is, we know that the element in the first cell of brick b_3 is smaller than any of the elements of σ to its right. Moreover, if there was a decrease between brick b_2 and b_3 , then there must be a $1324\dots p$ -match starting in cell $k + p - 3$. Since we are assuming there is not such a match this means that there is an increase between bricks b_2 and b_3 . Since the last element of b_2 must be the largest element in either brick b_1 or b_2 , it follows that $\{\sigma_1, \dots, \sigma_{k+p-2}\} = \{1, \dots, k + p - 2\}$. This forces that $\sigma_i = i$ for $i \leq k - 1$, $\sigma_k = k + 1$, $\sigma_{k+1} = k$, $\sigma_{k+2} = k + 2$, $\sigma_i = i$ for $k + 2 < i \leq k + p - 2$. Hence, the first

two bricks of (B, σ) are completely determined. It then follows that if we let O' be the result by removing the first $k + p - 2$ cells from (B, σ) and subtracting $k + p - 2$ from the numbers in the remaining cells, then O' will be a fixed point in $\mathcal{O}_{\Gamma_p, n-k-(p-2)}$. It then easily follows that fixed points in Subcase 2.A contribute $(-y)^2 U_{\Gamma_p, n-k-(p-2)}(y)$ to $U_{\Gamma_p, n}(y)$.

Subcase 2.b. There is a $1324 \dots p$ -match in (B, σ) starting at cell $k + p - 3$

In this case, it must be that $\sigma_{k+p-3} < \sigma_{k+p-1} < \sigma_{k+p-2}$ so that there is a decrease between bricks b_2 and b_3 . This means that the $1324 \dots p$ -match starting in cell $k + p - 3$ must be contained in bricks b_2 and b_3 . In particular, this means that $b_3 = p - 2$. In general, suppose that the bricks b_2, \dots, b_{m-1} all have exactly $p - 2$ cells and let $c_i = k + (i - 1)(p - 2) - 1$ for all $1 \leq i \leq m - 1$, so that c_i is the second-to-last cell of brick b_i . In addition, suppose there are $1324 \dots p$ -matches starting at cells c_1, c_2, \dots, c_{m-1} but there are no $1324 \dots p$ -match starting at cell $c_m = k - (m - 1)(p - 2) - 1$ in O . We then have the situation pictured in Figure 2.10 below.

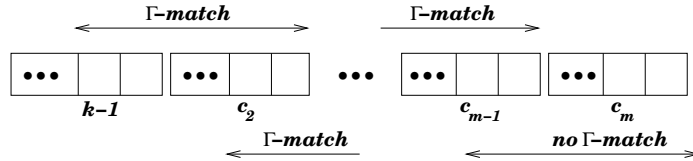


Figure 2.10: A fixed point with Γ_p -matches starting at c_i for $i = 1, \dots, m - 1$.

First, we claim that $\{\sigma_1, \sigma_2, \dots, \sigma_{c_{m+1}}\} = \{1, 2, \dots, c_{m+1}\}$. Since there is no Γ_p -match starting at σ_{c_m} in σ , it cannot be that there is decrease between brick b_m and b_{m+1} . Because the minimal elements in the bricks of B increase, reading from left to right, and the elements in each brick increase, it follows that $\sigma_{c_{m+2}}$, which is the first element of brick b_{m+1} , is smaller than all the elements to its right. On the

other hand, because there are $1324 \dots p$ -matches starting in σ starting at c_1, \dots, c_{m-1} it follows that $\sigma_{c_{m+1}}$, which is last cell in brick b_m , is greater than all elements of σ to its left. It follows that $\{\sigma_1, \sigma_2, \dots, \sigma_{c_{m+1}}\} = \{1, 2, \dots, c_{m+1}\}$.

Next we claim that we can prove by induction that $\sigma_{c_i} = c_i$ and $\{\sigma_1, \dots, \sigma_{c_i}\} = \{1, \dots, c_i\}$ for $1 \leq i \leq m$. Our arguments above show that $\sigma_i = i$ for $i = 1, \dots, k-1 = c_1$. Thus the base case holds. So assume that $\sigma_{c_{j-1}} = c_{j-1}$, for $1 \leq i \leq j$, and $\{\sigma_1, \sigma_2, \dots, \sigma_{c_{j-1}}\} = \{1, 2, \dots, c_{j-1}\}$. Since there is a $132 \dots p$ -match in σ starting at position c_{j-1} and $p \geq 5$, it must be the case that all the numbers $\sigma_{c_{j-1}}, \sigma_{c_{j-1}+1}, \dots, \sigma_{c_{j-1}+p-3}$ are all less than $\sigma_{c_j} = \sigma_{c_{j-1}+p-2}$. Since $\{\sigma_1, \sigma_2, \dots, \sigma_{c_{j-1}}\} = \{1, 2, \dots, c_{j-1}\}$, we must have $\sigma_{c_j} \geq c_j$. If $\sigma_{c_j} > c_j$, then let d be the smallest number from $\{1, 2, \dots, c_j\}$ that does not belong to the bricks b_1, \dots, b_j . Since the numbers in a brick increase and the first cells of the bricks form an increasing sequence, it must be the case that d is in the first cell of brick b_{j+1} , namely $\sigma_{c_{j+2}} = d$. We have two possibilities for j .

1. If $j < m$, then $\sigma_{c_{j+2}} = d < c_j \leq \sigma_{c_j}$. This contradicts the assumption that there is a $1324 \dots p$ -match starting from cell c_j in σ for σ_{c_j} needs to play the role of 1 in such a match.
2. If $j = m$, then there is a descent between the bricks b_m and b_{m+1} and there must be a $1324 \dots p$ -match that lies entirely in the cells of b_m and b_{m+1} in O . However, the only possible match must start from cell c_m , the second-to-last cell in b_m . This contradicts our assumption that there is no match starting from cell c_m in O .

Hence, $\sigma_{c_j} = c_j$ and $\{\sigma_1, \sigma_2, \dots, \sigma_{c_j}\} = \{1, 2, \dots, c_j\}$. for $1 \leq j \leq m$.

We claim that the values of σ_i are forced for $i \leq c_m + 1$. That is, consider the first $1324 \cdots p$ -match starting at position $k - 1$. Since $p \geq 5$, we know that $\sigma_{k+p-2} = k + p - 2 > \sigma_{k+2}$. This forces that $\sigma_k = k + 1$, $\sigma_{k+1} = k$, $\sigma_{k+2} = k + 2$ so that the values of σ_i for $i \leq k + p - 2$. This type of argument can be repeated for all the remaining $1324 \cdots p$ -matches starting at c_2, \dots, c_{m-1} . Thus if we remove the first $k + (m - 1)(p - 2)$ cells of O , we obtain a fixed point O' of I_{Γ_p} in $\mathcal{O}_{\Gamma_p, n-k-(m-1)(p-2)}$. On the other hand, suppose that we start with a fixed point (D, τ) of I_{Γ_p} in $\mathcal{O}_{\Gamma_p, n-k-(m-1)(p-2)}$ where $D = (d_1, \dots, d_r)$ and $\tau = \tau_1, \dots, \tau_{n-k-(m-1)(p-2)}$. Let $\bar{\tau} = \bar{\tau}_1 \cdots \bar{\tau}_{n-k-(m-1)(p-2)}$ be the result of adding $n - k - (m - 1)(p - 2)$ to every element of τ . Then it is easy to see that (B, σ) is a fixed point of I_{Γ_p} , where $B = (k, (p - 2)^m, d_1, \dots, d_r)$ and $\sigma = \sigma_1 \cdots \sigma_{k+(m-1)p-2} \bar{\tau}$ where $\sigma_1 \cdots \sigma_{k+(m-1)p-2}$ is the unique permutation in $S_{k+(m-1)(p-2)}$ with $1324 \cdots p$ -matches starting at positions c_1, \dots, c_{m-1} . It follows that the contribution of the fixed points in Case 2.b to $U_{\Gamma_p, n}(y)$ is $\sum_{m \geq 3} (-y)^m U_{\Gamma_p, n-k-(m-1)(p-2)}(y)$.

Hence, for any fixed point O_k that has k cells in the first brick, for $1 \leq k \leq p - 2$, the contribution of O_k to $U_{\Gamma_p, n}(y)$ is

$$(-y)U_{\Gamma_p, n-k}(y) + \sum_{m=2}^{\lfloor \frac{n-k}{p-2} \rfloor} (-y)^m U_{\Gamma_p, n-k-(m-1)(p-2)}(y).$$

Therefore, we obtain the following recursion for $U_{\Gamma_p, n}(y)$ as follows.

$$U_{\Gamma_p, n}(y) = \sum_{k=1}^{p-2} (-y)U_{\Gamma_p, n-k}(y) + \sum_{k=1}^{p-2} \sum_{m=2}^{\lfloor \frac{n-k}{p-2} \rfloor} (-y)^m U_{\Gamma_p, n-k-(m-1)(p-2)}(y).$$

This completes the proof of Theorem 7. \square

The results of this chapter is based on the paper by Bach and Remmel [3].

Chapter 3

The case of multiple descents

We first recall from the previous chapters that the two assumptions on Γ that allow the reciprocity method to work are that

- (A) all τ in Γ start with 1 and
- (B) all τ in Γ have at most one descent.

First, assumption (A) ensures that we can write $NM_\Gamma(t, x, y)$ in the form $\left(\frac{1}{U_\Gamma(t, y)}\right)^x$. Second, assumption (B) ensures that the involution I used to simplify the weighted sum over all filled, labeled brick tabloids that equals $n!\theta_\tau(h_n)$ is actually an involution and to ensure that the elements in any brick of a filled, labeled brick tabloids which is a fixed point of I must be increasing. Finally, (A) is used again to ensure that the minimal elements in bricks of any fixed point of I are increasing when read from left to right.

The main goal of this chapter is to study how we can apply the reciprocity method in the case where we no longer insist that all the $\tau \in \Gamma$ have at most one descent. We shall show that we can modify the definition of the involution used in

the early chapter to simplify the weighted sum over all filled, labeled brick tabloids that equals $n!\theta_\tau(h_n)$. However, the set of fixed points in such cases will be more complicated than in the case where Γ contains only permutations with at most one descent in that it will no longer be the case that, for fixed points of the involution, the fillings will be increasing in bricks and the minimal elements of the brick increase, reading from left to right. Nevertheless, we shall show that there still are a number of cases where we can successfully analyze the fixed points to prove that the polynomials $U_{\Gamma,n}(y)$ satisfy some simple recursions.

We note that our results allow us to compute $\text{NM}_\tau(t, x, y)$ in two cases where $\tau = \tau_1 \dots \tau_6$ and $\tau_1 = 1$, $\tau_3 = 2$, and $\tau_5 = 3$. Namely, the cases where $\tau = 162534$ and $\tau = 142536$. All such permutations have $\text{des}(\tau) = 2$.

3.1 A new involution

In Section 2.2, we defined the homomorphism θ_Γ on the ring of symmetric functions Λ by setting $\theta_\Gamma(e_0) = 1$ and, for $n \geq 1$,

$$\theta_\Gamma(e_n) = \frac{(-1)^n}{n!} \text{NM}_{\Gamma,n}(1, y).$$

Under this homomorphism, we proved that

$$n!\theta_\Gamma(h_n) = \sum_{O \in \mathcal{O}_{\Gamma,n}} \text{sgn}(O)W(O),$$

where the sum is over the set of all filled labeled brick tabloids described in the same section. The sign of each $O \in \mathcal{O}_{\Gamma,n}$ is given by $\text{sgn}(O) = (-1)^{\ell(\lambda)}$, and the weight

$W(O)$ of O is defined to be the product of all the labels y used in the brick.

Now we shall define a new sign-reversing, weight-preserving mapping $J_\Gamma : \mathcal{O}_{\Gamma,n} \rightarrow \mathcal{O}_{\Gamma,n}$ as follows. Let $(B, \sigma) \in \mathcal{O}_{\Gamma,n}$ where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \dots \sigma_n$. Then for any i , we let $\text{first}(b_i)$ be the element in the left-most cell of b_i and $\text{last}(b_i)$ be the element in the right-most cell of b_i . Then we read the cells of (B, σ) from left to right, looking for the first cell c such that either

Case I. cell c is labeled with a y in some brick b_j and either **(a)** $j = 1$ or **(b)** $j > 1$ with either **(b.1)** $\text{last}(b_{j-1}) < \text{first}(b_j)$ or **(b.2)** $\text{last}(b_{j-1}) > \text{first}(b_j)$ and there is τ -match contained in the cells of b_{j-1} and the cells b_j that end weakly to the left of cell c for some $\tau \in \Gamma$, or

Case II. cell c is at the end of brick b_i where $\sigma_c > \sigma_{c+1}$ and there is no Γ -match of σ that lies entirely in the cells of the bricks b_i and b_{i+1} .

In Case I, we define $J_\Gamma((B, \sigma))$ to be the filled labeled brick tabloid obtained from (B, σ) by breaking the brick b_j that contains cell c into two bricks b'_j and b''_j where b'_j contains the cells of b_j up to and including the cell c while b''_j contains the remaining cells of b_j . In addition, we change the label of cell c from y to $-y$. In Case II, $J_\Gamma((B, \sigma))$ is obtained by combining the two bricks b_i and b_{i+1} into a single brick b and changing the label of cell c from $-y$ to y . If neither case occurs, then we let $J_\Gamma((B, \sigma)) = (B, \sigma)$.

For example, suppose $\Gamma = \{\tau\}$ where $\tau = 14253$ and $(B, \sigma) \in \mathcal{O}_{\Gamma,19}$ pictured at the top of Figure 3.1. We cannot use cell $c = 4$ to define $J_\Gamma(B, \sigma)$, because if we combined bricks b_1 and b_2 , then $\text{red}(9\ 15\ 11\ 16\ 13) = \tau$ would be a τ -match contained in the resulting brick. Similarly, we cannot use cell $c = 6$ to apply the involution because it fails to meet condition (b.2). In fact the first c for which either Case I or

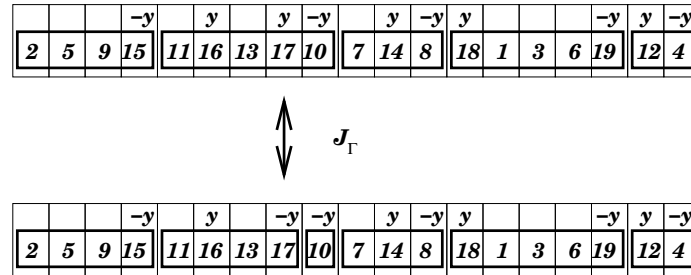


Figure 3.1: An example of the involution J_Γ .

Case II applies is cell $c = 8$ so that $J_\Gamma(B, \sigma)$ is equal to the (B', σ) pictured on the bottom of Figure 3.1.

We now prove that J_Γ is an involution by showing J_Γ^2 is the identity mapping. Let $(B, \sigma) \in \mathcal{O}_{\Gamma, n}$ where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \dots \sigma_n$. The key observation here is that applying the mapping J_Γ to a brick in Case I will produce one in Case II, and vice versa.

Suppose the filled, labeled brick tabloid (B, σ) is in Case I and its image $J_\Gamma((B, \sigma))$ is obtained by splitting some brick b_j after cell c into two bricks b'_j and b''_j . There are now two possibilities.

- (a) c is in the first brick b_1 . In this case, c must be the first cell which is labeled with y so that the elements in b'_1 will be increasing. Furthermore, since we are assuming there is no Γ -match in the cells of brick b_1 in (B, σ) , there cannot be any Γ -match that involves the cells of bricks b'_1 and b''_1 in $J_\Gamma((B, \sigma))$. Hence, when we consider $J_\Gamma((B, \sigma))$, the first possible cell where we can apply J_Γ will be cell c because we can now combine b'_1 and b''_1 . Thus, when we apply J_Γ to $J_\Gamma((B, \sigma))$, we will be in Case II using cell c so that we will recombine bricks b'_1 and b''_1 into b_1 and replace the label of $-y$ on cell c by y . Hence $J_\Gamma(J_\Gamma((B, \sigma))) = (B, \sigma)$ in this case.

(b) c is in brick b_j , where $j > 1$. Note that our definition of when a cell labeled y can be used in Case I to define J_Γ depends only on the cells and the brick structure to the left of that cell. Hence, we can not use any of the cells labeled y to the left of c to define $J_\Gamma(J_\Gamma((B, \sigma)))$. Similarly, if we have two bricks b_s and b_{s+1} which lie entirely to the left of cell c such that $\text{last}(b_s) = \sigma_d > \text{first}(b_{s+1}) = \sigma_{d+1}$, the criteria to use cell d in the definition of J_Γ on $J_\Gamma((B, \sigma))$ depends only on the elements in bricks b_s and b_{s+1} . Thus, the only cell d which we could possibly use to define J_Γ on $J_\Gamma((B, \sigma))$ that lies to the left of c is the last cell of b_{j-1} . However, our conditions that either $\text{last}(b_{j-1}) < \text{first}(b_j) = \text{first}(b'_j)$ or $\text{last}(b_{j-1}) > \text{first}(b_j) = \text{first}(b'_j)$ with a Γ -match contained in the cells of b_{j-1} and b'_j force the first cell that can be used to define J_Γ on $J_\Gamma((B, \sigma))$ to be cell c . Thus, when we apply J_Γ to $J_\Gamma((B, \sigma))$, we will be in Case II using cell c and we will recombine bricks b'_j and b''_j into b_j and replace the label of $-y$ on cell c by y . Thus $J_\Gamma(J_\Gamma((B, \sigma))) = (B, \sigma)$ in this case.

Suppose (B, σ) is in Case II and we define $J_\Gamma((B, \sigma))$ at cell c , where c is last cell of b_j and $\sigma_c > \sigma_{c+1}$. Then by the same arguments that we used in Case I, there can be no cell labeled y to the left of this cell c in either (B, σ) or $J(B, \sigma)$ which can be used to define the involution J_Γ . This follows from the fact that the brick structure before cell c is unchanged between (B, σ) and $J(B, \sigma)$. Similarly, there can be no two bricks that lie entirely to the left of cell c in $J_\Gamma((B, \sigma))$ that can be combined under J_Γ . Thus, the first cell that we can use to define J_Γ to $J_\Gamma((B, \sigma))$ is cell c and it is easy to check that it satisfies the conditions of Case I. Thus, when we apply J_Γ to $J_\Gamma((B, \sigma))$, we will be in Case I using cell c and we will combine bricks b_j and b_{j+1} into a single brick b and replaced the label on cell c by y . Then it is easy to see that

when applying J_Γ to $J_\Gamma((B, \sigma))$, we will split b back into bricks b_j and b_{j+1} and change the label on cell c back to $-y$. Thus, $J_\Gamma(J_\Gamma((B, \sigma))) = (B, \sigma)$ in this case.

Hence J_Γ is an involution. Also, it is clear that if $J_\Gamma(B, \sigma) \neq (B, \sigma)$, then $\text{sgn}(B, \sigma)W(B, \sigma) = -\text{sgn}(J_\Gamma(B, \sigma))W(J_\Gamma(B, \sigma))$. Hence, it follows from (2.4) that

$$U_{\Gamma, n}(y) = n! \theta_\Gamma(h_n) = \sum_{O \in \mathcal{O}_{\Gamma, n}} \text{sgn}(O)W(O) = \sum_{O \in \mathcal{O}_{\Gamma, n}, J_\Gamma(O)=O} \text{sgn}(O)W(O).$$

Therefore, to compute $U_{\Gamma, n}(y)$, we must analyze the fixed points of J_Γ . Our next lemma characterizes the fixed points of J_Γ .

Lemma 2. *Let $B = (b_1, \dots, b_k)$ be a brick tabloid of shape (n) and $\sigma = \sigma_1 \dots \sigma_n \in S_n$. Then (B, σ) is a fixed point of J_Γ if and only if it satisfies the following properties:*

- (a) *if $i = 1$ or $i > 1$ and $\text{last}(b_{i-1}) < \text{first}(b_i)$, then b_i can have no cell labeled y so that σ must be increasing in b_i ,*
- (b) *if $i > 1$ and $\sigma_e = \text{last}(b_{i-1}) > \text{first}(b_i) = \sigma_{e+1}$, then there must be a Γ -match contained in the cells of b_{i-1} and b_i which must necessarily involve σ_e and σ_{e+1} and there can be at most $k - 1$ cells labeled y in b_i , and*
- (c) *if Γ has the property that, for all $\tau \in \Gamma$ such that $\text{des}(\tau) = j \geq 1$, the bottom elements of the descents in τ are $2, \dots, j + 1$, when reading from left to right, then $\text{first}(b_1) < \text{first}(b_2) < \dots < \text{first}(b_k)$.*

Proof. Suppose (B, σ) is a fixed point of J_Γ . Then it must be the case that in (B, σ) , there is no cell c to which either Case I or Case II applies. That is, when attempting to apply the involution J_Γ to (B, σ) , we cannot split any brick at a cell labeled y and

we cannot combine two consecutive bricks where the last cell of the first brick is larger than the first cell of the second brick.

For (a), note that if there is a cell labeled y in b_i and c is the left-most cell of b_i labeled with y , then c satisfies the conditions of Case I. Thus, there can be no cell labeled y in b_i .

For (b), note that if there is no Γ -match contained in the cells of b_{i-1} and b_i , then e satisfies the conditions of Case II. Thus, there must be a Γ -match contained in the cells of b_{i-1} and b_i . If there are k or more cells labeled y in b_i , then let c be the k^{th} cell, reading from left to right, which is labeled with y . Then we know there is τ -match contained in the cells of b_{i-1} and b_i which must necessarily involve σ_e and σ_{e+1} for some $\tau \in \Gamma$. But this τ -match must end weakly before cell c since otherwise τ would have at least $k + 1$ descents. Thus c would satisfy the conditions to apply Case I of our involution. Hence there can be no such c which means that each such brick can contain at most $k - 1$ descents.

To prove (c), suppose for a contradiction that there exist two consecutive bricks b_i and b_{i+1} such that $\sigma_e = \text{first}(b_i) > \text{first}(b_{i+1}) = \sigma_f$. There are two cases.

Case A. σ is increasing in b_i .

In this case, $\sigma_{f-1} = \text{last}(b_i)$. If $\sigma_{f-1} < \sigma_f$, then we know that $\sigma_e \leq \sigma_{f-1} < \sigma_f$ which contradicts our choice of σ_e and σ_f . Thus it must be the case that $\sigma_{f-1} > \sigma_f$. But then there is $\tau \in \Gamma$ such that $\text{des}(\tau) = j \geq 1$ and there is a τ -match in the cells of b_i and b_{i+1} involving the σ_{f-1} and σ_f . By our assumptions, σ_f can only play the role of 2 in such a τ -match. Hence there must be some σ_g with $e \leq g \leq f - 2$ which plays the role of 1 in this τ -match. But then we would have $\sigma_e \leq \sigma_g < \sigma_f$ which contradicts our choice of σ_e and σ_f . Thus σ cannot be increasing in b_i .

Case B. σ is not increasing in b_i .

In this case, by part (a), we know that it must be the case that $\sigma_{e-1} = \text{last}(b_{i-1}) > \sigma_e = \text{first}(b_i)$ and, by (b), there is $\tau \in \Gamma$ such that $\text{des}(\tau) = j \geq 1$ and there is a τ -match in the cells of b_{i-1} and b_i involving the cells σ_{e-1} and σ_e . Call this τ -match α and suppose that cell h is the bottom element of the last descent in α . It cannot be that $\sigma_e = \sigma_h$. That is, there can be no cell labeled y that occurs after cell h in b_i since otherwise the left-most such cell c would satisfy the conditions of Case I of the definition of J_Γ . But this would mean that σ is increasing in b_i starting at σ_h so that if $\sigma_e = \sigma_h$, then σ would be increasing in b_i which contradicts our assumption in this case. Thus there is some $2 \leq i \leq j$ such that σ_e plays the role of i in the τ -match α and σ_h plays the role of $j + 1$ in the τ -match α . But this means that σ_e is the smallest element in brick b_i . That is, let σ_c be the smallest element in b_i . If $\sigma_e \neq \sigma_c$, then σ_c must be the bottom of some descent in b_i which implies that $c \leq h$. But then σ_c is part of the τ -match α which means that σ_c must be playing the role of one of $i + 1, \dots, j + 1$ in the τ -match α and σ_e is playing the role of i in the τ -match α which is impossible if $\sigma_e \neq \sigma_c$. It follows that $\sigma_e \leq \sigma_{f-1}$. Hence, it can not be that case that $\sigma_{f-1} < \sigma_f$ since otherwise $\sigma_e < \sigma_f$. Thus it must be the case that $\sigma_{f-1} > \sigma_f$. But this means that there exists some $\delta \in \Gamma$ such that $\text{des}(\delta) = p \geq 1$ and there is a δ -match in the cells of b_i and b_{i+1} involving the σ_{f-1} and σ_f . Call this δ -match β . By assumption, the bottom elements of the descents in δ are $2, 3, \dots, p + 1$ so that σ_f must be playing the role of $2, 3, \dots, p + 1$ in the δ -match β . Let σ_g be the element that plays the role of 1 in the δ -match β . σ_g must be in b_i since δ must start with 1. But then we would have that $\sigma_e \leq \sigma_g < \sigma_f$ since σ_e is the smallest element in b_i .

Thus, both Case A and Case B are impossible. Hence we must have that

first(b_1) < first(b_2) < \dots < first(b_k). □

We note that if condition (3) of the Lemma fails, it may be that the first elements of the bricks do not form an increasing sequence. For example, it is easy to check that if $\Gamma = \{15342\}$, then the (B, σ) pictured in Figure 3.2 is such a fixed point of J_Γ .

			$-y$		y		$-y$	$-y$			$-y$			y			$-y$	$-y$
1	2	3	19	15	16	4	5	6	7	8	17	13	14	9	10	11	12	18

Figure 3.2: A fixed point of $J_{\{15342\}}$.

3.2 Results of the new involution

In this section, we shall compute the generating functions $NM_\Gamma(t, x, y)$ when $\Gamma = \{14253, 15243\}$, $\Gamma = \{142536\}$, and when $\Gamma = \{\tau_a\}$ for any $a \geq 2$ where $\tau_a \in S_{2a}$ is the permutation such that $\tau_1\tau_3 \dots \tau_{2a-1} = 12 \dots a$ and $\tau_2\tau_4 \dots \tau_{2a} = (2a)(2a-1) \dots (a+1)$. In each case, the permutations have at least two descents. Below are the main results.

Theorem 10. *Let $\Gamma = \{14253, 15243\}$. Then*

$$NM_\Gamma(t, x, y) = \left(\frac{1}{U_\Gamma(t, y)} \right)^x \text{ where } U_\Gamma(t, y) = 1 + \sum_{n \geq 1} U_{\Gamma, n}(y) \frac{t^n}{n!},$$

with $U_{\Gamma, 1}(y) = -y$, and for $n \geq 2$,

$$U_{\Gamma, n}(y) = (1 - y)U_{\Gamma, n-1}(y) - y^2(n - 3) (U_{\Gamma, n-4}(y) + (1 - y)(n - 5)U_{\Gamma, n-5}(y)) - y^3(n - 3)(n - 5)(n - 6)U_{\Gamma, n-6}(y).$$

Let $C_n = \frac{1}{n+1} \binom{2n}{n}$ be the n -th Catalan number. Let M_n be the $n \times n$ matrix whose elements on the main diagonal equals C_2 , whose elements on j -th diagonal above the main diagonal are C_{3j+2} , whose elements on the sub-diagonal are -1 , and whose elements in diagonal below the sub-diagonal are 0. Thus,

$$M_k = \begin{vmatrix} C_2 & C_5 & C_8 & C_{11} & \cdots & C_{3k-4} & C_{3k-1} \\ -1 & C_2 & C_5 & C_8 & \cdots & C_{3k-7} & C_{3k-4} \\ 0 & -1 & C_2 & C_5 & \cdots & C_{3k-10} & C_{3k-7} \\ 0 & 0 & -1 & C_2 & \cdots & C_{3k-13} & C_{3k-10} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C_2 & C_5 \\ 0 & 0 & 0 & 0 & \cdots & -1 & C_2 \end{vmatrix}.$$

Let P_k be the matrix obtained from M_k by replacing each C_m in the last column by C_{m-1} . Thus,

$$P_k = \begin{vmatrix} C_2 & C_5 & C_8 & C_{11} & \cdots & C_{3k-4} & C_{3k-2} \\ -1 & C_2 & C_5 & C_8 & \cdots & C_{3k-7} & C_{3k-5} \\ 0 & -1 & C_2 & C_5 & \cdots & C_{3k-10} & C_{3k-8} \\ 0 & 0 & -1 & C_2 & \cdots & C_{3k-13} & C_{3k-11} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C_2 & C_4 \\ 0 & 0 & 0 & 0 & \cdots & -1 & C_1 \end{vmatrix}.$$

Theorem 11. *Let $\tau = 142536$. Then*

$$NM_\tau(t, x, y) = \left(\frac{1}{U_\tau(t, y)} \right)^x \text{ where } U_\tau(t, y) = 1 + \sum_{n \geq 1} U_{\tau, n}(y) \frac{t^n}{n!},$$

with $U_{\tau,1}(y) = -y$, and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\Gamma,n-1}(y) + \sum_{k=0}^{\lfloor (n-8)/6 \rfloor} \det(M_{k+1})y^{3k+3}U_{n-6k-7}(y) \\ + \sum_{k=0}^{\lfloor (n-6)/6 \rfloor} \det(P_{k+1})(-y^{3k+2}) [U_{\tau,n-6k-4}(y) + yU_{\tau,n-6k-5}(y)].$$

Theorem 12. For any $n \geq 2$, let $\tau = \tau_1 \dots \tau_{2a} \in S_{2a}$ where $\tau_1 \tau_3 \dots \tau_{2a-1} = 123 \dots a$ and $\tau_2 \tau_4 \dots \tau_{2a} = (2a)(2a-1) \dots (a+1)$. Then

$$NM_{\tau}(t, x, y) = \left(\frac{1}{U_{\tau}(t, y)} \right)^x \text{ where } U_{\tau}(t, y) = 1 + \sum_{n \geq 1} U_{\tau,n}(y) \frac{t^n}{n!},$$

with $U_{\tau,1}(y) = -y$, and for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) \\ - \sum_{k=0}^{\lfloor (n-2a)/(2a) \rfloor} \binom{n - (k+1)a - 1}{(k+1)a - 1} y^{(k+1)a-1} U_{\tau_a, n - (2(k+1)a) + 1}(y) \\ + \sum_{k=0}^{\lfloor (n-2a-2)/(2a) \rfloor} \binom{n - (k+1)a - 2}{(k+1)a} y^{(k+1)a} U_{\tau_a, n - (2(k+1)a) - 1}(y).$$

3.2.1 The case $\Gamma = \{14253, 15243\}$

We first consider the proof of Theorem 10 in the case where $\Gamma = \{14253, 15243\}$, which is the simplest of our examples. For convenience, we first restate the statement of Theorem 10 below.

Theorem. Let $\Gamma = \{14253, 15243\}$. Then

$$NM_{\Gamma}(t, x, y) = \left(\frac{1}{U_{\Gamma}(t, y)} \right)^x \text{ where } U_{\Gamma}(t, y) = 1 + \sum_{n \geq 1} U_{\Gamma,n}(y) \frac{t^n}{n!},$$

with $U_{\Gamma,1}(y) = -y$, and for $n \geq 2$,

$$U_{\Gamma,n}(y) = (1 - y)U_{\Gamma,n-1}(y) - y^2(n - 3) (U_{\Gamma,n-4}(y) + (1 - y)(n - 5)U_{\Gamma,n-5}(y)) \\ - y^3(n - 3)(n - 5)(n - 6)U_{\Gamma,n-6}(y).$$

Proof. Let $\Gamma = \{14253, 15243\}$, we need to show that the polynomials

$$U_{\Gamma,n}(y) = \sum_{O \in \mathcal{O}_{\Gamma,n}, J_{\Gamma}(O)=O} \text{sgn}(O)W(O)$$

satisfy the following properties:

1. $U_{\Gamma,1}(y) = -y$, and
2. for $n \geq 2$,

$$U_{\Gamma,n}(y) = (1 - y)U_{\Gamma,n-1}(y) - y^2(n - 3) (U_{\Gamma,n-4}(y) + (1 - y)(n - 5)U_{\Gamma,n-5}(y)) \\ - y^3(n - 3)(n - 5)(n - 6)U_{\Gamma,n-6}(y).$$

It is easy to see when $n = 1$, the only fixed point comes from brick tabloid that has a single brick of size 1 which contains 1 and the label on cell 1 is $-y$. Thus $U_{\Gamma,1}(y) = -y$.

For $n \geq 2$, let $O = (B, \sigma)$ be a fixed point of J_{Γ} where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$. First we show that 1 must be in the first cell of B . That is, if $1 = \sigma_c$ where $c > 1$, then $\sigma_{c-1} > \sigma_c$. We claim that whenever we have a descent $\sigma_i > \sigma_{i+1}$ in σ , then σ_i and σ_{i+1} must be part of a Γ -match in σ . That is, it is either the case that (i) there are bricks b_s and b_{s+1} such that σ_i is the last cell of b_s and σ_{i+1} is the first cell

of b_{s+1} or (ii) there is a brick b_s that contains both σ_i and σ_{i+1} . In case (i), condition 3 of Lemma 2 ensures that σ_i and σ_{i+1} must be part of Γ -match. In case (ii), we know that cell i is labeled with y . It follows from condition (2) of Lemma 2 that it can not be that either $s = 1$ so that $b_s = b_1$ or that $s > 1$ and $\text{last}(b_{s-1}) < \text{first}(b_s)$ because those conditions force that σ is increasing in b_s . Thus we must have that $s > 1$ and $\text{last}(b_{s-1}) > \text{first}(b_s)$. Since (B, σ) is a fixed point of J_Γ , it cannot be that there is a Γ -match in σ which includes $\text{last}(b_{s-1})$ and $\text{first}(b_s)$ that ends weakly to the left of σ_i because then cell i would satisfy Case I of our definition of J_Γ and, hence, (B, σ) would not be a fixed point of J_Γ . Thus the Γ -match which includes $\text{last}(b_{s-1})$ and $\text{first}(b_s)$ must involve σ_i and σ_{i+1} . However, there can be no Γ -match that involves σ_{c-1} and σ_c since $\sigma_c = 1$ can only play the role of 1 in a Γ -match and each element of Γ starts with 1. Thus, we must have $\sigma_1 = 1$.

Next we claim that 2 must be in either cell 2 or cell 3 in O . For a contradiction, assume that 2 is in cell c for $c > 3$. Then once again $\sigma_{c-1} > \sigma_c$ so that there must be a Γ -match in σ that involves the two cells $c - 1$ and c in (B, σ) . However, In this case, the number which is in cell $c - 2$ must be greater than σ_c so that the only possible Γ -match that involves 2 must start from cell c where 2 plays the role of 1 in the match. Thus there is no Γ -match in σ that involves σ_{c-1} and σ_c . We now have two cases.

Case 1. 2 is in cell 2 of O .

In this case there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of (B, σ) or (ii) brick b_1 is a single cell filled with 1, and 2 is in the first cell of the second brick b_2 of O . In either case, we know that 1 is not part of a Γ -match in σ . So if we remove cell 1 from O and subtract 1 from the elements in the remaining cells, we will obtain a fixed point O' of J_Γ in $\mathcal{O}_{\Gamma, n-1}$.

Moreover, we can create a fixed point $O = (B, \sigma) \in \mathcal{O}_n$ of J_Γ satisfying the three conditions of Lemma 2 where $\sigma_2 = 2$ by starting with a fixed point $(B', \sigma') \in \mathcal{O}_{\Gamma, n-1}$ of J_Γ , where $B' = (b'_1, \dots, b'_r)$ and $\sigma' = \sigma'_1 \cdots \sigma'_{n-1}$, and then letting $\sigma = 1(\sigma'_1 + 1) \cdots (\sigma'_{n-1} + 1)$, and setting $B = (1, b'_1, \dots, b'_r)$ or setting $B = (1 + b'_1, \dots, b'_r)$.

It follows that fixed points in Case 1 will contribute $(1 - y)U_{\Gamma, n-1}(y)$ to $U_{\Gamma, n}(y)$.

Case 2. 2 is in cell 3 of $O = (B, \sigma)$.

Since there is no decrease within the first brick b_1 of $O = (B, \sigma)$, it must be the case that 2 is in the first cell of brick b_2 and there must be either a 14253-match or a 15243-match that involves the cells of the first two bricks. Therefore, we know that brick b_2 has at least 3 cells. In addition, we claim that 3 is in cell 5 of O since otherwise, 3 must be in some cell c for $c > 6$ and there must be a Γ -match between the two cells $c - 1$ and c in O . By the previous argument, we can see that if 3 is too far away from 1 and 2, then it must play the role of 1 in any match that involves cell c . Thus, the only possible Γ -match that contains cell c must also start at c and can never involve both cells $c - 1$ and c . Also, 3 cannot be in cell 2 nor 4 in O since both σ_2 and σ_4 are greater than 3, due to the Γ -match starting from cell 1. We now have two subcases depending on whether or not there is a Γ -match in O starting at cell 3.

Subcase 2.a. There is no Γ -match in O starting at cell 3.

In this case, we first choose a number x to fill in cell 2 of O . There are $n - 3$ choices for x . For each choice of $\sigma_2 = x$, we let d be the smallest of the remaining numbers, that is,

$$d = \min(\{1, 2, \dots, n\} - \{1, 2, 3, \sigma_2\}).$$

We claim that d must be either in cell 4 or cell 6 in (B, σ) . First, d cannot be in cell 7 since otherwise there would be a Γ -match in σ starting at cell 3. Next d cannot be a

cell c where $c > 7$ since otherwise $\sigma_{c-1} > \sigma_c = d$ which means that there must be a Γ -match in σ which includes both σ_{c-1} and σ_c . However, in the case, we would also have $\sigma_{c-2} > \sigma_c$ which implies the only role that σ_c can play in a Γ -match is 1.

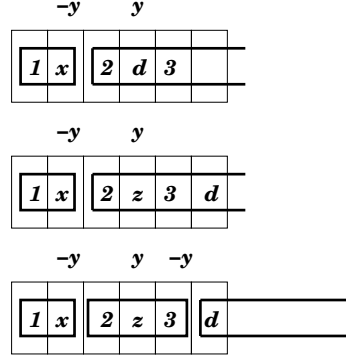


Figure 3.3: The possible choice for d in Subcase 2a.

This leaves us with three possibilities which are pictured in Figure 3.3. That is, either (i) d is in cell 4, (ii) d is in cell 6 and is in brick b_2 or (iii) d is in cell 6, but is the first element of brick b_3 . In case (i), we can remove that first four cells from B , reduce the remaining elements of σ to obtain a permutation $\alpha \in S_{n-4}$, and let $B' = (b_2 - 2, b_3, \dots, b_k)$ to obtain a fixed point (B', α) of J_Γ of size $n - 4$. Such fixed points will contribute $-y^2 U_{\Gamma, n-4}(y)$ to $U_{\Gamma, n}(y)$. In case (ii), we have $(n - 5)$ ways to choose the element z in cell 4. Then we can remove that first five cells from B , reduce the remaining elements of σ to obtain a permutation $\alpha \in S_{n-5}$, and let $B' = (b_2 - 3, b_3, \dots, b_k)$ to obtain a fixed point (B', α) of J_Γ of size $n - 5$. Such fixed points will contribute $-y^2 U_{\Gamma, n-5}(y)$ to $U_{\Gamma, n}(y)$. In case (iii), we have $(n - 5)$ ways to choose the element z in cell 4. Then we can remove that first five cells from B , reduce the remaining elements of σ to obtain a permutation $\alpha \in S_{n-5}$, and let $B' = (b_2 - 3, b_3, \dots, b_k)$ to obtain a fixed point (B', α) of J_Γ of size $n - 5$. Such fixed points will contribute $y^3 U_{\Gamma, n-5}(y)$ to $U_{\Gamma, n}(y)$. Therefore, the total contribution of the

fixed points from Subcase 2.a. is

$$-y^2(n-3)(U_{\Gamma, n-4}(y) + (1-y)(n-5)U_{\Gamma, n-5}(y)).$$

Subcase 2.b. There is a Γ -match in O starting at cell 3.

In this case, we first choose a number x to fill in cell 2 of O . There are $n-3$ choices for x . For each choice of σ_2 , let $d = \min(\{1, \dots, n\} - \{1, 2, 3, \sigma_2\})$. Then we claim that d must be in cell 7. That is, we can argue as in Subcase 2a that it cannot be that d in cell c for $c > 7$. But since there is a Γ -match starting at cell 3 we know $\sigma_4 > \sigma_7$ and $\sigma_6 > \sigma_7$ so that d cannot be in cells 4 or 6. We then have $(n-5)(n-6)$ ways to choose $\sigma_4 = z$ and $\sigma_6 = a$.

Next, by condition (b) of Lemma 2, we know that each brick in b in B can contain at most one descent. Since we know that b_2 must have size at least 3 because there is a Γ -match in σ starting at cell 1 which is contained in b_1 and b_2 , this means that either $b_2 = 3$ or $b_2 = 4$. We claim that b_2 is of size 4. That is, if $b_2 = 3$, then either (I) $a > d$ are in b_3 or (II) brick b_3 contains a single cell containing a and d is the first cell of b_4 . Case (I) cannot happen because then $\text{last}(b_2) = 3 < \text{first}(b_3) = a$ which implies that the elements in b_3 must be increasing by condition (a) of Lemma 2. Case (II) cannot happen because that $\text{last}(b_3) = a > \text{first}(b_4) = d$ which implies there must be a Γ -match contained in the cells of b_3 and b_4 which involves both $\sigma_6 = a$ and $\sigma_7 = d$ which is impossible since $a > d$. Thus we are in the situation pictured in Figure 3.4.

Then we can remove that first six cells from B , reduce the remaining elements of σ to obtain a permutation $\alpha \in S_{n-6}$, and let $B' = (b_3, \dots, b_k)$ to obtain a fixed point (B', α) of J_Γ of size $n-6$. Such fixed points will contribute $(n-3)(n-$

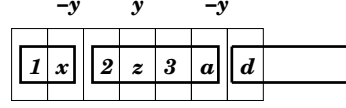


Figure 3.4: Subcase 2b.

5) $(n - 6)y^3U_{\Gamma,n-6}(y)$ to $U_{\Gamma,n}(y)$.

In total, we obtain the recursion for $U_{\Gamma,n}(y)$ as follows.

$$U_{\Gamma,n}(y) = (1 - y)U_{\Gamma,n-1}(y) - y^2(n - 3)(U_{\Gamma,n-4}(y) + (1 - y)(n - 5)U_{\Gamma,n-5}(y)) + y^3(n - 3)(n - 5)(n - 6)U_{\Gamma,n-6}(y).$$

This proves Theorem 10. □

Using Theorem 10, we computed the initial values of the $U_{\Gamma,n}(y)$ s which are given in Table 3.1.

Table 3.1: The polynomials $U_{\Gamma,n}(-y)$ for $\Gamma = \{14253, 15243\}$

n	$U_{\Gamma,n}(y)$
1	$-y$
2	$-y + y^2$
3	$-y + 2y^2 - y^3$
4	$-y + 3y^2 - 3y^3 + y^4$
5	$-y + 4y^2 - 4y^3 + 4y^4 - y^5$
6	$-y + 5y^2 - 2y^3 + 2y^4 - 5y^5 + y^6$
7	$-y + 6y^2 + 5y^3 - 28y^4 + 5y^5 + 6y^6 - y^7$
8	$-y + 7y^2 + 19y^3 - 123y^4 + 123y^5 - 19y^6 - 7y^7 + y^8$
9	$-y + 8y^2 + 42y^3 - 334y^4 + 588y^5 - 334y^6 + 42y^7 + 8y^8 - y^9$
10	$-y + 9y^2 + 76y^3 - 726y^4 + 1606y^5 - 1606y^6 + 726y^7 - 76y^8 - 9y^9 + y^{10}$

Using these initial values of the $U_{\Gamma,n}(y)$ s, one can then compute the initial values of $NM_{\Gamma,n}(x, y)$ which are given in Table 3.2.

Table 3.2: The polynomials $MN_{\Gamma,n}(x, y)$ for $\Gamma = \{14253, 15243\}$

n	$NM_{\Gamma,n}(x, y)$
1	xy
2	$xy + x^2y^2$
3	$xy + xy^2 + 3x^2y^2 + x^3y^3$
4	$xy + 4xy^2 + 7x^2y^2 + xy^3 + 4x^2y^3 + 6x^3y^3 + x^4y^4$
5	$xy + 11xy^2 + 15x^2y^2 + 9xy^3 + 30x^2y^3 + 25x^3y^3 + xy^4 + 5x^2y^4 + 10x^3y^4 + 10x^4y^4 + x^5y^5$
6	$xy + 26xy^2 + 31x^2y^2 + 58xy^3 + 146x^2y^3 + 90x^3y^3 + 22xy^4 + 79x^2y^4 + 120x^3y^4 + 65x^4y^4 + xy^5 + 6x^2y^5 + 15x^3y^5 + 20x^4y^5 + 15x^5y^5 + x^6y^6$
7	$xy + 57xy^2 + 63x^2y^2 + 282xy^3 + 588x^2y^3 + 301x^3y^3 + 252xy^4 + 770x^2y^4 + 896x^3y^4 + 350x^4y^4 + 51xy^5 + 210x^2y^5 + 364x^3y^5 + 350x^4y^5 + 140x^5y^5 + xy^6 + 7x^2y^6 + 21x^3y^6 + 35x^4y^6 + 35x^5y^6 + 21x^6y^6 + x^7y^7$

3.2.2 The case $\Gamma = \{142536\}$

In this section, we shall study the generating function $U_{\tau}(t, y)$ where $\tau = 142536$.

We let J_{τ} denote the involution J_{Γ} from Section 3.1 where $\Gamma = \{\tau\}$. We claim that the polynomials

$$U_{\tau,n}(y) = \sum_{O \in \mathcal{O}_{\tau,n}, J_{\tau}(O)=O} \text{sgn}(O)W(O)$$

satisfy the following properties:

1. $U_{\tau,1}(y) = -y$, and

2. for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\Gamma,n-1}(y) + \sum_{k=0}^{\lfloor (n-8)/6 \rfloor} \det(M_{k+1})y^{3k+3}U_{n-6k-7}(y) + \sum_{k=0}^{\lfloor (n-6)/6 \rfloor} \det(P_{k+1})(-y^{3k+2}) [U_{\tau,n-6k-4}(y) + yU_{\tau,n-6k-5}(y)].$$

It is easy to see when $n = 1$, the only fixed point comes from brick tabloid that has a single brick of size 1 which contains 1 and the label on cell 1 is $-y$. Thus

$$U_{\tau,1}(y) = -y.$$

For $n \geq 2$, let $O = (B, \sigma)$ be a fixed point of I_Γ where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \cdots \sigma_n$. First we show that 1 must be in the first cell of B . That is, if $1 = \sigma_c$ where $c > 1$, then $\sigma_{c-1} > \sigma_c$. We claim that whenever we have a descent $\sigma_i > \sigma_{i+1}$ in σ , then σ_i and σ_{i+1} must be part of a τ -match in σ . That is, it is either the case that (i) there are bricks b_s and b_{s+1} such that σ_i is the last cell of b_s and σ_{i+1} is the first cell of b_{s+1} or (ii) there is a brick b_s that contains both σ_i and σ_{i+1} . In case (i), condition 3 of Lemma 2 ensures that σ_i and σ_{i+1} must be part of τ -match. In case (ii), we know that cell i is labeled with y . It follows from condition (2) of Lemma 2 that it can not be that either $s = 1$ so that $b_s = b_1$ or that $s > 1$ and $\text{last}(b_{s-1}) < \text{first}(b_s)$ because those conditions force that σ is increasing in b_s . Thus we must have that $s > 1$ and $\text{last}(b_{s-1}) > \text{first}(b_s)$. Since (B, σ) is a fixed point of J_τ , it cannot be that there is a τ -match in σ which includes $\text{last}(b_{s-1})$ and $\text{first}(b_s)$ that ends weakly to the left of σ_i because then cell i would satisfy Case I of our definition of J_τ and, hence, (B, σ) would not be a fixed point of J_τ . Thus the τ -match which includes $\text{last}(b_{s-1})$ and $\text{first}(b_s)$ must involve σ_i and σ_{i+1} . However, there can be no τ -match that involves σ_{c-1} and σ_c since $\sigma_c = 1$ can only play the role of 1 in τ -match and τ starts with 1. Thus we must have $\sigma_1 = 1$.

Next we claim that 2 must be in either cell 2 or cell 3 in O . For a contradiction, assume that 2 is in cell c for $c > 3$. Then once again $\sigma_{c-1} > \sigma_c$ so that there must be a τ -match in σ that involves the two cells $c - 1$ and c in (B, σ) . However, since 2 is too far from 1 in B , the only possible 142536-match that involves 2 must start from cell c where 2 plays the role of 1 in the match. We then have two cases.

Case 1. 2 is in cell 2 of O .

In this case, there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of (B, σ) or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of (B, σ) . In either case, we know that 1 is not part of a τ -match in (B, σ) . So if we remove cell 1 from (B, σ) and subtract 1 from the elements in the remaining cells, we will obtain a fixed point (B', σ') of J_Γ in $\mathcal{O}_{\Gamma, n-1}$.

Moreover, we can create a fixed point $O = (B, \sigma) \in \mathcal{O}_n$ satisfying the three conditions of Lemma 2 where $\sigma_2 = 2$ by starting with a fixed point $(B', \sigma') \in \mathcal{O}_{\Gamma, n-1}$ of J_Γ , where $B' = (b'_1, \dots, b'_r)$ and $\sigma' = \sigma'_1 \cdots \sigma'_{n-1}$, and then letting $\sigma = 1(\sigma'_1 + 1) \cdots (\sigma'_{n-1} + 1)$, and setting $B = (1, b'_1, \dots, b'_r)$ or setting $B = (1 + b'_1, \dots, b'_r)$.

It follows that fixed points in Case 1 will contribute $(1-y)U_{\Gamma, n-1}(y)$ to $U_{\Gamma, n}(y)$.

Case 2. 2 is in cell 3 of $O = (B, \sigma)$.

Since there is no decrease within the first brick b_1 of $O = (B, \sigma)$, it must be the case that 2 is in the first cell of brick b_2 and there must be a 142536-match that involves the cells of the first two bricks. Therefore, we know that brick b_2 has at least 4 cells.

To analyze this case, it will be useful to picture $O = (B, \sigma)$ as a 2-line array $A(O)$ where the elements in the i -th column are σ_{2i-1} and σ_{2i} reading from bottom to top. In $A(O)$, imagine that we draw a directed arrow from the cell containing i to the cell containing $i + 1$. Then it is easy to see that a τ -match correspond to block of points as pictured in Figure 3.5

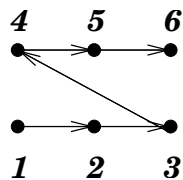


Figure 3.5: A 142536-match as a 2-line array.

Now imagine that $A(0)$ starts with series of τ -matches starting at positions $1, 3, 5, \dots$. We have pictured this situation at the top of Figure 3.6. Now consider the brick structure of $O = (B, \sigma)$. Since the elements of b_1 must be increasing and $\sigma_2 > \sigma_3$, it must be the case that $b_1 = 2$ and $b_2 \geq 4$. We claim that $b_2 = 4$ because if $b_2 > 4$, then $\sigma_6 > \sigma_7$ would be a descent in b_2 . Thus cell 6 would be labeled with a y . The τ -match starting at cell 1 ends a cell 6 so that cell 6 would satisfy Case I of our definition of J_τ which contracts that the fact that $O = (B, \sigma)$ is a fixed point of J_τ . Now the fact that $\sigma_6 > \sigma_7$ implies that $b_3 \geq 2$ since there must be a τ -match that involves σ_6 and σ_7 . Now if there is a τ -match starting at cell 7, then we can see that $\sigma_8 > \sigma_9$. It cannot be that σ_8 and σ_9 are both in b_3 because it would follow that cell 8 would be labeled with a y and the τ -match starting at σ_3 would end at cell 8. Thus cell 8 would be in Case I of our definition of J_τ which contracts that the fact that $O = (B, \sigma)$ is a fixed point of J_τ . Thus it must be the case that $b_3 = 2$. But the τ -match starting at cell 7 forces $\sigma_8 > \sigma_9$ so that there is a decrease between $\text{last}(b_3)$ and $\text{first}(b_4)$ which implies that there is τ contained in b_3 and b_4 , which then means that $b_4 \geq 4$. Now if there is a τ -matches starting at σ_9 , then it must be the case that $\sigma_{12} > \sigma_{13}$. Hence, it cannot be $b_4 > 4$ since otherwise cell 12 is labeled with a y . Since the τ -match starting a cell 7 ends at cell 12, then cell 12 would be in Case I of our definition of J_τ which contracts that the fact that $O = (B, \sigma)$ is a fixed point of J_τ . Thus it must be the case that $b_4 = 4$. We can continue to reason in this way to conclude that if there are τ -matches starting at cells $1, 3, 7, 9, \dots, 6k + 1, 6k + 3$, then $b_{2i-1} = 2$ for $i = 1, \dots, 2k + 1$ and $b_{2i} = 4$ for $i = 1, \dots, 2k$. Similarly, if there are τ -matches starting at cells $1, 3, 7, 9, \dots, 6k + 1$ but no τ -match starting at cell $6k + 3$, then $b_{2i-1} = 2$ for $i = 1, \dots, 2k$ and $b_{2i} = 4$ for $i = 1, \dots, 2k - 1$ and $b_{2k} \geq 4$.

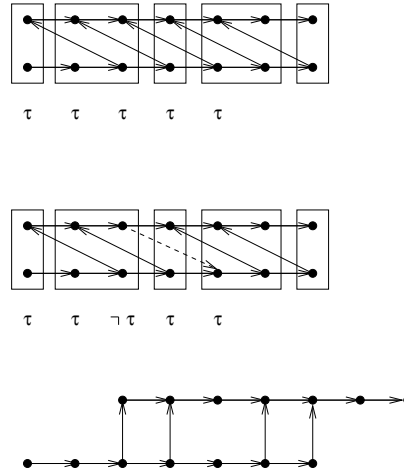


Figure 3.6: Fixed points that start with series of τ -matches.

Note that our arguments above did not use the fact that there were τ -matches starting at cells 5, 11, \dots . Indeed, these matches are not necessary to force the brick structure described above. For example, suppose that there were no τ -match starting at cell 5 but there were τ -matches starting at cell 7. We have pictured this situation on the second line of Figure 3.6 where we have written $\neg\tau$ below the position corresponding to cell 5 to indicate that there is not a τ -match starting at cell 5. Then one can see from the diagram pictured in the second line of Figure 3.6, that it must be the case that $\sigma_6 < \sigma_9$. It follows that if one looks at the requirements on σ to start with such a series of τ -matches, then σ must be a linear extension of poset whose Hasse diagram is pictured at the bottom of Figure 3.6.

There are now two cases depending on where the sequence of τ -matches starting at positions 1, 3, 7, 9, \dots ends.

Case 2.1. There are τ -matches in σ starting at positions 1, 3, 7, 9, $\dots, 6k + 3$, but there is no τ -match starting at position $6k + 7$. This situation is pictured in Figure 3.7 in the case where $k = 2$.

In this case, we claim that $\{\sigma_1, \dots, \sigma_{6k+8}\} = \{1, 2, \dots, 6k+8\}$. If not, then i be the least element in $\{1, 2, \dots, 6k+8\} - \{\sigma_1, \dots, \sigma_{6k+8}\}$. The question then becomes for which j is $\sigma_j = i$. It easy to see from the diagram at the top of Figure 3.7, that $\sigma_{6k+8} > \sigma_r$ for $r = 1, \dots, 6k+7$. This implies that $\sigma_{6k+8} \geq 6k+8$. But since $i \in \{1, 2, \dots, 6k+8\} - \{\sigma_1, \dots, \sigma_{6k+8}\}$, it must be the case that $\sigma_{6k+8} > 6k+8 \geq i$.

We claim that j cannot equal $6k+9$. That is, if $i = 6k+9$, then $\sigma_{6k+8} > \sigma_{6k+9}$. It cannot be that σ_{6k+8} and σ_{6k+9} are in brick b_{2k+3} because then σ_{6k+8} is labeled with y and there is a τ -match contained in bricks b_{2k+2} and b_{2k+3} that ends before cell $6k+8$ which means that cell $6k+8$ satisfies Case 1 of our definition of J_τ which violates our assumption that (B, σ) is fixed point of J_τ . If σ_{6k+9} starts brick b_{2k+4} , then brick b_{2k+3} must be of size 2 and there must be a τ -match contained in bricks b_{2k+3} and b_{2k+4} that involves σ_{6k+8} and σ_{6k+9} . But since $\sigma_{2k+8} > \sigma_{2k+9}$, that τ -match can only start at cell $6k+7$ which violates our assumption in this case.

Next we claim that j cannot be $\geq 6k+10$. That is, if $j \geq 6k+10$, then both σ_{j-2} and σ_{j-1} are greater than $\sigma_j = i$. Thus σ_{j-1} and σ_j must be part of τ -match in σ . But then the elements in two cells before cell j are bigger than that in cell j which means that the only role that σ_j can play in a τ -match is 1. Thus there can be no τ -match that includes σ_{j-1} and σ_j .

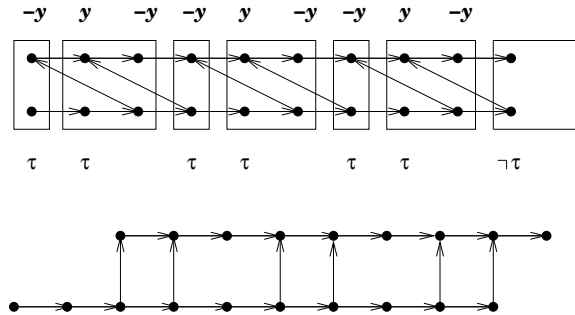


Figure 3.7: Fixed points that start with series of τ -matches in Case 2.1.

Let α be the permutation that is obtained from σ by removing the elements $1, \dots, 6k + 7$ and subtracting $6k + 7$ from the remaining elements. Let B' be the brick structure $(b_{2k+3} - 1, b_{2k+4}, \dots, b_k)$. Then it is easy to see that (B', α) is a fixed point of J_τ is size $n - 6k - 7$.

Vice versa, suppose we start with a fixed point (B', α) of J_τ whose size $n - 6k - 7$ where $B' = (d_1, d_2, \dots, d_s)$. Then we can obtain a fixed point (B, σ) of size n which has τ -matches in σ starting at positions $1, 3, 7, 9, \dots, 6k + 3$, but no τ -match starting at position $6k + 7$ by letting $\sigma_1 \dots \sigma_{6k+7}$ be any permutation of $1, \dots, 6k + 7$ which is a linear extension of the poset whose Hasse diagram is pictured at the bottom of Figure 3.7 and letting $\sigma_{6k+8} \dots \sigma_n$ be the sequence that results by adding $6k + 7$ to each element of α . Then let $B = (b_1, \dots, b_{2k+2}, d_1 + 1, d_2, \dots, d_s)$ where $b_{2i+1} = 2$ for $i = 0, \dots, k$ and $b_{2i} = 4$ for $i = 1, \dots, k + 1$.

It follows that contribution to $U_{\tau,n}(y)$ from the fixed points in Case 2.1 equal

$$\sum_{k=0}^{\lfloor \frac{n-8}{6} \rfloor} G_{6k+7} y^{3k+3} U_{\tau,n-6k-7},$$

where G_{6k+7} is the number of linear extensions of the poset pictured at the bottom of Figure 3.7 of size $6k + 7$.

Next, we want to compute the number of linear extensions of G_{6k+7} . It is easy to see that the left-most two elements at the bottom of the Hasse diagram of G_{6k+7} must be first two elements of the linear extension and the right-most element at the top of the Hasse diagram must be the largest element in any linear extension of G_{6k+7} . Thus the number of linear extensions of \bar{G}_{6k+4} which is the Hasse diagram of G_{6k+7} with those three elements removed, equals the number of linear extension of G_{6k+7} .

We have pictured the Hasse diagrams of \bar{G}_4 , \bar{G}_{10} and \bar{G}_{16} in Figure 3.8.

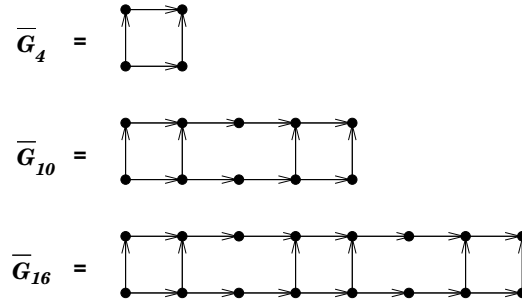


Figure 3.8: The Hasse diagram of \bar{G}_{6k+4} for $k = 0, 1, 2$.

Now let $A_0 = 1$ and A_{k+1} be the number of linear extensions of \bar{G}_{6k+4} for $k \geq 0$. It is easy to see that $A_1 = 2$. There is a natural recursion satisfied by the A_k , namely, for $k > 1$,

$$A_{k+1} = \sum_{j=0}^k C_{2+3j} A_{k-j}$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number. First, consider the number of linear extensions of the Hasse diagram of the poset D_n with n columns of the type pictured in Figure 3.9. It is easy to see that this is the number of standard tableaux of shape (n^2) which is well known to equal to C_n .

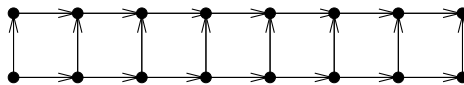


Figure 3.9: The Hasse diagram of D_n .

Next if we look at the Hasse diagram of \bar{G}_{6k+4} it is easy to see that there are no relation that is forced between the elements in columns $3i$ for $i = 1, \dots, k$. Now suppose that we partition the set of linear extensions of \bar{G}_{6k+4} by saying the bottom element in column $3i$ is less than the top element in column $3i$ for $i = 1, \dots, j$ and the top element of column $3j + 3$ is less than the bottom elements of column $3j + 3$. Then

we will have a situation as pictured in Figure 3.10 in the case where $k = 6$ and $j = 2$. One can see that when one straightens out the resulting Hasse diagram, it starts with the Hasse diagram of D_{2+3j} and all those elements must be less than the elements in the top part of Hasse diagram which is a copy of the Hasse diagram of $\bar{G}_{6(k-j-1)+4}$.

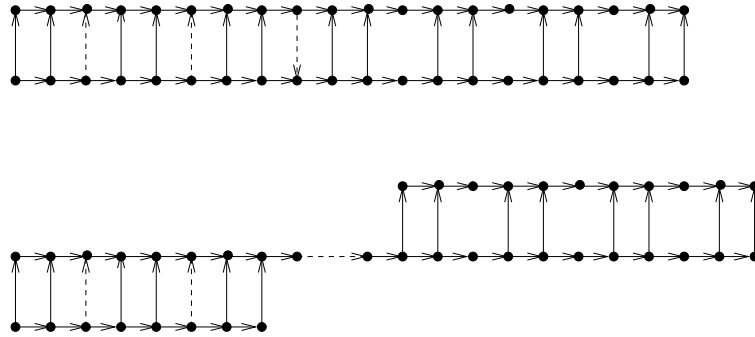


Figure 3.10: Partitioning the Hasse Diagram of \bar{G}_{6k+4} .

Now consider the determinant of the $n \times n$ matrix M_n whose elements on the main diagonal are C_2 , the elements on the j -diagonal above the main are C_{2+3j} for $j \geq 1$, the elements on the sub-diagonal are -1 , and the elements below the sub-diagonal are 0. For example we have pictured in M_7 in Figure 3.11. It is then easy to see that $\det(M_1) = C_2 = 2$. For $n > 1$ if we expand the determinant by minors about the first row, then we see that we have the recursion

$$\det(M_k) = \sum_{j=0}^{k-1} C_{2+3j} \det(M_{k-j-1}),$$

where we set $\det(M_0) = 1$.

For example, suppose that we expand the determinant M_7 pictured in Figure 3.11 about the element of C_8 in the first row. Then in the next two rows, we are forced to expand about the -1 's. It is easy to see that the total sign of these expansion is always $+1$ so that in this case, we would get a contribution of $C_8 \det(M_4)$ to $\det(M_7)$.

$$\begin{vmatrix} C_2 & C_5 & C_8 & C_{11} & C_{14} & C_{17} & C_{20} \\ -1 & C_2 & C_5 & C_8 & C_{11} & C_{14} & C_{17} \\ 0 & -1 & C_2 & C_5 & C_8 & C_{11} & C_{14} \\ 0 & 0 & -1 & C_2 & C_5 & C_8 & C_{11} \\ 0 & 0 & 0 & -1 & C_2 & C_5 & C_8 \\ 0 & 0 & 0 & 0 & -1 & C_2 & C_5 \\ 0 & 0 & 0 & 0 & 0 & -1 & C_2 \end{vmatrix}$$

Figure 3.11: The matrix M_7 .

Thus it follows that $A_n = \det(M_n)$ for all n .

Hence the contribution to $U_{\tau,n}$ from the fixed points in Case 1 equals

$$\sum_{k=0}^{\lfloor \frac{n-8}{6} \rfloor} \det(M_{k+1}) y^{3k+3} U_{\tau,n-6k-7}.$$

Case 2.2 There are τ -matches in σ starting at positions 1, 3, 7, 9, \dots , $6k + 1$, but there is no τ -match starting at position $6k + 3$. This situation is pictured in Figure 3.12 in the case where $k = 3$.

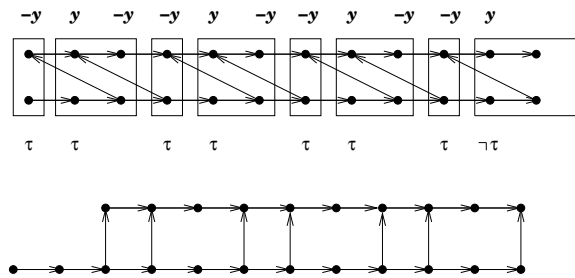


Figure 3.12: Fixed points that start with series of τ -matches in Case 2.2.

In this case, we claim that $\{\sigma_1, \dots, \sigma_{6k+5}\} = \{1, 2, \dots, 6k + 5\}$. If not, then let i be the least element in $\{1, 2, \dots, 6k + 5\} - \{\sigma_1, \dots, \sigma_{6k+5}\}$. The question then

becomes for which j is $\sigma_j = i$. It is easy to see from the diagram at the top of Figure 3.12, that $\sigma_{6k+6} > \sigma_r$ for $r = 1, \dots, 6k+5$ and that $\sigma_{6k+5} > \sigma_r$ for $r = 1, \dots, 6k+5$. This implies that $\sigma_{6k+5} \geq 6k+5$, but since $i \in \{1, 2, \dots, 6k+5\} - \{\sigma_1, \dots, \sigma_{6k+5}\}$, it follows that $6k+5 < \sigma_{6k+5} < \sigma_{6k+6}$.

It cannot be that $i = \sigma_{6k+7}$ because then $\sigma_{6k+6} > \sigma_{6k+7}$. Note that $\sigma_{6k+3}, \sigma_{6k+4}, \sigma_{6k+5}$, and σ_{6k+6} are elements of brick b_{2k+2} . If σ_{6k+7} was also an element of brick b_{2k+2} , then σ_{6k+6} would be marked with a y and there is a τ -match contained in bricks b_{2k+1} and b_{2k+2} that ends at cell $6k+6$ so that we could apply Case 1 of the involution J_τ at cell $6k+6$, which violates our assumption that (B, σ) was a fixed point of J_τ . If σ_{6k+7} starts brick b_{2k+3} , then there must be a τ -match that involves σ_{6k+6} and σ_{6k+7} and is contained in bricks b_{2k+2} and b_{2k+3} . Since we are assuming that there is no τ -match cannot starting at σ_{6k+3} , it must be the case that there is a τ -match starting at σ_{6k+5} . But then we have that situation pictured in Figure 3.13. In Figure 3.13, the dark arrows are forced by the τ -matches starting at σ_{6k+1} and σ_{6k+5} . However the top two elements in brick b_{2k+2} are σ_{6k+5} and σ_{6k+6} , which are both greater than i . This means that the dotted arrow is forced which implies that there is a τ -match starting at cell σ_{6k+3} .

Finally, it cannot be the case that $j > 6k+7$, because then it must be the case that $\sigma_{j-1} > \sigma_j$ so that σ_{j-1} and σ_j must be part of a τ -match in σ . But in this situation, the elements $1, \dots, i-1$ lie in cells that are more than 2 cells away from the cell containing i . This means that in any τ -match in σ containing the element i , i can only play the role of 1 in that τ -match. Thus, there could not be a τ -match containing σ_{j-1} and σ_j .

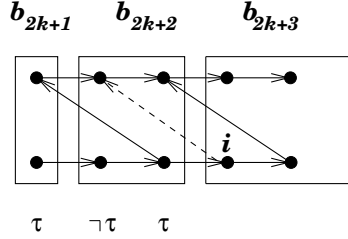


Figure 3.13: i starts brick b_{2k+3} .

Next, consider the possible j such that $\sigma_j = 6k + 6$. It cannot be that $j > 6k + 7$, because then it must be the case that $\sigma_{j-1} > \sigma_j$ so that σ_{j-1} and σ_j must be part of a τ -match in σ . But in this situation, the elements $1, \dots, 6k + 5$ lie in cells that are more than 2 cells away from the cell containing $6k + 6$. This means that in any τ -match containing the element $6k + 6$ in σ , $6k + 6$ can only play the role of 1 in that τ -match. Thus there could not be a τ -match in σ containing σ_{j-1} and σ_j . It follows that $6k + 6 = \sigma_{6k+6}$ or σ_{6k+7} . Let α be the permutation that is obtained from σ by removing the elements $1, \dots, 6k + 4$, setting $\alpha_1 = 1$, and letting $\alpha_2 \dots, \alpha_{n - (6k + 4)}$ be the result of subtracting $6k + 5$ from $\sigma_{6k+6} \dots \sigma_n$. Let B' be the brick structure $(b_{2k+2} - 2, b_{2k+3}, \dots, b_k)$. Then it is easy to see that (B', α) is a fixed point of J_τ is size $n - 6k - 4$ that starts with a brick of size at least 2.

Vice versa, suppose we start with a fixed point (B', α) of J_τ whose size $n - 6k - 4$ that starts with a brick of size at least 2 where $B' = (d_1, d_2, \dots, d_s)$. Then we can obtain a fixed point (B, σ) of size n which has τ -matches in σ starting at positions $1, 3, 7, 9, \dots, 6k + 1$, but no τ -match starting at position $6k + 3$, by letting $\sigma_1 \dots \sigma_{6k+5}$ be any permutation of $1, \dots, 6k + 5$ which is a linear extension of the poset whose Hasse diagram is pictured at the bottom of Figure 3.12 and letting $\sigma_{6k+6} \dots \sigma_n$ be the sequence that results by adding $6k + 5$ to each element of $\alpha_2 \dots \alpha_{n - (6k+4)}$. We let $B = (b_1, \dots, b_{2k+1}, d_1 + 2, d_2, \dots, d_s)$ where $b_{2i+1} = 2$ for $i = 0, \dots, k$ and $b_{2k} = 4$ for

$i = 1, \dots, k$.

Note that for any n , our arguments above show that the only fixed points (D, γ) of J_τ of size n where $D = (d_1, \dots, d_k)$ and $\sigma = \sigma_1 \dots \sigma_n$ which do not start with a brick of size at least 2 are the ones that start with a brick $b_1 = 1$ where $\sigma_1 = 1$ and $\sigma_2 = 2$. Clearly such fixed points are counted by $-yU_{n-1,y}$ because d_1 would have weight $-y$ and $((d_2, \dots, d_k), (\sigma_2 - 1)(\sigma_3 - 1) \dots (\sigma_n - 1))$ could be any fixed point of J_τ of size $n - 1$. It follows that sum of the weights of all fixed points of J_τ of size n which start with a brick of size at least 2 is equal to

$$U_{\tau,n} - (-yU_{n-1,\tau}) = U_{\tau,n} + yU_{n-1,\tau}.$$

It follows that contribution to $U_{\tau,n}$ from the fixed points in Case 2.2 equal

$$- \sum_{k=0}^{\lfloor \frac{n-6}{6} \rfloor} G_{6k+4} y^{3k+2} (U_{\tau,n-6k-4} + yU_{\tau,n-6k-5}),$$

where G_{6k+4} is the number of linear extensions of the poset pictured at the bottom of Figure 3.12 of size $6k + 4$.

Next we want to compute the number of linear extensions of G_{6k+4} . It is easy to see that the left-most two elements at the bottom of the Hasse diagram of G_{6k+4} must be first two elements of the linear extension. Thus the number of linear extensions of \bar{G}_{6k+2} which is the Hasse diagram of G_{6k+4} with those two elements removed, equals the number of linear extension of G_{6k+4} . We have pictured the Hasse diagrams of \bar{G}_2 , \bar{G}_8 and \bar{G}_{14} in Figure 3.14.

Now let $B_0 = 1$ and B_{k+1} be the number of linear extensions of \bar{G}_{6k+2} for $k \geq 0$. It is easy to see that $B_1 = 1$. Again there is a natural recursion satisfied by the B_k s,

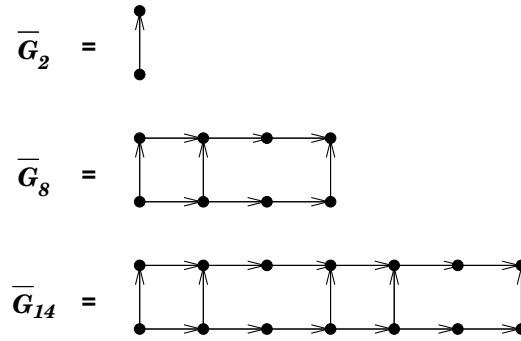


Figure 3.14: The Hasse diagram of \bar{G}_{6k+2} for $k = 0, 1, 2$.

namely, for $k > 1$,

$$B_{k+1} = C_{3k+1} + \sum_{j=0}^{k-1} C_{2+3j} B_{k-j-1},$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number.

As in the case of the posets \bar{G}_{6k+4} , there is no relations that is forced between the elements of the elements in columns $3i$ for $i = 1, \dots, k$. Now suppose that we partition the set of linear extensions of \bar{G}_{6k+2} by saying the bottom element in column $3i$ is less than the top element in column $3i$ for $i = 1, \dots, j$ and the top element of column $3j + 3$ is less than the bottom elements of column $3j + 3$. First if $j = k$, then we will have a copy of D_{3k+1} which gives a contribution of C_{3k+1} to the number of linear extensions of \bar{G}_{6k+4} . If $j < k$, then we will have a situation as pictured in Figure 3.15 in the case where $k = 6$ and $j = 2$. One can see that when one straightens out the resulting Hasse diagram, one obtains a diagram that starts with the Hasse diagram of D_{2+3j} and all those elements must be less than the elements in the top part of Hasse diagram which is a copy of the Hasse diagram of $\bar{G}_{6(k-j-1)+2}$.

Let P_n be the matrix that is obtained from the matrix M_n by replacing the elements C_m in the last column by C_{m-1} . For example we have pictured in P_7 in Figure 3.16. It is then easy to see that $\det(P_1) = 1$. For $n > 1$ if we expand the

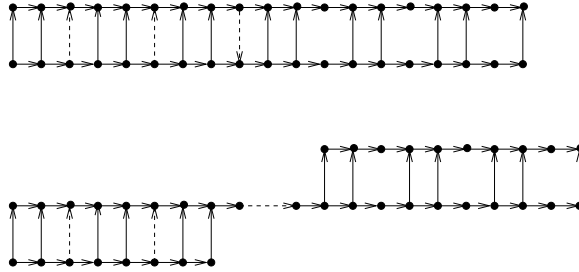


Figure 3.15: Partitioning the Hasse Diagram of \bar{G}_{6k+2} .

determinant by minors about the first row, then we see that we have the recursion

$$\det(P_k) = C_{3k-2} + \sum_{j=0}^{k-2} C_{2+3j} \det(P_{k-j-1}),$$

where we set $\det(P_0) = 1$.

For example, suppose that we expand the determinant P_7 pictured in Figure 3.16 about the element of C_{19} in the first row. Then in the next five rows, we would be forced to expand about the -1 's. It is easy to see that the total sign of these expansion is always $+1$ so that in this case, we would get a contribution of C_{19} to the $\det(P_7)$. Expanding the determinant about the other elements in the first row gives the remaining terms of the recursion just like it did in the expansion of the determinant of M_n .

Thus it follows that $B_n = \det(P_n)$ for all n .

Hence the contribution of fixed points of J_τ to $U_{\tau,n}(y)$ in the Case 2.2 equals

$$- \sum_{k=0}^{\lfloor \frac{n-6}{6} \rfloor} \det(P_{k+1}) y^{3k+2} (U_{\tau,n-6k-4} + y U_{\tau,n-6k-5}).$$

$$\begin{pmatrix} C_2 & C_5 & C_8 & C_{11} & C_{14} & C_{17} & C_{19} \\ (-1) & C_2 & C_5 & C_8 & C_{11} & C_{14} & C_{16} \\ 0 & (-1) & C_2 & C_5 & C_8 & C_{11} & C_{13} \\ 0 & 0 & (-1) & C_2 & C_5 & C_8 & C_{10} \\ 0 & 0 & 0 & (-1) & C_2 & C_5 & C_7 \\ 0 & 0 & 0 & 0 & (-1) & C_2 & C_4 \\ 0 & 0 & 0 & 0 & 0 & (-1) & 1 \end{pmatrix}$$

Figure 3.16: The matrix P_7 .

Therefore, we obtain the recursion for $U_{\tau,n}(y)$ for $\tau = 142536$ is as follows.

$$\begin{aligned} U_{\tau,n}(y) = & (1 - y)U_{\tau,n-1}(y) + \sum_{k=0}^{\lfloor (n-8)/6 \rfloor} \det(M_{k+1})y^{3k+3}U_{\tau,n-6k-7}(y) \\ & - \sum_{k=0}^{\lfloor (n-6)/6 \rfloor} \det(P_{k+1})y^{3k+2} [U_{\tau,n-6k-4}(y) + yU_{\tau,n-6k-5}(y)]. \end{aligned}$$

In Table 3.3, we computed $U_{142536,n}(y)$ for $n \leq 14$.

3.2.3 The proof of Theorem 12

Let $\tau_a = \tau = \tau_1 \dots \tau_{2a}$ where $\tau_1 \tau_3 \dots \tau_{2a-1} = 12 \dots a$ and $\tau_2 \tau_4 \dots \tau_{2a} = (2a)(2a-1) \dots (a+1)$. If we picture τ_a in a 2-line array like we did in the earlier section, then we will get a diagram as pictured in Figure 3.17

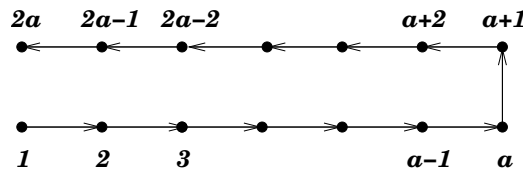


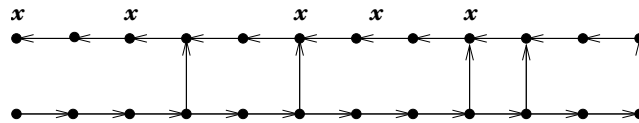
Figure 3.17: The Hasse diagram associated with τ_a .

The key property that τ_a has is that if $\sigma = \sigma_1 \dots \sigma_{2m}$ is permutation where we

Table 3.3: The polynomials $U_{\tau,n}(y)$ for $\tau = 142536$.

n	$U_{142536,n}(y)$
1	$-y$
2	$-y + y^2$
3	$-y + 2y^2 - y^3$
4	$-y + 3y^2 - 3y^3 + y^4$
5	$-y + 4y^2 - 6y^3 + 4y^4 - y^5$
6	$-y + 5y^2 - 9y^3 + 10y^4 - 5y^5 + y^6$
7	$-y + 6y^2 - 13y^3 + 18y^4 - 15y^5 + 6y^6 - y^7$
8	$-y + 7y^2 - 18y^3 + 27y^4 - 32y^5 + 21y^6 - 7y^7 + y^8$
9	$-y + 8y^2 - 24y^3 + 40y^4 - 54y^5 + 52y^6 - 28y^7 + 8y^8 - y^9$
10	$-y + 9y^2 - 31y^3 + 58y^4 - 85y^5 + 100y^6 - 79y^7 + 36y^8 - 9y^9 + y^{10}$
11	$-y + 10y^2 - 39y^3 + 82y^4 - 129y^5 + 170y^6 - 172y^7 + 114y^8 - 45y^9$ $+ 10y^{10} - y^{11}$
12	$-y + 11y^2 - 48y^3 + 113y^4 - 191y^5 + 289y^6 - 320y^7 + 278y^8$ $- 158y^9 + 55y^{10} - 11y^{11} + y^{12}$
13	$-y + 12y^2 - 58y^3 + 152y^4 - 277y^5 + 456y^6 - 578y^7 + 568y^8 - 427y^9$ $+ 212y^{10} - 66y^{11} + 12y^{12} - y^{13}$
14	$-y + 13y^2 - 69y^3 + 200y^4 - 394y^5 + 689y^6 - 1031y^7 + 1068y^8 + 956y^9$ $+ 629y^{10} - 277y^{11} + 78y^{12} - 13y^{13} + y^{14}$

have marked some of the τ_a -matches by placing an x at the start of a τ so that every element of σ is contained in some τ_a -match and any two consecutive marked τ_a in σ share at least one element, then it must be the case that $\sigma_1\sigma_3\dots\sigma_{2m-1} = 12\dots m$ and $\sigma_2\sigma_4\dots\sigma_{2m} = (2m)(2m-1)\dots(m+1)$. That is, it must be the case that $\sigma = \tau_m$. This can easily be seen from the picture of overlapping τ_a -matches like the one pictured in Figure 3.18 where $a = 4$ and $m = 12$. Note that in such a situation, we will in fact have τ_a matches starting at positions $1, 3, 5, \dots, 2(m-a) + 2$ in σ .

**Figure 3.18:** The Hasse diagram of overlapping τ_a -mathces.

We need to show that the polynomials

$$U_{\tau_a, n}(y) = \sum_{O \in \mathcal{O}_{\tau_a, n}, J_{\tau_a}(O) = O} \text{sgn}(O) W(O)$$

satisfy the following properties:

1. $U_{\tau, 1}(y) = -y$, and
2. for $n \geq 2$,

$$\begin{aligned} U_{\tau, n}(y) &= (1 - y)U_{\tau, n-1}(y) \\ &\quad - \sum_{k=0}^{\lfloor (n-2a)/(2a) \rfloor} \binom{n - (k+1)a - 1}{(k+1)a - 1} y^{(k+1)a-1} U_{\tau_a, n-(2(k+1)a)+1}(y) \\ &\quad + \sum_{k=0}^{\lfloor (n-2a-2)/(2a) \rfloor} \binom{n - (k+1)a - 2}{(k+1)a} y^{(k+1)a} U_{\tau_a, n-(2(k+1)a)-1}(y). \end{aligned}$$

Again, it is easy to see that when $n = 1$, $U_{\tau_a, 1}(y) = -y$. For $n \geq 2$, let $O = (B, \sigma)$ be a fixed point of J_{τ_a} where $B = (b_1, \dots, b_t)$ and $\sigma = \sigma_1 \cdots \sigma_n$. By the same argument as the previous sections, it must be the case that 1 is in the first cell of O and 2 must be in either cell of 2 or cell 3 in O . Thus, we now have two cases.

Case 1. 2 is in cell 2 of O .

Similar to Case 1 in the proof of Theorem 11, there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of (B, σ) or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of O . In either case, we can remove cell 1 from O and subtract 1 from the elements in the remaining cells, we will obtain a fixed point O' of J_{τ_a} in $\mathcal{O}_{\tau_a, n-1}$. So the fixed points in this case will contribute $(1 - y)U_{\tau_a, n-1}(y)$ to $U_{\tau_a, n}(y)$.

Case 2. 2 is in cell 3 of $O = (B, \sigma)$.

In this case, $\sigma_2 > \sigma_3 = 2$. Since σ must be increasing in b_1 , it follows that 2 is in the first cell of brick b_2 and there must be a τ_a match in the cells of b_1 and b_2 which can only start at cell 1. Thus it must be the case that brick b_2 has at least $2a - 2$ cells.

Again, we shall think of $O = (B, \sigma)$ as a two line array $A(0)$ where column i consists of σ_{2i-1} and σ_{2i} , reading from bottom to top. Now imagine that $A(0)$ starts with series of τ -matches starting at positions $1, 3, 5, \dots$. Our observation above shows that if this sequence of consecutive τ_a -matches covers cells $1, \dots, 2k$ for some k , then in the two line array $A(O)$, all in entries in the first row of the first k columns are less than all the entries in top row of the first k columns, the cells in the bottom row of the first k columns are increasing, reading from left to right, and the cells in top row are increasing, reading from right to left.

Next we consider the possible brick structures of $O = (B, \sigma)$. We claim that we are in one of two subcases: Subcase (2.A) where there is a $k \geq 0$ such that there are τ_a -matches in σ starting at cells $1, 3, 2a + 1, 2a + 3, \dots, 2(k - 1)a + 1, 2(k - 1)a + 3, 2ka + 1$, there is no τ_a -match in σ starting at cell $2ka + 3$, $2 = b_1 = b_3 = \dots = b_{2k-1}$, $2a - 2 = b_2 = b_4 = \dots = b_{2k}$, and $b_{2k+1} = 2$ and $b_{2k+2} \geq 2a - 2$ or Subcase (2.B) where there is a $k \geq 0$ such that there are τ_a -matches in σ starting at cells $1, 3, 2a + 1, 2a + 3, \dots, 2(k - 1)a + 1, 2(k - 1)a + 3, 2ka + 1, 2ka + 3$, there is no τ_a -match in σ starting at cell $2(k + 1)a + 1$, $2 = b_1 = b_3 = \dots = b_{2k-1} = b_{2k+1}$, $2a - 2 = b_2 = b_4 = \dots = b_{2k+2}$, and $b_{2k+3} \geq 2$. Subcase (2.A) is pictured at the top of Figure 3.19 and Subcase (2.B) is pictured at the bottom of Figure 3.19 in the case where $a = 4$ and $k = 2$. Note that by our remarks above, we also know the relative order of the elements involved in these τ_a -matches in σ which is indicated by the poset

whose Hasse diagram is pictured in Figure 3.19. We can prove this by induction. That is, suppose $k = 0$ and we are in Subcase (2.A). Then there is a τ_a -match in σ starting a cell 1 but no τ_a -match in σ starting at cell 3. Our argument above shows that $b_1 = 2$ and $b_2 \geq 2a - 2$. Next suppose that $k = 0$ and we are in Subcase (2.B) so that there are τ_a -matches in σ starting in cells 1 and 3 but there is no τ_a -match in σ starting at cell $2a + 1$. Then we claim we claim that $b_2 = 2a - 2$. That is, in such a situation we would know that $\sigma_{2a} > \sigma_{2a+1}$. Thus, if $b_2 > 2a - 2$, then $2a$ would be labeled with a y . The τ_a -match starting at cell 1 ends at cell $2a$ so that cell $2a$ would satisfy Case I of our definition of J_{τ_a} which contracts that the fact that $O = (B, \sigma)$ is a fixed point of J_{τ_a} . Thus, brick b_3 must start at cell $2a + 1$. Now the fact that $\sigma_{2a} > \sigma_{2a+1}$ implies that $b_3 \geq 2$ since there must be a τ_a -match that involves σ_{2a} and σ_{2a+1} and lies in cells of b_2 and b_3 .

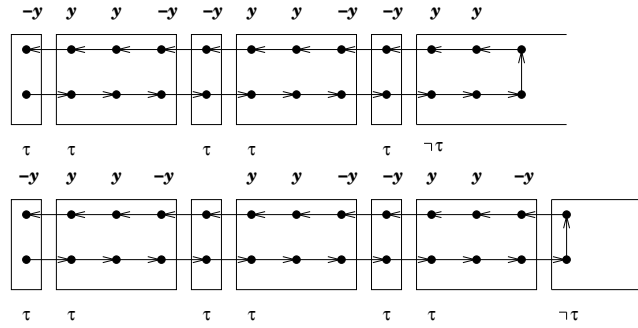


Figure 3.19: Subcases (2.A) and (2.B).

Now assume by induction that for $k \geq 1$, there are τ_a -matches in σ starting at cells $1, 3, 2a + 1, 2a + 3, \dots, 2(k - 1)a + 1, 2(k - 1)a + 3, 2 = b_1 = b_3 = \dots = b_{2k-1}, 2a - 2 = b_2 = b_4 = \dots = b_{2k-2}$, and $b_{2k} \geq 2a - 2$. Suppose we are in Subcase (2.A) so that there is τ_a -match starting at cell $2ka + 1$ but there is no τ_a starting at cell $2ka + 3$. Then we know that $\sigma_{2ka} > \sigma_{2ka+1}$ due to the τ_a -match in σ starting at cell $2(k - 1)a + 1$. It cannot be the case that $b_{2k} > 2a - 2$ since then cells $2ka$ and

$2ka + 1$ are contained in brick b_{2k} so that cell $2ka$ would be marked with a y . However, the τ_a -match starting at cell $2(k-1)a + 1$, which is the first cell of b_{2k} , ends at cell $2ka$ so that cell $2ka$ would satisfy Case I of our definition of J_{τ_a} which violates our assumption that (B, σ) is a fixed point of J_{τ_a} . This means that $b_{2k} = 2a - 2$ and b_{2k+1} starts at cell $2ka + 1$. Since $\sigma_{2ak} > \sigma_{2ak+1}$ due to the τ_a -match in σ starting at cell $2(k-1)a + 3$, we know that there must be a τ_a -match contained in the cells of b_{2k} and b_{2k+1} so that $b_{2k+1} \geq 2$. But then because of the τ_a -match in σ starting at cell $2ka + 1$, we know that $\sigma_{2ka+2} > \sigma_{2ka+3}$. It cannot be that cell $2ka + 3$ is in brick b_{2k+1} because then cell $2k + 2$ would be marked with a y and there is a τ_a -match in σ starting at cell $2(k-1)a + 3$ which ends at cell $2k + 2$ which is contained in the bricks b_{2k} and b_{2k+1} which means that cell $2ka + 2$ would satisfy Case 1 of our definition of J_{τ_a} which violates our assumption that (B, σ) is a fixed point of J_{τ_a} . Thus it must be the case that $b_{2k+1} = 2$ and brick b_{2k+2} starts at cell $2ka + 3$. But this means that there must be a τ_a -match in σ contained in the cells of b_{2k+1} and b_{2k+2} so that $b_{2k+2} \geq 2a - 2$. Now if there is also a τ_a -match in σ starting at cell $2ka + 3$, then we claim that $b_{2k+2} = 2a - 2$. That is, we know that $\sigma_{2(k+1)a} > \sigma_{2(k+1)a+1}$. It cannot be that $b_{2k+2} > 2a - 2$ because then cell $2(k+1)a$ would be labeled with a y and the τ_a -match in σ starting at cell $2ka + 1$ ends at cell $2(k+1)a$ and is contained in the bricks b_{2k+1} and b_{2k+2} so that cell $2(k+1)a$ would satisfy Case 1 of our definition of J_{τ_a} which would violate our assumption that (B, σ) is fixed point of J_{τ_a} . Thus $b_{2k+2} = 2a - 2$. But then due to the τ_a -match in σ starting at cell $2(k+1)a + 3$, we know that $\sigma_{2(k+1)a} > \sigma_{2(k+1)a+1}$ which means that there must be a τ_a match contained in bricks b_{2k+2} and b_{2k+3} . This means that $b_{2k+3} \geq 2$.

Thus we have two cases to consider.

Subcase (2.A) There is a $k \geq 0$ such that there are τ_a -matches in σ starting at cells $1, 3, 2a + 1, 2a + 3, \dots, 2(k - 1)a + 1, 2(k - 1)a + 3, 2ka + 1$, there is no τ_a -match in σ starting at cell $2ka + 3$, $2 = b_1 = b_3 = \dots = b_{2k-1}$, $2a - 2 = b_2 = b_4 = \dots = b_{2k}$, and $b_{2k+1} = 2$ and $b_{2k+2} \geq 2a - 2$.

Here, we claim that $\{1, \dots, (k + 1)a + 1\} = \{\sigma_1, \sigma_3, \dots, \sigma_{2(k+1)a-1}, \sigma_{2(k+1)a}\}$.

That is, if one considers the diagram at the top of Figure 3.19, then the elements in the bottom row are $1, 2, \dots, (k + 1)a$, reading from left to right, and the element at the top of column $(k + 1)a$ is equal to $(k + 1)a + 1$. If this is not the case, then let

$$i = \min(\{1, \dots, (k + 1)a + 1\} - \{\sigma_1, \sigma_3, \dots, \sigma_{2(k+1)a-1}, \sigma_{2(k+1)a}\}).$$

This means $\sigma_{2(k+1)a} > i$ and, hence one can see by the relative order of the elements in the first $(k + 1)a$ columns of $A(O)$ that i can not lie in the first $(k + 1)a$ columns. Then the question is for what j is $\sigma_j = i$. First we claim that it cannot be that $\sigma_{2(k+1)a+1} = i$. That is, in such a situation, $\sigma_{2(k+1)a} > \sigma_{2(k+1)a+1}$. Now it cannot be that $\sigma_{2(k+1)a}$ and $\sigma_{2(k+1)a+1}$ lie in brick b_{2k+2} because then the τ_a -match in σ that starts in the first cell of b_{2k+1} ends at cell $2(k + 1)a$ which means that cell $2(k + 1)a$ would be labeled with a y and satisfy Case I of our definition of J_{τ_a} which would violate our assumption that (B, σ) is fixed point of J_{τ_a} . Thus it must be the case that brick b_{2k+3} starts at cell $2(k + 1)a + 1$. But then there must be a τ_a -match in σ contained in the cells of bricks b_{2k+2} and b_{2k+3} which would imply that there is a τ_a -match in σ starting at cell $2ka + 3$ which violates our assumption in this case. Hence $j > 2(k + 1)a + 1$ which implies that both σ_{j-2} and σ_{j-1} are greater than $\sigma_j = i$. But then there could be no τ_a -match in σ which contains both σ_{j-1} and σ_j because the only role that i could play in τ_a -match in σ would be 1 under those circumstances.

It follows that if we remove the elements in $A(0)$ from the first $(k+1)a-1$ columns plus the bottom element of column $(k+1)a$, then (B', σ') , where $B' = (b_{2k+2} - (2a-1), b_{2k+3}, \dots, b_t)$ and $\sigma' = \text{red}(\sigma_{2(k+1)a} \dots \sigma_n)$, will be a fixed point of J_{τ_a} of size $n - (2(k+1)a) + 1$. Note that in such a situation, we will have $\binom{n-(k+1)a-1}{(k+1)a-1}$ ways to choose the elements of that lie in the top rows of the first $(k+1)a-1$ columns of $A(O)$. Note that the powers of y coming from the bricks b_1, \dots, b_{2k} is y^{ka} and the powers of y coming from bricks b_{2k+1} and b_{2k+2} is $-y^{a-1}$. It follows that the elements in Subcase (2.A) contribute

$$- \sum_{k=0}^{\lfloor (n-2a)/(2a) \rfloor} \binom{n-(k+1)a-1}{(k+1)a-1} y^{(k+1)a-1} U_{\tau_a, n-(2(k+1)a)+1}(y)$$

to $U_{\tau_a, n}(y)$.

Subcase (2.B). There is a $k \geq 0$ such that there are τ_a -matches in σ starting at cells $1, 3, 2a+1, 2a+3, \dots, 2(k-1)a+1, 2(k-1)a+3, 2ka+1, 2ka+3$, there is no τ_a -match in σ starting at cell $2(k+1)a+1$, $2 = b_1 = b_3 = \dots = b_{2k-1} = b_{2k+1}$, $2a-2 = b_2 = b_4 = \dots = b_{2k+2}$, and $b_{2k+3} \geq 2$.

Here, we claim that $\{1, \dots, (k+1)a+2\} = \{\sigma_1, \sigma_3, \dots, \sigma_{2(k+1)a+1}, \sigma_{2(k+1)a+2}\}$.

That is, if one considers the diagram at the bottom of Figure 3.19, then the elements in the bottom row are $1, 2, \dots, (k+1)a+1$, reading from left to right, and the element at the top of column $(k+1)a+1$ is equal to $(k+1)a+2$. If this is not the case, then let

$$i = \min(\{1, \dots, (k+1)a+2\} - \{\sigma_1, \sigma_3, \dots, \sigma_{2(k+1)a+1}, \sigma_{2(k+1)a+2}\}).$$

This means $\sigma_{2(k+1)a+2} > i$ and, hence one can see by the relative order of the elements in the first $(k+1)a+1$ columns of $A(O)$ that i can not lie in the first $(k+1)a+1$

columns. Then the question is for what j is $\sigma_j = i$. First we claim that it cannot be that $\sigma_{2(k+1)a+3} = i$. That is, in such a situation, $\sigma_{2(k+1)a+2} > \sigma_{2(k+1)a+3}$. Now it cannot be that $\sigma_{2(k+1)a+2}$ and $\sigma_{2(k+1)a+3}$ lie in brick b_{2k+3} because then the τ_a -match in σ that starts in the first cell of b_{2k+2} ends at cell $2(k+1)a+2$ which means that cell $2(k+1)a+2$ would be labeled with a y and satisfy Case I of our definition of J_{τ_a} which would violate our assumption that (B, σ) is fixed point of J_{τ_a} . Thus it must be the case $b_{2k+3} = 2$ that brick b_{2k+4} starts at cell $2(k+1)a+3$. But then there must be a τ_a -match in σ contained in the cells of bricks b_{2k+3} and b_{2k+4} which would imply that there is a τ_a -match in σ starting at cell $2(k+1)a+1$ which violates our assumption in this case. Hence $j > 2(k+1)a+3$ which implies that both σ_{j-2} and σ_{j-1} are greater than $\sigma_j = i$. But then there could be no τ_a -match in σ which contains both σ_{j-1} and σ_j because the only role that i could play in τ_a -match in σ would be 1 under those circumstances.

It follows that if we remove the elements in $A(0)$ from the first $(k+1)a+1$ columns plus the bottom element of column $(k+1)a+2$, then (B', σ') , where $B' = (b_{2k+3}-1, b_{2k+4}, \dots, b_t)$ and $\sigma' = \text{red}(\sigma_{2(k+1)a+2} \dots \sigma_n)$, will be a fixed point of J_{τ_a} of size $n - (2(k+1)a) - 1$. Note that in such a situation, we will have $\binom{n-(k+1)a-2}{(k+1)a}$ ways to choose the elements of that lie in the top rows of the first $(k+1)a-1$ columns of $A(O)$. Note that the powers of y coming from the bricks b_1, \dots, b_{2k_2} is $y^{(k+1)a}$. It follows that the elements in Subcase (2.B) contribute

$$\sum_{k=0}^{\lfloor (n-2a-2)/(2a) \rfloor} \binom{n-(k+1)a-2}{(k+1)a} y^{(k+1)a} U_{\tau_a, n-(2(k+1)a)-1}(y)$$

to $U_{\tau_a, n}(y)$.

Therefore, the recursion for the polynomials $U_{\tau,n}(y)$ is given by

$$\begin{aligned}
U_{\tau,n}(y) &= (1 - y)U_{\tau,n-1}(y) \\
&\quad - \sum_{k=0}^{\lfloor (n-2a)/(2a) \rfloor} \binom{n - (k+1)a - 1}{(k+1)a - 1} y^{(k+1)a-1} U_{\tau,n-(2(k+1)a)+1}(y) \\
&\quad + \sum_{k=0}^{\lfloor (n-2a-2)/(2a) \rfloor} \binom{n - (k+1)a - 2}{(k+1)a} y^{(k+1)a} U_{\tau,n-(2(k+1)a)-1}(y).
\end{aligned}$$

This concludes the proof of Theorem 12. \square

3.2.4 The remaining cases of $\tau = 152634$, $\tau = 152436$, $\tau = 162435$, and $\tau = 142635$

Our results in Sections 3.2.2 and 3.2.3 allows us to compute $NM_{\tau}(t, x, y)$ in two cases where $\tau = \tau_1 \dots \tau_6$ and $\tau_1 = 1$, $\tau_3 = 2$, and $\tau_5 = 3$. Namely, the case where $\tau = 142536$ is consider in Theorem 11 and the case where $\tau = 162534$ is a special case of Theorem 12 where $a = 3$. All such permutations have $\text{des}(\tau) = 2$. We now consider the other four cases where $\tau = 152634$, $\tau = 152436$, $\tau = 162435$, and $\tau = 142635$.

Case $\tau = 152634$.

Let $\tau = 152634$, we will show that the polynomials

$$U_{\Gamma,n}(y) = \sum_{O \in \mathcal{O}_{\Gamma,n}, J_{\Gamma}(O)=O} \text{sgn}(O)W(O)$$

satisfy the following properties:

1. $U_{\tau,1}(y) = -y$, and

2. for $n \geq 2$,

$$U_{\tau,n}(y) = (1-y)U_{\Gamma,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{4} \rfloor} (-1)^k y^{2k} (2k-1)!! \binom{n-2k-2}{2k} U_{\tau,n-4k-1}(y),$$

where $(2k-1)!!$ is the double factorial given by $(2k-1)!! = 1 \cdot 3 \cdot \dots \cdot (2k-3) \cdot (2k-1)$.

Again, it is easy to see that when $n = 1$, $U_{\tau,1}(y) = -y$. For $n \geq 2$, let $O = (B, \sigma)$ be a fixed point of I_{Γ} where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \dots \sigma_n$. By the same argument as before, it must be the case that 1 is in the first cell of O and 2 must be in either cell of 2 or cell 3 in O .

Case I. 2 is in cell 2 of O .

If 2 is in the second cell of O then, similar to Case 1 in the proof of Theorem 11, there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of (B, σ) or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of O . In either case, we can remove cell 1 from O and subtract 1 from the elements in the remaining cells, we will obtain a fixed point O' of J_{Γ} in $\mathcal{O}_{\Gamma,n-1}$. So the fixed points in this case will contribute $(1-y)U_{\Gamma,n-1}(y)$ to $U_{\Gamma,n}(y)$.

Case II. 2 is in cell 3 of O .

On the other hand, if 2 is in the third cell of $O = (B, \sigma)$ then we know that 2 must be in the first cell of brick b_2 and there must be a 152634-match starting from cell 1 that involves the cells of the first two bricks. Furthermore, it must be the case that brick b_2 has at least 4 cells and that 3 is in cell 5 of O . Unlike the previous two cases, the 152634-match that starts from cell 1 of O implies that $\sigma_6 < \sigma_2 < \sigma_4$ so we cannot have a 152634-match starting from the third cell of O . Thus, the next possible τ -match of

O can only start from cell 5. We now have two subcases.

Subcase II.a. There is no τ -match in O starting at cell 5.

By the same argument presented in the proof of Theorem 12, it must be the case that $\sigma_6 = 4$ and we have $\binom{n-4}{2}$ ways to choose two numbers to fill in the cells 2 and 4 of O . We can now remove the first five cells of O to obtain a fixed point O' of length $n - 5$. Hence, the contribution of Subcase II.a to $U_{\tau,n}$ is $-y^2 \binom{n-4}{2} U_{\tau,n-5}(y)$.

Subcase II.b. There is a τ -match in O starting at cell 5.

In this case, we cannot have a 152634-match starting from cell 7 since $\sigma_8 > \sigma_{10}$ (due to the match starting from cell 5). Therefore, the next possible match can only start from cell 9 in O . If there is no 152634-match in O starting from cell 9, then by the same argument of subcase 2.b in Section 4, we must have $\sigma_{2k-1} = k$ for $1 \leq k \leq 5$ and $\sigma_{10} = 6$. We have $\binom{n-6}{4}$ ways to fill in the cells 2, 4, 6, and 8 in O . In addition, the two 152634-matches that start from cell 1 and 5 in O imply that the entries in cells 2, 4, 6, and 8 must follow the Hasse diagram given in the left picture of Figure 3.20. Hence, there are 3 ways to arrange these chosen number so that the matches in the initial segment are satisfied. We can then remove the first nine cells of O and adjust the remaining entries to obtain a fixed point of length $n - 9$. This process, as showed above, is reversible and thus, the contribution of this case to $U_{\tau,n}$ is $3y^4 \binom{n-6}{4} U_{n-9}(y)$.

In the general case, suppose there are k 152634-matches starting from cells $1, 5, 7, \dots, 4k - 3$ but there is no 152634-match starting from cell $4k + 1$. Similar to the case for $\tau = 162534$ before, we know that within the first $4k + 2$ cells of O , all the numbers from $\{1, 2, \dots, 2k + 1\}$ must occupy the odd position cells and $\sigma_{4k+2} = 2k + 2$. Therefore, we will choose $2k$ entries to complete the initial segment and remove the first $4k + 1$ cells to obtain a fixed point of $\mathcal{O}_{\tau,n-6k-1}$. Hence, for any given value of k ,

the contribution of fixed points in this case is $(-1)^k y^{2k} M_k \binom{n-2k-2}{2k} U_{\tau, n-4k-1}(y)$, where M_k is the number of ways to arrange the chosen numbers into the even cells of the initial segment such that the 152634-matches between the cells are satisfied. This number in turn is given by the number of linear extensions of the right diagram in Figure 3.20 .

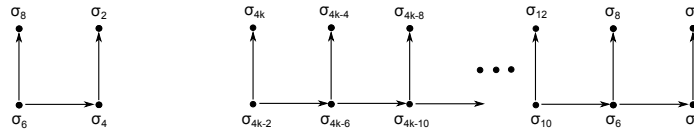


Figure 3.20: The ordering of $\{\sigma_2, \sigma_4, \dots, \sigma_{4k}\}$ for $k = 2$ (left) and for general k (right).

To count the number of linear extensions for the right diagram in Figure 3.20, we first observe that σ_{4k-2} is smaller than every other entries and thus, it must be the case that $\sigma_{4k-2} = \min\{\sigma_2, \sigma_4, \dots, \sigma_{4k}\}$. Next, we have $2k - 1$ choices for the value of σ_{4k} . For each of such choice, we then have M_{k-1} ways to arrange the remaining entries in to the Hasse diagram. Thus, the recursion for M_k is given by $M_k = (2k - 1)M_{k-1}$. From this recursion, it is easy to see that

$$M_k = (2k - 1)(2k - 3) \cdots 5 \cdot 3 = (2k - 1)!!.$$

Hence, the total contribution of Case II is

$$\sum_{k=1}^{\lfloor \frac{n-2}{4} \rfloor} (-1)^k y^{2k} (2k - 1)!! \binom{n - 2k - 2}{2k} U_{\tau, n-4k-1}(y).$$

Combining with the factor from Case I, we have proved the following recursion for the polynomials $U_{\tau,n}(y)$ in the case $\tau = 152634$.

$$U_{\tau,n}(y) = (1 - y)U_{\Gamma,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{4} \rfloor} (-1)^k y^{2k} (2k - 1)!! \binom{n - 2k - 2}{2k} U_{\tau,n-4k-1}(y).$$

Case $\tau = 152436$.

Let $\tau = 152436$, we want to derive a recursion for the polynomials

$$U_{\Gamma,n}(y) = \sum_{O \in \mathcal{O}_{\Gamma,n}, J_{\Gamma}(O)=O} \text{sgn}(O)W(O),$$

where the sum is over all the fixed point of the involution J_{τ} . Let $O = (B, \sigma) \in \mathcal{O}_{\tau,n}$ be a fixed point J_{τ} . Following the previous cases, it is still the case that $U_{\tau,1}(y) = -y$. In addition, similar to the above cases, it is easy to see that $\sigma_1 = 1$ and that either $\sigma_2 = 2$ or $\sigma_3 = 2$. As seen before, if $\sigma_2 = 2$ then the contribution of the fixed points in this case to $U_{\tau,n}(y)$ is $(1 - y)U_{\tau,n-1}(y)$.

If $\sigma_3 = 2$, then it must be the case that the first brick b_1 in O has exactly two cells, the second brick b_2 starts with 2 and has at least four cells, and there is a 152436-match that starts from cell 1 and involves the first six cells of O . In addition, we also know that $\sigma_5 = 3$. Similar to the case $\tau = 152634$, then 152436-match starting from cell 1 implies that $\sigma_4 > \sigma_6$ so there can be any 152436-match that starts from cell 4. Thus, the next possible τ -match must start from cell 5 in O .

Case A. There is no τ -match in O starting at cell 5.

In this case, following the same argument in Subcase 2.1 from the proof of Theorem 11, we can see that the first five integers must belong to the first five cells in O , i.e.

$\{1, 2, 3, 4, 5\} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ and that $\sigma_6 = 6$. Furthermore, the 152436-match that starts from cell 1 in O implies that $\sigma_2 = 5$ and $\sigma_4 = 4$. Therefore, just like in the proof of Theorem 11, we can remove that first five cells of O to obtain a fixed point of length $n - 5$ and thus, the contribution of this case to $U_{\tau,n}(y)$ is $-y^2U_{\tau,n-5}(y)$.

Case B. There is a τ -match in O starting at cell 5.

In this case, the second τ -match implies that $\sigma_8 < \sigma_{10}$ so there cannot be any 152436-match starting from cell 7 in O so the next possible match must start from cell 9. If there is no τ -match in O starting from cell 9, then by the previous argument, we can see that the first nine integers must belong to the first nine cells of O and that $\sigma_{10} = 10$. We can then remove the first nine cells of O and subtract 9 from the remaining entries to obtain a fixed point of length $n - 9$. The contribution of this case is then given by $y^4S_2U_{\tau,n-9}(y)$ where S_2 is the number of ways to arrange the integers $\{1, 2, \dots, 9, 10\}$ into the first ten cells of O such that the two τ -matches in the initial segment are satisfied.

Just like the previous sections, we will find S_2 through the number of linear extensions of a certain Hasse diagram. In this case, it is the diagram to the left of Figure 3.21 below. It is easy to see that $\sigma_1 = 1, \sigma_3 = 2, \sigma_5 = 3, \sigma_6 = 9$, and $\sigma_{10} = 10$ so we can simplify this diagram into the one to the right of the same figure. Hence, $S_2 = \binom{5}{2} = 10$. Thus, the contribution of this case to $U_{\tau,n}(y)$ is $10y^4U_{\tau,n-9}(y)$.

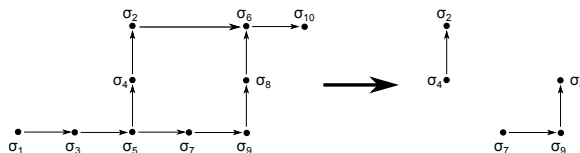


Figure 3.21: The ordering of $\{\sigma_1, \sigma_2, \dots, \sigma_{10}\}$ in S_2 .

In general, suppose there are k 152436-matches starting from cells $1, 5, 9, \dots, 4k -$

3 but there is no τ -match starting from cell $4k + 1$. Similar to the proof of Theorem 11, we first claim that the first $4k + 1$ integers belong to the first $4k + 1$ cells and $4k + 2$ must be in cell $4k + 2$ in O . To see this, let $m = \min(\{1, 2, \dots, 4k + 1\} - \{\sigma_1, \dots, \sigma_{4k+1}\})$. We know that m cannot be more than 2 cells away from cell $4k + 1$ in O . Furthermore, m cannot be in cell $4k + 2$ since the fact that τ ends with 6, the largest entry, implies that we need $\sigma_{4k+2} \geq 4k + 2 > m$ in order to satisfy the overlapping τ -matches in the initial segment. Thus, the only possible place for d is cell $4k + 3$ in O . This cell $4k + 3$ then must start a new brick and there is a decrease between cells $4k + 2$ and $4k + 3$. In order to prevent combining the bricks, we need a τ -match that involves both cells $4k + 2$ and $4k + 3$. However, such τ -match can only start from cell $4k + 1$, a contradiction. The fact that $\sigma_{4k+2} = 4k + 2$ also follows from a similar argument.

Now if we remove the first $4k + 1$ cells of O and subtract $4k + 1$ from the remaining numbers, then we will obtain a fixed point O' of length $n - 4k - 1$. As showed before, this process is reversible and thus the contribution of the fixed points in this case is $(-1)^k y^{2k} S_k U_{\tau, n-4k-1}(y)$, where S_k is the number of ways to arrange the chosen numbers into the even cells of the initial segment such that the 152436-matches between the cells are satisfied. In this case, it is given by the S_k , the number of linear extensions of the top Hasse diagram in Figure 3.22. Therefore, the contribution of fixed points in this case to $U_{\tau, n}(y)$ is $(-1)^k y^{2k} S_k U_{\tau, n-4k-1}(y)$. Hence, we obtain the recursion for the polynomials $U_{\tau, n}(y)$ for the case $\tau = 152436$ as follows.

$$U_{\tau, n}(y) = (1 - y)U_{\Gamma, n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{4} \rfloor} (-1)^k y^{2k} S_k U_{\tau, n-4k-1}(y).$$

To complete the recursion of $U_{152436, n}(y)$, we now need to compute S_k , which is given by the number of linear extensions of the top diagram in Figure 3.22. It is easy

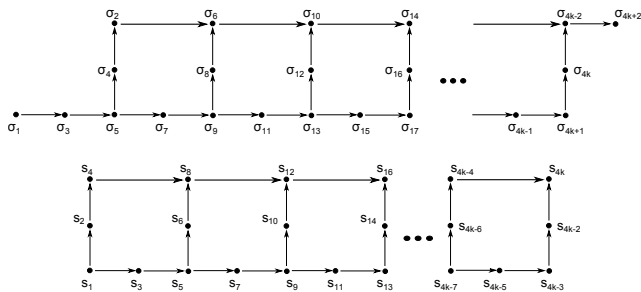


Figure 3.22: The ordering of $\{\sigma_1, \sigma_2, \dots, \sigma_{4k+2}\}$ in S_k (top) and its simplified structure (bottom).

to see that this diagram can be simplified in to the bottom Hasse diagram in the same figure. Unlike the previous cases, instead of a closed formula, we can only obtain a recursion for the number of linear extensions S_k . In [40], Pan and Remmel considered the generalized form of the bottom diagram in Figure 3.22 where we allow the first vertical line segment to have more than one vertices. Figure 3.23 below describes this generalized diagram whose number of linear extensions is defined by the authors of [40] as $L(k, n)$ for integers $n \geq 1$ and $k \geq 3$. They showed that $L(k, n)$ satisfy the following recursion

$$L(p, n) = \begin{cases} \binom{p+2}{3} & \text{for } n = 1 \\ \sum_{j=0}^{p-1} (p-j)(j+1)L(3+j, n-1) & \text{for all } n \geq 2. \end{cases}$$

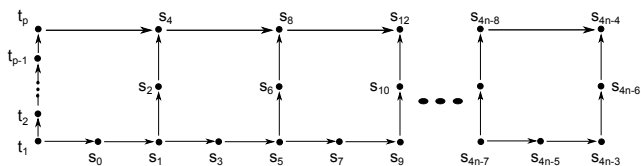


Figure 3.23: The generalized diagram for $L(k, n)$.

In our case, it is easy to see that $S_k = L(3, k - 1)$ and thus we obtain the

following recursion for S_k as follows.

$$S_k = \begin{cases} 1 & \text{for } n = 1 \\ \binom{5}{3} = 10 & \text{for } n = 2 \\ 3L(3, k-2) + 4L(4, n-2) + 3L(5, n-2) & \text{for all } n \geq 2. \end{cases} \quad (3.1)$$

The recursion in (3.1) allows us to compute the first few values of S_k . They are given in Table 3.4 below.

Table 3.4: The first eight values of S_k .

k	1	2	3	4	5	6	7	8
S_k	1	10	215	7200	328090	18914190	1318595475	107813147200

This concludes the proof for the case $\tau = 152436$ □

Case $\tau = 162435$

Suppose $\tau = 162435$, we will prove that the polynomials

$$U_{\Gamma, n}(y) = \sum_{O \in \mathcal{O}_{\Gamma, n}, J_{\Gamma}(O)=O} \text{sgn}(O)W(O)$$

satisfy the following recursion: $U_{\tau, 1}(y) = -y$, and for $n \geq 2$,

$$U_{\tau, n}(y) = (1-y)U_{\tau, n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{4} \rfloor} (-y)^k \binom{n-2k-3}{2k-1} U_{\tau, n-4k}(y) \prod_{i=1}^k (4i-3).$$

Again, it is easy to see that when $n = 1$, $U_{\tau, 1}(y) = -y$. When $n \geq 2$, using the same arguments as the previous sections, we can see that in any given fixed point

$O = (B, \sigma)$ of the involution, we always have $\sigma_1 = 1$ and either $\sigma_2 = 2$ or $\sigma_3 = 2$. When $\sigma_2 = 2$, namely 2 is in the second cell of O , we obtain the contribution $(1 - y)U_{\tau, n-1}(y)$ to $U_{\tau, n}(y)$.

When $\sigma_3 = 2$, we know that 2 must start the second brick b_2 and it must be the case that the first brick b_1 has exactly two cells and there is a 162435-match starting from cell 1 in O . Similar to the argument from the cases $\tau = 152634$ and $\tau = 152436$ before, we can see that the next possible 162435-match must start from cell 5 in O . If there is no 162435-match starting from cell 5, then it must be the case that $\sigma_1 = 1, \sigma_2 = 4, \sigma_3 = 2, \sigma_5 = 3$ and $\sigma_6 = 5$. In this case, we will choose an integer to fill in cell 4 and remove the first 4 cells to obtain a fixed point of length $n - 5$. It is easy to see that this process also is reversible and thus the contribution in this case to $U_{\tau, n}(y)$ is $-y(n - 5)U_{\tau, n-5}(y)$.

In general, we can assume there are k 162435-matches starting at the cells $1, 5, 9, \dots, 4k - 3$ but there is no τ -match starting from cell $4k + 1$ in O . Similar to the previous case, our scheme here is to fix some entries into the initial $4k + 2$ cells and then remove the first $4k + 1$ cells to obtain a fixed point of length $n - 4k - 1$. However, unlike the cases that τ ends with 6 where we can show that the first $4k + 2$ integers must belong to the first $4k + 2$ cells or the cases that τ ends with 4 and we have $\sigma_{2i-1} = i$, for all $1 \leq i \leq k + 1$ and $\sigma_{4k+2} = 2k + 2$, when τ ends with 5, none of the above properties hold. However, we will be able to show the following fact regarding the entries in the initial $4k + 2$ cells of O .

Lemma. *Let O be a fixed point of length n of the involution J_{162435} and suppose that there are 162435-matches in the cells of O starting from cells $1, 5, 9, \dots, 4k - 3$ but there is no τ -match starting from cell $4k + 1$ in O . Then*

- (i) The integers in $\{1, 2, \dots, 2k + 2, 2k + 3\}$ belong to the first $4k + 2$ cells of O , and
- (ii) If $\sigma_{4k+2} = d$ then d is smaller than every elements in O to its right.

Proof: For (i), let $m = \min(\{1, 2, \dots, 2k, 2k + 1\} - \{\sigma_1, \sigma_2, \dots, \sigma_{4k+2}\})$. Then m cannot be in the middle of brick b_{k+1} since there is already a descent within this brick locating at cell $4k$ due to the 162435-match starting from cell $4k - 3$ in O . In addition, this m cannot be in the middle of any brick b_j for $j > k + 1$ for m will then play the role of 1 in any match that lies to the right of cell $4k + 2$ and involve m . This shows that m must be at the start of some brick b_j for $j > k + 1$ and the only possibility is that m starts brick b_{k+2} since the initial elements of the bricks form an increasing sequence. Now recall that m cannot be more than three cells away from cell $4k + 1$ and thus, it must be the case that brick b_{k+1} has exactly four cells and there is a decrease between $last(b_{k+1})$ and $first(b_{k+2})$. Since O is a fixed point of the involution, in order to prevent combining the bricks, there must be a 162435-match that is contained in the cells of the two brick b_{k+1} and b_{k+2} and such match can start from cell $4k - 1$ or cell $4k + 1$ in O . It is easy to see that both cases are impossible since a 162435-match starting from cell $4k - 3$ will prevent one starting from cell $4k - 1$, and we have already assumed that there is no τ -match in O starting from cell $4k + 1$. Hence, all integers $\{1, 2, \dots, 2k + 2, 2k + 3\}$ must be in the initial $4k + 2$ cells of O .

For (ii), let $c = \min\{i < d = \sigma_{4k+2} : \sigma_j = i \text{ for some } j > 4k + 2\}$ and suppose that $\sigma_h = c$. We first observe that c cannot be in the same brick b_{k+1} as d since there already is a decrease in b_{k+1} occurring at cell $4k$. Furthermore, since c is the smallest number to the right of cell $4k + 2$, c cannot be in the middle of any brick and thus, it must be at the start of a brick, which in this case is b_{k+3} . This case is

also impossible since $c < d \leq \text{last}(b_{k+1})$ so in order to prevent combining the bricks, there must be a 162435-match that involves cells $h - 1$ and h . However, since c is the smallest number to the right of cell $4k + 2$, it will play the role of 1 in any such match, a contradiction. \square

Knowing the previous result, we will now fill in the remaining cells of the initial $4k + 2$ cells and there are $\binom{n-2k-3}{2k-1}$ ways to do that. Lastly, we will remove the first $4k + 1$ cells to obtain a fixed point of length $n - 4k - 1$. In total, for any given value of k , the contribution of the fixed points in this case is $(-y)^k \binom{n-2k-3}{2k-1} T_k U_{\tau, n-4k-1}(y)$ where T_k is the number of ways to arrange the numbers in the initial segment such that the 162435-matches between the bricks are satisfied.

Our last task for this proof is to count the number of such arrangements. Let \mathcal{D} be the set of $4k + 2$ integers chosen for the initial segment and label the elements of the set as $\mathcal{D} = \{d_1 < d_2 < d_3 < \dots < d_{4k+1} < d_{4k+2}\}$. As usual, we let σ_i denote the entry in cell i of O . It is easy to see that between any two consecutive 162435-matches, the number that plays the role of 6 in the latter match plays the role of 5 in the former. Thus, σ_2 , which plays the role of 6 in the very first 162435-match, will be the greatest number in the initial segment. Similarly, σ_6 , which plays the role of 5 in the first match and that of 6 in the second match, will be the second largest number. On the other hand, $\sigma_1 < \sigma_3 < \sigma_5$ are the three smallest number in the initial segment. Hence, when putting the numbers in \mathcal{D} into the cells, it must be the case that $\sigma_1 = d_1, \sigma_2 = d_{4k+2}, \sigma_3 = d_2, \sigma_5 = d_3$, and $\sigma_6 = d_{4k+1}$. Since there is no restriction between σ_4 and the rest of the initial segment, we will pick one of the remaining entries of \mathcal{D} to fill in this cell with $4k - 3$ choices for this. Once we make our choice for σ_4 , says $\sigma_4 = d_i$, we can place the three smallest numbers of

$\mathcal{D}_1 = \mathcal{D} \setminus \{d_1, d_2, d_3, d_i, d_{4k+1}, d_{4k+2}\}$ into cells 7 and 9, and place the greatest number in \mathcal{D}_1 in cell 10. We then make another choice for σ_8 and repeat the process with the remaining set of numbers. Recursively, we can see that the number of ways to arrange the numbers in \mathcal{D} into the initial segment is $\prod_{i=1}^k (4i - 3)$. Hence, $T_k = \prod_{i=1}^k (4i - 3)$.

In conclusion, for $n \geq 2$, the recursion of $U_{\tau,n}(y)$ is given by

$$U_{\tau,n}(y) = (1 - y)U_{\Gamma,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{4} \rfloor} (-y)^k \binom{n - 2k - 3}{2k - 1} U_{\tau,n-4k}(y) \prod_{i=1}^k (4i - 3).$$

□

Case $\tau = 142635$

The same proof for case $\tau = 162435$ also works for the case $\tau = 142635$ except we now have a different number of ways to arrange the chosen entries of \mathcal{D} into the initial $4k + 2$ cells such that the 142635-matches between the bricks are satisfied. In this case, when $n \geq 2$, the recursion for $U_{\tau,n}(y)$ is given by

$$U_{\tau,n}(y) = (1 - y)U_{\Gamma,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{4} \rfloor} (-y)^k \binom{n - 2k - 3}{2k - 1} U_{\tau,n-4k}(y) L_k,$$

where L_k is the number of linear extensions given by the Hasse diagram in Figure 3.24 below.

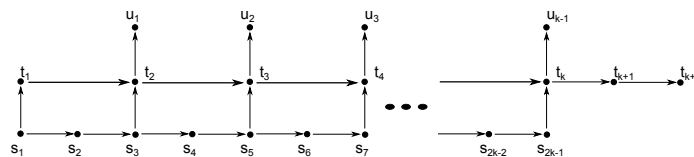


Figure 3.24: The ordering of the first $4k + 2$ cells of O for $\tau = 142635$.

In [40], Pan and Remmel obtained a more generalized result for the diagram in

Figure 3.23 by allowing each side of the squares in the diagram to contain any number of vertices. Suppose we have n squares in our diagram and let a_1, \dots, a_n, a_{n+1} be the number of vertices in each vertical line segment, not counting the endpoints. Similarly, let b_1, \dots, b_n and c_1, \dots, c_n be the number of internal vertices on the top and bottom horizontal line segments, respectively. This diagram is illustrated in Figure 3.25. Denote the number of linear extension of such diagram by $L(a_1, \dots, a_{n+1}; b_1, \dots, b_n; c_1, \dots, c_n)$. The authors of [40] showed that the number of linear extensions of the Hasse diagram in Figure 3.25 satisfies the recursion

$$\begin{aligned}
 &L(a_1, \dots, a_{n+1}; b_1, \dots, b_n; c_1, \dots, c_n) \\
 &= \sum_{k=0}^{a_1+b_1+1} \binom{a_2+k}{k} \binom{a_1+b_1+c_1+1-k}{c_1} L(a_2+k, \dots, a_{n+1}; b_2, \dots, b_n; c_2, \dots, c_n).
 \end{aligned} \tag{3.2}$$

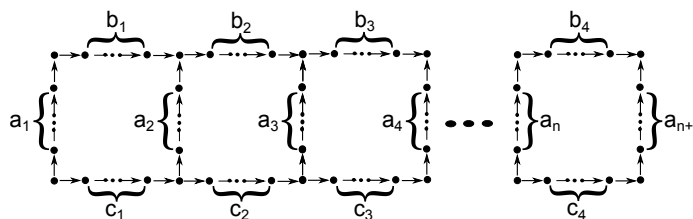


Figure 3.25: The Hasse diagram for $L(a_1, \dots, a_{n+1}; b_1, \dots, b_n; c_1, \dots, c_n)$.

Knowing the result in (3.2), our goal now will be to transform the Hasse diagram of Figure 3.24 into that of Figure 3.25 with $c_1 = \dots = c_n = 1$ and $a_1 = \dots = a_{n+1} = 0$. To this end, we will try to “shuffle” the vertices u_1, u_2, \dots, u_k of Figure 3.24 into the intervals $[t_2, t_3], [t_3, t_4], \dots, [t_{k-1}, t_k]$, and $[t_k, t_{k+2}]$. Starting with u_1 , it is easy to see that we can put this vertex in to any of the interval given by $[t_2, t_3], \dots, [t_{k-1}, t_k]$, and $[t_k, t_{k+2}]$. Let $i_1 = m$ denotes the fact that we insert u_1 into the interval that starts by t_m for $2 \leq m \leq k$. Consequently, this increase the value of b_m by 1. Similarly, we make

a choice to insert u_2 into one of the intervals $[t_3, t_4], \dots, [t_k, t_{k+2}]$ by choosing a value for i_2 between 3 and k . Again, if $i_2 = h$ then the number of internal vertices of the interval $[t_h, t_{h+1}]$ will be increased by 1. In general, for any sequence $(i_1, i_2, \dots, i_{k-1})$ with $j+1 \leq i_j \leq k$, we can transform the Hasse diagram of Figure 3.24 into a special case of the Figure 3.25 where $a_1 = \dots = a_k = 0$, $c_1 = \dots = c_{k-1} = 1$, and b_j equals the number of vertices u 's shuffled into the upper horizontal interval $[t_j, t_{j+1}]$. Lastly, for $2 \leq j \leq k-1$, within each horizontal line segment $[t_j, t_{j+1}]$ with b_j internal vertices, there are $b_j!$ ways to rearrange the entries; whereas in the interval $[t_k, t_{k+2}]$ with $b_k + 2$ vertices, there are $\binom{b_k+2}{2}(b_k+2)! = \frac{(b_k+2)!}{2}$ way to rearrange the entries. Therefore, we can express L_k in terms of $L(a_1, \dots, a_{n+1}; b_1, \dots, b_n; c_1, \dots, c_n)$ as follows.

$$L_k = \sum_{i_1=2}^k \cdots \sum_{i_{k-2}=k-1}^k \sum_{i_{k-1}=k}^k \left(\frac{(b_k+2)!}{2} \prod_{i=1}^{k-1} b_i! \right) L(0, \dots, 0; 0, b_1, b_2, \dots, b_{k-1}; 1, \dots, 1)$$

where for each $1 \leq j \leq k$,

$$b_j = \sum_{t=1}^{j-1} \chi(i_t = j).$$

Here, for any statement A , $\chi(A)$ is the indicator function given by $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false.

Again, we can use Mathematica to compute the first few terms of L_k . These values are given in Table 3.5 below.

Table 3.5: The first eight values of L_k .

k	1	2	3	4	5	6	7	8
L_k	1	9	210	8691	534474	44281890	4650892884	594362020995

The majority of the results presented in this chapter is based on [5], an

unpublished paper by Remmel and the dissertation author. However, the last part of the chapter consists of new material on computing $NM_\tau(t, x, y)$ in the other four cases where $\tau = \tau_1 \dots \tau_6$ and $\tau_1 = 1$, $\tau_3 = 2$, and $\tau_5 = 3$, in which the dissertation author is the main investigator.

Chapter 4

Refinements of the **c-Wilf** equivalent relation

The main goal of this chapter is to study refinements of the **c-Wilf** equivalence relation. For any permutation statistic **stat** on permutations and any pair of permutations α and β in S_j , we say that α is **stat-c-Wilf** equivalent to β if for all $n \geq 1$,

$$\sum_{\sigma \in \mathcal{NM}_n(\alpha)} x^{\mathbf{stat}(\sigma)} = \sum_{\sigma \in \mathcal{NM}_n(\beta)} x^{\mathbf{stat}(\sigma)}.$$

More generally, if $\mathbf{stat}_1, \dots, \mathbf{stat}_k$ are permutations statistics, then we say that α and β are $(\mathbf{stat}_1, \dots, \mathbf{stat}_k)$ -**c-Wilf** equivalent if for all $n \geq 1$,

$$\sum_{\sigma \in \mathcal{NM}_n(\alpha)} \prod_{i=1}^k x_i^{\mathbf{stat}_i(\sigma)} = \sum_{\sigma \in \mathcal{NM}_n(\beta)} \prod_{i=1}^k x_i^{\mathbf{stat}_i(\sigma)}.$$

The first question is whether there are interesting examples of **stat-c-Wilf** equivalent permutations. The answer is yes. There are a number of such examples in the case where $\mathbf{stat}(\sigma)$ is either $\text{inv}(\sigma)$, the number of inversions of σ , or $\text{coinv}(\sigma)$,

the number of co-inversions of σ . Here if $\sigma = \sigma_1 \dots \sigma_n \in S_n$, then

$$\begin{aligned} \text{inv}(\sigma) &= |\{(i, j) : 1 \leq i < j \leq n \ \& \ \sigma_i > \sigma_j\}| \text{ and} \\ \text{coinv}(\sigma) &= |\{(i, j) : 1 \leq i < j \leq n \ \& \ \sigma_i < \sigma_j\}|. \end{aligned}$$

Since for any permutation $\sigma \in S_n$, $\text{inv}(\sigma) + \text{coinv}(\sigma) = \binom{n}{2}$, it follows that

$$\sum_{\sigma \in \mathcal{NM}_n(\alpha)} x^{\text{inv}(\sigma)} = \sum_{\sigma \in \mathcal{NM}_n(\beta)} x^{\text{inv}(\sigma)}.$$

if and only if

$$\sum_{\sigma \in \mathcal{NM}_n(\alpha)} x^{\text{coinv}(\sigma)} = \sum_{\sigma \in \mathcal{NM}_n(\beta)} x^{\text{coinv}(\sigma)}.$$

Thus we will only consider inv-c-Wilf equivalence. It turns out that there are a large number of examples of α and β which are inv-c-Wilf equivalent when α and β are minimal overlapping permutations.

We say that a permutation $\tau \in S_j$ where $j \geq 3$ has the *minimal overlapping property*, or is *minimal overlapping*, if the smallest i such that there is a permutation $\sigma \in S_i$ with $\tau\text{-mch}(\sigma) = 2$ is $2j - 1$. This means that in any permutation $\sigma = \sigma_1 \dots \sigma_n$, any two τ -matches in σ can share at most one letter which must be at the end of the first τ -match and the start of the second τ -match. For example, $\tau = 123$ does not have the minimal overlapping property since $\tau\text{-mch}(1234) = 2$ and the τ -match starting at position 1 and the τ -match starting at position 2 share two letters, namely, 2 and 3. However, it is easy to see that the permutation $\tau = 132$ does have the minimal overlapping property. That is, the fact that there is an ascent starting at position 1 and descent starting at position 2 means that there cannot be two τ -matches in

a permutation $\sigma \in S_n$ which share two or more letters. If $\tau \in S_j$ has the minimal overlapping property, then the shortest permutations σ such that $\tau\text{-mch}(\sigma) = n$ will have length $n(j-1) + 1$. Thus, we let $\mathcal{MP}_{\tau, n(j-1)+1}$ be the set of permutations $\sigma \in S_{n(j-1)+1}$ such that $\tau\text{-mch}(\sigma) = n$. We shall refer to the permutations in $\mathcal{MP}_{\tau, n(j-1)+1}$ as *maximum packings* for τ . Then we let $mp_{\tau, n(j-1)+1} = |\mathcal{MP}_{\tau, n(j-1)+1}|$ and

$$mp_{\tau, n(j-1)+1}(p, q) = \sum_{\sigma \in \mathcal{MP}_{\tau, n(j-1)+1}} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}.$$

Duane and Remmel [16] proved the following theorem about minimal overlapping permutations.

Theorem 13. *If $\tau \in S_j$ has the minimal overlapping property, then*

$$\sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} p^{\text{coinv}(\sigma)} q^{\text{inv}(\sigma)} = \frac{1}{1 - (t + \sum_{n \geq 1} \frac{t^{n(j-1)+1}}{[n(j-1)+1]_{p,q}!} (x-1)^n mp_{\tau, n(j-1)+1}(p, q))}.$$

They also proved the following theorem.

Theorem 14. *Suppose that $\tau = \tau_1 \dots \tau_j$ where $\tau_1 = 1$ and $\tau_j = s$, then*

$$mp_{\tau, (n+1)(j-1)+1}(p, q) = p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)} p^{(s-1)n(j-1)} \begin{bmatrix} (n+1)(j-1) + 1 - s \\ j - s \end{bmatrix}_{p,q} mp_{\tau, n(j-1)+1}(p, q)$$

so that

$$mp_{\tau, (n+1)(j-1)+1}(p, q) = (p^{\text{coinv}(\tau)} q^{\text{inv}(\tau)})^{n+1} p^{(s-1)(j-1) \binom{n+1}{2}} \prod_{i=1}^{n+1} \begin{bmatrix} i(j-1) + 1 - s \\ j - s \end{bmatrix}_{p,q}.$$

An immediate consequence of Theorems 13 and 14 is the following theorem.

Corollary 15. *Suppose that $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are permutations in S_j such that $\alpha_1 = \beta_1 = 1$, $\alpha_j = \beta_j = s$, $\text{inv}(\alpha) = \text{inv}(\beta)$, and α and β have the minimal overlapping property. Then*

$$\sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\alpha\text{-mch}(\sigma)} q^{\text{inv}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\beta\text{-mch}(\sigma)} q^{\text{inv}(\sigma)}.$$

We shall give several examples of a pair of permutations that α and β that satisfy the hypotheses of Corollary 15 and thus are inv-c-Wilf equivalent. It is easy to see that there are no pairs α and β satisfying the hypothesis of Corollary 15 in S_4 . That is, there are only three possible pairs α and β which start with 1 and end with the same numbers, namely,

1. $\alpha = 1342$ and $\beta = 1432$,
2. $\alpha = 1243$ and $\beta = 1423$, and
3. $\alpha = 1234$ and $\beta = 1324$.

In each case, $\text{inv}(\alpha) \neq \text{inv}(\beta)$. However $\alpha = 14532$ and $\beta = 15342$ do satisfy the hypothesis of Corollary 15. Moreover it is easy to check that for any $n > 5$, any two permutations of the form $\alpha = 1453\sigma 2$ and $\beta = 1534\sigma 2$, where σ is the increasing sequence $678 \dots n$, satisfy the hypothesis of Corollary 15. Thus, there are non-trivial examples of inv-c-Wilf equivalence for all $n \geq 1$. In fact, Duane and Remmel proved an even stronger result than Theorem 14. That is, they proved the following theorem.

Theorem 16. *Suppose $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are minimal overlapping*

permutations in S_j and $\alpha_1 = \beta_1$ and $\alpha_j = \beta_j$, then for all $n \geq 1$,

$$mp_{\alpha, n(j-1)+1} = mp_{\beta, n(j-1)+1}.$$

If in addition, $p^{\text{coinv}(\alpha)} q^{\text{inv}(\alpha)} = p^{\text{coinv}(\beta)} q^{\text{inv}(\beta)}$, then

$$mp_{\alpha, n(j-1)+1}(p, q) = mp_{\beta, n(j-1)+1}(p, q).$$

Combining Theorems 13 and 16, we have the following theorem.

Theorem 17. *Suppose that $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are permutations in S_j such that $\alpha_1 = \beta_1$, $\alpha_j = \beta_j$, $\text{inv}(\alpha) = \text{inv}(\beta)$, and α and β have the minimal overlapping property. Then*

$$\sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\alpha\text{-mch}(\sigma)} q^{\text{inv}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\beta\text{-mch}(\sigma)} q^{\text{inv}(\sigma)}.$$

Theorem 17 above relaxes the condition that α and β both have to start with 1 and thus, introduces a stronger condition than just being *inv-c-Wilf* equivalent. In fact, we shall say that α and β are *strongly inv-c-Wilf equivalent* if they satisfy the hypotheses of Theorem 17. As an example, one can check that $\alpha = 241365$ and $\beta = 234165$ both start and end with the same element and have the same number of inversions. Therefore, they are strongly *inv-c-Wilf* equivalent.

Of course, one can make similar definitions in the case where we replace *c-Wilf* equivalence by *Wilf* equivalence. For example, we say that α is **stat**-*Wilf* equivalent to β if for all $n \geq 1$

$$\sum_{\sigma \in S_n(\alpha)} x^{\text{stat}(\sigma)} = \sum_{\sigma \in S_n(\beta)} x^{\text{stat}(\sigma)}.$$

Although this language has not been used, there are numerous examples in the literature where researchers have given a bijection $\phi_n : S_n(\alpha) \rightarrow S_n(\beta)$ to prove that α and β are Wilf equivalent where the bijection ϕ_n preserves other statistics. One example of this phenomenon is the work of Claesson and Kitaev [14] who gave a classification of various bijections between 321-avoiding and 132-avoiding permutations according to what statistics they preserved.

Our goal now is to give examples of α and β such that α and β are des- c -Wilf equivalent. The main result of this chapter is the following.

Theorem 18. *Suppose that $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are permutations in S_j such that $\alpha_1 = \beta_1 = 1$, $\alpha_j = \beta_j$, $\text{des}(\alpha) = \text{des}(\beta)$, and α and β have the minimal overlapping property. Then*

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\alpha)} x^{\text{des}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\beta)} x^{\text{des}(\sigma)}.$$

Thus α and β are des- c -Wilf equivalent.

If in addition, $\text{inv}(\alpha) = \text{inv}(\beta)$, then

$$\sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in \mathcal{NM}_n(\alpha)} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in \mathcal{NM}_n(\beta)} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)}.$$

Thus α and β are (des, inv)- c -Wilf equivalent.

In order to prove this theorem, we are interested in computing generating functions of the form

$$\text{INM}_\Gamma(t, q, z) = 1 + \sum_{n \geq 0} \frac{t^n}{[n]_q!} \text{INM}_{\Gamma, n}(q, z),$$

where $\text{INM}_{\Gamma,n}(q, z) = \sum_{\sigma \in \mathcal{NM}_n(\Gamma)} z^{\text{des}(\sigma)+1} q^{\text{inv}(\sigma)}$, which is a q -analogue of $\text{NM}_{\Gamma}(t, 1, y)$.

We shall write

$$\text{INM}_{\Gamma}(t, q, z) = \frac{1}{1 + \sum_{n \geq 1} \text{IU}_{\Gamma,n}(q, z) \frac{t^n}{[n]_q!}}$$

so that

$$\text{IU}(t, q, z) = 1 + \sum_{n \geq 1} \text{IU}_{\Gamma,n}(q, z) \frac{t^n}{[n]_q!} = \frac{1}{\text{INM}_{\Gamma}(t, q, z)}. \quad (4.1)$$

If $\Gamma = \{\tau\}$, we shall write $\text{INM}_{\tau}(t, q, z)$ for $\text{INM}_{\Gamma}(t, q, z)$, $\text{INM}_{\tau,n}(q, z)$ for $\text{INM}_{\Gamma,n}(q, z)$, $\text{IU}_{\tau}(t, q, z)$ for $\text{IU}_{\Gamma}(t, q, z)$, and $\text{IU}_{\tau,n}(q, z)$ for $\text{IU}_{\Gamma,n}(q, z)$. As before, we shall use the homomorphism method to give us a combinatorial interpretation of the right-hand side of (4.1) which can be used to develop recursions for $\text{IU}_{\Gamma,n}(q, z)$. In the case where α and β satisfy all the hypothesis of Theorem 18, then we will show that $\text{IU}_{\alpha,n}(q, z)$ and $\text{IU}_{\beta,n}(q, z)$ satisfy the same recursions so that $\text{INM}_{\alpha}(t, q, z) = \text{INM}_{\beta}(t, q, z)$.

Finally, there are stronger conditions on permutations α and β in S_j which will guarantee that α and β are des-c-Wilf equivalent, (des, inv)-c-Wilf equivalent, or (des, inv, LRmin)-c-Wilf equivalent. That is, we say that α and β are *mutually minimal overlapping* if α and β are minimal overlapping and the smallest n such that there exist a permutation $\sigma \in S_n$ with $\alpha\text{-mch}(\sigma) \geq 1$ and $\beta\text{-mch}(\sigma) \geq 1$ is $2j - 1$. This ensures that in any permutation σ , any pair of α -matches, any pair of β matches, and any pair of matches where one match is an α -match and one match is a β -match can share at most one letter. There are lots of examples of minimal overlapping permutations α and β in S_j such that α and β are mutually minimal overlapping. For example, we shall prove that any minimal overlapping pair of permutations α and β in S_j which start with 1 and end with 2 are automatically mutually minimal overlapping. We will also give examples of minimal overlapping permutations $\alpha = \alpha_1 \dots \alpha_j$ and

$\beta = \beta_1 \dots \beta_j$ in S_j such that $\alpha_1 = \beta_1 = 1$ and $\alpha_j = \beta_j$ which are not mutually minimal overlapping. Then we shall give a bijective proof the following theorem.

Theorem 19. *Suppose $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are permutations in S_j which are mutually minimal overlapping and there is an $1 \leq a < j$ such that $\alpha_i = \beta_i$ for all $i \leq a$, $\alpha_a = \beta_a = 1$, $\alpha_j = \beta_j$, and $\text{des}(\alpha) = \text{des}(\beta)$. Then*

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\alpha)} z^{\text{des}(\sigma)} u^{\text{LRmin}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\beta)} z^{\text{des}(\sigma)} u^{\text{LRmin}(\sigma)}.$$

Thus α and β are (des, LRmin)- c -Wilf equivalent.

If in addition, $\text{inv}(\alpha) = \text{inv}(\beta)$, then

$$\sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in \mathcal{NM}_n(\alpha)} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)} u^{\text{LRmin}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\beta)} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)} u^{\text{LRmin}(\sigma)}.$$

Thus α and β are (des, inv, LRmin)- c -Wilf equivalent.

4.1 The proof of Theorem 18

In this section, we shall prove Theorem 18. To remind the readers of the result, we shall restate the theorem below.

Theorem. *Suppose that $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are permutations in S_j such that $\alpha_1 = \beta_1 = 1$, $\alpha_j = \beta_j$, $\text{des}(\alpha) = \text{des}(\beta)$, and α and β have the minimal overlapping property. Then*

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\alpha)} x^{\text{des}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\beta)} x^{\text{des}(\sigma)}.$$

Thus α and β are des- c -Wilf equivalent.

If in addition, $\text{inv}(\alpha) = \text{inv}(\beta)$, then

$$\sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in \mathcal{NM}_n(\alpha)} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in \mathcal{NM}_n(\beta)} x^{\text{des}(\sigma)} q^{\text{inv}(\sigma)}.$$

Thus α and β are (des, inv)- c -Wilf equivalent.

This theorem is an immediate consequence of our next result.

Theorem 20. *Let $\tau = \tau_1 \tau_2 \cdots \tau_p \in S_p$ be such that $\tau_1 = 1, \tau_p = s$ where $2 \leq s < p$, and τ has the minimal overlapping property. Then*

$$\text{INM}_\tau(t, q, z) = \frac{1}{\text{IU}_\tau(t, q, z)} \text{ where } \text{IU}_\tau(t, q, z) = 1 + \sum_{n \geq 1} \text{IU}_{\tau, n}(q, z) \frac{t^n}{[n]_q!},$$

with $\text{IU}_{\tau, 1}(q, z) = -z$, and for $n \geq 2$,

$$\text{IU}_{\tau, n}(q, z) = (1 - z) \text{IU}_{\tau, n-1}(q, z) - z^{\text{des}(\tau)} q^{\text{inv}(\tau)} \begin{bmatrix} n - s \\ p - s \end{bmatrix}_q \text{U}_{\tau, n-p+1}(q, z).$$

Proof. Let

$$\text{INM}_{\Gamma, n}(q, z) = \sum_{\sigma \in \mathcal{NM}_n(\Gamma)} z^{\text{des}(\sigma)+1} q^{\text{inv}(\sigma)}. \quad (4.2)$$

We define a ring homomorphism θ_Γ on the ring of symmetric functions Λ by setting

$\theta_\Gamma(e_0) = 1$ and

$$\theta_\Gamma(e_n) = \frac{(-1)^n}{[n]_q!} \text{INM}_{\Gamma, n}(q, z) \quad (4.3)$$

for $n \geq 1$. It then follows that

$$\begin{aligned} \theta_\Gamma(H(t)) &= \sum_{n \geq 0} \theta_\Gamma(h_n) t^n = \frac{1}{\theta_\tau(E(-t))} = \frac{1}{1 + \sum_{n \geq 1} (-t)^n \theta_\Gamma(e_n)} \\ &= \frac{1}{1 + \sum_{n \geq 1} \frac{t^n}{[n]_q!} \text{INM}_{\Gamma,n}(q, z)} = \text{IU}_\Gamma(t, q, z). \end{aligned} \quad (4.4)$$

Using (1.4), we can compute

$$\begin{aligned} [n]_q! \theta_\Gamma(h_n) &= [n]_q! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \theta_\Gamma(e_\lambda) \\ &= [n]_q! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \sum_{(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{b_i}}{[b_i]_q!} \text{INM}_{\Gamma, b_i}(q, z) \\ &= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} \sum_{(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \left[\begin{matrix} n \\ b_1, \dots, b_{\ell(\lambda)} \end{matrix} \right]_q \prod_{i=1}^{\ell(\lambda)} \text{INM}_{\Gamma, b_i}(q, z). \end{aligned} \quad (4.5)$$

To give combinatorial interpretation to the right hand side of (4.5), we select a brick tabloid $B = (b_1, b_2, \dots, b_{\ell(\lambda)})$ of shape (n) filled with bricks whose sizes induce the partition λ . Given an ordered set partition $\mathcal{S} = (S_1, S_2, \dots, S_{\ell(\lambda)})$ of $\{1, 2, \dots, n\}$ such that $|S_i| = b_i$, for $i = 1, \dots, \ell(\lambda)$, let $S_1 \uparrow S_2 \uparrow \dots S_{\ell(\lambda)} \uparrow$ denote the permutation of S_n which results by taking the elements of S_i in increasing order and concatenating them from left to right. For example,

$$\{1, 5, 6\} \uparrow \{7, 9\} \uparrow \{2, 3, 4, 8\} \uparrow = 156792348.$$

It follows from [8] that we can interpret the q -multinomial coefficient $\left[\begin{matrix} n \\ b_1, \dots, b_{\ell(\lambda)} \end{matrix} \right]_q$ as the sum of $q^{\text{inv}(S_1 \uparrow S_2 \uparrow \dots S_{\ell(\lambda)} \uparrow)}$ over all ordered set partitions $\mathcal{S} = (S_1, S_2, \dots, S_{\ell(\lambda)})$ of $\{1, 2, \dots, n\}$ such that $|S_i| = b_i$, for $i = 1, \dots, \ell(\lambda)$. For each brick b_i , we then fill

the cells of b_i with numbers from S_i such that the entries in the brick reduce to a permutation $\sigma^{(i)} = \sigma_1 \cdots \sigma_{b_i}$ in $\mathcal{NM}_{b_i}(\Gamma)$. It follows that if we sum $q^{\text{inv}(\sigma)}$ over all possible choices of $(S_1, S_2, \dots, S_{\ell(\lambda)})$, we will obtain

$$\left[\begin{matrix} n \\ b_1, \dots, b_{\ell(\lambda)} \end{matrix} \right]_q \prod_{i=1}^{\ell(\mu)} q^{\text{inv}(\sigma^{(i)})}.$$

We label each descent of σ that occurs within each brick as well as the last cell of each brick by z . This accounts for the factor $z^{\text{des}(\sigma^{(i)})+1}$ within each brick. Finally, we use the factor $(-1)^{\ell(\lambda)}$ to change the label of the last cell of each brick from z to $-z$. We will denote the filled labeled brick tabloid constructed in this way as $\langle B, \mathcal{S}, (\sigma^{(1)}, \dots, \sigma^{(\ell(\lambda))}) \rangle$.

For example, when $n = 17, \Gamma = \{1324, 1423, 12345\}$, and $B = (9, 3, 5, 2)$, consider the ordered set partition $\mathcal{S} = (S_1, S_2, S_3, S_4)$ of $\{1, 2, \dots, 19\}$ where $S_1 = \{2, 5, 6, 9, 11, 15, 16, 17, 19\}$, $S_2 = \{7, 8, 14\}$, $S_3 = \{1, 3, 10, 13, 18\}$, $S_4 = \{4, 12\}$ and the permutations $\sigma^{(1)} = 1\ 2\ 4\ 6\ 5\ 3\ 7\ 9\ 8 \in \mathcal{NM}_9(\Gamma)$, $\sigma^{(2)} = 1\ 3\ 2 \in \mathcal{NM}_3(\Gamma)$, $\sigma^{(3)} = 5\ 1\ 2\ 4\ 3 \in \mathcal{NM}_5(\Gamma)$, and $\sigma^{(4)} = 2\ 1 \in \mathcal{NM}_2(\Gamma)$. Then the construction of $\langle B, \mathcal{S}, (\sigma^{(1)}, \dots, \sigma^{(4)}) \rangle$ is pictured in Figure 4.1.

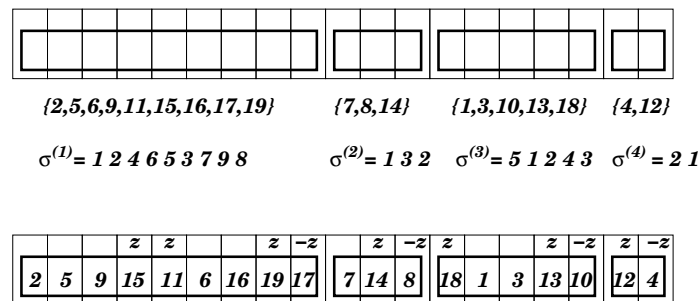


Figure 4.1: The construction of a filled, labeled brick tabloid.

We can then recover the triple $\langle B, (S_1, \dots, S_{\ell(\lambda)}), (\sigma^{(1)}, \dots, \sigma^{(\ell(\lambda))}) \rangle$ from B and

the permutation σ which is obtained by reading the entries in the cells from right to left. We let $\mathcal{O}_{\Gamma,n}$ denote the set of all filled labeled brick tabloids created this way.

That is, $\mathcal{O}_{\Gamma,n}$ consists of all pairs $O = (B, \sigma)$ where

1. $B = (b_1, b_2, \dots, b_{\ell(\lambda)})$ is a brick tabloid of shape n ,
2. $\sigma = \sigma_1 \cdots \sigma_n$ is a permutation in S_n such that there is no Γ -match of σ which lies entirely in a single brick of B , and
3. if there is a cell c such that a brick b_i contains both cells c and $c+1$ and $\sigma_c > \sigma_{c+1}$, then cell c is labeled with a z and the last cell of any brick is labeled with $-z$.

We define the sign of each O to be $\text{sgn}(O) = (-1)^{\ell(\lambda)}$. The weight $W(O)$ of O is defined to be $q^{\text{inv}(\sigma)}$ times the product of all the labels z used in the brick. Thus, the weight of the filled, labeled brick tabloid from Figure 4.1 above is $W(O) = z^{11}q^{84}$.

It follows that

$$[n]_q! \theta_{\Gamma}(h_n) = \sum_{O \in \mathcal{O}_{\Gamma,n}} \text{sgn}(O) W(O). \quad (4.6)$$

Then we can use the same sign-reversing, weight-preserving mapping $J_{\Gamma} : \mathcal{O}_{\Gamma,n} \rightarrow \mathcal{O}_{\Gamma,n}$ that we used in the previous sections to simplify 4.6). That is, let $(B, \sigma) \in \mathcal{O}_{\Gamma,n}$ where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \dots \sigma_n$. Then for any i , we let $\text{first}(b_i)$ be the element in the left-most cell of b_i and $\text{last}(b_i)$ be the element in the right-most cell of b_i . Then we read the cells of (B, σ) from left to right, looking for the first cell c that belongs to either one of the following two cases.

Case I. Either cell c is in the first brick b_1 and is labeled with a z , or it is in some brick b_j , for $j > 1$, with either

- i. $\text{last}(b_{j-1}) < \text{first}(b_j)$ or
- ii. $\text{last}(b_{j-1}) > \text{first}(b_j)$ and there is a τ in Γ such that there is a τ -match which ends weakly to the left of cell c and is contained in the cells of b_{j-1} and the cells b_j .

Case II. Cell c is at the end of brick b_i where $\sigma_c > \sigma_{c+1}$ and there is no Γ -match of σ that lies entirely in the cells of the bricks b_i and b_{i+1} .

In Case I, we define $J_\Gamma((B, \sigma))$ to be the filled, labeled brick tabloid obtained from (B, σ) by breaking the brick b_j that contains cell c into two bricks b'_j and b''_j where b'_j contains the cells of b_j up to and including the cell c while b''_j contains the remaining cells of b_j . In addition, we change the labeling of cell c from z to $-z$. In Case II, $J_\Gamma((B, \sigma))$ is obtained by combining the two bricks b_i and b_{i+1} into a single brick b and changing the label of cell c from $-z$ to z . If neither case occurs, then we let $J_\Gamma((B, \sigma)) = (B, \sigma)$.

It follows from our results in the previous chapter that J_Γ is an involution. That is, if $J_\Gamma(B, \sigma) \neq (B, \sigma)$, then $\text{sgn}(B, \sigma)W(B, \sigma) = -\text{sgn}(J_\Gamma(B, \sigma))W(J_\Gamma(B, \sigma))$. Thus, it follows from (4.6) that

$$[n]_q! \theta_\Gamma(h_n) = \sum_{O \in \mathcal{O}_{\Gamma, n}} \text{sgn}(O)W(O) = \sum_{O \in \mathcal{O}_{\Gamma, n}, J_\Gamma(O)=O} \text{sgn}(O)W(O). \quad (4.7)$$

Hence if all permutations in Γ start with 1, then

$$IU_{\Gamma, n}(q, z) = \sum_{O \in \mathcal{O}_{\Gamma, n}, I_\Gamma(O)=O} \text{sgn}(O)W(O). \quad (4.8)$$

Thus, to compute $IU_{\Gamma, n}(q, z)$, we must analyze the fixed points of J_Γ . Recall

that we have the following characterization of the fixed points of J_Γ .

Lemma 3. *Let $B = (b_1, \dots, b_k)$ be a brick tabloid of shape (n) and $\sigma = \sigma_1 \dots \sigma_n \in S_n$.*

Then (B, σ) is a fixed point of J_Γ if and only if it satisfies the following properties:

(a) if $i = 1$ or $i > 1$ and $\text{last}(b_{i-1}) < \text{first}(b_i)$, then b_i can have no cell labeled z so that σ must be increasing in b_i ,

(b) if $i > 1$ and $\sigma_e = \text{last}(b_{i-1}) > \text{first}(b_i) = \sigma_{e+1}$, then there must be a Γ -match contained in the cells of b_{i-1} and b_i which must necessarily involve σ_e and σ_{e+1} and there can be at most $k - 1$ cells labeled z in b_i , and

(c) if Γ has the property that, for all $\tau \in \Gamma$ such that $\text{des}(\tau) = j \geq 1$, the bottom elements of the descents in τ are $2, \dots, j + 1$, when reading from left to right, then

$$\text{first}(b_1) < \text{first}(b_2) < \dots < \text{first}(b_k).$$

In our case, we are considering the special case where $\Gamma = \{\tau\}$ where $\tau = \tau_1 \dots \tau_p$ and τ is a minimal overlapping permutation such that $\tau_1 = 1$ and $\tau_p = s$ where $2 \leq s \leq p$. Thus we shall use the notation J_τ for J_Γ in this case.

When $n = 1$, the only fixed point comes from the configuration that consists of a single cell filled with 1 and labeled $-z$. Therefore, it must be the case that $\text{IU}_{\tau,1}(q, z) = -z$.

For $n \geq 2$, let (B, σ) be a fixed point of J_τ where $B = (b_1, \dots, b_k)$ and $\sigma = \sigma_1 \dots \sigma_n$. We claim that 1 is in the first cell of (B, σ) . To see this, suppose 1 is in cell c where $c > 1$. Hence $\sigma_{c-1} > \sigma_c$. We claim that whenever $\sigma_{c-1} > \sigma_c$, σ_{c-1} and σ_c must be elements of some τ -match in σ . That is, c cannot be in brick b_1 because the elements in the first brick of any fixed point must be increasing. So we assume that

c is in brick b_i where $2 \leq i \leq k$. If c is the first cell of b_i , then $\text{last}(b_{i-1}) > \text{first}(b_i)$ and there must be a τ -match in the cells of b_{i-1} and b_i which involves cells $c - 1$ and c . If c is not the first cell of b_i , then we can not have that $\text{last}(b_{i-1}) < \text{first}(b_i)$ since this would force σ to be increasing in the cells of b_i . Thus, we must have that $\text{last}(b_{i-1}) > \text{first}(b_i)$ and there must be a τ -match in the cells of b_{i-1} and b_i . This τ -match cannot end before cell c since then c would satisfy the conditions of Case I of our definition of J_τ which would contradict the fact that (B, σ) is a fixed point of J_τ . Hence, cell c must be part of this τ -match. Thus if $\sigma_c = 1$ where $c > 1$, then σ_{c-1} and σ_c are elements of a τ -match in σ . But since τ starts with 1, the only role $\sigma_c = 1$ can play in a τ -match is 1 and hence σ_{c-1} and σ_c cannot be elements of a τ -match in σ . Hence, $\sigma_1 = 1$. We now have two cases.

Case 1. There is no τ -match in (B, σ) that starts from the first cell.

In this case, we claim that 2 must be in cell 2 of (B, σ) . By contradiction, suppose 2 is in cell c where $c \neq 2$. For any $c > 2$, it is easy to see that $\sigma_{c-1} > 2 = \sigma_c$ so there is a decrease between the two cells $c - 1$ and c in (B, σ) . By our argument above, there must exist a τ -match α that involves the two cells $c - 1$ and c . In this case, α must include 1 which is in cell 1 because it must be the case that 1 and 2 play the role of 1 and 2 in the τ -match α , respectively. This contradicts our assumption that there is no τ -match starting from the first cell. Hence, $\sigma_2 = 2$.

In this case there are two possibilities, namely, either (i) 1 and 2 are both in the first brick b_1 of (B, σ) or (ii) brick b_1 is a single cell filled with 1 and 2 is in the first cell of the second brick b_2 of (B, σ) . In either case, we know that 1 is not part of a τ -match in (B, σ) . So if we remove cell 1 from (B, σ) and subtract 1 from the elements in the remaining cells, we will obtain a fixed point $(B'\sigma')$ of J_Γ in $\mathcal{O}_{\Gamma, n-1}$.

Moreover, we can create a fixed point $O = (B, \sigma) \in \mathcal{O}_n$ satisfying the three conditions of Lemma 2 where $\sigma_2 = 2$ by starting with a fixed point $(B', \sigma') \in \mathcal{O}_{\Gamma, n-1}$ of J_Γ , where $B' = (b'_1, \dots, b'_r)$ and $\sigma' = \sigma'_1 \cdots \sigma'_{n-1}$, and then letting $\sigma = 1(\sigma'_1 + 1) \cdots (\sigma'_{n-1} + 1)$, and setting $B = (1, b'_1, \dots, b'_r)$ or setting $B = (1 + b'_1, \dots, b'_r)$.

It follows that fixed points in Case 1 will contribute $(1 - z)\text{IU}_{\Gamma, n-1}(q, z)$ to $\text{IU}_{\Gamma, n}(q, z)$.

Case 2. There is a τ -match in (B, σ) that starts from the first cell.

In this case, the τ -match that starts from the first cell of (B, σ) must involve the cells of the first two bricks b_1 and b_2 in (B, σ) . Since there is no decrease within the first brick b_1 of (B, σ) , it must be the case that the first brick b_1 has exactly d cells, where $1 < d < p$ is the position of the first descent in τ , and the brick b_2 has at least $p - d$ cells. Furthermore, we can see that the brick b_2 consists of exactly $\text{des}(\tau) - 1$ decreases, positioned according to their corresponding descents in τ . We first claim that all the integers in $\{1, \dots, s - 1, s\}$ must belong to the first p cells of (B, σ) . To see this, suppose otherwise and let $m = \min\{i : 1 \leq i \leq s, \sigma_k = i \text{ for some } k > p\}$. That is, m is the smallest integer from $\{1, \dots, s - 1, s\}$ that occupies a cell k strictly to the right of cell p in (B, σ) . It follows that m is the smallest number that occupies a cell strictly to the right of cell p in O and thus, it is the case that $\sigma_{k-1} \geq s > m = \sigma_k$. Then there are three possibilities:

- (i) brick b_2 has more than $p - d$ cells and m is in brick b_2 ,
- (ii) m starts some brick b_j for $j > 2$, or
- (iii) m is in the middle of some brick b_j for $j > 2$.

We will show that each of these cases contradicts our assumption (B, σ) is a fixed

point of J_Γ .

In case (i), since $\sigma_{k-1} > \sigma_k$, there is a decrease in brick b_2 that occurs strictly to the right of cell p . However, due to the τ -match starting from cell 1 of O , brick b_2 already has $\text{des}(\tau) - 1$ descents, the maximum number of allowed descents in a brick. Thus, by the second property of Lemma 2, this is a contradiction.

In case (ii), since $\text{last}(b_{j-1}) = \sigma_{k-1} > \sigma_k = \text{first}(b_j)$, by Lemma 2, there must be a τ -match that is contained in the cells of b_{j-1} and b_j and ends weakly to the left of cell k which contains m . Since τ is a minimal overlapping permutation, the only possible τ -match beside from the first one that starts from cell 1 in (B, σ) must occur weakly to the right of cell p in O . However, since m is the smallest number in the cells to the right of cell p and τ starts with 1, any match that involves m must also start from this cell. Thus, we can never have a τ -match in (B, σ) that involves both cells $k - 1$ and k in (B, σ) .

In case (iii), suppose that m occupies cell k that is in the middle of brick b_j . There are now two possibilities between the last cell of b_{j-1} and the first cell of brick b_j : either $\text{last}(b_{j-1}) < \text{first}(b_j)$ or $\text{last}(b_{j-1}) > \text{first}(b_j)$. If $\text{last}(b_{j-1}) < \text{first}(b_j)$ then we can simply break the brick b_j after cell $k - 1$, contradicting the fact that (B, σ) is a fixed point. On the other hand, if $\text{last}(b_{j-1}) > \text{first}(b_j)$ then by Lemma 2, there must be a τ -match that ends weakly to the left of cell k , and involves the two cells $k - 1$ and k . However, by previous argument, this cannot hold.

Hence, it must be the case that all the integers $\{1, 2, \dots, s - 1, s\}$ belong to the first p cells of (B, σ) . Furthermore, we only have one way to arrange these entries, according to their respective position within the τ -match. This also implies that $\sigma_p = s$. We will then choose $p - s$ numbers and fill these numbers in the empty cells

within the first p cells of (B, σ) such that $\text{red}(\sigma_1\sigma_2\cdots\sigma_p) = \tau$. There are $\binom{n-s}{p-s}$ ways to do this and keeping track of the inversions between our choice of $p-s$ numbers and the elements of (B, σ) which occurs after cell p , we obtain a factor of $\left[\begin{smallmatrix} n-s \\ p-s \end{smallmatrix} \right]_q$ from our possible choices. Then we have to count the inversions among the first p elements of (B, σ) , which contributes a factor of $q^{\text{inv}(\tau)}$. We notice that since τ has the minimal overlapping property, the next possible τ -match in (B, σ) must start from cell p that contains s . In addition, according to Lemma 2, any brick in a fixed point of the involution can have at most $\text{des}(\tau) - 1$ descents within the brick so there cannot be any descents in b_2 after cell p . By construction, $\sigma_p = s$ is less than the elements which occur to the right of cell p . Therefore, we can remove the first $p-1$ cells of (B, σ) and obtain a fixed point (B, σ') of length $n-p+1$.

This process is also reversible. Suppose $\tau \in S_p$ is a minimal overlapping permutation with $\tau_1 = 1, \tau_p = s$, and the first descent in τ occurs at position d . Given a fixed point (B', σ') of length $n-p+1$ where $B' = (b'_1, \dots, b'_r)$ and a choice T of $p-s$ elements from $\{s+1, \dots, n\}$, we let σ^* be the permutation of $\{1, \dots, s\} \cup T$ such that $\text{red}(\sigma^*) = \tau$ and σ^{**} be the permutation of $\{1, \dots, n\} - (\{1, \dots, s\} \cup T)$ such that $\text{red}(\sigma^*) = \sigma'$. Then if we let $\sigma = \sigma^*\sigma^{**}$ and $B = (d, p-d-1+b'_1, b'_2, \dots, b'_r)$, then (B, σ) will be a fixed point of J_Γ of length n that has τ -match starting in cell 1.

It follows that the contribution of the fixed points in Case 2 to $\text{IU}_{\tau,n}(q, z)$ is

$$-z^{\text{des}(\tau)} q^{\text{inv}(\tau)} \left[\begin{smallmatrix} n-s \\ p-s \end{smallmatrix} \right]_q \text{IU}_{\tau, n-p+1}(q, z).$$

Combining Cases 1 and 2, we see that for $n \geq 2$,

$$\text{IU}_{\tau,n}(q, z) = (1 - z)\text{IU}_{\tau,n-1}(q, z) - z^{\text{des}(\tau)}q^{\text{inv}(\tau)} \begin{bmatrix} n - s \\ p - s \end{bmatrix}_q \text{IU}_{\tau,n-p+1}(q, z)$$

which is what we wanted to prove. \square

It is easy to see that Theorem 18 follows immediately from Theorem 20. That is, Theorem 20 shows that for a minimal overlapping permutation $\tau \in S_j$ that starts with 1, the generating function

$$\text{INM}_{\tau}(t, 1, z) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} z^{\text{des}(\sigma)+1}$$

depends only on $s = \tau_j$ and $\text{des}(\tau)$. Thus if α and β are minimal overlapping permutations which start with 1 and end with s and $\text{des}(\alpha) = \text{des}(\beta)$, then $\text{INM}_{\alpha}(t, 1, z) = \text{INM}_{\beta}(t, 1, z)$ so that α and β are des-c -Wilf equivalent. Similarly, Theorem 20 shows that for a minimal overlapping permutation $\tau \in S_j$ that starts with 1, the generating function

$$\text{INM}_{\tau}(t, q, z) = 1 + \sum_{n \geq 1} \frac{t^n}{[n]_q!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} z^{\text{des}(\sigma)+1} q^{\text{inv}(\sigma)}$$

depends only on $s = \tau_j$, $\text{des}(\tau)$, and $\text{inv}(\tau)$. Thus if α and β are minimal overlapping permutations which start with 1 and end with s and $\text{des}(\alpha) = \text{des}(\beta)$ and $\text{inv}(\alpha) = \text{inv}(\beta)$, then $\text{INM}_{\alpha}(t, q, z) = \text{INM}_{\beta}(t, q, z)$ so that α and β are (des, inv) - c -Wilf equivalent.

There are lots of examples minimal overlapping permutations α and β for which the hypothesis of Theorem 18 apply. For example, consider $n = 5$. Since we are only interested in permutations that start with 1, we know that such a permutation

α starts with a rise. Then α cannot end in a rise since otherwise α is not minimal overlapping. Thus α must start with 1 and end in a descent. There are no such permutations that end in 5 and there are only two such permutations that end in 4, namely, 12354 and 13254 and these two permutations do not have the same number of descents. This leaves us 10 possible permutations to consider which we have listed in the following table. For each such σ , we have list $\text{des}(\sigma)$, $\text{inv}(\sigma)$, and indicated whether is minimal overlapping.

Table 4.1: The c-Wilf equivalent classes of length 5.

σ	$\text{des}(\sigma)$	$\text{inv}(\sigma)$	Is minimal overlapping?
12453	1	1	yes
12543	2	3	yes
14253	2	3	no
15243	2	4	no
13452	1	3	yes
13542	2	4	yes
14352	2	4	yes
14532	2	5	yes
15342	2	5	yes
15432	3	6	yes

Theorem 18 tells us that all the elements in the set $\{13542, 14352, 14532, 15342\}$ are des-c-Wilf equivalent. It also shows that the same set breaks up into 2 (des, inv)-c-Wilf equivalence classes, namely, $\{13542, 14352\}$ and $\{14532, 15342\}$.

Another natural question to ask is whether the size of (des, inv)-c-Wilf equivalence classes can get arbitrarily large as n goes to infinity. The answer to this question is yes. First, it is easy to see that if σ is a permutation that starts with 1 and ends with 2, it is automatically minimal overlapping. That is, if $\sigma = \sigma_1 \dots \sigma_n$ where $\sigma_1 = 1$ and $\sigma_n = 2$, then there can be no $2 \leq i \leq n - 1$ such that the first i elements of σ has

the same relative order as the last i elements of σ because in the first i elements of σ the smallest element is at the start while in the last i elements of σ , the smallest element is at the end.

Now consider three consecutive elements $x, x + 1, x + 2$. Then the sequences $t_1(x) = (x + 1)(x + 2)x$ and $t_2(x) = (x + 2)x(x + 1)$ each have one descent and two inversions. It follows that if we start with the permutation $\sigma = 1 t_1(3) t_1(6) t_1(9) \cdots t_1(3n) 2$, then we can replace any of the sequence $t_1(3k)$ by its corresponding sequence $t_2(3k)$ and it will keep the inversion number and the descent number of the permutation the same. Thus, the size of the (des, inv) -c-Wilf equivalence class of σ is at least 2^n .

There are lots of other examples of this type. For example, consider four consecutive elements $x, x + 1, x + 2, x + 3$. Then the sequences $s_1(x) = (x + 1)(x + 2)x(x + 3)$ and $s_2(x) = x(x + 3)(x + 1)(x + 2)$ each have one descent and two inversions. It follows that if we start with the permutation $\tau = 1 s_1(3) s_1(7) s_1(11) \cdots s_1(4n - 1) 2$, then we can replace any of the sequence $s_1(4k - 1)$ by its corresponding sequence $s_2(4k - 1)$ and it will keep the inversion number and the descent number of the permutation the same. This same argument can also be extended to permutations $\sigma \in S_n$ that start with $123 \cdots k$ and end with $k + 1$, for any $k > 0$. Hence, the size of the (des, inv) -c-Wilf equivalence class of τ is at least 2^n .

4.2 The proof of Theorem 19

In Theorem 19, we study the $(\text{des}, \text{LRmin})$ -c-Wilf equivalent relation and its q -analog which arise as another consequence of Theorem 20. First, we observe that for any permutation τ , $\text{NM}_\tau(t, 1, y) = \text{INM}_\tau(t, z, 1)$ and hence, $U_\tau(t, y) = \text{IU}_\tau(t, z, 1)$. Thus, if α and β are minimal overlapping permutations which start with 1 and end

with s with $\text{des}(\alpha) = \text{des}(\beta)$, then $U_\alpha(t, y) = U_\beta(t, y)$. This leads to

$$\text{NM}_\alpha(t, x, y) = \left(\frac{1}{U_\alpha(t, y)} \right)^x = \left(\frac{1}{U_\beta(t, y)} \right)^x = \text{NM}_\alpha(t, x, y).$$

Hence, if α and β are minimal overlapping permutations which start with 1 and end with s and $\text{des}(\alpha) = \text{des}(\beta)$, then α and β are (des, LRmin)-c-Wilf equivalent. In fact, by relaxing the condition that α and β start with 1, we can generalize this result for pairs of permutations α and β that satisfy the condition which we refer to as mutually minimal overlapping.

Before proceeding with the proof of Theorem 19, we first recall the definition of mutually minimal overlapping permutations. Here, we say that α and β are *mutually minimal overlapping* if α and β are minimal overlapping and the smallest n such that there exist a permutation $\sigma \in S_n$ such that $\alpha\text{-mch}(\sigma) \geq 1$ and $\beta\text{-mch}(\sigma) \geq 1$ is $2j - 1$. This ensures that in any permutation σ , any pair of α -matches, any pair of β matches, and any pair of matches where one match is an α -match and one match is a β -match can share at most one letter.

Note that if $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are minimal overlapping permutations in S_j that start with 1 and end with 2, then α and β are mutually minimal overlapping. That is, it cannot be that there is $1 < i < j$ such that the last i elements of α have the same relative order as the first i elements of β since the first i elements of α has its smallest element at the start while the last i elements of β has its smallest element at the end. Similarly, it can not be that there is $1 < i < j$ such that the last i elements of β have the same relative order as the first i elements of α . On the other

hand, if

$$\alpha = 1\ 9\ 3\ 8\ 2\ 7\ 6\ 5\ 4 \text{ and}$$

$$\beta = 1\ 3\ 9\ 8\ 7\ 5\ 2\ 6\ 4,$$

then one can check that α and β are minimal overlapping, $\text{des}(\alpha) = \text{des}(\beta) = 4$, and $\text{inv}(\alpha) = \text{inv}(\beta) = 19$. However α and β are not mutually minimal overlapping since the first 3 elements of α have the same relative order as that last three elements of β .

We shall give a bijective proof for a slightly stronger version of Theorem 18. In fact, Theorem 18 is the special case of the following result when $a = 1$.

Theorem. *Suppose $\alpha = \alpha_1 \dots \alpha_j$ and $\beta = \beta_1 \dots \beta_j$ are permutations in S_j which are mutually minimal overlapping and there is an $1 \leq a < j$ such that $\alpha_i = \beta_i$ for $i \leq a$, $\alpha_a = \beta_a = 1$, $\alpha_j = \beta_j$, and $\text{des}(\alpha) = \text{des}(\beta)$.*

Then

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\alpha)} x^{\text{des}(\sigma)} y^{\text{LRmin}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\beta)} x^{\text{des}(\sigma)} y^{\text{LRmin}(\sigma)}.$$

Thus α and β are (des, LRmin)-c-Wilf equivalent.

If in addition, $\text{inv}(\alpha) = \text{inv}(\beta)$, then

$$\sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in \mathcal{NM}_n(\alpha)} x^{\text{des}(\sigma)} y^{\text{LRmin}(\sigma)} q^{\text{inv}(\sigma)} = \sum_{n \geq 0} \frac{t^n}{[n]_q!} \sum_{\sigma \in \mathcal{NM}_n(\beta)} x^{\text{des}(\sigma)} y^{\text{LRmin}(\sigma)} q^{\text{inv}(\sigma)}.$$

Thus α and β are (des, LRmin, inv)-c-Wilf equivalent.

Proof. For any $n \geq 0$, we can partition the elements of S_n into four sets:

1. A_n equals the set of $\sigma \in S_n$ such that $\alpha\text{-mch}(\sigma) > 0$ and $\beta\text{-mch}(\sigma) = 0$,

2. B_n equals the set of $\sigma \in S_n$ such that $\beta\text{-mch}(\sigma) > 0$ and $\alpha\text{-mch}(\sigma) = 0$,
3. C_n equals the set of $\sigma \in S_n$ such that $\beta\text{-mch}(\sigma) > 0$ and $\alpha\text{-mch}(\sigma) > 0$,
4. D_n equals the set of $\sigma \in S_n$ such that $\beta\text{-mch}(\sigma) = 0$ and $\alpha\text{-mch}(\sigma) = 0$.

Clearly $\mathcal{NM}_n(\alpha) = D_n \cup B_n$ and $\mathcal{NM}_n(\beta) = D_n \cup A_n$. Thus, to prove that

$$\sum_{\sigma \in \mathcal{NM}_n(\alpha)} z^{\text{des}(\sigma)} u^{\text{LRmin}(\sigma)} = \sum_{\sigma \in \mathcal{NM}_n(\beta)} z^{\text{des}(\sigma)} u^{\text{LRmin}(\sigma)},$$

we need only prove that

$$\sum_{\sigma \in A_n} z^{\text{des}(\sigma)} u^{\text{LRmin}(\sigma)} = \sum_{\sigma \in B_n} z^{\text{des}(\sigma)} u^{\text{LRmin}(\sigma)}.$$

Thus, we need to define a bijection $\phi : A_n \rightarrow B_n$ such that for all $\sigma \in A_n$, $\text{des}(\sigma) = \text{des}(\phi(\sigma))$ and $\text{LRmin}(\sigma) = \text{LRmin}(\phi(\sigma))$. One simply replaces each α -match $\sigma_i \dots \sigma_{i+j-1}$ in σ by the β -match where we rearrange $\sigma_{i+1} \dots \sigma_{i+j-2}$ so that it matches β . Given our conditions on α and β , this means that we will simply rearrange $\sigma_{i+a} \dots \sigma_{i+j-2}$ to match the order of the elements $\beta_{a+1} \dots \beta_{j-1}$. Since α is minimal overlapping, the elements that we rearrange in any two α matches of σ are disjoint. Hence ϕ is well defined.

The fact that $\alpha_a = \beta_a = 1$ ensures that σ_{i+a-1} is less than each of the elements $\sigma_{i+a} \dots \sigma_{i+j-2}$ so that rearranging these can not effect the number of left-to-right minima. So $\text{LRmin}(\sigma) = \text{LRmin}(\phi(\sigma))$. The fact that $\text{des}(\alpha) = \text{des}(\beta)$ ensures that our rearrangement $\sigma_{i+1} \dots \sigma_{i+j-1}$ does not effect the number of descents so that $\text{des}(\sigma) = \text{des}(\phi(\sigma))$. Moreover, if $\text{inv}(\alpha) = \text{inv}(\beta)$, then our rearrangement $\sigma_{i+a} \dots \sigma_{i+j-2}$ does not effect the number of inversions so that $\text{inv}(\sigma) = \text{inv}(\phi(\sigma))$.

Next we claim the fact that α and β are mutually minimal overlapping ensures that $\phi(\sigma)$ is in B_n . That is, if $\phi(\sigma)$ has an α match, then it must have been the case that there was α -match $\sigma_i \dots \sigma_{i+j-1}$ in σ such that the rearrangement of $\sigma_{i+a} \dots \sigma_{i+j-2}$ or possibly two consecutive α -matches in σ $\sigma_i \dots \sigma_{i+2j-2}$ such that the rearrangement of $\sigma_{i+a} \dots \sigma_{i+j-2}$ and the rearrangement of $\sigma_{i+j-1+a} \dots \sigma_{i+2j-3}$ caused an α -match to appear. In either case, this would mean that there is an α -match in $\phi(\sigma)$ which shares more than 2 letters with a β -match in $\phi(\sigma)$. This is impossible since α and β are mutually minimal overlapping. Finally, it is clear how to define $\phi^{-1}(\sigma)$. One simply replaces each β -match $\sigma_i \dots \sigma_{i+j-1}$ in σ by the α -match where we rearrange $\sigma_{i+a} \dots \sigma_{i+j-2}$ so that it matches α . The same arguments will ensure that ϕ^{-1} is well defined and maps B_n into A_n . Thus ϕ proves theorem. \square

Lastly, we observe that our proof of Theorem 20 can also be modified to prove the following theorem which allows us to study the c-Wilf equivalent relations between families of permutations.

Theorem 21. *Suppose $\Gamma = \{\alpha^{(1)}, \dots, \alpha^{(k)}\}$ is a set of minimal overlapping permutations in S_p which all start with 1 and $\alpha^{(i)}$ and $\alpha^{(j)}$ are mutually minimal overlapping for all $1 \leq i < j \leq k$. For each $1 \leq i \leq k$, let s_i be the last element of $\alpha^{(i)}$. Then*

$$INM_{\Gamma}(t, q, z) = \frac{1}{IU_{\Gamma}(t, q, z)} \text{ where } IU_{\Gamma}(t, y) = 1 + \sum_{n \geq 1} IU_{\Gamma, n}(q, z) \frac{t^n}{[n]_q!},$$

with $IU_{\Gamma, 1}(q, z) = -z$, and for $n \geq 2$,

$$IU_{\Gamma, n}(q, z) = (1 - z)IU_{\Gamma, n-1}(q, z) - \sum_{i=1}^k z^{\text{des}(\alpha^{(i)})} q^{\text{inv}(\alpha^{(i)})} \begin{bmatrix} n - s_i \\ p - s_i \end{bmatrix}_q U_{\tau, n-p+1}(q, z).$$

The results in this chapter is based on a paper by Bach and Remmel [4].

Chapter 5

Generating Function for Initial and Final Descents

In this final chapter, we shall take a step away from the study of consecutive patterns in permutations and consider another application of the homomorphism method introduced in the first chapter to study the number of initial and final descents in permutations. The results of this chapter will appear in a future paper by Remmel and the dissertation author.

For each permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, we let

$$\text{indes}(\sigma) = \max\{i : \sigma_1 > \cdots > \sigma_i\}$$

be the number of *initial descents* in σ , and let

$$\text{findes}(\sigma) = \max\{n - j + 1 : \sigma_j > \sigma_{j+1} > \cdots > \sigma_n\}$$

be the number of *final descents* in σ . For example, if $\sigma = 983741652$ then $\text{indes}(\sigma) = 3$

and $\text{findes}(\sigma) = 2$.

The main goal of this chapter is to apply the homomorphism method through the identities in (1.5) and (1.8) to study the generating functions of the forms

$$\sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} y^{\text{indes}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}, \quad \text{and}$$

$$\sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} y^{\text{indes}(\sigma)} z^{\text{findes}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}.$$

Specifically, we shall prove the following two theorems.

Theorem 22.

$$\begin{aligned} \sum_{n \geq 1} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} y^{\text{indes}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \\ = \frac{(\mathbf{e}_{p,q}^{txy} - 1)(x-1)(y-1) + y(\mathbf{e}_{p,q}^{t(x-1)} - 1)}{y(xy - x + 1)(x - \mathbf{e}_{p,q}^{t(x-1)})} \end{aligned}$$

where $\mathbf{e}_{p,q}^x$ is the p, q -analogue of the exponential function e^x given by

$$\mathbf{e}_{p,q}^x = 1 + \sum_{n \geq 1} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} x^n.$$

Theorem 23.

$$\begin{aligned} \sum_{n \geq 2} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} y^{\text{indes}(\sigma)} z^{\text{findes}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \\ = G(x, y, z, t) + A(x, y, z, t) - (B(x, y, z, t) - C(x, y, z, t) - D(x, y, z, t)) \end{aligned}$$

where

$$\begin{aligned}
A(x, y, z, t) &= \frac{\mathbf{e}_{p,q}^{xyzt} - xyzt - 1}{xyz}, \\
B(x, y, z, t) &= \frac{(y-1)(\mathbf{e}_{p,q}^{xyt} - xyt - 1)}{xy(y-z)(xy-x+1)}, \\
C(x, y, z, t) &= \frac{(z-1)(\mathbf{e}_{p,q}^{xzt} - xzt - 1)}{xz(y-z)(xz-x+1)}, \\
D(x, y, z, t) &= \frac{\mathbf{e}_{p,q}^{t(x-1)} - t(x-1) - 1}{(x-1)(xy-x+1)(xz-x+1)}, \\
G(x, y, z, t) &= \frac{F(x, y, t)F(x, z, t)}{yz(x-1)(xy-x+1)(xz-x+1)(x - \mathbf{e}_{p,q}^{t(x-1)})} \quad \text{with} \\
F(x, y, t) &= (y-1)(x-1)\mathbf{e}_{p,q}^{xyt} + y\mathbf{e}_{p,q}^{t(x-1)} - (xy-x+1).
\end{aligned}$$

5.1 The proof of Theorem 22

Following the main idea of the homomorphism method, we first define a ring homomorphism φ by letting $\varphi(e_0) = 1$ and

$$\varphi(e_n) = \frac{(-1)^{n-1}}{[n]_{p,q}!} (x-1)^{n-1} q^{\binom{n}{2}},$$

for $n \geq 1$. We want to show that $[n]_{p,q}! \varphi(p_{n,\nu}) = \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} y^{\text{indeg}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}$.

Indeed, by applying the φ to both sides of (1.6), one can obtain the following.

$$\begin{aligned}
 [n]_{p,q}! \varphi(p_{n,\nu}) &= [n]_{p,q}! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \omega_\nu(B_{\lambda,n}) \varphi(e_\lambda) \\
 &= [n]_{p,q}! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \sum_{B=(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \omega_\nu(B) \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{b_i-1} q^{\binom{b_i}{2}}}{[b_i]_{p,q}!} (x-1)^{b_i-1} \\
 &= \sum_{\lambda \vdash n} \sum_{B=(b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}} \begin{bmatrix} n \\ b_1, b_2, \dots, b_{\ell(\lambda)} \end{bmatrix}_{p,q} q^{\sum \binom{b_i}{2}} \\
 &\quad \times \left((xy)^{b_1-1} - \sum_{k=0}^{b_1-2} (xy)^k (x-1)^{b_1-2-k} \right) \prod_{i=2}^{\ell(\lambda)} (x-1)^{b_i-1}.
 \end{aligned}
 \tag{5.1}$$

From the right hand side of (5.1), we create combinatorial objects by picking a partition λ of n . We then select a brick tabloid $B = (b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{\lambda,n}$ and label the terminal cell in each brick with 1. For each brick b_i for $2 \leq i \leq \ell(\lambda)$, we shall label the nonterminal cells with either x or -1 . These contribute to the term $\prod_{i=2}^{\ell(\lambda)} (x-1)^{b_i-1}$ in (5.1). For the first brick in B , we shall apply a different labeling scheme. That is, we can either label every nonterminal cells of b_1 with xy , or we can label the first k cells in b_1 with xy for where $0 \leq k \leq b_1 - 2$, followed by a -1 label in cell $k + 1$, and the rest of the nonterminal cells in b_1 with either x or -1 . Thus, the labeling of the first brick will contribute a factor of $\left((xy)^{b_1-1} - \sum_{k=0}^{b_1-2} (xy)^k (x-1)^{b_1-2-k} \right)$ to the right hand side of (5.1).

We shall consider the following lemma in order to interpret the p, q -multinomial coefficient in terms of the number of inversions and co-inversions. This is a generalization of the result by Carlitz in [13] with the proof given in the book [37].

Lemma 4. For positive integers b_1, \dots, b_k which sum to n ,

$$\begin{aligned} \left[\begin{matrix} n \\ b_1, \dots, b_k \end{matrix} \right]_{p,q} q^{\sum \binom{b_i}{2}} &= \sum_{r \in \mathcal{R}(1^{b_1}, \dots, k^{b_k})} q^{\text{inv}(r) + \sum \binom{b_i}{2}} p^{\text{coinv}(r)} \\ &= \sum_{\substack{\sigma \in S_n \text{ has descending runs} \\ \text{of lengths } b_1, b_2, \dots, b_k}} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \end{aligned}$$

where $\mathcal{R}(1^{b_1}, 2^{b_2}, \dots, k^{b_k})$ is the set of rearrangements of b_1 1's, b_2 2's, \dots , b_k k's and a descending run in σ is a consecutive decreasing subsequence of σ .

If the bricks in B have lengths $b_1, \dots, b_{\ell(\lambda)}$ when read from left to right, then Lemma 4 uses the powers of p and q to fill the cells of B with a permutation $\sigma \in S_n$ such that the elements in σ are decreasing within each brick. We also decorate each cell of B with $q^{\alpha_i} p^{\beta_i}$ where α_i counts the number of cells to the right of cell i which are filled with a number smaller than σ_i , and β_i counts the number of those that are larger than σ_i . This accounts for the term $\left[\begin{matrix} n \\ b_1, b_2, \dots, b_{\ell(\lambda)} \end{matrix} \right]_{p,q} q^{\sum \binom{b_i}{2}}$ in (5.1).

For example, one such combinatorial object created in the above manner is given in Figure 5.1 for $n = 14$ and $B = (6, 3, 1, 2, 2)$.

xy	xy	-1	x	-1	1	-1	x	1	1	x	1	-1	1
$q^{12}p^1$	q^9p^3	q^5p^6	q^4p^6	q^2p^7	q^0p^8	q^6p^1	q^5p^1	q^0p^5	q^1p^3	q^3p^0	q^2p^0	q^1p^0	q^0p^0
13	10	6	5	3	1	12	11	2	8	14	9	7	4

Figure 5.1: An example of the object created from equation (5.1).

This gives a configuration $T = (B, \sigma, \mathcal{L})$ where B is the brick tabloid of shape (n) and type λ , σ is the permutation used to fill in the cells of B , and \mathcal{L} keeps track of the $xy, x, \pm 1$ labels of B . We let \mathcal{T}_n be the set of all possible configuration T created under this process. That is, \mathcal{T}_n consists of all triples (B, σ, \mathcal{L}) where

- i. $B = (b_1, b_2, \dots, b_{\ell(\lambda)})$ is a brick tabloid in $\mathcal{B}_{\lambda, n}$
- ii. $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ is a permutation of S_n with descent runs of lengths $b_1, b_2, \dots, b_{\ell(\lambda)}$.
- iii. \mathcal{L} is the labeling of the brick tabloid B following the rule described above.

For each $T \in \mathcal{T}_n$, we define the weight of such combinatorial object to be the product of all x 's, y 's, ± 1 's, and the powers of p and q , then $[n]_{p,q}! \varphi(p_{n,\nu})$ is the weighted sum of all possible objects. For instance, the weight of the object given in Figure 5.1 is $x^5 y^2 q^{51} p^{40}$.

Now we define a mapping $I : \mathcal{T}_n \rightarrow \mathcal{T}_n$ as follows. Given a filled, labeled brick tabloid $O = (B, \sigma, \mathcal{L}) \in \mathcal{T}_n$, we read the cells of B from left to right, looking for the first cell c for which either

- (A) cell c is in the middle of some brick b_i and is labeled with -1 , or
- (B) cell c is at the end of brick b_j , cell $c + 1$ immediately to the right of it starts a new brick b_{j+1} , and there is a decrease between σ_c and σ_{c+1} .

If we are in case (A), then we define $I(O)$ to be the filled, labeled brick tabloid obtained from O by breaking the brick b_i that contains cell c into two bricks b'_i and b''_i where b'_i contains the cells of b_i up to and including the cell c while b''_i is the remaining cells of b_i . In addition, we change the labeling of cell c from -1 to 1 . If we are in case (B), then the image $I(O)$ is obtained by combining the two bricks b_j and b_{j+1} and change the label of cell c from 1 to -1 . For instance, the image of the brick tabloids from Figure 5.1 under this involution is given in Figure 5.2 below.

We claim that the mapping $I : \mathcal{T}_n \rightarrow \mathcal{T}_n$ defined above is indeed an involution, that is, $I(I(O)) = O$ for all $O \in \mathcal{T}_n$. To see this, let $O \in \mathcal{T}_n$ and suppose $I(O)$ is

xy	xy	1	x	-1	1	-1	x	1	1	x	1	-1	1
$q^{12}p^1$	q^9p^3	q^5p^6	q^4p^6	q^2p^7	q^0p^8	q^6p^1	q^5p^1	q^0p^5	q^1p^3	q^3p^0	q^2p^0	q^1p^0	q^0p^0
13	10	6	5	3	1	12	11	2	8	14	9	7	4

Figure 5.2: The image of the filled, labeled brick tabloid from Figure 5.1.

the image of O under the mapping. If we are in case (A) where $I(O)$ is obtained by breaking a brick b_i into two bricks b'_i and b''_i at some internal cell c which is labeled with -1 in O . Then it is the case that in $I(O)$, cell c is at the end of brick b'_i and there is a decrease between σ_c and σ_{c+1} . Furthermore, there cannot be any cell k where $k < c$ that is labeled with -1 since otherwise, we would not use cell c to define the mapping. Hence, when we reapply the mapping to $I(O)$, we have to combine the bricks b'_i and b''_i back into b_i and revert the label of cell c from 1 to -1. Suppose now we are in case (B) where cell c is at the end of brick b_j with $\sigma_c > \sigma_{c+1}$ and we combine the two bricks b_j and b_{j+1} . Then it must be the case that there is no cell labeled -1 that comes before cell c and that there is no decrease between any two consecutive brick before brick b_j . Thus, when we apply the mapping to $I(O)$, we will have to split the brick after cell c . In either case, we have $I(I(O)) = O$ which shows that I is an involution.

If neither case occurs, then we let $I(O) = O$, a fixed point of this involution. Let $\mathcal{F}_I(\mathcal{T}_n)$ be the set of all fixed points under the involution I . An example of an element of $\mathcal{F}_I(\mathcal{T}_n)$ is depicted in Figure 5.3 below. If $O = (B, \sigma, \mathcal{L}) \in \mathcal{F}_I(\mathcal{T}_n)$ then O cannot have any cell labeled -1 , and there must be an increase between any two consecutive bricks. This guarantees that within the first brick of any fixed point O , the last cell is labeled with 1 while the non-terminating cells are labeled with xy . In addition, all the bricks of O other than the first one have their last cell labeled with 1

while the non-terminating cells are labeled with x . Thus, if $O = (B, \sigma, \mathcal{L})$ then the power of x is $\text{des}(\sigma)$, while the power of y counts the number of initial descent in σ . Furthermore, the total powers of q and p in the cells counts the number of inversion and co-inversion of σ , respectively.

xy	xy	xy	xy	xy	1	x	x	1	1	x	1	x	1
$q^{12}p^1$	q^9p^3	q^5p^6	q^4p^6	q^2p^7	q^0p^8	q^6p^1	q^5p^1	q^0p^5	q^1p^3	q^3p^0	q^1p^1	q^1p^0	q^0p^0
13	10	6	5	3	1	12	11	2	8	14	7	9	4

Figure 5.3: A fixed point of the involution I .

Thus, we have

$$[n]_{p,q}! \theta(p_{n,\nu}) = \sum_{O \in \mathcal{T}_n} \omega(O) = \sum_{O \in \mathcal{F}_I(\mathcal{T}_n)} \omega(O) = \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} y^{\text{in}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)},$$

as desired.

To complete the proof and obtain the required generating function, we then use the relationship between $p_{n,\nu}$ and e_n given in (1.5), as follows.

$$\begin{aligned} \sum_{n \geq 1} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} y^{\text{in}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} &= \sum_{n=0}^{\infty} t^n \theta(p_{n,\nu}) \\ &= \theta \left(\sum_{n \geq 1} p_{n,\nu} t^n \right) \\ &= \theta \left(\frac{\sum_{n \geq 1} (-1)^{n-1} \nu(n) e_n t^n}{1 + \sum_{n \geq 1} e_n (-t)^n} \right) \\ &= \frac{\sum_{n \geq 1} (-1)^{n-1} \nu(n) \theta(e_n) t^n}{1 + \sum_{n \geq 1} \theta(e_n) (-t)^n}. \end{aligned} \tag{5.2}$$

The numerator of (5.2) then becomes

$$\begin{aligned}
\sum_{n \geq 1} (-1)^{n-1} \nu(n) \theta(e_n) t^n &= \sum_{n \geq 1} \frac{q^{\binom{n}{2}} t^n}{[n]_{p,q}!} \left((xy)^{n-1} - \sum_{k=0}^{n-2} (xy)^k (x-1)^{n-2-k} \right) \\
&= \sum_{n \geq 1} \frac{q^{\binom{n}{2}} t^n}{[n]_{p,q}!} \left((xy)^{n-1} - \frac{(xy)^{n-1} - (x-1)^{n-1}}{xy - (x-1)} \right) \\
&= \frac{1}{xy - x + 1} \sum_{n \geq 1} \frac{q^{\binom{n}{2}} t^n}{[n]_{p,q}!} ((xy)^n - x^n y^{n-1} + (x-1)^{n-1}) \\
&= \frac{1}{xy - x + 1} \sum_{n \geq 1} \left(\left(1 - \frac{1}{y}\right) \frac{q^{\binom{n}{2}} (txy)^n}{[n]_{p,q}!} + \frac{q^{\binom{n}{2}} (t(x-1))^n}{(x-1)[n]_{p,q}!} \right) \\
&= \frac{1}{xy - x + 1} \left(\frac{y-1}{y} (\mathbf{e}_{p,q}^{txy} - 1) + \frac{1}{x-1} (\mathbf{e}_{p,q}^{t(x-1)} - 1) \right) \\
&= \frac{(x-1)(y-1)(\mathbf{e}_{p,q}^{txy} - 1) + y(e^{t(x-1)} - 1)}{y(x-1)(xy - x + 1)}. \tag{5.3}
\end{aligned}$$

The denominator of (5.2) gives

$$\begin{aligned}
1 + \sum_{n \geq 1} \theta(e_n) (-t)^n &= 1 - \sum_{n \geq 1} \frac{q^{\binom{n}{2}} t^n}{[n]_{p,q}!} (x-1)^{n-1} \\
&= 1 - \frac{1}{x-1} \sum_{n \geq 1} \frac{q^{\binom{n}{2}} (t(x-1))^n}{[n]_{p,q}!} \\
&= 1 - \frac{1}{x-1} (\mathbf{e}_{p,q}^{t(x-1)} - 1) \\
&= \frac{x - \mathbf{e}_{p,q}^{t(x-1)}}{x-1}. \tag{5.4}
\end{aligned}$$

The statement of the theorem thus follows shortly from (5.3) and (5.4). \square

5.2 Generating function for the number of initial and final descents

The main goal of this section is to prove the following result of Theorem 23.

$$\begin{aligned} \sum_{n \geq 2} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} y^{\text{in}(\sigma)} z^{\text{fin}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \\ = G(x, y, z, t) + A(x, y, z, t) - (B(x, y, z, t) - C(x, y, z, t) - D(x, y, z, t)) \end{aligned}$$

where

$$\begin{aligned} A(x, y, z, t) &= \frac{\mathbf{e}_{p,q}^{xyzt} - xyzt - 1}{xyz}, \\ B(x, y, z, t) &= \frac{(y-1)(\mathbf{e}_{p,q}^{xyt} - xyt - 1)}{xy(y-z)(xy-x+1)}, \\ C(x, y, z, t) &= \frac{(z-1)(\mathbf{e}_{p,q}^{xzt} - xzt - 1)}{xz(y-z)(xz-x+1)}, \\ D(x, y, z, t) &= \frac{\mathbf{e}_{p,q}^{t(x-1)} - t(x-1) - 1}{(x-1)(xy-x+1)(xz-x+1)}, \\ G(x, y, z, t) &= \frac{F(x, y, t)F(x, z, t)}{yz(x-1)(xy-x+1)(xz-x+1)(x - \mathbf{e}_{p,q}^{t(x-1)})} \text{ for} \\ F(x, y, t) &= (y-1)(x-1)\mathbf{e}_{p,q}^{xyt} + y\mathbf{e}_{p,q}^{t(x-1)} - (xy-x+1). \end{aligned}$$

To this end, we shall use the same ring homomorphism $\theta : \Lambda \rightarrow \mathbb{Q}[x]$ from the proof of Theorem 22 given by $\varphi(e_0) = 1$ and $\varphi(e_n) = \frac{(-1)^{n-1}}{[n]_{p,q}!} (x-1)^{n-1} q^{\binom{n}{2}}$ for $n \geq 1$. However, unlike the case of Theorem 22 where we weight the brick tabloids only by the length of their first bricks, we now define the two weighting functions α_1 and α_2

on the first and last brick of every brick tabloid respectively by

$$\alpha_1(n) = \frac{(xy)^{n-1} - \sum_{k=0}^{n-2} (xy)^k (x-1)^{n-2-k}}{(x-1)^{n-1}}, \text{ and}$$

$$\alpha_2(n) = \frac{(xz)^{n-1} - \sum_{k=0}^{n-2} (xz)^k (x-1)^{n-2-k}}{(x-1)^{n-1}}.$$

Observe that for the first and last bricks, we also have the following results:

$$\nu(b_1)\theta(e_{b_1}) = \frac{(-1)^{b_1-1} q^{\binom{b_1}{2}}}{[b_1]_{p,q}!} \left((xy)^{b_1-1} - \sum_{k=0}^{b_1-2} (xy)^k (x-1)^{b_1-2-k} \right), \text{ and}$$

$$\nu(b_{\ell(\lambda)})\theta(e_{b_{\ell(\lambda)}}) = \frac{(-1)^{b_{\ell(\lambda)}-1} q^{\binom{b_{\ell(\lambda)}}{2}}}{[b_{\ell(\lambda)}]_{p,q}!} \left((xz)^{b_{\ell(\lambda)}-1} - \sum_{k=0}^{b_{\ell(\lambda)}-2} (xz)^k (x-1)^{b_{\ell(\lambda)}-2-k} \right).$$

Given a brick tabloid $B = (b_1, \dots, b_{\ell(\lambda)}) \in \mathcal{B}_{n,\lambda}$ with $\ell(\lambda) \geq 2$, the weight $w(B)$ under α_1 and α_2 is then given by $\omega_{\alpha_1, \alpha_2}(B) = \alpha_1(b_1)\alpha_2(b_{\ell(\lambda)})$. The new basis $P_{n;\alpha_1, \dots, \alpha_r}$ defined by Mendes, Remmel, and Riehl in (1.8) now becomes

$$p_{n;\alpha_1, \alpha_2} = \sum_{\lambda \vdash n; \ell(\lambda) \geq 2} (-1)^{n-\ell(\lambda)} \omega_{\alpha_1, \alpha_2}(B_{n,\lambda}) e_\lambda.$$

Similar to the argument of the previous section, we will start our proof by applying the homomorphism φ to $[n]_{p,q}! p_{n;\alpha_1, \alpha_2}$ to obtain the following result.

$$\begin{aligned}
[n]_{p,q}! \varphi(p_{n;\alpha_1,\alpha_2}) &= [n]_{p,q}! \sum_{\lambda \vdash n; \ell(\lambda) \geq 2} \omega_{\alpha_1,\alpha_2}(B_{n,\lambda}) \varphi(e_\lambda) \\
&= [n]_{p,q}! \sum_{\lambda \vdash n; \ell(\lambda) \geq 2} \sum_{B=(b_1,\dots,b_{\ell(\lambda)}) \in \mathcal{B}_{n,\lambda}} \omega_{\alpha_1,\alpha_2}(B) \prod_{i=1}^{\ell(\lambda)} \varphi(e_{b_i}) \\
&= [n]_{p,q}! \sum_{\lambda \vdash n; \ell(\lambda) \geq 2} \sum_{B=(b_1,\dots,b_{\ell(\lambda)}) \in \mathcal{B}_{n,\lambda}} (-1)^{n-\ell(\lambda)} \alpha_1(b_1) \alpha_2(b_{\ell(\lambda)}) \\
&\quad \times \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{b_i-1} (x-1)^{b_i-1}}{[b_i]_{p,q}!} \\
&= \sum_{\lambda \vdash n; \ell(\lambda) \geq 2} \sum_{B=(b_1,\dots,b_{\ell(\lambda)}) \in \mathcal{B}_{n,\lambda}} \left[\begin{matrix} n \\ b_1 \dots b_{\ell(\lambda)} \end{matrix} \right]_{p,q} \prod_{i=2}^{\ell(\lambda)-1} (x-1)^{b_i-1} \\
&\quad \times \left((xy)^{b_1-1} - \sum_{k=0}^{b_1-2} (xy)^k (x-1)^{b_1-2-k} \right) \\
&\quad \times \left((xz)^{b_{\ell(\lambda)}-1} - \sum_{k=0}^{b_{\ell(\lambda)}-2} (xz)^k (x-1)^{b_{\ell(\lambda)}-2-k} \right). \quad (5.5)
\end{aligned}$$

The combinatorial interpretation for the right hand side of (5.5) can be obtained from a similar manner to that of (5.1). First, we take a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}) \vdash n$ such that $\ell(\lambda) \geq 2$ and pick a brick tabloid $B = (b_1, b_2, \dots, b_{\ell(\lambda)})$ of shape n and type λ . Observe that the condition that $\ell(\lambda) \geq 2$ is quite important in this proof and will greatly affect our analysis for the fixed points. This fact will become more evident in the later part of this proof. We then fill the cells of B with a permutation $\sigma \in S_n$ such that σ has descending runs of lengths $b_1, \dots, b_{\ell(\lambda)}$ and also fill in each cell i of B with $q^{\alpha_i} p^{\beta_i}$ where α_i counts the number of cells to the right of cell i which are filled with number smaller than σ_i , and β_i counts those that are larger than σ_i . By Lemma 4, this is counted by $\left[\begin{matrix} n \\ b_1, b_2, \dots, b_{\ell(\lambda)} \end{matrix} \right]_{p,q} q^{\sum \binom{b_i}{2}}$ in (5.5). The terms $\prod_{i=2}^{\ell(\lambda)-1} (x-1)^{b_i-1}$

correspond to the labeling of the middle bricks $b_2, \dots, b_{\ell(\lambda)-1}$ in B with x and ± 1 . The term $(xy)^{b_1-1} - \sum_{k=0}^{b_1-2} (xy)^k (x-1)^{b_1-2-k}$ corresponds to the labeling of the first bricks where we can use one of the two schemes described in the above section. The same argument holds for the labeling of the last brick $b_{\ell(\lambda)}$, where we simply replace y with z to obtain the factor of $(xz)^{b_{\ell(\lambda)}-1} - \sum_{k=0}^{b_{\ell(\lambda)}-2} (xz)^k (x-1)^{b_{\ell(\lambda)}-2-k}$.

Thus, we obtain a configuration $M = (\bar{B}, \sigma, \mathcal{L})$ where \bar{B} is a brick tabloid of shape n and type λ where $\ell(\bar{B}) \geq 2$, $\sigma \in S_n$ is the permutation used to fill in the cells of \bar{B} , and \mathcal{L} keeps track of the x, y, z , and 1 labels of \bar{B} . We let \mathcal{M}_n be the set of all possible configurations M created under this process. For each $M \in \mathcal{M}_n$, we let the weight $w(M)$ to be the product of all the x, y, z , and -1 labels of M . Therefore, the identity in (5.5) now becomes

$$[n]_{p,q}! \theta(p_{n;\alpha_1, \alpha_2}) = \sum_{O \in \mathcal{M}_n} w(O). \quad (5.6)$$

Again, the set \mathcal{M}_n contains of all objects $(\bar{B}, \sigma, \mathcal{L})$ with both positive and negative weight. To eliminate the elements with a negative weight, we shall again apply an involution $I' : \mathcal{M}_n \rightarrow \mathcal{M}_n$. To be specific, we will modify the involution given in the previous section, as follows. Given a filled, labeled brick tabloid $O = (\bar{B}, \sigma, \mathcal{L}) \in \mathcal{M}_n$, we read the cells of \bar{B} from left to right, looking for the first cell c for which either

- (A) cell c is in the middle of some brick b_i and is labeled with -1 , or
- (B) cell c is at the end of brick b_j , cell $c+1$ immediately to the right of it starts a new brick b_{j+1} , there is a decrease between σ_c and σ_{c+1} , and the brick tabloid \bar{B} has more than two bricks within.

In case (A), we break the brick b_i that contains cell c into two bricks b'_i and b''_i

where b'_i contains the cells of b_i up to and including the cell c while b''_i is the remaining cells of b_i . In addition, we change the labeling of cell c from -1 to 1 . In case (B), we combine the two bricks b_j and b_{j+1} and change the label of cell c from 1 to -1 . Of course, if neither case occurs then we simply have a fixed point for the involution. If we let $\mathcal{F}'_I(\mathcal{M}_n)$ be the set of all fixed points of the involution I' then the identity in (5.6) now gives

$$[n]_{p,q}! \theta(p_n; \alpha_1, \alpha_2) = \sum_{O \in \mathcal{M}_n} \omega(O) = \sum_{O \in \mathcal{F}'_I(\mathcal{M}_n)} \omega(O).$$

However, unlike before, $\sum_{O \in \mathcal{F}'_I(\mathcal{M}_n)} \omega(O) \neq \sum_{\sigma \in \mathcal{S}_n} x^{\text{des}(\sigma)} y^{\text{indeg}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}$.

The new restriction in case (B) introduces a sets of configurations of \mathcal{M}_n where we are unable to combine the bricks as doing so will reduce the number of bricks in \bar{B} to below the minimum requirement of two, which then causes the co-domain of I' to no longer be \mathcal{M}_n . Thus, we now have two kinds of fixed points under this involution I' . The first kind consists of all "regular" configurations (B, σ, \mathcal{L}) where B has either

- (a.) at least three bricks with no cell labeled with -1 and there is an increase between every two consecutive bricks in B , or
- (b.) two bricks with no cell labeled -1 but there is an increase between the last cell of b_1 and the first cell of b_2 .

In either case, we can see that, for the corresponding permutation σ , all the initial descents of σ are labeled with xy , all the final descents in σ are labeled with xz , and all the other descents are labeled with x . The total weight of these objects corresponds

to the term

$$\sum_{\sigma \in S_n, n \geq 2} x^{\text{des}(\sigma)} y^{\text{inves}(\sigma)} z^{\text{findes}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)}$$

in the desired generating function. Figure 5.4 below gives an example of this kind of fixed points.

xy	xy	xy	xy	xy	1	x	x	1	xz	xz	xz	xz	1
$q^{12}p^1$	q^9p^3	q^5p^6	q^4p^6	q^2p^7	q^0p^8	q^6p^1	q^5p^1	q^0p^5	q^4p^0	q^3p^0	q^2p^0	q^1p^0	q^0p^0
13	10	6	5	3	1	12	11	2	14	9	8	7	4

Figure 5.4: A “regular” fixed point of the involution I' .

The second kind of fixed point for this involution consists of all configurations $(\bar{B}, \sigma, \mathcal{L})$ where \bar{B} has exactly two bricks with $last(b_1) > first(b_2)$ but we are unable to combine the bricks in order to maintain the minimum number of bricks being at least two. Then it is easy to see that the permutation σ used to fill in the cells of \bar{B} must be the reverse identity permutation of S_n , namely $\sigma = (n, n - 1, \dots, 1)$. Furthermore, since we are attempting to combine the bricks, it must be the case that the first brick b_1 of \bar{B} contains no -1 label. As a result, the first brick of \bar{B} must be labeled with 1 for its last cell and with xy for every non-terminal cell. Since we have already used the last cell of brick b_1 to define the mapping, the labeling of the last brick now becomes irrelevant. That is, we can label the last brick using any term from $(xz)^{b_2-1} - \sum_{k=0}^{b_2-2} (xz)^k (x-1)^{b_2-2-k}$. The numbers of inversions and co-inversions in this fixed point are given by $q^{\binom{n}{2}} p^0$. Therefore, if we let k be the length of the first

brick, then the total weight of these new objects is given by

$$\begin{aligned}
& \sum_{k=1}^{n-1} q^{\binom{n}{2}} (xy)^{k-1} \left((xz)^{n-k-1} - \sum_{i=0}^{n-k-2} (xz)^i (x-1)^{n-2-k-i} \right) \\
&= \sum_{k=1}^{n-1} q^{\binom{n}{2}} (xy)^{k-1} \left((xz)^{n-k-1} - \frac{(xz)^{n-k-1} - (x-1)^{n-k-1}}{xz - x + 1} \right) \\
&= q^{\binom{n}{2}} \left(\left(1 - \frac{1}{xz - x + 1} \right) \sum_{k=1}^{n-1} (xy)^{k-1} (xz)^{n-k-1} \right. \\
&\quad \left. + \frac{1}{xz - x + 1} \sum_{k=1}^{n-1} (xy)^{k-1} (x-1)^{n-k-1} \right) \\
&= q^{\binom{n}{2}} \left(\frac{(z-1)((xy)^{n-1} - (xz)^{n-1})}{(y-z)(xz-x+1)} + \frac{(xy)^{n-1} - (x-1)^{n-1}}{(xz-x+1)(xy-x+1)} \right) \\
&= q^{\binom{n}{2}} \left(\frac{(y-1)(xy)^{n-1}}{(y-z)(xy-x+1)} - \frac{(z-1)(xz)^{n-1}}{(y-z)(xz-x+1)} - \frac{(x-1)^{n-1}}{(xy-x+1)(xz-x+1)} \right).
\end{aligned}$$

In terms of the number of inversions, co-inversions, descents, initial descents, and final descents, the correct weight corresponding to the reverse identity permutation is $(xyz)^{n-1} q^{\binom{n}{2}}$. Hence,

$$\begin{aligned}
& [n]_{p,q}! \theta(p_{n;\alpha_1, \alpha_2}) \\
&= \sum_{B \in \mathcal{F}_1(\mathcal{M}_n)} \omega(B) \\
&= \sum_{\sigma \in S_n, n \geq 2} x^{\text{des}(\sigma)} y^{\text{indeg}(\sigma)} z^{\text{findes}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} - (xyz)^{n-1} q^{\binom{n}{2}} \\
&\quad + q^{\binom{n}{2}} \left(\frac{(y-1)(xy)^{n-1}}{(y-z)(xy-x+1)} - \frac{(z-1)(xz)^{n-1}}{(y-z)(xz-x+1)} - \frac{(x-1)^{n-1}}{(xz-x+1)(xy-x+1)} \right).
\end{aligned}$$

The generating function thus becomes

$$\begin{aligned}
& \sum_{n \geq 2} \frac{t^n}{[n]_{p,q}!} \sum_{\sigma \in S_n, n \geq 2} x^{\text{des}(\sigma)} y^{\text{indes}(\sigma)} z^{\text{findes}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \\
&= \sum_{n \geq 2} \theta(p_{n;\alpha_1, \alpha_2}) t^n + \frac{1}{xyz} \sum_{n \geq 2} \frac{q^{\binom{n}{2}} (xyzt)^n}{[n]_{p,q}!} - \frac{(y-1)}{xy(y-z)(xy-x+1)} \sum_{n \geq 2} \frac{q^{\binom{n}{2}} (xyt)^n}{n!} \\
&+ \frac{(z-1)}{xz(y-z)(xz-x+1)} \sum_{n \geq 2} \frac{q^{\binom{n}{2}} (xzt)^n}{[n]_{p,q}!} \\
&+ \frac{1}{(x-1)(xz-x+1)(xy-x+1)} \sum_{n \geq 2} \frac{q^{\binom{n}{2}} (t(x-1))^n}{[n]_{p,q}!}. \tag{5.7}
\end{aligned}$$

Each term in the right hand side of (5.7) then gives

$$\frac{1}{xyz} \sum_{n \geq 2} \frac{q^{\binom{n}{2}} (xyzt)^n}{[n]_{p,q}!} = \frac{\mathbf{e}_{p,q}^{xyzt} - xyzt - 1}{xyz} = A(x, y, z, t), \tag{5.8}$$

$$\frac{(y-1)}{xy(y-z)(xy-x+1)} \sum_{n \geq 2} \frac{q^{\binom{n}{2}} (xyt)^n}{[n]_{p,q}!} = \frac{(y-1)(\mathbf{e}_{p,q}^{xyt} - xyt - 1)}{xy(y-z)(xy-x+1)} = B(x, y, z, t), \tag{5.9}$$

$$\frac{(z-1)}{xz(y-z)(xz-x+1)} \sum_{n \geq 2} \frac{q^{\binom{n}{2}} (xzt)^n}{[n]_{p,q}!} = \frac{(y-1)(\mathbf{e}_{p,q}^{xzt} - xzt - 1)}{xz(y-z)(xz-x+1)} = C(x, y, z, t), \tag{5.10}$$

$$\begin{aligned}
& \frac{1}{(x-1)(xz-x+1)(xy-x+1)} \sum_{n \geq 2} \frac{q^{\binom{n}{2}} (t(x-1))^n}{[n]_{p,q}!} \\
&= \frac{\mathbf{e}_{p,q}^{t(x-1)} - t(x-1) - 1}{(x-1)(xy-x+1)(xz-x+1)} = D(x, y, z, t), \tag{5.11}
\end{aligned}$$

$$\sum_{n \geq 2} \theta(p_{n;\alpha_1, \alpha_2}) t^n = \frac{(\sum_{n \geq 1} (-1)^{n-1} \alpha_1(n) \theta(e_n) t^n) (\sum_{n \geq 1} (-1)^{n-1} \alpha_2(n) \theta(e_n) t^n)}{1 + \sum_{n \geq 1} (-1)^n \theta(e_n) (-t)^n}. \tag{5.12}$$

Similar to (5.4), the denominator of (5.12) becomes

$$1 + \sum_{n \geq 1} (-1)^n \theta(e_n) (-t)^n = \frac{x - \mathbf{e}_{p,q}^{t(x-1)}}{x-1}. \quad (5.13)$$

The first component of the numerator in (5.12) becomes

$$\begin{aligned} & \sum_{n \geq 1} (-1)^{n-1} \alpha_1(n) \theta(e_n) t^n \\ &= \sum_{n \geq 1} \frac{q^{\binom{n}{2}} t^n}{[n]_{p,q}!} \left((xy)^{n-1} - \frac{(xy)^{n-1} - (x-1)^{n-1}}{xy-x+1} \right) \\ &= \frac{y-1}{y(xy-x+1)} \sum_{n \geq 1} \frac{q^{\binom{n}{2}} (xyt)^n}{[n]_{p,q}!} + \frac{1}{(x-1)(xy-x+1)} \sum_{n \geq 1} \frac{q^{\binom{n}{2}} (t(x-1))^n}{[n]_{p,q}!} \\ &= \frac{y-1}{y(xy-x+1)} (\mathbf{e}_{p,q}^{xyt} - 1) + \frac{1}{(x-1)(xy-x+1)} (\mathbf{e}_{p,q}^{t(x-1)} - 1) \\ &= \frac{y-1}{y(xy-x+1)} \mathbf{e}_{p,q}^{xyt} + \frac{1}{(x-1)(xy-x+1)} \mathbf{e}_{p,q}^{t(x-1)} - \frac{1}{y(x-1)} \\ &= \frac{(y-1)(x-1) \mathbf{e}_{p,q}^{xyt} + y \mathbf{e}_{p,q}^{t(x-1)} - (xy-x+1)}{y(x-1)(xy-x+1)} \\ &= \frac{F(x, y, t)}{y(x-1)(xy-x+1)} \end{aligned} \quad (5.14)$$

By similar computation, the second component of the numerator of (5.12) becomes

$$\begin{aligned} \sum_{n \geq 1} (-1)^{n-1} \alpha_2(n) \theta(e_n) t^n &= \frac{(z-1)(x-1) \mathbf{e}_{p,q}^{xzt} + z \mathbf{e}_{p,q}^{t(x-1)} - (xz-x+1)}{z(x-1)(xz-x+1)} \\ &= \frac{F(x, z, t)}{z(x-1)(xz-x+1)} \end{aligned} \quad (5.15)$$

Then we combine the results from (5.12), (5.13), (5.14), and (5.15) to obtain

$$\begin{aligned}
\sum_{n \geq 2} \theta(p_{n:\alpha_1, \alpha_2}) t^n &= \frac{(\sum_{n \geq 1} (-1)^{n-1} \alpha_1(n) \theta(e_n) t^n) (\sum_{n \geq 1} (-1)^{n-1} \alpha_2(n) \theta(e_n) t^n)}{1 + \sum_{n \geq 1} (-1)^n \theta(e_n) (-t)^n} \\
&= \frac{F(x, y, t) F(x, z, t)}{yz(x-1)(xy-x+1)(xz-x+1)(x-e^{t(x-1)})} \\
&= G(x, y, z, t). \tag{5.16}
\end{aligned}$$

Lastly, we put the results from (5.8), (5.9), (5.10), (5.11), and (5.16) together to complete the proof for Theorem 23. \square

Finally, we notice that the same machinery presented in this chapter can also be applied to study the generation function for the number of initial and final descents in alternating permutations. These results will appear in a subsequent paper by Remmel and the dissertation author.

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