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On Connecting the Microscopic and Macroscopic Theories of Type II Superconductivity

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ABSTRACT

We present a simplified derivation of approximate solutions for the magnitude of, and the relations between, the three critical fields of a Type II superconductor done in the spirit of the Ginzburg-Landau equations. The appearance of an unexpected equation restricting the free choice of macroscopic parameters is shown to be a direct consequence of the "correctness" of Ginzburg-Landau theory, i.e., its derivability from the microscopic theory of Bardeen, Cooper, and Schrieffer.

INTRODUCTION

In most introductory courses dealing with superconductivity, the behavior of Type II superconductors is discussed entirely within the framework of the macroscopic semiphenomenological theory as developed by Ginzburg and Landau[1] and later extended by Abrikosov[2], and Gor'kov[3] (hereafter referred to collectively as (GLAG)).[4] In undergraduate survey courses, particularly, neither the microscopic theory of Bardeen, Cooper, and Schrieffer (BCS)[5] nor the Gor'kov proof of the derivability of the macroscopic parameters of Ginzburg and Landau from microscopic theory can be treated in sufficient detail to adequately bring out the physical equivalence of the BCS and GLAG treatments.[6] We have developed a particularly simple method for deriving a set of excellent approximations to the exact GLAG solutions for the relationships between the several critical fields of Type II superconductors by simple physical arguments which proceeds on the level of most introductory solid state physics texts. In addition to deriving expressions for the upper and lower critical fields H_{c1} and H_{c2}, we also obtain a fundamental equation relating the thermodynamic critical field H_c, and therefore the condensation energy, to the two characteristic lengths in the problem, the coherence length ξ and the penetration depth λ. The apparently paradoxical appearance of this "new" fundamental equation is clarified by appealing to the microscopic (BCS) definitions of λ and ξ, by which means it is quickly shown that the "new" equation reduces to the BCS equation for the condensation energy. The re-introduction of the microscopic theory at this unexpected point in an otherwise purely macroscopic treatment is both elegant and satisfying, and serves well the purpose of emphasizing the physical unity of the GLAG and BCS theories.

TYPE II SUPERCONDUCTORS: A SIMPLIFIED GLAG APPROACH

The GLAG theory was originally developed by Ginzburg and Landau to extend the macroscopic London[7] theory to correctly treat the behavior of superconductors in a magnetic field. Although, as discussed below, the original theory accounted well for the retention of the perfect diamagnetism of most elemental superconductors even in fields approaching the critical field H_c, it remained for Abrikosov to show that their theory also provided an explanation for the behavior of Type II superconductors, which retained their superconductivity up to very high fields while showing perfect diamagnetism only for quite small applied fields. The GLAG approach remains the favored way to treat the properties of Type II superconductors.[8] In this section we shall develop a set of approximations to the exact GLAG solutions by arguing from experimental observations in the spirit of their original derivation.

We begin with the usual definition of a Type I superconductor as being a zero-dc-resistance material exhibiting perfect diamagnetism (Meissner effect).[9] In the presence of an externally applied magnetic field H, the internal magnetic field $B = H + 4\pi M = 0$ everywhere except at the surface, where the field dies away exponentially with a characteristic length λ. This behavior persists until $H = H_c$, the critical field, at which point the superconductivity vanishes, and the field penetrates, so that $B \approx H$ for $H \geqslant H_c$. For $0 < H < H_c$, the free energy density $\mathcal{F}_S(H)$ of the superconductor will be raised above its zero-field value $\mathcal{F}_{SO}$ by the energy cost of excluding the external field. Taking the usual London model of the Meissner effect, the perfect diamagnetism is considered to be a superposition on the external field of a precisely equal but opposite magnetic field generated by the solenoidal current flowing within a penetration depth of the surface. Since this field must just cancel the homogeneous external field, its magnitude is H, and the increased energy per unit volume of the sample associated with the cancelling field is just $H^2/8\pi$. Therefore, as a consequence of the Meissner effect, the free energy density of the sample in an external field H must be

$$\mathcal{F}_S(H) = \mathcal{F}_{SO} + \frac{H^2}{8\pi} \tag{1}$$

per unit volume. Since $B \approx H$ in the normal state, the free energy density, $\mathcal{F}_n$, of the sample when normal is nearly independent of H. At the critical field H_c a transition to the normal state occurs; therefore $\mathcal{F}_S(H_c) = \mathcal{F}_n = \mathcal{F}_{SO} + H_c^2/8\pi$, or

$$\mathcal{F}_{SO} = \mathcal{F}_n - \frac{H_c^2}{8\pi} . \tag{2}$$

The quantity $H_c^2/8\pi$, then, measures the condensation energy, i.e., the free energy density difference between the superconducting and normal states in zero applied field.

Type II superconductors, although having the same fundamental mechanism for superconductivity, behave somewhat differently in a magnetic field: $B = 0$ for all $H < H_{c1}$, the lower critical field, and only here is the Meissner effect complete. For $H \geqslant H_{c2}$, the upper critical field (which may be several orders of magnitude larger than H_{c1}), the sample is normal. However, for $H_{c1} < H < H_{c2}$, $B \neq 0$ even though the sample is superconducting; the Meissner effect is incomplete.

The great power of the GLAG treatment of this case lies first in the idea that for $H_{c1} < H < H_{c2}$ the flux penetrates in the form of quantized flux lines or vortices, and second in the separation of all superconductors into Type I or Type II according to the single parameter $\kappa \equiv \lambda/\xi$ where λ is the magnetic field penetration depth and ξ is the coherence length, first used by Pippard[10] to introduce long range, non-local effects into the strictly local London theory. The parameter κ was in fact one of the goals of the Ginzburg-Landau theory, originally derived to provide a positive surface energy to prevent the formation of a normal-superconducting boundary when a magnetic field is applied. The necessity for this is made apparent by considering Eqs. (1) and (2) and the definition of λ.

Clearly the superconductor can lower its free energy by allowing the field to penetrate via a thin sheet of normal phase whose thickness is $d \ll \lambda$. The cost in condensation energy is only $(H_c^2/8\pi) \times \lambda$ per unit area, while the gain in energy due to field penetration is $-(H^2/8\pi) \times 2\lambda$. If the length of the superconducting block is L, and n sheets of normal phase enter, the free energy per unit area can be approximated by

$$\mathcal{F}_S'(H) \times L = \mathcal{F}_{S0} \times L + n\left(\frac{H_c^2}{8\pi} \times d - \frac{H^2}{8\pi} \times 2\lambda\right). \qquad (3)$$

For $d/\lambda \ll 1$, the free energy is minimized by maximizing n (within the restrictions of the boundary conditions) even for moderate values of H; this implies a reduced or even a vanishing Meissner effect. As this effect was demonstrably an experimental fact for many superconductors, Ginzburg and Landau sought a solution containing a positive interface energy which made the perfectly diamagnetic state stable against the formation of such normal intrusions. They did so by defining the coherence length, ξ, as the range of the superconducting interaction,[11] i.e., the minimum distance over which superconductivity could be destroyed. Therefore, the minimum thickness of a normal sheet would be ξ, and Eq. (3) becomes

$$\mathcal{F}_S'(H) \times L = \mathcal{F}_{SO} \times L + n\left(\frac{H_c^2}{8\pi} \times \xi - \frac{H^2}{8\pi} \times 2\lambda\right) \qquad (3')$$

and for $\xi > 2\lambda$ the Meissner solution ($n = 0$, $B = 0$) clearly has the lowest energy for all $H < H_c$. An exact GLAG calculation shows that for $\kappa < 1/\sqrt{2}$ the superconductor will be Type I and have a complete Meissner effect for all $H < H_c$, while for $\kappa > 1/\sqrt{2}$ there is some range of field over which it is favorable for the field to penetrate and the superconductor is Type II.

We can now derive H_{c1} in the standard[12] manner. As is well known, the flux will enter in the form of "vortices" or flux lines (properly flux tubes). Since H_{c1} is the minimum field for the breakdown of the Meissner effect, it must be just energetically favorable at $H = H_{c1}$ for the first line to enter. For such a flux line, we approximate the field distribution by a cylindrical tube of radius λ. The decrease in $\mathcal{F}_{SH}$ per unit length of line due to the penetration of this flux is, from Eq. (1),

$$F_{MAG} = -\frac{H^2}{8\pi} \times \pi\lambda^2. \qquad (4)$$

Similarly we approximate the normal core accompanying this flux by a cylinder of radius ξ. From Eq. (2), the free energy cost per unit length of producing this normal core in the vortex is

$$F_{CORE} = \frac{H_c^2}{8\pi} \times \pi\xi^2. \qquad (5)$$

The total energy of the line per unit length is then

$$F_{LINE} = F_{CORE} + F_{MAG} \approx \frac{1}{8} (H_c^2\xi^2 - H^2\lambda^2). \qquad (6)$$

For $F_{LINE} < 0$ the vortex is stable. The threshold field for stability is, by definition, H_{c1}, so that we have

$$H_{c1} \approx \frac{\xi}{\lambda} H_c = \frac{H_c}{\kappa} . \qquad (7)$$

For $H_{c1} < H < H_{c2}$, the sample is in the vortex state; as H is increased, more and more flux enters in the form of vortices. One of the most important results of the GLAG theory is that each flux line is identical to the first, and that the flux enters by the creation of more and more of such identical vortices in the superconductor. We now introduce the Abrikosov condition that the flux in the flux line is quantized and equal to $\Phi_o \equiv \frac{hc}{2e}$. The identity of vortices reduces to the statement that all flux lines contain one quantum of flux Φ_o. By our previous assumption, the field in the vortex is $\approx H_{c1}$, while its area is $\approx \pi\lambda^2$; therefore

$$H_{c1} \approx \frac{\Phi_o}{\pi\lambda^2} . \qquad (8)$$

As we approach the upper critical field H_{c2}, $B \approx H$, although zero dc resistance persists until H_{c2} is reached. Within our simple model we can say that the superconductivity will vanish at the field where the flux lines are packed so closely that the normal cores start to overlap, and there is no longer any continuous superconducting path through the material. Let us examine one of the normal cores in this case. The core has cross-sectional area $\pi\xi^2$. As the area of the flux tube is $\pi\lambda^2$, the selected core will have a flux contribution from every tube whose center lies within a distance λ. The number of tubes within λ is just the packing factor $PF \approx (\pi\lambda^2/\pi\xi^2)$ by our initial assumption, while the core intercepts a fractional flux $\phi \approx \Phi_o\ (\pi\xi^2/\pi\lambda^2)$ of each of these tubes. The total flux through the core is then $PF \times \phi = (\pi\lambda^2/\pi\xi^2) \times \Phi_o(\pi\xi^2/\pi\lambda^2) = \Phi_o$, and the field in the core is therefore $B = \Phi_o/\pi\xi^2$. Since $B \approx H$ at H_{c2}, the field is nearly homogeneous throughout and, therefore,

$$H_{c2} \approx \frac{\Phi_o}{\pi\xi^2} . \qquad (9)$$

EQUATIONS RELATING THE CRITICAL FIELDS

We may now obtain the equations which relate the several critical fields by using the definition of κ; $\kappa \equiv \lambda/\xi$. We may use Eq. (7) to rewrite Eq. (9) in terms of κ as

$$H_{c2} \approx \kappa^2\ H_{c1}. \qquad (10)$$

Taking this equation together with Eq. (7),

$$H_{c1} \approx \frac{H_c}{\kappa},$$

we obtain the geometrical mean field rule

$$(H_{c1}H_{c2})^{1/2} \approx H_c. \qquad (11)$$

We still have not adequately defined H_c in this case, for in the vortex state $H_{c1} < H < H_{c2}$ there is no observable effect at $H_c = (H_{c1}H_{c2})^{1/2}$. Therefore, we take a thermodynamic definition of H_c as being a measure of the condensation energy density, that is

$$\frac{H_c^2}{8\pi} \equiv \mathcal{F}_n - \mathcal{F}_{SO}$$

where H_c is now the thermodynamic critical field. From the usual thermodynamic considerations,[13] the magnetic contribution to the free energy density is given by

$$\frac{H_c^2}{8\pi} = -\frac{1}{4\pi}\int_0^{H_{c2}} (B-H)\,dH. \qquad (12)$$

So far we have merely performed a simple derivation of the relationship between the several critical fields for a large κ superconductor. Equations (8) and (9) are not surprising; the first is a consequence of the free energy consideration for penetration of the first fluxoid, and the second follows from the geometry of the vortex state and the definition of H_{c2}. However, substituting Eqs. (8) and (9) into Eq. (11) we obtain

$$H_c \approx \frac{\Phi_o}{\pi\lambda\xi} , \tag{13}$$

which appears to be a totally "new" equation restricting the allowed values of λ and ξ by connecting them via the condensation energy. Thus of the three parameters H_c, λ and ξ (or H_{c1},H_{c2},H_c) only two can be chosen independently. However, at no point in the preceding derivation was such a restriction explicitly imposed, nor does there appear to be any reason for doing so. The question then becomes: what physics have we overlooked, i.e., what have we been implicitly assuming about the nature of the superconducting state which, if stated, will supply us with a physical understanding of the nature and origin of Eq. (13)? It is at this point that one appeals to a "higher authority", the BCS microscopic theory of superconductivity.

AN APPEAL TO MICROSCOPIC THEORY

In the BCS pairing model of superconductivity, the superconducting charge carriers are taken to be electron pairs bound together by a pairing energy Δ per electron. The superconductivity arises from the binding of electrons within $\approx \Delta$ of the Fermi energy into pairs, resulting in an energy gap $E_g = 2\Delta$ for the creation of excitations which would damp the supercurrent. Taking N(0) to be the density of electrons of one spin at the Fermi energy, the number density of paired electrons is $\approx N(0)\Delta$. The pairing energy per electron is $\approx \Delta$, so that the pairing results in a lowering of the energy density of the superconducting state below that of the normal state by an amount $\delta E \approx N(0)\Delta^2$. By our previous definition of the thermodynamic critical field H_c (c.f. Eq. (12)) this must just be

$\delta E = \mathcal{F}_n - \mathcal{F}_{SO} = \frac{H_c^2}{8\pi}$ so that we obtain

$$\frac{H_c^2}{8\pi} \approx N(0)\Delta^2. \tag{14}$$

Let us compare this with Eq. (13) for H_c. We take the penetration depth to be just the London penetration depth $\lambda_L^2 = (mc^2/4n_s q^2)$. According to the pairing model n_s will be just half the number of conduction electrons, while q is twice the electronic charge, so that

$$\lambda_L = \left[\frac{3c^2}{8\ e^2 v_F^2 N(0)}\right]^{1/2} \tag{15}$$

where v_F is the velocity of an electron at the Fermi energy. ξ, on the other hand, arises directly from the pairing theory and the uncertainty principle, and must be independently derived.

Using a free-electron model, we define $E_F = p_F^2/2m$, where p_F is the momentum of an electron at the Fermi energy E_F. As a paired electron must be localized to within an energy Δ of E_F, it will have a maximum momentum δp given by $\Delta \approx p_F \delta p/m \approx v_F \delta p$, or $\delta p \approx \Delta/v_F$. From the uncertainty principle, the minimum range δr for the pairing interaction must then be $\delta r \approx \hbar/\delta p \approx \hbar v_F/\Delta$. Recalling our original definition of the coherence length, we identify δr with ξ by noting that δr will be the characteristic length for the superconducting behavior to decay at an interface. Thus we obtain

$$\xi \approx \frac{\hbar\ v_F}{\Delta}\ . \tag{16}$$

Substituting Eqs. (15) and (16) for λ and ξ into our macroscopically derived relation for H_c, Eq. (13), gives us

$$\frac{H_c^{\;2}}{8\ \pi} \approx \frac{\Phi_o^{\;2}}{8\pi^3 \lambda_L^{\;2} \xi^2} \approx \frac{N(0)\Delta^2}{3} \tag{17}$$

which should be compared with the purely microscopic expression given by Eq. (14).

SUMMARY AND CONCLUSION

The paradox of the surprising appearance of Eq. (13) in the macroscopic treatment, which restricts our ability to choose H_c, λ and ξ independently, can be resolved only by an appeal to microscopic theory. We conclude, therefore, that the physics which was overlooked in our derivation of Eq. (13) was the correct microscopic theory of the superconducting state. Although GLAG theory appears quite satisfying as a purely macroscopic, thermodynamic, semiphenomenological approach to superconductivity it must, if it is a correct theory, be equivalent to, and therefore derivable from, the microscopic BCS theory. From this point of view, the equivalence of Eq. (13) and Eq. (14) gives us the connection between the microscopic and macroscopic theories as expressed in Eq. (17). λ^{-2} is a normalized measure of the density of conduction electron states at the Fermi energy but is essentially a macroscopic parameter governing the penetration of the macroscopic magnetic field. ξ^{-2} is a normalized measure of the square of the energy gap and is an entirely microscopic parameter governing the range over which the superconductivity can be destroyed. Their appropriately normalized product gives the condensation energy correctly since the gap enters twice in the BCS model--once for counting the number of paired electrons and once for the binding energy of the pair. The appearance of their product in Eq. (17) is what informs us that the apparently purely macroscopic derivation of Eq. (13) is an illusion. Equation (13) in fact expresses the fundamental connection between the microscopic and macroscopic theories and thus the equivalence of the BCS and GLAG approaches.

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APPENDIX

In the preceding derivation we have tried to be consistent in our approach to the rough physical approximations. One can, of course, use more accurate values or estimates of H_c, ξ, or the BCS pairing energy. This will shift the coefficients of Eqs. (14) and (17) about somewhat, but it is extremely difficult to make the values differ by much more than the factor of 3 we obtain, and quite simple to manipulate the approximations to improve the comparison. We have obviously made no such attempt at convergence. It may also be (correctly!) pointed out that in a straightforward derivation of the GLAG equations following the original second-order phase transition method of Ginzburg and Landau, the parameter κ is actually defined by the equation $\kappa = 2\sqrt{2}\, eH_c\lambda^2/\hbar c$,[1,4,6] which is the exact formula corresponding to Eq. (13). In fact, as the Ginzburg-Landau equation for the free energy is expanded in terms of only two parameters, it follows necessarily that only two of the quantities H_c, λ, ξ can be chosen independently. This explanation is physically far from satisfactory unless Gor'kov's microscopic derivation of the Ginzburg-Landau parameters in terms of the BCS theory is also invoked. Although the proof of the equivalence of the two theories then becomes satisfactory, the connection between the macroscopic derivation of Eq. (13) and the microscopic theory is thereby rendered somewhat more remote than is necessary. It is far more elegant and satisfying to derive the connection directly.

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12. For example, see C. Kittel, Introduction to Solid State Physics, fourth ed. (Wiley, New York, 1971), chapter 12.

13. Kittel, ibid. Our notation differs slightly.

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