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Cournot Equilibrium in Factor Markets

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Abstract. The classic model of oligopoly is Cournot's case of two mineral water sellers whose products are perfect substitutes for each other. Cournot shows that in this case there is an equilibrium with positive profits in which each seller assumes the other's choice of *quantity* is invariant to his own. If, on the other hand, each producer assumes the other's *price* is invariant to his own price, the only equilibrium has zero profits. Cournot also analyzed a case of two factor-monopolists, a zinc producer and a copper producer, whose outputs are used in fixed proportions by a competitive brass industry. In this case there is an equilibrium with postitive rents to both factor monopolists if each assumes the other's *price is invariant* but if each assumes the other's *quantity* is invariant, then in equilibrium both receive zero rents. This paper unifies Cournot's two theories of duopoly as special cases of a more general technology in which the elasticity of substitution between the factors controlled by different oligopolists can take on values between zero and infinity. A neat duality is shown between Cournot equilibrium in quantity and Cournot equilibrium price and some applications are explored.

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Cournot Equilibrium in Factor Markets

Theodore C. Bergstrom

In his classic work, *Researches into the Mathematical Principles of the Theory of Wealth (1838)*, A. A. Cournot introduced two distinct theories of duopoly. The better known of these theories concerns the case of two sellers of the same product (mineral water). A less familiar analysis concerns the case of two monopolists whose outputs, (zinc and copper) are used in fixed proportions in the production of a final good (brass) which is produced competitively. In the former case, Cournot supposes that each monopolist sets his quantity and accepts the market determined price. In the latter case, each monopolist sets the price of the factor he controls and sells the quantity determined by derived demand from the competitively operated final goods industry.

Cournot's theory of duopolists producing identical goods was criticized by Bertrand (1883) and Edgeworth (1897) on the grounds that equilibrium as described by Cournot does not exist if firms assume constancy of their rivals' *prices* rather than quantities. Hotelling (1929) observed that the lack of a Cournot equilibrium in prices can be viewed as a consequence of the fact that when two firms produce an identical product, the demand for the output of either is discontinuous as a function of the other's price. He then demonstrates that where products are spatially differentiated, continuity is restored and Cournot equilibrium in prices can be found for an interesting class of models.

In this paper we study Cournot equilibria among monopolistic controllers of factor supplies where the production function for final goods is neoclassical and the final goods industry is competitive. This discussion unifies Cournot's two theories of duopoly as special cases of a more general theory and provides some perspective on the old question of the difference between Cournot equilibrium in prices and Cournot equilibrium in quantities. It is hoped that the analysis will contribute some insight into the workings of factor

markets in which several complementary or substitute factors are unionized or otherwise non-competitively supplied.

1. Production functions, cost of functions and equilibrium

Consider an industry in which a single output is produced using n factors as inputs. There are constant returns to scale and production possibilities are represented in the usual way by a twice differentiable concave production function $f(x)$, where $x = (x_1, \ldots, x_n)$ is the vector of inputs. Where $w = (w_1, \ldots, w_n)$, let $\breve{x}(w)$ be the cheapest vector of inputs, at the vector of wage rates, w, which is capable of producing one unit of output. Then $c(w) = w\breve{x}(w)$ is the unit cost function for the industry.

Let f_i and c_i denote the i'th partial derivatives of f and c. Let $D(p)$ be the demand function for the industry where p is the price of the industry output. Let $D^{-1}(q)$ be the inverse demand function, and let $\eta(p) = \frac{p}{D(p)} \frac{dD}{dp}$ be the elasticity of demand. We assume that the final goods industry operates competitively while the supply of each factor is monopolized.

Competitive operation of the final goods industry requires that if the vector of quantities employed is $x \gg 0$ and the vector of wages is $w \gg 0$ then the following equations must hold:

$$D(c(w)) = f(x) \tag{1}$$

$$c_i(w) = \frac{x_i}{f(x)} \tag{2a}$$

$$f_i(x) = \frac{w_i}{c(w)}. \tag{2b}$$

The first result holds since the competitive final goods industry must price at $c(w)$ and the market must clear. Result (2b) states that factors are paid their marginal value products and result (2a) (sometimes called Shepherd's Lemma) is a well known consequence of the envelope theorem as applied to the definition of a cost function (see e.g. Diewert, (1974)).

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We can now define:

$$\theta^i(w) \equiv \frac{w_i c_i(w)}{c(w)} \qquad (3a)$$

$$^i\theta(x) \equiv \frac{x_i f_i(x)}{f(x)}. \qquad (3b)$$

Whenever (2a) and (2b) hold, it must be that $\theta^i(w)$ and $^i\theta(x)$ are both equal to the share of industry revenue that goes to factor i. That is:

$$\theta^i(w) = {}^i\theta(x) = \frac{w_i x_i}{c(w)f(x)}. \qquad (4)$$

We can also write total industry revenue as a function either of x or of w. Thus define

$$R^t(w) = c(w)D(c(w)) \qquad (5a)$$

$$^t R(x) = f(x)D^{-1}(f(x)). \qquad (5b)$$

Where (1) holds, it follows that

$$R^t(w) = {}^t R(x) = c(w)f(x). \qquad (6)$$

From (4) and (6) it is apparent that revenue of factor i can be written either as a function of w or as a function of x. Thus:

$$R^i(w) \equiv w_i x_i = \theta^i(w)R^t(w) \qquad (7a)$$

$$^i R(x) \equiv w_i x_i = {}^i\theta(x)^t R(x). \qquad (7b)$$

Examinations of Equations 1 through 7a and 7b reveals that Cournot equilibrium in wages and Cournot equilibrium in quantities are formally dual to each other where we identify w with x, $c(\cdot)$ with $f(\cdot)$, and $D(\cdot)$ with $D^{-1}(\cdot)$. This generalizes an observation made by Sonnenschein (1968) who pointed out the formal duality between Cournot's two

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classical models of oligopoly. For each theorem that we prove about one type of Cournot equilibrium, we will be able to use "duality theory" to prove a dual, but not identical, theorem about the other.

A *Cournot equilibrium in wages* is a vector $\bar{w}$ of wage rates such that for all i,

$$R^i(\bar{w}) \geq R^i(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n) \text{ for all } w_i \geq 0$$

A *Cournot equilibrium in quantities* is a vector $\bar{x}$ of factor supplies such that for all i,

$$R^i(\bar{x}) \geq R^i(\bar{x}_1, \ldots, x_i, \ldots, \bar{x}) \text{ for all } x_i \geq 0.$$

2. Cournot equilibrium in wages

A. *Interior Cournot equilibrium in wages.*

An *interior Cournot* equilibrium in wages is a Cournot equilibrium in wages, $\bar{w}$, such that $R^i(\bar{w}) > 0$ for all i. If $R^i(\bar{w}) > 0$, for all i, then clearly, $\bar{w} \gg 0$. Therefore a necessary condition for an interior Cournot equilibrium in wages is:

$$\frac{\partial R^i(\bar{w})}{\partial w_i} \equiv \theta^i(\bar{w}) \frac{\partial R^t(\bar{w})}{\partial w_i} + R^t(\bar{w}) \frac{\partial \theta_i(\bar{w})}{\partial w_i} = 0. \qquad (8)$$

From Lemma 1 of the appendix we see that if there are just two factors of production or if there are n factors and the production function for the final good has constant elasticity of substitution, $\bar{\sigma}$, then (8) implies:

$$\frac{\partial R^i(\bar{w})}{\partial w_i} = \frac{\partial R^i(\bar{w})}{\bar{w}_i}[\bar{\theta}_i(1 + \bar{\eta}) + (1 - \bar{\theta}^i)(1 - \bar{\sigma})] \qquad (9)$$

where $\bar{\theta}^i \equiv \theta^i(\bar{w})$, $\bar{\eta} \equiv \eta(c(\bar{w}))$ and (in case there are two factors and σ is not constant) $\bar{\sigma}$ is the elasticity of substitution between the two factors when they are used in the proportions

$$\frac{x_1}{x_2} = \frac{c_1(\bar{w})}{c_2(\bar{w})}.$$

At an interior Cournot equilibrium we have $R^i(w)/w_i > 0$ for each i. Therefore (9) will hold only if

$$\bar{\theta}^i(\bar{\eta} + \bar{\sigma}) - (1 - \bar{\sigma}) = 0. \qquad (10)$$

From (10) we deduce the following rather striking result.

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Proposition 1a. *If there are constant returns to scale and either just two factors or n factors and a production function with constant elasticity of substitution (CES), $\bar{\sigma}$, where $0 \leq \bar{\sigma} < \infty$ then at an interior Cournot equilibrium in wages, either $\bar{\sigma} = 1$ and $\bar{\eta} = -1$ or $\bar{\theta}^1 = \bar{\theta}^2 = \cdots = \bar{\theta}^n = 1/n$. In either case, it must be that*

$$\bar{\theta} = (n-1)\bar{\sigma} - n. \tag{11}$$

An intuitive explanation of Proposition 1a is this. If the production function is Cobb-Douglas, so that $\bar{\sigma} = 1$ everywhere, then the share $\theta^i(w)$ of industry revenue received by any factor i is independent of the vector w. Thus revenue of any factor is maximized when industry revenue is maximized. Industry revenue is maximized at the wage vector $\bar{w}$ only if the price, $c(\bar{w})$ is such that $\eta(c(\bar{w})) = -1$. Thus, given the wage rates charged by the other factors, a revenue maximizing union should set its own wage rate so as to make unit production costs just equal to the revenue maximizing price for the final good. If, on the other hand, there is a constant elasticity $\sigma < 1$, then the higher its wages, the larger will be its share of industry revenue. Thus if $\sigma < 1$, factors will demand wages that force the price of the final goal above the revenue maximizing price into the region where demand for the final good is elastic. Similarly, if $\sigma > 1$, a factor can increase its share of industry revenue by reducing its wage. In this case, equilibrium wages would be low enough so that the final goods price is in the range where final demand is inelastic.

3. Existence and uniqueness

In order for the first order conditions (11) to be sufficient as well as necessary for an interior Cournot equilibrium, it must be that the first order conditions are satisfied only at a *global* maximum of $R^i(w)$ with respect to w_i (rather than, say, at a local maximum or minimum). Having shown that a solution to (11) is a Cournot equilibrium we would still need to know whether these conditions have one or more solutions. These issues can be very neatly settled where production functions are CES and where the elasticity of demand for the final good declines (increases in absolute value) as its price increases. Alfred Marshall (1890) argues that this latter condition is the usual case. We also have

occasion to assume that as price goes from zero to a price large enough to shut off demand completely, the elasticity of demand declines strictly monotonically from zero to minus infinity. This property holds for (downward sloping) linear demand curves and in fact for any demand curve which "touches both axes" with finite slope while $\frac{d\eta(p)}{dp} < 0$ for intermediate prices. Thus we have the following definitions.

Let $D(p)$ be the demand for a commodity as a function of its own price and let $\eta(p) = \frac{dD(p)}{dp}\frac{p}{D(p)}$. The elasticity of demand is said to be *Marshallian* if $\eta(p) < 0$ whenever $D(p) > 0$ and if $\frac{d\eta(p)}{dp} < 0$ for all p. The elasticity of demand is said to be *strongly Marshallian* if it is Marshallian and if in addition for any $y \leq 0$, there exists exactly one value of p such that $\eta(p) = -y$.

Proposition 2a. *If the elasticity of demand for the final goal is Marshallian, and the production function is CES, then the conditions of Proposition 1a are sufficient for $\bar{w} \gg 0$ to be a Cournot equilibrium in wages.*

We next examine the question of existence and uniqueness. If the elasticity of demand is strongly Marshallian, then the elasticity of demand for the final good will assume any specified negative value at precisely one price level. However the elasticity of demand must always be negative. Examining the equilibrium condition (11) in Proposition (1a) we see that if $(n-1)\sigma - n > 0$, then there is exactly one price level $\bar{p}$ at which $\bar{\eta} = \eta(\bar{p}) = (n-1)\sigma - n$. In the Appendix we demonstrate that if $\sigma \neq 1$, there is then a unique interior Cournot equilibrium in wages. If, on the other hand, $(n-1)\sigma - n \geq 0$, then (11) has no solution and therefore there can be no interior Cournot equilibrium in wages. These observations allow us to completely resolve the questions of uniqueness and existence of an interior Cournot equilibrium in wages when demand is strongly Marshallian and production is CES

Proposition 3a. *If the demand function for the final good is strongly Marshallian, if there are n factors and the production function has constant returns to scale and a constant elasticity of substitution, σ, then*

(i) If $0 \leq \sigma < \frac{n}{n-1}$ and $\sigma \neq 1$, there exists exactly one interior Cournot equilibrium in wages.

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(ii) *If $\sigma = 1$, there is a unique cost level $\bar{c}$ such that the set of interior Cournot equilibrium wage vectors is $\{w \gg 0 \mid c(w) = c\}$.*

(iii) *If $\sigma \geq \frac{n}{n-1}$ there does not exist an interior Cournot equilibrium in wages.*

4. Boundary equilibria

To complete our study of Cournot equilibria in wages we must examine the possibilities for boundary solutions. We will define our attention to cases where the elasticity of substitution is constant. If $\sigma > 1$ it is possible to produce arbitrarily large amounts of output using positive amounts of only one input. If this is so, let $\bar{w}$ be a wage vector where at least two factors have a wage of zero. Then the cost of production is zero and the revenue of all factors is zero. No factor can unilaterally achieve positive revenue by changing its wage rate, since the demand for any factor that attempts to charge a positive price will be zero. A different type of non-interior solution occurs where $\sigma > 1$. Then each factor is essential for production. In this case there can be a Cournot equilibrium in wages where all wages are so high that even if any factor unilaterally lowered its wage to zero, the unit cost of production would exceed the price at which demand is zero. The next proposition summarizes the situation.

Proposition 4a. *If there exists $\varepsilon > 0$ such that $D(p) > 0$ for all $p < \varepsilon$ and if the production function has constant elasticity of substitution, σ, then:*

(i) *If $1 < \sigma \leq \infty$, the set of non-interior Cournot equilibria in wages is the set*

$$\{w \geq 0 \mid w_j = w_k = 0 \text{ for at least two distinct factors } j \text{ and } k\}.$$

(ii) *If $0 \leq \sigma \leq 1$, the set of non-interior Cournot equilibria in wages is the set*

$$\{w \geq 0 \mid D(c(w')) = 0 \text{ if } w'_j = w_j \text{ for all but one of the factors } j \in \{1, \ldots, n\}\}.$$

This set will be nonempty if $D(p) = 0$ for p sufficiently large.

5. Cournot equilibrium in quantities

To each of the results of Section II corresponds a dual proposition dealing with Cournot equilibrium in quantities. Let $\hat{x}$ denote a Cournot equilibrium in quantities, $\hat{\sigma}$ the elasticity of substitution of the production function at $\hat{x}$. Let $\hat{\eta} = \eta(\hat{p})$ where $\hat{p} = D^{-1}(f(\hat{x}))$ and

$$i\hat{\theta} = \frac{\hat{x}_i f_i(\hat{x})}{f(\hat{x})}.$$

The dual propositions are as follows:

Proposition 1b. *If there are constant returns to scale and either just two factors or n factors and a production function with constant elasticity of substitution, $\hat{\sigma}$, where $0 \le \hat{\sigma} < \infty$ then at an interior Cournot equilibrium in quantities, either $\hat{\sigma} = 1$ and $\hat{\eta} = -1$ or ${}^1\hat{\theta} = {}^2\hat{\theta} = \ldots = {}^n\hat{\theta}$. In either case, it must be that*

$$\frac{1}{\hat{\eta}} = (n-1)\frac{1}{\hat{\sigma}} - n. \tag{12}$$

Proposition 2b. *If the elasticity of demand for the final good is Marshallian and the production function is CES, then the conditions of Proposition 1b are sufficient for $\hat{x}$ to be a Cournot equilibrium in quantities.*

Proposition 3b. *If the demand function for the final good is strongly Marshallian, if there are n factors, and the production function has constant returns to scale and a constant elasticity of substitution, σ, then:*

(i) If $\frac{n-1}{n} < \sigma \ne 1$, there exists exactly one interior Cournot equilibrium in quantities.

(ii) $\sigma = 1$, there is a unique output level $\hat{q}$ such that the set of interior Cournot equilibrium factor supply vectors is $\{x \mid f(x) = \hat{q}\}$.

(iii) If $\sigma \le \frac{n-1}{n}$ then there does not exist an interior Cournot equilibrium in quantities.

Proposition 4b. *If the production function has constant elasticity of substitution, σ, then*

(i) If $\sigma \le 1$, the set of non-interior Cournot equilibria in quantities is

$$\{x \ge 0 \mid x_j = 0, x_k = 0 \text{ for at least two distinct factors } j \text{ and } k\}.$$

(ii) If $\sigma > 1$ the set of non-interior Cournot equilibria is

$$\{x \ge 0 \mid D^{-1}(f(x')) = 0 \text{ if } x'_j = x_j \text{ for all but one of the } j \in 1, \ldots, n\}.$$

One can derive Propositions 1b-4b directly, using essentially the same methods used to demonstrate 1a-4a. A more elegant and interesting approach is to exploit the duality which we earlier remarked between Cournot equilibrium in prices and Cournot equilibrium in quantities. The way in which this is done is outlined in the appendix.

6. Comparing and classifying Cournot equilibrium in wages and quantities

Our results can readily be applied to Cournot's polar cases. In the case of the mineral water sellers, the two producers sell perfect substitute goods. This is the limiting case of our analysis as σ approaches infinity. Here, according to propositions (1b) and (3b), there is a unique interior Cournot equilibrium in quantities. In equilibrium the elasticity of demand for output is $1/n$. On the other hand, (3a) and (4a) inform us that there is no interior Cournot equilibrium in wages although there are boundary Cournot equilibrium at which at least two factors have zero wage. The first result agrees with Cournot's analysis. The second repeats observations made by Bertrand and Edgeworth in their critical remarks on Cournot's theory.

In his analysis of monopolized complementary factors, Cournot assumed that production requires the factors in fixed proportions. This is a limiting case of a CES production function with $\sigma = 0$. Propositions (1a) and (3a) inform us that in this case there will be a unique interior Cournot equilibrium in wages. In equilibrium the factors share equally in industry revenues and the elasticity of demand for the final good is $-n$. This, to be sure, is the same result obtained by Cournot. Cournot does not analyze Cournot equilibrium in quantities for this case. Indeed, there is no interior Cournot equilibrium in quantities, although there are boundary equilibria where at least two factors offer zero quantities. More generally, our results show that for constant elasticity of substitution $\sigma \leq \frac{n-1}{n}$ there exists an interior Cournot equilibrium in wages but not in quantities. If $\sigma \geq \frac{n}{n-1}$ there is an interior Cournot equilibrium in quantities but no in wages. If $\frac{n-1}{n} < \sigma < \frac{n}{n-1}$, then interior Cournot equilibria of both types exist and it is interesting to compare them. Rearranging terms in (11) and (12), we find:

$$\bar{\eta} + 1 = (n-1)(\sigma - 1) \tag{13}$$

$$\hat{\eta} + 1 = \frac{-\hat{\eta}}{\sigma}(n-1)(\sigma-1). \tag{14}$$

From (13) and (14) it is clear that if $\sigma < 1$, then at an interior Cournot equilibrium of either type, demand for the final good is inelastic, while if $\sigma > 1$, then in equilibrium the demand for the final good is elastic. Some further manipulation of (11) and (12) shows that:

$$\frac{\bar{\eta}}{\hat{\eta}} - 1 = -n(n-1)\frac{(1-\sigma)^2}{\sigma} < 0. \tag{15}$$

Since $\bar{\eta}$ and $\hat{\eta}$ are both negative, it follows that $\bar{\eta} \geq \hat{\eta}$ (with strict inequality if $\sigma \neq 1$). Thus if the elasticity of demand is Marshallian, we are assured that a Cournot equilibrium in quantities always has a higher price for the final good than a Cournot equilibrium in wages. For both kinds of equilibrium, demand for the final good is elastic if $\sigma < 1$ and inelastic if $\sigma > 1$. Therefore, of the two types of interior Cournot equilibria, equilibrium in wages, which results in the lower price, will yield the higher total revenue to the industry if $\sigma < 1$. If $\sigma > 1$ then equilibrium in quantities which results in the higher price for the final good will also lead to the higher total revenue for the industry and hence for each firm.

The next proposition summarizes these results where "bars" and "hats" denote respectively magnitudes associated with Cournot equilibrium in wages and quantities.

Proposition 5. *Let the production function have constant returns to scale and constant elasticity of substitution, σ. Let the elasticity of demand be strongly Marshallian. Then:*

(i) If $0 \leq \sigma \leq \frac{n-1}{n}$, there exists a unique interior Cournot equilibrium in wages. There is no interior Cournot equilibrium in quantities. There are boundary Cournot equilibria in quantities where at least two factors offer zero quantities. There are also boundary Cournot equilibria in wages where the unit cost of the final good is so great that if any single factor unilaterally reduced its wages to zero, total demand for the final good when priced at unit cost, would be zero.

(ii) If $\frac{n-1}{n} < \sigma < 1$, there exists a unique interior Cournot equilibrium of each kind. These compare as follows: $-1 > \hat{\eta} > \bar{\eta}, \bar{p} < \hat{p}$ and $\bar{R}^i > {}^i\hat{R}$ for all i. The situation regarding boundary Cournot equilibria is the same as in (i).

(iii) If $\sigma = 1$, there exist interior Cournot equilibria of both types. There is a single equilibrium price for the final good, $\bar{p} = \hat{p}$ and $R^i = {}^i R$ for all i. There are no boundary Cournot equilibria in wages. There are boundary Cournot equilibria in quantities where at least two factors offer zero quantity.

(iv) If $1 < \sigma < \frac{n}{n-1}$ there exists a unique interior Cournot equilibrium of each kind. These compare as follows: $4\bar{\eta} > \hat{\eta} > -1, \bar{p} < \hat{p}$ and ${}^i \hat{R} > \bar{R}^i$ for all i. There are boundary Cournot equilibria in quantities where the factors offer sufficient quantities that the market clearing demand price for the amount of final goods produced is zero and would continue to be zero if any factor unilaterally reduced its supply to zero. There are boundary Cournot equilibria in wages where at least two factors are offered at zero wages.

(v) If $\frac{n}{n-1} \le \sigma < \infty$, there is a unique interior Cournot equilibrium in quantities. There is no interior Cournot equilibrium in wages. The situation regarding boundary Cournot equilibria is the same as in (iv).

7. Asymmetric Cournot equilibrium

In case there are just two factors of production, it is of some interest to consider the case where the factors are asymmetric in their assumptions about the behavior of their rivals. Let us suppose, for example, that the monopolist for one factor, "labor" chooses its wage demand so as to maximize its total revenue on the assumption that the *quantity* of the other factor, "capital" is invariant to wages. Suppose that "capital" is invariant to wages. Suppose that "capital", on the other hand, is supplied by a monopolist who wishes to maximize his revenue and who believes that the wage rate set by "labor" is invariant to his own supply of "capital".

At this point it is worthwhile to observe a rather subtle point which often seems ignored in discussion of Cournot equilibrium. Where the behavior of the other firm is known as a function of ones own choice either of quantity or of price, it is immaterial whether we say that the decision-making firm chooses price or chooses quantity, since its choice for one will determine the other. What does matter is what it assumes about the behavior of the other factor supplier. Thus in Cournot equilibrium in wages, each firm assumes that the

other makes a wage demand that is invariant to its own choice of action. Implicitly, it then follows that a change in the decision maker's wage demand, say, forces the other factor to change its quantity in order to meet demand at an unaltered wage. A similar statement applies if the firm expects its rival to hold his quantity supplied invariant.

An interior *asymmetric Cournot equilibrium* for two factors is a vector of quantities $(\bar{x}_1, \bar{x}_2) \gg 0$ and a vector of wages $(\bar{w}_1, \bar{w}_2) \gg 0$ such that equations (1), (2a) and (2b) are satisfied and

$$^1R(\bar{x}_1, \bar{x}_2) \geq\, ^1R(x_1, \bar{x}_2) \text{ for all } x_1 \geq 0.$$

$$R^2(\bar{w}_1, \bar{w}_2) \geq R^2(\bar{w}_1, w_2) \text{ for all } w_2 \geq 0.$$

Here factors 1 and 2 play the roles described above for labor and capital respectively. Rather surprisingly, it turns out that there exists an interior asymmetric Cournot equilibrium for all non-negative values of σ. The situation is explained in Proposition 6, which is proved in the Appendix.

Proposition 6. *If there are two factors of production and consta   rns to scale, then necessary conditions for an asymmetric Cournot equilibrium are:*

$$\bar{\eta} = -\frac{1}{\bar{\sigma}}\, \bar{\theta}^1, = \frac{1}{1+\bar{\sigma}} \text{ and } \bar{\theta}^2 = \frac{\bar{\sigma}}{1+\bar{\sigma}} \tag{16}$$

where $\bar{\eta}$ and $\bar{\sigma}$ are the elasticity of demand and the elasticity of substitution in equilibrium. If production is CES and demand is strongly Marshallian, th   se conditions are sufficient as well as necessary. Furthermore, for all $\sigma \geq 0$, equilib   exists. If $\sigma \neq 1$, equilibrium is also unique. If $\sigma = 1$, the equilibrium price and quantity of the final good is unique.

8. The effect of numbers on price—an application

Since we were able to completely describe the Cournot equilibria for the cases treated above, we are now supplied with an interesting class of examples to use for studying

various conjectures about oligopolistic equilibrium. According to economic folklore, the addition of more oligopolists to an industry with homogeneous output will reduce price and increase total output. As the number of oligopolists becomes large, the Cournot equilibria should approach competitive equilibrium. See Novshek (10) and Gabszewicz and Vial (5). Gabszewicz and Thisse (6) pose the interesting question of whether in some sense this is also true when new entrants are added to an industry with differentiated products. We can use our results as a kind of testing ground for this conjecture. To do so we simply use equations (11) and (12) to determine the direction of effect of the number of factors on industry price and output. We find the following.

Proposition 7. *Let the production function be CES with constant returns to scale and let the elasticity of demand be strongly Marshallian.*

(i) If $\sigma > 1$ then as the number of factors is increased both the interior Cournot equilibrium in wages and the interior Cournot equilibrium in quantities occur at lower prices and greater output. In the limit as n becomes large, the interior Cournot equilibria in quantities approach the competitive equilibrium in which the price of the final good is zero. For large enough (but finite) n, there exists no interior Cournot equilibrium in wages, while the boundary Cournot equilibria in wages occur at the competitive price, zero.

(ii) If $\sigma < 1$, then as the number of factors is increased, both types of interior Cournot equilibrium occur at higher prices and lower quantities. In the limit as n approaches infinity, the interior Cournot equilibrium in wages has the price of the final good approaching the price at which demand is zero. For large enough but finite n, there exists no interior Cournot equilibria in quantities while the boundary Cournot equilibria in quantities occur where industry output is zero.

For the case $\sigma > 1$, the result of Proposition 7 supports the view that larger numbers tend to restore competition even if the substitute factors are not perfect substitutes. In case $\sigma < 1$, however, the addition of more factors results in prices that rise away from the competitive price.

Reflection about the nature of CES functions where $\sigma > 1$ and $\sigma < 1$, provides us with a clue to why this is the case. Where $\sigma > 1$, it is possible to produce positive output at

constant returns to scale even if the supplies of some factors are zero. In fact the restriction of a CES function to a subspace of its domain in which some of the factors must be zero, it itself a CES production function with positive marginal products. It is reasonable to think of additional factors as simply adding alternative ways to produce the output. Where $\sigma < 1$, this is not the case. Unless all factors supply a positive amount, output will be zero. Thus, among the CES production functions, the case where $\sigma > 1$ seems to be the relevant one for discussing what happens when more substitute factors appear.

Although we have thus far confined our discussion to production, it should be clear that many of the results also can be interpreted as applying to a case where the x_i's are consumer goods that are more or less good substitutes for each other. This is perhaps most obviously apparent if we think of $f(\cdot)$ as a "household production function".

9. Appendix — Proofs promised in text

Lemma 1 establishes results needed to deduce equation (9) of the text from equation (8)

Lemma.. *Let the production function have constant returns to scale.*

(i) If there are exactly two factors of production and $\sigma(w)$ is the elasticity of substitution of the production function at (x_1, x_2) when

$$\frac{x_1}{x_2} = \frac{c_1(w)}{c_2(w)},$$

then

$$\frac{\partial \theta^i(w)}{\partial w_i} = \frac{1}{w_i} \theta^i(w)(1 - \theta^i(w))(1 - \sigma(w)).$$

(ii) If there are n factors of production and the production function is CES with elasticity of substitution, σ, where $0 \le \sigma < \infty$,

$$\frac{\partial \theta^i(w)}{\partial w_i} = \frac{1}{w_i} \theta^i(w)(1 - \theta^i(w))(1 - \sigma).$$

(iii) For all $w \ge 0$,

$$\frac{\partial R^i(w)}{\partial w_i} = \frac{1}{w_i} R^i(w)(1 + \eta(c(w))).$$

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Proof of Lemma 1

Result (i)

Since there are two factors and constant returns to scale, it must be that

$$\frac{\theta^1(w)}{1 - \theta^1(w)} = \frac{w_1 x_1}{w_2 x_2}$$

where

$$\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \frac{w_1}{w_2}.$$

Taking logs of both sides and differentiating with respect to $\log w_1$ yields:

$$\frac{w_1 \frac{\partial \theta^1(w)}{\partial w_1}}{\theta^1(w)(1 - \theta^1(w))} = 1 + \frac{d \log\left(\frac{x_1}{x_2}\right)}{d \log w_i}.$$

Along the surface where

$$\frac{w_1}{w_2} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)},$$

we have, by definition,

$$-\sigma(w) = \frac{d \log\left(\frac{x_1}{x_2}\right)}{d \log\left(\frac{w_1}{w_2}\right)} = \frac{d \log\left(\frac{x_1}{x_2}\right)}{d \log w_1}.$$

Therefore

$$\frac{w_1 \frac{\partial \theta_1(w)}{\partial w_1}}{\theta^1(w)(1 - \theta^1(w))} = 1 - \sigma(w).$$

From this, Result (i) is immediate for factor 1. A symmetric argument takes care of factor 2.

Result (ii)

From Lemma 2, Result (ii) we see that for CES production function with constant returns to scale, where $0 \leq \sigma < 1$,

$$\frac{\theta^i(w)}{1 - \theta^i(w)} = \frac{\alpha_i^{\sigma-1} w_i^{1-\sigma}}{\sum_{j=1} \alpha_j^{\sigma-1} w_j^{1-\sigma}}$$

where the σ's are positive real numbers. Taking logs of both sides of this equation and differentiating with respect to $\log w_i$ yields:

$$\frac{w_i \frac{\partial \theta^i(w)}{\partial w_i}}{\theta^i(w)(1 - \theta^i(w))} = 1 - \sigma.$$

Result (ii) is immediate from rearranging this equation. ∎

Result (iii)

By definition, $R^t(w) = c(w)D(c(w))$. Taking logs of both sides and differentiating with respect to w_i yield

$$\frac{\partial R^t(w)}{\partial w_i} = R^t(w)\left[\frac{c_i(w)}{c(w)} + \frac{\partial D(c(w))}{\partial c}\frac{c_i(w)}{D(c(w))}\right]$$

$$= R^t(w)\frac{c_i(w)}{c(w)}[1 + \eta(c(w))]$$

$$= \frac{1}{w_i}R^t(w)\theta^i(w)[1 + \eta(c(w))]$$

$$= \frac{1}{w_i}R^i(w)[1 + \eta(c(w))].$$

Lemma 2 states a number of properties of CES production functions which we will find useful. Since these results are obtained by straight-forward computation and are also well established in the literature (see e.g. McFadden (1978)) we state them without proof. Part (i) of Lemma 2 may, if one wishes be taken as the definition of CES production functions.

Lemma 2

(i) The class of n factor production functions with constant returns to scale and constant elasticity of substitution where $0 < \sigma < \infty, \sigma \neq 1$ consists of all functions with domain the non-negative orthant in R^n and which take the form

$$f(x) = \left(\sum_{i=1}^{n}(\alpha_i x_i)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

$\sigma_i > 0$ for each i whenever $\sigma < 1$ or $x \gg 0$ and where if $\sigma < 1$ and $x_j = 0$ for some $j, f(x) = 0$. In addition the CES family includes limiting cases where $\sigma = 0, \sigma = \infty$, and $\sigma = 1$. In these cases, respectively, $f(x) = \min_{i\epsilon\{1,...,n\}}(\alpha_i x_i)$, $f(x) = \sum_{i=1}^{n}\alpha_i x_i$ and $f(x) = Kx_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$. In the last case we have the additional restrictions $K > 0$ and $\sum_{i=1}^{n}\alpha_i = 1$.

(ii) If the production function is CES with elasticity of substitution σ where $0 \leq \sigma < \infty$, then if $\breve{x}(w)$ minimizes $\sum_{i=1}^{n}w_i x_i$ subject to $f(x) = 1$:

$$\frac{\breve{x}_i(w)}{\breve{x}_j(w)} = \left(\frac{\alpha_i}{\alpha_j}\right)^{\sigma-1}\left(\frac{w_i}{w_j}\right)^{-\sigma}.$$

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Also

$$\frac{\theta^i(w)}{\theta^j(w)} = \left(\frac{\alpha_i}{\sigma_j}\right)^{\sigma-1} \left(\frac{\alpha_i}{\alpha_j}\right)^{\sigma-1} \left(\frac{w_i}{w_j}\right)^{1-\sigma}.$$

(iii) If the production function is CES with elasticity of substitution σ where $0 < \sigma < \infty$, and parameters $\alpha_1, \ldots, \alpha_n$ then the corresponding cost function where it is well-defined is CES with elasticity of substitution $1/\sigma$ and parameters $(1/\alpha_1, \ldots, 1/\alpha_n)$. This is also true for $\sigma = 0$ and $\sigma = \infty$ where we adopt the conventions $1/0 = \infty$ and $1/\infty = 0$. (The cost function is well-defined for all $w \geq 0$ if $\sigma \leq 1$ the cost function is well-defined only for $w \gg 0$. However, the function obtained by taking the infimum rather than minimum cost of producing a single unit is well defined everywhere and is CES as described above.

Lemma 3 states facts that are needed to establish Propositions 1b - 4b as duals to Propositions 1a - 4a.

Lemma 3

The cost function has constant returns to scale in w. The production function is CES with elasticity of substitution, σ, if and only if the cost function is CES with elasticity of substitution $1/\sigma$. If there are just two factors of production then the elasticity of substitution of the cost function at w is the inverse of the elasticity of substitution of the production function at $x(w)$. The elasticity of demand at p is the inverse of the elasticity of inverse demand at $D^{-1}(p)$. The elasticity of inverse demand is (strongly) Marshallian if and only if the elasticity of demand is (strongly) Marshallian.

Proof of Lemma 3

The first sentence of the lemma is a simple consequence of the definition of cost function. The second sentence is a restatement of part of Lemma 2. The third sentence is proved in McFadden (1978). Let $\eta(p) \equiv \frac{p}{D(p)}\frac{dD(p)}{dp}$ be the elasticity of demand at p and $\alpha(q) \equiv \frac{q}{D^{-1}(q)}\frac{dD^{-1}(q)}{dq}$ be the elasticity of inverse demand at q. If demand is Marshallian, then $\frac{dD(p)}{dp} < 0$ and $\frac{d\eta(p)}{dp} < 0$ Therefore, $\frac{dD^1(p)}{dp} < 0$. Also, since $\alpha(q) \equiv \frac{1}{\eta(D^{-1}(q))}$, it must be that

$$\frac{d\alpha(q)}{dq} = -\left(\frac{1}{\eta(p)^2}\right)^2 \frac{d\eta(p)}{dp}\frac{dD^{-1}(q)}{d(q)}$$

where $p = D^{-1}(q)$. Therefore demand is Marshallian if and only if $\frac{d\alpha(q)}{dq} < 0$ and $\frac{dD^{-1}(q)}{dq} < 0$ or equivalently if and only if inverse demand is Marshallian. If demand is strong Marshallian, then in addition $\eta(p)$ takes all any negative values for exactly one p. Since $\alpha(D(p)) = \frac{1}{\eta(p)}$, it is clear that η has this property if and only if α has it. It follows that the elasticity of demand is (strongly) Marshallian if and only if the elasticity of inverse demand is (strongly) Marshallian. ∎

Proof of Proposition 2a

We want to show for each i that Equation (11) of Proposition 1a implies that at $R^i(\bar{w})$, $\bar{w}_i$ maximizes $R^i(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n)$. Equivalently we could show for each i that $\ln R^i(\bar{w}_1, \ldots, e^{\ln w_i}, \ldots, \bar{w}_n)$ is maximized with respect to $\ln w_i$ at $\ln \bar{w}_i$.

From Equation (9) we see that

$$\frac{\partial \ln R_i(\bar{w})}{\partial \ln w_i} = \theta^i(\bar{w})[1 + \eta(c(\bar{w}))] + (1 - \bar{\theta}^i(w))(1 - \sigma)$$

$$= \theta^i(\bar{w})(\eta(c(\bar{w})) + \sigma) + (1 - \sigma).$$

Therefore Equation (11) of Proposition (1a) implies that $\frac{\partial \ln R_i(\bar{w})}{\partial \ln w_i} = 0$. One computes that

$$\frac{\partial^2 \ln R_i(\bar{w})}{\partial^2 \ln w_i} = \frac{\partial \theta^i(\bar{w})}{\partial w_i}(\eta(c(\bar{w})) + \sigma) + \theta^i(\bar{w})\frac{d\eta(c(\bar{w}))}{d \ln w_i}$$

$$= \theta^i(\bar{w})(1 - \theta^i(\bar{w}))(1 - \sigma)(\eta(c(\bar{w})) + \sigma) + \theta^i(\bar{w})w_ic_i(\bar{w})\frac{d\eta(c(\bar{w}))}{dc},$$

where the second equality is a consequence of Lemma 1. But when $\frac{\partial \ln R^i(\bar{w})}{\partial \ln w_i} = 0$, then $\theta^i(\bar{w})(\eta(c(w)) + \sigma) = \sigma - 1$, so that

$$\frac{\partial^2 \ln R^i(\bar{w})}{\partial^2 \ln w_i} = -(1 - \theta^i(\bar{w}))(1 - \sigma)^2 + \theta^i(\bar{w})w_ic_i(w)\frac{d\eta(c(w))}{dc}.$$

The first term on the right hand side of this expression must be non-positive. The second term must be negative since the elasticity of demand is assumed to be Marshallian. Therefore, $\frac{\partial^2 \ln R^i(\bar{w})}{\partial^2 \ln w_i^2} < 0$ whenever $\frac{\partial \ln R^i(\bar{w})}{\partial \ln w_i} = 0$. It is a simple fact of elementary calculus that if $f(\cdot)$ is a differentiable real valued function whose domain is a real interval, and if

$f'(x) = 0$ implies that $f''(x) < 0$ then any point $\bar{x}$ at which $f'(\bar{x}) = 0$ is a unique global maximizer of f. Applying this principle to $\ln R^i(\bar{w}, \ldots, e^{\ln w_i}, \ldots, \bar{w}_n)$ viewed as a function of $\ln w_i$ establishes Proposition 2a. ∎

Proposition 3a

If $\sigma \neq 1$, Proposition 1a and Lemma 2 imply that:

$$1 = \frac{\theta^i(\bar{w})}{\theta^j(\bar{w})} = \left(\frac{\alpha_i}{\alpha_j}\right)^{\sigma-1} \left(\frac{\bar{w}_i}{\bar{w}_j}\right)^{1-\sigma}.$$

Therefore $\frac{\bar{w}_i}{\bar{w}_j} = \frac{\alpha_i}{\alpha_j}$. Hence *relative* wages are determined by the parameters of the production function. If demand is strongly Marshallian, and $0 \leq \sigma < \frac{n}{n-1}$ there is exactly one price, $\bar{c}$, at which the necessary condition, $\eta(\bar{c}) = (n-1)\sigma - n$, is satisfied. Therefore, $c(\bar{w}) = \bar{c}$. Since $c(\cdot)$ is homogeneous of degree one in w and since relative wages are uniquely determined, there is exactly one wage vector $w \gg 0$ that satisfies the necessary conditions of Proposition 1a. According to Proposition 2a, these conditions are also sufficient when demand is Marshallian and production is CES Thus we have exactly one Cournot equilibrium in wages. ∎

Result (ii)

If $\sigma = 1$, Propositions 1a and 2a imply that $\bar{w} \gg 0$ is a Cournot equilibrium in wages if and only if $\eta(c(\bar{w})) = -1$. Since elasticity of demand for the final good is assumed to be strongly Marshallian, there is exactly one number, $\bar{c}$, such that $\eta(\bar{c}) = -1$. Thus the set of interior Cournot equilibria is the set $\{w \gg 0 \mid c(w) = \bar{c}\}$.

Result (iii)

A necessary condition for an interior Cournot equilibrium is $\eta(c(\bar{w})) = (n-1)\sigma - n$. If demand is Marshallian, it must be that $\eta(c(w)) < 0$ for all w. But if $\sigma \leq \frac{n}{n-1}$ then $(n-1)\sigma - n \geq 0$ so that the necessary condition can not be satisfied. ∎

Proof of 4a

Result (i)

The argument made in the text suffices to show that if $\bar{w} \in \{w \geq 0 \mid w_j = w_k = 0$ for at least two distinct factors j and $k\}$, then $\bar{w}$ is a non-interior Cournot equilibria. It remains to be shown that there are no others.

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If $\bar{w}_j > 0$ for all $j \neq i$ and $\bar{w}_i = 0$, then $R_i(\bar{w}) = 0$. But since $\sigma > 1$, it follows that $c(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n) > 0$ and hence $R_t(c(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n) > 0$. From Lemma 2 we see that for $\sigma < \infty$, $\theta^i(w) > 0$ for all $w \gg 0$. Also, for $\sigma = \infty$, $\theta^i(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n) = 1$ for w_i small enough. Therefore for $1 < \sigma \leq \infty$, $R^i(\bar{w}_i, \ldots, w_i, \ldots, \bar{w}_n) > 0$ for small positive w_i. Thus there cannot be a Cournot equilibrium in wages where just one factor has zero wage. The only remaining possibility is $\bar{w} \gg 0$ and $R^i(\bar{w}) = 0$ for some i. But if this is the case, we can again show that for w_i sufficiently small and positive $R^i(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n) > 0$. Therefore there can be no Cournot equilibrium in wages of this type. This establishes Result (i).

Result (ii)

If $\sigma = 1$ then the cost function is also a CES function with elasticity of substitution $1/\sigma = 1$. Therefore positive output requires a positive amount of each input but any factor can unilaterally make the cost of production arbitrarily small by setting its wage low enough. Thus for each i and any $\bar{w} \geq 0$, $R^i(\bar{w}, \ldots, w_i, \ldots, \bar{w}_n) > 0$ for w_i sufficiently small. Therefore there can be no Cournot equilibrium in wages with $R^i(\bar{w}) = 0$ for some i.

Result (iii)

If $w \geq 0$ and $D(c(w')) = 0$ whenever $w'_j = w_j$ for all but one of the factors $j = 1, \ldots, n$, then clearly $R^i(\bar{w}) = 0$ for all i and $R^i(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n) = 0$. Therefore if $\bar{w} \in \{w \geq 0 \mid D(c(w')) = 0$ if $w'_j = w_j$ for all but one of the factors $j\varepsilon\{1, \ldots, n\}\}$ then $\bar{w}$ is a boundary Cournot equilibrium.

Suppose $D(p) = 0$ for all $p \geq \bar{p} > 0$. If $0 \geq \sigma < 1$ then $c(w)$ must be CES with elasticity of substitution $1/\sigma > 1$. Therefore for any i, wage rates $\bar{w}_j$ can be found for all $j \neq i$ such that $c(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n) \geq \bar{p}$ for all $w_i \geq 0$. Therefore the set of boundary Cournot equilibria in wages is non-empty.

It remains to be shown that there are no other boundary Cournot equilibria in wages. If $\bar{w} \geq 0$ does not belong to the set of specified above, then for any i, and for w_i sufficiently small and positive $c(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n) > 0$, $D(c(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n)) > 0$ and hence $R^t(c(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n)) > 0$. Also, we see from Lemma 2 that if $w_i > 0$ then $\theta^i(w) > 0$. Therefore $R^i(\bar{w}_1, \ldots, w_i, \ldots, \bar{w}_n) > 0$. It follows that $\bar{w}$ cannot be a boundary Cournot equilibrium in wages. ∎

Proof of Theorems 1b - 4b by Duality

The formal isomorphism of Cournot equilibrium in prices and in quantities enables one to obtain true propositions from 1a - 4a by simply interchanging the entities which are identified in the two theories. This involves interchanging the following words and their corresponding symbols in the state of each proposition.

(i) "wage and quantity"

(ii) "cost function and production function"

(iii) "demand function and inverse demand function."

The resulting propositions, (call them 1a' - 4a') make statements and assumptions about such unconventional notions as the elasticity of substitution of the cost function and the elasticity of inverse demand. Thus, for example, Proposition 1a becomes:

Proposition 1a'

If there are constant returns to scale and there are either just two factors or the cost function is CES with elasticity of substitution $\tilde{\sigma}$ where $0 \leq \tilde{\sigma} < \infty$, then at an interior Cournot equilibrium in quantities, either $\bar{w} = 1$ and $\bar{\alpha} = 1$ (where $\bar{\alpha}$ is the elasticity of inverse demand) or $1_{\bar{\theta}} = \ldots = n_{\bar{\theta}} = \frac{1}{n}$. In either case it must be that

$$\bar{\sigma} = (n-1)\tilde{\sigma} - n.$$

From Lemma 3 we know that the elasticity of inverse demand is the inverse of the elasticity of demand and the elasticity of substitution of the cost function is the inverse of the elasticity of substitution of the production function. We also know that demand is (strongly) Marshallian if and only if inverse demand is (strongly) Marshallian and that the cost function is CES if and only if the production function is CES Using these facts, Propositions 1b - 4b are direct translations of Propositions 1a' - 4a'.

Proof of Proposition 6

At an interior asymmetric Cournot equilibrium, we must have:

$$\frac{\partial R^2(\bar{w}_1, \bar{w}_2)}{\partial w_2} = 0 \quad \text{and} \quad \frac{\partial^1 R(\bar{x}_1, \bar{x}_2)}{\partial x_1} = 0.$$

Equation 10 above informs us that the first equality is equivalent to:

$$(1 - \bar{\sigma}) + \bar{\theta}^2(\bar{\eta} + \bar{\sigma}) = 0.$$

The dual relation implied by the second equality is

$$(1 - \frac{1}{\bar{\sigma}}) +^1 \bar{\theta}(\frac{1}{\eta} + \frac{1}{\bar{\sigma}}) = 0.$$

Multiplying the second of these equations by $\bar{\eta}\bar{\sigma}$, adding it to the previous one and observing that $^1\bar{\theta} + \bar{\theta}^2 = 1$, we find $\bar{\eta}\bar{\sigma} = -1$. Substituting to eliminate $\bar{\eta}$ in the previous equations yields $\bar{\theta}^1 = \frac{1}{1+\sigma}$ and $\bar{\sigma}^2 = \frac{\sigma}{1+\sigma}$.

The sufficiency, uniqueness, and existence arguments are obtained by arguments almost identical to those used to establish Propositions 2a and 3a. There is no need to repeat these arguments here. ∎

Proof of Proposition 7

From Equations (13) and (14) it is easily seen that the equilibrium values of both $\bar{\eta}$ and $\hat{\eta}$ are increasing functions of n if $\sigma > 1$ and decreasing functions of n if $\sigma < 1$. If the elasticity of demand is Marshallian then it follows the equilibrium price is lower and output is higher. the greater is n. From (14) we see that if $\sigma > 1$ then $\lim_{\eta \to \infty} \hat{\eta} = 0$, so that $\lim_{\eta \to \infty} \hat{p} = 0$ where $\hat{p}$ is the price of the final good in a Cournot equilibrium in quantities. We also see that for large enough $n, (n-1)(\sigma-1) > 1$. But if this is so, then (13) cannot have a solution with $\hat{\eta} < 0$. This establishes result (i). From (13) we see that if $\sigma < 1$, then $\lim_{\eta \to \infty} = -\infty$. Therefore in the limit, the equilibrium price, $\bar{p}$, for the final good must approach the price at which demand for the final good is zero. It is also easily seen that for n large enough and $\sigma < 1$, (14) cannot have a solution for $\hat{\eta} < 0$. ∎

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