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Fundamentals of Treating Interference as Noise

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Fundamentals of Treating Interference as Noise

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

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by

Chunhua Geng

Dissertation Committee:
Professor Syed Jafar, Chair
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2016

DEDICATION

To my parents and wife

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ABSTRACT OF THE DISSERTATION

Fundamentals of Treating Interference as Noise

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Treating interference as noise (TIN) when it is sufficiently weak is one of the key principles of interference management for wireless networks. This dissertation revisits the optimality of TIN from an information theoretic perspective. It is shown that for K-user Gaussian interference channels, TIN achieves all points in the capacity region to within a constant gap, if for each user, the desired signal strength is no weaker than the sum of the strengths of the strongest interference caused by the user and the strongest interference suffered by the user (with all signal strengths measured in dB scale). We also extend the optimality of TIN to more general settings, including interference networks with general message sets, compound networks and MIMO interference channels, and characterize the secure capacity region within a constant gap for the identified TIN-optimal interference channels with secrecy constraints. Moreover, combining TIN with interference avoidance, we formulate a joint signal space and signal level optimization problem and propose a baseline decomposition approach.

Chapter 1

Introduction

1.1 Capacity of Gaussian Wireless Networks

For information theorists, the channel capacity of Gaussian wireless networks is the holy grail. For single-user point-to-point Gaussian channels, it is well known that the channel capacity C is characterized by [1, 2]

$$C = B\log(1 + \mathrm{SNR}) \tag{1.1}$$

where B denotes the channel bandwidth and SNR stands for the signal-to-noise ratio.

For multiuser Gaussian networks, the capacity is in general open and only known for some special cases. For instance, for the multiple access channel (MAC), where multiple transmitters intend to communicate with a common receiver, the capacity region is known [1, 2]. For its reciprocal channel, the broadcast channel (BC), where one transmitter intends to send independent messages to multiple receivers, if the channel state information at transmitters (CSIT) is perfect, the capacity region is fully established as well [2,3]. These two channels are widely used to model single cell uplink and downlink transmission, respectively, in cellular networks.

For Gaussian channel models with both multiple transmitters and multiple receivers, such as interference channels, the capacity is generally intractable. In Gaussian interference channels, each transmitter intends to deliver one independent message to its desired receiver. However, due to the broadcast nature of the wireless medium, each receiver also overhears the interfering signals from the other undesired transmitters. The interference channel can be used to model multi-cell networks and device-to-device networks, which are the current research frontier for the wireless community.

Although the exact capacity of Gaussian interference channels is still open in general, recent years have seen remarkable progress in our understanding of its first order approximation in the high SNR limit – degrees of freedom (DoF), which is spurred in part by exciting breakthroughs such as the idea of interference alignment [4–6]. These advances provide fascinating theoretical insights and show much promise under idealized conditions. The connection to practical settings, however, still remains elusive. This is in part due to the following two factors. First, because of the assumption of abundant CSIT, idealized studies often get caught in the minutiae of channel realizations, e.g., irrational versus rational values, which have little bearing in practice. Second, by focusing on the DoF of fully connected networks, these studies ignore the most critical aspect of interference management in practice – *the differences of signal strengths* due to path loss and fading (in short, network topology). Indeed, the DoF metric treats every channel as essentially equally strong (i.e., every channel is capable of carrying exactly one DoF). So the desired signal has to actively avoid *every* interferer, whereas in practice each user only needs to avoid a few significant interferers and the rest are weak enough to be ignored safely. Therefore, by trivializing the network topology, the DoF studies of fully connected networks make the problem much harder than it needs to be. Non-trivial solutions to this harder problem invariably rely on much more channel knowledge at transmitters than is available in practice. Thus, these two limiting factors in fact re-enforce each other.

1.2 A Progressive Refinement Path for Capacity Characterization

Recent research has shown that the following progressive refinement path is extremely helpful for pursuing the capacity limit of Gaussian wireless networks:

$$\text{DoF} \rightarrow \text{GDoF} \rightarrow \text{constant gap} \rightarrow \text{exact capacity}$$

In this path, DoF serves as the starting point. To characterize the DoF of Gaussian networks, we fix the values of channel coefficients and the local noise power at receivers, let the total transmit power approach infinity, and solve the capacity problem in the high SNR limit. As mentioned before, this coarse DoF metric suffers from severe limitations, i.e., it essentially treats all non-zero channels as equally strong (i.e., each link is capable of carrying exactly one DoF) in the high SNR limit, and thus totally ignores the strength distinctions of various signals, which are critical for interference management in practice. To avoid the pitfalls of DoF, the next progressive refinement goal is a more general metric – generalized degrees of freedom (GDoF), which refines the picture by adopting a model that maintains the ratio of signal strengths (in the dB scale) constant as the high SNR limit is approached. Therefore, the GDoF framework allows us to explore the channel settings with both weak and strong interference and offers insights into optimal schemes for those channels. It's worthwhile noting that unlike DoF, in the GDoF framework instead of fixing the channel realizations (i.e., the channel coefficients and noise variance are unchanged) and scaling the transmit power, we in fact study a class of *different* channel realizations as SNR approaches infinity. The reason these channels are studied together is because, when normalized by log(SNR), they all have (approximately) the same capacity. Hence the GDoF characterization simultaneously settles the capacity of all the channels in this class within a gap of $o(\log(\text{SNR}))$. Further, the GDoF characterization tends to be a stepping stone to capacity characterizations within a constant gap, i.e., a gap that does not depend on channel realizations or SNR values. As expected, our ultimate refinement goal is the exact capacity.

The state of affairs is exemplified by the evolution of the exact sum capacity of 2-user single-input single-output (SISO) Gaussian interference channels in the so-called "noisy interference regime" [7–9]. It is well known that the DoF of 2-user SISO Gaussian interference channel is only equal to 1 [10], which can be trivially achieved via single-user transmission. In [11], the authors fully characterize the GDoF region of 2-user interference channels and show that a simple Han-Kobayashi scheme achieves the entire channel capacity region within 1 bit. Remarkably, the authors also demonstrate that when the interference is sufficiently weak, treating interference as noise is optimal from the GDoF perspective and the outer bounds are derived from a non-trivial genie-aided argument. Motivated by [11], a noisy interference regime is established in [7–9] where treating interference as noise achieves the exact sum capacity of 2-user interference channels.

This dissertation will follow along this progressive refinement path. In the following chapters, we will show how to characterize GDoF for certain Gaussian interference networks and how to utilize the obtained insights to derive capacity characterizations within a constant gap.

1.3 Channel State Information at Transmitters

From the theoretical perspective, the capacity of Gaussian wireless networks is quite sensitive to channel uncertainty at transmitters (even in the high SNR limit).[1] Take the K-user

[1] Throughout this dissertation, we assume that channel state information at receivers (CSIR) is perfect.

interference channel as an example. If the CSIT is perfect, the sum DoF is characterized as $K/2$ in [5], which is achievable via interference alignment. However, if the CSIT is only available to finite precision, then the sum DoF collapses to 1 [12]. In recent years, we have seen remarkable advances in understanding the capacity limits of Gaussian wireless networks under the idealized assumption of abundant CSIT. While this reveals ingenious ways of exploiting the finer aspects of CSIT, it remains difficult to translate the obtained results into practice where CSIT is rarely abundant. Recognizing this challenge, recent research has started exploring a variety of settings with relaxed CSIT assumptions, e.g., compound CSIT [13–15], delayed CSIT [16, 17], mixed CSIT [18, 19], alternating CSIT [20], and CSIT with finite precision [12, 21]. Following this line of research, lots of clever schemes have emerged. Nevertheless, much of the theoretical insights are still too fragile to be applied to practice directly. Different from these studies, we take a complementary approach to address this challenge. In this dissertation, we consider the practical interference management schemes that are robust to channel uncertainty at transmitters (e.g., treating interference as noise) and study when these simple schemes are optimal from an information theoretic perspective.

1.4 Practical Interference Management Principles for Wireless Networks

Real-world wireless networks are mainly based upon two robust interference management principles — 1) *ignore* interference that is sufficiently weak, and 2) *avoid* interference that is not. In more technical terms, ignoring interference translates into treating it as noise, and avoiding interference translates into multiple access schemes such as TDMA/FDMA/CDMA. The intuitive appeal of these two principles lies in their robustness, and in particular, their minimal CSIT requirements. Recent studies have explored the optimality of both principles.

1.4.1 Topological Interference Management (TIM)

The optimality of *interference avoidance* has been investigated most recently by [22], as the topological interference management (TIM) problem. With CSIT limited to a coarse knowledge of network topology (which links are stronger/weaker than the effective noise floor), TIM is essentially an index coding problem [23]. TIM subsumes within itself the multiple access schemes, e.g., TDMA/FDMA/CDMA, as trivial special cases, but is in general much more capable than these conventional approaches. Remarkably, for the class of linear schemes, which are found to be optimal in most cases studied so far, and within which TIM is equivalent to the index coding problem, TIM is shown to be essentially an optimal allocation of signal vector spaces based on an interference alignment perspective [24]. Variants of the TIM problem have also been investigated, including TIM under short coherent time [25], TIM with alternating connectivity [26, 27], TIM with multiple antennas [28], and TIM with transmitter cooperation [29].

1.4.2 Treating Interference as Noise (TIN)

From a practical perspective, *treating interference as noise* (TIN) is attractive due to its low complexity and robustness to channel uncertainty. TIN involves the use of only point-to-point channel codes, which are quite practical and near-optimal to deal with unstructured noise. Moreover, it only requires a coarse channel knowledge of the signal to interference and noise ratio (SINR) at transmitters, thus the overhead associated with acquiring CSIT is minimal.

From a theoretical perspective, the optimality of TIN is also discussed extensively in previous work. As mentioned before, in [7–9], it is shown that in the noisy interference regime, TIN achieves the *exact* sum capacity of Gaussian interference channels. Then, extensions of the noisy interference regime are obtained for multiple-input multiple-output (MIMO) interference channels [30], parallel interference channels [31], and 2×2 X channels [32]. From the GDoF perspective, it has been shown that when the interference is sufficiently weak, TIN is optimal for 2-user (asymmetric) interference channels [11] and K-user *fully symmetric* interference channels [33].

1.5 Dissertation Outline

In the rest of this dissertation, we will revisit the optimality of TIN from an information theoretic perspective. Different from the previous work mentioned in Section 1.4.2, in this dissertation we mainly answer the question when TIN is optimal to achieve the channel capacity within a constant gap for various Gaussian channel models, including K-user *fully asymmetric* interference channels, $M \times N$ X channels, and compound networks. Note that in this dissertation, unless stated otherwise, we consider SISO interference networks, where all transmitters and receivers are equipped with one antenna. We discuss the MIMO case in Section 2.4. The material in this dissertation is presented in part in [34–41].

In Chapter 2, for K-user fully asymmetric Gaussian interference channels, we identify a broad condition under which TIN is optimal from the GDoF perspective and approaches the entire channel capacity region within a constant gap. Specifically, the identified TIN-optimality condition is "*for each user the desired signal strength is no less than the sum of the strengths of the strongest interference caused by this user and the strongest interference suffered by this user (all values in dB scale)*". Next, we fully characterize the achievable GDoF region via the TIN scheme (i.e., the TIN region) for interference channels with arbitrary channel strength and establish its duality. Moreover, we extend the TIN-optimality result to MIMO interference channels where all transmitters and receivers are equipped with the same number of antennas. For MIMO channels where transmitters and receivers have different antenna numbers, we demonstrate that there exist non-trivial parameter regimes where a simple scheme of zero-forcing strong interference and treating the others as noise achieves the sum GDoF value.

In Chapter 3, we extend the optimality of TIN to Gaussian interference networks with general message sets. The main result is that for TIN-optimal interference channels identified in Chapter 2, expanding the message set to include an independent message from each transmitter to each receiver (i.e., the X message setting) does not increase the sum GDoF, and operating the new channel as the original interference channel and treating interference as noise is optimal for the sum capacity up to a constant gap. We also generalize the optimality of TIN to X channels with arbitrary numbers of transmitters and receivers.

In Chapter 4, we generalize the optimality of TIN to Gaussian compound networks. First, we show that for K-user compound Gaussian interference channels, if in each possible network realization, the TIN-optimality condition of Chapter 2 is satisfied individually, then TIN achieves the entire GDoF region of the whole compound setting. Next, we investigate the power control problem for compound networks from the GDoF perspective. We demonstrate that for an arbitrary compound interference channel, we can always find a non-trivial counterpart regular interference channel, such that the two have the same TIN region, and the GDoF-optimal power control problems for the two are equivalent. Note that the regular interference channel has only one state for each receiver, which may be different from all of the original states. Then, taking the advantage of the simplification of the compound setting to the regular case, we develop GDoF-optimal power control schemes for compound networks.

In Chapter 5, we investigate Gaussian interference channels with information theoretic secrecy constraints. We demonstrate that if the TIN-optimality condition identified in Chapter 2 is satisfied, the secrecy constraints incur no penalty from the GDoF perspective, and a scheme based on Gaussian signaling, cooperative jamming, and smart power splitting achieves the whole secure capacity region within a constant gap.

In Chapter 6, by combining TIN with TIM, we formulate a joint optimization problem for signal vector space and signal power level allocations, and identify a baseline solution based on a decomposition approach.

In Chapter 7, we summarize this dissertation.

1.6 Notations and Abbreviations

Throughout this dissertation, $\mathbb{R}_+$ denotes the set of non-negative real numbers. For any positive integer Z, $\langle Z \rangle$ denotes the set $\{1, 2, ..., Z\}$, and for any real number a, $(a)^+$ and $\max\{0, a\}$ are used interchangeably. For two vectors $\mathbf{u}$ and $\mathbf{v}$, we say that $\mathbf{u}$ dominates $\mathbf{v}$ if $\mathbf{u} \geq \mathbf{v}$, where $\geq$ denotes componentwise inequality. For a matrix $\mathbf{A}$, $\mathrm{Tr}(\mathbf{A})$ stands for its trace, $|\mathbf{A}|$ denotes its determinant, $\mathrm{span}(\mathbf{A})$ represents the space spanned by the column vectors of $\mathbf{A}$, and $\mathcal{CN}(0, \mathbf{A})$ represents the distribution of a complex circularly symmetric Gaussian random vector with zero mean and covariance matrix $\mathbf{A}$. $\mathbf{I}_n$ stands for an $n \times n$ identity matrix. We also use $\mathbf{I}$ to denote an identity matrix if its size is clear from the context. In addition, unless otherwise stated, all logarithms are to the base 2.

The table on the next page lists the abbreviations used in this dissertation.

DoF	Degrees of Freedom
GDoF	Generalized Degrees of Freedom
SNR	Signal to Noise Ratio
INR	Interference to Noise Ratio
SINR	Signal to Interference and Noise Ratio
CSI	Channel State Information
CSIT	Channel State Information at Transmitters
CSIR	Channel State Information at Receivers
SISO	Single Input Single Output
SIMO	Single Input Multiple Output
MIMO	Multiple Input Multiple Output
MAC	Multiple Access Channel
BC	Broadcast Channel
TIN	Treating Interference as Noise
TIM	Topological Interference Management
ZF	Zero Forcing

Table 1.1: Table of abbreviations

Chapter 2

Optimality of TIN for Interference Channels

It is well known (see, e.g., [7–9]) that in Gaussian interference channels the simple TIN scheme is information theoretically optimal when the interference is sufficiently weak. Such results are remarkable since they provide exact capacity characterizations, which are rare in network information theory. However, the difficulty of pursuing the exact capacity metric manifests itself through various limitations — the results are typically limited to sum channel capacity (as opposed to the entire channel capacity region), power control is not involved (all transmitters use all available power), and the regime where the exact optimality of TIN is established tends to be rather small. In contrast, in this chapter, by pursuing approximate capacity characterizations, we identify a broad regime where TIN (with power control) is optimal from the GDoF perspective, and within a constant gap to the whole capacity region.

This chapter is organized as follows. In Section 2.1, we present the standard channel model of K-user Gaussian interference channels and translate it into an equivalent form which is more conducive for GDoF studies. In Section 2.2, for K-user fully asymmetric interference channels, we identify a broad condition under which power control and TIN is optimal from the GDoF perspective and approaches the entire channel capacity region within a constant gap. We also fully characterize the achievable TIN region for interference channels with arbitrary channel coefficients and establish its duality in Section 2.3. We extend the optimality of TIN to MIMO interference channels in Section 2.4 and summarize this chapter in Section 2.5.

2.1 Channel Model

Consider the following canonical model for general K-user Gaussian interference channels

$$Y_k(t) = \sum_{i=1}^{K} \tilde{h}_{ki}\tilde{X}_i(t) + Z_k(t), \quad \forall k \in \langle K \rangle, \tag{2.1}$$

where at each time index t, $Y_k(t)$ is the received signal of Receiver k, $\tilde{X}_i(t)$ is the transmitted symbol of Transmitter i, $\tilde{h}_{ki}$ is the complex channel gain value between Transmitter i and Receiver k, and $Z_k(t) \sim \mathcal{CN}(0,1)$ is the additive white Gaussian noise (AWGN) at Receiver k. All symbols are complex. Transmitter $i \in \langle K \rangle$ is subject to the average power constraint $\mathbb{E}[|\tilde{X}_i(t)|^2] \leq P_i$.

Next, we translate the standard channel model (2.1) into an equivalent normalized form that is more conducive for GDoF studies. Define the SNR of user i and INR of Transmitter i at Receiver k as follows.[1]

$$\text{SNR}_i \triangleq \max(1, |\tilde{h}_{ii}|^2 P_i), \quad \text{INR}_{ki} \triangleq \max(1, |\tilde{h}_{ki}|^2 P_i), \quad i \neq k, \quad i,k \in \langle K \rangle. \tag{2.2}$$

[1] It is not difficult to verify that assigning a value of 1 to SNR and INR that are less than 1, or equivalently, assigning a value of 0 to α_{ij} that might otherwise be negative, is only a matter of convenience, and has no impact on the GDoF or the constant gap result derived in this chapter (i.e., Theorem 2.1, 2.2 and 2.3). The details are relegated to Appendix A.

Following [11], for the GDoF metric, we preserve the ratios of different signal strengths in dB scale as all SNRs approach infinity. To this end, taking $P > 1$ as a nominal power value, we define

$$\alpha_{ii} \triangleq \frac{\log \mathrm{SNR}_i}{\log P}, \quad \alpha_{ki} \triangleq \frac{\log \mathrm{INR}_{ki}}{\log P}, \quad i \neq k, \quad i, k \in \langle K \rangle, \tag{2.3}$$

implying that for each user i, $\mathrm{SNR}_i = P^{\alpha_{ii}}$ and for any two distinct users i, k, $\mathrm{INR}_{ki} = P^{\alpha_{ki}}$.

Now according to (2.2) and (2.3), we represent the original channel model (2.1) in the following form

$$Y_k(t) = \sum_{i=1}^{K} \sqrt{P^{\alpha_{ki}}} e^{j\theta_{ki}} X_i(t) + Z_k(t), \quad \forall k \in \langle K \rangle. \tag{2.4}$$

In this equivalent channel model, $X_i(t) = \tilde{X}_i(t)/\sqrt{P_i}$ is the transmit symbol of Transmitter i, and the power constraint for each transmitter is normalized to unity (i.e., $\mathbb{E}[|X_i(t)|^2] \leq 1$, $\forall i \in \langle K \rangle$). Indeed, the transmit power in the original channel model is absorbed into the channel coefficients. $\sqrt{P^{\alpha_{ki}}}$ and θ_{ki} are the magnitude and the phase, respectively, of the channel between Transmitter i and Receiver k, $\forall i, k \in \langle K \rangle$. The exponent α_{ki} is called the *channel strength level* of the link between Transmitter i and Receiver k. In the rest of this chapter, unless otherwise stated, we will consider the equivalent channel model in (2.4).

2.1.1 Channel Capacity

In K-user interference channels, Transmitter i intends to send message W_i to Receiver i. The messages W_i are independent, $\forall i \in \langle K \rangle$. Denote the size of the message set of user i by $|W_i|$. For codewords spanning n channel uses, the rates $R_i = \frac{\log |W_i|}{n}, i \in \langle K \rangle$, are achievable if the probability of error at all receivers can be made arbitrarily small as n approaches infinity. The channel capacity region $\mathcal{C}$ is the closure of the set of all achievable rate tuples.

Collecting the channel strength levels and phases in the sets

$$\alpha \triangleq \{\alpha_{ki}\}, \quad \theta \triangleq \{\theta_{ki}\}, \quad \forall i, k \in \langle K \rangle, \tag{2.5}$$

the capacity region $\mathcal{C}$ is in fact a function of α, θ, and P.

2.1.2 Generalized Degrees of Freedom (GDoF)

The GDoF region of K-user interference channels in (2.4) is defined as

$$\mathcal{D} \triangleq \Big\{(d_1, ..., d_K): \ d_i = \lim_{P\to\infty} \frac{R_i}{\log P}, \ \forall i \in \langle K \rangle, \ (R_1, ..., R_K) \in \mathcal{C}\Big\}. \tag{2.6}$$

In general, the channel capacity (GDoF) region of complex Gaussian interference channel may depend on both the channel strength levels α, and the channel phases θ. Later our results will show that when TIN is optimal from the GDoF perspective, the capacity (GDoF) inner and outer bounds we derived in this chapter depend *only* on α. As such, our results hold regardless of whether or not the channel phase information is available at transmitters.

Remark 2.1. *As we mentioned before, it is notable that unlike DoF, the scaling with P in the GDoF framework does not correspond to a scaling of transmit powers for a fixed channel realization, because of the disparate power scaling exponents α_{ij}. Instead, each P value in (2.4) defines a new channel. These channels are studied together, because when normalized by* $\log(P)$*, they all have (approximately) the same capacity. Therefore, a GDoF characterization simultaneously settles the capacity of all the channels in this class within a gap of* $o(\log(P))$.

2.1.3 Channel Capacity within a Constant Gap

Following the same definitions in [11] and [42], an achievable rate region is said to be within b ($b \geq 0$) bits of the capacity region if for any tuple $(R_1, R_2, ..., R_K)$ on the boundary of the achievable rate region, the tuple $(R_1 + b, R_2 + b, ..., R_K + b)$ is outside the channel capacity region.

2.2 TIN-optimality Condition for Interference Channels

The main result of this chapter is the following theorem.

Theorem 2.1. *In a K-user interference channel, if the following condition is satisfied*

$$\alpha_{ii} \geq \max_{j:j\neq i}\{\alpha_{ji}\} + \max_{k:k\neq i}\{\alpha_{ik}\}, \quad \forall i, j, k \in \langle K \rangle, \tag{2.7}$$

then power control and TIN achieves the entire GDoF region. The GDoF region is the set of all K-tuples $(d_1, ..., d_K)$ satisfying

$$0 \leq d_i \leq \alpha_{ii}, \qquad \forall i \in \langle K \rangle \tag{2.8}$$

$$\sum_{j=1}^{m} d_{i_j} \leq \sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}), \quad \forall (i_1, i_2, ..., i_m) \in \Pi_K, \ \forall m \in \{2, 3, ..., K\}, \tag{2.9}$$

where Π_K is the set of all possible cyclic sequences[2] *of all subsets of $\langle K \rangle$ with cardinality no less than 2, and the modulo-m arithmetic is implicitly used on user indices, e.g., $i_m = i_0$.*

In words, condition (2.7) can be stated as — *for each user the desired signal strength is no*

[2]Each cyclic sequence in Π_K is essentially a cyclically ordered subset of user indices, without repetitions. In Π_K, there exist $\sum_{m=2}^{K} \binom{K}{m}(m-1)!$ cyclic sequences.

less than the sum of the strengths of the strongest interference caused by this user and the strongest interference suffered by this user (all values in dB scale). Theorem 2.1 claims that under this condition, for K-user fully asymmetric interference channels, TIN is optimal from the GDoF perspective. For brevity, in the rest of this dissertation, we call the condition (2.7) the TIN-optimality condition, and refer to interference channels satisfying this TIN-optimality condition as TIN-optimal interference channels.

Remark 2.2. *Note that both the TIN-optimality condition (2.7) and the GDoF region specified by (2.8)-(2.9) display a natural duality in the sense that they are both unchanged if the roles of the transmitters and receivers are switched, i.e., if all α_{ij} values are switched with α_{ji} values. In other words, for the same channel strengths, if we consider the reciprocal channel (in the same sense as a BC being the reciprocal of a MAC), then under condition (2.7), TIN is optimal from the GDoF perspective, and the GDoF region is the same as in the original channel.*

Remark 2.3. *In [34], we conjecture that condition (2.7) is also necessary for TIN to be GDoF-optimal for K-user interference channels except for a set of channel gain values of measure zero.*

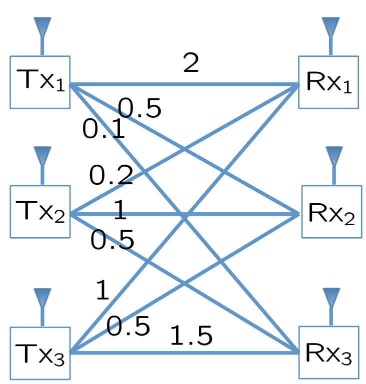

Figure 2.1: A 3-user interference channel

Example 2.1. *To help interpret the results in Theorem 2.1, we derive the GDoF region for a 3-user interference channel where the TIN-optimality condition (2.7) is satisfied. Consider* *the 3-user channel in Fig. 2.1. In this channel, the value on each link denotes the channel strength level. For the case of* $K = 3$, $\Pi_K = \{(1,2),(1,3),(2,3),(1,2,3),(1,3,2)\}$. *According to Theorem 2.1, the GDoF region is fully characterized by*

$$
\begin{aligned}
0 \leq d_1 &\leq 2 \\
0 \leq d_2 &\leq 1 \\
0 \leq d_3 &\leq 1.5 \\
d_1 + d_2 &\leq 2.3 \\
d_1 + d_3 &\leq 2.4 \\
d_2 + d_3 &\leq 1.5 \\
d_1 + d_2 + d_3 &\leq 3.7 \\
d_1 + d_2 + d_3 &\leq 2.5
\end{aligned}
$$

which is achievable via power control and TIN.

2.2.1 Achievability Proof of Theorem 2.1

We proceed to prove the achievability of Theorem 2.1 through the following steps. First, we introduce a polyhedral version of the TIN scheme called polyhedral TIN. We illustrate that the achievable GDoF region by polyhedral TIN (i.e., the polyhedral TIN region) is no larger than the achievable GDoF region by the original TIN scheme (i.e., the TIN region). In fact, later our results will show that *under the TIN-optimality condition (2.7), the polyhedral TIN region is the same as the TIN region.* Next, we demonstrate that the polyhedral TIN region can be characterized by checking the existence of a potential function for an induced fully-connected digraph, with vertexes representing the source-destination pairs (i.e., transmitter-receiver pairs or user pairs) in the interference channel (with an additional "ground" vertex) and a specific length assignment to the edges of the digraph. Finally, we derive the characterization of the polyhedral TIN region based on the potential theorem in [43], which essentially conducts *Fourier-Motzkin elimination* for the power allocation variables.

We start with the original TIN scheme. Assume that Transmitter $i \in \langle K \rangle$ uses a transmit power of P^{r_i}. Due to the unit power constraint in the channel model of (2.4), we have $r_i \leq 0$. By treating all the incoming interference as noise, user i achieves any rate R_i such that

$$R_i \leq \log\left(1 + \frac{P^{\alpha_{ii}+r_i}}{1+\sum_{j\neq i} P^{\alpha_{ij}+r_j}}\right) \tag{2.10}$$

So the achievable GDoF value of user i is given by

$$0 \leq d_i \leq \max\{0, \alpha_{ii} + r_i - \max\{0, \max_{j:j\neq i}(\alpha_{ij} + r_j)\}\} \tag{2.11}$$

The TIN region, which is denoted by $\mathcal{P}^*$, is the set of all K-tuples $(d_1, d_2, ..., d_K)$ for which there exist r_i's, $r_i \leq 0$, $i \in \langle K \rangle$, such that (2.11) holds for all $i \in \langle K \rangle$.

Next, we introduce a polyhedral version of the TIN scheme, which is called *polyhedral TIN*. Specifically, by requiring that $\alpha_{ii} + r_i - \max\{0, \max_{j:j\neq i}(\alpha_{ij} + r_j)\}$ is no less than 0 for each i, we can ignoring the first $\max\{0, ...\}$ term in (2.11). With this modification, the obtained polyhedral TIN region, which is denoted by $\mathcal{P}$, is the set of all K-tuples $(d_1, d_2, ..., d_K)$ for which there exist r_i's, $i \in \langle K \rangle$, such that

$$r_i \leq 0, \qquad \forall i \in \langle K \rangle \tag{2.12}$$

$$0 \leq d_i \leq \alpha_{ii} + r_i - \max\{0, \max_{j:j\neq i}(\alpha_{ij} + r_j)\}, \ \forall i \in \langle K \rangle \tag{2.13}$$

This region $\mathcal{P}$ is always a polyhedron (see Example 2.1), which is why the scheme is called polyhedral TIN. The above modification actually puts more constraints on the power exponents r_i's besides the constraints of $r_i \leq 0$. It can only shrink the achievable GDoF region of the TIN scheme. Thus in general we have $\mathcal{P} \subseteq \mathcal{P}^*$.

Example 2.2. *Consider a 2-user interference channel with $\alpha_{ij} = 1$, $\forall i, j \in \{1, 2\}$. In the polyhedral TIN scheme, we require that*

$$1 + r_1 - \max\{0, 1 + r_2\} \geq 0,$$
$$1 + r_2 - \max\{0, 1 + r_1\} \geq 0.$$

Combining with the constraints of $r_i \leq 0$, it is easy to verify that the valid power exponents r_i's for polyhedral TIN satisfy $r_1 = r_2$ and $r_1, r_2 \in [-1, 0]$, and the polyhedral TIN region $\mathcal{P}$ is a single point (0,0). While in the original TIN scheme, according to (2.11), the TIN region $\mathcal{P}^*$ is the union of two line segments, i.e., $\mathcal{P}^* = \{(d_1, d_2) : 0 \leq d_1 \leq 1, d_2 = 0\} \cup \{(d_1, d_2) : d_1 = 0, 0 \leq d_2 \leq 1\}$. Therefore, in this example $\mathcal{P} \subset \mathcal{P}^*$.

Remark 2.4. *As we mentioned, later our results will show that when the TIN-optimality condition (2.7) holds, compared with the original TIN scheme, polyhedral TIN incurs no loss. In other words, when condition (2.7) is satisfied, the TIN region $\mathcal{P}^*$ is same as the polyhedral TIN region $\mathcal{P}$.*

Rewriting (2.12) and (2.13), we obtain that the polyhedral TIN region $\mathcal{P}$ is fully characterized by the following linear inequalities

$$d_i \geq 0, \qquad \forall i \in \langle K \rangle \tag{2.14}$$

$$r_i \leq 0, \qquad \forall i \in \langle K \rangle \tag{2.15}$$

$$d_i \leq \alpha_{ii} + r_i \Leftrightarrow r_i \geq d_i - \alpha_{ii}, \qquad \forall i \in \langle K \rangle \tag{2.16}$$

$$d_i \leq \alpha_{ii} + r_i - (\alpha_{ij} + r_j) \Leftrightarrow r_i - r_j \geq \alpha_{ij} + (d_i - \alpha_{ii}), \ \forall i, j \in \langle K \rangle, i \neq j. \tag{2.17}$$

As we will show, by essentially a *Fourier-Motzkin elimination* of the power allocation variables r_i, $i \in \langle K \rangle$, the region $\mathcal{P}$ is fully characterized by (2.8)-(2.9). In the following, we complete this elimination by applying the potential theorem in [43]. Towards this end, for an arbitrary K-user interference channel, we construct a fully-connected digraph $D_p = (\mathcal{V}, \mathcal{E})$ called the *potential graph*, where $\mathcal{V}$ and $\mathcal{E}$ are the sets of vertices and edges, respectively, and

$$\mathcal{V} = \{v_1, v_2, ..., v_K, u\}$$
$$\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$$
$$\mathcal{E}_1 = \{(v_i, v_j) : i, j \in \langle K \rangle, i \neq j\}$$
$$\mathcal{E}_2 = \{(v_i, u) : i \in \langle K \rangle\}$$
$$\mathcal{E}_3 = \{(u, v_i) : i \in \langle K \rangle\}$$

We assign a length $l(e)$ to every edge $e \in \mathcal{E}$ as follows.

$$l(v_i, v_j) = \alpha_{ii} - d_i - \alpha_{ij}$$
$$l(v_i, u) = \alpha_{ii} - d_i$$
$$l(u, v_i) = 0.$$

Note that we denote by (a, b) the edge from vertex a to vertex b. As an example, the potential graph D_p for the 3-user interference channel in Example 2.1 is given in Fig. 2.2.

In the potential graph D_p, according to [43], a function $p : \mathcal{V} \to \mathbb{R}$ is called a potential if for every two vertexes $a, b \in \mathcal{V}$ such that $(a, b) \in \mathcal{E}$, $l(a, b) \geq p(b) - p(a)$. One can find that these inequalities only depend on the *difference* between potential function values. Therefore, without loss of generality, if there exists a valid potential function for the potential graph D_p, we can make one vertex, say the vertex u, *ground* (i.e., let $p(u) = 0$). Also letting $p(v_i) = r_i$, the potential function values should satisfy

$$\alpha_{ii} - d_i - \alpha_{ij} \geq r_j - r_i, \qquad \forall i, j \in \langle K \rangle, i \neq j \tag{2.18}$$

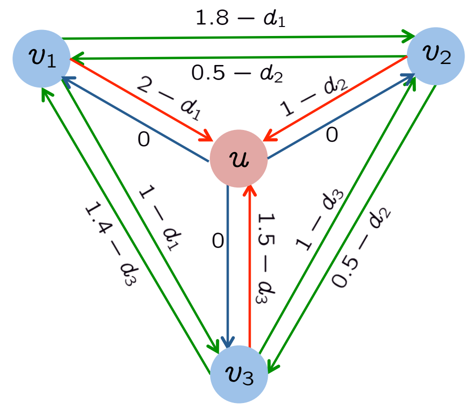

Figure 2.2: The potential graph D_p for the 3-user interference channel in Fig. 2.1

$$\alpha_{ii} - d_i \geq -r_i, \qquad \forall i \in \langle K \rangle \tag{2.19}$$

$$0 \geq r_i, \qquad \forall i \in \langle K \rangle. \tag{2.20}$$

which exactly match (2.15)-(2.17). Therefore, *for a K-user interference channel, a GDoF tuple $(d_1, d_2, ..., d_K) \in \mathbb{R}_+^K$ (i.e., the non-negative orthant of the K-dimensional Euclidean space) is in the region $\mathcal{P}$ if and only if there exists a valid potential function for its potential graph D_p.*

Equipped with the above observation, we invoke the following theorem in [43] to complete the characterization of the polyhedral TIN region $\mathcal{P}$.

Potential Theorem (Theorem 8.2 of [43])*: There exists a potential function for a digraph D if and only if each directed circuit in D has non-negative length.*

According to potential theorem, we conclude that $(d_1, d_2, ..., d_K) \in \mathbb{R}_+^K$ is in the polyhedral TIN region $\mathcal{P}$ if and only if each directed circuit in D_p has a non-negative length. Therefore, it remains to interpret the conditions of non-negative length for the circuits in D_p. We categorize these circuits in the following three classes:

- Circuits in the form of $(u \to v_i \to u)$, $\forall i \in \langle K \rangle$. For these circuits, we have

$$\alpha_{ii} - d_i \geq 0 \Leftrightarrow d_i \leq \alpha_{ii}. \tag{2.21}$$

- Circuits in the form of $(v_{i_0} \to v_{i_1} \to ... \to v_{i_m})$, where $i_0 = i_m$, $\forall (i_1, i_2, ..., i_m) \in \Pi_K$, $\forall m \in \{2, 3, ..., K\}$ (i.e., the circuits that do not include the ground vertex u). For these circuits, the non-negative length condition yields

$$\sum_{j=0}^{m-1} (\alpha_{i_j i_j} - d_{i_j} - \alpha_{i_j i_{j+1}}) \geq 0 \Leftrightarrow \sum_{j=0}^{m-1} d_{i_j} \leq \sum_{j=0}^{m-1} (\alpha_{i_j i_j} - \alpha_{i_j i_{j+1}}) \tag{2.22}$$

$$\overset{(a)}{\Leftrightarrow} \sum_{j=1}^{m} d_{i_j} \leq \sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}) \tag{2.23}$$

where in (a) we reorder the terms in the right hand side and use the fact that $i_m = i_0$.

- Circuits in the form of $(u \to v_{i_1} \to ... \to v_{i_m} \to u)$, $\forall (i_1, i_2, ..., i_m) \in \Pi_K$, $\forall m \in \{2, 3, ..., K\}$. For these circuits, we get

$$\sum_{j=1}^{m-1} (\alpha_{i_j i_j} - d_{i_j} - \alpha_{i_j i_{j+1}}) + (\alpha_{i_m i_m} - d_{i_m}) \geq 0 \tag{2.24}$$

Since $\alpha_{i_m i_1} \geq 0$, we have $\alpha_{i_m i_m} - d_{i_m} \geq \alpha_{i_m i_m} - d_{i_m} - \alpha_{i_m i_1}$. Therefore, given (2.23), the conditions for this class of circuits are redundant.

Consequently, we end up with conditions (2.21) and (2.23). Adding the non-negativity constraint on d_i explictly, we obtain (2.8)-(2.9), which is in fact the polyhedral TIN region $\mathcal{P}$ for interference channels *with arbitrary channel strength levels.* In other words, the region $\mathcal{P}$ specified by (2.8)-(2.9) is always achievable through polyhedral TIN for interference channels with arbitrary channel strength levels.

2.2.2 Converse Proof of Theorem 2.1

To prove the converse, we first derive the following lemma.

Lemma 2.1. *For K-user interference channels with channel input-output relationship in (2.1), the capacity region is included in the set of rate tuples $(R_1, R_2, ..., R_K)$ such that*

$$R_i \leq \log(1 + |\tilde{h}_{ii}|^2 P_i), \quad \forall i \in \langle K \rangle \tag{2.25}$$

$$\sum_{j=1}^{m} R_{i_j} \leq \sum_{j=1}^{m} \log\left(1 + |\tilde{h}_{i_j i_{j+1}}|^2 P_{i_{j+1}} + \frac{|\tilde{h}_{i_j i_j}|^2 P_{i_j}}{1 + |\tilde{h}_{i_{j-1} i_j}|^2 P_{i_j}}\right),$$
$$\forall (i_1, i_2, ..., i_m) \in \Pi_K, \quad \forall m \in \{2, 3, ..., K\}, \tag{2.26}$$

where the modulo-m arithmetic is implicitly used on user indices, e.g., $i_m = i_0$.

Proof of Lemma 2.1: The proof mainly follows [11, 42]. First, each *individual bound* (2.25) simply comes from the single user capacity bound. Next, consider the *cycle bound* (2.26). For any cyclic sequence $(i_1, i_2, ..., i_m) \in \Pi_K$, we start with the fully connected K-user interference channel with input-output relationship (2.1), and go through the following steps:

- Eliminate all users $i \in \langle K \rangle \backslash \{i_1, i_2, ..., i_m\}$ and their desired messages;
- Remove all the interfering links but the links from Transmitter i_j to Receiver i_{j-1}, $\forall j \in \langle m \rangle$.

We end up with the m-user cyclic interference channel depicted in Fig. 2.3. The above two steps cannot hurt the rates of the remaining messages. Therefore, the sum rate of users $i \in \{i_1, i_2, ..., i_m\}$ in the original K-user interference channel is upper bounded by that of the remaining m-user cyclic interference channel. Define

$$S_{i_j}(t) = \tilde{h}_{i_{j-1} i_j} \tilde{X}_{i_j}(t) + Z_{i_{j-1}}(t), \quad \forall j \in \langle m \rangle$$

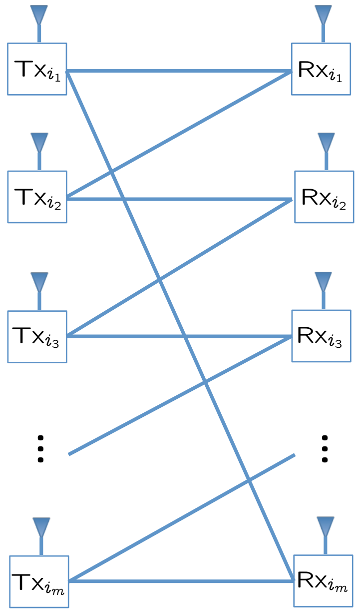

Figure 2.3: The m-user cyclic interference channel

For Receiver i_j, provide $S^n_{i_j}$ through a genie. From Fano's inequality, we have

$$
\begin{aligned}
& n(R_{i_j} - \epsilon) \\
& \leq I(W_{i_j}; Y^n_{i_j}, S^n_{i_j}) \\
& = h(Y^n_{i_j}, S^n_{i_j}) - h(Y^n_{i_j}, S^n_{i_j} | W_{i_j}) \\
& = h(S^n_{i_j}) + h(Y^n_{i_j} | S^n_{i_j}) - h(S^n_{i_j} | W_{i_j}) - h(Y^n_{i_j} | S^n_{i_j}, W_{i_j}) \\
& = h(S^n_{i_j}) + h(Y^n_{i_j} | S^n_{i_j}) - h(Z^n_{i_{j-1}}) - h(S^n_{i_{j+1}})
\end{aligned}
$$

Taking the sum of $n(R_{i_j} - \epsilon)$ for all $j \in \langle m \rangle$, we get

$$
\begin{aligned}
n \sum_{j=1}^{m} (R_{i_j} - \epsilon) & \leq \sum_{j=1}^{m} \left[h(Y^n_{i_j} | S^n_{i_j}) - h(Z^n_{i_j}) \right] \\
& \leq \sum_{t=1}^{n} \sum_{j=1}^{m} \left[h(Y_{i_j}(t) | S_{i_j}(t)) - h(Z_{i_j}(t)) \right],
\end{aligned}
$$

where the last inequality follows chain rule and the fact that dropping conditioning does not reduce entropy. Finally, using the fact that the circularly symmetric Gaussian distribution maximizes conditional differential entropy under a given covariance constraint, we obtain the desired outer bound (2.26). ■

Equipped with Lemma 2.1, we can proceed to complete the converse of Theorem 2.1 as follows. The individual bounds in (2.8) follow directly from (2.25).

$$d_i = \lim_{P\to\infty} \frac{R_i}{\log P} \le \lim_{P\to\infty} \frac{\log(1+P^{\alpha_{ii}})}{\log P} = \alpha_{ii}, \ \forall i \in \langle K \rangle$$

The cycle bounds in (2.9) follow from (2.26). For any cycle $(i_1, i_2, ..., i_m) \in \Pi_K$, we have

$$\begin{aligned}
\sum_{j=1}^{m} d_{i_j} &= \lim_{P\to\infty} \frac{\sum_{j=1}^{m} R_{i_j}}{\log P} \le \lim_{P\to\infty} \frac{\sum_{j=1}^{m} \log\left(1 + P^{\alpha_{i_j i_{j+1}}} + \frac{P^{\alpha_{i_j i_j}}}{1+P^{\alpha_{i_{j-1} i_j}}}\right)}{\log P} \\
&= \sum_{j=1}^{m} \max\{0, \alpha_{i_j i_{j+1}}, \alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}\} = \sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j})
\end{aligned}$$

where the last equality holds due to the TIN-optimality condition (2.7). This completes the converse proof of Theorem 2.1.

Remark 2.5. *In fact, Theorem 2.1 indicates that when condition (2.7) is satisfied, the GDoF region $\mathcal{D}$ is the polyhedral TIN region $\mathcal{P}$. As a consequence, in this regime we have $\mathcal{D} = \mathcal{P} = \mathcal{P}^*$. Recall that $\mathcal{P}$ is a convex polyhedron. This also means that when condition (2.7) holds, time-sharing cannot help enlarge the achievable GDoF region via the TIN scheme.*

2.2.3 Constant Gap Characterization

In this section, based on the insight obtained in the GDoF study, we derive a capacity characterization within a constant gap for the TIN-optimal interference channels identified in Theorem 2.1. The main result is given in the following theorem.

Theorem 2.2. *In a K-user interference channel, if condition (2.7) holds, then power control and TIN achieves to within* $\log(3K)$ *bits of the entire capacity region at any finite SNR.*

For the converse of Theorem 2.2, applying Lemma 2.1 to the equivalent channel model (2.4), we obtain the following outer bounds.

$$R_i \leq \log(1 + P^{\alpha_{ii}}), \quad \forall i \in \langle K \rangle \tag{2.27}$$

$$\sum_{j=1}^{m} R_{i_j} \leq \sum_{j=1}^{m} \log\left(1 + P^{\alpha_{i_j i_{j+1}}} + \frac{P^{\alpha_{i_j i_j}}}{1 + P^{\alpha_{i_{j-1} i_j}}}\right),$$
$$\forall (i_1, i_2, ..., i_m) \in \Pi_K, \quad \forall m \in \{2, 3, ..., K\}. \tag{2.28}$$

Since $P > 1$, it follows that

$$R_i \leq \log(1 + P^{\alpha_{ii}}) \leq \alpha_{ii} \log P + 1, \ \forall i \in \langle K \rangle, \tag{2.29}$$

$$\sum_{j=1}^{m} R_{i_j} \leq \sum_{j=1}^{m} \log\left(1 + P^{\alpha_{i_j i_{j+1}}} + \frac{P^{\alpha_{i_j i_j}}}{1 + P^{\alpha_{i_{j-1} i_j}}}\right) \tag{2.30}$$

$$< \sum_{j=1}^{m} \log\left(1 + P^{\alpha_{i_j i_{j+1}}} + \frac{P^{\alpha_{i_j i_j}}}{P^{\alpha_{i_{j-1} i_j}}}\right) \tag{2.31}$$

$$= \sum_{j=1}^{m} \log\left(\frac{P^{\alpha_{i_{j-1} i_j}} + P^{\alpha_{i_j i_{j+1}} + \alpha_{i_{j-1} i_j}} + P^{\alpha_{i_j i_j}}}{P^{\alpha_{i_{j-1} i_j}}}\right) \tag{2.32}$$

$$\leq \sum_{j=1}^{m} \log\left(\frac{3P^{\alpha_{i_j i_j}}}{P^{\alpha_{i_{j-1} i_j}}}\right) \tag{2.33}$$

$$= \sum_{j=1}^{m} [(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}) \log P + \log 3], \tag{2.34}$$

for all cycles $(i_1, i_2, ..., i_m) \in \Pi_K$, $\forall m \in \{2, 3, ..., K\}$.

Next, consider the achievability of Theorem 2.2. Assume that the power allocated to Transmitter $i \in \langle K \rangle$ is P^{r_i}, where $r_i \leq 0$. Through the TIN scheme, user i can achieve the rate

$$R_{i,\mathrm{TIN}} = \log\left(1 + \frac{P^{r_i+\alpha_{ii}}}{1+\sum_{j\neq i} P^{r_j+\alpha_{ij}}}\right). \tag{2.35}$$

From the proof of Theorem 2.1, we know that under condition (2.7), if d_i's satisfy (2.8) and (2.9), then there exist r_i's such that

$$r_i + \alpha_{ii} - \max_{j\neq i}\{0, r_j + \alpha_{ij}\} \geq d_i, \quad \forall i, j \in \langle K\rangle, \tag{2.36}$$

$$r_i \leq 0, \quad \forall i \in \langle K\rangle. \tag{2.37}$$

Therefore, we have

$$R_{i,\mathrm{TIN}} = \log\left(1 + \frac{P^{r_i+\alpha_{ii}}}{1+\sum_{j\neq i} P^{r_j+\alpha_{ij}}}\right) \geq \log\left(\frac{P^{r_i+\alpha_{ii}}}{P^0+\sum_{j\neq i} P^{r_j+\alpha_{ij}}}\right) \tag{2.38}$$

$$\geq \log\left(\frac{P^{r_i+\alpha_{ii}}}{KP^{r_i+\alpha_{ii}-d_i}}\right) = d_i \log P + \log\left(\frac{1}{K}\right). \tag{2.39}$$

In other words, when d_i's satisfy (2.8) and (2.9), the rates in (2.39) are always achievable by TIN, $\forall i \in \langle K\rangle$. Thus it is not hard to get that the achievable rate region by TIN includes the rate tuples $(R_{1,\mathrm{TIN}}, R_{2,\mathrm{TIN}}, ..., R_{K,\mathrm{TIN}})$ satisfying

$$0 \leq R_{i,\mathrm{TIN}} \leq \max\left\{0, \alpha_{ii}\log P + \log\left(\frac{1}{K}\right)\right\}, \ \forall i \in \langle K\rangle, \tag{2.40}$$

$$\sum_{j=1}^{m} R_{i_j,\mathrm{TIN}} \leq \max\left\{0, \sum_{j=1}^{m}\left[(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j})\log P + \log\left(\frac{1}{K}\right)\right]\right\} \tag{2.41}$$

for all cycles $(i_1, i_2, ..., i_m) \in \Pi_K$, $\forall m \in \{2, 3, ..., K\}$.

With both converse and achievability, to complete the proof of Theorem 2.2, we need to show that each of the rate constraints in (2.40) and (2.41) is within $\log(3K)$ bits of its corresponding outer bound in (2.29) and (2.34), i.e.,

$$\sigma_{R_i} < \log(3K), \qquad \forall i \in \langle K\rangle, \tag{2.42}$$

$$\sigma_{\sum_{j=1}^{m} R_{i_j}} \leq m\log(3K), \quad \forall(i_1, i_2, ..., i_m) \in \Pi_K, \ \forall m \in \{2, 3, ..., K\}, \tag{2.43}$$

where $\sigma_{(.)}$ denotes the difference between the achievable rate in (2.40) and (2.41) and its corresponding outer bound in (2.29) and (2.34). For σ_{R_i}, $\forall i \in \langle K \rangle$, consider the following two cases.

- $\alpha_{ii}\log P + \log\left(\frac{1}{K}\right) \leq 0$: In this case, we obtain

$$\sigma_{R_i} = \alpha_{ii}\log P + 1 \leq \log K + 1 < \log(3K).$$

- $\alpha_{ii}\log P + \log\left(\frac{1}{K}\right) > 0$: In this case, we have

$$\sigma_{R_i} = \left[\alpha_{ii}\log P + 1\right] - \left[\alpha_{ii}\log P + \log\left(\frac{1}{K}\right)\right] = 1 + \log K < \log(3K).$$

Similarly, for $\sigma_{\sum_{j=1}^{m} R_{i_j}}$, $\forall(i_1, i_2, ..., i_m) \in \Pi_K$, $\forall m \in \{2, 3, ..., K\}$, consider the following two cases.

- $\sum_{j=1}^{m}\left[(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j})\log P + \log\left(\frac{1}{K}\right)\right] \leq 0$: In this case, we obtain

$$\begin{aligned}\sigma_{\sum_{j=1}^{m} R_{i_j}} &= \sum_{j=1}^{m}\left[(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j})\log P + \log 3\right] \\ &\leq \sum_{j=1}^{m}[\log 3 + \log K] = m\log(3K).\end{aligned}$$

- $\sum_{j=1}^{m}\left[(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j})\log P + \log\left(\frac{1}{K}\right)\right] > 0$: In this case, we have

$$\begin{aligned}\sigma_{\sum_{j=1}^{m} R_{i_j}} = &\sum_{j=1}^{m}\left[(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j})\log P + \log 3\right] \\ &- \sum_{j=1}^{m}\left[(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j})\log P + \log\left(\frac{1}{K}\right)\right]\end{aligned}$$

$$= \sum_{j=1}^{m} [\log 3 + \log K] = m \log(3K).$$

Combining the above results together, we get (2.42) and (2.43) and complete the proof of Theorem 2.2.

2.3 Achievable TIN Region for General Interference Channels

Based on the characterization of the polyhedral TIN region $\mathcal{P}$ given in Section 2.2.1, it is not hard to further characterize the TIN region $\mathcal{P}^*$ for K-user interference channels with arbitrary channel strength levels. The main result is stated in the following theorem.

Theorem 2.3. *In a K-user interference channel, the TIN region $\mathcal{P}^*$ is given by*

$$\mathcal{P}^* = \bigcup_{\mathcal{S} \subseteq \langle K \rangle} \mathcal{P}_{\mathcal{S}}, \tag{2.44}$$

where $\mathcal{P}_{\mathcal{S}}$, $\mathcal{S} \subseteq \langle K \rangle$, is defined as

$$\begin{aligned} \mathcal{P}_{\mathcal{S}} = \{ & (d_1, d_2, ..., d_K) : d_i = 0, \forall i \in \mathcal{S}, 0 \leq d_j \leq \alpha_{jj}, \forall j \in \mathcal{S}^c, \\ & \sum_{j=1}^{m} d_{i_j} \leq \sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}), \forall (i_1, i_2, ..., i_m) \in \Pi_{\mathcal{S}^c} \}, \end{aligned}$$

and $\Pi_{\mathcal{S}^c}$ is the set of all possible cyclic sequences of all subsets of $\mathcal{S}^c$ (i.e., the complement of $\mathcal{S}$ in $\langle K \rangle$) with cardinality no less than 2.

In words, the TIN region $\mathcal{P}^*$ is the union of 2^K polyhedral TIN regions $\mathcal{P}_{\mathcal{S}}$, each of which corresponds to the case where the users in the set $\mathcal{S}$ are silent. Note that $\mathcal{P}_{\phi}$ is actually the polyhedral TIN region $\mathcal{P}$. Except for the polyhedral TIN region $\mathcal{P}$, all the other $\mathcal{P}_{\mathcal{S}}$

have zero volume in $\mathbb{R}^K$ since in each of them the users in $\mathcal{S}$ always have zero GDoF value. Therefore, the TIN region $\mathcal{P}^*$ is almost the same as the polyhedral TIN region $\mathcal{P}$ in the sense that the measure of the difference of these two sets is zero in $\mathbb{R}^K$.

2.3.1 Duality of Achievable TIN Region

From Remark 2.2, we have already known that for TIN-optimal interference channels identified in Theorem 2.1, the GDoF region (which is the polyhedral TIN region $\mathcal{P}$) satisfies duality. Theorem 2.3 demonstrates that for K-user interference channels with arbitrary channel strengths, the TIN region $\mathcal{P}^*$ satisfies duality as well. More specifically, for an arbitrary K-user interference channel and its reciprocal channel, where the roles of transmitters and receivers are switched, they always have the same TIN region. This result is quite remarkable from a theoretical perspective. In previous work, a similar duality relationship is only established for the symmetric achievable rate [44]. Our result establishes that from the GDoF perspective, the duality holds for the entire achievable TIN region.

2.3.2 Convexity of Achievable TIN Region

Recall that for TIN-optimal interference channels, the TIN region $\mathcal{P}^* = \mathcal{P}$, which is always convex without time-sharing. However, according to Theorem 2.3, the TIN region $\mathcal{P}^*$ may not be convex in general, and if time-sharing is allowed, the achievable region may become substantially larger. In other words, Theorem 2.3 reveals that *when the TIN-optimality condition (2.7) is violated, time-sharing may help enlarge the achievable GDoF region of TIN.*

Example 2.3. *Consider a 2-user interference channel with* $\alpha_{11} = \alpha_{22} = 1$, $\alpha_{12} = \alpha_{21} = 0.6$. *Clearly, this channel does not satisfy the TIN-optimality condition (2.7).*

First, if both the users are active, we have the polyhedral TIN region as follows.

$$\mathcal{P}_{\emptyset} = \big\{(d_1, d_2) : d_1 \geq 0, d_2 \geq 0, d_1 + d_2 \leq 0.8\big\},$$

which is in fact the region $\mathcal{P}$*. Next, consider the cases in which one user is made silent and hence has a GDoF value of 0, and the other user is active.*

$$\mathcal{P}_{\{1\}} = \big\{(d_1, d_2) : \; d_1 = 0, \; 0 \leq d_2 \leq 1\big\}$$
$$\mathcal{P}_{\{2\}} = \big\{(d_1, d_2) : \; d_2 = 0, \; 0 \leq d_1 \leq 1\big\}$$

We also have

$$\mathcal{P}_{\{1,2\}} = \big\{(d_1, d_2) : \; d_1 = d_2 = 0\big\}$$

Finally, according to Theorem 2.3, the TIN region $\mathcal{P}^*$ *is*

$$\mathcal{P}^* = \mathcal{P}_{\emptyset} \cup \mathcal{P}_{\{1\}} \cup \mathcal{P}_{\{2\}} \cup \mathcal{P}_{\{1,2\}} = \mathcal{P}_{\emptyset} \cup \mathcal{P}_{\{1\}} \cup \mathcal{P}_{\{2\}} \tag{2.45}$$

It is easy to verify that the region $\mathcal{P}^*$ *is not convex. Therefore, time-sharing can help enlarge the TIN region for this 2-user interference channel.*

2.4 Extension to MIMO Interference Channels

2.4.1 MIMO Channel Model

Consider K-user MIMO interference channels with M_i and N_i antennas at Transmitter i and Receiver i, respectively, $\forall i \in \langle K \rangle$. The channel input-output relationship is described by the following equation

$$\mathbf{Y}_i(t) = \sum_{j=1}^{K} \sqrt{P^{\alpha_{ij}}}\mathbf{H}_{ij}\mathbf{X}_j(t) + \mathbf{Z}_i(t), \quad \forall i \in \langle K \rangle \tag{2.46}$$

Here, over channel use t, $\mathbf{X}_j(t)$ is the $M_j \times 1$ input signal vector of Transmitter j, $\mathbf{Y}_i(t)$ is the $N_i \times 1$ received signal vector at Receiver i, $\mathbf{H}_{ij}$ is the constant $N_i \times M_j$ channel matrix between Transmitter j and Receiver i, and the noise term $\mathbf{Z}_i(t) \sim \mathcal{CN}(0, \mathbf{I}_{N_i})$. We assume that the channel input signal $\mathbf{X}_j(t)$ satisfies the unit average power constraint. We also assume that the entries of the channel matrix $\mathbf{H}_{ij}$ are drawn from a continuous and unitarily invariant distribution [45] to avoid degenerate channel conditions, and the Frobenius norm of $\mathbf{H}_{ij}$ is normalized to unity (i.e., $||\mathbf{H}_{ij}|| = 1$). The channel strength from Transmitter j to Receiver i is characterized by $\sqrt{P^{\alpha_{ij}}}$, where $P > 1$ is a nominal parameter and $\alpha_{ij} \in \mathbb{R}_+$. Following the notation in the SISO case, we call α_{ij} the channel strength level between Transmitter j and Receiver i. Similar to the SISO case, the codebooks, probability of error, achievable rate tuples $(R_1, ..., R_K)$, and the capacity region $\mathcal{C}$ are also defined in the standard Shannon sense. Again, the GDoF region $\mathcal{D}$ is defined as

$$\mathcal{D} \triangleq \left\{(d_1, ..., d_K) : \ d_k = \lim_{P\to\infty} \frac{R_k}{\log P}, \ \forall k \in \langle K \rangle, \quad (R_1, ..., R_K) \in \mathcal{C}\right\} \tag{2.47}$$

In the following sections of this chapter, we consider MIMO interference channel where all transmitters are equipped with M antennas and all receivers are equipped with N antennas. For notation brevity, we denote this channel as $(M, N)^K$ MIMO interference channel. Also, for any K-user interference channel, the modulo-K arithmetic is implicitly used on user indices.

2.4.2 TIN-optimality Condition for MIMO Channels

In this section, we consider the K-user MIMO interference channels where all transmitters and receivers are equipped with the same number of antennas M, i.e., the $(M, M)^K$ MIMO interference channel. We establish the optimality of TIN for such MIMO channels in the following theorem.

Theorem 2.4. *In an $(M, M)^K$ MIMO interference channel, if the following condition is satisfied*

$$\alpha_{ii} \geq \max_{j:j\neq i}\{\alpha_{ji}\} + \max_{k:k\neq i}\{\alpha_{ik}\}, \quad \forall i, j, k \in \langle K \rangle, \tag{2.48}$$

then power control and TIN achieves the whole GDoF region. The GDoF region is the set of all K-tuples $(d_1, d_2, ..., d_K)$ satisfying

$$0 \leq d_i \leq M\alpha_{ii}, \qquad \forall i \in \langle K \rangle \tag{2.49}$$

$$\sum_{j=1}^{m} d_{i_j} \leq M \sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}), \ \forall (i_1, i_2, ..., i_m) \in \Pi_K, \quad \forall m \in \{2, 3, ..., K\} \tag{2.50}$$

where the modulo-m arithmetic is implicitly used on user indices, e.g., $i_m = i_0$.

From Theorem 2.4, one can find that for TIN-optimal Gaussian interference channels identified in Theorem 2.1, its entire GDoF region scales uniformly with spatial dimensions almost surely. In other words, for TIN-optimal interference channels, if the number of antennas at each node is scaled by a common constant factor, then the whole GDoF region scales by the same factor almost surely, and the TIN scheme remains optimal from the GDoF perspective. Remarkably, the spatial scale invariance of TIN-optimal interference channels holds even if only coarse channel strength information (but no phase information) is available at transmitters.

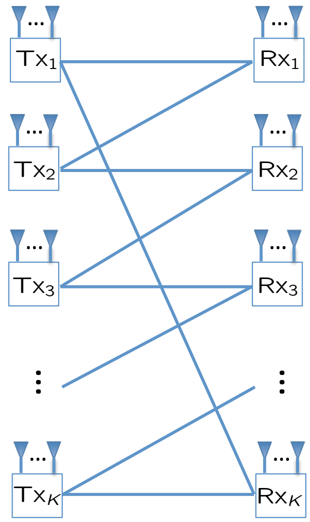

Figure 2.4: An $(M,N)^K$ MIMO cyclic interference channel, where transmitters and receivers are equipped with M and N antennas, respectively

In the following, we proceed to prove Theorem 2.4. First, consider the converse. Again, the individual bound (2.49) comes from the single user capacity. To get the cycle bound (2.50), we need the following lemma.

Lemma 2.2. *In an $(M,N)^K$ MIMO cyclic interference channel as shown in Fig. 2.4 where $\frac{N}{2} < M < 2N$, if the condition (2.48) is satisfied, then its sum GDoF is upper bounded by*

$$d_\Sigma \leq \begin{cases} N\sum_{k=1}^K \alpha_{kk} - (2N-M)\sum_{k=1}^K \alpha_{k-1,k}, & N \leq M \\ M\sum_{k=1}^K \alpha_{kk} - (2M-N)\sum_{k=1}^K \alpha_{k-1,k}, & M < N \end{cases}$$

Proof of Lemma 2.2: Note that for the partially connected channel given in Fig. 2.4, the condition (2.48) is reduced to

$$\alpha_{kk} \geq \alpha_{k,k+1} + \alpha_{k-1,k}, \quad \forall k \in \langle K \rangle$$

Define

$$\mathbf{S}_k(t) = \sqrt{P^{\alpha_{k-1,k}}}\mathbf{H}_{k-1,k}\mathbf{X}_k(t) + \mathbf{Z}_{k-1}(t), \quad \forall k \in \langle K \rangle \tag{2.51}$$

$$\mathbf{Q}_k(t) = \mathbb{E}(\mathbf{X}_k(t)\mathbf{X}_k^{\dagger}(t)), \qquad \forall k \in \langle K \rangle \tag{2.52}$$

$$\bar{\mathbf{H}}_{k-1,k} = \sqrt{P^{\alpha_{k-1,k}}}\mathbf{H}_{k-1,k}, \qquad \forall k \in \langle K \rangle \tag{2.53}$$

For Receiver $k \in \langle K \rangle$, provide $\mathbf{S}_k(t)$ through a genie. From Fano's inequality, we have

$$\begin{aligned}
n(R_k - \epsilon) &\leq I(W_k; \mathbf{Y}_k^n, \mathbf{S}_k^n) \\
&= h(\mathbf{Y}_k^n, \mathbf{S}_k^n) - h(\mathbf{Y}_k^n, \mathbf{S}_k^n | W_k) \\
&= h(\mathbf{S}_k^n) + h(\mathbf{Y}_k^n | \mathbf{S}_k^n) - h(\mathbf{S}_k^n | W_k) - h(\mathbf{Y}_k^n | \mathbf{S}_k^n, W_k) \\
&= h(\mathbf{S}_k^n) + h(\mathbf{Y}_k^n | \mathbf{S}_k^n) - h(\mathbf{Z}_{k-1}^n) - h(\mathbf{S}_{k+1}^n)
\end{aligned}$$

According to the above inequalities, the sum rate R_Σ satisfies

$$\begin{aligned}
nR_\Sigma - nK\epsilon &\leq \sum_{k=1}^{K} \left[h(\mathbf{Y}_k^n | \mathbf{S}_k^n) - h(\mathbf{Z}_k^n) \right] \\
&\leq \sum_{t=1}^{n} \sum_{k=1}^{K} \left[h(\mathbf{Y}_k(t) | \mathbf{S}_k(t)) - h(\mathbf{Z}_k(t)) \right] \qquad (2.54) \\
&\leq \sum_{t=1}^{n} \sum_{k=1}^{K} \left[h(\mathbf{Y}_k^G(t) | \mathbf{S}_k^G(t)) - h(\mathbf{Z}_k(t)) \right], \qquad (2.55)
\end{aligned}$$

where the last inequality follows from Lemma 1 in [46], and the superscript G denotes that the corresponding inputs are independent $\mathbf{X}_i^G(t) \sim \mathcal{CN}(0, \mathbf{Q}_i(t))$, $\forall i \in \langle K \rangle$. In the following, we omit the time index t for notation brevity. We have

$$h(\mathbf{Y}_k^G | \mathbf{S}_k^G) = \log \left| \pi e \Sigma_{\mathbf{Y}_k^G | \mathbf{S}_k^G} \right| \tag{2.56}$$

and

$$
\begin{aligned}
\Sigma_{\mathbf{Y}_k^G|\mathbf{S}_k^G} &= E(\mathbf{Y}_k^G\mathbf{Y}_k^{G\dagger}) - E(\mathbf{Y}_k^G\mathbf{S}_k^{G\dagger})E(\mathbf{S}_k^G\mathbf{S}_k^{G\dagger})E(\mathbf{S}_k^G\mathbf{Y}_k^{G\dagger}) && (2.57)\\
&= \mathbf{I} + P^{\alpha_{k,k+1}}\mathbf{H}_{k,k+1}\mathbf{Q}_{k+1}\mathbf{H}_{k,k+1}^\dagger + P^{\alpha_{kk}}\mathbf{H}_{kk}\mathbf{Q}_k\mathbf{H}_{kk}^\dagger \\
&\quad - P^{\alpha_{kk}+\alpha_{k-1,k}}\mathbf{H}_{kk}\mathbf{Q}_k\mathbf{H}_{k-1,k}^\dagger(\mathbf{I}+P^{\alpha_{k-1,k}}\mathbf{H}_{k-1,k}\mathbf{Q}_k\mathbf{H}_{k-1,k}^\dagger)^{-1}\mathbf{H}_{k-1,k}\mathbf{Q}_k\mathbf{H}_{kk}^\dagger && (2.58)\\
&= \mathbf{I} + P^{\alpha_{k,k+1}}\mathbf{H}_{k,k+1}\mathbf{Q}_{k+1}\mathbf{H}_{k,k+1}^\dagger \\
&\quad + P^{\alpha_{kk}}\mathbf{H}_{kk}\mathbf{Q}_k^{\frac{1}{2}}\big[\mathbf{I} - \mathbf{Q}_k^{\frac{1}{2}}\bar{\mathbf{H}}_{k-1,k}^\dagger(\mathbf{I}+\bar{\mathbf{H}}_{k-1,k}\mathbf{Q}_k\bar{\mathbf{H}}_{k-1,k}^\dagger)^{-1}\bar{\mathbf{H}}_{k-1,k}\mathbf{Q}_k^{\frac{1}{2}}\big]\mathbf{Q}_k^{\frac{1}{2}}\mathbf{H}_{kk}^\dagger \\
&= \mathbf{I} + P^{\alpha_{k,k+1}}\mathbf{H}_{k,k+1}\mathbf{Q}_{k+1}\mathbf{H}_{k,k+1}^\dagger \\
&\quad + P^{\alpha_{kk}}\mathbf{H}_{kk}\mathbf{Q}_k^{\frac{1}{2}}\big(\mathbf{I} + \mathbf{Q}_k^{\frac{1}{2}}\bar{\mathbf{H}}_{k-1,k}^\dagger\bar{\mathbf{H}}_{k-1,k}\mathbf{Q}_k^{\frac{1}{2}}\big)^{-1}\mathbf{Q}_k^{\frac{1}{2}}\mathbf{H}_{kk}^\dagger && (2.59)\\
&\leq \mathbf{I} + P^{\alpha_{k,k+1}}\mathbf{H}_{k,k+1}\mathbf{H}_{k,k+1}^\dagger + P^{\alpha_{kk}}\mathbf{H}_{kk}\big(\mathbf{I} + P^{\alpha_{k-1,k}}\mathbf{H}_{k-1,k}^\dagger\mathbf{H}_{k-1,k}\big)^{-1}\mathbf{H}_{kk}^\dagger && (2.60)
\end{aligned}
$$

where (2.59) follows from the Woodbury identity, i.e.,

$$(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1}+\mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}$$

and (2.60) follows the proof of Lemma 1 in [46]. Next, based on similar argument used in the proof of Theorem 1 in [45], when (2.48) is satisfied we obtain

$$h(\mathbf{Y}_k^G|\mathbf{S}_k^G) \leq \begin{cases} \big[N\alpha_{kk} - (2N-M)\alpha_{k-1,k}\big]\log P + \mathcal{O}(1), & N \leq M < 2N \\ \big[M(\alpha_{kk}-\alpha_{k-1,k}) + (N-M)\alpha_{k,k+1}\big]\log P + \mathcal{O}(1), & \frac{N}{2} < M < N \end{cases} \tag{2.61}$$

Plugging (2.61) into (2.55), we complete the proof. ■

Next, similar to the SISO case, for any cyclic sequence $(i_1, i_2, ..., i_m) \in \Pi_K$, we start with the fully connected K-user MIMO interference channel, and go through the following steps:

- Eliminate all the users $i \in \langle K\rangle\backslash\{i_1, i_2, ..., i_m\}$ and their desired messages;

- Remove all the interfering links but the links from Transmitter i_j to Receiver i_{j-1}, $\forall j \in \langle m \rangle$.

We obtain an m-user MIMO cyclic interference channel. The above operations cannot hurt the rates of the remaining messages. Therefore, the sum GDoF of users $i \in \{i_1, i_2, ..., i_m\}$ in the original K-user MIMO interference channel is upper bounded by that of the remaining m-user MIMO cyclic interference channel. Applying Lemma 2.2 to the m-user MIMO cyclic channel and letting $M = N$, we end up with the desired cycle bounds (2.50).

In the sequel, we proceed to present the achievability proof. At Transmitter $k \in \langle K \rangle$, for the channel input signal $\mathbf{X}_k \sim \mathcal{CN}(0, \mathbf{Q}_k)$, let $\mathbf{Q}_k = \frac{P^{r_k}}{M}\mathbf{I}_M$, where $r_k \leq 0$. At receivers, by treating all incoming interference as noise, user $k \in \langle K \rangle$ achieves any rate $R_k \in \mathbb{R}_+$ such that

$$\begin{aligned} R_k &\leq I(\mathbf{X}_k; \mathbf{Y}_k) = h(\mathbf{Y}_k) - h(\mathbf{Y}_k|\mathbf{X}_k) \\ &= \log \left| \mathbf{I}_M + \frac{1}{M}\sum_{i=1}^{K} P^{\alpha_{ki}+r_i}\mathbf{H}_{ki}\mathbf{H}_{ki}^{\dagger} \right| - \log \left| \mathbf{I}_M + \frac{1}{M}\sum_{i=1, i\neq k}^{K} P^{\alpha_{ki}+r_i}\mathbf{H}_{ki}\mathbf{H}_{ki}^{\dagger} \right| + o(\log P) \end{aligned}$$

According to [45, 46], we get the achievable GDoF value $d_k \in \mathbb{R}_+$ of user k,

$$d_k \leq M \left[\max_{i \in \langle K \rangle}\{0, \alpha_{ki} + r_i\} - \max_{i \in \langle K \rangle \setminus \{k\}}\{0, \alpha_{ki} + r_i\} \right] \tag{2.62}$$

$$= M \max \left\{ 0, \alpha_{kk} + r_k - \max_{i \in \langle K \rangle \setminus \{k\}}\{0, \alpha_{ki} + r_i\} \right\} \tag{2.63}$$

Then following the same achievability argument for the SISO case, we obtain that TIN achieves the whole GDoF region that is specified by (2.49) and (2.50), and hence complete the proof of Theorem 2.4.

2.4.3 Optimality of Zero-forcing and TIN for MIMO Channels

In this section, we consider $(M,N)^K$ MIMO interference channels where $M \neq N$, and show that in this case there exist non-trivial parameter regimes where a simple scheme of zero-forcing strong interference and treating the others as noise achieves the sum GDoF. The main result is the following theorem.

Theorem 2.5. *In an $(M,N)^K$ MIMO interference channel where $(K-1)M \leq N < KM$, if for any permutation $(i_1, i_2, ..., i_K) \in \Pi_K$, the following condition is satisfied,*

$$\alpha_{i_j i_j} \geq \alpha_{i_j i_{j+1}} + \alpha_{i_{j+1} i_j}, \qquad \forall j \in \langle K \rangle \tag{2.64}$$

$$\alpha_{i_j i_{j+m}} \geq \alpha_{i_j i_{j+1}} + \alpha_{i_{j+1} i_{j+m}}, \quad \forall j \in \langle K \rangle, \forall m \in \{2, ..., K-1\} \tag{2.65}$$

then its sum GDoF value is given by

$$d_\Sigma = M \sum_{k=1}^{K} \alpha_{kk} - (KM - N) \sum_{j=1}^{K} \alpha_{i_j i_{j+1}} \tag{2.66}$$

which is achievable via ZF and TIN.

Example 2.4. *Consider the 3-user case of Theorem 2.5. Without loss of generality, let $i_1 = 1$, $i_2 = 2$ and $i_3 = 3$. The optimality conditions (2.64) and (2.65) become $\alpha_{11} \geq \alpha_{12} + \alpha_{21}$, $\alpha_{13} \geq \alpha_{12} + \alpha_{23}$, $\alpha_{22} \geq \alpha_{23} + \alpha_{32}$, $\alpha_{21} \geq \alpha_{23} + \alpha_{31}$, $\alpha_{33} \geq \alpha_{31} + \alpha_{13}$, and $\alpha_{32} \geq \alpha_{31} + \alpha_{12}$. A 3-user $(1,2)^3$ SIMO channel satisfying the above conditions is depicted in Fig. 2.5. According to Theorem 2.5, ZF and TIN achieves the sum GDoF value*

$$d_\Sigma = (\alpha_{11} + \alpha_{22} + \alpha_{33}) - (\alpha_{12} + \alpha_{23} + \alpha_{31}) = 2.2$$

To obtain the above result, we first zero-force the stronger interference link (the red links in Fig. 2.5) for each receiver, which leads to a SISO cyclic interference channel satisfying the TIN-optimality condition (2.7). Then it is easy to verify that for the remaining cyclic *channel the TIN scheme achieves the sum GDoF value* 2.2.

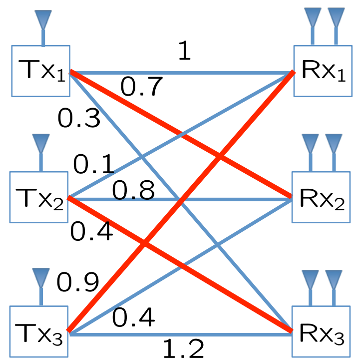

Figure 2.5: A 3-user $(1, 2)^3$ SIMO interference channel where ZF and TIN achieves the sum GDoF

In the following, we proceed to prove Theorem 2.5. For the achievability, we invoke the following lemma.

Lemma 2.3. *In an $(M, N)^K$ MIMO cyclic interference channel as shown in Fig. 2.4 where $\frac{N}{2} < M < 2N$, define $\kappa^- \triangleq \min\{M, N\}$ and $\kappa^+ \triangleq \max\{M, N\}$. If the following condition holds,*

$$\alpha_{kk} \geq \alpha_{k,k+1} + \alpha_{k-1,k}, \quad \forall k \in \langle K \rangle \tag{2.67}$$

then the GDoF region is characterized by

$$0 \leq d_i \leq \kappa^- \alpha_{ii} \tag{2.68}$$

$$\sum_{i=1}^{K} d_i \leq \kappa^- \sum_{i=1}^{K} \alpha_{ii} - (2\kappa^- - \kappa^+) \sum_{i=1}^{K} \alpha_{i-1,i} \tag{2.69}$$

$$\sum_{i=l}^{l+s-1} d_i \leq \kappa^- \sum_{i=l}^{l+s-1} \alpha_{ii} - (2\kappa^- - \kappa^+) \sum_{i=l+1}^{l+s-1} \alpha_{i-1,i}, \quad \forall l \in \langle K \rangle, \forall s \in \langle K-1 \rangle \tag{2.70}$$

which is achievable by ZF and TIN.

Proof of Lemma 2.3: The converse follows from the single-user capacity bound and Lemma

2.2. For the achievability, it is easy to verify that the naive TIN scheme is not optimal any more. Here first consider the case where $\frac{N}{2} < M < N$. Transmitter $k \in \langle K \rangle$ sends out M data streams $s_{k,1}$, ..., $s_{k,M}$ (intended for Receiver k) with linearly independent unit-norm beamforming vectors $\mathbf{v}_{k,1}$, ..., $\mathbf{v}_{k,M}$ and associated power allocations $\frac{1}{M}P^{r_{k,1}}$, ..., $\frac{1}{M}P^{r_{k,M}}$, where $r_{k,m} \leq 0$, $\forall m \in \langle M \rangle$. The factor $1/M$ guarantees that the unit transmit power constraint is satisfied. Set the power levels $r_{k,1} = ... = r_{k,N-M} = \bar{r}_k$ and $r_{k,N-M+1} = ... = r_{k,M} = r_k$, $\forall k \in \langle K \rangle$. In the N dimensional space, Receiver k receives $2M > N$ data streams. Receiver k first decodes the $2M - N$ desired data streams $\{s_{k,N-M+1}, ..., s_{k,M}\}$ by zero-forcing the $2(N-M)$ data streams $\{s_{k,1}, ..., s_{k,N-M}\}$ and $\{s_{k+1,1}, ..., s_{k+1,N-M}\}$ and treating the remaining $2M - N$ interfering data streams $\{s_{k+1,N-M+1}, ..., s_{k+1,M}\}$ as noise. The ZF operation is possible since we have $(2M - N) + 2(N - M) = N$. The achievable GDoF value d'_k of the $2M - N$ data streams $\{s_{k,N-M+1}, ..., s_{k,M}\}$ satisfies

$$0 \leq d'_k \leq (2M - N) \max\left\{0, \alpha_{kk} + r_k - \max\{0, \alpha_{k,k+1} + r_{k+1}\}\right\}$$

After decoding the $2M - N$ desired data streams $\{s_{k,N-M+1}, ..., s_{k,M}\}$, Receiver k subtracts them from the received signal and then decodes the remaining desired $N - M$ data streams $\{s_{k,1}, ..., s_{k,N-M}\}$ by zero-forcing the M interfering data streams from Transmitter $k + 1$. The achievable GDoF value of the data streams $\{s_{k,1}, ..., s_{k,N-M}\}$ satisfies

$$0 \leq d''_k \leq (N - M)\alpha_{kk}$$

Adding d'_k and d''_k together, we obtain the achievable GDoF value of user k. To complete the achievability proof, we construct the potential graph for d'_k and then apply the potential theorem following Section 2.2.1. The details are omitted to avoid repetition. The achievability proof for $N \leq M < 2N$ follows similarly. The only difference is that when $M > N$, ZF precoding or beamforming at transmitters is needed. ■

To achieve the sum GDoF value in (2.66), Receiver i_j first zero-forces the interfering links from Transmitter i_{j+m}, $\forall m \in \{2,3,...,K-1\}$, leading to an $(M, N-(K-2)M)^K$ MIMO cyclic interference channel. For the remaining links, it is easy to verify that $\alpha_{i_j i_j} \geq \alpha_{i_j i_{j+1}} + \alpha_{i_{j-1} i_j}$. Applying Lemma 2.3 to the remaining cyclic channel, we complete the achievability proof.

Next, consider the converse of Theorem 2.5, which is a a non-trivial generalization of [47]. Without loss of generality, assume that $i_j = j$, $\forall j \in \langle K \rangle$. So the conditions in Theorem 2.5 become

$$\alpha_{jj} \geq \alpha_{j,j+1} + \alpha_{j+1,j}, \quad \forall j \in \langle K \rangle \tag{2.71}$$

$$\alpha_{j,j+m} \geq \alpha_{j,j+1} + \alpha_{j+1,j+m}, \ \forall j \in \langle K \rangle, \forall m \in \{2, ..., K-1\} \tag{2.72}$$

Let $\mathbf{S}_{i,\mathcal{A}} = \sum_{j \in \mathcal{A}} \sqrt{P^{\alpha_{ij}}} \mathbf{H}_{ij} \mathbf{X}_j + \mathbf{Z}_i$, where $\mathcal{A}$ is a subset of user indexes, i.e., $\mathcal{A} \subseteq \langle K \rangle$. For Receiver 1, provide $\mathbf{S}_{2,\langle K \rangle \backslash \{2\}}$ through a genie. Start with Fano's inequality.

$$\begin{aligned}
& n(R_1 - \epsilon) \leq I(W_1; \mathbf{Y}_1^n, \mathbf{S}^n_{2,\langle K \rangle \backslash \{2\}}) \\
& = h(\mathbf{Y}_1^n, \mathbf{S}^n_{2,\langle K \rangle \backslash \{2\}}) - h(\mathbf{Y}_1^n, \mathbf{S}^n_{2,\langle K \rangle \backslash \{2\}} | W_1) \\
& = h(\mathbf{Y}_1^n | \mathbf{S}^n_{2,\langle K \rangle \backslash \{2\}}) + h(\mathbf{S}^n_{2,\langle K \rangle \backslash \{2\}}) - h(\mathbf{S}^n_{1,\langle K \rangle \backslash \{1\}}, \mathbf{S}^n_{2,\langle K \rangle \backslash \{1,2\}}) \\
& = h(\mathbf{Y}_1^n | \mathbf{S}^n_{2,\langle K \rangle \backslash \{2\}}) + h(\mathbf{S}^n_{2,\langle K \rangle \backslash \{2\}}) - h(\mathbf{S}^n_{1,\langle K \rangle \backslash \{1\}}) - h(\mathbf{S}^n_{2,\langle K \rangle \backslash \{1,2\}} | \mathbf{S}^n_{1,\langle K \rangle \backslash \{1\}}) \\
& = h(\mathbf{Y}_1^n | \mathbf{S}^n_{2,\langle K \rangle \backslash \{2\}}) + h(\mathbf{S}^n_{2,\langle K \rangle \backslash \{2\}}) - h(\mathbf{S}^n_{1,\langle K \rangle \backslash \{1\}}) + n \ o(\log P)
\end{aligned}$$

The last step holds due to $N \geq (K-1)M$ and $\alpha_{1,1+m} \geq \alpha_{2,1+m}$, $\forall m \in \{2,3,...,K-1\}$ (see (2.72)). For all the other Receivers $i \in \{2,3,...,K\}$, provide $\mathbf{S}_{i+1,\langle K \rangle \backslash \{i+1\}}$ through a genie and conduct similar manipulations. Adding all the obtained terms together, we get

$$n(\sum_{k=1}^{K} R_k - K\epsilon) \leq \sum_{i=1}^{K} \left[h(\mathbf{Y}_i^n | \mathbf{S}^n_{i+1,\langle K \rangle \backslash \{i+1\}}) + n \ o(\log P) \right] \tag{2.73}$$

For each term in the right hand side of (2.73), we have

$$
\begin{aligned}
& h(\mathbf{Y}_i^n|\mathbf{S}^n_{i+1,\langle K\rangle\setminus\{i+1\}}) \\
& = h(\mathbf{Y}_i^n|\mathbf{S}^n_{i+1,1},...,\mathbf{S}^n_{i+1,i},\mathbf{S}^n_{i+1,i+2},...,\mathbf{S}^n_{i+1,K}) + n\ o(\log P) && (2.74)\\
& \leq nh(\mathbf{Y}_i^G|\mathbf{S}^G_{i+1,1},...,\mathbf{S}^G_{i+1,i},\mathbf{S}^G_{i+1,i+2},...,\mathbf{S}^G_{i+1,K}) + n\ o(\log P) && (2.75)\\
& \leq n\Bigg(M\sum_{j=1,j\neq i+1}^{K}(\alpha_{ij}-\alpha_{i+1,j}) + [N-(K-1)M]\alpha_{i,i+1}\Bigg)\log P + n\ o(\log P) && (2.76)
\end{aligned}
$$

where (2.74) holds since $N \geq (K-1)M$ and Receiver $i+1$ has enough spatial space to recover the signals from the other transmitters individually within bounded variance noise distortion, (2.76) follows from the Woodbury matrix identity, the proof of Lemma 1 in [46], the proof of Theorem 1 in [45], and the conditions (2.71)-(2.72). Combining all the above results together, we establish the upper bound for the sum rate of all users, which leads to the desired sum GDoF outer bound.

Remark 2.6. *For* $(1,N)^{N+1}$ $(N \geq 2)$ *SIMO interference channels, in [48] it has been shown that in the fully symmetric case, ZF and TIN is always suboptimal in terms of sum GDoF. Interestingly, Theorem 2.5 illustrates that when the* $(1,N)^{N+1}$ $(N \geq 2)$ *SIMO interference channel is asymmetric (i.e., (2.65) indicates that for each user some interfering link is much weaker than others), under certain conditions ZF and TIN achieves the sum GDoF value. This is also noted by [47] for the case of N=2 with cyclically symmetric channel parameters.*

2.5 Summary

In this chapter, for K-user fully asymmetric interference channels, we identify a broad TIN-optimality condition (2.7) under which power control and TIN achieves the entire GDoF region and approaches the whole channel capacity region within a constant gap. In words, the TIN-optimality condition is *for each user, the desired signal strength is no weaker than* *the sum of the strengths of the strongest interference caused by the user and the strongest interference suffered by the user (with all signal strengths measured in dB scale).* To obtain this result, the key is the GDoF characterization. Remarkably, the achievability (i.e., the polyhedral TIN region) is obtained by essentially a Fourier-Motzkin elimination of the power control variables (accomplished by applying the potential theorem in [43]). Note that this approach is particularly useful because while the achievable rate regions of the TIN scheme have been investigated for decades due to their obvious practical significance, the main complication has been the coupling of achievable rates and the transmit powers, which requires joint optimization over both. De-coupling the rate (GDoF) region from power allocation variables allows direct rate optimizations, seemingly ignoring power control variables while in fact automatically optimizing over those as well. Moreover, we fully characterize the achievable TIN region for interference channels with arbitrary channel strengths and establish the duality of the TIN region as a byproduct. We also extend the optimality of TIN to MIMO interference channels where all transmitters and receivers have the same number of antennas. For MIMO networks where transmitters and receivers are with different antenna numbers, we show that there exist non-trivial parameter regimes where ZF and TIN achieves the sum GDoF value. Finally, note that due to the aforementioned Fourier-Motzkin elimination of the power control variables in the achievability proof, it is still unclear how to obtain the optimal power allocation for given GDoF tuples. This GDoF-based power control problem will be discussed in Chapter 4 in details.

Chapter 3

Optimality of TIN for General Message Sets

In Chapter 2, we demonstrate the optimality of TIN for K-user Gaussian interference channels. In this chapter, we extend the result of Chapter 2 and explore the sum-rate optimality of TIN when the message set is expanded to include an independent message from each transmitter to each receiver, i.e., the X message setting [49, 50]. Related previous works on the X setting have primarily focused on the case with 2 transmitters and 2 receivers, i.e., 2×2 X channels [32, 51]. In [32], the authors characterize the sum GDoF for symmetric X channels and identify sufficient conditions for TIN to achieve exact sum capacity in the general asymmetric case. In [51], the authors characterize the capacity for asymmetric X channels within a constant gap subject to an outage set. The main contribution of this chapter is to show that, for the K-user TIN-optimal interference channels identified in Theorem 2.1, even if the message set is increased to the X message setting, operating as the original interference channel and treating interference as noise at each receiver is still optimal for the *sum* capacity up to a constant gap. We also extend the optimality result of TIN to general X channels with arbitrary numbers of transmitters and receivers.

Chapter 3 is organized as follows. In Section 3.1, we describe the system model for X channels. In Section 3.2 and 3.3, we demonstrate the optimality of TIN for $K \times K$ X channels and general X channels with arbitrary numbers of transmitters and receivers, respectively. We summarize this chapter in Section 3.4.

3.1 Channel Model

Consider the wireless channel with M transmitters and N receivers,

$$Y_k(t) = \sum_{i=1}^{M} \tilde{h}_{ki}\tilde{X}_i(t) + Z_k(t), \quad \forall k \in \langle N \rangle \tag{3.1}$$

Similar to the K-user interference channel presented in Section 2.1, in (3.1) $\tilde{h}_{ki}$ is the complex channel gain value from Transmitter i to Receiver k. $\tilde{X}_i(t)$, $Y_k(t)$ and $Z_k(t)$ are the transmitted symbol of Transmitter i, the received signal of Receiver k, and the AWGN with zero mean and unit variance seen by Receiver k, respectively, at each time index t. Again, all the symbols are complex, and Transmitter $i \in \langle M \rangle$ is subject to the power constraint $\mathbb{E}[|\tilde{X}_i(t)|^2] \leq P_i$.

Following similar approaches in Section 2.1, we translate the original channel model (3.1) to the following form,

$$Y_k(t) = \sum_{i=1}^{M} h_{ki}X_i(t) + Z_k(t) = \sum_{i=1}^{M} \sqrt{P^{\alpha_{ki}}}e^{j\theta_{ki}}X_i(t) + Z_k(t),\ \forall k \in \langle N \rangle \tag{3.2}$$

Recall that $X_i(t) = \tilde{X}_i(t)/\sqrt{P_i}$ is the normalized transmit symbol of Transmitter i, subject to the unit power constraint.

In the K-user interference channel where $M = N = K$, each transmitter intends to send one independent message to its corresponding receiver. Because in this chapter we wish to prove the negative result that additional messages do not add to the sum GDoF in the TIN-optimal network identified in Theorem 2.1, the strongest result corresponds to the case where messages from every transmitter to every receiver are included. Therefore, we will only consider the X channel setting in the following. In the general $M \times N$ X channel, Transmitter i intends to send message W_{ki} to Receiver k, and the MN messages are mutually independent, $\forall i \in \langle M \rangle, \forall k \in \langle N \rangle$. The size of the message set $\{W_{ki}\}$ is denoted by $|W_{ki}|$. For codewords spanning n channel uses, the rates $R_{ki} = \frac{\log |W_{ki}|}{n}$ are achievable if the probability of error of all messages can be made arbitrarily small simultaneously by choosing an appropriately large n. The channel capacity region $\mathcal{C}$ is the closure of the set of all achievable rate tuples. The sum channel capacity is defined as

$$C_{\Sigma,X} \triangleq \max \left\{ \sum_{i=1}^{M} \sum_{k=1}^{N} R_{ki} : (R_{11}, R_{12}, ..., R_{NM}) \in \mathcal{C} \right\} \tag{3.3}$$

The GDoF region of the X channel in (3.2) is given by

$$\begin{aligned} \mathcal{D} \triangleq \Big\{ (d_{11}, d_{12}, ..., d_{NM}) : \ d_{ki} = \lim_{P \to \infty} \frac{R_{ki}}{\log P}, \quad \forall i \in \langle M \rangle, \forall k \in \langle N \rangle, \\ (R_{11}, R_{12}, ..., R_{NM}) \in \mathcal{C} \Big\}, \end{aligned} \tag{3.4}$$

and its sum GDoF value is

$$d_{\Sigma,X} \triangleq \max \left\{ \sum_{i=1}^{M} \sum_{k=1}^{N} d_{ki} : (d_{11}, d_{12}, ..., d_{NM}) \in \mathcal{D} \right\} \tag{3.5}$$

3.2 TIN-optimality Condition for $K \times K$ X Channels

The main result of this section is the following theorem.

Theorem 3.1. *In a K-user interference channel, when the following condition is satisfied,*

$$\alpha_{ii} \geq \max_{j:j\neq i}\{\alpha_{ji}\} + \max_{k:k\neq i}\{\alpha_{ik}\}, \quad \forall i,j,k \in \langle K \rangle \tag{3.6}$$

then even if the message set is increased to the X channel setting, operating the new channel as the original interference channel and treating interference as noise at each receiver still achieves the sum GDoF. Furthermore, the same scheme is also optimal for the sum channel capacity up to a constant gap of no more than $K\log[K(K+1)]$ bits.

The above theorem shows that for TIN-optimal interference channels identified in Theorem 2.1, expanding message set does not increase sum GDoF. For this theorem, the achievability argument follows directly from Chapter 2 because it is based only on operating the target network as an interference channel and treating interference as noise. The main difficulty lies in deriving tight information theoretical outer bounds. Recall that for interference channels, the converse is based on reducing the channel to a cyclic network. Each such reduction produces an outer bound and collectively these outer bounds suffice for the GDoF characterization in the setting of interference channels. However, this is no longer true when the message set is expanded. While one can similarly obtain outer bounds on the sum rates of subsets of messages by considering all cyclic subnetworks, it is easy to verify that these bounds do not suffice for our purpose.

Example 3.1. *Consider the* 3*-user TIN-optimal interference channel in Example 2.1, where each transmitter intends to send an independent message to its corresponding receiver. There are* 3 *messages in this setting. It is not hard to verify that the sum GDoF value of this interference channel is*

$$d_{\Sigma,IC} = d_1 + d_2 + d_3 = 2.5$$

For this channel, we expand the message set to the X channel setting as shown in Fig. 3.1,

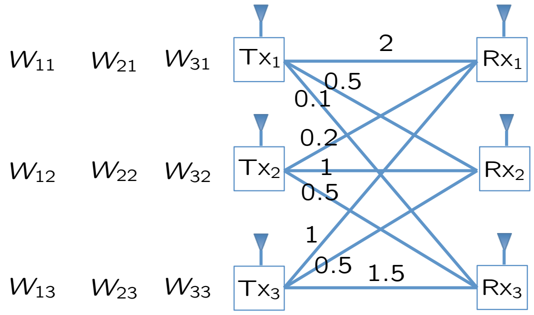

Figure 3.1: A 3×3 X channel with 9 messages

where each transmitter intends to send an independent message to each receiver. There are totally 9 messages in this X channel. Theorem 3.1 claims that for this 3×3 X channel, the sum GDoF value is still

$$d_{\Sigma,X} = \sum_{i=1}^{3} \sum_{k=1}^{3} d_{ki} = 2.5$$

which can be achieved by setting $W_{ki} = \phi$ ($i \neq k$, $\forall i, k \in \{1, 2, 3\}$), sending only $\{W_{11}, W_{22}, W_{33}\}$ through the channel and treating interference as noise at each receiver.

3.2.1 Proof for the GDoF Result

To prove the GDoF results in Theorem 3.1, we go through the following two steps. First, we show that when condition (3.6) is satisfied, for all individual and cycle bounds of the TIN-optimal K-user interference channel (see Theorem 2.1), if each d_i ($\forall i \in \langle K \rangle$) is replaced by $\hat{d}_i = \sum_{j=1}^{K} d_{ij}$, these bounds still hold for its counterpart X channel. Next, based on the first step, we prove that under condition (3.6), the K-user interference channel and its counterpart X channel have the same sum GDoF. Therefore, according to Theorem 2.1, we establish that power control and TIN achieves the sum GDoF of the $K \times K$ X channel when condition (3.6) holds.

Let's start with the first step. For the individual bounds in the K-user TIN-optimal interference channel

$$d_i \leq \alpha_{ii}, \quad \forall i \in \langle K \rangle, \tag{3.7}$$

in its counterpart X channel, the corresponding bound comes from the MAC consisting of all transmitters and Receiver i,

$$\sum_{j=1}^{K} R_{ij} \leq \log(1 + \sum_{j=1}^{K} P^{\alpha_{ij}}) \tag{3.8}$$

According to the condition (3.6), in the GDoF sense we obtain

$$\hat{d}_i = \sum_{j=1}^{K} d_{ij} \leq \alpha_{ii} \tag{3.9}$$

For any cycle bound in the interference channel

$$\sum_{j=1}^{m} d_{i_j} \leq \sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}), \qquad \forall (i_1, i_2, ..., i_m) \in \Pi_K, \quad \forall m \in \{2, 3, ..., K\}, \tag{3.10}$$

consider the subnetwork consisting of all transmitters and Receivers $\{i_1, i_2, ..., i_m\}$. Eliminate all other receivers and their desired messages, which does not hurt the rates of the remaining messages. For such a $K \times m$ X channel, define $\mathcal{W} \triangleq \{W_{i_j i_k}\}$, $\mathcal{W}^*_{i_j} \triangleq \{W_{i_j i_1}, W_{i_j i_2}, ..., W_{i_j i_K}\}$, $\mathcal{W}^\dagger_{i_k} \triangleq \{W_{i_1 i_k}, W_{i_2 i_k}, ..., W_{i_m i_k}\}$, and $\mathcal{W}^c_{\mathcal{S}} \triangleq \mathcal{W} \backslash \mathcal{W}_{\mathcal{S}}$, where $\forall j \in \langle m \rangle$, $\forall k \in \langle K \rangle$, and $\mathcal{S}$ is any subset of message indices. In words, the sets $\mathcal{W}$, $\mathcal{W}^*_{i_j}$, and $\mathcal{W}^\dagger_{i_k}$ represent all the messages delivered in this $K \times m$ X channel, all the messages intended to Receiver i_j, and all the messages coming from Transmitter i_k, respectively, and $\mathcal{W}^c_{\mathcal{S}}$ is the complement of $\mathcal{W}_{\mathcal{S}}$ in $\mathcal{W}$. For example, when $j, k \in \{1, 2\}$ and $\mathcal{S} = \{i_1 i_1, i_1 i_2\}$, then $\mathcal{W}_{\mathcal{S}} = \{W_{i_1 i_1}, W_{i_1 i_2}\}$ and $\mathcal{W}^c_{\mathcal{S}} = \{W_{i_2 i_1}, W_{i_2 i_2}\}$. Modulo-$m$ arithmetic is used on receiver indices, e.g., $i_0 = i_m$. Also define

$$S_{i_j}(t) = h_{i_{j-1}i_j} X_{i_j}(t) + Z_{i_{j-1}}(t), \quad \forall j \in \langle m \rangle \tag{3.11}$$

For Receiver i_1, provide $S_{i_1}^n$, $\mathcal{W}_{i_2i_2}^c \backslash \mathcal{W}_{i_1}^*$ through a genie. From Fano's inequality, we have

$$\begin{aligned}
& n(\sum_{k=1}^{K} R_{i_1i_k} - \epsilon) \\
& \leq I(\mathcal{W}_{i_1}^*; Y_{i_1}^n, S_{i_1}^n, \mathcal{W}_{i_2i_2}^c \backslash \mathcal{W}_{i_1}^*) && (3.12) \\
& = I(\mathcal{W}_{i_1}^*; Y_{i_1}^n, S_{i_1}^n | \mathcal{W}_{i_2i_2}^c \backslash \mathcal{W}_{i_1}^*) && (3.13) \\
& = I(\mathcal{W}_{i_1}^*; S_{i_1}^n | \mathcal{W}_{i_2i_2}^c \backslash \mathcal{W}_{i_1}^*) + I(\mathcal{W}_{i_1}^*; Y_{i_1}^n | S_{i_1}^n, \mathcal{W}_{i_2i_2}^c \backslash \mathcal{W}_{i_1}^*) && (3.14) \\
& = h(S_{i_1}^n | \mathcal{W}_{i_2i_2}^c \backslash \mathcal{W}_{i_1}^*) - h(S_{i_1}^n | \mathcal{W}_{i_2i_2}^c) + h(Y_{i_1}^n | S_{i_1}^n, \mathcal{W}_{i_2i_2}^c \backslash \mathcal{W}_{i_1}^*) - h(Y_{i_1}^n | S_{i_1}^n, \mathcal{W}_{i_2i_2}^c) && (3.15) \\
& \leq h(S_{i_1}^n | \mathcal{W}_{i_1}^\dagger \backslash W_{i_1i_1}) - h(Z_{i_0}^n) + h(Y_{i_1}^n | S_{i_1}^n) - h(S_{i_2}^n | \mathcal{W}_{i_2}^\dagger \backslash W_{i_2i_2}) && (3.16)
\end{aligned}$$

where (3.13) follows because all messages are mutually independent, and in (3.16) we use the fact that dropping conditioning does not reduce entropy.

Similarly, for Receivers i_j, $\forall j \in \{2, 3, ..., m-1\}$, provide $S_{i_j}^n$, $\mathcal{W}_{i_{j+1}i_{j+1}}^c \backslash \mathcal{W}_{i_j}^*$ through a genie. We get

$$n(\sum_{k=1}^{K} R_{i_ji_k} - \epsilon) \leq h(S_{i_j}^n | \mathcal{W}_{i_j}^\dagger \backslash W_{i_ji_j}) - h(Z_{i_{j-1}}^n) + h(Y_{i_j}^n | S_{i_j}^n) - h(S_{i_{j+1}}^n | \mathcal{W}_{i_{j+1}}^\dagger \backslash W_{i_{j+1}i_{j+1}}) \tag{3.17}$$

Finally, for Receiver i_m, we provide $S_{i_m}^n$, $\mathcal{W}_{i_1i_1}^c \backslash \mathcal{W}_{i_m}^*$ through a genie and obtain

$$n(\sum_{k=1}^{K} R_{i_mi_k} - \epsilon) \leq h(S_{i_m}^n | \mathcal{W}_{i_m}^\dagger \backslash W_{i_mi_m}) - h(Z_{i_{m-1}}^n) + h(Y_{i_m}^n | S_{i_m}^n) - h(S_{i_1}^n | \mathcal{W}_{i_1}^\dagger \backslash W_{i_1i_1}) \tag{3.18}$$

Taking the sum of $n(\sum_{k=1}^{K} R_{i_j i_k} - \epsilon)$ for all $j \in \langle m \rangle$, we have

$$n(\sum_{j=1}^{m}\sum_{k=1}^{K} R_{i_j i_k} - m\epsilon) \leq \sum_{j=1}^{m}[h(Y_{i_j}^n|S_{i_j}^n) - h(Z_{i_j}^n)] \tag{3.19}$$

$$\leq \sum_{t=1}^{n}\sum_{j=1}^{m}[h(Y_{i_j}(t)|S_{i_j}(t)) - h(Z_{i_j}(t))] \tag{3.20}$$

where (3.20) follows the chain rule and the fact that dropping conditioning does not reduce entropy. Using the fact that the circularly symmetric complex Gaussian distribution maximizes conditional differential entropy for a given covariance constraint and the condition (3.6), we can obtain the following desired outer bound in the GDoF sense after some simple manipulations,

$$\sum_{j=1}^{m} \hat{d}_{i_j} = \sum_{j=1}^{m}\sum_{k=1}^{K} d_{i_j i_k} \leq \sum_{j=1}^{m}(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}) \tag{3.21}$$

Now we proceed to the second step to prove that under condition (3.6), the K-user interference channel and its counterpart $K \times K$ X channel have the same sum GDoF. According to Theorem 2.1, for the K-user interference channel, under condition (3.6), to obtain its sum GDoF value $d_{\Sigma,IC}$, we need to solve the following linear programming (LP) problem

$$\max \sum_{i=1}^{K} d_i \tag{3.22}$$

$$\text{s.t.} \quad 0 \leq d_i \leq \alpha_{ii} \qquad \forall i \in \langle K \rangle \tag{3.23}$$

$$\sum_{j=1}^{m} d_{i_j} \leq \sum_{j=1}^{m}(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}), \qquad \forall (i_1, i_2, ..., i_m) \in \Pi_K, \quad \forall m \in \{2, 3, ..., K\} \tag{3.24}$$

To get the sum GDoF of its counterpart X channel $d_{\Sigma,X}$, we need to consider a similar LP problem. For this LP problem with the objective function $\sum_{i=1}^{K} \hat{d}_i$, from the first step, we know that at least it needs to follow two similar constraints to (3.23) and (3.24), where each d_i is replaced by $\hat{d}_i$. Thus we have $d_{\Sigma,IC} \geq d_{\Sigma,X}$. Obviously, the sum GDoF of the

K-user interference channel must be no larger than that of its counterpart X channel, i.e., $d_{\Sigma,IC} \leq d_{\Sigma,X}$. Therefore, we establish that under condition (3.6), the K-user interference channel and its counterpart X channel have the same sum GDoF and complete the proof.

3.2.2 Proof for the Constant Gap Result

Based on the insight obtained in the proof of the GDoF results, for TIN-optimal $K \times K$ X channels, we can further characterize the sum channel capacity to within a constant gap of no larger than $K \log[K(K+1)]$ bits. The achievability is the same as that of Theorem 2.2. By operating the $K \times K$ X channel as an interference channel, where Transmitter i sends one independent message W_i to Receiver i ($\forall i \in \langle K \rangle$), power control and TIN achieves the rate tuples $(R_{1,\text{TIN}}, R_{2,\text{TIN}}, ..., R_{K,\text{TIN}})$ satisfying

$$0 \leq R_{i,\text{TIN}} \leq \max\left\{0, \alpha_{ii} \log P + \log(\frac{1}{K})\right\}, \quad \forall i \in \langle K \rangle, \tag{3.25}$$

$$\sum_{j=1}^{m} R_{i_j,\text{TIN}} \leq \max\left\{0, \sum_{j=1}^{m} [(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}) \log P + \log(\frac{1}{K})]\right\}, \tag{3.26}$$

for all cyclic sequences $(i_1, i_2, ..., i_m) \in \Pi_K$, $\forall m \in \{2, 3, ..., K\}$.

Next, consider the converse. Start with the individual bounds,

$$\hat{R}_i = \sum_{j=1}^{K} R_{ij} \leq \log(1 + \sum_{j=1}^{K} P^{\alpha_{ij}}) \leq \log[(K+1)P^{\alpha_{ii}}] \tag{3.27}$$

$$= \alpha_{ii} \log P + \log(K+1) \tag{3.28}$$

For the cycle bounds, from (3.20) it is easy to obtain

$$\sum_{j=1}^{m} \hat{R}_{i_j} \leq \sum_{j=1}^{m} \log\left[\frac{(K+1)P^{\alpha_{i_j i_j}}}{P^{\alpha_{i_{j-1} i_j}}}\right]$$

$$= \sum_{j=1}^{m} \left[(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}) \log P + \log(K+1) \right] \tag{3.29}$$

for all cyclic sequences $(i_1, i_2, ..., i_m) \in \Pi_K$, $\forall m \in \{2, 3, ..., K\}$.

Denote the difference between the achievable rate in (3.25) and the outer bound in (3.28) as $\delta_{\hat{R}_i}$. Compare (3.25) with (3.28). We have the following two cases.

- $\alpha_{ii} \log P + \log(\frac{1}{K}) > 0$: In this case, we obtain

$$\begin{aligned}
\delta_{\hat{R}_i} &= [\alpha_{ii} \log P + \log(K+1)] - \max\left\{0, \alpha_{ii} \log P + \log(\frac{1}{K})\right\} \\
&= [\alpha_{ii} \log P + \log(K+1)] - \left[\alpha_{ii} \log P + \log(\frac{1}{K})\right] \\
&= \log[K(K+1)]
\end{aligned}$$

- $\alpha_{ii} \log P + \log(\frac{1}{K}) \leq 0$: In this case, we get

$$\begin{aligned}
\delta_{\hat{R}_i} &= [\alpha_{ii} \log P + \log(K+1)] - \max\left\{0, \alpha_{ii} \log P + \log(\frac{1}{K})\right\} \\
&= \alpha_{ii} \log P + \log(K+1) \\
&\leq \log K + \log(K+1) \\
&= \log[K(K+1)]
\end{aligned}$$

In both cases, we have

$$\delta_{\hat{R}_i} \leq \log[K(K+1)], \quad \forall i \in \langle K \rangle \tag{3.30}$$

Similarly, denote the difference between the achievable rate in (3.26) and the outer bound in (3.29) as $\delta_{\sum_{j=1}^m \hat{R}_{i_j}}$. Comparing (3.26) with (3.29), we always have

$$\delta_{\sum_{j=1}^m \hat{R}_{i_j}} \leq m \log[K(K+1)], \quad \forall (i_1, i_2, ..., i_m) \in \Pi_K, \forall m \in \{2, 3, ..., K\} \tag{3.31}$$

Therefore, we characterize the sum channel capacity to within a constant gap of no more than $K \log[K(K+1)]$ bits.

3.3 TIN-optimality Condition for General $M \times N$ X Channels

While the K-user interference channel is naturally associated with a $K \times K$ X channel, the X channel setting allows for unequal numbers of transmitters and receivers. One may wonder whether a generalization of the TIN-optimality result is possible for $M \times N$ X channels where $M \neq N$. The following theorem provides such a generalization.

Theorem 3.2. *In an $M \times N$ X channel, if there exist two permutations Π^T and Π^R for the transmitter and receiver indices, respectively, such that*

$$\alpha_{\Pi_i^R \Pi_i^T} \geq \max_{j:j \neq i}\{\alpha_{\Pi_j^R \Pi_i^T}\} + \max_{k:k \neq i}\{\alpha_{\Pi_i^R \Pi_k^T}\}, \quad \forall i \in \langle \kappa \rangle, \forall j \in \langle N \rangle, \forall k \in \langle M \rangle, \tag{3.32}$$

where $\kappa \triangleq \min\{M, N\}$, Π_i^T (Π_i^R) denotes the i-th element in the permutation of transmitters (receivers) Π^T (Π^R), then operating the channel as a κ-user interference channel, where Transmitter Π_i^T intends to deliver an independent message to Receiver Π_i^R, $\forall i \in \langle \kappa \rangle$, and treating interference as noise at each receiver achieves the sum GDoF.

Example 3.2. *In the 3-user TIN-optimal interference channel in Example 2.1, add another transmitter (i.e., Transmitter 4) as depicted in Fig. 3.2. The value on each link denotes its channel strength level. Consider the X message setting. In this 4×3 X channel, the*

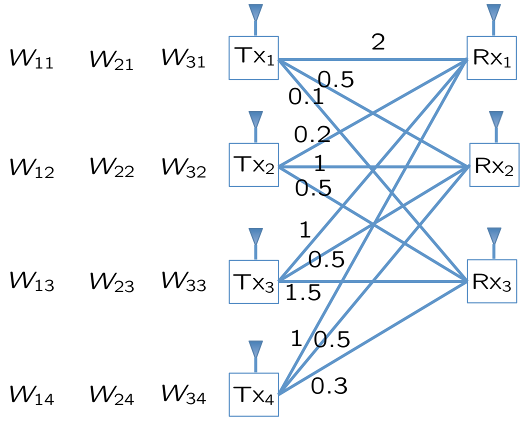

Figure 3.2: A 4×3 X channel with 12 messages

number of the messages increases to 12. *It's easy to verify that the condition (3.32) holds. According to Theorem 3.2, for this X channel and its reciprocal channel, the TIN scheme used in Example 3.1 is optimal to achieve the sum GDoF value, which remains as* 2.5.

3.3.1 Proof Sketch of Theorem 3.2

In this section, we sketch the proof for Theorem 3.2. Without loss of generality, we assume that the two permutations Π^T and Π^R satisfying the condition (3.32) are $\Pi^T = \langle M \rangle$ and $\Pi^R = \langle N \rangle$. In other words, the following condition is satisfied,

$$\alpha_{ii} \geq \max_{j:j \neq i}\{\alpha_{ji}\} + \max_{k:k \neq i}\{\alpha_{ik}\}, \quad \forall i \in \langle \kappa \rangle, \forall j \in \langle N \rangle, \forall k \in \langle M \rangle \tag{3.33}$$

It is not hard to verify that when $M \geq N$, by defining $\hat{d}_i = \sum_{j=1}^{M} d_{ij}$ ($\forall i \in \langle N \rangle$) and following the same argument as in the proof of Theorem 3.1, we can complete the proof. Therefore, in the following we only consider the case where $\kappa = M < N$.

When $\kappa = M < N$, the key step is to show that when (3.33) is satisfied, for each individual bound and cycle bound in the M-user interference channel consisting of transmitters $\langle M \rangle$

and receivers $\langle M \rangle$, if each d_i is replaced by $\bar{d}_i = \sum_{j=1}^{N} d_{ji}$, the resulting bounds still hold in the $M \times N$ X channel. Then based on the same argument of Theorem 3.1, we can complete the proof.

For the individual bounds, consider the degraded BC comprised of Transmitter $i \in \langle M \rangle$ and all receivers, eliminating all the other transmitters and their associated messages. According to (3.33), Receiver i is the strongest receiver, which can decode all the messages coming from Transmitter i. Thus in the GDoF sense, we have

$$\bar{d}_i = \sum_{j=1}^{N} d_{ji} \leq \alpha_{ii}, \quad \forall i \in \langle M \rangle \tag{3.34}$$

The challenge comes from recovering the cycle bounds in the X setting, i.e., after replacing d_i with $\bar{d}_i$, all the cycle bounds still hold. To help understand the main idea of the proof, in the sequel we present an intuitive proof sketch for the following cycle bound in a 2×4 X channel where (3.33) is satisfied,

$$\bar{d}_1 + \bar{d}_2 \leq (\alpha_{11} + \alpha_{22}) - (\alpha_{12} + \alpha_{21}) \tag{3.35}$$

An intuitive sketch of proof for the cycle bound (3.35): In the 2×4 X channel, assume that $\alpha_{31} \geq \alpha_{41}$ and $\alpha_{42} \geq \alpha_{32}$. The proof for all the other cases follows similarly. Define the message set $\widetilde{\mathcal{W}} \triangleq \{W_{ki}\}$, $\forall i \in \{1, 2\}$, $\forall k \in \{1, 2, 3, 4\}$. Also define

$$S_1(t) = h_{21} X_1(t) + Z_2(t)$$
$$S_2(t) = h_{12} X_2(t) + Z_1(t)$$

For Receiver 1, provide S_1^n and W_{21} through a genie. From Fano's inequality, we get

$n(R_{11} + R_{12} - \epsilon)$

$$\leq I(W_{11}, W_{12}; Y_1^n, S_1^n, W_{21}) \tag{3.36}$$

$$= I(W_{11}, W_{12}; Y_1^n, S_1^n | W_{21}) \tag{3.37}$$

$$= h(Y_1^n, S_1^n | W_{21}) - h(Y_1^n, S_1^n | W_{21}, W_{11}, W_{12}) \tag{3.38}$$

$$= h(S_1^n | W_{21}) + h(Y_1^n | S_1^n, W_{21}) - h(Y_1^n | W_{21}, W_{11}, W_{12}) - h(S_1^n | Y_1^n, W_{21}, W_{11}, W_{12}) \tag{3.39}$$

$$\begin{aligned} &\leq h(S_1^n | W_{21}) + h(Y_1^n | S_1^n) - h(Y_1^n | W_{21}, W_{11}, W_{12}) \\ &\quad - h(S_1^n | Y_1^n, W_{21}, W_{11}, W_{12}, W_{31}, W_{41}) \end{aligned} \tag{3.40}$$

$$\begin{aligned} &= h(S_1^n | W_{21}) + h(Y_1^n | S_1^n) - h(Y_1^n | W_{21}, W_{11}, W_{12}) \\ &\quad - h(S_1^n | Y_1^n, W_{21}, W_{11}, W_{12}, W_{31}, W_{41}, X_1^n) \end{aligned} \tag{3.41}$$

$$= h(S_1^n | W_{21}) + h(Y_1^n | S_1^n) - h(Y_1^n | W_{21}, W_{11}, W_{12}) - h(Z_2^n) \tag{3.42}$$

$$\leq h(S_1^n | W_{21}) + h(Y_1^n | S_1^n) - h(Y_3^n | \widetilde{\mathcal{W}}^c_{\{31,41\}}) - h(S_2^n | W_{12}) - h(Z_2^n) - n\ o(\log(P)) \tag{3.43}$$

where $\widetilde{\mathcal{W}}^c_{\{31,41\}}$ denotes the complement of $\{W_{31}, W_{41}\}$ in $\widetilde{\mathcal{W}}$, (3.37) holds since W_{21} is independent of W_{11} and W_{12}, (3.40) follows that dropping conditioning (in the second term) does not reduce entropy and adding conditioning (in the last term) does not increase entropy, and (3.41) holds since we can reconstruct X_1^n from W_{11}, W_{21}, W_{31} and W_{41}. *The last inequality in (3.43) is the key step of the proof.* Intuitively, it is due to the fact that out of the $\alpha_{11}\log(P)$ bit levels of Y_1 that are above the noise floor, S_2 is contained in the lowest $\alpha_{12}\log(P)$ bit levels of Y_1, whereas only the top $\alpha_{31}\log(P)$ bit levels are seen by Receiver 3. Since $\alpha_{11} \geq \alpha_{12} + \alpha_{31}$, these bit levels do not overlap, i.e., they can be recovered from Y_1 within a bounded entropy gap. As shown later in Section 3.3.2 and 3.3.3, this argument becomes evident in a deterministic approach.

The following proof is straightforward. Consider the degraded BC comprised of Transmitter 1 and Receiver 3 and 4. Since $\alpha_{31} \geq \alpha_{41}$, we obtain

$$n(R_{31} + R_{41} - \epsilon) \leq I(W_{31}, W_{41}; Y_3^n | \widetilde{\mathcal{W}}^c_{\{31,41\}}) = h(Y_3^n | \widetilde{\mathcal{W}}^c_{\{31,41\}}) - h(Z_3^n) \tag{3.44}$$

Adding (3.43) and (3.44) together, we have

$$n(R_{11} + R_{12} + R_{31} + R_{41} - 2\epsilon) \le h(S_1^n|W_{21}) + h(Y_1^n|S_1^n) - h(S_2^n|W_{12}) - n\ o(\log(P)) \tag{3.45}$$

Similarly, we have

$$n(R_{21} + R_{22} + R_{32} + R_{42} - 2\epsilon) \le h(S_2^n|W_{12}) + h(Y_2^n|S_2^n) - h(S_1^n|W_{21}) - n\ o(\log(P)) \tag{3.46}$$

Finally, combining (3.45) and (3.46) together, we establish the desired outer bound,

$$\begin{aligned} n(R_\Sigma - 4\epsilon) &\le h(Y_1^n|S_1^n) + h(Y_2^n|S_2^n) - n\ o(\log(P)) \\ \Rightarrow \bar{d}_1 + \bar{d}_2 &\le (\alpha_{11} + \alpha_{22}) - (\alpha_{12} + \alpha_{21}) \end{aligned}$$

In the sequel, in order to make the intuitive justification of the key step (3.43) rigorous, we adopt a *deterministic* approach [32, 51–56]. In Section 3.3.2, we first present a deterministic channel model which is mainly inspired by the Avestimehr-Diggavi-Tse (ADT) linear deterministic model and the truncated deterministic model in [53]. Next, we show that the sum capacity of the original complex Gaussian X channel is upper bounded by the sum capacity of this deterministic channel up to a constant gap. In Section 3.3.3, by upper bounding this deterministic channel, we obtain the desired converse for the original Gaussian case. Such a deterministic approach has been shown instrumental to provide approximate capacity characterization for various Gaussian networks, e.g., 2-user interference channels [54], K-user interference channels [55], and $K \times K \times K$ interference networks [56].

3.3.2 Deterministic Channel Model

In the original complex Gaussian $M \times N$ X channel, denote

$$X_k(t) = X_k^R(t) + jX_k^I(t)$$
$$h_{ik} = \sqrt{P^{\alpha_{ik}}}e^{j\theta_{ik}} = h_{ik}^R + jh_{ik}^I$$

The channel input-output relationship can be rewritten as

$$\begin{aligned} Y_i(t) &= \sum_{k=1}^{M} h_{ik}X_k(t) + Z_i(t) && (3.47)\\ &= \sum_{k=1}^{M} \Big[\big(h_{ik}^R X_k^R(t) - h_{ik}^I X_k^I(t)\big) + j\big(h_{ik}^I X_k^R(t) + h_{ik}^R X_k^I(t)\big)\Big] + Z_i(t), \quad \forall i \in \langle N\rangle && (3.48)\end{aligned}$$

where $\mathbb{E}[|X_i(t)|^2] \leq 1$ and $Z_i(t) \sim \mathcal{CN}(0,1)$. To facilitate the deterministic approach, by scaling the output, we may set

$$\mathbb{E}[|X_i(t)|^2] \leq 2,\ Z_i(t) \sim \mathcal{CN}(0,2),$$

which does not affect the channel capacity of the Gaussian model (3.48).

In this dissertation, we consider the following deterministic model,

$$\begin{aligned} \hat{Y}_i(t) = \sum_{k=1}^{M} \Big[&\Big(\lfloor \mathrm{sign}(\bar{X}_k^R(t)) h_{ik}^R \sum_{b=1}^{m_{ik}^R} \bar{X}_{k,b}^R(t)2^{-b}\rfloor - \lfloor \mathrm{sign}(\bar{X}_k^I(t)) h_{ik}^I \sum_{b=1}^{m_{ik}^I} \bar{X}_{k,b}^I(t)2^{-b}\rfloor\Big) \\ &+ j\Big(\lfloor \mathrm{sign}(\bar{X}_k^R(t)) h_{ik}^I \sum_{b=1}^{m_{ik}^I} \bar{X}_{k,b}^R(t)2^{-b}\rfloor + \lfloor \mathrm{sign}(\bar{X}_k^I(t)) h_{ik}^R \sum_{b=1}^{m_{ik}^R} \bar{X}_{k,b}^I(t)2^{-b}\rfloor\Big)\Big], \quad \forall i \in \langle N\rangle \end{aligned} \tag{3.49}$$

where $\lfloor x \rfloor$ is the truncation function *which maps x to its integer part*, $m_{ik}^R \triangleq \lfloor \log |h_{ik}^R| \rfloor$, $m_{ik}^I \triangleq \lfloor \log |h_{ik}^I| \rfloor$, the real and imaginary parts of the input signal $\bar{X}_i(t) = \bar{X}_i^R(t) + j\bar{X}_i^I(t)$ both satisfy the unit *peak* power constraint, and $\bar{X}_{i,b}^R(t)$ ($\bar{X}_{i,b}^I(t)$) is the b-th bit in the fractional part of $|\bar{X}_i^R(t)|$ ($|\bar{X}_{i,b}^I(t)|$) in the binary expansion.[1] In the following, to distinguish the deterministic model in (3.49) from others, we call it the *truncated binary-expansion deterministic model*. The following lemma shows that the sum capacity of the Gaussian X channel in (3.48) is upper bounded by that of the truncated binary-expansion deterministic model in (3.49) up to a constant gap.

Lemma 3.1. *The sum capacity of the complex Gaussian $M \times N$ X channel is upper bounded by the sum capacity of its corresponding truncated binary-expansion deterministic model up to a constant gap.*

Proof of Lemma 3.1: The proof mainly follows from [54]. We start with a general complex Gaussian $M \times N$ X channel, and convert it to the corresponding truncated binary-expansion deterministic channel step-by-step. In each step, we show that only a loss of constant bits is introduced. For the sake of simplicity, define $\mathcal{W}_i^\star \triangleq \{W_{i1}, W_{i2}, ..., W_{iM}\}$. The time index t is suppressed if no confusion would be caused.

- Step 1: **Average power constraint to peak power constraint.** Recall that in the original complex Gaussian channels, we scale the output and set

$$\mathbb{E}[|X_i(t)|^2] \leq 2, \;\; Z_i(t) \sim \mathcal{CN}(0, 2)$$

For each input $X_i = X_i^R + jX_i^I$, we truncate both the real and imaginary parts to satisfy the peak power constraint of 1. Define the part of input X_i^R that exceeds the

[1]We can write the real-valued signal $|\bar{X}_i^R|$ ($|\bar{X}_i^R| \leq 1$) in terms of its binary expansion as

$$|\bar{X}_i^R| = \sum_{b=1}^{\infty} \bar{X}_{i,b}^R 2^{-b} = 0.\bar{X}_{i,1}^R \bar{X}_{i,2}^R \bar{X}_{i,3}^R ...$$

unit peak power constraint as

$$\tilde{X}_i^R = \lfloor X_i^R \rfloor = \text{sign}(X_i^R) \sum_{b=-\infty}^{0} X_{i,b}^R 2^{-b}$$

and the remaining signal as

$$\bar{X}_i^R = X_i^R - \tilde{X}_i^R = \text{sign}(X_i^R) \sum_{b=1}^{\infty} X_{i,b}^R 2^{-b}$$

For the imaginary part of the input, we have the similar definitions for X_i^I with I replacing R. So $\bar{X}_i^R$ and $\bar{X}_i^I$ satisfy the unit peak power constraint. Also define

$$\bar{X}_i = \bar{X}_i^R + j\bar{X}_i^I \tag{3.50}$$

$$\tilde{X}_i = \tilde{X}_i^R + j\tilde{X}_i^I \tag{3.51}$$

Let $\bar{Y}_i$ be the channel output at Receiver i due to the truncated input $\bar{X}_i$, and $\tilde{Y}_i$ be the difference between Y_i and $\bar{Y}_i$. For Receiver $i \in \langle N \rangle$, we have

$$I(\mathcal{W}_i^\star; Y_i^n) \leq I(\mathcal{W}_i^\star; \bar{Y}_i^n, \tilde{Y}_i^n) \tag{3.52}$$

$$= I(\mathcal{W}_i^\star; \bar{Y}_i^n) + I(\mathcal{W}_i^\star; \tilde{Y}_i^n | \bar{Y}_i^n) \tag{3.53}$$

$$\leq I(\mathcal{W}_i^\star; \bar{Y}_i^n) + H(\tilde{Y}_i^n) \tag{3.54}$$

$$\leq I(\mathcal{W}_i^\star; \bar{Y}_i^n) + \sum_{k=1}^{M} H(\tilde{X}_k^n) \tag{3.55}$$

$$\leq I(\mathcal{W}_i^\star; \bar{Y}_i^n) + n \times \text{constant} \tag{3.56}$$

where the last inequality follows from Lemma 6 in [54].

- Step 2: **Truncate signals at noise level and remove noise.** Recall $\lfloor \log |h_{ik}^R| \rfloor = m_{ik}^R$

and $\lfloor \log |h_{ik}^I| \rfloor = m_{ik}^I$. We have

$$\hat{Y}_i = \sum_{k=1}^{M} \Big[(\lfloor \text{sign}(X_k^R) h_{ik}^R \sum_{b=1}^{m_{ik}^R} X_{k,b}^R 2^{-b} \rfloor - \lfloor \text{sign}(X_k^I) h_{ik}^I \sum_{b=1}^{m_{ik}^I} X_{k,b}^I 2^{-b} \rfloor) + j(\lfloor \text{sign}(X_k^R) h_{ik}^I \sum_{b=1}^{m_{ik}^I} X_{k,b}^R 2^{-b} \rfloor + \lfloor \text{sign}(X_k^I) h_{ik}^R \sum_{b=1}^{m_{ik}^R} X_{k,b}^I 2^{-b} \rfloor) \Big] \tag{3.57}$$

Next, let ε_i be the difference between $\bar{Y}_i$ and $\hat{Y}_i$, i.e.,

$$\begin{aligned} \varepsilon_i =& \bar{Y}_i - \hat{Y}_i \\ =& \sum_{k=1}^{M} \Bigg\{ \Big[\text{sign}(X_k^R) h_{ik}^R \sum_{b=m_{ik}^R+1}^{\infty} X_{k,b}^R 2^{-b} - \text{sign}(X_k^I) h_{ik}^I \sum_{b=m_{ik}^I+1}^{\infty} X_{k,b}^I 2^{-b} \\ &+ \text{frac}(\text{sign}(X_k^R) h_{ik}^R \sum_{b=1}^{m_{ik}^R} X_{k,b}^R 2^{-b}) - \text{frac}(\text{sign}(X_k^I) h_{ik}^I \sum_{b=1}^{m_{ik}^I} X_{k,b}^I 2^{-b}) \Big] \\ &+ j \Big[\text{sign}(X_k^R) h_{ik}^I \sum_{b=m_{ik}^I+1}^{\infty} X_{k,b}^R 2^{-b} + \text{sign}(X_k^I) h_{ik}^R \sum_{b=m_{ik}^R+1}^{\infty} X_{k,b}^I 2^{-b} \\ &+ \text{frac}(\text{sign}(X_k^R) h_{ik}^I \sum_{b=1}^{m_{ik}^I} X_{k,b}^R 2^{-b}) + \text{frac}(\text{sign}(X_k^I) h_{ik}^R \sum_{b=1}^{m_{ik}^R} X_{k,b}^I 2^{-b}) \Big] \Bigg\} + Z_i \\ =& \sum_{k=1}^{M} \hat{X}_k + Z_i \end{aligned} \tag{3.58}$$

where frac(x) denotes the fractional part of x. Also note

$$|h_{ik}^R \sum_{b=m_{ik}^R+1}^{\infty} X_{k,b}^R 2^{-b}| \leq 2^{m_{ik}^R+1} 2^{-(m_{ik}^R)} = 2$$

Similarly, we have

$$|h_{ik}^I \sum_{b=m_{ik}^I+1}^{\infty} X_{k,b}^I 2^{-b}| \leq 2$$

$$|h_{ik}^I \sum_{b=m_{ik}^I+1}^{\infty} X_{k,b}^R 2^{-b}| \leq 2$$

$$|h_{ik}^R \sum_{b=m_{ik}^R+1}^{\infty} X_{k,b}^I 2^{-b}| \leq 2$$

Finally, we obtain

$$\begin{aligned}
&I(\mathcal{W}_i^\star;\bar{Y}_i^n) \\
&\leq I(\mathcal{W}_i^\star;\hat{Y}_i^n,\varepsilon_i^n) && (3.59)\\
&= I(\mathcal{W}_i^\star;\hat{Y}_i^n) + I(\mathcal{W}_i^\star;\varepsilon_i^n|\hat{Y}_i^n) && (3.60)\\
&= I(\mathcal{W}_i^\star;\hat{Y}_i^n) + h(\varepsilon_i^n|\hat{Y}_i^n) - h(\varepsilon_i^n|\hat{Y}_i^n,\mathcal{W}_i^\star) && (3.61)\\
&\leq I(\mathcal{W}_i^\star;\hat{Y}_i^n) + h(\varepsilon_i^n) - h(Z_i^n) && (3.62)\\
&= I(\mathcal{W}_i^\star;\hat{Y}_i^n) + I(\hat{X}_1^n,\hat{X}_2^n,...,\hat{X}_M^n;\varepsilon_i^n) && (3.63)\\
&\leq I(\mathcal{W}_i^\star;\hat{Y}_i^n) + n\times \text{constant} && (3.64)
\end{aligned}$$

where the last inequality holds since $\hat{X}_1,\hat{X}_2,...,\hat{X}_M \mapsto \varepsilon_i$ forms a complex Gaussian MAC with a finite SNR independent of P for each transmitter [54]. ■

3.3.3 Cycle Bound Proof Based on a Deterministic Approach

Define $m_{ij} \triangleq \lfloor \frac{1}{2}\log P^{\alpha_{ij}} \rfloor$. Note that when $\alpha_{ii} \geq \alpha_{ij}+\alpha_{ki}$, $\forall i \notin \{j,k\}$, since $P>1$, we have

$$\begin{aligned}
&\lfloor \frac{\alpha_{ii}}{2}\log P \rfloor \geq \lfloor \frac{(\alpha_{ij}+\alpha_{ki})}{2}\log P \rfloor && (3.65)\\
\Rightarrow &\lfloor \frac{\alpha_{ii}}{2}\log P \rfloor \geq \lfloor \frac{\alpha_{ij}}{2}\log P \rfloor + \lfloor \frac{\alpha_{ki}}{2}\log P \rfloor && (3.66)\\
\Rightarrow &m_{ii} \geq m_{ij}+m_{ki} \quad \forall i,j,k, \quad i\notin\{j,k\} && (3.67)
\end{aligned}$$

In the following, in order to convey the key ingredients of the deterministic approach, we first consider the *real* Gaussian 2×4 X channel as an example, where the condition (3.33) is satisfied. For the 2×4 X channel, we intend to prove the following cycle bound

$$\bar{d}_1 + \bar{d}_2 \leq \frac{1}{2}[(\alpha_{11} + \alpha_{22}) - (\alpha_{12} + \alpha_{21})]$$

where the factor $\frac{1}{2}$ is due to the fact that the Gaussian X channel is real-valued. Here still assume that $\alpha_{31} \geq \alpha_{41}$ and $\alpha_{42} \geq \alpha_{32}$. As previously mentioned, the proof for all the other cases follows similarly. Again, define $\widetilde{\mathcal{W}} \triangleq \{W_{ki}\}$, $\forall i \in \{1,2\}$, $\forall k \in \{1,2,3,4\}$.

Start with the corresponding truncated binary-expansion deterministic model. Define

$$\hat{S}_1(t) = \lfloor \text{sign}(\bar{X}_1(t)) h_{21} \sum_{b=1}^{m_{21}} \bar{X}_{1,b}(t) 2^{-b} \rfloor \tag{3.68}$$

$$\hat{S}_2(t) = \lfloor \text{sign}(\bar{X}_2(t)) h_{12} \sum_{b=1}^{m_{12}} \bar{X}_{2,b}(t) 2^{-b} \rfloor \tag{3.69}$$

Also define

$$\bar{X}_{31,S}(t) = \text{sign}(\bar{X}_1(t)) \sum_{b=1}^{m_{31}} \bar{X}_{1,b}(t) 2^{-b} \tag{3.70}$$

The channel output at Receiver 1 can be written as

$$\hat{Y}_1(t) = \lfloor \text{sign}(\bar{X}_1(t)) h_{11} \sum_{b=1}^{m_{11}} \bar{X}_{1,b}(t) 2^{-b} \rfloor + \lfloor \text{sign}(\bar{X}_2(t)) h_{12} \sum_{b=1}^{m_{12}} \bar{X}_{2,b}(t) 2^{-b} \rfloor \tag{3.71}$$

$$= \lfloor \text{sign}(\bar{X}_1(t)) h_{11} \sum_{b=1}^{m_{31}} \bar{X}_{1,b}(t) 2^{-b} \rfloor + \lfloor \text{sign}(\bar{X}_1(t)) h_{11} \sum_{b=m_{31}+1}^{m_{11}} \bar{X}_{1,b}(t) 2^{-b} \rfloor + \hat{S}_2(t) + \hat{C}_1(t) \tag{3.72}$$

$$= \underbrace{\lfloor h_{11} \bar{X}_{31,S}(t) \rfloor}_{\hat{Y}_{1,u}(t)} + \underbrace{\lfloor \text{sign}(\bar{X}_1(t)) h_{11} \sum_{b=m_{31}+1}^{m_{11}} \bar{X}_{1,b}(t) 2^{-b} \rfloor + \hat{S}_2(t)}_{\hat{Y}_{1,l}(t)} + \hat{C}_1(t) \tag{3.73}$$

where $\hat{C}_1(t)$ may take a value from $\{-1, 0, 1\}$.

For Receiver 1, we have

$$
\begin{aligned}
&n(R_{11}+R_{12}-\epsilon) \\
&\le I(W_{11},W_{12};\hat{Y}^n_{1,u},\hat{Y}^n_{1,l},\hat{C}^n_1,\hat{S}^n_1|W_{21}) && (3.74)\\
&= H(\hat{Y}^n_{1,u},\hat{Y}^n_{1,l},\hat{C}^n_1,\hat{S}^n_1|W_{21}) - H(\hat{Y}^n_{1,u},\hat{Y}^n_{1,l},\hat{C}^n_1,\hat{S}^n_1|W_{21},W_{11},W_{12}) && (3.75)\\
&= H(\hat{S}^n_1|W_{21}) + H(\hat{Y}^n_{1,u},\hat{Y}^n_{1,l},\hat{C}^n_1|\hat{S}^n_1,W_{21}) - H(\hat{Y}^n_{1,u},\hat{Y}^n_{1,l},\hat{C}^n_1|W_{21},W_{11},W_{12}) \\
&\quad - H(S^n_1|\hat{Y}^n_{1,u},\hat{Y}^n_{1,l},\hat{C}^n_1,W_{21},W_{11},W_{12}) && (3.76)\\
&\le H(\hat{S}^n_1|W_{21}) + H(\hat{Y}^n_{1,u},\hat{Y}^n_{1,l},\hat{C}^n_1,|\hat{S}^n_1) - H(\hat{Y}^n_{1,u},\hat{Y}^n_{1,l},\hat{C}^n_1,|W_{21},W_{11},W_{12}) && (3.77)
\end{aligned}
$$

where (3.77) follows that dropping conditioning does not reduce entropy. Now consider the last term in (3.77).

$$
\begin{aligned}
&H(\hat{Y}^n_{1,u},\hat{Y}^n_{1,l},\hat{C}^n_1|W_{21},W_{11},W_{12}) && (3.78)\\
&= H(\hat{Y}^n_{1,u}|W_{21},W_{11},W_{12}) + H(\hat{Y}^n_{1,l},\hat{C}^n_1|\hat{Y}^n_{1,u},W_{21},W_{11},W_{12}) && (3.79)\\
&= H(\bar{X}^n_{31,S}|W_{21},W_{11},W_{12}) + H(\hat{Y}^n_{1,l},\hat{C}^n_1|\hat{Y}^n_{1,u},W_{21},W_{11},W_{12}) && (3.80)\\
&\ge H(\bar{X}^n_{31,S}|\widetilde{\mathcal{W}}^c_{\{31,41\}}) + H(\hat{S}^n_2|\hat{Y}^n_{1,u},W_{21},W_{11},W_{12}) && (3.81)\\
&= H(\hat{Y}^n_3|\widetilde{\mathcal{W}}^c_{\{31,41\}}) + H(\hat{S}^n_2|W_{12}) && (3.82)
\end{aligned}
$$

where (3.80) holds since the function $f : \bar{X}_{31,S} \to \hat{Y}_{1,u}$ is bijective when $m_{11} \ge m_{31}$, and (3.82) follows that conditioning on the messages $\widetilde{\mathcal{W}}^c_{\{31,41\}}$, the function $f : \bar{X}_{31,S} \to \hat{Y}_3$ is bijective. Note that the equations (3.78)-(3.82) correspond to the key step (3.43) in the intuitive proof given in Section 3.3.1.

Plugging (3.82) into (3.77), we get

$$
n(R_{11}+R_{12}-\epsilon) \le H(\hat{S}^n_1|W_{21}) + H(\hat{Y}^n_{1,u},\hat{Y}^n_{1,l},\hat{C}^n_1|\hat{S}^n_1) - H(\hat{S}^n_2|W_{12}) - H(\hat{Y}^n_3|\widetilde{\mathcal{W}}^c_{\{31,41\}}) \tag{3.83}
$$

Next, consider the degraded BC comprised of Transmitter 1 and Receiver 3 and 4. Since $m_{31} \geq m_{41}$, we have

$$n(R_{31} + R_{41} - \epsilon) \leq I(W_{31}, W_{41}; \hat{Y}_3^n | \widetilde{\mathcal{W}}^c_{\{31,41\}}) = H(\hat{Y}_3^n | \widetilde{\mathcal{W}}^c_{\{31,41\}}) \tag{3.84}$$

Combining (3.83) and (3.84), we obtain

$$n(R_{11} + R_{12} + R_{31} + R_{41} - 2\epsilon) \leq H(\hat{S}_1^n | W_{21}) + H(\hat{Y}_{1,u}^n, \hat{Y}_{1,l}^n, \hat{C}_1^n | \hat{S}_1^n) - H(\hat{S}_2^n | W_{12}) \tag{3.85}$$

Similarly, considering Receiver 2 and the degraded BC comprised of Transmitter 2 and Receivers 3 and 4, we obtain

$$n(R_{21} + R_{22} + R_{32} + R_{42} - 2\epsilon) \leq H(\hat{S}_2^n | W_{12}) + H(\hat{Y}_{2,u}^n, \hat{Y}_{2,l}^n, \hat{C}_2^n | \hat{S}_2^n) - H(\hat{S}_1^n | W_{21}) \tag{3.86}$$

Adding (3.85) and (3.86) together, the sum capacity of this deterministic 2×4 X channel is upper bounded by

$$\begin{aligned} n(R_{\Sigma,D} - 4\epsilon) &\leq H(\hat{Y}_{1,u}^n, \hat{Y}_{1,l}^n, \hat{C}_1^n | \hat{S}_1^n) + H(\hat{Y}_{2,u}^n, \hat{Y}_{2,l}^n, \hat{C}_2^n | \hat{S}_2^n) & (3.87) \\ &\leq \sum_{t=1}^{n} \Big[H(\hat{Y}_{1,u}(t) | \hat{S}_1(t)) + H(\hat{Y}_{1,l}(t) | \hat{S}_1(t)) + H(\hat{C}_1(t)) \\ &\quad + H(\hat{Y}_{2,u}(t) | \hat{S}_2(t)) + H(\hat{Y}_{2,l}(t) | \hat{S}_2(t)) + H(\hat{C}_2(t)) \Big] & (3.88) \end{aligned}$$

where the last inequality follows from the chain rule and the fact that dropping conditioning does not reduce entropy.

For the term $H(\hat{Y}_{1,u}(t) | \hat{S}_1(t)) + H(\hat{Y}_{1,l}(t) | \hat{S}_1(t))$, we consider the following two cases.

- $m_{21} \geq m_{31}$: In this case, we have

$$H(\hat{Y}_{1,u}(t)|\hat{S}_1(t)) + H(\hat{Y}_{1,l}(t)|\hat{S}_1(t)) \leq 0 + (m_{11} - m_{21}) + \text{constant} \tag{3.89}$$

$$= (m_{11} - m_{21}) + \text{constant} \tag{3.90}$$

where (3.89) follows that conditioning on $\hat{S}_1$, out of the received signal $\hat{Y}_{1,l}$, both the signals from Transmitter 1 and 2 have at most $m_{11} - m_{21}$ bit-levels, and the signs of the signals and carry-overs due to the sum of two such signals can only induce a loss of constant bits.

- $m_{21} < m_{31}$: In this case, similarly we have

$$H(\hat{Y}_{1,u}(t)|\hat{S}_1(t)) + H(\hat{Y}_{1,l}(t)|\hat{S}_1(t)) \leq (m_{31} - m_{21}) + (m_{11} - m_{31}) + \text{constant} \tag{3.91}$$

$$= (m_{11} - m_{21}) + \text{constant} \tag{3.92}$$

Due to symmetry, we always have

$$H(\hat{Y}_{2,u}(t)|\hat{S}_2(t)) + H(\hat{Y}_{2,l}(t)|\hat{S}_2(t)) \leq (m_{22} - m_{12}) + \text{constant} \tag{3.93}$$

Therefore, we obtain

$$n(R_{\Sigma,D} - 4\epsilon) \leq \sum_{t=1}^{n} [(m_{11} - m_{21}) + (m_{22} - m_{12}) + \text{constant}] \tag{3.94}$$

According to Lemma 3.1, for the sum capacity of the original Gaussian X channel $R_{\Sigma,G}$, we have

$$R_{\Sigma,G} \leq R_{\Sigma,D} + \text{constant} \tag{3.95}$$

$$\leq (m_{11} - m_{21}) + (m_{22} - m_{12}) + \text{constant} \tag{3.96}$$

$$\leq \frac{1}{2}[(\alpha_{11} - \alpha_{21}) + (\alpha_{22} - \alpha_{12})] \log P + \text{constant} \tag{3.97}$$

Finally, we obtain the following desired GDoF cycle bound and complete the proof via the deterministic approach

$$\bar{d}_1 + \bar{d}_2 \leq \frac{1}{2}[(\alpha_{11} - \alpha_{21}) + (\alpha_{22} - \alpha_{12})]. \tag{3.98}$$

Next, consider the general *complex* Gaussian case. The proof is similar to the above example in the real Gaussian case. To avoid repetition, we will focus on the differences. Without loss of generality, assume that in the complex Gaussian case, we intend to prove the following bound

$$\sum_{j=1}^{m} \bar{d}_j \leq \sum_{j=1}^{m} (\alpha_{jj} - \alpha_{j-1,j}) \tag{3.99}$$

where $m \in \{2, 3, ..., M\}$ and modulo-m arithmetic is used on user indices. Consider the $m \times N$ X subnetwork consisting of all the receivers and Transmitters $i \in \langle m \rangle$, where all other transmitters and their associated messages are eliminated. Define the message set $\widetilde{\mathcal{W}} \triangleq \{W_{kj}\}$, $\forall j \in \langle m \rangle$, $\forall k \in \langle N \rangle$. Also define $\mathcal{W} \triangleq \{W_{kj}\}$, $\mathcal{W}_k^* \triangleq \{W_{k1}, W_{k2}, ..., W_{km}\}$, and $\mathcal{W}_{j'} \triangleq \{W_{m+1,j}, W_{m+2,j}, ..., W_{Nj}\}$, $\forall j, k \in \langle m \rangle$. Similarly, $\mathcal{W}_{\mathcal{S}}^c$ denotes $\mathcal{W} \backslash \mathcal{W}_{\mathcal{S}}$, where $\mathcal{S}$ is a subset of message indices.

To simplify the proof, we construct the following channel as shown in Fig. 3.3, which upper bounds the sum channel capacity of the above $m \times N$ complex Gaussian X channel:

- Step 1: Start with an $m \times m$ X channel with channel coefficients h_{kj}, $\forall k, j \in \langle m \rangle$.
- Step 2: For Transmitter $j \in \langle m \rangle$, create another $N - m$ virtual receivers. The virtual receiver with index kj, $\forall k \in \{m+1, m+2, ..., N\}$, only connects to Transmitter j with the channel coefficient h_{kj} and desires the message W_{kj} from Transmitter j. Note that after adding these virtual receivers, there are $m \times N$ messages in the network.

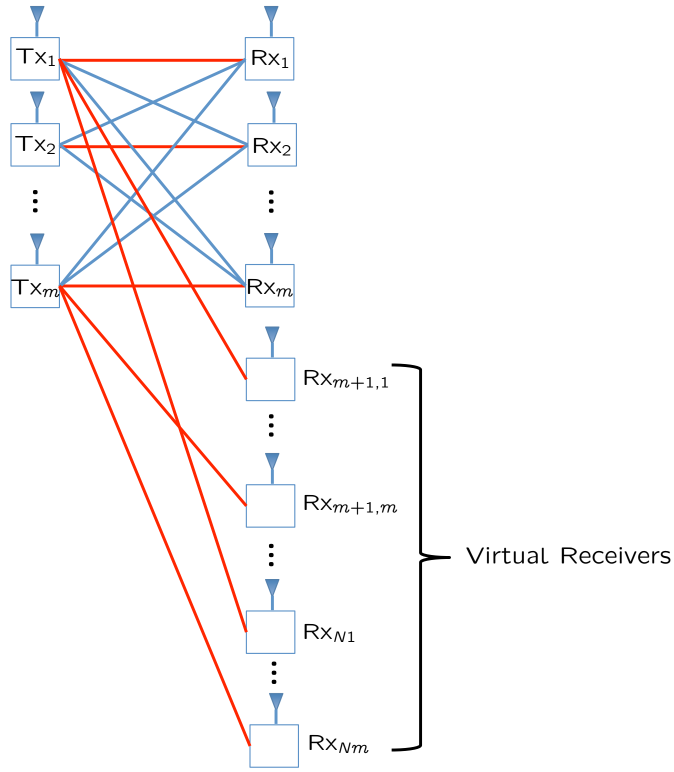

Figure 3.3: The constructed channel upper-bounding the sum capacity of the $m \times N$ X channel, where the red links are real-valued by rotating the phase of the received signal at the corresponding receivers.

- Step 3: For Receiver $k \in \langle m \rangle$, rotate its channel output by multiplying it with $e^{-j\theta_{kk}}$ to make the link from Transmitter k to Receiver k real-valued, which does not affect the capacity of this channel. Similarly, for virtual receiver with index kj, $j \in \langle m \rangle$, $k \in \{m+1, m+2, ..., N\}$, rotate its channel output by multiplying it with $e^{-j\theta_{kj}}$ to make its only connected link real-valued, which again does not affect the channel capacity. Therefore, without loss of generality, we assume that in Fig. 3.3, all the links in red are real-valued.

- Step 4: Finally, assume that the input signal $X_j(t)$ satisfies the power constraint $\mathbb{E}[|X_j(t)|^2] \leq 2$, $\forall j \in \langle m \rangle$, and the AWGN at all receivers are independent and with zero mean and variance 2.

According to Lemma 3.1, to obtain the desired cycle bound in (3.99), we consider the truncated binary-expansion deterministic model of the constructed channel in Fig. 3.3. For Receiver $j \in \langle m \rangle$, the channel output can be written in the following matrix form,

$$\hat{Y}_j(t) = \begin{pmatrix} \lfloor \text{sign}(\bar{X}_j^R(t)) h_{jj}^R \sum_{b=1}^{m_{jj}^R} \bar{X}_{j,b}^R(t) 2^{-b} \rfloor \\ \lfloor \text{sign}(\bar{X}_j^I(t)) h_{jj}^R \sum_{b=1}^{m_{jj}^R} \bar{X}_{j,b}^I(t) 2^{-b} \rfloor \end{pmatrix}$$
$$+ \sum_{k=1, k\neq j}^{m} \begin{pmatrix} \lfloor \text{sign}(\bar{X}_k^R(t)) h_{jk}^R \sum_{b=1}^{m_{jk}^R} \bar{X}_{k,b}^R(t) 2^{-b} \rfloor - \lfloor \text{sign}(\bar{X}_k^I(t)) h_{jk}^I \sum_{b=1}^{m_{jk}^I} \bar{X}_{k,b}^I(t) 2^{-b} \rfloor \\ \lfloor \text{sign}(\bar{X}_k^R(t)) h_{jk}^I \sum_{b=1}^{m_{jk}^I} \bar{X}_{k,b}^R(t) 2^{-b} \rfloor + \lfloor \text{sign}(\bar{X}_k^I(t)) h_{jk}^R \sum_{b=1}^{m_{jk}^R} \bar{X}_{k,b}^I(t) 2^{-b} \rfloor \end{pmatrix} \tag{3.100}$$

For the virtual receiver with index kj, the channel output is

$$\hat{Y}_{kj}(t) = \begin{pmatrix} \lfloor \text{sign}(\bar{X}_j^R(t)) h_{kj}^R \sum_{b=1}^{m_{kj}^R} \bar{X}_{j,b}^R(t) 2^{-b} \rfloor \\ \lfloor \text{sign}(\bar{X}_j^I(t)) h_{kj}^R \sum_{b=1}^{m_{kj}^R} \bar{X}_{j,b}^I(t) 2^{-b} \rfloor \end{pmatrix}, \quad \forall j \in \langle m \rangle, \ \forall k \in \{m+1, m+2, ..., N\}. \tag{3.101}$$

Next, for $j \in \langle m \rangle$, define

$$\hat{S}_j(t) = \begin{pmatrix} \text{sign}(\bar{X}_j^R(t)) \sum_{b=1}^{\max\{m_{j-1,j}^R, m_{j-1,j}^I\}} \bar{X}_{j,b}^R(t) 2^{-b} \\ \text{sign}(\bar{X}_j^I(t)) \sum_{b=1}^{\max\{m_{j-1,j}^R, m_{j-1,j}^I\}} \bar{X}_{j,b}^I(t) 2^{-b} \end{pmatrix}, \tag{3.102}$$

$$\hat{S}'_j(t) = \begin{pmatrix} \lfloor \text{sign}(\bar{X}_j^R(t)) h_{j-1,j}^R \sum_{b=1}^{m_{j-1,j}^R} \bar{X}_{j,b}^R(t) 2^{-b} \rfloor - \lfloor \text{sign}(\bar{X}_j^I(t)) h_{j-1,j}^I \sum_{b=1}^{m_{j-1,j}^I} \bar{X}_{j,b}^I(t) 2^{-b} \rfloor \\ \lfloor \text{sign}(\bar{X}_j^R(t)) h_{j-1,j}^I \sum_{b=1}^{m_{j-1,j}^I} \bar{X}_{j,b}^R(t) 2^{-b} \rfloor + \lfloor \text{sign}(\bar{X}_j^I(t)) h_{j-1,j}^R \sum_{b=1}^{m_{j-1,j}^R} \bar{X}_{j,b}^I(t) 2^{-b} \rfloor \end{pmatrix} \tag{3.103}$$

where the modulo-m arithmetic is implicitly used on the user indices. We have the following lemma.

Lemma 3.2. *$f : \hat{S}_j(t) \to \hat{S}'_j(t)$ is bijective.*

Proof of Lemma 3.2: For notation brevity, define

$$\breve{X}_R \triangleq \text{sign}(\bar{X}_j^R(t)) \sum_{b=1}^{\max\{m_{j-1,j}^R, m_{j-1,j}^I\}} \bar{X}_{j,b}^R(t) 2^{-b} \tag{3.104}$$

$$\breve{X}_I \triangleq \text{sign}(\bar{X}_j^I(t)) \sum_{b=1}^{\max\{m_{j-1,j}^R, m_{j-1,j}^I\}} \bar{X}_{j,b}^I(t) 2^{-b} \tag{3.105}$$

$$\breve{S}_R \triangleq \underbrace{\lfloor \text{sign}(\bar{X}_j^R(t)) h_{j-1,j}^R \sum_{b=1}^{m_{j-1,j}^R} \bar{X}_{j,b}^R(t) 2^{-b} \rfloor}_{\breve{S}_{R,1}} - \underbrace{\lfloor \text{sign}(\bar{X}_j^I(t)) h_{j-1,j}^I \sum_{b=1}^{m_{j-1,j}^I} \bar{X}_{j,b}^I(t) 2^{-b} \rfloor}_{\breve{S}_{R,2}} \tag{3.106}$$

$$\breve{S}_I \triangleq \underbrace{\lfloor \text{sign}(\bar{X}_j^R(t)) h_{j-1,j}^I \sum_{b=1}^{m_{j-1,j}^I} \bar{X}_{j,b}^R(t) 2^{-b} \rfloor}_{\breve{S}_{I,1}} + \underbrace{\lfloor \text{sign}(\bar{X}_j^I(t)) h_{j-1,j}^R \sum_{b=1}^{m_{j-1,j}^R} \bar{X}_{j,b}^I(t) 2^{-b} \rfloor}_{\breve{S}_{I,2}} \tag{3.107}$$

Note that $(\breve{X}_R, \breve{X}_I)$ and $(\breve{S}_R, \breve{S}_I)$ can be regarded as the input and output of the deterministic channel, respectively. Apparently, given one input $(\breve{X}_R, \breve{X}_I)$, we can only produce one output $(\breve{S}_R, \breve{S}_I)$.

Next, we proceed to prove the other direction by contradiction. Assume that there exist two distinct inputs $(\breve{X}_R^*, \breve{X}_I^*)$ and $(\breve{X}_R^{**}, \breve{X}_I^{**})$ that can generate the same output, i.e., $(\breve{S}_R^*, \breve{S}_I^*) = (\breve{S}_R^{**}, \breve{S}_I^{**})$. In the following, without loss of generality, assume

$$|h_{j-1,j}^R| \geq |h_{j-1,j}^I| \Rightarrow m_{j-1,j}^R \geq m_{j-1,j}^I. \tag{3.108}$$

First, consider the case where $\text{sign}(h_{j-1,j}^R) = \text{sign}(h_{j-1,j}^I)$. For the term $\breve{S}_R$, we have the following sub-cases.

- $\breve{S}_{R,1}^* = \breve{S}_{R,1}^{**}$ and $\breve{S}_{R,2}^* = \breve{S}_{R,2}^{**}$. In this case, for the term $\breve{S}_I$, if $|h_{j-1,j}^R| > |h_{j-1,j}^I|$, since $(\breve{X}_R^*, \breve{X}_I^*)$ and $(\breve{X}_R^{**}, \breve{X}_I^{**})$ are different, we have $\breve{S}_{I,1}^* = \breve{S}_{I,1}^{**}$ and $\breve{S}_{I,2}^* \neq \breve{S}_{I,2}^{**}$, which contradicts the assumption that $(\breve{X}_R^*, \breve{X}_I^*)$ and $(\breve{X}_R^{**}, \breve{X}_I^{**})$ generate the same output; if $|h_{j-1,j}^R| = |h_{j-1,j}^I|$, since $(\breve{X}_R^*, \breve{X}_I^*)$ and $(\breve{X}_R^{**}, \breve{X}_I^{**})$ generate the same $(\breve{S}_R, \breve{S}_I)$, we have $(\breve{X}_R^*, \breve{X}_I^*) = (\breve{X}_R^{**}, \breve{X}_I^{**})$, which contradicts the assumption that $(\breve{X}_R^*, \breve{X}_I^*)$ and $(\breve{X}_R^{**}, \breve{X}_I^{**})$ are different.

- $\breve{S}_{R,1}^* > \breve{S}_{R,1}^{**}$ and $\breve{S}_{R,2}^* > \breve{S}_{R,2}^{**}$. In this case, for the term $\breve{S}_I$, we have $\breve{S}_{I,1}^* \geq \breve{S}_{I,1}^{**}$ and $\breve{S}_{I,2}^* > \breve{S}_{I,2}^{**}$, which contradicts the assumption that $(\breve{X}_R^*, \breve{X}_I^*)$ and $(\breve{X}_R^{**}, \breve{X}_I^{**})$ generate the same output.

- $\breve{S}_{R,1}^* < \breve{S}_{R,1}^{**}$ and $\breve{S}_{R,2}^* < \breve{S}_{R,2}^{**}$. In this case, for the term $\breve{S}_I$, we have $\breve{S}_{I,1}^* \leq \breve{S}_{I,1}^{**}$ and $\breve{S}_{I,2}^* < \breve{S}_{I,2}^{**}$, which contradicts the assumption that $(\breve{X}_R^*, \breve{X}_I^*)$ and $(\breve{X}_R^{**}, \breve{X}_I^{**})$ generate the same output.

For the other case where $\text{sign}(h_{j-1,j}^R) = -\text{sign}(h_{i_{j-1}i_j}^I)$, we can follow the same argument above and obtain the same conclusion. ■

Consider Receiver $j \in \langle m \rangle$ again. Its channel output can be rewritten as

$$\hat{Y}_j(t) = \begin{pmatrix} \hat{Y}_j^R(t) \\ \hat{Y}_j^I(t) \end{pmatrix} \tag{3.109}$$

$$= \underbrace{\begin{pmatrix} \lfloor \text{sign}(\bar{X}_j^R(t)) h_{jj}^R \sum_{b=1}^{m_{j^*j}^R} \bar{X}_{j,b}^R(t) 2^{-b} \rfloor \\ \lfloor \text{sign}(\bar{X}_j^I(t)) h_{jj}^R \sum_{b=1}^{m_{j^*j}^R} \bar{X}_{j,b}^I(t) 2^{-b} \rfloor \end{pmatrix}}_{\hat{Y}_{j,u}(t)} \tag{3.110}$$

$$+ \begin{pmatrix} \lfloor \text{sign}(\bar{X}_j^R(t)) h_{jj}^R \sum_{b=m_{j^*j}^R+1}^{m_{jj}^R} \bar{X}_{j,b}^R(t) 2^{-b} \rfloor \\ \lfloor \text{sign}(\bar{X}_j^I(t)) h_{jj}^R \sum_{b=m_{j^*j}^R+1}^{m_{jj}^R} \bar{X}_{j,b}^I(t) 2^{-b} \rfloor \end{pmatrix} \tag{3.111}$$

$$+ \sum_{k \neq j} \begin{pmatrix} \lfloor \text{sign}(\bar{X}_k^R(t)) h_{jk}^R \sum_{b=1}^{m_{jk}^R} \bar{X}_{k,b}^R(t) 2^{-b} \rfloor - \lfloor \text{sign}(\bar{X}_k^I(t)) h_{jk}^I \sum_{b=1}^{m_{jk}^I} \bar{X}_{k,b}^I(t) 2^{-b} \rfloor \\ \lfloor \text{sign}(\bar{X}_k^R(t)) h_{jk}^I \sum_{b=1}^{m_{jk}^I} \bar{X}_{k,b}^R(t) 2^{-b} \rfloor + \lfloor \text{sign}(\bar{X}_k^I(t)) h_{jk}^R \sum_{b=1}^{m_{jk}^R} \bar{X}_{k,b}^I(t) 2^{-b} \rfloor \end{pmatrix} \tag{3.112}$$

$$+ \underbrace{\begin{pmatrix} \hat{C}_j^R(t) \\ \hat{C}_j^I(t) \end{pmatrix}}_{\hat{C}_j(t)} \tag{3.113}$$

where the virtual receiver with index j^*j is the strongest virtual receiver connected to Transmitter j (i.e., $|h_{j^*j}| = \max_{i\in\{m+1,\dots,N\}}\{|h_{ij}|\}$), and $\hat{C}_j^R(t)$ and $\hat{C}_j^I(t)$ can both take values from $\{-1, 0, 1\}$. Define

$$\bar{X}^R_{j^*j,S}(t) = \text{sign}(\bar{X}^R_j(t)) \sum_{b=1}^{m^R_{j^*j}} \bar{X}^R_{j,b}(t) 2^{-b} \tag{3.114}$$

$$\bar{X}^I_{j^*j,S}(t) = \text{sign}(\bar{X}^I_j(t)) \sum_{b=1}^{m^R_{j^*j}} \bar{X}^I_{j,b}(t) 2^{-b} \tag{3.115}$$

Also define the sum of (3.111) and (3.112) as $\hat{Y}_{j,l}(t)$. Then we have

$$\hat{Y}_j(t) = \hat{Y}_{j,u}(t) + \hat{Y}_{j,l}(t) + \hat{C}_j(t) \tag{3.116}$$

For Receiver 1, start from Fano's inequality.

$$\begin{aligned}
&n(\sum_{j=1}^{m} R_{1j} - \epsilon) && (3.117)\\
&\leq I(\mathcal{W}_1^*; \hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1, \hat{S}^n_1 | \mathcal{W}^c_{22} \backslash \mathcal{W}^*_1) && (3.118)\\
&= H(\hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1, \hat{S}^n_1 | \mathcal{W}^c_{22} \backslash \mathcal{W}^*_1) - H(\hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1, \hat{S}^n_1 | \mathcal{W}^c_{22}) && (3.119)\\
&= H(\hat{S}^n_1 | \mathcal{W}^c_{22} \backslash \mathcal{W}^*_1) + H(\hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1 | \hat{S}^n_1, \mathcal{W}^c_{22} \backslash \mathcal{W}^*_1) - H(\hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1 | \mathcal{W}^c_{22})\\
&\quad - H(\hat{S}^n_1 | \hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1, \mathcal{W}^c_{22}) && (3.120)\\
&\leq H(\hat{S}^n_1 | W_{21}, W_{31}, \dots, W_{m1}) + H(\hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1 | \hat{S}^n_1) - H(\hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1 | \mathcal{W}^c_{22}) && (3.121)
\end{aligned}$$

where the last inequality follows from the fact that dropping conditioning does not reduce entropy. Proceed to consider the last term in (3.121),

$$\begin{aligned}
&H(\hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1 | \mathcal{W}^c_{22})\\
&= H(\hat{Y}^n_{1,u} | \mathcal{W}^c_{22}) + H(\hat{Y}^n_{1,l}, \hat{C}^n_1 | \hat{Y}^n_{1,u}, \mathcal{W}^c_{22}) && (3.122)\\
&\geq H(\hat{Y}^n_{1,u} | \mathcal{W}^c_{22}) + H(\hat{S}'^n_2 | \hat{Y}^n_{1,u}, \mathcal{W}^c_{22}) && (3.123)
\end{aligned}$$

$$= H(\bar{X}^{R\,n}_{1^*1,S}, \bar{X}^{I\,n}_{1^*1,S}|\mathcal{W}^c_{22}) + H(\hat{S}^n_2|\hat{Y}^n_{1,u}, \mathcal{W}^c_{22}) \tag{3.124}$$

$$\geq H(\hat{Y}^n_{1^*1}|\widetilde{\mathcal{W}}\backslash\mathcal{W}_{1'}) + H(\hat{S}^n_2|W_{12}, W_{32}, ..., W_{m2}) \tag{3.125}$$

where (3.124) follows from Lemma 3.2, i.e., both functions $f : \hat{Y}_{1,u} \to \bar{X}^R_{1^*1,S} \times \bar{X}^I_{1^*1,S}$ and $f : \hat{S}'_2 \to \hat{S}_2$ are bijective.

Plugging (3.125) into (3.121), we have

$$\begin{aligned} n(\sum_{j=1}^{m} R_{1j} - \epsilon) \leq & H(\hat{S}^n_1|W_{21}, W_{31}, ..., W_{m1}) + H(\hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1|\hat{S}^n_1) \\ & - H(\hat{S}^n_2|W_{12}, W_{32}, ..., W_{m2}) - H(\hat{Y}^n_{1^*1}|\widetilde{\mathcal{W}}\backslash\mathcal{W}_{1'}) \end{aligned} \tag{3.126}$$

Next, consider the BC comprised of Transmitter 1 and all its connected virtual receivers. Since the virtual receiver with index 1*1 is the strongest one that is able to decode all the messages from Transmitter 1 (to all the connected virtual receivers), we have

$$n(\sum_{j=m+1}^{N} R_{j1} - \epsilon) \leq I(\mathcal{W}_{1'}; \hat{Y}^n_{1^*1}|\widetilde{\mathcal{W}}\backslash\mathcal{W}_{1'}) = H(\hat{Y}^n_{1^*1}|\widetilde{\mathcal{W}}\backslash\mathcal{W}_{1'}) \tag{3.127}$$

Adding (3.126) and (3.127) together, we obtain

$$\begin{aligned} & n(\sum_{j=1}^{m} R_{1j} + \sum_{j=m+1}^{N} R_{j1} - 2\epsilon) \\ & \leq H(\hat{S}^n_1|W_{21}, W_{31}, ..., W_{m1}) + H(\hat{Y}^n_{1,u}, \hat{Y}^n_{1,l}, \hat{C}^n_1|\hat{S}^n_1) - H(\hat{S}^n_2|W_{12}, W_{32}, ..., W_{m2}) \end{aligned} \tag{3.128}$$

Similarly, for $k \in \{2, 3, ..., m-1\}$ we have

$$\begin{aligned}
&n(\sum_{j=1}^{m} R_{kj} + \sum_{j=m+1}^{N} R_{jk} - 2\epsilon) \\
&\leq H(\hat{S}_k^n | W_{1k}, ..., W_{k-1,k}, W_{k+1,k}, ..., W_{mk}) + H(\hat{Y}_{k,u}^n, \hat{Y}_{k,l}^n, \hat{C}_k^n | \hat{S}_k^n) \\
&\quad - H(\hat{S}_{k+1}^n | W_{1,k+1}, ..., W_{k,k+1}, W_{k+2,k+1} ..., W_{m,k+1})
\end{aligned} \tag{3.129}$$

From Receiver m and the degraded BC comprised of Transmitter m and all its connected virtual receivers, we obtain

$$\begin{aligned}
&n(\sum_{j=1}^{m} R_{mj} + \sum_{j=m+1}^{N} R_{jm} - 2\epsilon) \\
&\leq H(\hat{S}_m^n | W_{1m}, W_{2m} ..., W_{m-1,m}) + H(\hat{Y}_{m,u}^n, \hat{Y}_{m,l}^n, \hat{C}_m^n | \hat{S}_m^n) - H(\hat{S}_1^n | W_{21}, W_{31} ..., W_{m1})
\end{aligned} \tag{3.130}$$

Adding all the terms in (3.128), (3.129) and (3.130) together, we get

$$n(R_{\Sigma,D} - 2m\epsilon) \leq \sum_{k=1}^{m} H(\hat{Y}_{k,u}^n, \hat{Y}_{k,l}^n, \hat{C}_k^n | \hat{S}_k^n) \tag{3.131}$$

$$\leq \sum_{k=1}^{m} \sum_{t=1}^{n} [H(\hat{Y}_{k,u}(t) | \hat{S}_k(t)) + H(\hat{Y}_{k,l}(t) | \hat{S}_k(t)) + H(\hat{C}_k(t))] \tag{3.132}$$

$$\leq n \sum_{k=1}^{m} \left[2(m_{kk} - m_{k-1,k}) + \text{constant}\right] \tag{3.133}$$

$$\leq n \sum_{k=1}^{m} \left[(\alpha_{kk} - \alpha_{k-1,k}) \log P + \text{constant}\right] \tag{3.134}$$

where (3.133) holds since $m_{kj} - 1 \leq \max\{m_{kj}^R, m_{kj}^I\} \leq m_{kj}$. Finally, according to Lemma 3.1, we obtain the desired GDoF cycle bound (3.99) for the original complex Gaussian X channel.

3.4 Summary

In this chapter, we first show that for the K-user TIN-optimal interference channels identified in Theorem 2.1, even if the message set is expanded to the X setting, operating as the original interference channel and treating interference as noise at each receiver is still optimal for the *sum* capacity up to a constant gap. Next, we extend the optimality of TIN to general X channels with arbitrary numbers of transmitters and receivers. In both cases, the achievability argument follows directly from Chapter 2 for the setting of interference channels. The main challenge lies in deriving tight information theoretical outer bounds. As illustrated in this chapter, to obtain desired outer bounds, the genie signal provided to each receiver should be chosen more judiciously. Notably, to complete the generalization to X channels where the number of receivers is larger than that of transmitters, due to the added difficulty, we also resort to deterministic channel models [32, 51–54], where certain combinatorial structure can be exploited to simplify the proof.

Chapter 4

Optimality of TIN for Compound Networks

In this chapter, we generalize the optimality of TIN to compound networks [13–15], where each receiver is associated with a set of states (a receiver state is identified by the channel realizations associated with that receiver). In the compound interference channel, the sets of possible states for each receiver are globally known a-priori, however the transmitters are unaware of the particular realization chosen from within these sets. Therefore, a reliable coding scheme must guarantee vanishing error probability for each possible realization of every receiver. Equivalently, the compound interference channel can be regarded as having potentially multiple receivers for each message, which is known as the multiple groupcast setting [24] and is of interest in its own right. Here of particular interest are compound interference channels where each possible network state individually satisfies the TIN-optimality condition of Theorem 2.1. Is TIN still optimal for such a compound setting as a whole, when all possible states are considered simultaneously? Moreover, the implications of the compound setting on the achievable TIN region and GDoF-optimal power control are also of interest and will be explored in this chapter as well.

This chapter is organized as follows. The channel model for the compound interference channel is presented in Section 4.1. In Section 4.2, we show that if each possible network realization satisfies the TIN-optimality condition of Theorem 2.1, TIN achieves the entire GDoF region of the compound setting. The TIN region is also fully characterized for general compound interference channels in this section. In Section 4.3, we study the power control problem for compound networks. We demonstrate that from the GDoF perspective, the power control and TIN problems for compound and regular (where the network has only one state) interference channels are equivalent. Remarkably, the regular counterpart might be different from all the possible network realizations of the compound channel. Then by taking advantage of the simplification of the compound setting to the regular case, we develop several GDoF-based power control algorithms for compound networks. A summarization is given in Section 4.4.

4.1 Channel Model

Consider K-user Gaussian compound interference channels. The channel coefficients associated with Receiver $k \in \langle K \rangle$ are denoted as a vector $(\tilde{h}_{k1}, \tilde{h}_{k2}, ..., \tilde{h}_{kK})$, which is drawn from a finite set $\mathcal{L}_k$ with cardinality L_k. Assume that the channel coefficients remain fixed during the transmission. In addition, while the transmitters are unaware of the specific channel realizations, knowledge of $\mathcal{L}_k$ is assumed to be globally available. Note that, as always, the receivers are assumed to have perfect CSI. In this compound setting, reliable communications need to be guaranteed simultaneously for all possible channel realizations. At Receiver k, the channel output in state l_k is given by

$$Y_k^{[l_k]}(t) = \sum_{i=1}^{K} \tilde{h}_{ki}^{[l_k]} \tilde{X}_i(t) + Z_k^{[l_k]}(t), \quad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle \tag{4.1}$$

where $\tilde{h}_{ki}^{[l_k]}$ is the channel gain value from Transmitter i to Receiver k, $\tilde{X}_i(t)$ and $Z_k^{[l_k]}(t)$ are the transmitted symbol of Transmitter i and the additive white circularly symmetric Gaussian noise with zero mean and unit variance at Receiver k, respectively, in the t-th channel use. Similar to previous chapters, all symbols are complex, and Transmitter $i \in \langle K \rangle$ is subject to the average power constraint $\mathbb{E}[|\tilde{X}_i(t)|^2] \leq P_i$.

Again, to facilitate the GDoF studies, the standard channel model (4.1) is translated into an equivalent normalized form following similar approaches in previous chapters. Define

$$\alpha_{ki}^{[l_k]} \triangleq \frac{\log(\max\{1, |\tilde{h}_{ki}^{[l_k]}|^2 P_i\})}{\log P}, \quad \forall i, k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle \tag{4.2}$$

where $P > 1$ is a nominal power value. The original channel model (4.1) is then presented in the following form,

$$Y_k^{[l_k]}(t) = \sum_{i=1}^{K} h_{ki}^{[l_k]} X_i(t) + Z_k^{[l_k]}(t) \tag{4.3}$$

$$= \sum_{i=1}^{K} \sqrt{P^{\alpha_{ki}^{[l_k]}}} e^{j\theta_{ki}^{[l_k]}} X_i(t) + Z_k^{[l_k]}(t), \quad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle \tag{4.4}$$

where $X_i(t)$ is the normalized transmit symbol of Transmitter i, subject to the unit power constraint (i.e., $\mathbb{E}[|X_i(t)|^2] \leq 1$).

In this K-user compound interference channel, Transmitter i intends to deliver one independent message W_i to Receiver i, $\forall i \in \langle K \rangle$. The size of the message set $\{W_i\}$ is denoted by $|W_i|$. For codewords spanning n channel uses, the rates $R_i = \frac{\log |W_i|}{n}$ are achievable if all messages can be decoded simultaneously with arbitrarily small error probability as n grows to infinity regardless of the channel realizations. The channel capacity region $\mathcal{C}$ is the closure of the set of all achievable rate tuples. The GDoF region of the K-user compound interference channel is given by

$$\mathcal{D} \triangleq \left\{ (d_1, d_2, ..., d_K) : \ d_k = \lim_{P \to \infty} \frac{R_k}{\log P}, \ \forall k \in \langle K \rangle, \ (R_1, R_2, ..., R_K) \in \mathcal{C} \right\} \tag{4.5}$$

It is notable that the compound interference channel can be modeled in different ways. The model used in this chapter assumes that each receiver has multiple possible states, i.e.,

$$(\tilde{h}_{k1}, ..., \tilde{h}_{kK}) \in \mathcal{L}_k, \quad \forall k \in \langle K \rangle \tag{4.6}$$

Another way is to allow for all the channel coefficients $\tilde{h}_{ki}$ to be jointly taken from a finite set $\mathcal{L}$, i.e.,

$$(\tilde{h}_{11}, ..., \tilde{h}_{1K}, ..., \tilde{h}_{K1}, ..., \tilde{h}_{KK}) \in \mathcal{L} \tag{4.7}$$

As illustrated in [15] (see Proposition 2 in [15]), since the receivers cannot cooperate with each other, it turns out that *the latter is equivalent to the model adopted in this chapter*. On the other hand, a less general model for the compound setting than the one adopted here, is to assume that each channel coefficient $\tilde{h}_{ij}$ can take any value over its own set of allowed values $\mathcal{S}_{ij}$ *independently* of other channel coefficients. From the TIN perspective, this model is rather trivial, since only a readily attainable "worst case" matters, where for $i = j$, $\tilde{h}_{ij} = \min\{|s_{ij}| : s_{ij} \in \mathcal{S}_{ij}\}$, and for $i \neq j$, $\tilde{h}_{ij} = \max\{|s_{ij}| : s_{ij} \in \mathcal{S}_{ij}\}$.

4.2 TIN-optimality Condition for Compound Networks

In this section, we are mainly interested in the compound interference channel where each possible state of the network individually satisfies the TIN-optimality condition of Theorem 2.1. The natural question is whether the simple TIN scheme is still optimal for such a compound setting when all possible states are considered simultaneously.

4.2.1 Challenge Posed by the Compound Setting

We start with an overview of what makes the optimality of TIN for compound networks nontrivial. Denote by $\mathcal{H}$ the set of all the possible network realizations, $\mathcal{P}_A$ the set of all the valid power allocations, and $\mathcal{D}(\mathbf{P}, \mathbf{H})$ the achievable GDoF region through the TIN scheme for the network realization $\mathbf{H} \in \mathcal{H}$ with power allocation $\mathbf{P} \in \mathcal{P}_A$.

First, consider the achievability. Since in the TIN scheme the rate for each message is limited by the minimum SINR across all states of the intended receiver, it is evident that for any valid power allocation $\mathbf{P} \in \mathcal{P}_A$, $\cap_{\mathbf{H}\in\mathcal{H}}\mathcal{D}(\mathbf{P}, \mathbf{H})$ is achievable. Taking the union of achievable rates over all the valid power allocations, we obtain the following inner bound on the GDoF region.

$$\text{Inner Bound} = \cup_{\mathbf{P}\in\mathcal{P}_A} \cap_{\mathbf{H}\in\mathcal{H}} \mathcal{D}(\mathbf{P}, \mathbf{H}) \tag{4.8}$$

Next, consider the converse. Since the compound network satisfies the TIN-optimality condition of Theorem 2.1 in each possible state, accordingly for a given network state $\mathbf{H} \in \mathcal{H}$ its entire GDoF region can be expressed as $\cup_{\mathbf{P}\in\mathcal{P}_A}\mathcal{D}(\mathbf{P}, \mathbf{H})$. Since the GDoF region for each state is a natural outer bound for the GDoF region of the whole compound setting, by taking the intersection of the GDoF regions of all possible network states, we get the following outer bound on the GDoF region.

$$\text{Outer Bound} = \cap_{\mathbf{H}\in\mathcal{H}} \cup_{\mathbf{P}\in\mathcal{P}_A} \mathcal{D}(\mathbf{P}, \mathbf{H}) \tag{4.9}$$

Notice that while both the inner and outer bounds for the GDoF region involve union over power allocations and intersection over network states, the inner bound is the union of intersections whereas the outer bound is the intersection of unions. In general, the right hand side of (4.8) (i.e., a union of intersections) is no larger than that of (4.9) (i.e., an intersection of unions), consistent with their roles as inner and outer bounds, respectively.[1] So the main challenge in settling the optimality of TIN for compound interference channels is to prove that the two are indeed identical in our context.

4.2.2 Polyhedral TIN for Compound Interference Channels

Similar to the regular setting in Chapter 2, polyhedral TIN plays a key role for help establish the optimality of TIN in the compound channels. In the section, we generalize polyhedral TIN to the compound setting.

In the K-user compound interference channel, assume that the power allocation of Transmitter $k \in \langle K \rangle$ is P^{r_k}, where $r_k \leq 0$. In the TIN scheme, the achievable rate of user k is limited by the smallest SINR across all states possible for this user. So user k achieves any rate R_k such that

$$R_k \leq \min_{l_k \in \langle L_k \rangle} \left\{ \log \left(1 + \frac{P^{r_k} \times P^{\alpha_{kk}^{[l_k]}}}{1 + \sum_{j=1, j \neq k}^{K} P^{r_j} \times P^{\alpha_{kj}^{[l_k]}}} \right) \right\}, \quad \forall k \in \langle K \rangle \tag{4.10}$$

In the GDoF sense, we have

$$0 \leq d_k \leq \min_{l_k \in \langle L_k \rangle} \left\{ \max\{0, \alpha_{kk}^{[l_k]} + r_k - (\max_{j:j \neq k} \{\alpha_{kj}^{[l_k]} + r_j\})^+\} \right\}, \quad \forall k \in \langle K \rangle \tag{4.11}$$

The TIN region $\mathcal{P}^*$ is the set of all GDoF tuples $(d_1, ..., d_K)$ for which there exist r_k's, $r_k \leq 0$, $\forall k \in \langle K \rangle$, such that (4.11) holds for all $k \in \langle K \rangle$.

Similar to Chapter 2, in the compound setting we also introduce polyhedral TIN, i.e., a

[1] Consider an example where $\mathcal{P}_A = \{\mathbf{P_1}, \mathbf{P_2}\}$, $\mathcal{H} = \{\mathbf{H_1}, \mathbf{H_2}\}$, $\mathcal{D}(\mathbf{P_1}, \mathbf{H_1}) = \{1\}$, $\mathcal{D}(\mathbf{P_1}, \mathbf{H_2}) = \{2\}$, $\mathcal{D}(\mathbf{P_2}, \mathbf{H_1}) = \{2\}$, and $\mathcal{D}(\mathbf{P_2}, \mathbf{H_2}) = \{1\}$. It is easy to check that the right hand sides of (4.8) and (4.9) are ϕ and $\{1, 2\}$, respectively, which are not equal to each other.

polyhedral version of the TIN scheme. By requiring $\min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]} + r_k - (\max_{j:j\neq k}\{\alpha_{kj}^{[l_k]} + r_j\})^+\}$ to be non-negative for all users, we can ignore the first $\max\{0,.\}$ term in (4.11) and obtain

$$0 \leq d_k \leq \min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]} + r_k - (\max_{j:j\neq k}\{\alpha_{kj}^{[l_k]} + r_j\})^+\}, \quad \forall k \in \langle K \rangle \tag{4.12}$$

According to (4.12), the polyhedral TIN region $\mathcal{P}$ in the compound setting includes all GDoF tuples $(d_1, ..., d_K)$ for which there exist r_k's, $k \in \langle K \rangle$, such that

$$\begin{aligned} r_k &\leq 0, & &\forall k \in \langle K \rangle \\ d_k &\geq 0, & &\forall k \in \langle K \rangle \\ d_k \leq \alpha_{kk}^{[l_k]} + r_k \Leftrightarrow r_k &\geq d_k - \alpha_{kk}^{[l_k]}, & &\forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle \\ d_k \leq \alpha_{kk}^{[l_k]} + r_k - (\alpha_{kj}^{[l_k]} + r_j) \Leftrightarrow r_k - r_j &\geq (\alpha_{kj}^{[l_k]} - \alpha_{kk}^{[l_k]}) + d_k, & &\forall k, j \in \langle K \rangle, k \neq j, \forall l_k \in \langle L_k \rangle \end{aligned}$$

Similar to the regular setting, with the above modification we put more constraints on the power exponents r_i besides $r_i \leq 0$, which can only shrink the achievable GDoF region of the TIN scheme. In other words, we have $\mathcal{P} \subseteq \mathcal{P}^*$.

4.2.3 Main Result

The following theorem settles the question on the optimality of TIN for compound interference channels.

Theorem 4.1. *In a K-user compound interference channel, if the following condition is satisfied,*

$$\alpha_{ii}^{[l_i]} \geq \max_{j:j\neq i}\{\alpha_{ji}^{[l_j]}\} + \max_{k:k\neq i}\{\alpha_{ik}^{[l_i]}\}, \quad \forall i, j, k \in \langle K \rangle, \forall l_i \in \langle L_i \rangle, \forall l_j \in \langle L_j \rangle \tag{4.13}$$

then power control and TIN achieves the entire GDoF region, which is the intersection of the GDoF regions of all the possible network realizations. Moreover, the GDoF region of the compound channel is fully characterized by

$$0 \leq d_i \leq \alpha_{ii}^{[l_i]}, \qquad \forall i \in \langle K \rangle, \forall l_i \in \langle L_i \rangle \tag{4.14}$$

$$\sum_{j=1}^{m} d_{i_j} \leq \sum_{j=1}^{m} (\alpha_{i_j i_j}^{[l_{i_j}]} - \alpha_{i_j i_{j+1}}^{[l_{i_j}]}), \ \forall (i_1, i_2, ..., i_m) \in \Pi_K, \forall m \in \{2, 3, ..., K\}, \forall l_{i_j} \in \langle L_{i_j} \rangle \tag{4.15}$$

where the modulo-m arithmetic is implicitly used on user indices, e.g., $i_m = i_0$.

The proof of Theorem 4.1 is relegated to Section 4.2.4. Theorem 4.1 demonstrates that if in each possible network realization the channel satisfies the TIN-optimality condition of Theorem 2.1 individually, then TIN is, indeed, GDoF-optimal for the entire compound setting. In the above theorem, setting $L_k = 1$, $\forall k \in \langle K \rangle$, Theorem 2.1 is readily recovered for regular interference channels. Similar to Theorem 2.1, (4.14) and (4.15) fully characterize the polyhedral TIN region $\mathcal{P}$ of general compound interference channel (even if the condition (4.13) is not satisifed), which is in fact the intersection of the polyhedral TIN regions for all possible network states. In other words, *for a general compound interference channel, the intersection of the polyhedral TIN regions for all its potential network states is always achievable via power control and TIN.*

Based on the results in Theorem 4.1, we further obtain the TIN region $\mathcal{P}^*$ for general K-user compound interference channels. Similar to Theorem 2.3, we get that in general $\mathcal{P}^*$ is a union of 2^K polyhedral TIN regions $\mathcal{P}_{\mathcal{S}}$, each of which corresponds to the case where the users in $\mathcal{S}$ (any subset of $\langle K \rangle$) are silent and thus removed from the network. Note that for general compound interference channels, $\mathcal{P}_\phi$ (i.e., the polyhedral TIN region for the case where all users are active) is the same as the polyhedral TIN region $\mathcal{P}$ defined in Section 4.2.2, i.e., $\mathcal{P} = \mathcal{P}_\phi$. When (4.13) is satisfied, the polyhedral TIN region $\mathcal{P}_\phi$ subsumes all the others and the TIN scheme is GDoF-optimal, i.e., $\mathcal{D} = \mathcal{P}^* = \mathcal{P}$.

Remark 4.1. *It is not hard to generalize the result of Theorem 4.1 to the sum GDoF optimality of TIN for $M \times N$ compound X channels following Chapter 3. In addition, based on the bounding techniques given in Chapter 2 and 3, it is easy to show that under the same condition (4.13), the TIN scheme is sufficient to achieve a constant gap to the capacity region of K-user compound interference channels and the sum capacity of compound X channels.*

4.2.4 Proof of Theorem 4.1

In the section, we present the proof of Theorem 4.1.

Start with the converse argument. Consider each possible network realization individually. Since in each network realization, the channel satisfies the TIN-optimality condition identified in Theorem 2.1, it is easy to characterize its GDoF region. Taking the intersection of the GDoF regions of all the possible network realizations, we obtain the desired outer bounds.

Next, consider the achievability argument. The proof is derived from a non-trivial argument based on the potential theorem in [43], which builds upon the application of potential theorem in Chapter 2. We first generalize the potential graph D_p defined in Chapter 2 to the compound setting. For an arbitary K-user compound interference channel, we construct a fully connected digraph $D_p = (\mathcal{V}, \mathcal{E})$ with $\sum_k L_k + 1$ vertices, where $\mathcal{V}$ and $\mathcal{E}$ are the sets of vertices and edges, respectively, and

$$
\begin{aligned}
\mathcal{V} &= \{u\} \cup \bigcup_{k \in \langle K \rangle} \{v_k^{[1]}, ..., v_k^{[L_k]}\} \\
\mathcal{E} &= \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \\
\mathcal{E}_1 &= \{(v_k^{[l_k]}, v_k^{[l'_k]})\}, && \forall k \in \langle K \rangle, \forall l_k, l'_k \in \langle L_k \rangle, l_k \neq l'_k \\
\mathcal{E}_2 &= \{(v_k^{[l_k]}, v_j^{[l_j]})\}, && \forall k, j \in \langle K \rangle, k \neq j, \forall l_k \in \langle L_k \rangle, \forall l_j \in \langle L_j \rangle \\
\mathcal{E}_3 &= \{(v_k^{[l_k]}, u)\}, && \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle
\end{aligned}
$$

$$\mathcal{E}_4 = \{(u, v_k^{[l_k]})\}, \qquad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle$$

We assign a length $l(e)$ to each edge $e \in \mathcal{E}$ as follows.

$$l(v_k^{[l_k]}, v_k^{[l'_k]}) = 0, \qquad \forall k \in \langle K \rangle, \forall l_k, l'_k \in \langle L_k \rangle, l_k \neq l'_k \tag{4.16}$$

$$l(v_k^{[l_k]}, v_j^{[l_j]}) = (\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]}) - d_k, \qquad \forall k, j \in \langle K \rangle, k \neq j, \forall l_k \in \langle L_k \rangle, \forall l_j \in \langle L_j \rangle \tag{4.17}$$

$$l(v_k^{[l_k]}, u) = \alpha_{kk}^{[l_k]} - d_k, \qquad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle \tag{4.18}$$

$$l(u, v_k^{[l_k]}) = 0, \qquad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle \tag{4.19}$$

This complete digraph D_p is the potential graph for the compound interference channel, which is a generalization for the regular case. In the potential graph D_p, denote the length of the shortest path from the vertex u to each vertex $v_k^{[1]}$ as $l_{k,\text{dst}}$, $\forall k \in \langle K \rangle$. It is notable that due to (4.16) the lengths of the shortest paths from u to $v_k^{[l_k]}$ and $v_k^{[l'_k]}$ ($l_k \neq l'_k$) are always the same. In addition, according to (4.19), $l_{k,\text{dst}} \leq 0$, $\forall k \in \langle K \rangle$.

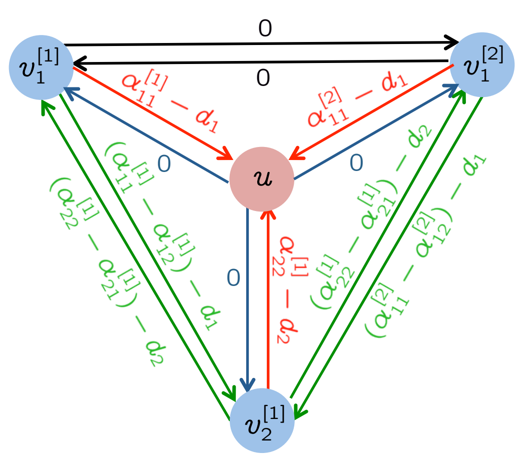

Figure 4.1: The potential graph D_p for a 2-user compound interference channel with $L_1 = 2$ and $L_2 = 1$

Example 4.1. *Consider a 2-user compound interference channel where $L_1 = 2$ and $L_2 = 1$. Its potential graph $D_p = (\mathcal{V}, \mathcal{E})$ is depicted in Fig. 4.1. It is comprised of $\sum_{k=1}^{2} L_k + 1 = 4$*

vertices, black edges $\mathcal{E}_1$*, green edges* $\mathcal{E}_2$*, red edges* $\mathcal{E}_3$ *and blue edges* $\mathcal{E}_4$*, where*

$$\begin{aligned}
\mathcal{V} &= \{v_1^{[1]}, v_1^{[2]}, v_2^{[1]}, u\} \\
\mathcal{E} &= \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \\
\mathcal{E}_1 &= \{(v_1^{[1]}, v_1^{[2]}), (v_1^{[2]}, v_1^{[1]})\} \\
\mathcal{E}_2 &= \{(v_1^{[1]}, v_2^{[1]}), (v_1^{[2]}, v_2^{[1]}), (v_2^{[1]}, v_1^{[1]}), (v_2^{[1]}, v_1^{[2]})\} \\
\mathcal{E}_3 &= \{(v_1^{[1]}, u), (v_1^{[2]}, u), (v_2^{[1]}, u)\} \\
\mathcal{E}_4 &= \{(u, v_1^{[1]}), (u, v_1^{[2]}), (u, v_2^{[1]})\}
\end{aligned}$$

The length $l(e)$ *for each edge* $e \in \mathcal{E}$ *is denoted in Fig. 4.1 as well.*

Recall that for K-user compound interference channels, the polyhedral TIN region $\mathcal{P}$ is the set of all GDoF tuples $(d_1, d_2, ..., d_K)$ for which there exist r_k's, $k \in \langle K \rangle$, such that

$$d_k \geq 0, \qquad \forall k \in \langle K \rangle \tag{4.20}$$

$$r_k \leq 0, \qquad \forall k \in \langle K \rangle \tag{4.21}$$

$$r_k \geq d_k - \alpha_{kk}^{[l_k]}, \qquad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle \tag{4.22}$$

$$r_k - r_j \geq (\alpha_{kj}^{[l_k]} - \alpha_{kk}^{[l_k]}) + d_k, \ \forall k, j \in \langle K \rangle, k \neq j, \forall l_k \in \langle L_k \rangle \tag{4.23}$$

Setting

$$r_k^{[l_k]} = r_k, \quad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle, \tag{4.24}$$

it is easy to verify that the set of the above inequalities (4.20)-(4.23) is equivalent to the following ones

$$d_k \geq 0, \qquad \forall k \in \langle K \rangle \tag{4.25}$$

$$r_k^{[l_k]} \leq 0, \qquad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle \tag{4.26}$$

$$r_k^{[l_k]} - r_k^{[l_k']} \leq 0, \qquad \forall k \in \langle K \rangle, \forall l_k, l_k' \in \langle L_k \rangle, l_k \neq l_k' \tag{4.27}$$

$$r_k^{[l_k]} \geq d_k - \alpha_{kk}^{[l_k]}, \qquad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle \tag{4.28}$$

$$r_k^{[l_k]} - r_j^{[l_j]} \geq (\alpha_{kj}^{[l_k]} - \alpha_{kk}^{[l_k]}) + d_k, \ \forall k, j \in \langle K \rangle, k \neq j, \forall l_k \in \langle L_k \rangle, \forall l_j \in \langle L_j \rangle \tag{4.29}$$

Therefore, the polyhedral TIN region $\mathcal{P}$ is also fully specified by (4.25)-(4.29). In other words, a GDoF tuple $(d_1, d_2, ..., d_K) \in \mathbb{R}_+^K$ is in the polyhedral TIN region $\mathcal{P}$ if and only if there exists $r_k^{[l_k]}$, $k \in \langle K \rangle$, $l_k \in \langle L_k \rangle$, such that (4.26)-(4.29) hold.

Now consider the potential graph D_p for the K-user compound interference channel. Similar to the regular case, without loss of generality, if there exists a valid potential function for the potential graph D_p, we can make the vertex u ground (i.e., $p(u) = 0$). Then let $p(v_k^{[l_k]}) = r_k^{[l_k]}$, $\forall k \in \langle K \rangle$, $\forall l_k \in \langle L_k \rangle$. It is not hard to verify that the potential function values should satisfy the inequalities (4.26) - (4.29). Therefore, *in a K-user compound interference channel, a GDoF tuple $(d_1, d_2, ..., d_K) \in \mathbb{R}_+^K$ is in the region $\mathcal{P}$ if and only if there exists a valid potential function for its potential graph D_p.* Again, based on the potential theorem in [43], we conclude that a GDoF tuple $(d_1, d_2, ..., d_K) \in \mathbb{R}_+^K$ is in the region $\mathcal{P}$ if and only if the length of each directed circuit in D_p is non-negative. We categorize all the directed circuits of D_p into the following three classes:

- Circuits in the form of $(u \rightarrow v_k^{[l_k]} \rightarrow u)$. For these circuits, the non-negative length condition yields

$$\alpha_{kk}^{[l_k]} - d_k \geq 0 \Leftrightarrow d_k \leq \alpha_{kk}^{[l_k]} \tag{4.30}$$

- Circuits in the form $(v_{i_0}^{[l_{i_0}]} \rightarrow v_{i_1}^{[l_{i_1}]} \rightarrow ... \rightarrow v_{i_m}^{[l_{i_m}]})$, where $i_0 = i_m$ and $(i_1, i_2, ..., i_m) \in$

Π_K. For these circuits, the non-negative length condition indicates

$$\sum_{j=1}^{m}(\alpha_{i_j i_j}^{[l_{i_j}]} - \alpha_{i_j i_{j+1}}^{[l_{i_j}]} - d_{i_j}) \geq 0 \Leftrightarrow \sum_{j=1}^{m} d_{i_j} \leq \sum_{j=1}^{m}(\alpha_{i_j i_j}^{[l_{i_j}]} - \alpha_{i_j i_{j+1}}^{[l_{i_j}]}) \tag{4.31}$$

- All the other circuits. For these remaining circuits, it is not hard to check that given (4.30) and (4.31), the inequalities derived from the non-negative length condition are all redundant.

Consequently, we end up with (4.30)-(4.31). Explicitly adding the non-negative constraint on d_i in (4.30)-(4.31), we obtain the polyhedral TIN region $\mathcal{P}$, which is fully characterized by (4.14)-(4.15) and turns out to be the intersection of the polyhedral TIN regions for all the possible network realizations. Clearly, under condition (4.13), the polyhedral TIN region $\mathcal{P}$ is matched with the derived outer bounds. Therefore, we complete the proof of Theorem 4.1.

4.3 GDoF-based Power Control for Compound Networks

Previous results in this dissertation have shown that under certain conditions, power control and TIN is optimal from the GDoF perspective for various channel settings, e.g., regular interference channels, X channels, and compound networks. However, the optimal power control solution, i.e., the optimal power exponent r_i at each transmitter for the target GDoF tuple, is not given explicitly due to the procedure of Fourier-Motzlin elimination (which is accomplished via the application of potential theorem). In this section, we will explore the power control problem from the GDoF perspective for general compound channels (note that the regular channel is only a special case of the compound setting). Compared with the regular setting, the compound setting in fact adds another layer of complexity on the optimal power allocation problem, since it is not clear a-priori where the bottlenecks lie. To identify the bottlenecks, simplify the compound network setting as much as possible, and to allocate power optimally to achieve any desired GDoF objective function are the goals that we pursue in this section.

4.3.1 Previous Work on Power Control

Transmit power control is of paramount importance for the TIN scheme. There is abundant literature on the optimization of transmit power allocations. In [57, 58], Zander considers the carrier-to-interference (C/I) balancing problem and intends to maximize the minimal achievable C/I ratios of all users with the minimized overall power consumption. For a given feasible SINR target, distributed power control algorithms are proposed in [59–61] to obtain the optimal power allocation. In particular, there is much interest in joint SINR (or rate) assignment (e.g., for sum-rate maximization) and power control problem, studied recently in [62–65]. This joint optimization problem is quite challenging because: 1) although subclasses solvable in polynomial time are identified, it is non-convex and NP-hard in general [64] (one standard approach to deal with this issue is using high SINR approximations to formulate a convex optimization problem [63]); 2) the feasible SINR region is highly coupled with power allocations across all users in the network. There are several studies attempting to derive the entire feasible SINR (or rate) region [66–68], and it is well-known that the feasible SINR region is log-convex [69].[2] So far the sum-rate maximization problem is only solved in 2-user asymmetric interference channels [70, 71] and K-user fully symmetric interference channels [72]. For the general K-user asymmetric setting, various algorithms based on

[2]It is notable that the feasible SINR region studied in [69] corresponds to the case where all users in the network are active and thus attain strictly positive SINRs. In the GDoF framework, this feasible SINR region in the log scale essentially corresponds to the polyhedral TIN region $\mathcal{P}$, which is also always convex. However, as shown in Chapter 2, in general the TIN region $\mathcal{P}^*$ is not convex without time-sharing, if we allow certain users in the network to be silent.

geometric programming, game theory, etc. (see [73,74] and references therein), are developed.

Within the GDoF framework, most relevant are the results of Chapter 2 for fully asymmetric K-user interference channels, where the entire TIN (GDoF) region is fully characterized. A remarkable advantage of this GDoF approach is that using (essentially) Fourier-Motzkin Elimination, the transmit power allocation variables are entirely eliminated and thus the feasible GDoF region characterization and power control problem are *decoupled*. Therefore, a closed-form feasible GDoF region depending only upon the effective channel gains (i.e., the channel strength level α) is derived, with which the optimal GDoF assignment under a given objective function (e.g., to maximize the achievable sum GDoF) is relatively easy to solve. The only problem left is power control, i.e., how to allocate transmit powers (efficiently, since multiple solutions may be possible in general) to achieve the target GDoF tuples.

4.3.2 Preliminaries on GDoF Based Power Control

From the GDoF perspective, we refer to the power exponent vector $(r_1, r_2, ..., r_K)$ as the transmit power allocation. Also notice that without loss of generality, for the GDoF-based power control problem we will only consider the GDoF tuples $(d_1, d_2, ..., d_K)$ where $d_i > 0$, $\forall i \in \langle K \rangle$. Apparently, if for certain user i the target GDoF value $d_i = 0$, we only need to set $r_i = -\infty$ and then remove this user i from the network without affecting others. Therefore, in the following, we only deal with the polyhedral TIN scheme, as the polyhedral TIN region $\mathcal{P}$ includes all the GDoF tuples we are interested in. And implicitly we only need to consider a subset of the valid power allocations for the polyhedral TIN scheme, i.e., the power exponent vectors which guarantee the right hand side of (4.12) to be positive for all users.

For the GDoF-based power control problem, we are primarily interested in the *globally optimal* power allocation, in which to achieve the target GDoF tuple none of the users can lower their transmit powers. The desired *globally optimal* solution should be *locally optimal* at first, in the sense that no single user can *unilaterally* lower its own transmit power and still achieve its target GDoF value.[3] Note that for one GDoF tuple, while there may exist multiple locally optimal power allocations in general, there is only one globally optimal power control solution, which is also the unique Pareto optimal solution as shown in [37].

4.3.3 Equivalence of Compound and Regular Interference Channels

In this section, we show how to simplify the GDoF-based power control problem for general compound interference channels. Denote the K-user compound interference channel defined in (4.4) as $\mathcal{IC}_C$. Based on $\mathcal{IC}_C$, we construct a counterpart K-user regular interference channel $\mathcal{IC}_R$ with the following input-output relationship

$$Y_k(t) = \sum_{i=1}^{K} \sqrt{P^{\bar{\alpha}_{ki}}} e^{j\bar{\theta}_{ki}} X_i(t) + \bar{Z}_k(t), \quad \forall k \in \langle K \rangle, \tag{4.32}$$

where[4]

$$\bar{\alpha}_{kk} = \min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]}\}, \qquad \forall k \in \langle K \rangle \tag{4.33}$$

$$\bar{\alpha}_{kj} = \min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]}\} - \min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]}\}, \quad \forall k, j \in \langle K \rangle, j \neq k, \tag{4.34}$$

[3]For example, in a 3-user interference channel, with the power allocation (r_1^*, r_2^*, r_3^*), through the TIN scheme we obtain a GDoF tuple (d_1^*, d_2^*, d_3^*), where $d_i^* > 0$ for $i \in \{1, 2, 3\}$. Then the power allocation (r_1^*, r_2^*, r_3^*) is locally optimal for the tuple (d_1^*, d_2^*, d_3^*). Using the terminology of game theory, each locally optimal power allocation refers to a *Nash equilibrium*.

[4]In this chapter, for a compound interference channel, we denote the channel strength level of its regular counterpart by $\bar{\alpha}_{ij}$ to highlight the corresponding relationship between these two. In the context where this corresponding relationship is not our focus, we still denote the channel strength level of regular interference channels by α_{ij} for brevity.

$\bar{\theta}_{ki}$ can take any value (note that the channel phases do not affect the power control results), and $\bar{Z}_k(t) \sim \mathcal{CN}(0,1)$. We can rewrite (4.34) as follows

$$\bar{\alpha}_{kk} - \bar{\alpha}_{kj} = \min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]}\}, \qquad \forall k, j \in \langle K \rangle, j \neq k \tag{4.35}$$

Denote $\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]}$ as the power level "gain" of user k against user j under state l_k, $k, j \in \langle K \rangle$. From (4.33) and (4.35), one can find that for general K-user compound interference channels, in its regular counterpart, the channel strength level of the direct link for user k is equal to that of the weakest direct link among all states of user k in the compound channel, and the power level gain of user k against user j (i.e., $\bar{\alpha}_{kk} - \bar{\alpha}_{kj}$) is equal to the minimal power level gain of user k against user j among all states of user k in the compound setting. It is notable that in general for user k of the original compound channel, the state with the weakest direct link and the state with the minimal power level gain of user k against user j are not the same. As a consequence, the regular interference channel $\mathcal{IC}_R$ is a non-trivial mixture of states of the compound channel $\mathcal{IC}_C$.

The following theorem establishes that for a K-user compound interference channel $\mathcal{IC}_C$ and its regular counterpart $\mathcal{IC}_R$, the GDoF-based power control problems are equivalent.

Theorem 4.2. *The K-user compound interference channel $\mathcal{IC}_C$ and its counterpart K-user regular interference channel $\mathcal{IC}_R$ have the same TIN region $\mathcal{P}^*$ and the same polyhedral TIN region $\mathcal{P}$. Moreover, for any feasible GDoF tuple in $\mathcal{P}$, $\mathcal{IC}_C$ and $\mathcal{IC}_R$ have the same set of locally optimal power allocations.*

Example 4.2. *In Fig. 4.2, we show a 3-user compound interference channel (Receiver 1 has two possible states) and its counterpart 3-user regular interference channel, where the values on each link denotes the channel strength level. According to Theorem 4.2, not only the two channels in Fig. 4.2 have the same TIN region, but also solving the GDoF-based power control problem for one is equivalent to solving the problem for the other. Note that in*

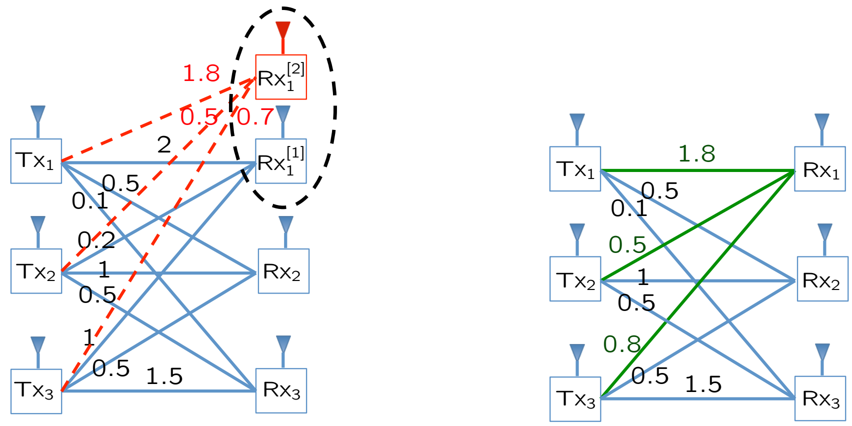

Figure 4.2: A 3-user compound interference channel and its regular counterpart

Fig. 4.2, the regular channel is different from either of the two possible network realizations for the compound channel.

We first prove that $\mathcal{IC}_C$ and $\mathcal{IC}_R$ have the same TIN region and the same polyhedral TIN region. In fact, there are several ways to complete the proof. First, we give an explanation based on potential graph. From the potential graph D_p of $\mathcal{IC}_C$, we can construct another complete digraph $\bar{D}_p = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$ with $K+1$ vertices, where

$$
\begin{aligned}
\bar{\mathcal{V}} &= \{\bar{v}_1, \bar{v}_2, ..., \bar{v}_K, u\} \\
\bar{\mathcal{E}} &= \bar{\mathcal{E}}_1 \cup \bar{\mathcal{E}}_2 \cup \bar{\mathcal{E}}_3 \\
\bar{\mathcal{E}}_1 &= \{(\bar{v}_k, \bar{v}_j)\}, \qquad && \forall k, j \in \langle K \rangle, k \neq j \\
\bar{\mathcal{E}}_2 &= \{(\bar{v}_k, u)\}, && \forall k \in \langle K \rangle \\
\bar{\mathcal{E}}_3 &= \{(u, \bar{v}_k)\}, && \forall k \in \langle K \rangle
\end{aligned}
$$

The length $l(\bar{e})$ of each edge $\bar{e} \in \bar{\mathcal{E}}$ is assigned as follows,

$$
l(\bar{v}_k, \bar{v}_j) = \min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]}\} - d_k, \qquad \forall k, j \in \langle K \rangle, k \neq j
$$

$$l(\bar{v}_k, u) = \min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]}\} - d_k, \qquad \forall k \in \langle K \rangle$$

$$l(u, \bar{v}_k) = 0, \qquad \forall k \in \langle K \rangle$$

It is easy to verify that $\bar{D}_p$ is the potential graph of $\mathcal{IC}_R$. It is also not hard to check that the length of the shortest path from the vertex $v_k^{[l_k]}$ to the vertex $v_j^{[l_j]}$ in D_p is equal to that of the shortest path from $\bar{v}_k$ to $\bar{v}_j$ in $\bar{D}_p$, $\forall k, j \in \langle K \rangle$, $j \neq k$. Similarly, the length of the shortest path from u (or $v_k^{[l_k]}$) to $v_k^{[l_k]}$ (or u) in D_p is equal to that of the shortest path from u (or $\bar{v}_k$) to $\bar{v}_k$ (or u) in $\bar{D}_p$, $\forall k \in \langle K \rangle$. Denote the polyhedral TIN regions of $\mathcal{IC}_C$ and $\mathcal{IC}_R$ as $\mathcal{P}_C$ and $\mathcal{P}_R$, respectively. In $\mathcal{IC}_C$, for any GDoF tuple $(d_1^*, d_2^*, ..., d_K^*) \in \mathcal{P}_C$, according to the proof of Theorem 4.1, we have known that in the potential graph D_p, there exists no circuit with a negative length. Then it is easy to check that for $\mathcal{IC}_R$, when the target GDoF tuple is $(d_1^*, d_2^*, ..., d_K^*)$, all the circuits in $\bar{D}_p$ have non-negative lengths as well. Thus, $(d_1^*, d_2^*, ..., d_K^*) \in \mathcal{P}_R$ and $\mathcal{P}_C \subseteq \mathcal{P}_R$. Similarly, we have $\mathcal{P}_R \subseteq \mathcal{P}_C$. Therefore, we establish that $\mathcal{IC}_C$ and $\mathcal{IC}_R$ have the same polyhedral TIN region. Further, it is not hard to argue that they also have the same TIN region.

The second approach is more straightforward. To prove $\mathcal{IC}_C$ and $\mathcal{IC}_R$ have the same TIN region, we only need to show that with the same power allocation $\mathbf{r}^* = (r_1^*, r_2^*, ..., r_K^*)$ in $\mathcal{IC}_C$ and $\mathcal{IC}_R$, we obtain the same GDoF tuple. In the compound channel $\mathcal{IC}_C$, when the power allocation is $(r_1^*, r_2^*, ..., r_K^*)$, we have

$$d_k^\dagger = \min_{l_k \in \langle L_k \rangle} \left\{\alpha_{kk}^{[l_k]} + r_k^* - (\max_{j \neq k}\{\alpha_{kj}^{[l_k]} + r_j^*\})^+\right\} \tag{4.36}$$

$$= \min_{l_k \in \langle L_k \rangle} \left\{\alpha_{kk}^{[l_k]} + r_k^* - \max\{0, \max_{j \neq k}\{\alpha_{kj}^{[l_k]} + r_j^*\}\}\right\} \tag{4.37}$$

$$= \min_{l_k \in \langle L_k \rangle} \left\{\alpha_{kk}^{[l_k]} + r_k^* - \max_{j \neq k}\{\alpha_{kj}^{[l_k]} + r_j^*, 0\}\right\} \tag{4.38}$$

$$= r_k^* + \min_{l_k \in \langle L_k \rangle} \left\{\alpha_{kk}^{[l_k]} - \max_{j \neq k}\{\alpha_{kj}^{[l_k]} + r_j^*, 0\}\right\} \tag{4.39}$$

$$= r_k^* + \min_{l_k \in \langle L_k \rangle} \left\{\alpha_{kk}^{[l_k]} + \min_{j \neq k}\{-\alpha_{kj}^{[l_k]} - r_j^*, 0\}\right\} \tag{4.40}$$

$$=r_k^* + \min_{l_k \in \langle L_k \rangle} \big\{ \min_{j \neq k} \{\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]} - r_j^*, \alpha_{kk}^{[l_k]}\}\big\} \tag{4.41}$$

For user $k \in \langle K \rangle$ in $\mathcal{IC}_C$, the achievable GDoF value is

$$d_k = \min_{l_k \in \langle L_k \rangle} \big\{ \max\{0, \alpha_{kk}^{[l_k]} + r_k - (\max_{j:j \neq k}\{\alpha_{kj}^{[l_k]} + r_j\})^+\}\big\} \tag{4.42}$$

$$= \max\{0, d_k^{\dagger}\} \tag{4.43}$$

$$= \max \left\{ 0, r_k^* + \min_{l_k \in \langle L_k \rangle} \big\{ \min_{j \neq k} \{\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]} - r_j^*, \alpha_{kk}^{[l_k]}\}\big\} \right\} \tag{4.44}$$

In its regular counterpart $\mathcal{IC}_R$, with the same power allocation $(r_1^*, r_2^*, ..., r_K^*)$, we obtain

$$d_k^{\dagger\dagger} = \bar{\alpha}_{kk} + r_k^* - (\max_{j \neq k}\{\bar{\alpha}_{kj} + r_j^*\})^+ \tag{4.45}$$

$$= \bar{\alpha}_{kk} + r_k^* - \max_{j \neq k}\{\bar{\alpha}_{kj} + r_j^*, 0\} \tag{4.46}$$

$$= \bar{\alpha}_{kk} + r_k^* + \min_{j \neq k}\{-\bar{\alpha}_{kj} - r_j^*, 0\} \tag{4.47}$$

$$= r_k^* + \min_{j \neq k}\{\bar{\alpha}_{kk} - \bar{\alpha}_{kj} - r_j^*, \bar{\alpha}_{kk}\} \tag{4.48}$$

$$= r_k^* + \min_{j \neq k} \big\{ \min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]} - r_j^*\}, \min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]}\}\big\} \tag{4.49}$$

$$= r_k^* + \min_{j \neq k} \big\{ \min_{l_k \in \langle L_k \rangle} \{\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]} - r_j^*, \alpha_{kk}^{[l_k]}\}\big\} \tag{4.50}$$

$$= r_k^* + \min_{l_k \in \langle L_k \rangle} \big\{ \min_{j \neq k} \{\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]} - r_j^*, \alpha_{kk}^{[l_k]}\}\big\} \tag{4.51}$$

For user $k \in \langle K \rangle$ in $\mathcal{IC}_R$, the achievable GDoF value is

$$d_k = \max\{0, d_k^{\dagger\dagger}\} \tag{4.52}$$

$$= \max \left\{ 0, r_k^* + \min_{l_k \in \langle L_k \rangle} \big\{ \min_{j \neq k} \{\alpha_{kk}^{[l_k]} - \alpha_{kj}^{[l_k]} - r_j^*, \alpha_{kk}^{[l_k]}\}\big\} \right\} \tag{4.53}$$

Comparing (4.44) with (4.53), we establish that $\mathcal{IC}_C$ and $\mathcal{IC}_R$ have the same TIN region. Based on the above proof, we can further prove that $\mathcal{IC}_C$ and $\mathcal{IC}_R$ have the same polyhedral TIN region. Again, denote by $\mathcal{P}_C$ and $\mathcal{P}_R$ the polyhedral TIN regions of $\mathcal{IC}_C$ and $\mathcal{IC}_R$, respectively. Also denote by $\mathcal{S}_C$ and $\mathcal{S}_R$ the sets of all valid power allocations for the polyhedral TIN scheme of $\mathcal{IC}_C$ and $\mathcal{IC}_R$, respectively. In $\mathcal{IC}_C$, with any power allocation $\mathbf{r}^* = (r_1^*, r_2^*, ..., r_K^*) \in \mathcal{S}_C$, the obtained GDoF value of user $k \in \langle K \rangle$ is $d_k^\dagger$ in (4.41) and $d_k^\dagger \geq 0$. And the obtained GDoF tuple is $\mathbf{d} = \{d_1^\dagger, d_2^\dagger, ..., d_K^\dagger\} \in \mathcal{P}_C$. According to (4.41) and (4.51), in $\mathcal{IC}_R$ with the same power allocation $\mathbf{r}^*$, for user $k \in \langle K \rangle$ the obtained GDoF value is $d_k^{\dagger\dagger} = d_k^\dagger \geq 0$. Thus in $\mathcal{IC}_R$, $\mathbf{r}^* \in \mathcal{S}_R$ and $\mathbf{d} \in \mathcal{P}_R$. So we obtain that $\mathcal{P}_C \subseteq \mathcal{P}_R$. Similarly, we can argue that $\mathcal{P}_R \subseteq \mathcal{P}_C$. Therefore, $\mathcal{P}_C = \mathcal{P}_R$.

Finally, through contradictions we can prove that for any feasible GDoF tuple in the polyhedral TIN region $\mathcal{P}$, $\mathcal{IC}_C$ and $\mathcal{IC}_R$ have the same set of locally optimal power allocations. For any feasible GDoF tuple $\mathbf{d} \in \mathcal{P}$, denote by $\mathcal{L}_C$ and $\mathcal{L}_R$ the sets of locally optimal solutions of $\mathcal{IC}_C$ and $\mathcal{IC}_R$, respectively. We assume that $\mathcal{L}_C \neq \mathcal{L}_R$. Then we have a power allocation $\mathbf{r}$ such that $\mathbf{r} \in \mathcal{L}_C$ and $\mathbf{r} \notin \mathcal{L}_R$ (or $\mathbf{r} \notin \mathcal{L}_C$ and $\mathbf{r} \in \mathcal{L}_R$), which clearly contradicts the fact that with the same power allocation, $\mathcal{IC}_C$ and $\mathcal{IC}_R$ obtain the same GDoF tuple. Therefore, we complete the proof of Theorem 4.2.

Remark 4.2. *It should be noted that the compound channel and its regular counterpart are only equivalent in terms of the achievability of the TIN scheme. It is not hard to verify that if the compound interference channel* $\mathcal{IC}_C$ *satisfies the condition (4.13), its regular counterpart* $\mathcal{IC}_R$ *is a TIN-optimal interference channel. However, the converse is not true. In [37], we have shown that for a compound interference channel, even if its regular counterpart satisfies the TIN-optimality condition of Theorem 2.1, there may exist other achievable schemes outperforming the TIN scheme in the original compound setting.*

4.3.4 Properties of Potential Graph

In this section, we explore some properties of the potential graph, which will help us develop efficient GDoF-based power control algorithms. We present a useful lemma first.

Lemma 4.1. *In a K-user compound interference channel, for any achievable GDoF tuple $(d_1, d_2, ..., d_K)$ in the polyhedral TIN region $\mathcal{P}$, using $(l_{1,\text{dst}}, l_{2,\text{dst}}, ..., l_{K,\text{dst}})$ in the potential graph D_p as the transmit power allocation and treating interference as noise at each receiver, we obtain a GDoF tuple $(\hat{d}_1, \hat{d}_2, ..., \hat{d}_K) \in \mathcal{P}$, which dominates $(d_1, d_2, ..., d_K)$, i.e., $\hat{d}_k \geq d_k$, $\forall k \in \langle K \rangle$.*

Proof of Lemma 4.1: In a K-user compound interference channel, start with any achievable GDoF tuple $(d_1, d_2, ..., d_K) \in \mathcal{P}$. According to the proof of Theorem 4.1, for all the achievable GDoF tuples in $\mathcal{P}$, there exist valid potential functions in the potential graph D_p, and all the directed circuits in D_p have non-negative lengths. It is easy to verify that if in D_p each directed circuit has a non-negative length, the length of the shortest path starting from the ground vertex u to each vertex in D_p is a valid potential function. Specifically, we have a valid potential function

$$p(v_k^{[l_k]}) = l_{k,\text{dst}} \leq 0, \quad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle, \tag{4.54}$$

and

$$p(u) = 0. \tag{4.55}$$

According to the definition of potential function (i.e., for any edge $(a, b) \in \mathcal{E}$, $l(a, b) \geq p(b) - p(a)$), from (4.17), (4.18), (4.54) and (4.55) we have

$$d_k \leq \alpha_{kk}^{[l_k]} + l_{k,\text{dst}} - (\alpha_{kj}^{[l_k]} + l_{j,\text{dst}}), \quad \forall k, j \in \langle K \rangle, k \neq j, \forall l_k \in \langle L_k \rangle, \tag{4.56}$$

$$d_k \leq \alpha_{kk}^{[l_k]} + l_{k,\mathrm{dst}}, \qquad \forall k \in \langle K \rangle, \forall l_k \in \langle L_k \rangle \tag{4.57}$$

Due to (4.56) and (4.57), we obtain

$$d_k \leq \min_{l_k \in \langle L_k \rangle} \left\{ \alpha_{kk}^{[l_k]} + l_{k,\mathrm{dst}} - (\max_{j:j \neq k} \{ \alpha_{kj}^{[l_k]} + l_{j,\mathrm{dst}} \})^+ \right\}, \quad \forall k \in \langle K \rangle \tag{4.58}$$

Using $(l_{1,\mathrm{dst}}, l_{2,\mathrm{dst}}, ..., l_{K,\mathrm{dst}})$ as the transmit power allocation, which satisfies the unit power constraint for each user according to (4.54), in the TIN scheme the achievable GDoF value of user $k \in \langle K \rangle$ is given by

$$\hat{d}_k = \min_{l_k \in \langle L_k \rangle} \left\{ \max\{0, \alpha_{kk}^{[l_k]} + l_{k,\mathrm{dst}} - (\max_{j:j \neq k} \{ \alpha_{kj}^{[l_k]} + l_{j,\mathrm{dst}} \})^+ \} \right\} \tag{4.59}$$

$$\geq \min_{l_k \in \langle L_k \rangle} \left\{ \alpha_{kk}^{[l_k]} + l_{k,\mathrm{dst}} - (\max_{j:j \neq k} \{ \alpha_{kj}^{[l_k]} + l_{j,\mathrm{dst}} \})^+ \right\} \tag{4.60}$$

$$\geq d_k \tag{4.61}$$

So we complete the proof. ■

As discussed before, to get the shortest path results in the potential graph D_p of the original K-user compound channel $\mathcal{IC}_C$, we only need to deal with a corresponding single-source shortest path problem in $\bar{D}_p$ (i.e., the potential graph of the regular counterpart $\mathcal{IC}_R$). This problem can be solved efficiently via the classical Bellman-Ford algorithm with complexity $O(K^3)$. It is also well known that Bellman-Ford algorithm can detect the negative-length circuits in a digraph with arbitrary-length edges [43]. Recall that for a K-user compound interference channel, when the target GDoF tuple is out of the polyhedral TIN region $\mathcal{P}$ (i.e., it is not achievable via the polyhedral TIN scheme), there exist negative-length circuits in D_p and $\bar{D}_p$. Therefore, Bellman-Ford algorithm can also serve as the feasibility test for the target GDoF tuples. Correspondingly, in the conventional SINR-based power control algorithms for regular interference channels, the feasibility of SINR targets can be checked through centralized computations based on non-negative matrix theory [61]. How to test the feasibility in a distributed fashion is still open.

To optimize the system performance, we are primarily interested in the *Pareto optimal* GDoF tuples in the feasible GDoF region. If a GDoF tuple $(d_1, d_2, ..., d_K)$ is Pareto optimal for the polyhedral TIN region $\mathcal{P}$, it indicates that in $\mathcal{P}$ there does not exist another distinct tuple $(\hat{d}_1, \hat{d}_2, ..., \hat{d}_K)$ such that $\hat{d}_k \geq d_k$, $\forall k \in \langle K \rangle$. Therefore, based on Theorem 4.2 and Lemma 4.1, we obtain the following theorem straightforwardly.

Theorem 4.3. *In a K-user compound interference channel $\mathcal{IC}_C$, for any Pareto optimal GDoF tuple in its polyhedral TIN region $\mathcal{P}$, $(l_{1,\text{dst}}, l_{2,\text{dst}}, ..., l_{K,\text{dst}})$ in the potential graph $\bar{D}_p$ of its regular counterpart $\mathcal{IC}_R$ is a locally optimal transmit power allocation.*

Since in a K-user compound interference channel where condition (4.13) holds, its GDoF region is exactly the polyhedral TIN region $\mathcal{P}$, we have the following corollary.

Corollary 4.1. *In a K-user compound interference channel $\mathcal{IC}_C$, if the condition (4.13) is satisfied, for any Pareto optimal GDoF tuple in its GDoF region, $(l_{1,\text{dst}}, l_{2,\text{dst}}, ..., l_{K,\text{dst}})$ in the potential graph $\bar{D}_p$ of its regular counterpart $\mathcal{IC}_R$ is a locally optimal transmit power allocation.*

4.3.5 Fixed-point Power Control Algorithm

In the following, we will demonstrate how to obtain the optimal power allocation for an arbitrary achievable GDoF tuple in the polyhedral TIN region $\mathcal{P}$. It is notable that there exist remarkable differences between the power control problems for target *rate* tuples and *GDoF* tuples. For an achievable *rate* tuple, there only exists a unique locally optimal transmit power vector, which is the globally optimal power control solution naturally. However, as mentioned before, for one feasible *GDoF* tuple, in general there are multiple locally optimal power allocations. It is also well known that for any feasible target *rate* tuple, a celebrated synchronous fixed-point power control algorithm developed by Foschini and Miljanic [59] provides the globally optimal power allocation. In this section, we will show that for an achievable *GDoF* tuple, through a similar GDoF-based synchronous fixed-point power control (GSFPC) algorithm, we obtain a locally optimal solution, which is *not* globally optimal in general. The GSFPC algorithm for general K-user compound interference channels is specified as follows.[5]

Algorithm 1 GDoF-based synchronous fixed-point power control (GSFPC)

1) Initialize: set $r_k(0) = l_{k,dst}$ for $k \in \langle K \rangle$;
2) Update:

$$r_k(n+1) = d_k - \min_{l_k \in \langle L_k \rangle} \left\{ \alpha_{kk}^{[l_k]} - (\max_{j:j \neq k} \{ \alpha_{kj}^{[l_k]} + r_j(n) \})^+ \right\}, \quad k \in \langle K \rangle \tag{4.62}$$

where n indexes discrete time slots.

The following theorem demonstrates the convergence of the GSFPC algorithm.

Theorem 4.4. *In a K-user compound interference channel, for any achievable GDoF tuple in its polyhedral TIN region $\mathcal{P}$, the GSFPC algorithm converges to a locally optimal transmit power allocation. Further, if there are multiple locally optimal power allocations $\mathbf{r}^*$ satisfying $\mathbf{r}^* \leq (l_{1,dst}, l_{2,dst}, ..., l_{K,dst})$, denote by $\mathcal{R}_l$ the set of all such locally optimal solutions. Then, $\mathbf{r}^{**}$, the locally optimal power allocation obtained from the GSFPC algorithm, satisfies $\mathbf{r}^{**} \geq \mathbf{r}^*$, $\forall \mathbf{r}^* \in \mathcal{R}_l$.*

For conventional SINR-based fixed-point power control algorithms, the convergence is usually proved through the framework of standard interference function developed in [60], which turns out to be inapplicable to the GDoF setting. To proceed with the proof of Theorem

[5]Note that to solve the power control problem for compound channels, according to Theorem 4.2, we can also apply the GSFPC algorithm to its regular counterpart.

4.4, define $\mathbf{r}(n) = (r_1(n), r_2(n), ..., r_K(n))$, and rewrite (4.62) in the vector form

$$\mathbf{r}(n+1) = f(\mathbf{r}(n)) \tag{4.63}$$

where $f : \mathbb{R}^K \to \mathbb{R}^K$ is the update function in the GSFPC algorithm.

We first demonstrate the convergence of the GSFPC algorithm. To this end, we go through the following three steps:

- First, we show that $\{\mathbf{r}(n)\}_{n=0}^{\infty}$ is a decreasing sequence through induction. Due to (4.58), we have

$$d_k \leq \min_{l_k \in \langle L_k \rangle} \left\{ \alpha_{kk}^{[l_k]} + l_{k,\text{dst}} - (\max_{j:j\neq k}\{\alpha_{kj}^{[l_k]} + l_{j,\text{dst}}\})^+ \right\} \tag{4.64}$$

$$\Leftrightarrow l_{k,\text{dst}} \geq d_k - \min_{l_k \in \langle L_k \rangle} \left\{ \alpha_{kk}^{[l_k]} - (\max_{j:j\neq k}\{\alpha_{kj}^{[l_k]} + l_{j,\text{dst}}\})^+ \right\} \tag{4.65}$$

$$\Leftrightarrow r_k(0) \geq d_k - \min_{l_k \in \langle L_k \rangle} \left\{ \alpha_{kk}^{[l_k]} - (\max_{j:j\neq k}\{\alpha_{kj}^{[l_k]} + r_j(0)\})^+ \right\} \tag{4.66}$$

  Writing the above inequality in the vector form, we get

$$\mathbf{r}(0) \geq f(\mathbf{r}(0)) = \mathbf{r}(1) \tag{4.67}$$

  Next, assume $\mathbf{r}(k) \leq \mathbf{r}(k-1)$. Since f is an increasing function, we obtain

$$\mathbf{r}(k+1) = f(\mathbf{r}(k)) \leq f(\mathbf{r}(k-1)) = \mathbf{r}(k) \tag{4.68}$$

  Therefore, the sequence $\{\mathbf{r}(n)\}_{n=0}^{\infty}$ is decreasing.

- Second, we prove that there exists a fixed point $\mathbf{r}^* = (r_1^*, r_2^*, ..., r_K^*) \leq \mathbf{r}(0)$ such that $\mathbf{r}^* = f(\mathbf{r}^*)$. It is not hard to verify that the following conditions are satisfied: (i) f is a continuous increasing function, since the maximum/minimum of continuous functions is still continuous; (ii) According to (4.67), the set $\mathcal{S}_{\mathbf{r}} = \{\mathbf{r} : \mathbf{r} \geq f(\mathbf{r})\}$ is not empty; (iii) The set $\mathcal{S}_{\mathbf{r}}$ is bounded from below (i.e., for each power allocation $\bar{\mathbf{r}} \in \mathcal{S}_r$, its i-th entry $\bar{r}_i$ is bounded away from negative infinity). Therefore, according to the fixed point theorem in [75] (Proposition 6 in [75]), we establish that for $\mathbf{r}(0) \in \mathcal{S}_{\mathbf{r}}$, there exists a $\mathbf{r}^* \leq \mathbf{r}(0)$ such that $\mathbf{r}^* = f(\mathbf{r}^*)$.

- Finally, we illustrate that the sequence $\{\mathbf{r}(n)\}_{n=0}^{\infty}$ is bounded by induction. From the second step, we have known that $\mathbf{r}(0) \geq \mathbf{r}^*$. Next, assume $\mathbf{r}(k) \geq \mathbf{r}^*$. We have

$$\mathbf{r}(k+1) = f(\mathbf{r}(k)) \geq f(\mathbf{r}^*) = \mathbf{r}^* \tag{4.69}$$

  Therefore, the sequence $\{\mathbf{r}(n)\}_{n=0}^{\infty}$ is bounded from below by $\mathbf{r}^*$.

According to the above steps, we have known that the sequence $\{\mathbf{r}(n)\}_{n=0}^{\infty}$ is decreasing and bounded, thus convergent. As a fixed-point algorithm is able to estimate a fixed point if and only if $\{\mathbf{r}(n)\}_{n=0}^{\infty}$ converges, and clearly the obtained fixed point solution is a locally optimal power allocation for the target GDoF tuple, we complete the proof for the convergence of the GSFPC algorithm.

It is also not hard to prove that the locally optimal power allocation yielded by the GSFPC algorithm, i.e., $\mathbf{r}^{**}$, satisfies $\mathbf{r}^{**} \geq \mathbf{r}^*$, $\forall \mathbf{r}^* \in \mathcal{R}_l$. Let $\mathbf{r}^*$ be any locally optimal power allocation satisfying $\mathbf{r}^* \leq \mathbf{r}(0)$. In the proof above, we have known that $\{\mathbf{r}(n)\}_{n=0}^{\infty}$ is bounded from below by $\mathbf{r}^*$. Therefore, we establish that $\mathbf{r}^{**} = \lim_{n\to\infty} \mathbf{r}(n) \geq \mathbf{r}^*$, $\forall \mathbf{r}^* \in \mathcal{R}_l$.

4.3.6 GDoF-optimal Power Control Algorithm

In this section, we show how to obtain the *globally optimal* power allocations for all the feasible GDoF tuples in the polyhedral TIN region $\mathcal{P}$. We start with a motivating example.

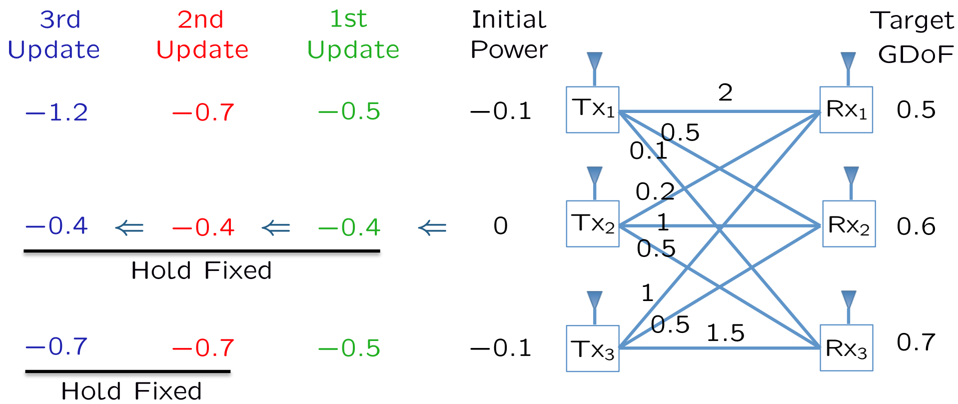

Figure 4.3: The transmit power updates for a 3-user interference channel

Example 4.3. *Consider the 3-user interference channel in Fig. 4.3. In this channel, we intend to achieve the GDoF tuple* $(d_1, d_2, d_3) = (0.5, 0.6, 0.7)$*. Start with the initial power allocation given by the shortest path result in the potential graph* $\big(r_1(0), r_2(0), r_3(0)\big) = (-0.1, 0, -0.1)$*. It is easy to find that if all users reduce their transmit powers by the same amount below a threshold* $\Delta(0)$*, we still achieve an acceptable GDoF tuple,*[6] *where*

$$\Delta(0) = \min_i \big\{r_i(0) + \alpha_{ii} - d_i\big\} = 0.4, \quad i \in \{1, 2, 3\}$$

Thus in the first update, each user lowers its transmit power by 0.4*. The resulting transmit power allocation becomes* $\big(r_1(1), r_2(1), r_3(1)\big) = (-0.5, -0.4, -0.5)$*, and the achieved GDoF tuple is* $(1, 0.6, 0.9)$*. At this point, user 2 cannot reduce its transmit power further while maintaining the desired GDoF value* 0.6 *(if* $r_2 < -0.4$*, then* $r_2 + \alpha_{22} < 0.6$*). The consequence is that in the following updates, Transmitter* 2 *exerts fixed interference levels to Receiver* 1 *and* 3*, whose strengths are* $\max\{0, \alpha_{12} + r_2(1))\} = 0$ *and* $\max\{0, \alpha_{32} + r_2(1)\} = 0.1$*, respectively. In addition, it is easy to check that the resulting power allocation after the first update is not the desired globally optimal power allocation. So we proceed the transmit power updates. In*

[6]In the GDoF-based power control problem, if for each user the achievable GDoF value is no less than the target value, then this GDoF tuple is referred to as an acceptable GDoF tuple.

the second update, the other two users (i.e., user 1 and 3) can further reduce their transmit power simultaneously by the same amount below a threshold $\Delta(1)$ *to obtain an acceptable GDoF tuple, where*

$$\Delta(1) = \min_i \big\{ r_i(1) + \alpha_{ii} - d_i - \max\{0, \alpha_{i2} + r_2(1))\} \big\} = 0.2, \quad i \in \{1, 3\}$$

After the second update, by keeping the transmit power of user 2 *as* -0.4 *and reducing the transmit powers of user* 1 *and* 3 *by* 0.2*, we obtain the power allocation* $\big(r_1(2), r_2(2), r_3(2)\big) = (-0.7, -0.4, -0.7)$ *and its corresponding achieved GDoF tuple* $(1, 0.6, 0.7)$*. Now user* 3 *cannot further reduce its transmit power while maintaining the desired GDoF value* 0.7*. Thus in the following update, Transmitter* 3 *exerts a fixed interference level to Receiver* 1 *with strength* $\max\{0, \alpha_{13} + r_3(2)\} = 0.3$*. To obtain an acceptable GDoF tuple, at this point only user* 1 *can further lower its transmit power by an amount below the threshold*

$$\Delta(2) = r_1(2) + \alpha_{11} - d_1 - \max\{0, \alpha_{1j} + r_j(2))\} = 0.5, \quad j \in \{2, 3\}$$

Finally, we get the power allocation $\big(r_1(3), r_2(3), r_3(3)\big) = (-1.2, -0.4, -0.7)$*, which is in fact the globally optimal power allocation for the target GDoF tuple* $(0.5, 0.6, 0.7)$*.*

For K-user interference channels, to obtain the globally optimal power allocation for an achievable GDoF tuple $(d_1, d_2, ..., d_K) \in \mathcal{P}$, we develop an iterative algorithm with at most K updates, which is called GDoF-based globally-optimal power control (GGPC) algorithm and specified at the top of next page.

In the GGPC algorithm, we can use Bellman-Ford algorithm to calculate the lengths of the shortest paths $l_{i,dst}$ $(i \in \langle K \rangle)$ in the initialize phase, which also serves as the feasibility test for the target GDoF tuple. For the feasible GDoF tuple, we repeat the update phases until $\mathcal{I} = \phi$ to obtain its globally optimal power control solution. Starting from the initial power allocation, in each update the GGPC algorithm reduces the transmit powers of certain users

Algorithm 2 GDoF-based globally-optimal power control (GGPC)

1) Initialize: set $\mathcal{I} = \langle K \rangle$, $\mathcal{M} = \phi$, and $r_i(0) = l_{i,dst}$ for $i \in \langle K \rangle$;
2) Update:

$$\Delta(n) = \min_i \Big\{ r_i(n) + \alpha_{ii} - d_i - \max_{m \neq i} \{0, \alpha_{im} + r_m(n)\} \Big\}, \quad i \in \mathcal{I}, m \in \mathcal{M} \tag{4.70}$$

$$\mathcal{N} = \arg\min_i \Big\{ r_i(n) + \alpha_{ii} - d_i - \max_{m \neq i} \{0, \alpha_{im} + r_m(n)\} \Big\}, \quad i \in \mathcal{I}, m \in \mathcal{M} \tag{4.71}$$

$$r_i(n+1) = r_i(n) - \Delta(n), \quad i \in \mathcal{I} \tag{4.72}$$

$$r_m(n+1) = r_m(n), \quad m \in \mathcal{M} \tag{4.73}$$

$$\mathcal{I} = \mathcal{I} \backslash \mathcal{N}, \mathcal{M} = \mathcal{M} \cup \mathcal{N} \tag{4.74}$$

where n indexes discrete time slots. The update phase terminates when $\mathcal{I} = \phi$.

to their limits (these users also achieve the target GDoF value exactly after the update) and still guarantees to achieve an acceptable GDoF tuple. Then these users hold their powers fixed, which in turn exert fixed interference levels for the remaining users participating in the following updates. These fixed interference levels will limit how much power can be further reduced by the remaining users. The optimality of the GGPC algorithm is illustrated in the following theorem.

Theorem 4.5. *In a K-user interference channel, for any achievable GDoF tuple in its polyhedral TIN region $\mathcal{P}$, the GGPC algorithm yields the globally optimal transmit power allocation.*

The proof of Theorem 4.5 is relegated to Appendix B. Combining Theorem 4.2 with Theorem 4.5, we get the following corollary on compound interference channels.

Corollary 4.2. *In a K-user compound interference channel $\mathcal{IC}_C$, for any achievable GDoF tuple in its polyhedral TIN region $\mathcal{P}$, applying the GGPC algorithm to its regular counterpart $\mathcal{IC}_R$ yields the globally optimal transmit power allocation for both $\mathcal{IC}_C$ and $\mathcal{IC}_R$.*

4.4 Summary

In this chapter, we first generalize the optimality of TIN to compound interference channels. We demonstrate that for a K-user compound Gaussian interference channel, if in each possible network state, the channel satisfies the TIN-optimality condition identified in Theorem 2.1, then its GDoF region is the intersection of the GDoF regions of all possible network states, which is achievable via the TIN scheme. We also fully characterize the TIN region for compound interference channels with arbitrary channel strengths. Next, the power control problem is investigated from the GDoF perspective for compound networks. Notably, we show that to solve the GDoF-based power control problem for a K-user compound interference channel with arbitrary number of states for each receiver, we only need to construct its nontrivial counterpart K-user regular interference channel, and solve the power control problem in this new channel. Finally, for general K-user compound interference channels, we develop several power control algorithms to obtain the optimal transmit power allocations for feasible GDoF tuples.

Note that combining the TIN region characterization with GDoF-based power control algorithms, we also solve the joint GDoF assignment and power control problem for general K-user compound interference channels. For channels where power control and TIN is optimal from the GDoF perspective, this approach provides a constant gap guarantee for the information-theoretically optimal result (e.g., the sum capacity at any finite SNR). As a byproduct, it provides an alternative perspective to deal with the challenging joint rate assignment and power control problem as well. In [37, 76], some numerical results indicate that this GDoF-based approach may attain close performance to its sophisticated SINR-based counterpart in the finite SNR setting. This line of research is still in its infancy, and substantial efforts are needed to evaluate the performance of GDoF-based power control schemes for general network settings with finite SNR values.

Chapter 5

TIN-optimal Interference Channels with Confidential Messages

In this chapter, we study Gaussian networks with information theoretical secrecy constraints. A canonical model is the Gaussian wiretap channel [77, 78], where the secure capacity has been characterized as the difference between the capacities of the main and the wiretap channels, implying that there exists a capacity penalty for ensuring the secrecy. In multi-user settings (e.g., interference channels), since the exact capacity is in general intractable even without secrecy constraints, much of the recent progress has come from the pursuit of capacity approximations along the progressive refinement path presented in Chapter 1 (i.e., starting with the coarse DoF metric, this approach progressively targets finer metrics in the form of GDoF, capacity within a constant gap, and ultimately the exact capacity). For K-user interference channels with confidential messages, the secure sum DoF value has been established in [79, 80] as $\frac{K(K-1)}{2K-1}$. Compared with the conventional setting without secrecy constraints (where the sum DoF value is $\frac{K}{2}$), there is a DoF penalty as expected. Following the progressive refinement approach, the next goals are GDoF and constant gap characterizations. In this chapter, we will mainly consider the TIN-optimal Gaussian interference channels (identified in Theorem 2.1) with secrecy constraints.

This chapter is organized as follows. The system model of Gaussian interference channels with confidential messages is described in Section 5.1. In Section 5.2, we show the main result of this chapter, i.e., for TIN-optimal interference channels identified in Theorem 2.1, secrecy constraints on unintended messages incur no GDoF penalty, which also leads to a secure capacity region characterization within a constant gap. In Section 5.3, we consider interference channels with both confidential messages and external eavesdroppers. We summarize this chapter in Section 5.4.

5.1 Channel Model

Consider K-user complex Gaussian interference channels *with confidential messages*. The channel input-output relationship is the same as that of the interference channel (2.4) in Chapter 2, i.e.,

$$Y_k(t) = \sum_{i=1}^{K} \sqrt{P^{\alpha_{ki}}} e^{j\theta_{ki}} X_i(t) + Z_k(t), \quad \forall k \in \langle K \rangle \tag{5.1}$$

Recall that in the K-user interference channel, Transmitter i intends to send an independent message W_i to Receiver i, $\forall i \in \langle K \rangle$. For user i, the size of the message set is denoted by $|W_i|$, and the rate of message W_i is defined as $R_i \triangleq \frac{\log |W_i|}{n}$, where n is the number of channel uses. Receiver i decodes its desired message as $\hat{W}_i$ based on the channel output. In the conventional setting without secrecy constraints (or confidential messages), the rate tuple $(R_1, R_2, ..., R_K)$ is achievable if for any $\epsilon > 0$, there exist n-length codes such that the decoding error probabilities at all the receivers are no larger than ϵ, i.e.,

$$\max_i \Pr\big(W_i \neq \hat{W}_i\big) \leq \epsilon \tag{5.2}$$

For interference channels with confidential messages, the difference is that each message W_i is also required to be kept secret against its unintended receivers. Define $W_{-i}^K \triangleq \{W_k : \forall k \in \langle K \rangle \backslash \{i\}\}$. The information theoretic secrecy constraint is given by

$$H(W_{-i}^K | Y_i^n) \geq H(W_{-i}^K) - n\epsilon, \quad \forall i \in \langle K \rangle \tag{5.3}$$

Essentially, secrecy requires that the channel output Y_i^n is almost independent of W_{-i}^K. In this setting, a secure rate tuple $(R_1^s, R_2^s, ..., R_K^s)$ is achievable if there exist n-length codes such that both the secrecy constraint (5.3) and the decodability constraint (5.2) are satisfied. The secure channel capacity region $\mathcal{C}^s$ is the closure of the set of all achievable secure rate tuples and the secure GDoF region $\mathcal{D}^s$ is defined as

$$\mathcal{D}^s \triangleq \left\{(d_1^s, d_2^s, ..., d_K^s) : d_i^s = \lim_{P \to \infty} \frac{R_i^s}{\log P}, \ \forall i \in \langle K \rangle, (R_1^s, R_2^s, ..., R_K^s) \in \mathcal{C}^s \right\} \tag{5.4}$$

5.2 Results on TIN-optimal Interference Channels

To obtain the main result of this chapter, we first present a useful lemma, where an achievable secure GDoF region is identified for Gaussian interference channels with arbitrary channel strengths.

Lemma 5.1. *In a general K-user Gaussian interference channel, all the GDoF tuples in the polyhedral TIN region $\mathcal{P}$ are achievable under the secrecy constraint (5.3).*

Proof of Lemma 5.1: To complete the proof, we need to invoke the following theorem.

Theorem 5.1. *(Theorem 2 of [80]) In K-user interference channels with confidential mes-*

sages and one external eavesdropper, the following rate region is achievable[1]

$$R_i^s \geq I(V_i; Y_i) - \max_{j \in \{0, \langle K \rangle\} \backslash \{i\}} I(V_i; Y_j | V_{-i}^K), \ \forall i \in \langle K \rangle \tag{5.5}$$

where $V_{-i}^K \triangleq \{V_j : \forall j \in \langle K \rangle \backslash \{i\}\}$. *The auxiliary random variables* V_i *are mutually independent, and for each* i, *we have the following Markov chain* $V_i \to X_i \to (Y_1, Y_2, ..., Y_K)$.

Consider any GDoF tuple $(d_1, d_2, ..., d_K)$ in the polyhedral TIN region $\mathcal{P}$. According to the power control schemes in Chapter 4, we can obtain its optimal power allocation $(r_1, r_2, ..., r_K)$, where $r_i \leq 0$, $i \in \langle K \rangle$. Recall that for $(d_1, d_2, ..., d_K) \in \mathcal{P}$, we have

$$d_i = \alpha_{ii} + r_i - \max\{0, \max_{j:j \neq i}(\alpha_{ij} + r_j)\} \Leftrightarrow r_i - d_i + \alpha_{ii} = \max\{0, \max_{j:j \neq i}(\alpha_{ij} + r_j)\} \tag{5.6}$$

In the following, we show how to obtain the same GDoF tuple $(d_1, d_2, ..., d_K)$ under the secrecy constraint (5.3). Let $X_i = V_i + J_i$, $\forall i \in \langle K \rangle$, where $V_i \sim \mathcal{CN}(0, \frac{P^{r_i}}{2})$ is the message-carrying signal, $J_i \sim \mathcal{CN}(0, \frac{P^{r_i - d_i}}{2})$ is the random jamming signal, and V_i, J_i are independent, $\forall i \in \langle K \rangle$. Note that with the above power allocations, each user satisfies the unit power constraint. The received signal of user k is given by

$$Y_k = \sum_{i=1}^{K} \sqrt{P^{\alpha_{ki}}} e^{j\theta_{ki}} (V_i + J_i) + Z_k, \quad \forall k \in \langle K \rangle \tag{5.7}$$

Consider the first term in the right hand side of (5.5).

$$\begin{aligned} I(V_i; Y_i) &= h(Y_i) - h(Y_i | V_i) && (5.8) \\ &= \log\left(1 + \frac{\frac{1}{2}P^{r_i + \alpha_{ii}}}{1 + \frac{1}{2}\sum_{j \neq i} P^{r_j + \alpha_{ij}} + \frac{1}{2}\sum_j P^{r_j - d_j + \alpha_{ij}}}\right) && (5.9) \\ &> \log\left(\frac{\frac{1}{2}P^{r_i + \alpha_{ii}}}{P^0 + \sum_{j \neq i} P^{r_j + \alpha_{ij}} + \frac{1}{2}P^{r_i - d_i + \alpha_{ii}}}\right) && (5.10) \end{aligned}$$

[1] In Theorem 5.1, Y_0 denotes the output of the external eavesdropper (see equation (5.24)). In the case where we only consider the confidential messages, we simply remove the eavesdropper and ignore this term.

$$\geq \log\left(\frac{\frac{1}{2}P^{r_i+\alpha_{ii}}}{P^{r_i-d_i+\alpha_{ii}}+(K-1)P^{r_i-d_i+\alpha_{ii}}+\frac{1}{2}P^{r_i-d_i+\alpha_{ii}}}\right) \tag{5.11}$$

$$= \log\left(\frac{P^{r_i+\alpha_{ii}}}{(2K+1)P^{r_i-d_i+\alpha_{ii}}}\right) \tag{5.12}$$

$$= d_i \log P - \log(2K+1) \tag{5.13}$$

where (5.10) follows $P > 1$ and $d_i \geq 0$, $i \in \langle K \rangle$, and (5.11) holds due to (5.6).

Next, consider the second term in the right hand side of (5.5), i.e., the secrecy penalty.

$$I(V_i; Y_j | V_{-i}^K) = h(Y_j | V_{-i}^K) - h(Y_j | V_1, V_2, ..., V_K) \tag{5.14}$$

$$= \log\left(1 + \frac{\frac{1}{2}P^{r_i+\alpha_{ji}}}{1+\frac{1}{2}\sum_k P^{r_k-d_k+\alpha_{jk}}}\right) \tag{5.15}$$

$$\leq \log\left(1 + \frac{\frac{1}{2}P^{r_i+\alpha_{ji}}}{\frac{1}{2}P^{r_j-d_j+\alpha_{jj}}}\right) \leq 1 \tag{5.16}$$

where $i, j \in \langle K \rangle$, $i \neq j$, and (5.16) follows (5.6).

Finally, plugging (5.13) and (5.16) into (5.5), we obtain

$$R_i^s \geq d_i \log P - \log 2(2K+1), \ \forall i \in \langle K \rangle \tag{5.17}$$

$$\Rightarrow d_i^s \geq \lim_{P \to \infty} \frac{d_i \log P - \log 2(2K+1)}{\log P} = d_i, \ \forall i \in \langle K \rangle$$

which implies that $(d_1, d_2, ..., d_K)$ is still achievable under the secrecy constraint (5.3). ■

To achieve the polyhedral TIN region $\mathcal{P}$ under the secrecy constraint (5.3), in the above proof of Lemma 5.1, we adopt a Gaussian-based Cooperative Jamming (GCJ) scheme, where each user splits its transmit signal into two parts: the first part carries the desired message based on a Gaussian codebook, and the second part is a randomly generated Gaussian jamming signal helping reduce the information leakage at each receiver. For each user, the power allocation of these two parts is derived from the GDoF-optimal power control solution presented in Chapter 4. Note that the GCJ scheme is a generalization of the Artificial Noise scheme for 2-user Gaussian interference channels [81].

Based on Lemma 5.1, we get the main result of this chapter.

Theorem 5.2. *In a K-user Gaussian interference channel with confidential messages, if the TIN-optimality condition (2.7) is satisfied, then the secure GDoF region is equal to the polyhedral TIN region $\mathcal{P}$ and the GCJ scheme achieves the entire secure capacity region to within a constant gap of no larger than* $\log[6(2K+1)]$ *bits per user.*

According to Theorem 5.2, when condition (2.7) holds, for K-user interference channels, $\mathcal{D}^s = \mathcal{P} = \mathcal{D}$, i.e., the secrecy constraint (5.3) does not introduce GDoF penalty. For Theorem 5.2, in terms of GDoF, the converse follows trivially from Theorem 2.1 since adding a confidentiality constraint cannot help, and the achievability follows from Lemma 5.1 directly. Here we give some intuitive explanations to help interpret this result. Viewed in terms of signal strength levels in the log scale, recall that in the TIN scheme without secrecy constraints, for each user the achievable GDoF value is the amount by which the desired signal strength exceeds the interference strength, i.e., the desired signal levels that are seen above the interference floor. Let's call these the *useful* signal levels for brevity. The desired signal levels that are received above the noise floor but below the interference floor are not useful for communicating the desired message and are therefore left unused in the TIN scheme. This allows receivers to decode and subtract the desired signal and then cleanly observe the remaining interference levels that are received above the noise floor, thus compromising confidentiality. When the secrecy constraints are enforced, the essential difference is that now in the GCJ scheme each transmitter jams all the signal levels below the useful levels, which were previously unused, by sending random noise at all these levels, thus raising the noise floor at its desired receiver to match the interference floor. Therefore, all interfering signals are hidden by the jamming noise. Further, notice that because the jamming noise levels are always below the signal levels, and the signal levels are already below the useful levels of all undesired receivers, the jamming signal does not hurt undesired receivers. Thus, the secrecy constraints are incorporated into the TIN scheme with no GDoF penalty.

Based on the insight obtained from the GDoF study, we are also able to show that the GCJ scheme achieves the entire secure capacity region to within a constant gap for the TIN-optimal interference channels with confidential messages. For the converse, apply the outer bounds in Theorem 2.2 for TIN-optimal interference channels without any secrecy constraints.

$$R_i^s \leq \alpha_{ii} \log P + 1, \quad \forall i \in \langle K \rangle \tag{5.18}$$

$$\sum_{j=1}^{m} R_{i_j}^s < \sum_{j=1}^{m} \Big[(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}) \log P + \log 3 \Big]$$

$$\forall (i_1, i_2, ..., i_m) \in \Pi_K, \quad \forall m \in \{2, 3, ..., K\} \tag{5.19}$$

Next, consider the achievability. For any GDoF tuple $(d_1, d_2, ..., d_K)$ in the polyhedral TIN region $\mathcal{P}$, we have known that under the secrecy constraint (5.3), the rate in (5.17) is always achievable. Then it is easy to verify that the achievable secure rate region includes the tuples $(\bar{R}_1^s, \bar{R}_2^s, ..., \bar{R}_K^s) \in \mathbb{R}_K^+$ satisfying

$$\bar{R}_i^s \leq (\alpha_{ii} \log P - \log 2(2K+1))^+, \forall i \in \langle K \rangle \tag{5.20}$$

$$\sum_{j=1}^{m} \bar{R}_{i_j}^s \leq \left(\sum_{j=1}^{m} (\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}) \log P - m \log 2(2K+1) \right)^+$$

$$\forall (i_1, i_2, ..., i_m) \in \Pi_K, \quad \forall m \in \{2, 3, ..., K\} \tag{5.21}$$

Comparing (5.18), (5.19) with (5.20), (5.21), it is not hard to get that

$$\sigma_{R_i^s} < \log 6(2K+1), \qquad \forall i \in \langle K \rangle \tag{5.22}$$

$$\sigma_{\sum_{j=1}^{m} R_{i_j}^s} \leq m \log 6(2K+1), \quad \forall (i_1, i_2, ..., i_m) \in \Pi_K, \forall m \in \{2, 3, ..., K\}, \tag{5.23}$$

where $\sigma_{(.)}$ denotes the difference between the achievable rate in (5.20) and (5.21) and its corresponding outer bound in (5.18) and (5.19). Therefore, we complete the proof of Theorem 5.2.

Remark 5.1. *Based on Lemma 5.1, the result of no GDoF penalty under the confidentiality constraints can be extended to a class of partially connected TIN-optimal interference channels identified in [76]. Also the results in Theorem 5.2 can be generalized to TIN-optimal $K \times K$ X channels directly.*[2]

Remark 5.2. *Note that for the networks where TIN achieves the exact capacity, in general there is a non-zero capacity penalty for ensuring secrecy. In [82], the authors have shown that for the many-to-one interference channel, in the noisy interference regime where TIN is exactly capacity optimal [7–9], the sum rate upper bound under secrecy constraints is less than the sum rate achieved by the TIN scheme (without secrecy constraints) in some cases.*

5.3 Extension to Channels with External Eavesdroppers

In this section, we mainly investigate the performance of the GCJ scheme in K-user Gaussian interference channels with confidential messages and *an external eavesdropper*. The channel output of the eavesdropper is denoted by

$$Y_0(t) = \sum_{i=1}^{K} \sqrt{P^{\alpha_{0i}}} e^{j\theta_{0i}} X_i(t) + Z_0(t) \tag{5.24}$$

where $Z_0(t) \sim \mathcal{CN}(0,1)$ is the noise term at the eavesdropper. In this channel, besides the secrecy constraint (5.3), we also require that all messages are kept secret against the

[2] In the $K \times K$ X message setting, the secrecy constraint is given by $H(W_{i-}^{K\times K}|Y_i^n) \geq H(W_{i-}^{K\times K}) - n\epsilon, \quad \forall i \in \langle K\rangle$, where $W_{i-}^{K\times K} \triangleq \{W_{jk} : \forall j \in \langle K\rangle\backslash\{i\}, \forall k \in \langle K\rangle\}$. Recall that W_{jk} is the message from Transmitter k to Receiver j.

eavesdropper, i.e.,

$$H(W_1, W_2, ..., W_K|Y_0^n) \geq H(W_1, W_2, ..., W_K) - n\epsilon \tag{5.25}$$

The main result of this section is the following theorem.

Theorem 5.3. *In a K-user Gaussian interference channel, assume that for the GDoF tuple $(d_1, d_2, ..., d_K) \in \mathcal{P}$, the optimal power allocation of the TIN scheme is $(r_1, r_2, ..., r_K)$. For the same interference channel with both confidential messages and an external eavesdropper, define*

$$i^* = \arg\max_i \{r_i + \alpha_{0i}\}, \tag{5.26}$$

then the secure GDoF tuple $(d_1, ..., d_{i^*-1}, 0, d_{i^*+1}, ..., d_K)$ is achievable through the GCJ scheme. In other words, for the GCJ scheme, the GDoF loss due to the external eavesdropper is at most the achievable secure GDoF value of one user i^*.

In the following, we proceed to prove Theorem 5.3. To ensure secrecy against the added external eavesdropper, we still use the GCJ scheme. For user i^*, let it only generate a random jamming signal, i.e., $X_{i^*} = J_{i^*}$, where $J_{i^*} \sim \mathcal{CN}(0, \frac{P^{r_{i^*}}}{2})$. For user $i \neq i^*$, still let $X_i = V_i + J_i$, where $V_i \sim \mathcal{CN}(0, \frac{P^{r_i}}{2})$ and $J_i \sim \mathcal{CN}(0, \frac{P^{r_i - d_i}}{2})$. Next, we invoke Theorem 5.1 to derive the achievable secure GDoF tuple. For user i^*, obviously its achievable secure GDoF value is 0. For the other users $i \neq i^*$, following the proof of Lemma 5.1, it is not hard to get

$$I(V_i; Y_i) \geq d_i \log P - o(\log P), \ \forall i \in \langle K \rangle, i \neq i^*, \tag{5.27}$$

$$I(V_i; Y_j|V_{-i}^K) \leq o(\log P), \forall i, j \in \langle K \rangle, i \neq j, i \neq i^* \tag{5.28}$$

According to the secrecy constraint (5.25), we also need to consider the secrecy penalty induced by the eavesdropper. For user $i \neq i^*$, we obtain

$$I(V_i; Y_0|V_{-i}^K) = h(Y_0|V_{-i}^K) - h(Y_0|V_1, V_2, ..., V_K) \tag{5.29}$$

$$= \log\left(1 + \frac{\frac{1}{2}P^{r_i+\alpha_{0i}}}{1 + \frac{1}{2}P^{r_{i^*}+\alpha_{0i^*}} + \frac{1}{2}\sum_{k\neq i^*} P^{r_k-d_k+\alpha_{0k}}}\right) \tag{5.30}$$

$$< \log\left(1 + \frac{\frac{1}{2}P^{r_i+\alpha_{0i}}}{\frac{1}{2}P^{r_{i^*}+\alpha_{0i^*}}}\right) \leq \log(1+1) = 1 \tag{5.31}$$

Combining (5.27), (5.28) and (5.31) together, we establish that the secure GDoF value d_i is still achievable for user $i \neq i^*$, which completes the proof of Theorem 5.3.

Remark 5.3. *It is notable that the GDoF loss introduced by the external eavesdropper is unavoidable in the worst-case where, e.g., the eavesdropper sees the same (or better) channel conditions as one of the users. Similarly, with $k \leq K$ eavesdroppers, the worst case would result in the penalty of GDoF of k users. However, the argument given in Theorem 5.3 demonstrates that for the GCJ scheme, the penalty due to k eavesdroppers is never more than loss of achievable GDoF values for k users.*

5.4 Summary

In this chapter, we characterize the secure GDoF region for TIN-optimal interference channels identified in Theorem 2.1. We demonstrate that for such channels, if we impose a confidentiality constraint on the messages such that they are required to remain secure against unintended receivers, the GDoF region is unchanged. We also prove that for such TIN-optimal interference channels, a GCJ scheme with smart power splitting between message signals and jamming signals achieves the entire secure capacity region within a constant gap. Finally, we discuss the performance of the GCJ scheme for interference channels with both confidential messages and external eavesdroppers. It should be noted that for Gaussian interference channels with secrecy constraints, in general the GCJ scheme is not optimal from the GDoF perspective, and the interference alignment principle can help improve the performance in certain channel parameter regimes [80, 83].

Chapter 6

Multilevel Topological Interference Management

As mentioned in Chapter 1, existing wireless networks are mainly based upon two robust interference management principles – 1) ignoring interference that is sufficiently weak, and 2) avoiding interference that is not. These two principles – avoiding versus ignoring interference – which are mapped to TIM and TIN, respectively, naturally correspond to interference management in terms of *signal vector spaces* and *signal power levels*. Essentially, TIM uses the interference alignment perspective to optimally allocate signal vector subspaces among the strong interferers. On the other hand, TIN optimally allocates signal levels among users by setting the transmit power level at each transmitter and the noise floor level at each receiver. While interference management approaches based on signal spaces and signal levels have each been extensively studied in a variety of settings, combining the two has still been a challenge, especially when the schemes involved are rather fragile because of their extreme sensitivity to the precise channel realizations. However, due to the minimal channel knowledge requirements in the TIM and TIN settings, a robust combination of these two is promising. In this chapter, we will explore how to associate the robust TIM and TIN principles together within a *multilevel topological interference management* framework.

Chapter 6 is organized as follows. In Section 6.1, we briefly introduce the background. The joint TIM-TIN problem is formulated in Section 6.2. A baseline decomposition approach is presented in Section 6.3. Section 6.4 summarizes this chapter.

6.1 Background

As discussed in Chapter 1, although recent years have seen a rapid progress in the capacity characterization of Gaussian wireless networks, most of current studies are limited by two factors: the idealized assumption of precise CSIT and the DoF metric. In short, in practice the precise CSIT is rarely available, and DoF ignores the diverse channel strengths (or network topology). Evidently, in order to avoid these pitfalls, we should shift our focus away from optimal ways of exploiting precise CSI, and toward optimal ways of exploiting a coarse knowledge of interference network topology. This line of thought motivates robust models of interference networks where *only a coarse knowledge of channel strength levels is available to transmitters and no channel phase knowledge is assumed*. This is the *multilevel* topological interference management framework. It is a generalization of the elementary TIM framework introduced in [22], where transmitters can only distinguish between channels that are connected (strong) and disconnected (weak). In this multilevel setting, because of the minimal CSIT requirements for TIM and TIN, a robust combination of these two presents itself. Associating TIM with signal vector space allocations and TIN with signal power level allocations within the multilevel TIM framework, we refer to the joint allocation of signal vector spaces and signal power levels as the TIM-TIN problem.

TIM-TIN Problem: With only a coarse channel strength knowledge available at transmitters, we intend to carefully allocate not only the directions of beamforming vectors (i.e., signal vector spaces) but also the transmit powers (i.e., signal power levels) to each of those beamforming vectors. The necessity of a joint TIM-TIN perspective is evident as follows. In the vector space allocation schemes used in DoF studies, the signal space containing the interference is entirely zero-forced. This is typically fine for linear DoF studies because all signals are essentially equally strong (i.e., every substream carries exactly one DoF), so any desired signal projected into the interference space always achieves a DoF value of 0. However, in the GDoF framework we account for the difference in signal strengths. The signal vector space dimensions occupied by interference may not be *fully* occupied in terms of power levels when the interference is weak. Thus, non-zero GDoF values can be achieved by desired signals projected into the same dimensions as occupied by the interference, if the interference is weaker than the desired signal. It is this aspect that we intend to exploit in the multilevel TIM framework.

6.2 Problem Formulation

In this section, we formulate the joint TIM-TIN problem. Consider general K-user interference channels presented in (2.4), i.e.,

$$Y_k(t) = \sum_{i=1}^{K} \sqrt{P^{\alpha_{ki}}} e^{j\theta_{ki}} X_i(t) + Z_k(t), \quad \forall k \in \langle K \rangle.$$

Suppose that over n channel uses, user $k \in \langle K \rangle$ sends out b_k independent scalar data streams, each of which carries one symbol $s_{k,l}$ and is transmitted along the $n \times 1$ beamforming vector $\mathbf{v}_{k,l}$, $l \in \langle b_k \rangle$. The symbols $s_{k,l}$ come from independent Gaussian codebooks, each with zero mean and unit power, and the beamforming vectors $\mathbf{v}_{k,l}$ are scaled to have unit norm. Over

n channel uses, the output of Receiver k is an $n \times 1$ vector

$$\mathbf{y}_k = \sum_{i=1}^{K} \sum_{l=1}^{b_i} \sqrt{P^{\alpha_{ki}}} e^{j\theta_{ki}} \sqrt{P^{r_{i,l}}} \mathbf{v}_{i,l} s_{i,l} + \mathbf{z}_k \tag{6.1}$$

where $\mathbf{z}_k$ is the $n \times 1$ AWGN vector at Receiver k, and $P^{r_{i,l}}$ is the power allocated to the l-th data stream of user i. Note that $r_{i,l} \leq 0$ due to the power constraint.[1] At Receiver k, the covariance matrix of the desired signal is give by

$$\mathbf{Q}_k^D = \sum_{l=1}^{b_k} (\mathbf{v}_{k,l} \mathbf{v}_{k,l}^\dagger) P^{r_{k,l} + \alpha_{kk}} \tag{6.2}$$

The covariance matrix of the interference from Transmitter $i \neq k$ is

$$\mathbf{Q}_{ki} = \sum_{l=1}^{b_i} (\mathbf{v}_{i,l} \mathbf{v}_{i,l}^\dagger) P^{r_{i,l} + \alpha_{ki}} \tag{6.3}$$

Thus the covariance matrix of the net interference-plus-noise is

$$\mathbf{Q}_k^{N+I} = \sum_{i \neq k} \mathbf{Q}_{ki} + \mathbf{I} \tag{6.4}$$

For user $k \in \langle K \rangle$, given the beamforming vectors and power allocation for each data stream, as in the TIM-TIN problem the receivers do not attempt to decode the interference, the achievable rate per channel use is

$$R_k = \frac{1}{n} I(s_{k,1}, s_{k,2}, ..., s_{k,b_k}; \mathbf{y}_k) \tag{6.5}$$

$$= \frac{1}{n} \Big[h(\mathbf{y}_k) - h(\mathbf{y}_k | s_{k,1}, s_{k,2}, ..., s_{k,b_k}) \Big] \tag{6.6}$$

$$= \frac{1}{n} \Big\{ \log \Big| \mathbf{Q}_k^D + \mathbf{Q}_k^{N+I} \Big| - \log \Big| \mathbf{Q}_k^{N+I} \Big| \Big\} \tag{6.7}$$

[1]Rigorously speaking, with multiple b_i data streams at Transmitter i, to satisfy the unit power constraint, we may assume the power allocated to each data stream as $\frac{1}{b_i} P^{r_{i,l}}$, where $r_{i,l} \leq 0$. Note that the constant $1/b_i$ does not affect the GDoF results. Thus in this chapter we ignore $1/b_i$ for simplicity.

and the achievable GDoF value d_k is

$$d_k = \lim_{P\to\infty} \frac{R_k}{\log P} = \lim_{P\to\infty} \frac{\log \left|\mathbf{Q}_k^D + \mathbf{Q}_k^{N+I}\right| - \log \left|\mathbf{Q}_k^{N+I}\right|}{n \log P} \tag{6.8}$$

Next, we proceed to simplify the achievable GDoF result (6.8) into a more intuitive form. Consider a term of the type $\log |\mathbf{I} + P^{\kappa_1}\mathbf{v}_1\mathbf{v}_1^\dagger + P^{\kappa_2}\mathbf{v}_2\mathbf{v}_2^\dagger + ... + P^{\kappa_m}\mathbf{v}_m\mathbf{v}_m^\dagger|$, where $\mathbf{v}_i$, $i \in \langle m\rangle$ are $n \times 1$ beamforming vectors. Without loss of generality, assume $\kappa_1 \geq \kappa_2 \geq ... \geq \kappa_m \geq 0$. Consider the beamforming vectors one by one. For $\mathbf{v}_1$, relabel it as $\mathbf{v}_{\Pi(1)}$ and correspondingly its associated power exponent κ_1 as $\kappa_{\Pi(1)}$. For $\mathbf{v}_2$, if it falls into $\text{span}(\mathbf{v}_{\Pi(1)})$, remove it and then proceed to $\mathbf{v}_3$; otherwise, relabel it as $\mathbf{v}_{\Pi(2)}$ and correspondingly its associated power exponent κ_2 as $\kappa_{\Pi(2)}$. Repeat this operation for each beamforming vector. Specifically, for $\mathbf{v}_i$, if it falls into $\text{span}(\mathbf{v}_{\Pi(1)}, \mathbf{v}_{\Pi(2)}, ..., \mathbf{v}_{\Pi(l)})$, which is the span of all previous linearly independent vectors obtained from $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{i-1}\}$, remove it and then proceed to $\mathbf{v}_{i+1}$; otherwise, relabel it as $\mathbf{v}_{\Pi(l+1)}$ and correspondingly its associated power exponent κ_i as $\kappa_{\Pi(l+1)}$. Finally, we end up with $\gamma \leq n$ linearly independent beamforming vectors $\mathcal{V}_\Pi = \{\mathbf{v}_{\Pi(1)}, \mathbf{v}_{\Pi(2)}, ..., \mathbf{v}_{\Pi(\gamma)}\}$ and their associated power exponents $\mathcal{P}_\Pi = \{\kappa_{\Pi(1)}, \kappa_{\Pi(2)}, ..., \kappa_{\Pi(\gamma)}\}$. Based on the above definitions, we present the following lemma.

Lemma 6.1. *Suppose* $\mathbf{v}_i$, $i \in \langle m\rangle$ *are* $n \times 1$ *vectors, and* $\kappa_1 \geq \kappa_2 \geq ... \geq \kappa_m \geq 0$, *then*

$$\log \left|\mathbf{I} + P^{\kappa_1}\mathbf{v}_1\mathbf{v}_1^\dagger + P^{\kappa_2}\mathbf{v}_2\mathbf{v}_2^\dagger + ... + P^{\kappa_m}\mathbf{v}_m\mathbf{v}_m^\dagger\right| = \sum_{i=1}^{\gamma} \kappa_{\Pi(i)} \log P + o(\log(P)) \tag{6.9}$$

Proof of Lemma 6.1: Let $x_i \sim \mathcal{CN}(0, P^{\kappa_i})$ be independent Gaussian variables. Denote by $\mathbf{z}$ the $n \times 1$ zero mean unit variance circularly symmetric Gaussian vector. When $m > \gamma$, denote the vectors $\mathbf{v}_i \not\subseteq \mathcal{V}_\Pi$ as $\mathbf{v}_{\Pi'(j)}$, $j \in \langle m - \gamma\rangle$. We have

$$\log \left|\mathbf{I} + P^{\kappa_1}\mathbf{v}_1\mathbf{v}_1^\dagger + P^{\kappa_2}\mathbf{v}_2\mathbf{v}_2^\dagger + ... + P^{\kappa_m}\mathbf{v}_m\mathbf{v}_m^\dagger\right|$$

$$= h\Big(\sum_{i=1}^{m} \mathbf{v}_i x_i + \mathbf{z}\Big) + o(\log(P)) \tag{6.10}$$

$$= h\Big(\sum_{i=1}^{\gamma} \mathbf{v}_{\Pi(i)} x_{\Pi(i)} + \sum_{j=1}^{m-\gamma} \mathbf{v}_{\Pi'(j)} x_{\Pi'(j)} + \mathbf{z}\Big) + o(\log(P)) \tag{6.11}$$

$$= h\Big(\sum_{i=1}^{\gamma} \mathbf{v}_{\Pi(i)} x_{\Pi(i)} + \mathbf{z}\Big) + o(\log(P)) \tag{6.12}$$

$$= \log\left|\mathbf{I} + \sum_{i=1}^{\gamma} P^{\kappa_{\Pi(i)}} \mathbf{v}_{\Pi(i)} \mathbf{v}_{\Pi(i)}^{\dagger}\right| + o(\log(P)) \tag{6.13}$$

where (6.12) follows from the facts that $\mathbf{v}_{\Pi'(j)}, \forall j \in \langle m-\gamma \rangle$, is a linear combination of the vectors in $\mathcal{V}_{\Pi}$, and the term $\sum_{j=1}^{m-\gamma} \mathbf{v}_{\Pi'(j)} x_{\Pi'(j)}$ becomes insignificant when P approaches infinity. More specifically, as $P \to \infty$, for the term $\mathbf{v}_i(x_i + x_j + ... + x_k)$ $(i < j < ... < k)$, only the symbol x_i with the dominant power exponent κ_i matters, implying that for the vector $\mathbf{v}_i$ we can ignore all the other independent symbols with equal or smaller power exponents in the limit of $P \to \infty$. Next, we follow the proof of Lemma 1 in [48]. Define $\mathbf{V}_{\Pi} \triangleq [\mathbf{v}_{\Pi(1)}\ \mathbf{v}_{\Pi(2)}\ ...\ \mathbf{v}_{\Pi(\gamma)}\ ]$ with size $n \times \gamma$, and the diagonal matrix $\mathbf{P}_{\Pi} \triangleq \mathrm{diag}[P^{\kappa_{\Pi(1)}}\ P^{\kappa_{\Pi(2)}}\ ...\ P^{\kappa_{\Pi(\gamma)}}]$ with size $\gamma \times \gamma$. We have

$$\log\left|\mathbf{I} + \sum_{i=1}^{\gamma} P^{\kappa_{\Pi(i)}} \mathbf{v}_{\Pi(i)} \mathbf{v}_{\Pi(i)}^{\dagger}\right|$$

$$= \log\left|\mathbf{I} + \mathbf{V}_{\Pi}\mathbf{P}_{\Pi}\mathbf{V}_{\Pi}^{\dagger}\right| \tag{6.14}$$

$$= \log\left|\mathbf{I} + \mathbf{V}_{\Pi}^{\dagger}\mathbf{V}_{\Pi}\mathbf{P}_{\Pi}\right| \tag{6.15}$$

$$= \log|\mathbf{P}_{\Pi}| + \log\left|\mathbf{P}_{\Pi}^{-1} + \mathbf{V}_{\Pi}^{\dagger}\mathbf{V}_{\Pi}\right| \tag{6.16}$$

$$= \sum_{i=1}^{\gamma} \kappa_{\Pi(i)} \log P + \mathcal{O}(1). \tag{6.17}$$

Plugging (6.17) into (6.13), we complete the proof. ■

Applying Lemma 6.1 to (6.8), the TIM-TIN problem is simplified into a form where the dependence on the assigned vector spaces and power levels is explicit.

To help further understand the encoding/decoding scheme, according to the chain rule for the mutual information, we rewrite the achievable rate of user k in (6.5) as follows.

$$R_k = \frac{1}{n} \sum_{i=1}^{b_k} I(s_{k,i}; \mathbf{y}_k | s_{k,1}, ..., s_{k,i-1}) \tag{6.18}$$

From the right hand side of (6.18), we obtain the achievable GDoF value for each data stream $s_{k,i}$, i.e.,

$$d_{k,i} = \lim_{P \to \infty} \frac{I(s_{k,i}; \mathbf{y}_k | s_{k,1}, ..., s_{k,i-1})}{n \log P} \tag{6.19}$$

$$= \lim_{P \to \infty} \frac{h(\mathbf{y}_k | s_{k,1}, ..., s_{k,i-1}) - h(\mathbf{y}_k | s_{k,1}, ..., s_{k,i-1}, s_{k,i})}{n \log P} \tag{6.20}$$

Applying Lemma 6.1 to (6.20) and summing up the achievable GDoF value for each data stream $d_{k,i}$, $i \in \langle b_k \rangle$, we obtain the same result d_k as applying Lemma 6.1 to (6.8). Interestingly, (6.18)-(6.20) indicate that d_k can be obtained through successive interference cancellation. Specifically, we first decode $s_{k,1}$ from the received signal at Receiver k, whose achievable rate is given by $I(s_{k,1}; \mathbf{y}_k)$. Then according to (6.20), we obtain the achievable GDoF value $d_{k,1}$. After decoding $s_{k,1}$, Receiver k can subtract it from the received signal and proceed to decode $s_{k,2}$, whose achievable rate is equal to $I(s_{k,2}; \mathbf{y}_k | s_{k,1})$. Similarly, according to (6.20), we obtain $d_{k,2}$. We repeat this decode-and-subtract procedure to get the achievable GDoF values of all desired data streams for user k, which lead to the final result d_k.

Example 6.1. *Consider a* 3*-user interference channel. Over* 2 *channel uses, Transmitter* 1*,* 2*, and* 3 *send* 2*,* 2*, and* 1 *data streams, respectively, to their desired receivers. Assume that given the beamforming vectors, the transmitted power allocation of each symbol and the channel strength level for each link, the received signal at Receiver* 1 *is given in Fig. 6.1, where* $\mathbf{v}_{2,1}$ *and* $\mathbf{v}_{3,1}$ *are aligned along one direction. The length of the vector denotes the*

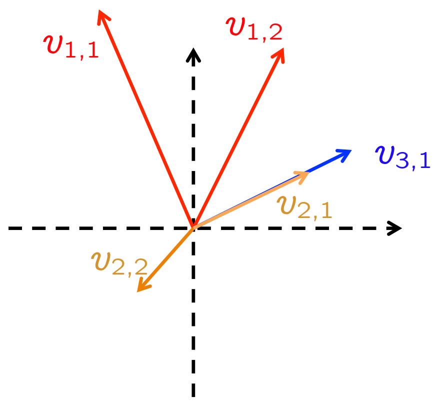

Figure 6.1: The received signal at Receiver 1, where the length of the vector represents the received power of the carried symbol.

received power of the carried symbol. We have

$$r_{1,1} + \alpha_{11} > r_{1,2} + \alpha_{11} > r_{3,1} + \alpha_{13} > r_{2,1} + \alpha_{12} > r_{2,2} + \alpha_{12} > 0.$$

Define

$$d'_k = \lim_{P\to\infty} \frac{\log \left|\mathbf{Q}_k^D + \mathbf{Q}_k^{N+I}\right|}{\log P}, \quad d''_k = \lim_{P\to\infty} \frac{\log \left|\mathbf{Q}_k^{N+I}\right|}{\log P} \tag{6.21}$$

Applying Lemma 6.1 to the above two terms, we get

$$d'_1 = r_{1,1} + \alpha_{11} + r_{1,2} + \alpha_{11}, \quad d''_1 = r_{3,1} + \alpha_{13} + r_{2,2} + \alpha_{12} \tag{6.22}$$

So the achievable GDoF value of user 1 *is*

$$d_1 = \frac{d'_1 - d''_1}{2} = \frac{1}{2}[(r_{1,1} + \alpha_{11} + r_{1,2} + \alpha_{11}) - (r_{3,1} + \alpha_{13} + r_{2,2} + \alpha_{12})] \tag{6.23}$$

We can obtain the same GDoF result through successive interference cancellation. To decode $s_{1,1}$*, we first zero-force the strongest interference* $s_{1,2}$ *and then treat all the other interference* *as noise. The achievable GDoF value of data stream* $s_{1,1}$ *is*

$$d_{1,1} = \frac{(r_{1,1} + \alpha_{11} - \max\{r_{3,1} + \alpha_{13}, r_{2,1} + \alpha_{12}, r_{2,2} + \alpha_{12}\})}{2} \tag{6.24}$$

$$= \frac{1}{2}(r_{1,1} + \alpha_{11} - r_{3,1} - \alpha_{13}) \tag{6.25}$$

After recovering $s_{1,1}$*, we subtract it off from the received signal and then decode* $s_{1,2}$*. Similarly, in this step we first zero-force the strongest interference* $s_{3,1}$ *(and the aligned interfering data stream* $s_{2,1}$ *simultaneously) and then treat all the other interference as noise. The achievable GDoF value of data stream* $s_{1,2}$ *is*

$$d_{1,2} = \frac{1}{2}(r_{1,2} + \alpha_{11} - r_{2,2} - \alpha_{12}) \tag{6.26}$$

Adding (6.25) and (6.26) together, the achievable GDoF value for user 1 *is*

$$d_1 = \frac{1}{2}[(r_{1,1} + \alpha_{11} + r_{1,2} + \alpha_{11}) - (r_{3,1} + \alpha_{13} + r_{2,2} + \alpha_{12})] \tag{6.27}$$

which is the same as (6.23). Note that the achievable GDoF value does not depend on the decoding order, i.e., if we take a reverse decoding order for $s_{1,1}$ *and* $s_{1,2}$*, we still achieve the same GDoF value for user 1 through successive interference cancellation.*

6.3 A Baseline Decomposition Approach

In general, the TIM-TIN problem remains open. In this section, we present a baseline decomposition approach. It is based on a decomposition of the network into two components, which can be solved separately and then combined to produce an achievable GDoF region for the original network. Specifically, given an arbitrary interference network, in the decomposition approach, two copies of the network are created, which are called the TIM component and the TIN component. The desired links are copied in both networks. However, each interfering link is mapped to either the TIM component or the TIN component (but not both). Note that for one network, different TIM-TIN decompositions are possible.

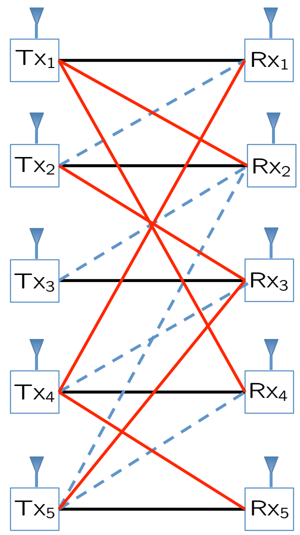

Figure 6.2: A 5-user interference channel

Example 6.2. *Consider the 5-user interference channel in Fig. 6.2, where the channel strength level of solid black and red links is 1 and the channel strength level of dashed blue links is 0.5. Fig. 6.3 shows one possible decomposition of this channel into a TIN component and a TIM component, where the blue (weak) interference links are mapped to the TIN component and the red (strong) ones to the TIM component.*

The purpose of the decomposition approach is to simplify the TIM-TIN problem by solving the TIM and TIN components separately. First, consider the TIM component only. We assume that all the non-zero links that are mapped to the TIM component are equally strong (even if they are not) and find a linear TIM solution to obtain the GDoF tuple $(d_{1,\text{TIM}}, d_{2,\text{TIM}}, ..., d_{K,\text{TIM}})$, which identifies the fraction of the interference-free *signal space* that is available to each user. Next, consider the TIN component only. Suppose that through appropriate power control, the GDoF tuple $(d_{1,\text{TIN}}, d_{2,\text{TIN}}, ..., d_{K,\text{TIN}})$ is achievable

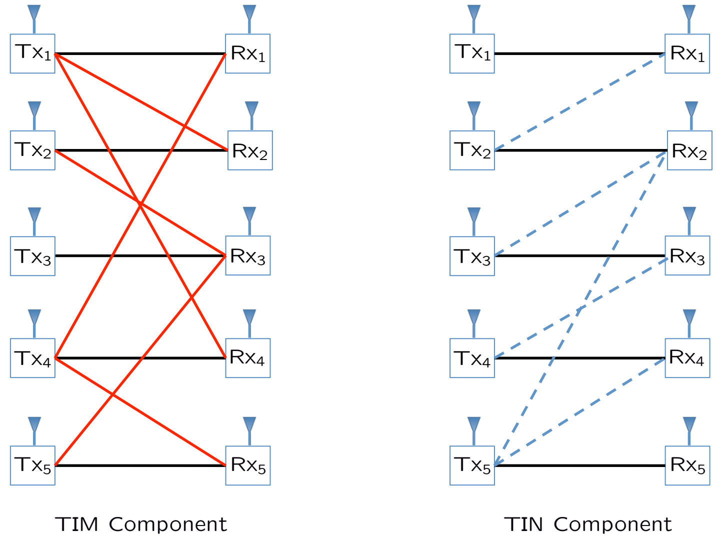

Figure 6.3: One possible decomposition for the interference channel in Fig. 6.2

via the TIN scheme, which identifies the fraction of the available *signal power level* to each user. It turns out that the product of the two fractions for each user, i.e., the GDoF tuple $(d_{1,\text{TIN}} \times d_{1,\text{TIM}}, d_{2,\text{TIN}} \times d_{2,\text{TIM}}, ..., d_{K,\text{TIN}} \times d_{K,\text{TIM}})$ is achievable in the original network, which identifies the *net signal dimensions* available to each user. Finally, the convex hull of all similarly achieved GDoF tuples corresponding to different decompositions is achievable through time-sharing.

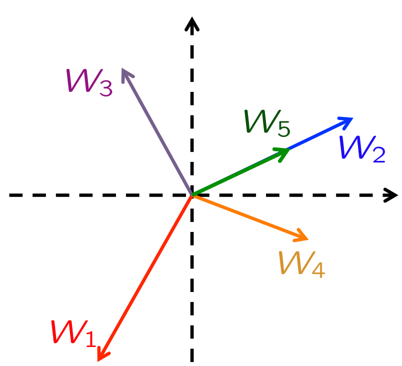

Figure 6.4: The achievable scheme to achieve the symmetric GDoF value 0.3 for the interference channel in Fig. 6.2

To help understand the decomposition approach, we proceed to revisit Example 6.2. In the TIM component of Fig. 6.3, according to [22], it is not hard to verify that a symmetric GDoF value 0.5 is achievable. In the TIN component of Fig. 6.3, where the TIN optimality condition of Theorem 2.1 is satisfied, according to Theorem 2.1, a symmetric GDoF value 0.6 is achievable. Therefore, in the original network, a symmetric GDoF value $0.6 \times 0.5 = 0.3$ is achievable through the decomposition approach. Specifically, the achievable scheme is given explicitly in Fig. 6.4. It uses 2 dimensional space (i.e., 2 channel uses) and 4 beamforming vectors, where any two of them are linearly independent. Each user sends out one data stream, and the data streams carrying the messages W_2 and W_5 are aligned along the same vector. From the GDoF perspective, the transmit power r_i for user i is selected as $r_1 = 0$, $r_2 = -0.1$, $r_3 = -0.2$, $r_4 = -0.3$ and $r_5 = -0.4$. It is not hard to verify that every user achieves a GDoF value of 0.3:

- Receiver 1 first zero-forces the interference from Transmitter 4. Then, in the remaining signal dimension, it treats the interference from Transmitter 2 as noise. Thus the achievable GDoF value for user 1 is $(1 - 0.4)/2 = 0.3$.
- Receiver 2 first zero-forces the interference from Transmitter 1 and then treats the interference from Transmitter 3 and 5 as noise, which achieves $(0.9 - 0.3)/2 = 0.3$ GDoF.
- Receiver 3 first zero-forces the interference from Transmitter 2 and 5 and then treats the interference from Transmitter 4 as noise, which achieves $(0.8 - 0.2)/2 = 0.3$ GDoF.
- Receiver 4 first zero-forces the interference from Transmitter 1 and then treats the interference from Transmitter 5 as noise, which achieves $(0.7 - 0.1)/2 = 0.3$ GDoF.
- Receiver 5 only needs to zero-force the interference from Transmitter 4 to achieve $0.6/2 = 0.3$ GDoF.

Note that in the achievable scheme given in Fig. 6.4, the transmit power allocation policy comes from the power control solution (which achieves the symmetric GDoF value 0.6) for the TIN component in Fig. 6.3, and the alignment relationship for the beamforming vectors comes from the TIM solution (which achieves the symmetric GDoF value 0.5) for the TIM component in Fig. 6.3.

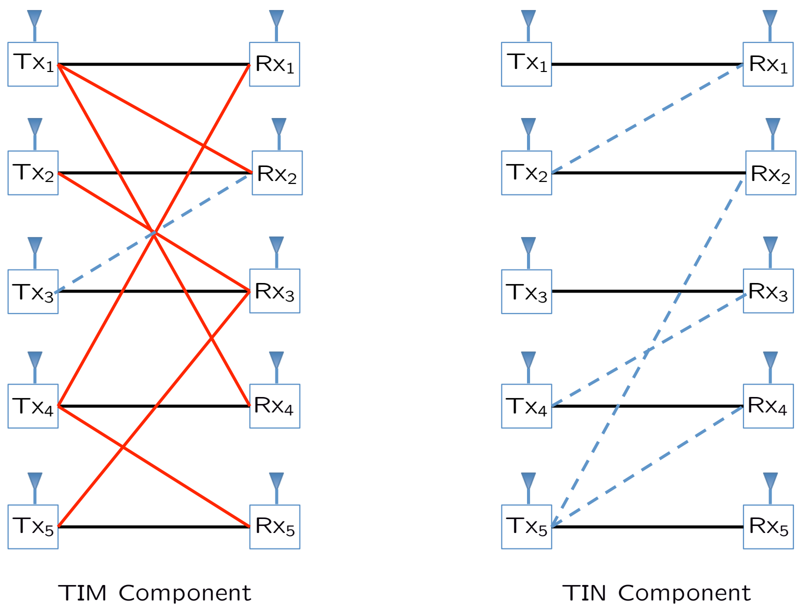

Figure 6.5: Another possible decomposition for the interference channel in Fig. 6.2

Finally, recall that the decomposition approach is quite flexible, i.e., any interfering link can be mapped into either TIM or TIN components. In general, one would expect that to obtain a "good" achievable GDoF region, the TIM component should contain all the "strong" interfering links and the TIN component should contain all the "weak" interfering links. This is, however, not always the case. Consider the 5-user interference channel in Example 6.2 again. It is not difficult to verify that if the interfering link between Transmitter 3 and Receiver 2 (note that for this link $\alpha_{23} = 0.5$) is moved from the TIN component to the TIM component as shown in Fig. 6.5, then the new TIN and TIM component achieve a symmetric GDoF value of 2/3 and 1/2, respectively. Therefore, the achievable symmetric

GDoF value via the decomposition approach can be improved to 1/3. As shown in Fig. 6.6, the corresponding achievable scheme uses a 2-dimensional vector space and 3 beamforming vectors, any two of which are linearly independent. Each user sends out one data stream. The beamforming vectors carrying the messages W_1 and W_3 are aligned along one direction, and the beamforming vectors carrying W_2 and W_5 are aligned along another direction. The transmit powers are $r_1 = r_3 = 0$, $r_2 = r_4 = -1/6$, and $r_5 = -1/3$.

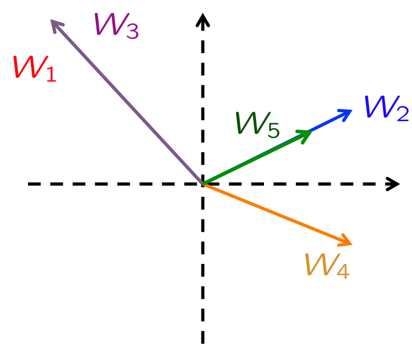

Figure 6.6: The achievable scheme to achieve the symmetric GDoF value 1/3 for the interference channel in Fig. 6.2

6.4 Summary

In this chapter, under the assumption that only a coarse knowledge of channel strengths and no knowledge of channel phases is available to transmitters, we formulate a joint signal vector space and signal power level optimization problem (i.e., the TIM-TIN problem). This problem is still open in general. In the first step of solving this problem, our focus here is not on optimality, but rather on simplicity and robustness. In particular, we propose a natural baseline decomposition approach to address this joint optimization problem, which decomposes a network into TIN and TIM components, allocates the signal power levels to each user in the TIN component, allocates signal vector space dimensions to each user in the TIM component, and guarantees that the product of the two is an achievable number of signal dimensions available to each user in the original network.

Chapter 7

Conclusion

In this dissertation, we mainly demonstrate the optimality of TIN for various Gaussian interference networks from an information theoretic perspective. The contributions are summarized as follows:

- In Chapter 2, for K-user Gaussian interference channels we identify a broad condition under which TIN is optimal from the GDoF perspective and within a constant gap to the exact capacity region. In words, this TIN-optimality condition is "for each user the desired signal strength is no less than the sum of the strengths of the strongest interference from this user and the strongest interference to this user (all values in dB scale)". Moreover, for K-user Gaussian interference channels with arbitrary channel strengths, we fully characterize the achievable GDoF region via the TIN scheme (i.e., the TIN region) and establish its duality as a byproduct.

- In Chapter 2, we also extend the optimality of TIN to MIMO interference channels where all transmitters and receivers are equipped with the same number of antennas. For MIMO channels with different transmit and receive antenna numbers, we show that there exist non-trivial parameter regimes where a simple scheme of zero-forcing strong interference and treating the others as noise achieves the sum GDoF value.

- In Chapter 3, we show that for TIN-optimal interference channels identified in Theorem 2.1, expanding the message set to include an independent message from each transmitter to each receiver does not increase sum GDoF, and that operating the new channel as the original interference channel and treating interference as noise is optimal for the sum capacity up to a constant gap. We also extend the sum GDoF optimality of TIN to general X channels with arbitrary numbers of transmitters and receivers.

- In Chapter 4, we prove that for K-user compound Gaussian interference channels, if in each possible network realization, the TIN-optimality condition of Theorem 2.1 is satisfied individually, then TIN achieves the entire GDoF region of the whole compound setting, which is the intersection of the GDoF regions of all possible network realizations.

- In Chapter 4, we also show that in terms of GDoF, the power control and TIN problems for compound and regular interference channels are equivalent. Remarkably, the equivalent regular counterpart may be different from all the possible network realizations of the compound channel. In addition, we develop several power control algorithms from the GDoF perspective for compound networks.

- In Chapter 5, we demonstrate that for Gaussian interference channels with confidential messages, if the TIN-optimality condition identified in Theorem 2.1 is satisfied, the secrecy constraints incur no penalty from the GDoF perspective, and that a scheme based on Gaussian signaling, cooperative jamming, and smart power splitting achieves the entire secure capacity region within a constant gap.

- In Chapter 6, combining TIM with TIN, we formulate a joint signal vector space and signal power level optimization problem and propose a baseline decomposition approach.

An interesting future direction is to determine both sufficient and necessary conditions for the optimality of TIN. It is noteworthy that in previous work, from both exact capacity and GDoF perspectives, the existing TIN-optimality results are primarily in the form of sufficient conditions. In most cases, the necessity of these optimality conditions remains undetermined. In [7–9], for 2-user interference channels, it is shown that when the interference strength is below certain threshold, TIN achieves the exact sum capacity, but the optimal interference threshold is still open. For K-user interference channels, it is conjectured that the TIN-optimality condition identified in Theorem 2.1 is also necessary for TIN to be optimal for the whole GDoF region except for a set of channel gain values with measure zero [34]. For X channels and compound interference channels, we have shown that the identified conditions in this dissertation is only sufficient but not necessary for the optimality of TIN [36, 37].

Another research direction is to apply the theoretical insights obtained in this dissertation to real-world wireless network design. Our work has motived several scheduling and power control algorithms for heterogeneous networks. In [84], inspired by the TIN-optimality condition of Theorem 2.1, Naderializadeh and Avestimehr proposed a distributed scheduling algorithm called ITLinQ for device-to-device networks. The numerical results in [84] show that ITLinQ significantly outperforms other state-of-the-art scheduling schemes (e.g., FlashLinQ [85]) in some practical settings. Similarly, motived by the TIN-optimality condition, in [86] Adhikary, Dhillon, and Caire developed an interference coordination scheme for heterogeneous cellular networks. In terms of power control, besides our work presented in Chapter 4, Yi and Caire designed several power control schemes by reformulating the TIN problem from a combinatorial perspective, and also studied the joint scheduling and power control problem in [76]. This line of research is still in its infancy. It is interesting from both theoretical and practical points of view.

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Appendix A

Replacing $\alpha_{ij} < 0$ with $\alpha_{ij} = 0$

For exposition simplicity, in this appendix we refer to the channel with potentially negative α_{ij} $(i, j \in \langle K \rangle)$ as the original channel, and the channel with all negative α_{ij} replaced by 0 as the modified channel. To prove the claim that replacing $\alpha_{ij} < 0$ with $\alpha_{ij} = 0$ does not impact the GDoF or the constant gap results in Chapter 2, we go through the following steps:

- First, we establish that the capacity region of the original channel is within a constant gap to that of the modified channel, which indeed illustrates that the two channels have the same GDoF region. The proof requires two directions, namely

$$\mathcal{C}_{\text{original}} \subseteq \mathcal{C}_{\text{modified}} + \text{constant},$$

  and

$$\mathcal{C}_{\text{modified}} \subseteq \mathcal{C}_{\text{original}} + \text{constant}.$$

  In the following, for clarity denote by $\bar{\alpha}_{ij}$ the channel strength level of the link between

Transmitter j and Receiver i in the original channel, $\forall i, j \in \langle K \rangle$. The channel input-output relationship for the original channel is then described by

$$\bar{Y}_k(t) = \sum_{i=1}^{K} \sqrt{P^{\bar{\alpha}_{ki}}} e^{j\theta_{ki}} X_i(t) + \bar{Z}_k(t), \quad \forall k \in \langle K \rangle,$$

where $\bar{Z}_k(t) \sim \mathcal{CN}(0,1)$ and some $\bar{\alpha}_{ki}$ might be negative. Define $\alpha_{ij} \triangleq \bar{\alpha}_{ij}^{+}$, $\forall i, j \in \langle K \rangle$. The received signal of user $k \in \langle K \rangle$ in the modified channel is

$$\begin{aligned}
Y_k(t) &= \sum_{i=1}^{K} \sqrt{P^{\alpha_{ki}}} e^{j\theta_{ki}} X_i(t) + Z_k(t) \\
&= \sum_{i=1}^{K} \sqrt{P^{\bar{\alpha}_{ki}^{+}}} e^{j\theta_{ki}} X_i(t) + Z_k(t) \\
&= \sum_{i \in \mathcal{N}_k} e^{j\theta_{ki}} X_i(t) + \sum_{i \notin \mathcal{N}_k} \sqrt{P^{\bar{\alpha}_{ki}}} e^{j\theta_{ki}} X_i(t) + Z_k(t),
\end{aligned}$$

where $Z_k(t) \sim \mathcal{CN}(0,1)$ is independent of $\bar{Z}_k(t)$, $\mathcal{N}_k$ is the set of transmitter indices whose link to Receiver k is with negative channel strength level in the original channel (i.e., $\mathcal{N}_k = \big\{ i \in \langle K \rangle : \bar{\alpha}_{ki} < 0 \big\}$).

First, we prove $\mathcal{C}_{\text{original}} \subseteq \mathcal{C}_{\text{modified}} + \text{constant}$. Define $\mathcal{W} \triangleq \{W_1, W_2, ..., W_K\}$, and let

$$\hat{Y}_k(t) = \bar{Y}_k(t) - Y_k(t) = \sum_{i \in \mathcal{N}_k} (\sqrt{P^{\bar{\alpha}_{ki}}} - 1) e^{j\theta_{ki}} X_i(t) + \bar{Z}_k(t) - Z_k(t).$$

Then, we have

$$\begin{aligned}
& I(W_k; \bar{Y}_k^n) \\
& \leq I(W_k; Y_k^n, \hat{Y}_k^n) \\
& = I(W_k; Y_k^n) + I(W_k; \hat{Y}_k^n | Y_k^n) \\
& = I(W_k; Y_k^n) + h(\hat{Y}_k^n | Y_k^n) - h(\hat{Y}_k^n | Y_k^n, W_k) \\
& \overset{(a)}{\leq} I(W_k; Y_k^n) + h(\hat{Y}_k^n) - h(\hat{Y}_k^n | Y_k^n, \mathcal{W})
\end{aligned}$$

$$
\begin{aligned}
&= I(W_k; Y_k^n) + h(\hat{Y}_k^n) - h(\bar{Z}_k^n - Z_k^n | \mathcal{W}, Z_k^n) \\
&\overset{(b)}{\leq} I(W_k; Y_k^n) + \sum_{t=1}^{n} h(\hat{Y}_k(t)) - h(\bar{Z}_k^n) \\
&\overset{(c)}{\leq} I(W_k; Y_k^n) + n \log[\pi e(K+2)] - n \log(\pi e) \\
&= I(W_k; Y_k^n) + n \log(K+2),
\end{aligned}
$$

where step (a) follows the facts that dropping conditioning does not reduce entropy (for the second term) and adding conditioning does not increase entropy (for the third term), step (b) follows the chain rule and the fact that dropping conditioning does not reduce entropy, and step (c) holds since $|\mathcal{N}_k| \leq K$ and Gaussian distribution maximizes differential entropy under a given variance constraint. This implies that $\mathcal{C}_{\text{original}} \subseteq \mathcal{C}_{\text{modified}} + \text{constant}$.

Similarly, we can prove the other direction, i.e., $\mathcal{C}_{\text{modified}} \subseteq \mathcal{C}_{\text{original}} + \text{constant}$, as follows.

$$
\begin{aligned}
&I(W_k; Y_k^n) \\
&\leq I(W_k; \bar{Y}_k^n, \hat{Y}_k^n) \\
&= I(W_k; \bar{Y}_k^n) + I(W_k; \hat{Y}_k^n | \bar{Y}_k^n) \\
&= I(W_k; \bar{Y}_k^n) + h(\hat{Y}_k^n | \bar{Y}_k^n) - h(\hat{Y}_k^n | \bar{Y}_k^n, W_k) \\
&\leq I(W_k; \bar{Y}_k^n) + h(\hat{Y}_k^n) - h(\hat{Y}_k^n | \bar{Y}_k^n, \mathcal{W}) \\
&= I(W_k; \bar{Y}_k^n) + h(\hat{Y}_k^n) - h(\bar{Z}_k^n - Z_k^n | \mathcal{W}, \bar{Z}_k^n) \\
&\leq I(W_k; \bar{Y}_k^n) + \sum_{t=1}^{n} h(\hat{Y}_k(t)) - h(Z_k^n) \\
&\leq I(W_k; \bar{Y}_k^n) + n \log(K+2).
\end{aligned}
$$

- Next, we prove that the original and modified channels always have the same TIN region $\mathcal{P}^*$. To this end, we only need to show that with the same transmit power vector $(P^{r_1}, P^{r_2}, ..., P^{r_K})$, user $i \in \langle K \rangle$ in both channels achieves the same GDoF value through the TIN scheme. In the modified channel, through the TIN scheme, the rate achieved by user i is

$$R_i = \log\left(1 + \frac{P^{\alpha_{ii}+r_i}}{1+\sum_{j\neq i} P^{\alpha_{ij}+r_j}}\right),$$

and the achievable GDoF by user i through TIN equals

$$d_i = \max\{0, \alpha_{ii} + r_i - \max\{0, \max_{j:j\neq i}(\alpha_{ij} + r_j)\}\}. \tag{A.1}$$

Now consider the original channel. Similarly, the achievable rate of user i is

$$\bar{R}_i = \log\left(1 + \frac{P^{\bar{\alpha}_{ii}+r_i}}{1+\sum_{j\neq i} P^{\bar{\alpha}_{ij}+r_j}}\right).$$

In the original channel, denote the set of user indices whose direct link is with negative channel strength level as $\mathcal{U}$. For all the users $i \in \mathcal{U}$, it is easy to verify that the achievable GDoF through TIN is

$$\bar{d}_i = 0, \tag{A.2}$$

while for the users $i \notin \mathcal{U}$, we have

$$\begin{aligned}
\bar{d}_i &= \max\{0, \bar{\alpha}_{ii} + r_i - \max\{0, \max_{j:j\neq i}(\bar{\alpha}_{ij} + r_j)\}\} \\
&\overset{(d)}{=} \max\{0, \bar{\alpha}_{ii}^{+} + r_i - \max\{0, \max_{j:j\neq i}(\bar{\alpha}_{ij} + r_j)\}\} \\
&\overset{(e)}{=} \max\{0, \bar{\alpha}_{ii}^{+} + r_i - \max\{0, \max_{j:j\neq i}(\bar{\alpha}_{ij}^{+} + r_j)\}\} \\
&= \max\{0, \alpha_{ii} + r_i - \max\{0, \max_{j:j\neq i}(\alpha_{ij} + r_j)\}\},
\end{aligned} \tag{A.3}$$

where step (d) follows from the fact that $\bar{\alpha}_{ii}^{+} = \bar{\alpha}_{ii}$ for users $i \notin \mathcal{U}$, and step (e) holds since when $\bar{\alpha}_{ij} < 0$, we have $\bar{\alpha}_{ij} + r_j < 0$, $\bar{\alpha}_{ij}^{+} + r_j \leq 0$, and replacing the former with the latter does not impact the final result.

Combining (A.2) and (A.3), we obtain that for user $i \in \langle K \rangle$

$$\bar{d}_i = \max\{0, \alpha_{ii} + r_i - \max\{0, \max_{j:j\neq i}(\alpha_{ij} + r_j)\}\} \tag{A.4}$$

Comparing (A.1) with (A.4), we establish that the original and modified channels have the same TIN region $\mathcal{P}^*$.

- Finally, we demonstrate that in the original channel, if the following condition holds

$$\bar{\alpha}_{ii}^{+} \geq \max_{j:j\neq i}\{\bar{\alpha}_{ji}^{+}\} + \max_{k:k\neq i}\{\bar{\alpha}_{ik}^{+}\}, \quad \forall i \in \langle K \rangle \tag{A.5}$$

TIN achieves the capacity region to within $\log(3K)$ bits.

Start with the converse. For the original channel, when condition (A.5) is satisfied, based on Lemma 2.1, we have

$$R_i \leq \log(1 + P^{\bar{\alpha}_{ii}}) \leq \log(1 + P^{\bar{\alpha}_{ii}^{+}}) \leq \bar{\alpha}_{ii}^{+} \log P + 1 = \alpha_{ii} \log P + 1, \ \forall i \in \langle K \rangle \tag{A.6}$$

$$\begin{aligned}
\sum_{j=1}^{m} R_{i_j} &\leq \sum_{j=1}^{m} \log\left(1 + P^{\bar{\alpha}_{i_j i_{j+1}}} + \frac{P^{\bar{\alpha}_{i_j i_j}}}{1 + P^{\bar{\alpha}_{i_{j-1} i_j}}}\right) && \text{(A.7)}\\
&\leq \sum_{j=1}^{m} \log\left(1 + P^{\bar{\alpha}_{i_j i_{j+1}}} + \frac{P^{\bar{\alpha}_{i_j i_j}^{+}}}{1 + P^{\bar{\alpha}_{i_{j-1} i_j}}}\right) && \text{(A.8)}\\
&= \sum_{j=1}^{m} \log\left(1 + P^{\bar{\alpha}_{i_j i_{j+1}}} + \frac{P^{\bar{\alpha}_{i_j i_j}^{+}}}{P^0 + P^{\bar{\alpha}_{i_{j-1} i_j}}}\right) && \text{(A.9)}\\
&< \sum_{j=1}^{m} \log\left(1 + P^{\bar{\alpha}_{i_j i_{j+1}}} + \frac{P^{\bar{\alpha}_{i_j i_j}^{+}}}{P^{\bar{\alpha}_{i_{j-1} i_j}^{+}}}\right) && \text{(A.10)}\\
&= \sum_{j=1}^{m} \log\left(\frac{P^{\bar{\alpha}_{i_{j-1} i_j}^{+}} + P^{\bar{\alpha}_{i_j i_{j+1}} + \bar{\alpha}_{i_{j-1} i_j}^{+}} + P^{\bar{\alpha}_{i_j i_j}^{+}}}{P^{\bar{\alpha}_{i_{j-1} i_j}^{+}}}\right) && \text{(A.11)}
\end{aligned}$$

$$\leq \sum_{j=1}^{m} \log\left(\frac{3P^{\bar{\alpha}^{+}_{i_j i_j}}}{P^{\bar{\alpha}^{+}_{i_{j-1} i_j}}}\right) \tag{A.12}$$

$$= \sum_{j=1}^{m} [(\bar{\alpha}^{+}_{i_j i_j} - \bar{\alpha}^{+}_{i_{j-1} i_j}) \log P + \log 3] \tag{A.13}$$

$$= \sum_{j=1}^{m} [(\alpha_{i_j i_j} - \alpha_{i_{j-1} i_j}) \log P + \log 3], \tag{A.14}$$

for all cycles $(i_1, i_2, ..., i_m) \in \Pi_K$, $\forall m \in \{2, 3, ..., K\}$. Comparing (2.29) and (2.34) with (A.6) and (A.14), we find that the modified and original channels have the same outer bounds.

Next, consider the achievability. For the modified channel, denote the achievable rate region through TIN under condition (2.7) as $\mathcal{R}_{\text{TIN}}$. In the modified channel, for any rate tuple $\mathbf{R}_{\text{TIN}} = (R_1, R_2, ..., R_K) \in \mathcal{R}_{\text{TIN}}$, we have a corresponding transmit power vector $\mathbf{P}_{\text{TIN}} = (P^{r_1}, P^{r_2}, ..., P^{r_K})$. Denote by $\mathcal{P}_{\text{TIN}}$ the set of the transmit power vectors for all the rate tuples in $\mathcal{R}_{\text{TIN}}$. In the original channel, applying the same set of transmit power vectors $\mathcal{P}_{\text{TIN}}$ for transmitters and treating interference as noise at each receiver, we obtain an achievable TIN region $\bar{\mathcal{R}}_{\text{TIN}}$ such that (i) any user $k \notin \mathcal{U}$ achieves a rate no less than that in the modified channel when the same transmit power vector is utilized, as for that user the interfering links in the original channel are no stronger than those in the modified channel, which indicates that for users $k \notin \mathcal{U}$ the constant gap cannot increase in the original channel; (ii) the constant gap for any user $k \in \mathcal{U}$ is at most 1 bit, since according to (A.6) the achievable rate of that user is upper bounded by 1 bit. Therefore, combining with the constant gap result for the modified channel (see Theorem 2.2), we show that for the original channel, when condition (A.5) holds, TIN achieves to within $\log(3K)$ bits of the entire capacity region.

Combining the above steps, we establish that assigning a value of 0 to negative channel strength levels (in the original channel) does not impact the GDoF or the constant gap results in Chapter 2 (i.e., Theorem 2.1, 2.2 and 2.3).

Appendix B

Proof of Theorem 4.5

We prove Theorem 4.5 through contradiction. For clarity, we first explain how the GGPC algorithm works in details. Recall that in the initialize step, we obtain the initial power allocation $r_i(0) = l_{i,dst}$, $\forall i \in \langle K \rangle$. According to Lemma 4.1, with this initial power allocation, the achievable GDoF tuple $(d_1(0), d_2(0), ...d_K(0))$ dominates the target GDoF tuple $(d_1, d_2, ..., d_K)$. Also notice that at this point, the effective noise floor at each receiver is 0. Then in the first update of the GGPC algorithm, each transmitter reduces its power by

$$\Delta(0) = \min_i \{r_i(0) + \alpha_{ii} - d_i\}, \tag{B.1}$$

which is no less than 0, as $r_i(0) + \alpha_{ii} \geq d_i(0) \geq d_i$, $\forall i \in \langle K \rangle$. Without loss of generality, assume

$$\arg\min_i \{r_i(0) + \alpha_{ii} - d_i\} = 1 \tag{B.2}$$

After the first update, the transmit power of user 1 is

$$r_1(1) = r_1(0) - \Delta(0) = d_1 - \alpha_{11} \tag{B.3}$$

With the updated power allocation, user 1 achieves the following GDoF value

$$d_1(1) = \max\left\{0, r_1(0) - \Delta(0) + \alpha_{11} - \max_{j\neq 1}\{0, r_j(0) - \Delta(0) + \alpha_{1j}\}\right\} \tag{B.4}$$

$$= \max\left\{0, r_1(0) - [r_1(0) + \alpha_{11} - d_1] + \alpha_{11}\right\} \tag{B.5}$$

$$= \max\{0, d_1\} \tag{B.6}$$

$$= d_1 \tag{B.7}$$

where (B.5) follows from (B.1), (B.2) and the argument below

$$d_1(0) \geq d_1 \tag{B.8}$$

$$\Rightarrow r_1(0) + \alpha_{11} - \max_{j\neq 1}\{0, r_j(0) + \alpha_{1j}\} \geq d_1 \tag{B.9}$$

$$\Rightarrow r_1(0) + \alpha_{11} - d_1 \geq \max_{j\neq 1}\{0, r_j(0) + \alpha_{1j}\} \tag{B.10}$$

$$\Rightarrow \Delta(0) \geq \max_{j\neq 1}\{0, r_j(0) + \alpha_{1j}\} \tag{B.11}$$

$$\Rightarrow \Delta(0) \geq \max_{j\neq 1}\{r_j(0) + \alpha_{1j}\} \tag{B.12}$$

$$\Rightarrow \max_{j\neq 1}\{0, r_j(0) - \Delta(0) + \alpha_{1j}\} = 0 \tag{B.13}$$

From (B.13), we find that after the first update, at Receiver 1 the interference from others is all below the effective noise floor 0, and Transmitter 1 cannot further lower its power in (B.3) due to the effective noise floor. In other words, user 1 cannot further reduce its transmit power while maintaining the target GDoF value, since it "hits" the effective noise floor.

For other users $j \neq 1$, after the first power allocation update the achievable GDoF value is

$$d_j(1) = \max\left\{0, r_j(0) - \Delta(0) + \alpha_{jj} - \max_{i\neq j}\{0, r_i(0) - \Delta(0) + \alpha_{ji}\}\right\} \tag{B.14}$$

Consider the following two cases.

- $\max_{i\neq j}\{r_i(0)-\Delta(0)+\alpha_{ji}\}\leq 0$: In this case, from (B.14), we obtain

$$\begin{aligned}
d_j(1) =& \max\left\{0, r_j(0)-\Delta(0)+\alpha_{jj}\right\}\\
=& \max\left\{0, r_j(0)+\alpha_{jj}-\min_i\{r_i(0)+\alpha_{ii}-d_i\}\right\}\\
\geq& \max\left\{0, r_j(0)+\alpha_{jj}-[r_j(0)+\alpha_{jj}-d_j]\right\}\\
=& \max\{0, d_j\}\\
=& d_j
\end{aligned}$$

- $\max_{i\neq j}\{r_i(0)-\Delta(0)+\alpha_{ji}\}> 0$: In this case, from (B.14), we get

$$\begin{aligned}
d_j(1) &= \max\left\{0, r_j(0)-\Delta(0)+\alpha_{jj}-\max_{i\neq j}\{r_i(0)-\Delta(0)+\alpha_{ji}\}\right\}\\
&= \max\left\{0, r_j(0)+\alpha_{jj}-\max_{i\neq j}\{r_i(0)+\alpha_{ji}\}\right\}\\
&= \max\left\{0, r_j(0)+\alpha_{jj}-\max_{i\neq j}\{0, r_i(0)+\alpha_{ji}\}\right\}\\
&= \max\{0, d_j(0)\}\\
&= d_j(0)\\
&\geq d_j
\end{aligned}$$

Combining the above two cases together, we demonstrate that after the first update, user $j\neq 1$ still achieves an GDoF value which dominates the target one. To summarize, after the first update of transmit power allocations, user 1 obtains the exact target GDoF value d_1 and achieves its transmit power limit due to the effective noise floor, and $(d_1(1), d_2(1), ..., d_K(1))$ dominates the target GDoF tuple.

We proceed the transmit power updates. In the second update, according to the GGPC algorithm, the transmit power of user 1 is fixed and the transmit powers of others are updated. It is notable that in the following updates the fixed power of user 1 exerts a constant interference level to other users. Therefore, the effective noise floor at Receiver $j \neq 1$ becomes $\max\{0, \alpha_{j1} + r_1(1)\}$. Also note that in the sequel updates, user $j \neq 1$ can only reduce its transmit power, so the achieved GDoF value of user 1 remains as d_1. Next, we only need to repeat the argument of the first update for the following updates till the powers of all users are fixed by the GGPC algorithm. More specifically, in each update, transmit powers of some users are reduced to the limits for achieving the target GDoF values, since they "hit" their own effective noise floor, which is determined by the users whose transmit powers are fixed in the previous updates. And after each update, we always obtain an acceptable GDoF tuple dominating the target one.

Equipped with the above observations, now we proceed to prove Theorem 4.5 by contradiction. Assume that for a target GDoF tuple $(d_1, d_2, ..., d_k)$, the power control solution $\mathbf{r}^* = (r_1^*, r_2^*, ..., r_K^*)$ yielded by the GGPC algorithm is not globally optimal. So there exists another power allocation $\mathbf{r}^\dagger = (r_1^\dagger, r_2^\dagger, ..., r_K^\dagger)$ that also achieves the target GDoF tuple, and there is at least one $i_0 \in \langle K \rangle$ such that $r_{i_0}^\dagger < r_{i_0}^*$. For the GGPC algorithm, in the update step determining the final power allocation r_i^* for user $i \in \langle K \rangle$, denote by $\mathcal{U}_i$ the set of users whose transmit powers have been fixed already. For instance, if r_i^* is determined in the first update, then $\mathcal{U}_i = \phi$.

Assume that $r_{i_0}^*$ is limited by user $i_1 \in \mathcal{U}_{i_0}$. In other words, after the previous updates, among all users in the set $\mathcal{U}_{i_0}$, user $i_1 \in \mathcal{U}_{i_0}$ with transmit power $r_{i_1}^*$ yields the strongest interference level to user i_0, which is larger than 0.[1] Apparently, if after the previous updates, the strongest interference level from the users in $\mathcal{U}_{i_0}$ to user i_0 is no larger than 0, then according to the GGPC algorithm, $r_{i_0}^* = d_{i_0} - \alpha_{i_0 i_0}$. In this case, $r_{i_0}^*$ cannot be reduced further while maintaining the desired GDoF value d_{i_0}, which contradicts that $r_{i_0}^\dagger < r_{i_0}^*$ still achieves d_{i_0}. As $r_{i_0}^*$ can be reduced without affecting the achievable GDoF value d_{i_0}, the transmit power of user i_1 should also be reduced in order to lower the interference level to user i_0. Thus, we

[1]If there are multiple users yielding the same strongest interference level to user i_0, we can pick up user i_1 as any one of them to proceed the proof.

have $r^{\dagger}_{i_1} < r^*_{i_1}$. Then apply the same argument to user i_1. Similarly, we will obtain a user $i_2 \in \mathcal{U}_{i_1}$ such that $r^{\dagger}_{i_2} < r^*_{i_2}$. Repeat the same argument. More specifically, for user i_{n-1}, we have known that $r^{\dagger}_{i_{n-1}} < r^*_{i_{n-1}}$. Then among all the users in the set $\mathcal{U}_{i_{n-1}}$, there exists a user $i_n \in \mathcal{U}_{i_{n-1}}$ yielding the strongest interference level to user i_{n-1}, which is larger than 0. Since $r^*_{i_{n-1}}$ can be reduced without affecting the achievable GDoF value $d_{i_{n-1}}$, the transmit power of user i_n should also be reduced in order to lower the interference level to user i_{n-1}. Hence we have $r^{\dagger}_{i_n} < r^*_{i_n}$. Assume that for user i_m, it final power allocation is determined in the first update and holds fixed afterwards. Finally, we obtain $r^{\dagger}_{i_m} < r^*_{i_m}$. Recall that in the GGPC algorithm, after the first update, $r^*_{i_m} = d_{i_m} - \alpha_{i_m i_m}$, which cannot be reduced further while maintaining the target GDoF value. We end up with a contradiction that $r^{\dagger}_{i_m} < r^*_{i_m}$ still achieves the target GDoF value for user i_m. Therefore, we complete the proof of Theorem 4.5 and establish that the GGPC algorithm indeed yields the globally optimal transmit power allocation for any feasible GDoF tuple in the polyhedral TIN region.