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SHELLING THE $m = 1$ AMPLITUHEDRON

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Abstract. The amplituhedron $\mathcal{A}_{n,k,m}$ was introduced by Arkani-Hamed and Trnka (2014) in order to give a geometric basis for calculating scattering amplitudes in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. It is a projection inside the Grassmannian $\mathrm{Gr}_{k,k+m}$ of the totally nonnegative part of $\mathrm{Gr}_{k,n}$. Karp and Williams (2019) studied the $m = 1$ amplituhedron $\mathcal{A}_{n,k,1}$, giving a regular CW decomposition of it. Its face poset $R_{n,l}$ (with $l := n-k-1$) consists of all projective sign vectors of length n with exactly l sign changes. We show that $R_{n,l}$ is EL-shellable, resolving a problem posed by Karp and Williams. This gives a new proof that $\mathcal{A}_{n,k,1}$ is homeomorphic to a closed ball, which was originally proved by Karp and Williams. We also give explicit formulas for the f-vector and h-vector of $R_{n,l}$, and show that it is rank-log-concave and strongly Sperner. Finally, we consider a related poset $P_{n,l}$ introduced by Machacek (2019), consisting of all projective sign vectors of length n with at most l sign changes. We show that it is rank-log-concave, and conjecture that it is Sperner.

Keywords. Amplituhedron, shellability, Eulerian number, log concavity, Sperner property

Mathematics Subject Classifications. 06A07, 14M15, 81T60, 05A19

1. Introduction

Let $\mathrm{Gr}_{k,n}^{\geq 0}$ denote the *totally nonnegative Grassmannian* [Pos07, Lus94], comprised of all k-dimensional subspaces of $\mathbb{R}^n$ whose Plücker coordinates are nonnegative. Motivated by the physics of scattering amplitudes, Arkani-Hamed and Trnka [AHT14] introduced a generalization of $\mathrm{Gr}_{k,n}^{\geq 0}$, called the *(tree) amplituhedron* and denoted $\mathcal{A}_{n,k,m}(Z)$. It is defined as the image of $\mathrm{Gr}_{k,n}^{\geq 0}$ under (the map induced by) a linear surjection $Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$ whose $(k+m) \times (k+m)$ minors are all positive. While the definition of $\mathcal{A}_{n,k,m}(Z)$ depends on the choice of Z, it is expected that its geometric and combinatorial properties only depend on n, k, and m. The amplituhedron may be regarded as a generalization of both a cyclic polytope

(which we obtain when $k = 1$) and the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ (which we obtain when $k + m = n$).

When $m = 4$, the amplituhedron $\mathcal{A}_{n,k,4}(Z)$ gives a geometric basis for computing tree-level scattering amplitudes in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, but it is an interesting mathematical object for any m. In [KW19], Karp and Williams carried out a detailed study of the $m = 1$ amplituhedron $\mathcal{A}_{n,k,1}(Z)$. They gave a regular CW decomposition of $\mathcal{A}_{n,k,1}(Z)$, whose face poset, which we denote by $R_{n,l}$ (with $l := n-k-1$), can be described as follows.[1] The elements of $R_{n,l}$ are *projective sign vectors* of length n (i.e. elements of $\{0, +, -\}^n \setminus \{(0, \ldots, 0)\}$ modulo the relation $(v_1, \ldots, v_n) \sim (-v_1, \ldots, -v_n)$) with exactly l sign changes. The order relation in $R_{n,l}$ is such that

$$(v_1, \ldots, v_n) \leqslant (w_1, \ldots, w_n) \quad \Longleftrightarrow \quad v_i \in \{0, w_i\} \text{ for } 1 \leqslant i \leqslant n. \tag{1.1}$$

For example, $R_{3,1}$ is depicted in Figure 1.1.

Karp and Williams posed the problem [KW19, Problem 6.19] of showing that the poset $R_{n,l}$ is shellable. We resolve this problem:

Theorem 1.1. *The poset $R_{n,l}$ with a minimum and a maximum adjoined is EL-shellable.*

The motivation behind [KW19, Problem 6.19] was the following. Karp and Williams showed that the $m = 1$ amplituhedron $\mathcal{A}_{n,k,1}(Z)$ is a regular CW complex which can be identified with the bounded complex of a certain generic arrangement of n hyperplanes in $\mathbb{R}^k$ (namely, a *cyclic arrangement*). It then follows from a general result of Dong [Don08] that $\mathcal{A}_{n,k,1}(Z)$ is homeomorphic to a k-dimensional closed ball. Karp and Williams observed that rather than appealing to [Don08], one could reach the same conclusion by showing that the face poset $R_{n,l}$ is shellable, using a result of Björner [Bjö84, Proposition 4.3(c)]. (This relies on the regular CW decomposition, along with the fact that every cell of codimension one is contained in the closure of at most two maximal cells.) Therefore, as a consequence of Theorem 1.1, we obtain a new proof that $\mathcal{A}_{n,k,1}(Z)$ is homeomorphic to a closed ball:

Corollary 1.2 ([KW19, Corollary 6.18])**.** *The $m = 1$ amplituhedron $\mathcal{A}_{n,k,1}(Z)$ is homeomorphic to a k-dimensional closed ball.*

We expect that for any $m \geqslant 1$, the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ has a shellable regular CW decomposition and is homeomorphic to a closed ball, thereby generalizing the situation which holds when $n = k + m$. Indeed, in this case $\mathcal{A}_{n,k,n-k}(Z)$ is the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$; Williams [Wil07] showed that the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ is EL-shellable, and Galashin, Karp, and Lam [GKL22b, GKL22a] showed that $\mathrm{Gr}_{k,n}^{\geq 0}$ is a regular CW complex homeomorphic to a closed ball. See Remark 2.10 for further discussion of related work. In the case we consider here, $m = 1$, we make use of the explicit description of the face poset $R_{n,l}$ of a cell decomposition of $\mathcal{A}_{n,k,m}(Z)$. No such description is known as yet for general m. For work in

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[1]For simplicity, we use an equivalent but slightly different labeling of the face poset than in [KW19]. Namely, in [KW19], the face poset is denoted $\overline{\mathbb{P}\mathrm{Sign}}_{n,k,1}$, and is obtained from $R_{n,l}$ by applying the involution $(v_1, \ldots, v_n) \mapsto (v_1, -v_2, v_3, -v_4, \ldots, (-1)^{n-1}v_n)$.

$$(+,-,-) \qquad (+,+,-) \qquad\qquad\qquad (\{1\},\{2,3\}) \quad (\{1,2\},\{3\})$$

$$(+,-,0) \qquad (+,0,-) \qquad (0,+,-) \qquad (\{1\},\{2\}) \qquad (\{1\},\{3\}) \qquad (\{2\},\{3\})$$
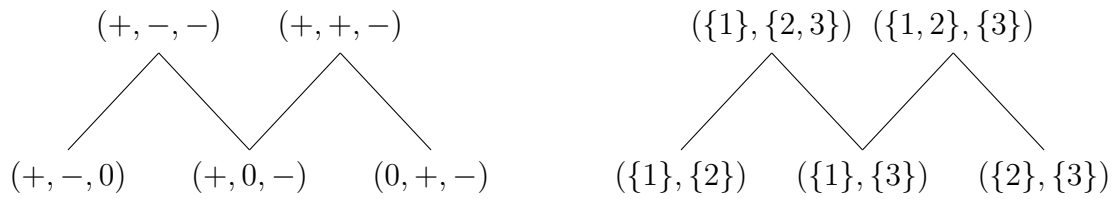
Figure 1.1: The Hasse diagram of the poset $R_{3,1}$, with elements labeled as sign vectors (left) and as tuples of sets (right).

this direction, see [KWZ20, EZLT21] for the case $m = 4$, and [Łuk19, BH19, ŁPW20] for the case $m = 2$.

Another consequence of Theorem 1.1 is that the poset $R_{n,l}$ has a nonnegative h-vector. In particular, by a result of Björner [Bjö80] and Stanley [Sta72], h_i equals the number of maximal chains of $R_{n,l}$ with exactly i descents with respect to the EL-labeling of Theorem 1.1 (see Theorem 3.5). We give an alternative description of the h-vector using generating functions (see Theorem 3.14), which is explicit but non-positive.

We observe that when $l = 0$, the poset $R_{n,l}$ is the Boolean algebra B_n (consisting of all subsets of $\{1, \ldots, n\}$ ordered by containment) with the minimum removed. Maximal chains of $R_{n,0}$ correspond to permutations of $\{1, \ldots, n\}$ with the usual notion of descent, and h_i is the Eulerian number $\left\langle {n \atop i} \right\rangle$ (see Proposition 3.9). Therefore the h-vector of $R_{n,l}$ provides a generalization of the Eulerian numbers.

Two further well-known properties of the Boolean algebra B_n are that its rank sizes form a *log-concave sequence* and that it is *strongly Sperner* (see e.g. [Eng97]). We show that $R_{n,l}$ also has these properties:

Theorem 1.3. *The poset $R_{n,l}$ is rank-log-concave. It also admits a normalized flow, and hence is strongly Sperner.*

Finally, we consider a poset closely related to $R_{n,l}$, denoted $P_{n,l}$, introduced by Machacek [Mac19]. It consists of projective sign vectors of length n with at most (rather than exactly) l sign changes, under the relation (1.1). For example, $P_{3,1}$ is depicted in Figure 5.1. The poset $P_{n,l}$ can be regarded as a quotient of the face poset of a certain simplicial complex $\mathcal{B}(l, n)$ studied by Klee and Novik [KN12]. Notice that $P_{n,l}$ also specializes to B_n when $l = 0$. Machacek [Mac19] showed that the order complex of $P_{n,l}$ is a manifold with boundary which is homotopy equivalent to $\mathbb{RP}^l$, and homeomorphic to $\mathbb{RP}^{n-1}$ when $l = n - 1$. Although $\hat{P}_{n,l}$ is not shellable in general, Bergeron, Dermenjian, and Machacek [BDM20] showed that when l is even or $l = n - 1$, the order complex of $P_{n,l}$ is partitionable. This is a weaker property which still implies that the h-vector is nonnegative, and they showed that the h-vector counts certain type-D permutations with respect to type-B descents.

We prove that $P_{n,l}$ is rank-log-concave (see Theorem 5.2), and we conjecture that it is Sperner (see Conjecture 5.3). We prove this conjecture when l equals 0, 1, or $n - 1$ by constructing a normalized flow (see Proposition 5.4).

The remainder of this paper is organized as follows. In Section 2 we give some background on poset topology and prove Theorem 1.1 (see Theorem 2.8). In Section 3 we consider the f-vector and h-vector of $R_{n,l}$. In Section 4 we give background on unimodality, log-concavity, and the Sperner property, and prove Theorem 1.3. In Section 5 we consider the poset $P_{n,l}$.

2. EL-labeling

2.1. Notation and background

We let $\mathbb{N}$ denote $\{0, 1, 2, \dots\}$. For $n \in \mathbb{N}$ we define $[n] := \{1, 2, \dots, n\}$, and for $0 \leqslant k \leqslant n$ we let $\binom{[n]}{k}$ denote the set of k-element subsets of $[n]$. We let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$.

We assume the reader has some familiarity with posets; we refer to [Sta12, Wac07] for further background. We use $\lessdot$ to denote cover relations in a poset, i.e., $x \lessdot y$ if and only if $x < y$ and there does not exist z such that $x < z < y$.

Definition 2.1. Let P be a finite poset. We say that P is *graded* (or *pure*) if every maximal chain has the same length d, which we call the *rank* of P.

Definition 2.2. Let P be a poset. We define the *bounded extension* as the poset $\hat{P}$ obtained from P by adjoining a new minimum $\hat{0}$ and a new maximum $\hat{1}$.

We now recall the definition of an EL-labeling, due to Björner [Bjö80, Definition 2.1]. We slightly modify the original definition, following Wachs [Wac07]; see [Wac07, Remark 3.2.5] for further discussion.

Definition 2.3 ([Wac07, Definition 3.2.1])**.** Let P be a finite graded poset. An *edge labeling* of P is a function λ from the set of edges of the Hasse diagram of P (i.e. the cover relations of P) to a poset $(\Lambda, \preceq)$. An *increasing chain* is a saturated chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_r$ in P whose edge labels strictly increase in Λ:

$$\lambda(x_0 \lessdot x_1) \prec \lambda(x_1 \lessdot x_2) \prec \cdots \prec \lambda(x_{r-1} \lessdot x_r).$$

We call λ an *EL-labeling* of P if the following properties hold for every closed interval $[x, y]$ in P:

(EL1) there exists a unique increasing maximal chain C_0 in $[x, y]$; and

(EL2) if $x \lessdot z \leqslant y$ such that $z \neq x_1$, where $x \lessdot x_1$ is the first edge of C_0, then $\lambda(x \lessdot x_1) \prec \lambda(x \lessdot z)$.[2]

Björner showed that a finite graded poset with an EL-labeling is shellable [Bjö80, Theorem 2.3].

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[2]One may replace (EL2) by the condition that C_0 is lexicographically minimal among all maximal chains of $[x, y]$; see [Bjö80, Proposition 2.5].

2.2. Edge labeling

We now study the bounded extension $\hat{R}_{n,l}$ of $R_{n,l}$. Recall that $R_{n,l}$ consists of all projective sign vectors of length n with exactly l sign changes, under the relation (1.1). We begin by giving an alternative definition of $R_{n,l}$.

Definition 2.4. Let $0 \leqslant l < n$. We may equivalently define $R_{n,l}$ as follows. Its elements are $(l+1)$-tuples $(A_1, \ldots, A_{l+1})$ of nonempty subsets of $[n]$ (called *blocks*) such that $\max(A_i) < \min(A_{i+1})$ for all $i \in [l]$. The order relation on $(l+1)$-tuples is given by component-wise containment:

$$(A_1, \ldots, A_{l+1}) \leqslant (B_1, \ldots, B_{l+1}) \qquad \Longleftrightarrow \qquad A_i \subseteq B_i \text{ for } i \in [l+1].$$

We may verify that this is equivalent to the definition of $R_{n,l}$ from (1.1), where an $(l+1)$-tuple of subsets records the positions of the consecutive runs of $+$'s and $-$'s in a sign vector. That is, $(A_1, \ldots, A_{l+1})$ corresponds to the sign vector $(v_1, \ldots, v_n)$ such that for $1 \leqslant i \leqslant n$,

$$v_i = \begin{cases} (-1)^{j-1}, & \text{if } i \in A_j \text{ for some } j \in [l+1]; \\ 0, & \text{if } i \notin A_1 \cup \cdots \cup A_{l+1}. \end{cases}$$

For example, in $R_{9,2}$, the tuple of sets $(\{1,2,4\}, \{6,8\}, \{9\})$ corresponds to the sign vector $(+, +, 0, +, 0, -, 0, -, +)$. Also see Figure 1.1.

We observe that the bounded extension $\hat{R}_{n,l}$ of $R_{n,l}$ is graded. Explicitly, the minimum $\hat{0}$ has rank 0, the maximum $\hat{1}$ has rank $n - l + 1$, and $(A_1, \ldots, A_{l+1})$ has rank $|A_1| + \cdots + |A_l| - l$.

We now divide the cover relations of $R_{n,l}$ into two types; see Remark 2.7 for motivation.

Definition 2.5. Let $0 \leqslant l < n$, and let $x = (A_1, \ldots, A_{l+1}) \in R_{n,l}$. Note that the elements of $R_{n,l}$ which cover x are precisely those that can be obtained from it by adding some element $a \in [n] \setminus (A_1 \cup \cdots \cup A_{l+1})$ to the ith block A_i, where $i \in [l+1]$ such that

$$\max(A_{i-1}) < a < \min(A_{i+1}).$$

(We take the inequality above to be $a < \min(A_2)$ when $i = 1$, and $\max(A_l) < a$ when $i = l+1$.) We say that such a cover relation is of *type* α if $a < \max(A_i)$, and of *type* β if $a > \max(A_i)$.

Definition 2.6. Let $0 \leqslant l < n$. We define a total order $(\Lambda_{n,l}, \preceq)$ on the disjoint union of $\{\alpha, \beta\} \times [l+1] \times [n+1]$ and $\binom{[n]}{l+1}$, as follows (where $*$ denotes an arbitrary number):

- $(\alpha, *, *) \prec I \prec (\beta, *, *)$ for all $I \in \binom{[n]}{l+1}$;
- $(\alpha, i, *) \prec (\alpha, j, *)$ and $(\beta, i, *) \succ (\beta, j, *)$ for all $i < j$ in $[l+1]$;
- $(\alpha, i, a) \prec (\alpha, i, b)$ and $(\beta, i, a) \prec (\beta, i, b)$ for all $i \in [l+1]$ and $a < b$ in $[n+1]$; and
- $\binom{[n]}{l+1}$ is ordered lexicographically: $\{1, \ldots, l+1\} \prec \cdots \prec \{n-l, \ldots, n\}$.

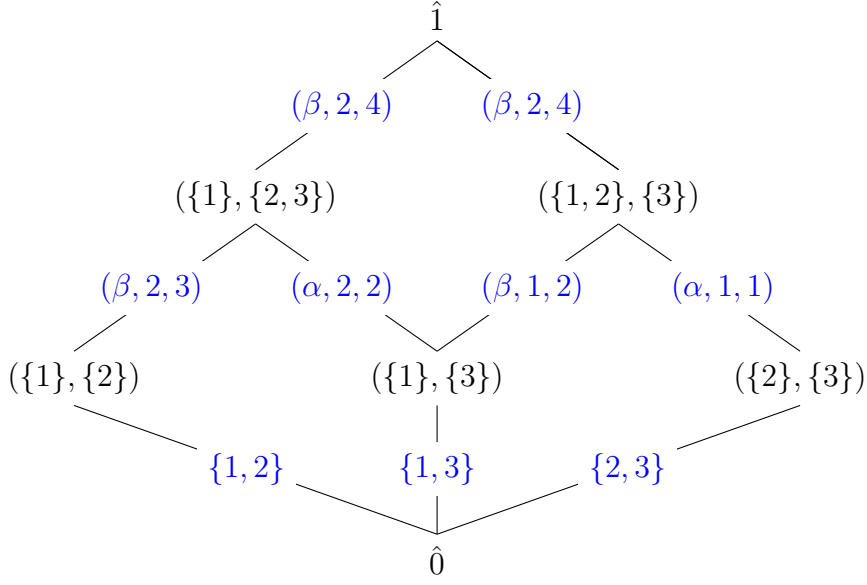We define an edge labeling on $\hat{R}_{n,l}$, with label set $\Lambda_{n,l}$, as follows.

Figure 2.1: The edge labeling of $\hat{R}_{3,1}$ in Definition 2.6.

(i) We label the edge $\hat{0} \lessdot (\{a_1\}, \ldots, \{a_{l+1}\})$ by $\{a_1, \ldots, a_{l+1}\} \in \binom{[n]}{l+1}$.

(ii) Let $\hat{0} < x \lessdot y < \hat{1}$. Then as in Definition 2.5, y is obtained from x by adding some element a to the ith block of x, in a cover relation of type γ (where $\gamma \in \{\alpha, \beta\}$). We label the edge $x \lessdot y$ by (γ, i, a).

(iii) We label the edge $x \lessdot \hat{1}$ by $(\beta, l+1, n+1)$.

For example, see Figure 2.1.

Remark 2.7. We were led to the construction in Definition 2.6 in part so that the following property holds (though we will not end up using it). Let $x \in \hat{R}_{n,l}$ such that x is not covered by $\hat{1}$, and let $y_1, \ldots, y_r$ be the elements of $\hat{R}_{n,l}$ which cover x, ordered so that the labels of $x \lessdot y_1, \ldots, x \lessdot y_r$ are increasing in $(\Lambda_{n,l}, \preceq)$. Then $y_1, \ldots, y_r$ are in increasing order in the lexicographic order on $(l+1)$-tuples. For example, see Figure 2.2. In fact, one can show that ordering the atoms of $[x, \hat{1}]$ lexicographically for all $x \in \hat{R}_{n,l}$ defines a *recursive atom ordering* of $\hat{R}_{n,l}$ (see e.g. [Wac07, Section 4.2]), where the order of the atoms of $[x, \hat{1}]$ does not depend on a choice of maximal chain of $[\hat{0}, x]$. Li [Li21, Lemma 1.1] showed that any finite, bounded, and graded poset admitting such a recursive atom ordering is EL-shellable, so this provides an alternative way to prove Theorem 1.1. We omit the proof of this fact, and instead find it simplest to work only with the edge labeling in Definition 2.6.

Theorem 2.8. *The edge labeling of $\hat{R}_{n,l}$ in Definition 2.6 is an EL-labeling.*

Proof. We must verify that (EL1) and (EL2) hold for every closed interval $[x, y]$ in $\hat{R}_{n,l}$. We consider four cases, depending on whether $x = \hat{0}$ and $y = \hat{1}$. When $x \neq \hat{0}$ we write $x =$

$$(124, 6, 8) \quad (234, 6, 8) \quad (24, 56, 8) \quad (24, 6, 78) \quad (24, 6, 89) \quad (24, 67, 8) \quad (245, 6, 8)$$

$$(\alpha, 1, 1) \quad (\alpha, 1, 3) \quad (\alpha, 2, 5) \quad (\alpha, 3, 7) \quad (\beta, 3, 9) \quad (\beta, 2, 7) \quad (\beta, 1, 5)$$
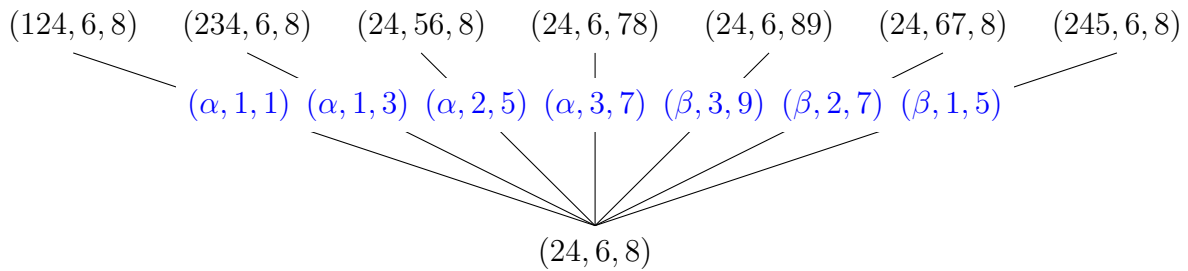
$$(24, 6, 8)$$

Figure 2.2: The element $(\{2, 4\}, \{6\}, \{8\}) \in \hat{R}_{9,2}$, and the elements covering it ordered by increasing edge label (equivalently, ordered lexicographically as $(l + 1)$-tuples).

$(A_1, \ldots, A_{l+1})$, and when $y \neq \hat{1}$ we write $y = (B_1, \ldots, B_{l+1})$. In each case, we explicitly describe the unique maximal chain of $[x, y]$, thereby proving (EL1). It will then be apparent from the form of this maximal chain that (EL2) holds.

Case 1: $x \neq \hat{0}$, $y \neq \hat{1}$. The maximal chains of $[x, y]$ are obtained by adding, in some order, all the elements of $B_i \setminus A_i$ to the ith block (for $i \in [l + 1]$). The unique increasing chain is given by adding these elements in the following order:

- for $i = 1, \ldots, l + 1$, we add the elements of $B_i \setminus A_i$ which are less than $\max(A_i)$ to the ith block, in increasing order (in cover relations of type α);

- for $i = l + 1, \ldots, 1$, we add the elements of $B_i \setminus A_i$ which are greater than $\max(A_i)$ to the ith block, in increasing order (in cover relations of type β).

We see that (EL2) holds.

Case 2: $x = \hat{0}$, $y \neq \hat{1}$. The first edge of any maximal chain of $[\hat{0}, y]$ is labeled by an element of $\binom{[n]}{l+1}$, and so if it is increasing, after the first edge it must pass through edges only of type β. Therefore the unique increasing maximal chain of $[\hat{0}, y]$ begins with the edge $\hat{0} \lessdot (\{b_1\}, \ldots, \{b_{l+1}\})$, where $b_i := \min(B_i)$ for $i \in [l + 1]$ (whence (EL2) is satisfied), and after that follows the unique increasing chain from $(\{b_1\}, \ldots, \{b_{l+1}\})$ to y, as in Case 1.

Case 3: $x \neq \hat{0}$, $y = \hat{1}$. The last edge of any maximal chain of $[x, \hat{1}]$ is labeled by $(\beta, l+1, n+1)$, and so if it is increasing, before the final edge it must pass through edges only of type α or with a label $(\beta, l + 1, *)$. Therefore the unique increasing maximal chain of $[x, \hat{1}]$ ends with the edge $(C_1, \ldots, C_{l+1}) \lessdot \hat{1}$, where

$$C_1 := \{1, 2, \ldots, \max(A_1)\}, C_2 := \{\max(A_1) + 1, \max(A_1) + 2, \ldots, \max(A_2)\}, \ldots,$$
$$C_{l+1} := \{\max(A_l) + 1, \max(A_l) + 2, \ldots, n\},$$

and before that follows the unique increasing chain from x to $(C_1, \ldots, C_{l+1})$, as in Case 1. We see that (EL2) holds.

Case 4: $x = \hat{0}$, $y = \hat{1}$. Reasoning as in Cases 2 and 3, the unique increasing maximal chain of $[\hat{0}, \hat{1}]$ begins with the edge $\hat{0} \lessdot (\{1\}, \ldots, \{l+1\})$, ends with the edge $(\{1\}, \ldots, \{l\}, \{l+1, \ldots, n\}) \lessdot \hat{1}$, and in between follows the unique increasing chain as in Case 1. As in Case 2, (EL2) holds. $\qquad\square$

Remark 2.9. There are results in the literature which imply that various special families of posets are shellable. However, as far as we know, $\hat{R}_{n,l}$ is not contained in such a family. For example, Provan and Billera [PB80, Section 3.4.2] showed that all *distributive lattices* (cf. [Sta12, Section 3.4]) are shellable. While $\hat{R}_{n,l}$ is a lattice, it is not distributive unless $l = 0$ (in which case $R_{n,l}$ is the Boolean algebra B_n with the minimum removed) or $l = n - 1$ (in which case $R_{n,l}$ has a single element). For example, one can see from Figure 2.1 that $\hat{R}_{3,1}$ is not distributive. Also, Björner [Bjö80, Theorem 3.1] showed that all *semimodular lattices* (cf. [Sta12, Section 3.3]) are shellable. However, $\hat{R}_{n,l}$ is not upper-semimodular unless $l = 0$ or $l = n - 1$, and $\hat{R}_{n,l}$ is not lower-semimodular unless $l = 0$, $l = n - 1$, or $(n, l) = (3, 1)$. For example, one can see from Figure 2.1 that $\hat{R}_{3,1}$ is not upper-semimodular. We omit the proofs of these claims.

Remark 2.10. Recall that $R_{n,l}$ is the face poset of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ when $m = 1$. Another interesting special case of $\mathcal{A}_{n,k,m}(Z)$ is $n = k + m$, whence it becomes the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$. Williams [Wil07] and Bao and He [BH21, Theorem 4.1] showed that the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ with a minimum $\hat{0}$ adjoined is EL-shellable, and Knutson, Lam, and Speyer [KLS13, Section 3.5] showed that the face poset (without $\hat{0}$ adjoined) is dual EL-shellable. We point out that $R_{n,l}$ with $\hat{0}$ adjoined (but not $\hat{1}$) is an induced subposet of the face poset of $\mathrm{Gr}_{k,n}^{\geq 0}$ with $\hat{0}$ adjoined [KW19, Theorem 5.17], and so it is EL-shellable by [Wil07, BH21]. Therefore the main difficulty in proving Theorem 1.1 is in dealing with the adjoined maximum $\hat{1}$. Our EL-labeling of $\hat{R}_{n,l}$ does not use the labelings of [Wil07, KLS13, BH21], and it is not clear to us how our labeling is related to these. We plan to study this further in future work.

3. f-vector and h-vector

In this section we examine the f-vector and h-vector of $R_{n,l}$, as well as their refinements by ranks, namely the flag f-vector and flag h-vector. We give a combinatorial interpretation for the h-vector in terms of the EL-labeling of Section 2.2, and also prove explicit formulas for the f-vector and h-vector.

3.1. Background

We refer to [Sta96, Sta12] for background on the f-vector and h-vector.

Definition 3.1 ([Sta12, Section 3.13]). Let P be a finite graded poset of rank $d - 1$, with ranks labeled from 1 to d. For $S \subseteq [d]$, we let α_S be the number of chains of P supported exactly at the ranks in S; we call α the *flag f-vector* of P. We also define the *flag h-vector* β of P by

$$\beta_S := \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_T, \quad \text{or equivalently}, \quad \alpha_S =: \sum_{T \subseteq S} \beta_T \quad (S \subseteq [d]).$$

Alternatively, let P_S denote the induced subposet of P consisting of all elements whose rank lies in S. Then α_S is the number of maximal chains of P_S, and $(-1)^{|S|+1}\beta_S$ is the Möbius invariant $\mu(\hat{P}_S)$ of the bounded extension of P_S.

We define the f-vector $(f_{-1}, f_0, \ldots, f_{d-1})$ and h-vector $(h_0, \ldots, h_d)$ of P by

$$f_{i-1} := \sum_{S \in \binom{[d]}{i}} \alpha_S \quad \text{and} \quad h_i := \sum_{S \in \binom{[d]}{i}} \beta_S \quad (0 \leqslant i \leqslant d).$$

Defining the generating functions[3]

$$F(t) := \sum_{i=0}^{d} f_{i-1} t^i \quad \text{and} \quad H(t) := \sum_{i=0}^{d} h_i t^i,$$

the f-vector and h-vector are related by the equation

$$H(t) = (1-t)^d F(\tfrac{t}{1-t}). \tag{3.1}$$

Remark 3.2. Let P be a finite graded poset of rank $d - 1$, with ranks labeled from 1 to d. Let $\hat{P}$ denote the bounded extension of P, with ranks labeled from 0 to $d + 1$. Then for all $S \subseteq \{0, \ldots, d + 1\}$, we have [Sta12, p. 294]

$$\alpha_S(\hat{P}) = \alpha_{S \setminus \{0, d+1\}}(P) \quad \text{and} \quad \beta_S(\hat{P}) = \begin{cases} 0, & \text{if } 0 \in S \text{ or } d+1 \in S; \\ \beta_S(P), & \text{otherwise.} \end{cases}$$

In particular, P and $\hat{P}$ have the same (flag) h-vector, and the (flag) f-vector of $\hat{P}$ is easily determined from P. Therefore enumerative results for $R_{n,l}$ apply as well to $\hat{R}_{n,l}$, and vice-versa. Keeping this connection in mind, we will label the ranks of $R_{n,l}$ from 1 to $n - l$ (rather than from 0 to $n - l - 1$).

Example 3.3. Consider the poset $R_{3,1}$, shown in Figure 1.1. Then $d = 2$, and

$$\alpha_\varnothing = 1, \quad \alpha_{\{1\}} = 3, \quad \alpha_{\{2\}} = 2, \quad \alpha_{\{1,2\}} = 4, \quad (f_{-1}, f_0, f_1) = (1, 5, 4), \quad F(t) = 1 + 5t + 4t^2;$$
$$\beta_\varnothing = 1, \quad \beta_{\{1\}} = 2, \quad \beta_{\{2\}} = 1, \quad \beta_{\{1,2\}} = 0, \quad (h_0, h_1, h_2) = (1, 3, 0), \quad H(t) = 1 + 3t.$$

3.2. Combinatorial interpretations

Björner [Bjö80, Theorem 2.7], based on work of Stanley [Sta72, Theorem 1.2], gave a combinatorial interpretation for the flag h-vector of any poset with an edge labeling satisfying (EL1). We state it here in the special case of $\hat{R}_{n,l}$, with the edge labeling defined in Definition 2.6.

Definition 3.4. Given a maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{n-l} \lessdot x_{n-l+1} = \hat{1}$ of $\hat{R}_{n,l}$, we say that $i \in [n - l]$ is a *descent* of C when $\lambda(x_{i-1} \lessdot x_i) \succ \lambda(x_i \lessdot x_{i+1})$.[4]

---

[3]Our $F(t)$ and $H(t)$ are the reverses of the generating functions in [Sta12].

[4]For edge labelings of general posets, one should replace '$\succ$' with '$\not\prec$' in the definition. There is no difference for our edge labeling of $\hat{R}_{n,l}$, since the label set $\Lambda_{n,l}$ is totally ordered and no label is repeated in any maximal chain.
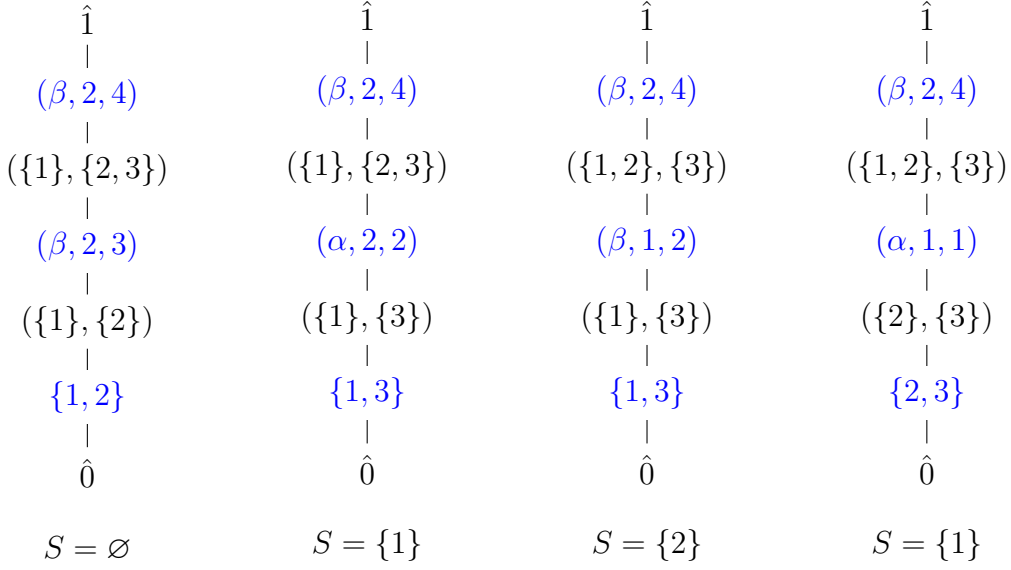
$$
\begin{array}{cccc}
\hat{1} & \hat{1} & \hat{1} & \hat{1} \\
| & | & | & | \\
(\beta, 2, 4) & (\beta, 2, 4) & (\beta, 2, 4) & (\beta, 2, 4) \\
| & | & | & | \\
(\{1\}, \{2, 3\}) & (\{1\}, \{2, 3\}) & (\{1, 2\}, \{3\}) & (\{1, 2\}, \{3\}) \\
| & | & | & | \\
(\beta, 2, 3) & (\alpha, 2, 2) & (\beta, 1, 2) & (\alpha, 1, 1) \\
| & | & | & | \\
(\{1\}, \{2\}) & (\{1\}, \{3\}) & (\{1\}, \{3\}) & (\{2\}, \{3\}) \\
| & | & | & | \\
\{1, 2\} & \{1, 3\} & \{1, 3\} & \{2, 3\} \\
| & | & | & | \\
\hat{0} & \hat{0} & \hat{0} & \hat{0} \\
\\
S = \varnothing & S = \{1\} & S = \{2\} & S = \{1\}
\end{array}
$$

Figure 3.1: The maximal chains of $\hat{R}_{3,1}$ and their descent sets S.

Theorem 3.5 (Björner and Stanley; cf. [Sta12, Theorem 3.14.2][5]). *Recall the edge labeling of $\hat{R}_{n,l}$ in Definition 2.6. For all $S \subseteq [n-l]$, we have that β_S equals the number of maximal chains of $\hat{R}_{n,l}$ with descent set S. Thus for all $0 \leqslant d \leqslant n-l$, we have that h_d equals the number of maximal chains of $\hat{R}_{n,l}$ with exactly d descents.*

Example 3.6. The maximal chains of $\hat{R}_{3,1}$ and their descent sets are shown in Figure 3.1. According to Theorem 3.5, we have $\beta_\varnothing = 1$, $\beta_{\{1\}} = 2$, $\beta_{\{2\}} = 1$, and $\beta_{\{1,2\}} = 0$, consistent with Example 3.3.

We also have the following explicit description of all the maximal chains of $R_{n,l}$ (and hence also $\hat{R}_{n,l}$):

Proposition 3.7. *The number of maximal chains of $R_{n,l}$ is $f_{n-l-1} = \binom{n+l}{2l+1}(n-l-1)!$. Explicitly, given $A \in \binom{[n+l]}{2l+1}$ and a permutation $\pi \in \mathfrak{S}_{n-l-1}$, we associate a maximal chain $C(A, \pi)$ of $R_{n,l}$ as follows:*

- *writing $A = \{c_1 < \cdots < c_{2l+1}\}$, we set $a_i := c_{2i-1} - i + 1$ for $1 \leqslant i \leqslant l + 1$ and $b_i := c_{2i} - i$ for $1 \leqslant i \leqslant l$;*

- *we take $C(A, \pi)$ to have minimal element $x := (\{a_1\}, \ldots, \{a_{l+1}\})$ and maximal element $y := (\{1, \ldots, b_1\}, \{b_1 + 1, \ldots, b_2\}, \ldots, \{b_l + 1, \ldots, n\})$;*

---

[5]Our conventions differ slightly from those in [Sta12], since in (EL1) we require edge labels to strictly (rather than weakly) increase. Nevertheless, the result [Sta12, Theorem 3.14.2] and its proof transfer easily to our setting.

- *writing $[n] \setminus \{a_1, \ldots, a_{l+1}\} = \{a'_1, \ldots, a'_{n-l-1}\}$ (in increasing order), $C(A, \pi)$ is given by adding the elements $a'_{\pi(1)}, \ldots, a'_{\pi(n-l-1)}$ to x (in that order), each to the appropriate block (determined by y).*

Proof. We can verify that the map $(A, \pi) \mapsto C(A, \pi)$ gives a bijection from $\binom{[n+l]}{2l+1} \times \mathfrak{S}_{n-l-1}$ to the set of maximal chains of $R_{n,l}$. $\square$

For example, the maximal chains in Figure 3.1 are (from left to right)

$$C(\{1, 2, 3\}, 1), \; C(\{1, 2, 4\}, 1), \; C(\{1, 3, 4\}, 1), \; \text{and} \; C(\{2, 3, 4\}, 1).$$

While Proposition 3.7 gives a simple description of the maximal chains of $\hat{R}_{n,l}$, we are not able in general to translate Definition 2.6 into a simple description of the descents of $C(A, \pi)$ in terms of A and π. However, in the special case $l = 0$, we do have such a simple description: maximal chains correspond to permutations of $[n]$ with the usual notion of descent, as we now explain.

Definition 3.8. Given $\pi \in \mathfrak{S}_n$, we say that $r \in [n-1]$ is a *descent* of π if $\pi(r) > \pi(r+1)$. For $0 \leqslant d \leqslant n$, we define the *Eulerian number* $\left\langle {n \atop d} \right\rangle$ as the number of permutations in $\mathfrak{S}_n$ with exactly d descents.

For example, $\left\langle {3 \atop 1} \right\rangle = 4$, corresponding to the permutations (in one-line notation) 132, 213, 231, and 312. We refer to [Pet15] for further details about Eulerian numbers.

Proposition 3.9. *There is a bijection between maximal chains of $\hat{R}_{n,0}$ and permutations in $\mathfrak{S}_n$ which preserves descent sets. In particular, by Theorem 3.5, we have $h_d = \left\langle {n \atop d} \right\rangle$ for $0 \leqslant d \leqslant n$.*

Proof. The bijection sends the permutation $\pi \in \mathfrak{S}_n$ to the maximal chain

$$\hat{0} \lessdot (\{\pi(1)\}) \lessdot (\{\pi(1), \pi(2)\}) \lessdot \cdots \lessdot (\{\pi(1), \ldots, \pi(n)\}) \lessdot \hat{1}.$$

We can verify that the notions of descent in Definition 2.6 and Definition 3.8 agree. $\square$

3.3. Explicit formulas

We now turn to giving explicit formulas for the f-vector and h-vector of $R_{n,l}$.

Proposition 3.10. *The flag f-vector of $R_{n,l}$ is given by*

$$\alpha_S = \binom{l + r_1 - 1}{l} \binom{n}{l + r_d} \binom{2l + r_d}{r_d - r_1} \binom{r_d - r_1}{r_2 - r_1, \ldots, r_d - r_{d-1}}$$

for all $S = \{r_1 < \cdots < r_d\} \subseteq [n-l]$.

Proof. We enumerate the chains $x_1 < \cdots < x_d$ of $R_{n,l}$ supported at ranks $r_1, \ldots, r_d$ as follows. Write $x_1 = (A_1, \ldots, A_{l+1})$ and $x_d = (B_1, \ldots, B_{l+1})$. Since $|B_1 \cup \cdots \cup B_{l+1}| = l + r_d$, the number of ways to choose $B_1 \cup \cdots \cup B_{l+1}$ is

$$\binom{n}{l + r_d}.$$

After relabeling the set $[n]$, we may assume that $B_1 \cup \cdots \cup B_{l+1} = [l + r_d]$.

Let s_i denote the size of A_i (for $1 \leqslant i \leqslant l + 1$), so that $s_i \geqslant 1$ and $s_1 + \cdots + s_{l+1} = l + r_1$. The number of ways to choose $s_1, \ldots, s_{l+1}$ is

$$\binom{l + r_1 - 1}{l}.$$

For $1 \leqslant i \leqslant l + 1$, write $A_i = \{a_{i,1} < \cdots < a_{i,s_i}\}$, and set $b_i := \max(B_i)$. Then the $a_{i,j}$'s and b_i's are arbitrary elements of $[l + r_d]$ subject to

$$a_{1,1} < \cdots < a_{1,s_1} \leqslant b_1 < a_{2,1} < \cdots < a_{2,s_2} \leqslant b_2 < \cdots \leqslant b_l < a_{l+1,1} < \cdots < a_{l+1,s_{l+1}}.$$

The number of ways to choose the $a_{i,j}$'s and b_i's is

$$\binom{2l + r_d}{2l + r_1} = \binom{2l + r_d}{r_d - r_1},$$

at which point x_1 and x_d are fixed.

Finally, the elements $x_2, \ldots, x_{d-1}$ are determined by a set composition of $(B_1 \cup \cdots \cup B_{l+1}) \setminus (A_1 \cup \cdots \cup A_{l+1})$ with blocks of respective sizes $r_2 - r_1, \ldots, r_d - r_{d-1}$. The number of choices is

$$\binom{r_d - r_1}{r_2 - r_1, \ldots, r_d - r_{d-1}}. \qquad \square$$

We now use Proposition 3.10 to give a formula for the f-vector of $R_{n,l}$. The following formula allows us to simplify the resulting sum, at the cost of introducing minus signs.

Lemma 3.11 ([Sta12, (1.94a)])**.** *Let $s \in \mathbb{N}$ and $d \in \mathbb{Z}_{>0}$. Then*

$$\sum_{\substack{i_1, \ldots, i_d \geqslant 1, \\ i_1 + \cdots + i_d = s}} \binom{s}{i_1, \ldots, i_d} = \sum_{i=0}^{d} (-1)^i \binom{d}{i} (d - i)^s.$$

Proof. This follows from [Sta12, (1.94a)], since both sides equal $d! \, S(s, d)$, where $S(s, d)$ is a Stirling number of the second kind. Alternatively, we can prove this directly from the inclusion-exclusion principle [Sta12, Theorem 2.1.1]. $\qquad \square$

Corollary 3.12. *Let $0 \leqslant l < n$ and $0 \leqslant d \leqslant n - l - 1$. The number of chains of $R_{n,l}$ of length d which begin at rank r and end at rank $r + s$ equals*

$$\sum_{i=0}^{d} (-1)^i \binom{d}{i} \binom{l + r - 1}{l} \binom{n}{l + r + s} \binom{2l + r + s}{s} (d - i)^s.$$

Then f_d is given by summing the quantity above over all $r \geqslant 1$ and $s \geqslant 0$ (or alternatively $s \geqslant d$).

Proof. This follows from Proposition 3.10, using Lemma 3.11. $\qquad \square$

Example 3.13. Taking $d = 0$ in Corollary 3.12, we obtain the number of elements of $R_{n,l}$:

$$f_0 = \sum_{r=1}^{n-l} \binom{l+r-1}{l}\binom{n}{l+r}.$$

Finally, we use Corollary 3.12 to obtain the generating functions for the f- and h-vectors:

Theorem 3.14. *The generating functions for the f- and h-vectors of $R_{n,l}$ are given by*

$$F(t) = 1 + \sum_{j,r,s \geqslant 0} \binom{l+r}{l}\binom{n}{l+r+s+1}\binom{2l+r+s+1}{s} j^s \left(\frac{t}{1+t}\right)^{j+1}$$

and

$$H(t) = (1-t)^{n-l}\left(1 + \sum_{j,r,s \geqslant 0} \binom{l+r}{l}\binom{n}{l+r+s+1}\binom{2l+r+s+1}{s} j^s t^{j+1}\right).$$

We then obtain an explicit formula (albeit with negative signs) for h_i by taking the coefficient of t^i in $H(t)$.

Proof. By Corollary 3.12 (replacing $r - 1$ by r), and then writing $d = i + j$ and applying the negative binomial theorem, we obtain

$$\begin{aligned}
F(t) &= 1 + \sum_{d \geqslant 0} \sum_{i,r,s \geqslant 0} (-1)^i \binom{d}{i}\binom{l+r}{l}\binom{n}{l+r+s+1}\binom{2l+r+s+1}{s}(d-i)^s t^{d+1} \\
&= 1 + \sum_{i,j,r,s \geqslant 0} (-1)^i \binom{i+j}{i}\binom{l+r}{l}\binom{n}{l+r+s+1}\binom{2l+r+s+1}{s} j^s t^{i+j+1} \\
&= 1 + \sum_{j,r,s \geqslant 0} \binom{l+r}{l}\binom{n}{l+r+s+1}\binom{2l+r+s+1}{s} j^s t^{j+1}(1+t)^{-(j+1)}.
\end{aligned}$$

This proves the first equation. The second equation follows by applying (3.1). $\square$

Example 3.15. Let us set $l = 0$ in Theorem 3.14 to obtain the generating function for the h-vector of $R_{n,0}$:

$$\begin{aligned}
H(t) &= (1-t)^n\left(1 + \sum_{j,r,s \geqslant 0} \binom{n}{r+s+1}\binom{r+s+1}{s} j^s t^{j+1}\right) \\
&= (1-t)^n\left(1 + \sum_{j,r,s \geqslant 0} \binom{n}{s}\binom{n-s}{r+1} j^s t^{j+1}\right) \\
&= (1-t)^n\left(1 + \sum_{j,s \geqslant 0} \binom{n}{s}(2^{n-s}-1) j^s t^{j+1}\right)
\end{aligned}$$

$$= (1-t)^n \left(1 + \sum_{j \geqslant 0} \left((j+2)^n - (j+1)^n \right) t^{j+1} \right)$$

$$= (1-t)^{n+1} \sum_{j \geqslant 0} (j+1)^n t^j,$$

where we applied the binomial theorem twice. This yields a well-known generating function for the Eulerian numbers [Pet15, (1.10)], in agreement with Proposition 3.9.

Example 3.16. Let us use Theorem 3.14 to find h_1 for $R_{n,l}$, by taking the coefficient of t in $H(t)$:

$$h_1 = l - n + \sum_{r \geqslant 0} \binom{l+r}{l} \binom{n}{l+r+1}.$$

We can compute the latter sum using the identity

$$\sum_{r \geqslant 0} \binom{l+r}{l} \binom{n}{l+r+1} u^r = \frac{1}{l!} \frac{d^l}{du^l} \left(\frac{(1+u)^n - 1}{u} \right)$$

$$= (-1)^{l+1} u^{-(l+1)} + \sum_{i=0}^{l} (-1)^{l-i} \binom{n}{i} (1+u)^{n-i} u^{-(l-i+1)};$$

the first equality above follows from the binomial theorem, and the second equality follows from the product rule for the derivative. Setting $u = 1$ gives

$$h_1 = l - n + (-1)^{l+1} + \sum_{i=0}^{l} (-1)^{l-i} \binom{n}{i} 2^{n-i}.$$

For example, when $l = 0$ we obtain $\left\langle {n \atop 1} \right\rangle = h_1 = 2^n - n - 1$.

4. Normalized flow

In this section we prove Theorem 1.3, which states that $R_{n,l}$ is rank-log-concave and strongly Sperner. We prove the former in Proposition 4.3, and the latter in Theorem 4.7 using Harper's notion of a *normalized flow* [Har74].

4.1. Background

We provide some background on unimodal and log-concave sequences, the strongly Sperner property, and normalized flows, following [Sta12, Eng97, Har74].

Definition 4.1. Let $s = (s_1, \ldots, s_d)$ be a sequence of nonnegative real numbers. We say that s is *unimodal* if for some $1 \leqslant j \leqslant d$, we have

$$s_1 \leqslant \cdots \leqslant s_{j-1} \leqslant s_j \geqslant s_{j+1} \geqslant \cdots \geqslant s_d.$$

We say that s is *log-concave* if

$$s_{i-1}s_{i+1} \leqslant s_i^2 \quad \text{for } 2 \leqslant i \leqslant d-1.$$

One can verify that if s is a log-concave sequence of nonnegative real numbers and has no internal zeros, then s is unimodal. We also observe that the entry-wise product of two log-concave sequences is log-concave.

Definition 4.2. Let P be a finite graded poset of rank $d-1$, with ranks labeled from 1 to d. For $1 \leqslant r \leqslant d$, the rth *Whitney number of the second kind* W_r is defined to be the number of elements of P of rank r. In terms of the flag f-vector, we have $W_r = \alpha_{\{r\}}$. We say that P is *rank-unimodal* (respectively, *rank-log-concave*) if the sequence $(W_1, \ldots, W_r)$ is unimodal (respectively, log-concave). We observe that if P is rank-log-concave, then it is rank-unimodal.

For example, from Figure 1.1 we see that for $R_{3,1}$, we have $(W_1, W_2) = (3, 2)$. For general $R_{n,l}$, we can read off W_r from Proposition 3.10:

Proposition 4.3. *The Whitney numbers of the second kind of $R_{n,l}$ (with ranks labeled from 1 to $n-l$) are*

$$W_r = \binom{l+r-1}{l}\binom{n}{l+r} \quad \text{for } 1 \leqslant r \leqslant n-l.$$

In particular, $R_{n,l}$ is rank-log-concave.

Proof. The formula for W_r follows by taking $S = \{r\}$ in Proposition 3.10. The sequence $(W_1, \ldots, W_{n-l})$ is log-concave because it is the entry-wise product of the log-concave sequences

$$\left(\binom{l+r-1}{l}\right)_{r=1}^{n-l} \quad \text{and} \quad \left(\binom{n}{l+r}\right)_{r=1}^{n-l}. \qquad \square$$

We now introduce the (strongly) Sperner property and normalized flows. Recall that an *antichain* in a poset is a subset of pairwise incomparable elements.

Definition 4.4. Let P be a finite graded poset of rank $d-1$, with ranks labeled from 1 to d. Given $j \geqslant 1$, we say that P is j-*Sperner* if the maximum size of a union of j antichains is realized by taking the j largest ranks, i.e.,

$$|A_1 \cup \cdots \cup A_j| \leqslant \max_{1 \leqslant r_1 < \cdots < r_j \leqslant d} W_{r_1} + \cdots + W_{r_j} \quad \text{for all antichains } A_1, \ldots, A_j \subseteq P.$$

We say that P is *Sperner*[6] if P is 1-Sperner, and we say that P is *strongly Sperner* if P is j-Sperner for all $j \geqslant 1$.

Definition 4.5 ([Har74]; [Eng97, p. 150]). Let P be a finite graded poset of rank $d-1$, with ranks labeled from 1 to d. A *normalized flow* is an edge labeling f (of the edges of the Hasse diagram of P) taking values in $\mathbb{R}_{\geqslant 0}$, such that the following conditions hold for $1 \leqslant r \leqslant d-1$:[7]

---

[6]The term is so named because Sperner showed that the Boolean algebra B_n has the Sperner property [Spe28]. In fact, B_n is strongly Sperner (cf. [Eng97, Example 4.6.2]).

[7]The original definition also requires that the sum of $f(x \lessdot y)$ over all cover relations $x \lessdot y$ between ranks r and $r+1$ equals 1. Given an f satisfying (NF1) and (NF2), we can achieve this additional constraint by rescaling all such $f(x \lessdot y)$ by the same appropriate positive constant (depending on r).

(NF1) $\sum_{y,\, x \lessdot y} f(x \lessdot y)$ is the same positive number for all $x \in P$ of rank r; and

(NF2) $\sum_{x,\, x \lessdot y} f(x \lessdot y)$ is the same positive number for all $y \in P$ of rank $r+1$.

Harper [Har74, Theorem p. 55] showed that if P admits a normalized flow, then P is strongly Sperner. In fact, it follows from work of Kleitman [Kle74] (see [Eng97, Theorem 4.5.1]) that such a P satisfies the stronger *LYM inequality*:

$$\sum_{x \in A} \frac{1}{W_{\mathrm{rank}(x)}} \leqslant 1 \quad \text{for all antichains } A.$$

4.2. Construction of the normalized flow

We define a normalized flow on $R_{n,l}$. Our definition will manifestly satisfy (NF1), and we will then check carefully that (NF2) holds.

Definition 4.6. Let $0 \leqslant l < n$. We define an edge labeling f on $R_{n,l}$, with label set $\mathbb{R}_{\geqslant 0}$, as follows. Let $x = (A_1, \dots, A_{l+1}) \in R_{n,l}$, and let $a \in [n] \setminus (A_1 \cup \cdots \cup A_{l+1})$. Consider all elements $y \gtrdot x$ obtained from x by adding a to some block; there are exactly 1 or 2 such y. There is a unique such y if and only if $a < \max(A_1)$ or $a > \min(A_{l+1})$, in which case, we set

$$f(x \lessdot y) := 1.$$

Otherwise, we have $\max(A_i) < a < \min(A_{i+1})$ for some $1 \leqslant i \leqslant l$. We can add a either to the ith block or to the $(i+1)$th block, forming, say, y_1 and y_2, respectively. We then set

$$f(x \lessdot y_1) := \frac{|A_1 \cup \cdots \cup A_i|}{|A_1 \cup \cdots \cup A_{l+1}|} \quad \text{and} \quad f(x \lessdot y_2) := \frac{|A_{i+1} \cup \cdots \cup A_{l+1}|}{|A_1 \cup \cdots \cup A_{l+1}|}.$$

Note that in either case, given x and a, the sum of $f(x \lessdot y)$ over all y obtained from x by adding a to some block equals 1. For example, see Figure 4.1 and Figure 4.2.

Theorem 4.7. *The edge labeling of $R_{n,l}$ in Definition 4.6 is a normalized flow. In particular, $R_{n,l}$ is strongly Sperner.*

Proof. Fix $1 \leqslant r \leqslant n - l - 1$. Let $x \in R_{n,l}$ have rank r, so that $x = (A_1, \dots, A_{l+1})$ with $|A_1| + \cdots + |A_{l+1}| = l + r$. Then by construction, we have

$$\sum_{y,\, x \lessdot y} f(x \lessdot y) = n - l - r,$$

which is positive and depends only on r. Therefore (NF1) holds.

Now we prove (NF2). Let $y \in R_{n,l}$ have rank $r+1$, and write $y = (B_1, \dots, B_{l+1})$. Let $s_i := |B_i|$ for $1 \leqslant i \leqslant l+1$, so that $s_1 + \cdots + s_{l+1} = l + r + 1$. Note that the elements $x \lessdot y$ are precisely those obtained from y by selecting some block i ($1 \leqslant i \leqslant l+1$) with $s_i \geqslant 2$, and removing some element b of B_i. The value $f(x \lessdot y)$ is determined according to the following three cases:
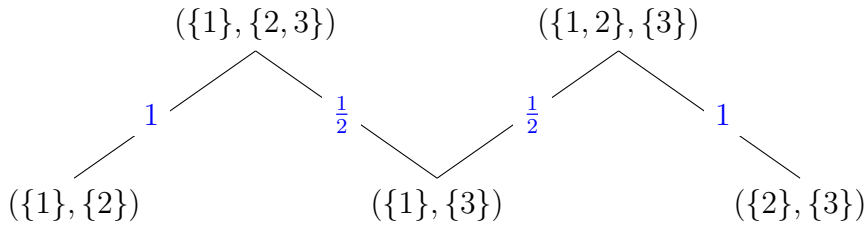
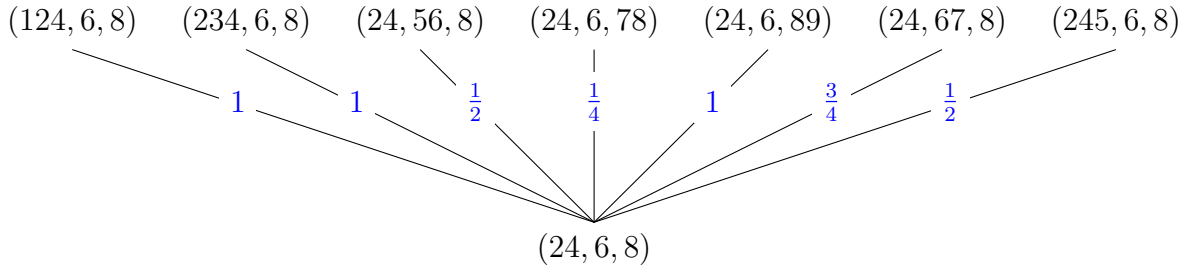Figure 4.1: The normalized flow on $R_{3,1}$ defined in Definition 4.6.

Figure 4.2: The element $(\{2,4\},\{6\},\{8\}) \in R_{9,2}$, the elements covering it, and the values of the normalized flow defined in Definition 4.6.

(i) if $b = \min(B_i)$, then $f(x \lessdot y) = \frac{s_i + \cdots + s_{l+1} - 1}{l+r}$;

(ii) if $\min(B_i) < b < \max(B_i)$, then $f(x \lessdot y) = 1$; and

(iii) if $b = \max(B_i)$, then $f(x \lessdot y) = \frac{s_1 + \cdots + s_i - 1}{l+r}$.

The sum of the values $f(x \lessdot y)$, over all b in all three cases above (with y and i fixed), equals

$$\frac{s_i + \cdots + s_{l+1} - 1}{l+r} + (s_i - 2) + \frac{s_1 + \cdots + s_i - 1}{l+r} = \frac{l+r+1}{l+r}(s_i - 1).$$

Note that this formula also gives the desired sum (i.e. 0) when $s_i = 1$. Therefore we obtain

$$\sum_{x,\, x \lessdot y} f(x \lessdot y) = \sum_{i=1}^{l+1} \frac{l+r+1}{l+r}(s_i - 1) = \frac{r(l+r+1)}{l+r},$$

which is positive and depends only on r. This completes the proof. $\qquad\square$

5. The poset $P_{n,l}$

In this section we consider the poset $P_{n,l}$. Recall that $P_{n,l}$ is the poset of projective sign vectors of length n with at most l sign changes, under the relation (1.1) (see Figure 5.1).

It is natural to ask which properties of $R_{n,l}$ carry over to $P_{n,l}$. First we consider shellability. Since $P_{n,0} = R_{n,0}$, by e.g. Theorem 1.1, we have that $\hat{P}_{n,0}$ is EL-shellable. We can also verify
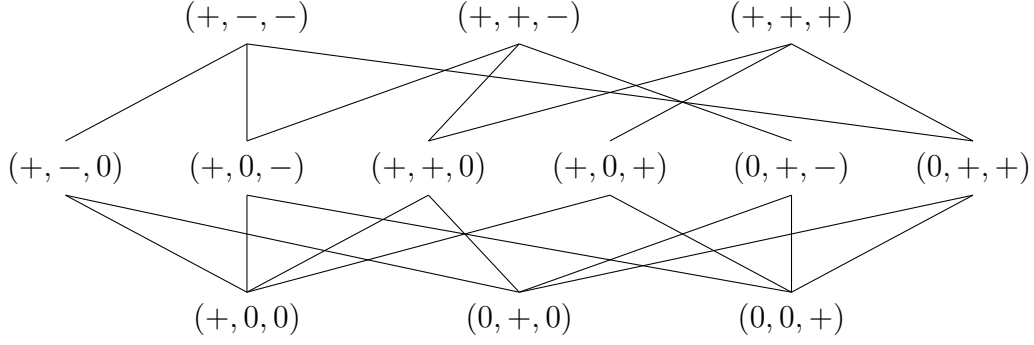
Figure 5.1: The Hasse diagram of the poset $P_{3,1}$.

directly that $\hat{P}_{2,1}$ is EL-shellable. We claim that in the remaining cases, $\hat{P}_{n,l}$ is not shellable. Indeed, if it were shellable, then the order complex of $P_{n,l}$ would be homeomorphic to a sphere or a closed ball of dimension $n-1$ [Bjö84, Proposition 4.3]. On the other hand, Machacek [Mac19] showed that the order complex of $P_{n,l}$ is homotopy equivalent to $\mathbb{RP}^l$, which is homeomorphic to the sphere S^1 when $l = 1$, and is not homotopy equivalent to a sphere or a closed ball when $l \geqslant 2$.

We now show that $P_{n,l}$, like $R_{n,l}$, is rank-log-concave. We will use the following lemma, which appeared in talk slides of Mani [Man09]. We give a proof following an argument of Semple and Welsh [SW08, Example 2.2], who showed that a similar sequence is log-concave.

Lemma 5.1. *Let $l \in \mathbb{N}$. Then the sequence $(s_1, s_2, \dots)$ is log-concave, where*

$$s_r := \sum_{i=0}^{l} \binom{r-1}{i} \quad \text{for } r \geqslant 1.$$

Proof. We must show $s_{r+1}s_{r+3} \leqslant s_{r+2}^2$ for $r \geqslant 0$. Using Pascal's identity $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$, we get

$$s_{r+2} = 2s_{r+1} - \binom{r}{l} \quad \text{and} \quad s_{r+3} = 4s_{r+1} - 3\binom{r}{l} - \binom{r}{l-1}.$$

Therefore we can rewrite the inequality $s_{r+1}s_{r+3} \leqslant s_{r+2}^2$ as

$$\binom{r}{l}\left(s_{r+1} - \binom{r}{l}\right) \leqslant \binom{r}{l-1}s_{r+1}.$$

This follows by summing the inequalities

$$\binom{r}{l}\binom{r}{i-1} \leqslant \binom{r}{l-1}\binom{r}{i} \quad \text{for } 0 \leqslant i \leqslant l. \qquad \square$$

Theorem 5.2. *Let $0 \leqslant l < n$. The Whitney numbers of the second kind of $P_{n,l}$ (with ranks labeled from 1 to n) are*

$$W_r = \binom{n}{r}\sum_{i=0}^{l}\binom{r-1}{i} \quad \text{for } 1 \leqslant r \leqslant n.$$

The sequence $(W_1, \ldots, W_n)$ is log-concave, i.e., $P_{n,l}$ is rank-log-concave.

Proof. The set of elements of $P_{n,l}$ is the disjoint union of $R_{n,i}$ for $0 \leqslant i \leqslant l$, where rank s of $R_{n,i}$ appears in $P_{n,l}$ in rank $s + i$. Therefore by Proposition 4.3, we have

$$W_r(P_{n,l}) = \sum_{i=0}^{\min(l, r-1)} W_{r-i}(R_{n,i}) = \binom{n}{r} \sum_{i=0}^{l} \binom{r-1}{i} \quad \text{for } 1 \leqslant r \leqslant n.$$

This proves the formula for W_r. Now note that $(W_1, \ldots, W_n)$ is the product of the two sequences

$$\left(\binom{n}{r} \right)_{r=1}^{n} \quad \text{and} \quad \left(\sum_{i=0}^{l} \binom{r-1}{i} \right)_{r=1}^{n}.$$

We can verify that the first sequence is log-concave, and the second sequence is log-concave by Lemma 5.1. Therefore $(W_1, \ldots, W_n)$ is log-concave. $\qquad\square$

We conjecture that $P_{n,l}$, like $R_{n,l}$, is Sperner:

Conjecture 5.3. For $0 \leqslant l < n$, the poset $P_{n,l}$ is Sperner.

We have verified that Conjecture 5.3 holds for all $0 \leqslant l < n \leqslant 8$. We also show that it holds when l equals 0, 1, or $n - 1$:

Proposition 5.4. *The posets $P_{n,0}$, $P_{n,1}$, and $P_{n,n-1}$ admit a normalized flow, and hence are strongly Sperner.*

Proof. For $P_{n,0} = R_{n,0}$, this follows from Theorem 4.7. For $P_{n,n-1}$, the constant function 1 is a normalized flow. This is because $P_{n,n-1}$ is *biregular*, i.e., any two elements of $P_{n,n-1}$ of the same rank have the same up-degree and the same down-degree in the Hasse diagram.

Finally, we construct a normalized flow f on $P_{n,1}$, similar to the one defined on $R_{n,1}$ in Definition 4.6. Let $x \in P_{n,1}$, and let $a \in [n]$ such that $x_a = 0$. Consider the elements covering x obtained by changing entry a to either $+$ or $-$; there are exactly one or two of them. If there is one such element, say y, we set $f(x \lessdot y) := 1$. If there are two such elements, say y_1 and y_2, we set $f(x \lessdot y_i) := \frac{1}{2}$ for $i = 1, 2$. Then if x has rank r (with $1 \leqslant r \leqslant n-1$), there are exactly $n - r$ possible values of a, so

$$\sum_{y, \, x \lessdot y} f(x \lessdot y) = n - r.$$

This is positive and depends only on r, which proves (NF1).

Now we verify that (NF2) holds. Let $1 \leqslant r \leqslant n - 1$, and let $y \in P_{n,l}$ have rank $r + 1$. Given $a \in [n]$ such that $y_a \neq 0$, let $x \lessdot y$ be obtained from y by changing entry a to 0, and let z be the sign vector obtained from y by flipping entry a (from $+$ to $-$ or vice versa). If z has at most one sign change, then $f(x \lessdot y) = \frac{1}{2}$, while if z has at least two sign changes, then $f(x \lessdot y) = 1$. We observe that the first case occurs for exactly 2 values of a, while the second case occurs for the remaining $r - 1$ values of a. Therefore

$$\sum_{x, \, x \lessdot y} f(x \lessdot y) = 2(\tfrac{1}{2}) + (r - 1) = r,$$

which is positive and depends only on r. This proves (NF2). $\qquad\square$

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