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ADVANCES IN DOUBLET MECHANICS: III. UNIQUENESS AND SOLUTION METHODS IN PLANE ELASTOSTATICS

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Advances in Doublet Mechanics: III. Uniqueness and Solution Methods in Plane Elastostatics

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Abstract

A uniqueness theorem is presented for linear elastic doublet mechanics. Restricting attention to plane problems in elastostatics a correspondence between solutions in doublet and continuum mechanics is achieved, thus allowing a technique for generating solutions. A micro-stress function is introduced in analogy with the Airy stress function in continuum mechanics. Utilizing these solution methodologies a sampling of problems in plane elastostatic doublet mechanics is solved. In particular, the fundamental problems of Flamant, Kelvin and stress concentration due to a circular hole are considered.

1 Introduction

Recognizing the discrete nature of natural materials Granik and Ferrari (1993) introduced the theory of doublet mechanics (DM), wherein the body is modeled as a lattice of discrete points rather than a continuum. A DM body is thus a collection of discrete points separated by "small," but finite, distances. Since its inception, the theory has been successfully applied to failure theories (Ferrari and Granik, 1994, 1995), viscoelasticity (Maddalena and Ferrari, 1995), elastic wave propogation (Granik and Ferrari, 1995; Zhang and Ferrari, 1995) and thermomechanics (Mon and Ferrari, 1995).

We begin with a uniqueness theorem in linear elastic doublet mechanics. With this tool in hand we present two methods of obtaining solutions. The first stems from a correspondence between problems in DM and continuum mechanics (CM), which allows the generation of a solution in one theory given a solution in the other. The second methodology involves combining the micro-stress equilibrium and micro-strain compatibility requirements into a single condition, from which stems a micro-stress function (MSF) analogous to the CM Airy stress function (ASF). While an admissible ASF is any bi-harmonic function, the equation governing the MSF is a more general fourth order differential equation dependent on the lattice geometry. We show that for a specific choice of the DM lattice the MSF a bi-harmonic function.

We illustrate the advantages of the two methodologies by solving several problems in plane elastostatics We begin by considering homogeneous deformations and then obtain solutions to the classical problems of Flamant and Kelvin. Finally we obtain the stress concentrations due to a circular hole in an infinite plate subjected to bi-axial tension. Our purpose is not to provide a catalog of solutions in DM, but rather to elucidate the techniques.

As an application, Granik and Ferrari (1993) considered the DM equivalent of Flamant's problem: a concentrated force acting normal to the free-surface boundary of a planar elastic half-space. Though the qualitative description of the existing DM solution is accurate, its quantitative inconsistencies are corrected in this paper.

DM is a scale dependent theory but for the purposes of this paper we consider only the non-scale subcase. The reader is referred to Granik and Ferrari (1993, 1995) for background and the general theory of DM.

2 Uniqueness Theorem

The objective of this section is to establish a uniqueness theorem in linear elastic DM. Consider a DM body $\mathcal{B}$ with boundary $\partial\mathcal{B}$. The body $\mathcal{B}$ is subjected to a body force field $\mathbf{b}$. The boundary is partitioned into $\{\partial\mathcal{B}_u, \partial\mathcal{B}_T\}$ such that the displacement field $\mathbf{u}$ is prescribed on $\partial\mathcal{B}_u$ and tractions $\mathbf{T}$ are prescribed on $\partial\mathcal{B}_T$. Similarly, the boundary is partitioned into $\{\partial\mathcal{B}_\phi, \partial\mathcal{B}_M\}$ such that the infinitesimal rotation vector field $\boldsymbol{\phi}$ is prescribed on $\partial\mathcal{B}_\phi$ and the couple traction $\mathbf{M}$ is prescribed on $\partial\mathcal{B}_M$.

The kinematical vector fields $\mathbf{u}$ and $\boldsymbol{\phi}$ give rise to the micro-strain quantities ϵ_α, μ_α, and $\boldsymbol{\gamma}_\alpha$ corresponding to elongation, torsion and shear, respectively, of the doublet. Greek subscripts distinguish doublets and range from 1 to n where n is the number of doublets. Summation convention is not enforced with respect to greek subscripts. The work conjugate micro-stresses p_α, m_α and $\mathbf{t}_\alpha$, corresponding to elongation, torsion and shear, respectively, are assumed to be derivable from a stored strain energy function $W = W(\epsilon_\alpha, \mu_\alpha, \gamma_{\alpha i})$ to exclude the possibility of generating energy through a closed cycle of deformation. Unless otherwise noted, Latin subscripts will denote the component of the quantity expressed with respect to a Cartesian coordinate system. For example, $\gamma_{\alpha i}$ are the Cartesian components of $\boldsymbol{\gamma}_\alpha$ with respect to the orthonormal basis $\mathbf{e}_i$. The summation convention is enforced for repeated Latin subscripts where the range of the index is $\{1, 2, 3\}$ unless otherwise noted. For the most general linear elastic response, the micro-stresses are related to the micro-strains through the following linear constitutive relation:

$$p_\alpha = \sum_{\beta=1}^{n} (A_{\alpha\beta}\,\epsilon_\beta + B_{\alpha\beta}\,\mu_\beta + C_{\alpha\beta i}\,\gamma_{\beta i}) \tag{1}$$

$$m_\alpha = \sum_{\beta=1}^{n} (B_{\beta\alpha}\,\epsilon_\beta + E_{\alpha\beta}\,\mu_\beta + F_{\alpha\beta i}\,\gamma_{\beta i}) \tag{2}$$

$$t_{\alpha i} = \sum_{\beta=1}^{n} (C_{\beta\alpha i}\,\epsilon_\beta + F_{\beta\alpha i}\,\mu_\alpha + I_{\alpha\beta ij}\,\gamma_{\beta j}) \tag{3}$$

where $A_{\alpha\beta} = A_{\beta\alpha}$, $E_{\alpha\beta} = E_{\beta\alpha}$ and $I_{\alpha\beta ij} = I_{\beta\alpha ji}$. Further restrictions on the form of the constitutive relations are addressed by Mou and Ferrari (1995). For the material stability we take the strain energy function W be non-negative at all points for all compatible micro-strain fields. We shall return to this point below. Due to the resulting quadratic form, W achieves a minimum when $\epsilon_\alpha = 0$, $\mu_\alpha = 0$ and $\gamma_{\alpha i} = 0$; Without loss of generality we take this minimum to be zero. It follows that $W = 0$ if and only if $\epsilon_\alpha = \mu_\alpha = \gamma_{\alpha i} = 0$.

The internal energy W_{int} is given by (Granik and Ferrari, 1993)

$$2\,W_{\text{int}} = \sum_{\alpha=1}^{n} \int_{\mathcal{B}} (p_\alpha\,\epsilon_\alpha + m_\alpha\,\mu_\alpha + \mathbf{t}_\alpha \cdot \boldsymbol{\gamma}_\alpha)\,dV \tag{4}$$

and the external energy W_{ext} is given by

$$2\,W_{\text{ext}} = \int_{\mathcal{B}} \mathbf{b}\cdot\mathbf{u}\,dV + \int_{\partial\mathcal{B}} (\mathbf{T}\cdot\mathbf{u} + \mathbf{M}\cdot\boldsymbol{\phi})\,dS. \tag{5}$$

At equilibrium, $W_{\text{int}} = W_{\text{ext}}$.

Let $\{\mathbf{u}^1, \phi^1, \mu^1_\alpha, \gamma^1_\alpha, p^1_\alpha, m^1_\alpha, \mathbf{t}^1_\alpha\}$ and $\{\mathbf{u}^2, \phi^2, \mu^2_\alpha, \gamma^2_\alpha, p^2_\alpha, m^2_\alpha, \mathbf{t}^2_\alpha\}$ denote two sets of fields which satisfy the governing equations. Furthermore, let $(\tilde{\cdot})$ denote the difference in the quantity $(\cdot)$ between the two solutions. For example, $\tilde{\mathbf{u}} = \mathbf{u}^2 - \mathbf{u}^1$. It follows that

$$2\tilde{W}_{\text{ext}} = \int_B \tilde{\mathbf{b}} \cdot \tilde{\mathbf{u}}\, dV + \int_{\partial B} \left(\tilde{\mathbf{T}} \cdot \tilde{\mathbf{u}} + \tilde{\mathbf{M}} \cdot \tilde{\phi}\right) dS = 0. \tag{6}$$

Thus, at equilibrium $\tilde{W}_{\text{int}} = 0$ from which it follows that $\tilde{\epsilon}_\alpha = \tilde{\mu}_\alpha = \tilde{\gamma}_{\alpha i} = 0$ which then implies that the micro-strains and thus also the micro-stresses are the same for the two solutions, i.e., $\epsilon^1_\alpha = \epsilon^2_\alpha, \ldots, \gamma^1_{\alpha i} = \gamma^2_{\alpha i}$. It follows that the two sets of kinematical fields differ by at most a motion which is strain-free. That is, the micro-strains associated with the kinematical fields $\tilde{\mathbf{u}}$ and $\tilde{\phi}$ are zero. We call these micro-strain-free motions rigid body motions. The infinitesimal displacement $\mathbf{u}$ and rotation ϕ fields are thus unique to within at most a rigid body motion. This concludes the uniqueness proof.

It was mentioned above that we demand the strain energy function W to be greater than or equal to zero when evaluated at any point within the body for all admissible motions. This was to assure material stability. In other words, W is to be non-negative for all physically realizable occurences of it arguements, the micro-strain.. In the realm of linear elastic CM, the strain energy functional W^c is taken to be a function of the linearized CM strain measure ε^c_{ij}. To determine over what set W^c must be non-negative consider an $\varepsilon^\times_{ij} \in \Re^6$. We now ask if there exists a displacement field u^c_i which when evaluated and *some* point yields the strain measure $\varepsilon^\times_{ij}$? The answer is in the affirmative; Take the displacement field $u^c_i = \varepsilon^\times_{ij} x_j$ which yields a strain measure ε^c_{ij} at *all* points in the body. Thus, W^c must be non-negative for all $\varepsilon_{ij} \in \Re^6$. The quadratic form of W^c in linear elasticity allows one to stipulate that $W^c = 0$ if and only if $\varepsilon^c_{ij} = 0$. As a result, W^c is required to be positive definite. In DM $(\epsilon_\alpha, \mu_\alpha, \gamma_{\alpha i}) \in \Re^{8n}$ but it is not clear that it is necessary to demand that W be non-negative with respect to $\Re^{8n}$—though this would be sufficient.

3 Inversion Technique

In this subsection we develop a connection between solutions in DM and CM. This connection proves useful in that it allows, given a solution in one realm, the generation of a solution in the other realm. Below we present conditions which are sufficient to allow for this connection.

Consider two mathematical models of the same physical body, one linear elastic DM and the other linear elastic CM. For the DM model we consider the non-scale theory with no infinitesimal rotational kinematical vector field ϕ. We assume the material to be incapable of supporting micro-torsional and micro-shear stresses. The equilibrium equations for the DM model are

$$\sum_{\alpha=1}^{n} \tau^\circ_{\alpha i}\, \tau^\circ_{\alpha j}\, p_{\alpha,j} + b_i = 0 \qquad \text{in } B \tag{7}$$

where $p_{\alpha,j} := \partial p_\alpha / \partial x_j$ and $\tau^\circ_{\alpha i}$ is the direction cosine of the τ°_α doublet with the x_i-axis. The boundary conditions are

$$n_j \sum_{\alpha=1}^{n} \tau^\circ_{\alpha i}\, \tau^\circ_{\alpha j}\, p_\alpha = \bar{T}_i \qquad \text{on } \partial B_T \tag{8}$$

and

$$u^d_i = \bar{u}_i \qquad \text{on } \partial B_u \tag{9}$$

where ∂B_T and ∂B_u form a partition of the boundary ∂B and $\bar{T}_i$ and $\bar{u}_i$ denote prescribed quantities. The micro-constitutive relation is

$$p_\alpha = \sum_{\beta=1}^{n} A_{\alpha\beta}\, \epsilon_\beta \tag{10}$$

where $A_{\alpha\beta}$ is symmetric and positive definite. We shall often make use of a specific form of the constitutive relation (10), namely, taking $A_{\alpha\beta} = A_o\, \delta_{\alpha\beta}$ where A_o is a scalar and $\delta_{\alpha\beta}$ is the Kronecker delta, to yield

$$p_\alpha = A_o\, \epsilon_\alpha. \tag{11}$$

This special constitutive relation is termed "non-polar." Compatibility is given by the identity

$$\varepsilon^d_{11,22} - 2\,\varepsilon^d_{12,12} + \varepsilon^d_{22,11} = 0. \tag{12}$$

where $\varepsilon^d_{ij} = (u^d_{i,j} + u^d_{j,i})/2$.

The equilibrium equations for the CM model are the familiar equations

$$\sigma^c_{ij,j} + b_i = 0 \qquad \text{in } \mathcal{B}. \tag{13}$$

The boundary conditions are

$$\sigma^c_{ij}\, n_j = \bar{T}_i. \qquad \text{on } \partial\mathcal{B}_T. \tag{14}$$

and

$$u^c_i = \bar{u}_i \qquad \text{on } \partial\mathcal{B}_u. \tag{15}$$

The constitutive relation is

$$\sigma^c_{ij} = C_{ijkl}\,\varepsilon^c_{kl}, \tag{16}$$

where $C_{ijkl} = C_{klij} = C_{jikl}$ and $\mathbf{C}$ is positive definite.

The transition from micro- to macro-stresses is given by the relation

$$\sigma^d_{ij} = \sum_{\alpha=1}^{n} \tau^\circ_{\alpha i}\,\tau^\circ_{\alpha j}\,p_\alpha. \tag{17}$$

As noted by Granik and Ferrari (1993) it is possible to define equivalent macro-stresses and -strains in terms of the micro-stresses and -strains, respectively. The micro- to macro-strain relation is given by

$$\epsilon_\alpha = \tau^\circ_{\alpha i}\,\tau^\circ_{\alpha j}\,u^d_{i,j}. \tag{18}$$

The micro to macro stress relation (17) permits a convenient representation of the DM governing equations. Substituting eqn (17) into the DM equilibrium eqn (7) yields

$$\sigma^d_{ij,j} + b_i = 0 \qquad \text{in } \mathcal{B} \tag{19}$$

provided that the doublet directions do not vary spatially. Substituting eqn (17) into the traction boundary condition (8) yields the expression

$$\sigma^d_{ij}\, n_j = \bar{T}_i. \qquad \text{on } \partial\mathcal{B}_T. \tag{20}$$

The form equivalence between eqns (19) and (13) and between eqns (20) and (14) gives rise to the following result. If σ^c_{ij} is an admissible stress field then any set of micro-stresses p_α which yield $\sigma^d_{ij} \equiv \sigma^c_{ij}$ satisfy both the micro-stress equilibrium and the micro-stress traction boundary condition. Conversely, if $\{p_\alpha\}$ is an admissible set of micro-stresses then $\sigma^c_{ij} = \sigma^d_{ij}$ is an admissible CM stress field.

Similarly, if ε^c_{ij} is an admissible strain field then micro-strains obtained from eqn (18) using $\varepsilon^d_{ij} = \varepsilon^c_{ij}$ is an admissible micro-strain field. Conversely, if ϵ_α is an admissible micro-strain field consistent with the equivalent macroscopic strain field ε^d_{ij} then $\varepsilon^c_{ij} = \varepsilon^d_{ij}$ is an admissible CM strain field.

We now introduce some matrix notation which will allow for convenient representation of some of the above relations and later manipulations. Let $\hat{\epsilon} := \{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}^T$ be the column vector of axial micro-strains and let $\hat{p} := \{p_1, p_2, \ldots, p_n\}^T$ be the column vector of axial micro-stresses. The micro-constitutive relation (10) may thus be expressed as $\hat{\mathbf{p}} = \mathbf{A}\,\hat{\epsilon}$.

We now restrict attention to planar problems. The developments below, however, are easily extended to three dimensions. In this context, let $\hat{\sigma} := \{\sigma_{11}, \sigma_{22}, \sigma_{12}\}^T$ be the column vector of in-plane stresses. Likewise, let $\hat{\varepsilon} := \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12} + \varepsilon_{21}\}^T$. The constitutive relation (16) takes the form $\hat{\sigma} = \hat{\mathbf{C}}\,\hat{\varepsilon}$ where $\hat{\mathbf{C}}$ is the appropriate matrix representation of $\mathbf{C}$ for the type of planar problem under consideration, i.e., plane strain or plane stress.

The micro-macro relation (17) may be expressed as

$$\hat{\sigma}^d = \mathbf{M}\,\hat{\mathbf{p}}, \tag{21}$$

which implicitly defines the $3 \times n$ matrix $\mathbf{M}$. Similarly, the micro-macro strain relation (18) can be expressed as

$$\hat{\epsilon} = \mathbf{M}^T \hat{\epsilon}^d. \tag{22}$$

From the developments given above it follows directly that

$$\hat{\mathbf{p}} = \mathbf{M}^{-1} \hat{\sigma}^c \tag{23}$$

is an admissible micro-stress field provided that σ^c_{ij} is admissible and $\mathbf{M}$ is invertible. The matrix $\mathbf{M}$ is invertible if $n = 3$ and none of the three doublets are collinear. For $n = 3$, $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} (\tau^\circ_{11})^2 & (\tau^\circ_{21})^2 & (\tau^\circ_{31})^2 \\ (\tau^\circ_{21})^2 & (\tau^\circ_{22})^2 & (\tau^\circ_{32})^2 \\ \tau^\circ_{11}\,\tau^\circ_{12} & \tau^\circ_{21}\,\tau^\circ_{22} & \tau^\circ_{31}\,\tau^\circ_{32} \end{bmatrix}. \tag{24}$$

Furthermore,

$$\hat{\epsilon} = \mathbf{M}^T \hat{\epsilon}^c \tag{25}$$

is an admissible micro-strain field provided that ϵ^c_{ij} is admissible. Thus if ϵ^c_{ij} and σ^c_{ij} are the equilibrium solution fields to the CM problem then eqns (23) and (25) yield admissible micro-stress and micro-strain fields, respectively. In order for these micro-strain and micro-stress fields to be the solution to the DM problem they must be related through the constitutive relation (10) or equivalently,

$$\mathbf{M}\,\mathbf{A}\,\mathbf{M}^{\mathbf{T}} = \hat{\mathbf{C}}. \tag{26}$$

The proof follows by substituting (23) and (25) into $\hat{\mathbf{p}} = \mathbf{A}\,\hat{\epsilon}$ which yields

$$\hat{\sigma}^c = \mathbf{M}\,\mathbf{A}\,\mathbf{M}^{\mathbf{T}}\,\hat{\epsilon}^c \tag{27}$$

which is true if $\mathbf{M}\,\mathbf{A}\,\mathbf{M}^T = \hat{\mathbf{C}}$. If the micro-constitutive relation is non-polar then it can be proven that $\mathbf{M}\,\mathbf{A}\,\mathbf{M}^T = A_o \mathbf{M}\,\mathbf{M}^T$ is isotropic for all values of θ if and only if $\gamma = \pi/3$.

In summary, if we desire a DM solution to a planar problem with three doublets and $\gamma = \pi/3$ then the solution can be calculated directly from the associated CM problem with an isotropic material. Again, we emphasize that this method of inversion of the macro-stresses has been presented only for the planar case but it is applicable to 3-D problems as well. It should be noted, however, that there is no arrangement of 6 doublets in 3 dimensions that with a non-polar micro-constitutive relation yields $\mathbf{MAM}^T$ to be isotropic.

3.1 *Homogeneous Deformations*

In this section we present some homogeneous deformations of discrete materials. We begin by considering $n = 3$ with $\gamma = \pi/3$. Consider a plate of DM material modeled with three in-plane doublets with a structural angle of $\gamma = \pi/3$ (cf. Figure 1a). We now subject the material square to uniaxial tension (cf. Figure 1b) and pure shear (cf. Figure 1c). Using the method of macro-stress inversion the micro-stresses can be computed for arbitrary angles of θ—the inclination of the τ°_3 doublet with the x_1-axis. The results are presented as follows. In Figure 2 the micro-stresses are presented for uniaxial tension as a function of the angle θ. Note that compressive micro-stresses are achieved in distinction to the macroscopic principal stresses which are everywhere non-compressive. In Figure 3 the micro-stresses for the pure shear loading case is presented. Note that the micro-stresses exceed the magnitude of the applied shear stress.

We now consider the effect of $\gamma \neq \pi/3$ for arbitrary θ. To simplify the presentation of results we will consider the energy stored in a plate of material subjected to shear. The stored energy is indicative of the magnitude of the stresses in the set of doublets. It is observed that γ has a significant effect on the micro-stress field. In general, the energy grows unbounded as $\gamma \to 0$ and as $\gamma \to \pi/2$ indicating that at least one of the micro-stresses grows without bound. When $\gamma = \pi/3$ the internal stored energy is the same for all angles θ thus all curves illustrated in Figure 4 pass through the point (1/3, 4/3). For each value of θ there exists an angle γ which minimizes the internal stored energy. For a given value of θ the value of γ which minimizes the stored energy varies with the state of applied tractions.

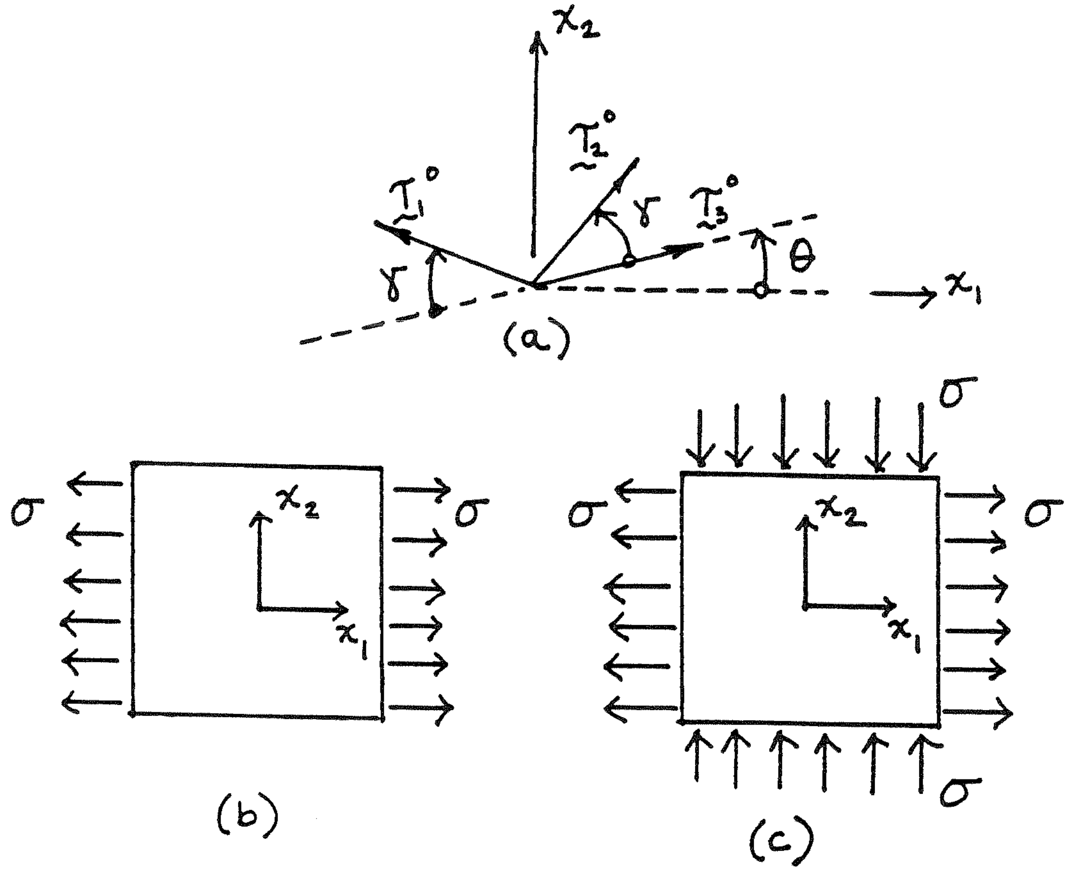

Figure 1: Doublet geometry for homogeneous deformations

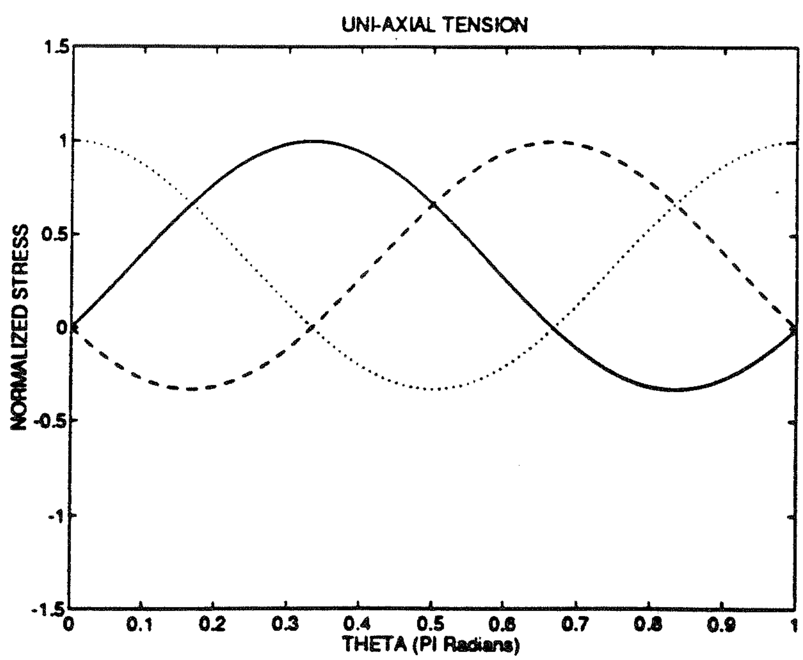

Figure 2: Normalized micro-stress of a plate under uni-axial tension: solid-line is p_1/σ; dashed-line is p_2/σ; dotted-line is p_3/σ.

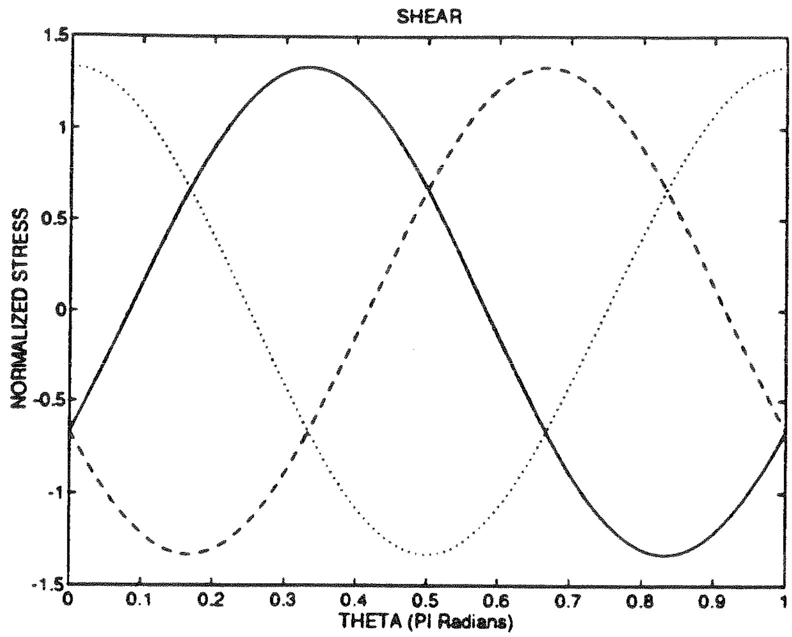

Figure 3: Normalized micro-stress of a plate under shear: solid-line is p_1/σ; dashed-line is p_2/σ; dotted-line is p_3/σ.

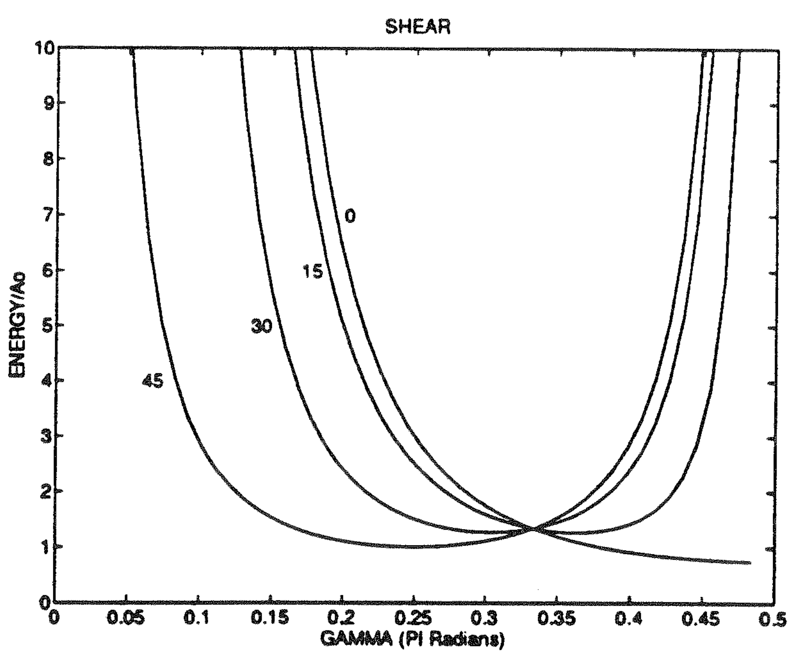

Figure 4: Stored energy in a plate under shear for different values of θ (degrees).

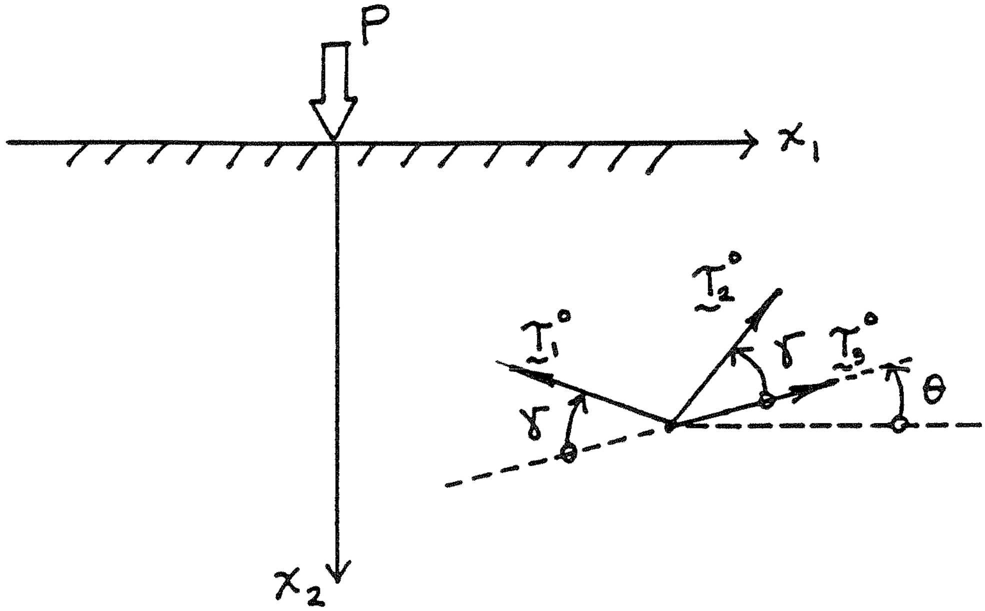

Figure 5: Doublet geometry for Flamant's problem

3.2 *Flamant Problem*

Consider the classical problem of Flamant: A penetrating point force P acting normal to the straight boundary of a semi-infinite plate of isotropic material. The classical CM solution is characterized by a stress field whose principal stresses are everywhere non-positive. In this section we consider the DM solution to Flamant's problem with three doublets with a structural angle of $\gamma = \pi/3$ and for arbitrary θ. This problem has been treated previously by Granik and Ferrari (1993) for the case $\theta = 0$. Their solution contains some quantitative inconsistencies which are corrected here.

Since the DM domain consists of three doublets with $\gamma = \pi/3$ we can use the classical result of Flamant to obtain the DM solution using the method detailed in section 3. Flamant's solution reads

$$\sigma_{11}^c = -\frac{2P}{\pi}\frac{x^2 y}{(x^2+y^2)^2} \tag{28}$$

$$\sigma_{12}^c = \sigma_{21}^c = -\frac{2P}{\pi}\frac{x y^2}{(x^2+y^2)^2} \tag{29}$$

$$\sigma_{22}^c = -\frac{2P}{\pi}\frac{y^3}{(x^2+y^2)^2} \tag{30}$$

and

$$u_1^c = \frac{P}{2\pi\mu}\left[\frac{\mu}{\lambda'+\mu}\arctan\left(\frac{y}{x}\right) - \frac{\pi}{2} + \frac{x\,y}{x^2+y^2}\right] \tag{31}$$

$$u_2^c = -\frac{P}{2\pi\mu}\left[\frac{\lambda'+2\mu}{2(\lambda'+\mu)}\log(x^2+y^2) + \frac{x^2}{x^2+y^2}\right]. \tag{32}$$

where $\lambda' := 2\lambda\mu/(\lambda+2\mu)$ and λ and μ are the Lamé constants of the isotropic continuum.

The micro-stresses for $\theta = 0$ are evaluated to be

$$p_1 = -\frac{4P}{3\pi}\frac{y^2(\sqrt{3}\,x+y)}{(x^2+y^2)^2} \tag{33}$$

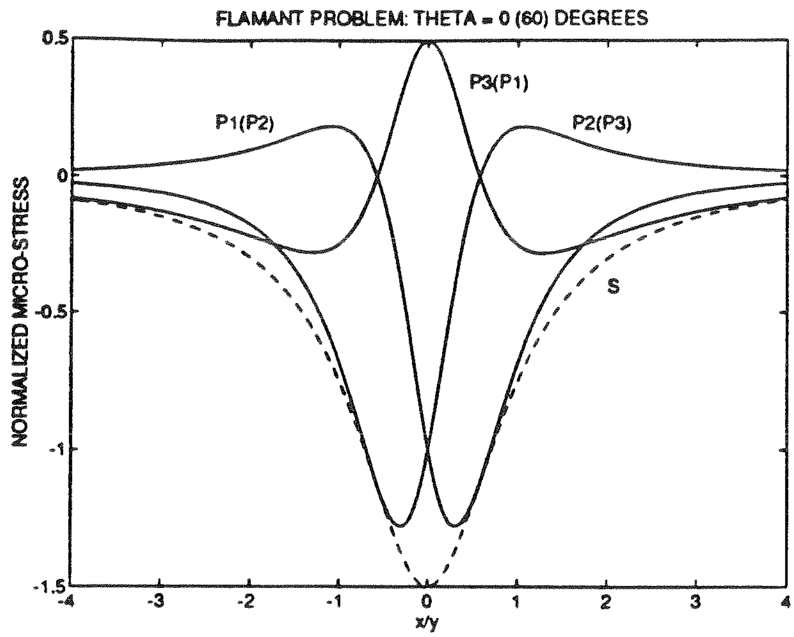

Figure 6: Normalized micro- and macro-stresses for the Flamant problem with $\theta = 0$ ($\theta = \pi/3$).

$$p_2 = \frac{4P}{3\pi}\frac{y^2(\sqrt{3}\,x - y)}{(x^2+y^2)^2} \tag{34}$$

$$p_3 = -\frac{2P}{3\pi}\frac{y(3x^2 - y^2)}{(x^2+y^2)^2}. \tag{35}$$

The classical solution due to Flamant is characterized by a stress field for which the principal stresses are nowhere tensile for a penetrating applied load. Unlike the Flamant stresses, the micro-stresses are found to be tensile within particular regions of the domain. For instance, the p_3 micro-stress associated with the τ_3 doublet is tensile within the sector defined by $y > \sqrt{3}\,|x|$. The p_1 and p_2 micro-stresses are also tensile within specific regions. To graphically illustrate these characteristics it is convenient to normalize eqns (33)–(35). Let $\bar{x} := x/y$. It follows that eqns (33)–(35) can be recast in the following form:

$$P1 := \frac{3\pi y}{4P}p_1 = -\frac{\sqrt{3}\,\bar{x}+1}{(1+\bar{x}^2)^2} \tag{36}$$

$$P2 := \frac{3\pi y}{4P}p_2 = \frac{\sqrt{3}\,\bar{x}-1}{(1+\bar{x}^2)^2} \tag{37}$$

$$P3 := \frac{3\pi y}{4P}p_3 = \frac{1-3\bar{x}^2}{2(1+\bar{x}^2)^2}P3. \tag{38}$$

In addition, the only non-zero principal macro-stress σ^p can be expressed as

$$S := \frac{3\pi y}{4P}\sigma^p = -\frac{3}{2(1+\bar{x}^2)}. \tag{39}$$

The normalized quantities $P1$, $P2$, $P3$ and S are presented in Figure 6 for $\theta = 0$.

We now provide the in-plane displacement field. Using the non-polar constitutive relation it may be verified that the components of the in-plane displacement field are given by

$$u_1^d = \frac{2P}{3\pi A_o}\left[\arctan\left(\frac{y}{x}\right) - \frac{\pi}{2} + \frac{2\,x\,y}{x^2+y^2}\right] \tag{40}$$

$$u_2^d = -\frac{2P}{3\pi A_o}\left[\frac{3}{2}\log(x^2+y^2) + \frac{2\,x^2}{x^2+y^2}\right]. \tag{41}$$

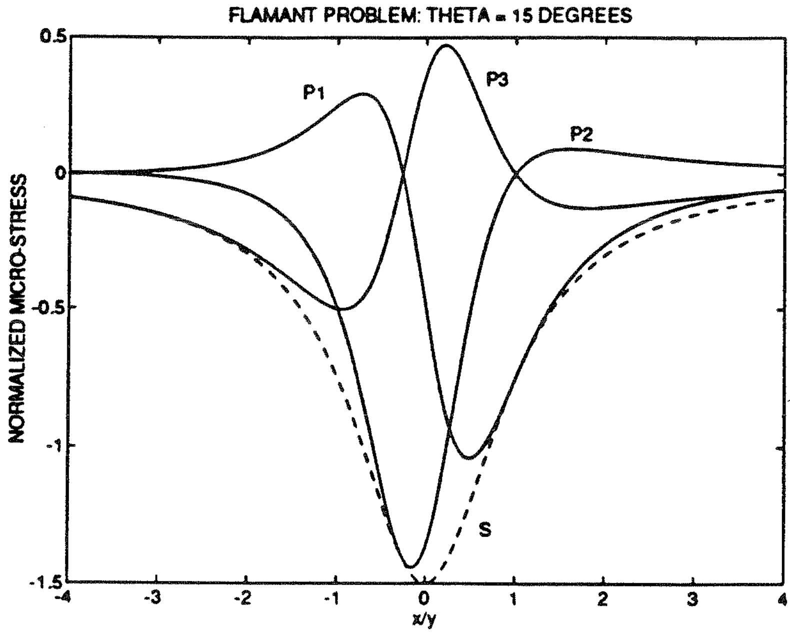

Figure 7: Normalized micro- and macro-stresses for the Flamant problem with $\theta = 15$ degrees.

In eqn (41), log denotes the natural logarithm. We note that the displacement field does not tend to zero at infinite distances from the point of application of the load as is the case in the Flamant solution (see e.g., Love (1944, p.211)). In general, the displacement field given by eqns (40) and (41) is not the same displacement field as the Flamant solution. This observation is intuitive since the material domain of Flamant is characterized by two constitutive parameters while the microstructured material considered herein is characterized by only one constitutive parameter, namely A_o, while the stress field is independent of the constitutive relation. Let (λ, μ) denote the Lamé constants of the elastic continuum utilized be Flamant. When $(\lambda, \mu) \mapsto (3A_o/4, 3A_o/8)$ (i.e., Poisson's ratio is 1/3) it may be shown that the two displacement fields are equivalent.

To illustrate the effect of θ on the micro-stresses, plots similar to that in Figure 6 are given for $\theta = 15, 30, 45,$ degrees, in Figures 7, 8 and 9, respectively. Closed form expressions for the micro-stresses in terms of θ are very lengthy but for special values of θ simple expressions do exist. We have seen this above for $\theta = 0$. When $\theta = \pi/6$ similar expressions exist. The normalized micro-stresses when $\theta = \pi/6$ are given by

$$P1 := \frac{3\pi y}{4P} p_1 = -\frac{\bar{x}(\sqrt{3} + \bar{x})}{(1 + \bar{x}^2)^2} \tag{42}$$

$$P2 := \frac{3\pi y}{4P} p_2 = \frac{\bar{x}^2 - 3}{2(1 + \bar{x}^2)^2} \tag{43}$$

$$P3 := \frac{3\pi y}{4P} p_3 = \frac{\bar{x}(\sqrt{3} - \bar{x})}{(1 + \bar{x}^2)^2}. \tag{44}$$

These are the functions plotted in Figure 8.

We note that the components of the displacement field for the DM solutions for all values of θ are given by eqns (40) and (41).

3.3 *Kelvin's Problem*

The problem of Flamant treated in section 3.2 is a Green's function for elasticity problems with normal surface tractions. We now present another Green's function: a point force acting in the plane of an infinite

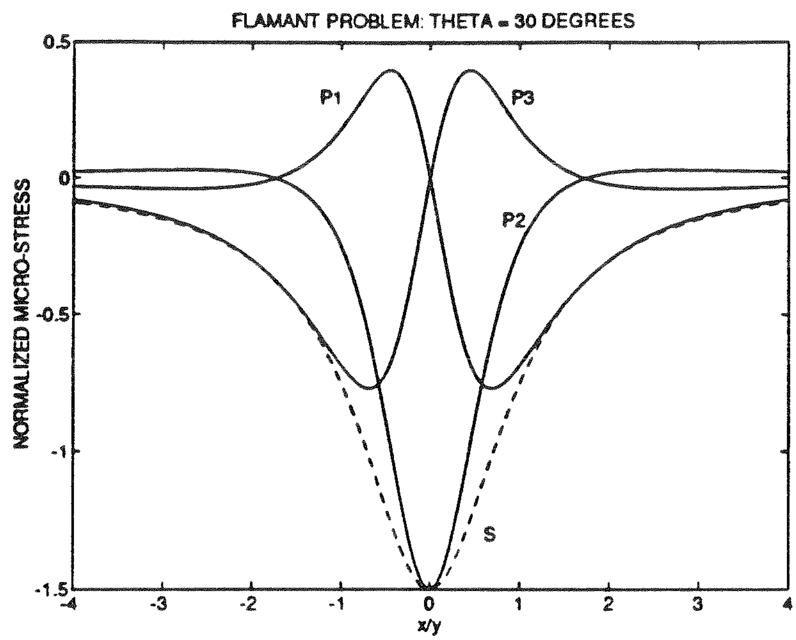

Figure 8: Normalized micro- and macro-stresses for the Flamant problem with $\theta = 30$ degrees.

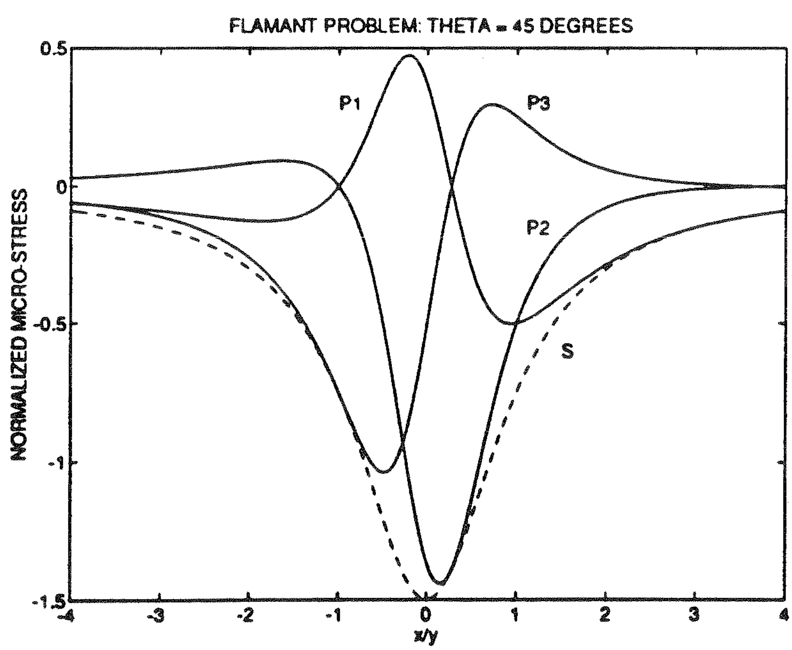

Figure 9: Normalized micro- and macro-stresses for the Flamant problem with $\theta = 45$ degrees.

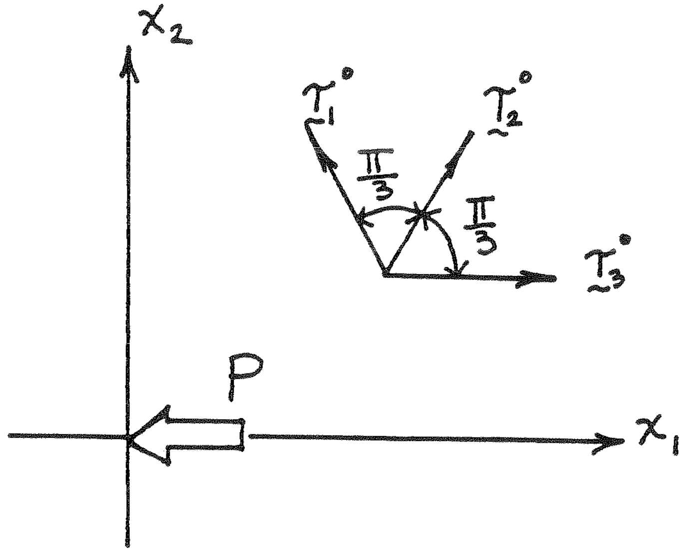

Figure 10: Doublet geometry for Kelvin's problem

plate, originally solved by Lord Kelvin. This solution enables one to solve problems with arbitrary body force distributions by integration. In particular, let us apply a point force of magnitude P at the origin of the plate in the negative x_1 direction, with the x_1- and x_2-axes lying in the plane of the plate (cf. Figure 10). From classical CM mechanics, the macrostress state is given by (Love, 1944)

$$\sigma_{11}^c = \frac{P}{8\pi}\frac{x_1}{r^2}\left[7 - 6\left(\frac{x_2}{r}\right)^2\right] \tag{45}$$

$$\sigma_{22}^c = \frac{P}{8\pi}\frac{x_1}{r^2}\left[6\left(\frac{x_2}{r}\right)^2 - 1\right] \tag{46}$$

$$\sigma_{12}^c = \frac{P}{8\pi}\frac{x_2}{r^2}\left[6\left(\frac{x_1}{r}\right)^2 + 1\right] \tag{47}$$

where $r^2 := x_1^2 + x_2^2$.

The micro-stresses are obtained by substituting the CM macro-stresses (45)–(47) into equ (23) yielding

$$p_1 = -\frac{P}{12\pi r^4}\left[x_1^3 - 5\,x_1\,x_2^2 + 7\sqrt{3}\,x_1^2\,x_2 + \sqrt{3}\,x_2^3\right] \tag{48}$$

$$p_2 = -\frac{P}{12\pi r^4}\left[x_1^3 - 5\,x_1\,x_2^2 - 7\sqrt{3}\,x_1^2\,x_2 - \sqrt{3}\,x_2^3\right] \tag{49}$$

$$p_3 = \frac{P\,x_1}{12\pi\,r^4}\left[11\,x_1^2 - x_2^2\right] \tag{50}$$

where we have assumed a three doublet DM lattice with $\gamma = \pi/3$ and $\theta = 0$. Expectedly, equs (48)–(50) satisfy the equilibrium equs (7), and the microstrains derived via (11) satisfy the equation of compatibility (12).

3.4 *Stress Concentration*

In addition to obtaining DM Green's functions, the Inversion technique can be used to find microstress concentration factors. As an example, we consider a circular void within the infinite plate subjected to

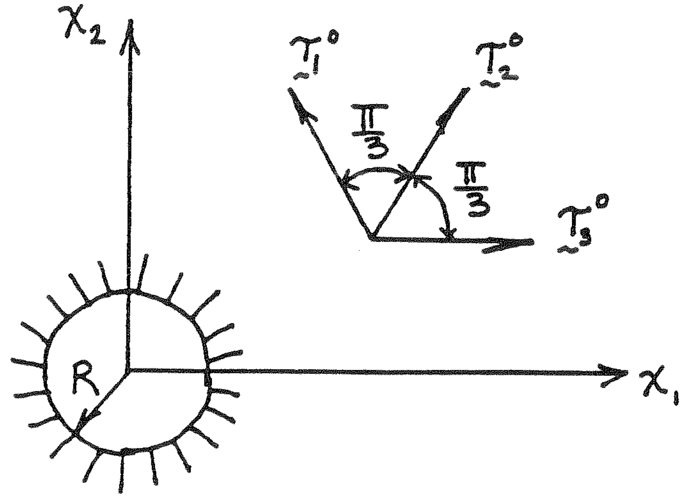

Figure 11: Circular hole in an infinite plate.

hydrostatic tractions, $\mathbf{T} = \sigma\, \mathbf{n}$, at infinity, as illustrated in Figure 11. The CM solution in polar coordinates is given by

$$\sigma^c_{rr} = \sigma\,[1 - (R/r)^2] \quad (51)$$
$$\sigma^c_{\theta\theta} = \sigma\,[1 + (R/r)^2] \quad (52)$$
$$\sigma^c_{r\theta} = 0 \quad (53)$$

where r and θ are polar coordinates and the circular void is centered at $r = 0$ with radius R.

To obtain the microstresses from eqns (51)–(53) we must first rewrite the macrostresses in terms of Cartesian coordinates, since the DM lattice is defined with respect to this latter coordinate system. Equation (23) can then be utilized to give the microstresses around the void

$$p_1 = \frac{2\,\sigma}{3}\left[1 - \frac{R^2\,(x_2^2 - 2\sqrt{3}\,x_1\,x_2 - x_1^2)}{r^4}\right] \quad (54)$$
$$p_2 = \frac{2\,\sigma}{3}\left[1 - \frac{R^2\,(x_2^2 + 2\sqrt{3}\,x_1\,x_2 - x_1^2)}{r^4}\right] \quad (55)$$
$$p_3 = \frac{2\,\sigma}{3}\left[1 - \frac{2\,R^2\,(x_1^2 - x_2^2)}{r^4}\right]. \quad (56)$$

Comparison of the macro- and microstresses shows that while CM predicts solely compressive principal stresses arising from applied compressive tractions, the microstresses are tensile in certain regions adjacent to the void. It should thus not be surprising if a granular body under hydrostatic pressure develops tensile openings, analogous to the paradox associated with Flamant's Problem. We find that the microstresses in the vicinity of the void vary from -1 to 3 times the far-field microstresses while the macrostresses vary between 0 and 2 times the far-field equivalent macro-stress.

4 Micro-stress Function

Consider a non-polar medium with no body forces and with lattice geometry as shown in, Figure 1a. The equilibrium eqns (7) take the form

$$(p_1 + p_2 + \csc^2\gamma\, p_3)_{,1} + \tan\gamma\,(p_2 - p_1)_{,2} = 0 \tag{57}$$

$$(p_2 - p_1)_{,1} + \tan\gamma\,(p_1 + p_2)_{,2} = 0. \tag{58}$$

The Integrability theorem applied to eqn (57) implies the existence of a function $\Psi = \Psi(x_1, x_2)$ such that

$$\Psi_{,1} = \tan\gamma\,(p_2 - p_1) \tag{59}$$

$$\Psi_{,2} = -(p_1 + p_2 + \csc^2\gamma\, p_3)_{,1} + \tan\gamma\,(p_2 - p_1). \tag{60}$$

Similarly, the Integrability theorem applied to eqn (58) implies the existence of a function $\Theta = \Theta(x_1, x_2)$ such that

$$\Theta_{,1} = -\tan^2\gamma\,(p_1 + p_2) \tag{61}$$

$$\Theta_{,2} = \tan\gamma\,(p_2 - p_1). \tag{62}$$

From eqns (59) and (62) follows the relation $\Psi_{,1} = \Theta_{,2}$ which itself implies the existence of a function $\chi = \chi(x_1, x_2)$ such that $\chi_{,1} = \Theta$ and $\chi_{,2} = \Psi$. Solving for the micro-stresses in terms of the second partial derivatives of χ yields

$$p_1 = -\frac{1}{2}\cot^2\gamma\;(\tan\gamma\,\chi_{,12} + \chi_{,11}) \tag{63}$$

$$p_2 = -\frac{1}{2}\cot^2\gamma\;(\tan\gamma\,\chi_{,12} - \chi_{,11}) \tag{64}$$

$$p_3 = \cot^2\gamma\;(\cos^2\gamma\,\chi_{,11} - \sin^2\gamma\,\chi_{,22}) \tag{65}$$

The micro-stresses (63)–(65) thus satisfy equilibrium for a sufficiently differentiable function χ. Compatibility is now addressed. Compatibility in terms of the micro-stresses is obtained by substituting the constitutive relation $p_\alpha = A_\alpha \epsilon_\alpha$ into eqn (12):

$$0 = \left[\sec^2\gamma\,(p_1 + p_2) - 2\cot^2\gamma\, p_3\right]_{,11} + 2\,p_{3,22} + \csc\gamma\,\sec\gamma\,(p_1 - p_2)_{,12}. \tag{66}$$

Substitution of the micro-stresses (63)–(65) into the compatibility relation (66) yields

$$0 = \cot^2\gamma\,\csc^2\gamma\,(1 + \cos^4\gamma)\chi_{,1111} + \csc^2\gamma(1 - 4\cos^4\gamma)\chi_{,1122} + 2\cos^2\gamma\,\chi_{,2222} \tag{67}$$

Any function χ which satisfies eqn (67) thus yields a micro-stress field which satisfies equilibrium and compatibility. When $\gamma = \pi/3$, eqn (67) simplifies to

$$0 = \chi_{,1111} + 2\,\chi_{,1122} + \chi_{,2222} =: \nabla^2 \cdot \nabla^2\chi =: \nabla^4\chi \tag{68}$$

which is the bi-harmonic equation.

Substituting the micro-stresses (63)–(65) into the micro- to macro-stress relation (17) yields

$$\sigma^d_{11} = \bar{\chi}_{,22} \tag{69}$$

$$\sigma^d_{22} = \bar{\chi}_{,11} \tag{70}$$

$$\sigma^d_{12} = -\bar{\chi}_{,12} \tag{71}$$

where $\bar{\chi} := -\cos^2\gamma\,\chi$. Equations (69)–(71) are form equivalent to the equations for the classical continuum stresses in terms of the Airy stress function. When $\gamma = \pi/3$ the connection between DM and CM is further elucidated, since χ is governed by the bi-harmonic equation, as the stresses are obtained in CM from equations which are form equivalent to eqns (69)–(71).

In sections 4.1 through 4.4 below, we investigate the solutions generated by the family of third-order polynomial $\chi(x_1, x_2)$ functions, and we present the MSF for the problems solved above via the Inversion technique of section 3/

4.1 *Third-order Polynomial χ Functions*

Considering the combined equilibrium/compatibility relation (67) shows that any third-order polynomial $\chi(x_1, x_2)$ represents an admissible DM solution. Let us consider the DM lattice with $\gamma = \pi/3$ and postulate a MSF of the form:

$$\chi(x_1, x_2) = A_1\, x_1 + A_2\, x_1^2 + A_3\, x_1^3 + B_1\, x_2 + B_2\, x_2^2 + B_3\, x_2^3 + C_1\, x_1\, x_2 + C_2\, x_1^2\, x_2 + C_3\, x_1\, x_2^2 + D. \quad (72)$$

We substitute eqn (72) into eqns (63)–(65) to obtain the resulting microstresses:

$$p_1 = -\frac{1}{3}\left[\left(\sqrt{3}\,C_2 + 3\,A_3\right) x_1 + \left(\sqrt{3}\,C_3 + C_2\right) x_2 + \frac{\sqrt{3}}{2}\,C_1 + A_2\right] \quad (73)$$

$$p_2 = \frac{1}{3}\left[\left(\sqrt{3}\,C_2 - 3\,A_3\right) x_1 + \left(\sqrt{3}\,C_3 - C_2\right) x_2 + \frac{\sqrt{3}}{2}\,C_1 - A_2\right] \quad (74)$$

$$p_3 = \frac{1}{6}\left[3\,(A_3 - C_3)\, x_1 + (C_2 - 9\,B_3)\, x_2 - 3\,B_2 + A_2\right]. \quad (75)$$

Substituting eqns (73)–(75) into the relation (17) yields the equivalent macrostresses:

$$\sigma_{11}^d = -\frac{1}{2}\left[C_3\, x_1 + 3\,B_3\, x_2 + B_2\right] \quad (76)$$

$$\sigma_{22}^d = -\frac{1}{2}\left[3\,A_3\, x_1 + C_2\, x_2 + A_2\right] \quad (77)$$

$$\sigma_{12}^d = \frac{1}{4}\left[2\,C_2\, x_1 + 2\,C_3\, x_2 + C_1\right]. \quad (78)$$

Analysing either the micro- or macrostresses, we see that χ given by eqn (72) will provide solutions to problems with homogeneous or linearly varying stress states. Examples include uniaxial or biaxial tension and compression, pure shear and beams under pure bending. Note that shear stresses arise only from the seventh through the ninth terms of (72). Hence, if the three C_i coefficients are zero, the x_1- and x_2-axes are the pincipal axes; and conversely, states of shear are given by only the non-zero C_i's. The coefficients A_1, B_1 and D have no effect on the stress state and thus represent superfluous information.

4.2 *Flamant's Problem*

The family of third-order polynomials given by (72) represents only a subset of admissible χ functions. We now consider a more involved MSF containing a trigonometric term:

$$\chi(x_1, x_2) = \frac{4\,P\,x_1}{\pi}\left[1 - \arctan(x_2/x_1)\right] \quad (79)$$

which represents an admissible DM solution when $\gamma = \pi/3$. The microstresses derived from (79) are identical with (33)–(35), indicating that we have found the MSF for Flamant's Problem. Note, however, that for this particular application of the MSF a slightly varied form of (63)–(65) is necessary, since the coordinate system has been changed to that shown in Figure 5.

4.3 *Kelvin's Problem*

The point force in the infinite plane is a similar problem in some respects to Flamant's Problem. Hence, we start our search for the appropriate MSF with modifications to the family of arc tangent functions. The final result is

$$\chi(x_1, x_2) = \frac{P}{4\,\pi}\left[x_1\,\log(x_1^2 + x_2^2) + 8\,x_2\arctan(x_1/x_2) - 8\,x_1\right]. \quad (80)$$

Substituting eqn (80) in eqns (63)–(65) yields microstresses identical to eqns (48)–(50).

4.4 *Stress Concentration*

The problem of a circular void within the hydrostatically stressed plate is the superposition of a homogeneous stress state with the concentration stresses arising around the hole. Thus, the MSF should be a sum of a second-order polynomial and a term or terms that account for the effects of the void. The result we find is

$$\chi(x_1, x_2) = -2\,\sigma\left[x_1^2 + x_2^2 - R^2 \log\left(x_1^2 + x_2^2\right)\right] \tag{81}$$

which exactly recovers eqns (54)–(56). Expectedly, the logarithmic term acts to satisfy the zero-traction boundary condition at surface of the void.

5 Conclusion

We initiated our study by presenting a uniqueness theorem in linear elastic DM. We then developed two methods for obtaining solutions in plane elastostatics.

In the first, we noted a correspondence between DM and CM for specific DM lattice geometries, namely, a three doublet arrangement with $\gamma = \pi/3$. The result of this connection between DM and CM is that given a solution in either one of the two regimes, one can generate an equivalent solution in the other. We demonstrated the utility of this technique by obtaining DM solutions to homogeneous deformation problems, the classic problems of Flamant and Kelvin, and stress concentration around a hole. In the first two applications, we extended the study to a lattice rotated by an angle θ with respect to the original coordinate system. In the case of homogeneous deformations, we also analyzed the general three double lattice where $\gamma \neq \pi/3$.

The second technique which was developed arises when the micro-stress equilibrium and micro-strain compatibility requirements are manipulated to yield the microstress function (MSF). In illustrating the use of this second technique, we studied the solutions generated by the family of third order polynomial MSF's, and we derived the MSF's corresponding to the three problems considered via the Inversion technique.

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