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Topics in Nonsupersymmetric Scattering Amplitudes in Gauge and Gravity Theories

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Topics in Nonsupersymmetric Scattering Amplitudes in Gauge and Gravity Theories

A dissertation submitted in partial satisfaction
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by

Joshua David Nohle

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Abstract of the Dissertation

Topics in Nonsupersymmetric Scattering Amplitudes in Gauge and Gravity Theories

by

Joshua David Nohle

Doctor of Philosophy in Physics

University of California, Los Angeles, 2015

Professor Zvi Bern, Chair

In Chapters 1 and 2, we introduce and review the duality between color and kinematics in Yang-Mills theory uncovered by Bern, Carrasco and Johansson (BCJ). In addition to revealing interesting structures in Yang-Mills theory, this conjectured duality immensely simplifies the computation of scattering amplitudes in theories of gravity.

In Chapter 3, we provide evidence in favor of the conjectured duality between color and kinematics for the case of nonsupersymmetric pure Yang-Mills amplitudes by constructing a form of the one-loop four-point amplitude of this theory that makes the duality manifest. Our construction is valid in any dimension. We also describe a duality-satisfying representation for the two-loop four-point amplitude with identical four-dimensional external helicities. We use these results to obtain corresponding gravity integrands for a theory containing a graviton, dilaton, and antisymmetric tensor, simply by replacing color factors with specified diagram numerators. Using this, we give explicit forms of ultraviolet divergences at one loop in four, six, and eight dimensions, and at two loops in four dimensions.

In Chapter 4, we extend the four-point one-loop nonsupersymmetric pure Yang-Mills discussion of Chapter 3 to include fermions and scalars circulating in the loop with all external gluons. This gives another nontrivial loop-level example showing that the duality between color and kinematics holds in nonsupersymmetric gauge theory. The construction is valid in any spacetime dimension and written in terms of formal polarization vectors. We

also convert these expressions into a four-dimensional form with explicit external helicity states. Using this, we compare our results to one-loop duality-satisfying amplitudes that are already present in literature.

In Chapter 5, we switch from the topic of color-kinematics duality to discuss the recently renewed interest in the soft behavior of gravitons and gluons. Specifically, we discuss the subleading low-energy behavior.

Cachazo and Strominger recently proposed an extension of the soft-graviton theorem found by Weinberg. In addition, they proved the validity of their extension at tree level. This was motivated by a Virasoro symmetry of the gravity S-matrix related to BMS symmetry. As shown long ago by Weinberg, the leading soft behavior is not corrected by loops. In contrast, we show in Chapter 6 that with the standard definition of soft limits in dimensional regularization, the subleading behavior is anomalous and modified by loop effects. We argue that there are no new types of corrections to the first subleading behavior beyond one loop and to the second subleading behavior beyond two loops. To facilitate our investigation, we introduce a new momentum-conservation prescription for defining the subleading terms of the soft limit. We discuss the loop-level subleading soft behavior of gauge-theory amplitudes before turning to gravity amplitudes.

In Chapter 7, we show that at tree level, on-shell gauge invariance can be used to fully determine the first subleading soft-gluon behavior and the first two subleading soft-graviton behaviors. Our proofs of the behaviors for n-gluon and n-graviton tree amplitudes are valid in D dimensions and are similar to Low's proof of universality of the first subleading behavior of photons. In contrast to photons coupling to massive particles, in four dimensions the soft behaviors of gluons and gravitons are corrected by loop effects. We comment on how such corrections arise from this perspective. We also show that loop corrections in graviton amplitudes arising from scalar loops appear only at the second soft subleading order. This case is particularly transparent because it is not entangled with graviton infrared singularities. Our result suggests that if we set aside the issue of infrared singularities, soft-graviton Ward identities of extended BMS symmetry are not anomalous through the first subleading order.

Finally, in Chapter 8, we conclude this dissertation with a discussion of the evanescent effects on nonsupersymmetric gravity at two loops. Evanescent operators such as the Gauss-Bonnet term have vanishing perturbative matrix elements in exactly $D = 4$ dimensions. Similarly, evanescent fields do not propagate in $D = 4$; a three-form field is in this class, since it is dual to a cosmological-constant contribution. In this chapter, we show that evanescent operators and fields modify the leading ultraviolet divergence in pure gravity. To analyze the divergence, we compute the two-loop identical-helicity four-graviton amplitude and determine the coefficient of the associated (non-evanescent) R^3 counterterm studied long ago by Goroff and Sagnotti. We compare two pairs of theories that are dual in $D = 4$: gravity coupled to nothing or to three-form matter, and gravity coupled to zero-form or to two-form matter. Duff and van Nieuwenhuizen showed that, curiously, the one-loop conformal anomaly — the coefficient of the Gauss-Bonnet operator — changes under p-form duality transformations. We concur, and also find that the leading R^3 divergence changes under duality transformations. Nevertheless, in both cases the physical renormalized two-loop identical-helicity four-graviton amplitude can be chosen to respect duality. In particular, its renormalization-scale dependence is unaltered.

The dissertation of Joshua David Nohle is approved.

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University of California, Los Angeles

2015

To my father, Rick Nohle; my mother, Roberta Nohle; my sister, Katie Nohle-Kyde; my brother, Chris Nohle; and my friends, especially those who have become so close during our time at UCLA

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Acknowledgments

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Publications

Z. Bern, C. Cheung, H. H. Chi, S. Davies, L. Dixon and J. Nohle, "Evanescent Effects Can Alter Ultraviolet Divergences in Quantum Gravity without Physical Consequences," arXiv:1507.06118 [hep-th].

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Part I

Color-Kinematics Duality in Yang-Mills Theory and Gravity

CHAPTER 1

Introduction to Color-Kinematics Duality

Recent years have seen remarkable progress in computing and understanding scattering processes in gauge and gravity theories, both for phenomenological and theoretical applications. (For various reviews see Refs. [6, 7, 8].) In particular, various new structures have been uncovered in the amplitudes of these theories (see, for example, Ref. [9]). One such structure is the duality between color and kinematics found by Bern, Carrasco, and Johansson [10, 11]. This Bern-Carrasco-Johansson (BCJ) duality is conjectured to hold at all loop orders in Yang-Mills theory and its supersymmetric counterparts. Besides imposing strong constraints on gauge-theory amplitudes, whenever a form of a gauge-theory loop integrand is obtained where the duality is manifest, we obtain corresponding gravity integrands simply by replacing color factors by specified gauge-theory kinematic numerator factors.

The duality between color and kinematics was noticed long ago for four-point tree-level Feynman diagrams as a possible way to explain certain radiation zeros [12]. For higher points or at loop level, the duality is rather nontrivial and no longer holds for ordinary Feynman diagrams, but requires nontrivial rearrangements to display it. The duality has been confirmed in numerous tree-level studies [13, 14, 15, 16, 17, 18], including the construction of explicit representations for an arbitrary number of external legs [19]. At loop level, the duality remains a conjecture, but there is already nontrivial evidence in its favor, especially for supersymmetric theories [11, 20, 21, 22, 23, 24, 25]. Beyond these explicit duality-satisfying constructions, the duality implies nontrivial relations amongst gauge-theory color-ordered partial tree amplitudes [10, 26]. There is also an partial understanding of the duality at the level of the Lagrangian [15, 17, 27]. The BCJ duality also points to new hidden symmetries. In particular, in the self-dual case an underlying infinite-dimensional Lie algebra corresponding to area-preserving diffeomorphisms has been shown to be responsible for the duality [17, 28]. Even after carrying out loop integrations, the duality points to strong links between gravity and gauge-theory amplitudes [29, 30, 20, 31, 32].

In Chapter 3 and Chapter 4 we provide further evidence in favor of the duality at loop level, explicitly showing that it holds for Yang-Mills one-loop four-point amplitudes (with and without matter in the loop) for all polarization states in D dimensions. We also present a duality-satisfying representation of the two-loop four-point identical-helicity amplitude of pure Yang-Mills. In order to construct the one-loop four-point Yang-Mills amplitude, we use a D-dimensional variant [33, 34] of the unitarity method [35]. Our construction begins by finding an ansatz for the amplitude constrained to satisfy the duality. Since the amplitude is fully determined from its D-dimensional unitarity cuts, we obtain a form of the amplitude with the duality manifest by enforcing that the ansatz has the correct unitarity cuts. The existence of such a form where both the duality and the cuts are simultaneously satisfied is rather nontrivial. We do not use helicity states tied to specific dimensions but instead use formal polarization vectors because we wish to have an expression for the amplitude valid in any dimension and for all states. The price for this generality is that the expressions are lengthier. Since the constructed integrand has manifest BCJ duality, the double-copy construction immediately gives the corresponding gravity amplitude in a theory with a graviton, dilaton, and antisymmetric tensor.

We use these results to study the ultraviolet divergences of the corresponding gravity amplitudes. Recent years have seen a renaissance in the study of ultraviolet divergences in gravity theories, in a large measure due to the greatly improved ability to carry out explicit multiloop computations in gravity theories [36, 37, 11, 29, 30, 20, 38, 31, 32, 39]. The unitarity method also has revealed hints that multiloop supergravity theories may be better behaved in the ultraviolet than power-counting arguments based on standard symmetries suggest [40]. Even pure Einstein gravity at one loop exhibits surprising cancellations as the number of external legs increases [41]. The question of whether it is possible to construct a finite supergravity is still an open one, though there has been enormous progress on this question in recent years, including new computations and a much better understanding of the consequences of supersymmetry and duality symmetry (see e.g. Refs. [42, 43]). In half-maximal supergravity [44], two- and three-loop examples are known where the divergence vanishes, yet the understanding of the possible symmetry behind this vanishing is incomplete [43, 38, 45, 31]. The duality between color and kinematics and its associated double-copy formula offer a new angle on the ultraviolet divergences in supergravity theories [11, 20, 38, 31, 46].

CHAPTER 2

Review of Color-Kinematics Duality and Gravity as a Double Copy of Yang-Mills Theory

An m-point L-loop gauge-theory amplitude in D dimensions, with all particles in the adjoint representation, may be written as

$$\mathcal{A}_m^{L\text{-loop}} = i^L g^{m-2+2L} \sum_{\mathcal{S}_m} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{c_j n_j}{\prod_{\alpha_j} p_{\alpha_j}^2} \,, \tag{2.1}$$

where g is the gauge coupling constant. The first sum runs over the $m!$ permutations of the external legs, denoted by $\mathcal{S}_m$. The S_j symmetry factor removes any overcounting from these permutations and also from any internal automorphism symmetries of graph j. The j-sum is over the set of distinct, nonisomorphic, m-point L-loop graphs with only *cubic* (i.e., trivalent) vertices. These graphs are sufficient because any diagram with quartic or higher vertices can be converted to a diagram with only cubic vertices by multiplying and dividing by the appropriate propagators. The propagators appearing in the graph are $1/\prod_{\alpha_j} p_{\alpha_j}^2$. The nontrivial kinematic information is contained in the numerators n_j and depends on momenta, polarizations, and spinors. In supersymmetric cases it will depend also on Grassmann parameters, if a superspace form is used. The loop integral is over L independent D-dimensional loop momenta, p_l. Finally, c_j denotes the color factor, obtained by dressing every vertex in graph j with the group-theory structure constant, $\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \mathrm{Tr}([T^a, T^b]T^c)$, where the hermitian generators of the gauge group are normalized via $\mathrm{Tr}(T^aT^b) = \delta^{ab}$.

The numerators appearing in Eq. (2.1) are by no means unique because of freedom in moving terms between different diagrams. Utilizing this freedom, the BCJ conjecture is that

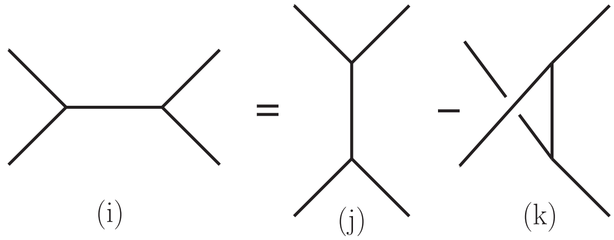

Figure 2.1: The basic Jacobi relation for either color or numerator factors. These three diagrams can be embedded in a larger diagram, including loops.

to all loop orders, representations of the amplitude exist where kinematic numerators obey the same algebraic relations that the color factors obey [10, 11]. In ordinary gauge theories, this is simply the Jacobi identity,

$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k\,, \tag{2.2}$$

where i, j, and k label three diagrams whose color factors obey the Jacobi identity. The basic Jacobi identity is displayed in Fig. 2.1. The identity generalizes to any loop order with any number of external legs by embedding it in larger diagrams, where the other parts of the diagrams are identical for the three diagrams. Furthermore, if the color factor of a diagram is antisymmetric under a swap of legs, we require that the numerator obey the same antisymmetry,

$$c_i \to -c_i \;\Rightarrow\; n_i \to -n_i\,. \tag{2.3}$$

The duality was noticed long ago for tree-level four-point Feynman diagrams [12]; beyond this, it is rather nontrivial and no longer holds for ordinary Feynman diagrams. We note that the numerator relations are nontrivial functional relations because they depend on momenta, polarizations, and spinors, as discussed in some detail in Refs. [20, 8, 18].

While a complete understanding of the duality and its consequences is still lacking, a variety of studies have elucidated it, especially at tree level. In particular, this duality leads to nontrivial relations between gauge-theory color-ordered partial tree amplitudes [10, 26]. The duality (2.3) has also been studied in string theory [47, 13, 16]. In the self-dual case, light-cone gauge Feynman rules have been shown to exhibit the duality [17]. Explicit forms of n-point tree amplitudes satisfying the duality have been found [19]. Although we do not yet have a complete Lagrangian understanding, some progress in this direction can be found in Refs. [15, 17]. The duality (2.2) does not need to be expressed in terms of group structure constants but can alternatively be expressed in terms of a trace-based representation [48]. Progress has also been made in understanding the underlying infinite-dimensional Lie algebra [17, 28] responsible for the duality. The duality between color and kinematics also appears to hold in three-dimensional theories based on three algebras [49], as well as in some cases with higher-dimension operators [50]. Some initial studies of duality and its implications for gravity in the high-energy limit have also been carried out [51].

At loop level, the duality remains a conjecture, but there is already nontrivial evidence in its favor, especially for supersymmetric theories. At present, the list of loop-level cases where duality-satisfying forms of the amplitude are known to hold includes:

- Up to four loops for four-point $\mathcal{N} = 4$ super-Yang-Mills [11, 20] in a form valid in D dimensions;
- up to two loops for five external gluons in $\mathcal{N} = 4$ super-Yang-Mills theory [21];
- up to seven points for one-loop amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory [22];
- up to two loops for four-point identical-helicity pure Yang-Mills amplitudes [11];
- through n points for one-loop all-plus- or single-minus-helicity pure Yang-Mills amplitudes [23];
- through four loops for a two-point (Sudakov) form factor in $\mathcal{N} = 4$ super-Yang-Mills theory [24];
- one-loop four-point amplitudes in Yang-Mills theories with less than maximally supersymmetric amplitudes [25].

In Chapter 3 and Chapter 4, we add Yang-Mills one-loop four-point amplitudes (with and without matter in the loop) in D dimensions to this list. Besides direct constructions, we note that the duality also appears to be consistent with loop-level infrared properties of both gauge and gravity theories [52].

Another significant aspect of the duality is the ease with which gravity loop integrands can be obtained from gauge-theory ones, once the duality is made manifest [10, 11]. One simply replaces the color factor with a kinematic numerator from a second gauge theory,

$$c_i \;\rightarrow\; \tilde{n}_i\,. \tag{2.4}$$

This immediately gives the double-copy form of gravity amplitudes,

$$\mathcal{M}_m^{L\text{-loop}} = i^{L+1}\left(\frac{\kappa}{2}\right)^{m-2+2L}\sum_{\mathcal{S}_m}\sum_j\int\prod_{l=1}^{L}\frac{d^Dp_l}{(2\pi)^D}\frac{1}{S_j}\frac{\tilde{n}_j n_j}{\prod_{\alpha_j}p^2_{\alpha_j}}\,, \tag{2.5}$$

where $\tilde{n}_j$ and n_j are gauge-theory numerator factors. Only one of the two sets of numerators needs to satisfy the duality (2.2) [11, 15] in order for the double-copy form (2.5) to be valid. The double-copy formalism has been studied at loop level in some detail in a variety of cases [11, 29, 21, 20, 38, 31, 52, 22].

We briefly mention that the four-point one-loop amplitudes of Eq. (2.1)—which we will be concerned with in Chapter 3 and Chapter 4—can be written as [53]:

$$\mathcal{A}_4^{(1)}(1,2,3,4) = g^4\left[c^{(1)}_{1234}A_4^{(1)}(1,2,3,4)+c^{(1)}_{1423}A_4^{(1)}(1,4,2,3)+c^{(1)}_{1342}A_4^{(1)}(1,3,4,2)\right]. \tag{2.6}$$

The $A_4^{(1)}$'s are the one-loop color-ordered amplitudes [54], which are independently gauge invariant. The color factors, $c^{(1)}_{1i_2i_3i_4}$, are given by dressing the vertices of the one-loop box diagram that has the external-leg ordering $(1,i_2,i_3,i_4)$ with the structure constants $\tilde{f}^{abc}$.

CHAPTER 3

Color-Kinematics Duality for Pure Yang-Mills and Gravity at One and Two Loops

3.1 Introduction

In this chapter, we present duality-satisfying representations for pure Yang-Mills one-loop four-point amplitudes for all polarization states in D dimensions as well as for the two-loop four-point identical-helicity amplitude of pure Yang-Mills. The latter amplitude was first given in Ref. [55] in a non-duality-satisfying representation, while Ref. [11] noted the existence of a duality-satisfying form. Here, we explicitly give the full duality-satisfying form, including contributions from diagrams absent from Ref. [55] that vanish under integration but are necessary to make the duality manifest.

We use the gravity integrands constructed via the double-copy property to determine the exact form of the ultraviolet divergences. We do so at one loop in dimensions $D = 4, 6, 8$. The ultraviolet properties of one-loop four-point gravity amplitudes have already been studied in some detail over the years, including cases with scalars or antisymmetric tensors coupling to gravity [56, 57, 58, 59, 60, 41], so no surprises should be expected, at least at four points. Nevertheless, it is useful to look in some detail at the ultraviolet properties to understand them from the double-copy perspective. Here we examine the four-point amplitudes in a theory of gravity coupled to a dilaton and an antisymmetric tensor, corresponding to the double copy of pure Yang-Mills theory. While related calculations have been carried out, we are unaware of any calculations of the ultraviolet properties in the theory corresponding to the double-copy theory.

We find that in $D = 4$, there are no one-loop divergences in amplitudes involving external gravitons, though there are divergences in the remaining amplitudes involving only external dilatons or antisymmetric tensors, as expected from simple counterterm arguments [56]. By two loops, even the four-graviton amplitudes contain divergences, as we demonstrate by computing the form and numerical coefficient of the divergence. In the two-loop case, the divergence is proportional to a unique R^3 operator which gives a divergence in the identical-helicity four-point amplitude. This means that the identical-helicity four-point amplitude is sufficient for determining the coefficient of the R^3 divergence. In $D = 6$ and $D = 8$, we find one-loop divergences in the four-external-graviton amplitudes. These results are not surprising and are in line with the earlier studies. Our conclusion is that, by itself, the double-copy structure is insufficient to render a gravity theory finite in $D = 4$ and requires additional ultraviolet cancellations, such as those from supersymmetry.

This chapter is organized as follows. In Section 3.2, we present the construction of the duality-satisfying pure Yang-Mills numerators at one and two loops. Then in Section 3.3, we study the ultraviolet properties of gravity coupled to a dilaton and an antisymmetric tensor at one loop in four, six, and eight dimensions. In the same section, we also present the ultraviolet properties at two loops in four dimensions. Finally, in Section 3.4 we give our conclusions. Appendices evaluating two-loop integrals needed in Section 3.3.2 are included. Appendix A focuses on extracting the divergences in dimensional regularization. This procedure mixes infrared and ultraviolet divergences; so, in Appendix B we give the infrared divergences that must be subtracted to obtain the ultraviolet ones. Appendix C evaluates the integrals using an alternative method for obtaining the ultraviolet divergences more directly, by introducing a mass to separate out the infrared divergences from the ultraviolet ones.

3.2 Construction of Duality-Satisfying Integrands

We now describe the construction of a duality-satisfying representation of the one-loop four-point amplitude in pure Yang-Mills. Since we want the form to be valid in all dimensions and for all $D-2$ gluon states, we use formal polarizations instead of helicity states. This complicates the expression for the amplitude, but has the advantage that it allows us to straightforwardly study the amplitude and its gravity double copy in various dimensions. In this section, we also present a form of the two-loop pure Yang-Mills identical-helicity amplitude given in Ref. [55] that satisfies BCJ duality after some rearrangement and addition of diagrams that integrate to zero. In Section 3.3.2, we use this amplitude to show that although four-graviton amplitudes are ultraviolet finite in $D=4$ at one loop, they diverge at two loops, in accordance with expectations.

3.2.1 One Loop

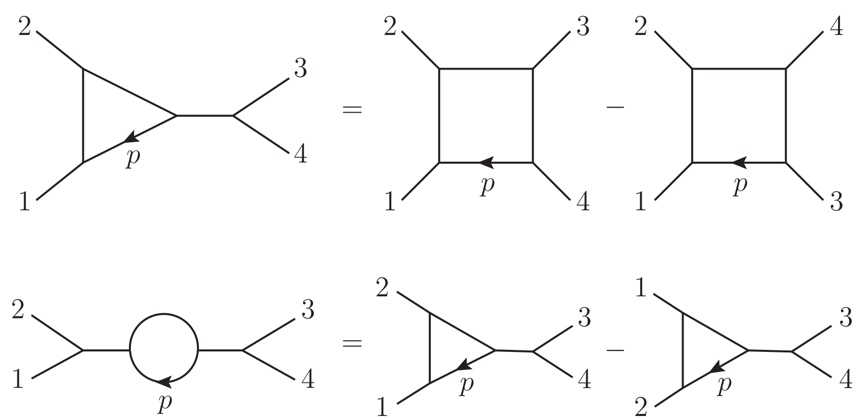

Figure 3.1: The Jacobi relations determining either color or kinematic numerators of the four-point diagrams containing either a triangle or internal bubble.

For a one-loop n-point amplitude, the duality (2.2) can be used to express kinematic numerators of any diagram directly in terms of n-gon numerators. In particular, for the four-point case we have two basic relations determining triangle and bubble contributions from box numerators as illustrated in Fig. 3.1,

$$\begin{aligned} n_{12(34);p} &= n_{1234;p} - n_{1243;p} \,, \\ n_{(12)(34);p} &= n_{12(34);p} - n_{21(34);p} \,. \end{aligned} \tag{3.1}$$

The labels $1,2,3,4$ refer to the momenta and states of each external leg, while the label p

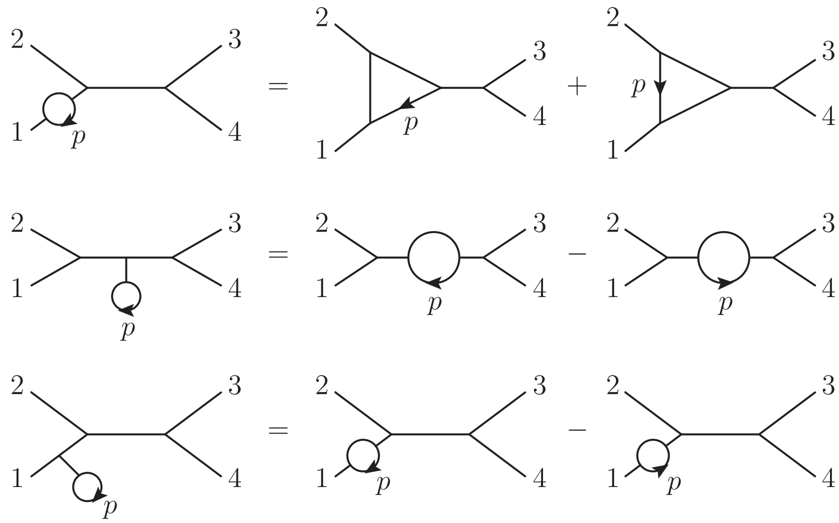

Figure 3.2: The color or kinematic Jacobi relations involving a bubble on an external leg or a tadpole. These diagrams have vanishing contribution to the integrated amplitude.

denotes the loop momentum of the leg indicated in Fig. 3.1. (The parentheses in the subscript of the numerators indicate which external legs are pinched off to form a tree attached to the loop.) Note that in the figure the momentum of each internal leg of each diagram is the same as in the other two diagrams *except* for the single internal leg that differs between the diagrams. In general, the bubble and triangle contributions are nonvanishing; indeed, this explicitly holds for the BCJ representation of the one-loop four-point amplitude of pure Yang-Mills theory that we construct.

Besides the diagrams in Fig. 3.1, there are diagrams with a bubble on an external leg and diagrams with a tadpole, as shown in Fig. 3.2. The duality also determines the numerators of these diagrams via

$$\begin{aligned} n_{1(234);p} &= n_{12(34);p} + n_{1(43)2;p} \, , \\ n_{(1234);p} &= n_{(12)(34);p} - n_{(12)(34);-p} \, , \\ n_{(\hat{1}234);p} &= n_{1(234);p} - n_{1(234);-p} \, , \end{aligned} \tag{3.2}$$

corresponding respectively to the three relations in Fig. 3.2. (On the final line in Eq. (3.2), the hat marks leg 1 as the location where the tadpole is attached.) We use these equations to impose the auxiliary constraint that the tadpole numerators determined by BCJ duality vanish identically and that all terms in the bubble-on-external-leg diagrams integrate to zero as they do for Feynman diagrams. Thus, these diagrams are not necessary for determining the integrated amplitudes (though in $D = 4$ the bubble-on-external-leg diagrams do affect the Yang-Mills ultraviolet divergence).

Once we impose the BCJ conditions, the amplitude is entirely specified by the box numerators. Our task is then to find an expression for the box numerators such that we obtain the correct amplitude. It is useful to impose a few auxiliary constraints to help simplify the one-loop construction:

1. The box diagrams should have no more than four powers of loop momenta in the pure Yang-Mills case, matching the usual power count of Feynman-gauge Feynman diagrams.

2. Each numerator written in terms of formal polarization vectors respects the symmetries of the diagrams. In particular, this condition implies that once a box diagram with one ordering of external legs is specified, the other orderings are obtained simply by relabeling.

3. The numerators of tadpole diagrams vanish prior to integration.

4. All terms in the bubble-on-external-leg diagrams integrate to zero, as they do for Feynman diagrams.

While it is not necessary to impose these conditions, they greatly simplify the construction. They ensure that the type of terms that appear in the ansatz are similar to those of ordinary Feynman-gauge Feynman diagrams, avoiding unnecessarily complicated terms. (Using generalized gauge invariance, one can always introduce arbitrarily complicated terms into amplitudes, which cancel at the end.)

The first three conditions simplify the construction by restricting the number of terms that appear. The purpose of the fourth auxiliary constraint is a bit more subtle. While bubble-on-external-leg Feynman diagrams are well defined in the on-shell limit, the freedom to reassign terms used in the construction of BCJ numerators can introduce ill-defined terms into such diagrams. As a simple example, consider the effect of the term $(k_1+k_2)^2\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1\varepsilon_3\cdot\varepsilon_4$ when added to the numerator of the first diagram of Fig. 3.2 (with k_i and ε_i external momenta and polarizations). Even after integration, this contribution to the diagram is ill-defined because of the on-shell intermediate propagator. Such singular contributions would need to be regularized by an appropriate off-shell continuation to ensure that the introduced singularities cancel properly against singularities of other diagrams. While in principle we can introduce such a regulator, it is best to avoid this complication altogether. The fourth condition ensures that we can treat the bubble-on-external-leg contributions in the same way as for Feynman diagrams. In particular, with the constraint imposed, the bubble-on-external-leg contributions match the Feynman-diagram property that they are proportional to $(k_i^2)^{(D-4)/2}$, after accounting for the intermediate on-shell propagator, and hence vanish in $D>4$, for k_i on shell. We note that even with the fourth constraint, near $D=4$ we encounter the same subtlety encountered with Feynman diagrams: Although bubble-on-external-leg contributions are set to zero in dimensional regularization, they can carry ultraviolet divergences. Such ultraviolet divergences cancel against infrared ones leaving a vanishing result for on-shell bubble-on-external-leg diagrams. The net effect is that in gauge theory, we need to account for such contributions to obtain the correct ultraviolet divergences. In contrast, in gravity even near $D=4$ there are neither infrared nor ultraviolet divergences hiding in the bubble-on-external-leg contributions because an extra two powers of numerator momenta give rise to an additional vanishing.

We start the construction with an ansatz containing all possible products of $\varepsilon_i\cdot\varepsilon_j$, $p\cdot\varepsilon_i$, $k_i\cdot\varepsilon_j$, $p\cdot k_i$, $p\cdot p$, s, and t, where the k_i are three independent external momenta, p is the loop momentum, ε_i are external polarization vectors, and

$$s=(k_1+k_2)^2\,,\qquad\qquad t=(k_2+k_3)^2\,,\tag{3.3}$$

are the usual Mandelstam invariants. By dimensional analysis, each numerator term must contain four momenta in addition to being linear in all four ε_i's. We also set $k_i \cdot \varepsilon_i = 0$ and impose momentum conservation with $k_4 = -k_1 - k_2 - k_3$ and $k_1 \cdot \varepsilon_4 = -k_2 \cdot \varepsilon_4 - k_3 \cdot \varepsilon_4$. This yields 468 terms, each with a coefficient to be determined.

Our first constraint on the coefficients comes from demanding that the box numerator obey the rotation and reflection symmetries of the box diagram. This leaves us with 81 free coefficients. An ansatz for the full amplitude is then obtained by using the duality relations (3.1), (3.2) to determine numerators for all other diagrams.

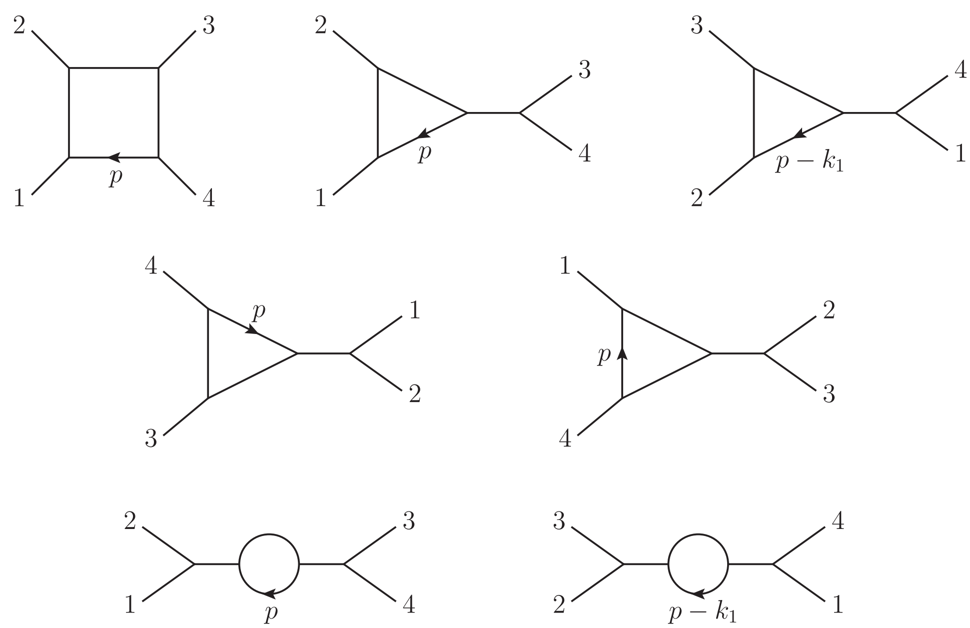

Figure 3.3: The seven diagrams for the color-ordered amplitude with ordering $(1, 2, 3, 4)$.

The next step is to determine coefficients in the ansatz by matching to the unitarity cuts of the amplitude. It is convenient to use a color-ordered form of the amplitude [54] for this matching. The seven diagrams contributing to the color-ordered amplitude, that is the coefficient of the color trace $N_c \text{Tr}[T^{a_1} T^{a_2} T^{a_3} T^{a_4}]$, are shown in Fig. 3.3. The other color-ordered amplitudes are simple relabelings of this one. For the one-loop four-point amplitude, the s- and t-channel unitarity cuts—shown in Fig. 3.4—are sufficient to determine this color-ordered amplitude up to terms that integrate to zero. One straightforward means

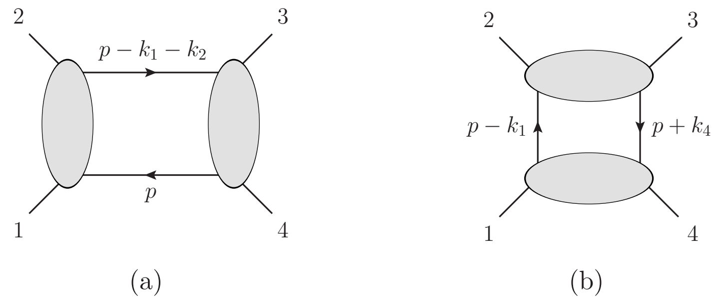

Figure 3.4: The (a) s-channel and (b) t-channel unitarity cuts used to determine the amplitude. The exposed intermediate legs are on shell.

for determining the cuts is to construct the amplitude in Feynman gauge and then take its unitarity cuts at the integrand level prior to integration. This automatically gives us an expression for the cuts valid in D dimensions without any spurious denominators (such as light-cone denominators from physical state projectors). This matching procedure nontrivially rearranges the amplitude so that BCJ duality is manifest. After matching the cuts, we also impose the fourth auxiliary condition to tame the bubble-on-external-leg contributions. Finally we impose that the tadpole numerators vanish. Including all the auxiliary constraints with these conditions, we can solve for all but five free coefficients. Because the s- and t-channel unitarity cuts are independent of these parameters, the integrated amplitude should not depend on them.

Using the shorthand notation,

$$
\begin{aligned}
&p_1 = p\,, \qquad p_2 = p - k_1\,, \qquad p_3 = p - k_1 - k_2\,, \qquad p_4 = p + k_4\,, \\
&\mathcal{E}_{ij} = \varepsilon_i \cdot \varepsilon_j\,, \qquad \mathcal{P}_{ij} = p_i \cdot \varepsilon_j\,, \qquad \mathcal{K}_{ij} = k_i \cdot \varepsilon_j\,,
\end{aligned}
\tag{3.4}
$$

and setting the free parameters to zero for simplicity, the box numerator is

$$
\begin{aligned}
n_{1234;p} \; = -i\Big[& \tfrac{D_s-2}{8}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_1^2\,p_3^2 + \tfrac{D_s-2}{24}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,p_1^2\,p_3^2 - \tfrac{D_s-2}{24}\,\mathcal{E}_{12}\,\mathcal{E}_{34}\,p_1^2\,p_3^2 - \tfrac{2}{3}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_3^2\,s \\
& - \tfrac{2}{3}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,p_2^2\,s + \tfrac{2}{3}\,\mathcal{E}_{12}\,\mathcal{E}_{34}\,p_2^2\,s + \tfrac{2}{3}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_2^2\,s + \tfrac{1}{2}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,s^2 + 2\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,p_3^2 \\
& + \tfrac{D_s-74}{24}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{34}\,p_3^2 + \tfrac{D_s-74}{24}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{41}\,p_3^2 - \tfrac{D_s-26}{3}\,\mathcal{E}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{34}\,p_3^2 \\
& - \tfrac{D_s-26}{6}\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{K}_{42}\,p_3^2 - \tfrac{D_s-26}{2}\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,p_3^2 - \tfrac{D_s-26}{2}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{K}_{41}\,p_3^2 \\
& + \tfrac{D_s-26}{12}\,\mathcal{E}_{34}\,\mathcal{K}_{31}\,\mathcal{K}_{42}\,p_3^2 + \tfrac{5(D_s-26)}{24}\,\mathcal{E}_{24}\,\mathcal{K}_{13}\,\mathcal{K}_{41}\,p_3^2 - \tfrac{D_s-26}{8}\,\mathcal{E}_{24}\,\mathcal{K}_{13}\,\mathcal{K}_{31}\,p_3^2 \\
& - \tfrac{11(D_s-26)}{24}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{31}\,p_3^2 - \tfrac{D_s-26}{24}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{K}_{31}\,p_3^2 + \tfrac{D_s-30}{2}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{24}\,p_3^2 \\
& - \tfrac{D_s-14}{6}\,\mathcal{E}_{13}\,\mathcal{K}_{34}\,\mathcal{K}_{42}\,p_3^2 + \tfrac{D_s-38}{6}\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,p_3^2 - \tfrac{5(D_s-26)}{24}\,\mathcal{E}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,p_3^2 \\
& - \tfrac{11(D_s-26)}{24}\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,p_3^2 + \tfrac{13D_s-290}{24}\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{K}_{42}\,p_3^2 + (D_s-24)\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,p_3^2 \\
& + \tfrac{11(D_s-26)}{24}\,\mathcal{E}_{14}\,\mathcal{K}_{13}\,\mathcal{K}_{42}\,p_3^2 + \tfrac{11(D_s-26)}{24}\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,p_3^2 - \tfrac{D_s-26}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,p_3^2 \\
& - \tfrac{D_s-26}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{31}\,\mathcal{K}_{34}\,p_3^2 - \tfrac{D_s-50}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{41}\,p_3^2 - 4\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,s - 2\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,s \\
& - 2\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,s - 2\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,s - 2\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,s - 2\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,s \\
& + \tfrac{7D_s-230}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{31}\,\mathcal{P}_{44}\,p_3^2 + \tfrac{7D_s-230}{24}\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,p_3^2 + \tfrac{7D_s-230}{24}\,\mathcal{E}_{23}\,\mathcal{K}_{41}\,\mathcal{P}_{44}\,p_3^2 \\
& + \tfrac{7D_s-230}{24}\,\mathcal{E}_{13}\,\mathcal{K}_{34}\,\mathcal{P}_{22}\,p_3^2 + \tfrac{7D_s-230}{24}\,\mathcal{E}_{24}\,\mathcal{K}_{41}\,\mathcal{P}_{33}\,p_3^2 - \tfrac{7(D_s-26)}{24}\,\mathcal{E}_{24}\,\mathcal{K}_{13}\,\mathcal{P}_{11}\,p_3^2 \\
& + \tfrac{7(D_s-26)}{24}\,\mathcal{E}_{12}\,\mathcal{K}_{13}\,\mathcal{P}_{44}\,p_3^2 - \tfrac{7D_s-230}{24}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11}\,p_3^2 - \tfrac{7D_s-230}{24}\,\mathcal{E}_{12}\,\mathcal{K}_{24}\,\mathcal{P}_{33}\,p_3^2 \\
& - \tfrac{7D_s-230}{24}\,\mathcal{E}_{34}\,\mathcal{K}_{42}\,\mathcal{P}_{11}\,p_3^2 - \tfrac{11D_s-238}{24}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{P}_{44}\,p_3^2 - \tfrac{11D_s-238}{24}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{P}_{11}\,p_3^2 \\
& + 2\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{P}_{44}\,p_3^2 + 2\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{P}_{11}\,p_3^2 - \tfrac{D_s-14}{6}\,\mathcal{E}_{13}\,\mathcal{K}_{42}\,\mathcal{P}_{44}\,p_3^2 - \tfrac{3(D_s-26)}{8}\,\mathcal{E}_{34}\,\mathcal{K}_{31}\,\mathcal{P}_{22}\,p_3^2 \\
& - \tfrac{3(D_s-26)}{8}\,\mathcal{E}_{24}\,\mathcal{K}_{31}\,\mathcal{P}_{33}\,p_3^2 - \tfrac{2(D_s-29)}{3}\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,p_3^2 - \tfrac{2(D_s-29)}{3}\,\mathcal{E}_{12}\,\mathcal{K}_{34}\,\mathcal{P}_{33}\,p_3^2 \\
& + \tfrac{13D_s-290}{24}\,\mathcal{E}_{14}\,\mathcal{K}_{42}\,\mathcal{P}_{33}\,p_3^2 + \tfrac{13D_s-290}{24}\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{P}_{33}\,p_3^2 + \tfrac{13D_s-290}{24}\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{P}_{22}\,p_3^2 \\
& + \tfrac{2(D_s-29)}{3}\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{22}\,p_3^2 - 2\,\mathcal{E}_{14}\,\mathcal{K}_{42}\,\mathcal{P}_{33}\,s - 2\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,s - 2\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{P}_{33}\,s \\
& - 2\,\mathcal{E}_{12}\,\mathcal{K}_{24}\,\mathcal{P}_{33}\,s - 2\,\mathcal{E}_{12}\,\mathcal{K}_{34}\,\mathcal{P}_{33}\,s - 2\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{P}_{22}\,s + 2\,\mathcal{E}_{13}\,\mathcal{K}_{34}\,\mathcal{P}_{22}\,s + 2\,\mathcal{E}_{24}\,\mathcal{K}_{41}\,\mathcal{P}_{33}\,s \\
& + 2\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{22}\,s - (D_s-2)\,\mathcal{E}_{23}\,\mathcal{P}_{11}\,\mathcal{P}_{44}\,p_3^2 - \tfrac{D_s-2}{6}\,\mathcal{E}_{13}\,\mathcal{P}_{22}\,\mathcal{P}_{44}\,p_3^2 - \tfrac{D_s-2}{6}\,\mathcal{E}_{24}\,\mathcal{P}_{33}\,\mathcal{P}_{11}\,p_3^2 \\
& + \tfrac{D_s-2}{6}\,\mathcal{E}_{12}\,\mathcal{P}_{33}\,\mathcal{P}_{44}\,p_3^2 + \tfrac{D_s-2}{6}\,\mathcal{E}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22}\,p_3^2 - 4\,\mathcal{E}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22}\,s + 2\,\mathcal{E}_{13}\,\mathcal{P}_{22}\,\mathcal{P}_{44}\,s \\
& + 2\,\mathcal{E}_{24}\,\mathcal{P}_{33}\,\mathcal{P}_{11}\,s + 4\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{31} + 4\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31} + 2\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{31}\,\mathcal{K}_{34} \\
& + 4\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{31}\,\mathcal{K}_{34} + \,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,\mathcal{K}_{42} + 2\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{41} - 4\,\mathcal{K}_{12}\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,\mathcal{P}_{33} \\
& + 4\,\mathcal{K}_{31}\,\mathcal{K}_{34}\,\mathcal{K}_{42}\,\mathcal{P}_{33} + 4\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,\mathcal{K}_{42}\,\mathcal{P}_{33} + 4\,\mathcal{K}_{34}\,\mathcal{K}_{41}\,\mathcal{K}_{42}\,\mathcal{P}_{33} + 4\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,\mathcal{K}_{42}\,\mathcal{P}_{33} \\
& - 8\,\mathcal{K}_{34}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,\mathcal{P}_{33} - 8\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,\mathcal{P}_{33} + 4\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,\mathcal{P}_{11}\,\mathcal{P}_{33} \\
& + (D_s-2)\,\mathcal{P}_{11}\,\mathcal{P}_{22}\,\mathcal{P}_{33}\,\mathcal{P}_{44}\Big] + \text{cyclic}\,, \qquad (3.5)
\end{aligned}
$$

where D_s is a state-counting parameter, so that $D_s - 2$ is the number of gluon states circulating in the loop. The notation '+ cyclic' indicates that one should include the three additional cyclic permutations of indices, giving a total of four permutations $(1, 2, 3, 4)$,

$(2,3,4,1)$, $(3,4,1,2)$, $(4,1,2,3)$ of all variables $\varepsilon_i, k_i, p_i, s, t$. Plain-text, computer-readable versions of the full expressions for the numerators, including also gluino- and scalar-loop contributions, can be found online [61]. In Eq. (3.5), we have written the expression for the box numerator in a different form than that available online in order to exhibit the cyclic symmetry.

We have explicitly checked that after reducing the pure Yang-Mills amplitude to an integral basis,[1] the expression is free of arbitrary parameters and in $D = 4$ matches the known expression for the amplitude in Ref. [62], after accounting for the fact that the expression in that chapter is renormalized. The reduction for four-dimensional external states was carried out by expanding the external polarizations in terms of the external momenta plus a dual vector [63].

As another simple cross check, we have extracted the ultraviolet divergences in $D = 6, 8$ and compared them to the known forms. In $D = 6, 8$, with our fourth auxiliary constraint there are no ultraviolet contributions from bubbles on external legs. This allows us to directly extract the ultraviolet divergences by introducing a mass regulator and then expanding in small external momenta using the methods of Ref. [64]. We find complete agreement with both earlier evaluations in Ref. [65]. We have also compared this to an extraction of the ultraviolet divergences directly using dimensional regularization without introducing an additional mass regulator and again find agreement.

3.2.2 Two Loops

We now turn to two loops. As we shall discuss in Section 3.3, the four-graviton amplitude in the double-copy theory is ultraviolet finite at one loop. To test whether this continues at two loops, we need the two-loop amplitude. As it turns out, the identical-helicity amplitude is sufficient for our purposes because the divergence comes from an R^3 operator whose coefficient is fixed by this amplitude. We therefore now turn to finding a form of the two-

[1] We thank R. Roiban for cross-checking our computation.

loop identical-helicity amplitude where BCJ duality is manifest. It would be interesting to obtain a general two-loop construction valid for all states in D dimensions, but we do not do so here.

The identical-helicity pure Yang-Mills amplitude has previously been constructed in Ref. [55]. There the amplitude is given in the following representation:

$$\mathcal{A}_4^{(2)}(1^+,2^+,3^+,4^+) = g^6\,\frac{1}{4}\sum_{\mathcal{S}_4}\Big[C^{\rm P}_{1234}A^{\rm P'}_{1234} + C^{\rm NP}_{12;34}A^{\rm NP}_{12;34}\Big]\,, \tag{3.6}$$

where the sum runs over all 24 permutations of the external legs. We will describe the all-plus-helicity case; the all-negative-helicity case follows from parity conjugation. The prefactor of 1/4 accounts for the overcount due to symmetries of the diagrams. $C^{\rm P}_{1234}$ and $C^{\rm NP}_{12;34}$ are the color factors obtained from the planar double-box and nonplanar double-box diagrams shown in Fig. 3.5(a) and (b), respectively, by dressing each vertex with an $\tilde{f}^{abc}$ and summing over the contracted color indices. $A^{\rm P'}_{1234}$ and $A^{\rm NP}_{12;34}$ are then the associated partial amplitudes. These partial amplitudes are [55]

$$\begin{aligned}
A^{\rm P'}_{1234} = i\mathcal{T}\Big\{ & s\,\mathcal{I}_4^{\rm P}\big[(D_s-2)\left(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2\right)+16\left((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2\right)\big](s,t)\\
&+4(D_s-2)\mathcal{I}_4^{\text{bow-tie}}\big[\left(\lambda_p^2+\lambda_q^2\right)(\lambda_p\cdot\lambda_q)\big](s)\\
&+\frac{(D_s-2)^2}{s}\mathcal{I}_4^{\text{bow-tie}}\big[\lambda_p^2\lambda_q^2((p+q)^2+s)\big](s,t)\Big\}\,,\\
A^{\rm NP}_{12;34} = i\mathcal{T}\,s\,\mathcal{I}_4^{\rm NP} & \big[(D_s-2)\left(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2\right)+16\left((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2\right)\big](s,t)\,,
\end{aligned} \tag{3.7}$$

where the permutation-invariant kinematic prefactor is given by

$$\mathcal{T} \equiv \frac{[1\,2][3\,4]}{\langle 1\,2\rangle\langle 3\,4\rangle}\,, \tag{3.8}$$

where the angle and square brackets are standard spinor inner products. For the all negative-helicity case, the angle and square products should be swapped. The planar double-box

(Fig. 3.5(a)), nonplanar double-box (Fig. 3.5(b)), and bow-tie integrals (Fig. 3.6) are

$$\begin{aligned}
&\mathcal{I}_4^{\rm P}[\mathcal{P}(\lambda_i, p, q, k_i)](s,t) \\
&\qquad \equiv \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{\mathcal{P}(\lambda_i, p, q, k_i)}{p^2 q^2 (p+q)^2 (p-k_1)^2 (p-k_1-k_2)^2 (q-k_4)^2 (q-k_3-k_4)^2}\,, \\
&\mathcal{I}_4^{\rm NP}[\mathcal{P}(\lambda_i, p, q, k_i)](s,t) \\
&\qquad \equiv \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{\mathcal{P}(\lambda_i, p, q, k_i)}{p^2 q^2 (p+q)^2 (p-k_1)^2 (q-k_2)^2 (p+q+k_3)^2 (p+q+k_3+k_4)^2}\,, \\
&\mathcal{I}_4^{\text{bow-tie}}[\mathcal{P}(\lambda_i, p, q, k_i)](s,t) \\
&\qquad \equiv \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{\mathcal{P}(\lambda_i, p, q, k_i)}{p^2 q^2 (p-k_1)^2 (p-k_1-k_2)^2 (q-k_4)^2 (q-k_3-k_4)^2}\,,
\end{aligned} \tag{3.9}$$

where λ_p, λ_q, and λ_{p+q} represent the (-2ϵ)-dimensional components of loop momenta p, q, and $(p+q)$.

Ref. [11] notes that a representation where the numerators satisfy the BCJ duality can be obtained directly from the representation of the amplitude given in Ref. [55]. Here we describe this in more detail, including additional diagrams that integrate to zero and are undetectable in ordinary unitarity cuts, but are needed to make the duality manifest.

We begin with a rearranged form of the identical-helicity amplitude,

$$\mathcal{A}_4^{(2)}(1^+,2^+,3^+,4^+) = g^6 \sum_{\mathcal{S}_4} \left[\frac{1}{4} C^{\rm P}_{1234} A^{\rm P}_{1234} + \frac{1}{4} C^{\rm NP}_{12;34} A^{\rm NP}_{12;34} + \frac{1}{8} C^{\rm DT}_{1234} A^{\rm DT}_{1234} \right]. \tag{3.10}$$

$C^{\rm DT}_{1234}$ is the color factor obtained from the stretched bow-tie or double-triangle diagram in

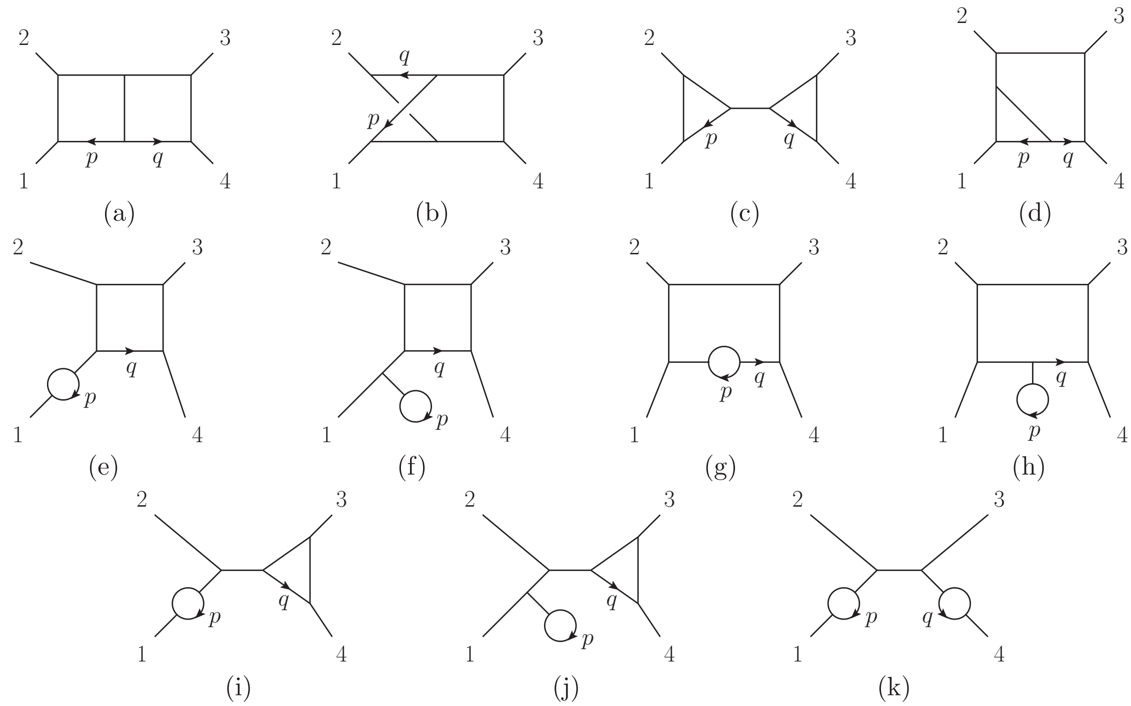

Figure 3.5: The diagrams needed to describe an integrand for the identical helicity-amplitude where the duality between color and kinematics is manifest. When integrated all diagrams, except the (a) planar double-box, (b) nonplanar double-box, and (c) double-triangle integrals, vanish.

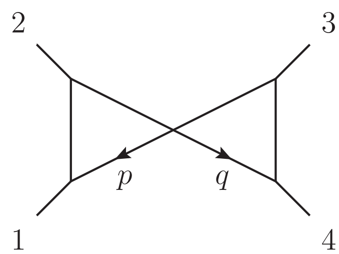

Figure 3.6: The bow-tie integral appearing in the identical-helicity pure Yang-Mills amplitude.

Fig. 3.5(c). $A^{\rm NP}_{12;34}$ is given in Eq. (3.7), while $A^{\rm P}_{1234}$ and $A^{\rm DT}_{1234}$ are

$$
\begin{aligned}
A^{\rm P}_{1234} = i\mathcal{T}\,\mathcal{I}^{\rm P}_4 \bigg[& \frac{(D_s-2)^2}{2}(p+q)^2\lambda_p^2\lambda_q^2 + 16s\left((\lambda_p\cdot\lambda_q)^2 - \lambda_p^2\lambda_q^2\right) \\
& + (D_s-2)\Big(s\left(\lambda_p^2\lambda_q^2 + \lambda_p^2\lambda_{p+q}^2 + \lambda_q^2\lambda_{p+q}^2\right) + 4(p+q)^2(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)\Big)\bigg](s,t)\,, \\
A^{\rm DT}_{1234} = i\mathcal{T}\,\mathcal{I}^{\rm DT}_4 \bigg[& \frac{(D_s-2)^2}{2}\left(4p\cdot q + 2(p-q)\cdot(k_1+k_2) - s\right)\lambda_p^2\lambda_q^2 \\
& + 8(D_s-2)\left(\lambda_p^2+\lambda_q^2\right)(\lambda_p\cdot\lambda_q)\left(p^2+q^2-(p-q)\cdot(k_1+k_2)+s\right)\bigg](s,t)\,.
\end{aligned}
\tag{3.11}
$$

The double-triangle integral displayed in Fig. 3.5(c) is simply

$$\mathcal{I}_4^{\rm DT}[\mathcal{P}(\lambda_i,p,q,k_i)](s,t) = \frac{1}{s}\,\mathcal{I}_4^{\text{bow-tie}}[\mathcal{P}(\lambda_i,p,q,k_i)](s,t)\,, \tag{3.12}$$

so that all integrals in the new representation of the amplitude are given by trivalent graphs.

This form of the amplitude differs from Eq. (3.6) by absorbing the bow-tie contribution depicted in Fig. 3.6 into both the planar double box in Fig. 3.5(a) and the double triangle in Fig. 3.5(c). When moving terms into the double box (a), we must multiply by a factor of $(p+q)^2$ in the numerator to cancel the central propagator, while in the double triangle (c), we must multiply by a factor of s. In this rearrangement we have also included terms that integrate to zero. In particular, the second term in the double-triangle contribution in Eq. (3.11) proportional to $(\lambda_p \cdot \lambda_q)$ integrates to zero and does not contribute to the integrated amplitude. We are therefore free to drop it. We can also modify the first term in the double-triangle integral into the form appearing in Ref. [55] by using the fact that the substitution,

$$(4p\cdot q + 2(p-q)\cdot(k_1+k_2) - s) \to 2(p+q)^2 + s\,, \tag{3.13}$$

does not alter the value of the integrated amplitude: All terms that are proportional to p^2, q^2, $(p-k_1-k_2)^2$, and $(q-k_3-k_4)^2$ yield scale-free integrals that integrate to zero. Finally, to see the equivalence of the two representations, we note that the double triangle (c) has a different color factor from that of the planar double box (a). However, we can convert the double-triangle (c) color factor to the double-box (a) color factor via the color Jacobi identity $C^{\rm DT}_{1234} = C^{\rm P}_{1234} - C^{\rm P}_{2134}$. This matches the color assignment used in Ref. [55]. Although not manifest, the kinematic numerator reflects the antisymmetry of the Jacobi relations so that the additional terms picked up by $A^{\rm P}_{1234}$ and $A^{\rm P}_{2134}$ are simply related by relabelings. Thus, after integration our representation in Eq. (3.10) is equivalent to the one in Eq. (3.6), which comes from Ref. [55].

The integrand in Eq. (3.10) satisfies BCJ duality once we include additional contributions that integrate to zero. To find the full form, we consider Jacobi relations (2.2) around each internal propagator of the planar double box, the nonplanar double box, and the double triangle, as well as all resultant integrals that arise from these Jacobi relations. Duality relations where all three numerators are nonvanishing are depicted in Fig. 3.7. The need for additional nonvanishing numerators depicted in Fig. 3.5(d)-(m) arises from these dual-Jacobi relations. Other sample Jacobi relations where one of the numerators vanishes are shown in Fig. 3.8. Up to relabelings, there are in total 16 such relations involving two nonvanishing numerators and one vanishing numerator. A fully duality-satisfying form is given by the numerators,

$$
\begin{aligned}
\mathcal{P}^{\mathrm{P}}(\lambda_i,p,q,k_i) &= \frac{(D_s-2)^2}{2}(p+q)^2\lambda_p^2\lambda_q^2 + 16s\left((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2\right) \\
&\quad + (D_s-2)\left(s\left(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2\right)+4(p+q)^2\left(\lambda_p^2+\lambda_q^2\right)(\lambda_p\cdot\lambda_q)\right)\,, \\
\mathcal{P}^{\mathrm{NP}}(\lambda_i,p,q,k_i) &= (D_s-2)s\left(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2\right)+16s\left((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2\right)\,, \\
\mathcal{P}^{\mathrm{DT}}(\lambda_i,p,q,k_i) &= \frac{(D_s-2)^2}{2}\left(4p\cdot q+2(p-q)\cdot(k_1+k_2)-s\right)\lambda_p^2\lambda_q^2 \\
&\quad + 8(D_s-2)\left(\lambda_p^2+\lambda_q^2\right)(\lambda_p\cdot\lambda_q)\left(p^2+q^2-(p-q)\cdot(k_1+k_2)+s\right)\,, \\
\mathcal{P}^{(\mathrm{d})}(\lambda_i,p,q,k_i) &= \frac{(D_s-2)^2}{2}(p+q)^2\lambda_p^2\lambda_q^2+4(D_s-2)(p+q)^2\left(\lambda_p^2+\lambda_q^2\right)(\lambda_p\cdot\lambda_q)\,, \\
\mathcal{P}^{(\mathrm{e})}(\lambda_i,p,q,k_i) &= (D_s-2)^2\left(p^2+q^2-(p-q)\cdot k_1\right)\lambda_p^2\lambda_q^2 \\
&\quad + 8(D_s-2)\left(2p\cdot q+(p-q)\cdot k_1\right)\left(\lambda_p^2+\lambda_q^2\right)(\lambda_p\cdot\lambda_q) \\
\mathcal{P}^{(\mathrm{f})}(\lambda_i,p,q,k_i) &= -2(D_s-2)^2(p\cdot k_1)\lambda_p^2\lambda_q^2-16(D_s-2)(q\cdot k_1)\left(\lambda_p^2+\lambda_q^2\right)(\lambda_p\cdot\lambda_q) \\
\mathcal{P}^{(\mathrm{g})}(\lambda_i,p,q,k_i) &= \frac{(D_s-2)^2}{2}\left((p+q)^2\lambda_p^2+p^2\lambda_{p+q}^2\right)\lambda_q^2 \\
&\quad + 4(D_s-2)\left((p+q)^2\left(\lambda_p^2+\lambda_q^2\right)(\lambda_p\cdot\lambda_q)-p^2\left(\lambda_q^2+\lambda_{p+q}^2\right)(\lambda_q\cdot\lambda_{p+q})\right)\,, \\
\mathcal{P}^{(\mathrm{h})}(\lambda_i,p,q,k_i) &= 2(D_s-2)^2\left((p\cdot q)\lambda_p^2+p^2(\lambda_p\cdot\lambda_q)\right)\lambda_q^2 \\
&\quad - 8(D_s-2)\left(3p^2\lambda_q^2-q^2\left(\lambda_p^2+\lambda_q^2\right)\right)(\lambda_p\cdot\lambda_q)\,, \\
\mathcal{P}^{(\mathrm{i})}(\lambda_i,p,q,k_i) &= -\frac{(D_s-2)^2}{2}(4q\cdot k_2+s)\lambda_p^2\lambda_q^2-4(D_s-2)(4p\cdot k_2-s)\left(\lambda_p^2+\lambda_q^2\right)(\lambda_p\cdot\lambda_q)\,, \\
\mathcal{P}^{(\mathrm{j})}(\lambda_i,p,q,k_i) &= 8(D_s-2)s\left(\lambda_p^2+\lambda_q^2\right)(\lambda_p\cdot\lambda_q)\,, \\
\mathcal{P}^{(\mathrm{k})}(\lambda_i,p,q,k_i) &= (D_s-2)^2 t\,\lambda_p^2\lambda_q^2\,,
\end{aligned}
\tag{3.14}
$$

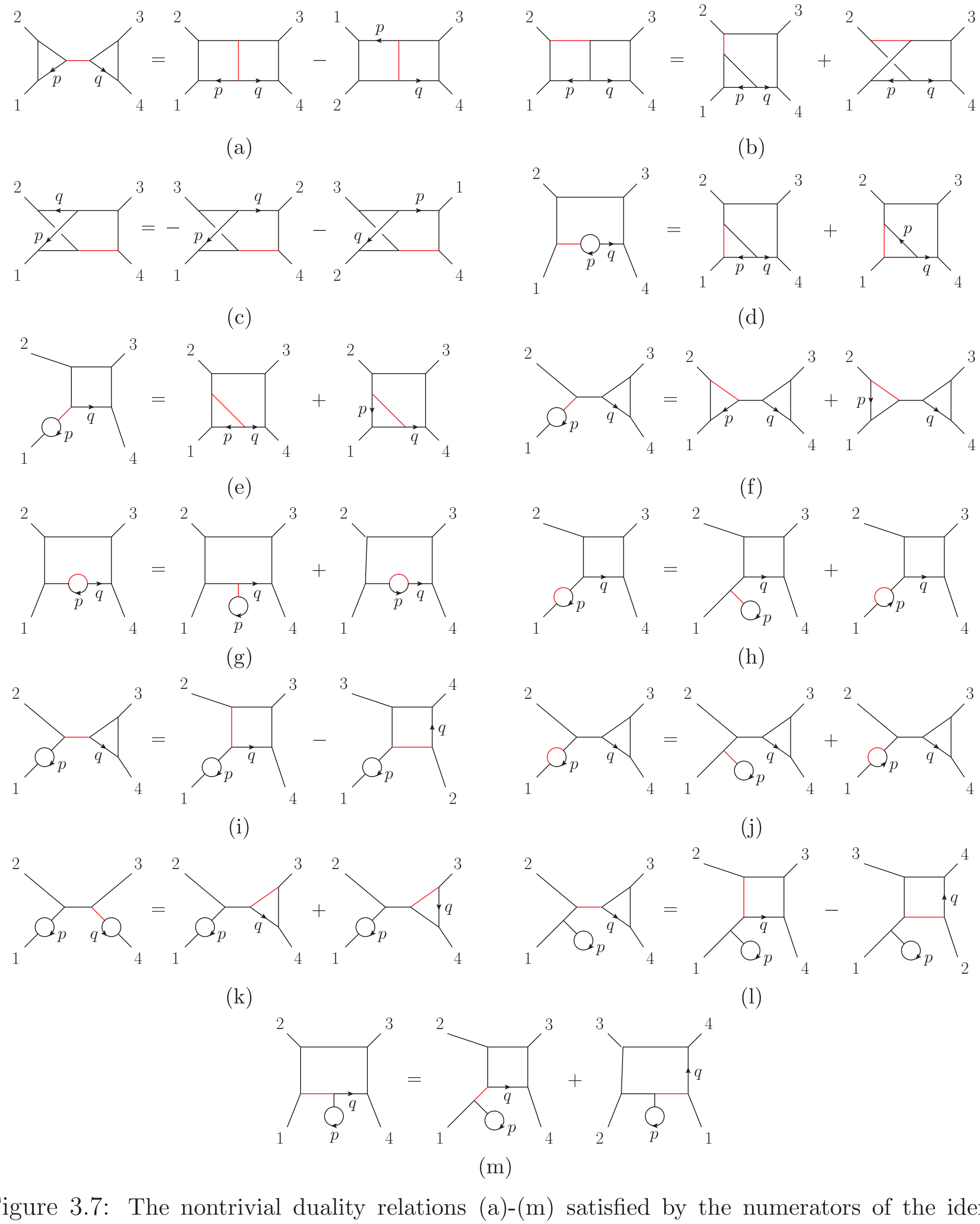

Figure 3.7: The nontrivial duality relations (a)-(m) satisfied by the numerators of the identical-helicity two-loop amplitude. The shaded (red) leg marks the central leg of the applied Jacobi identity.

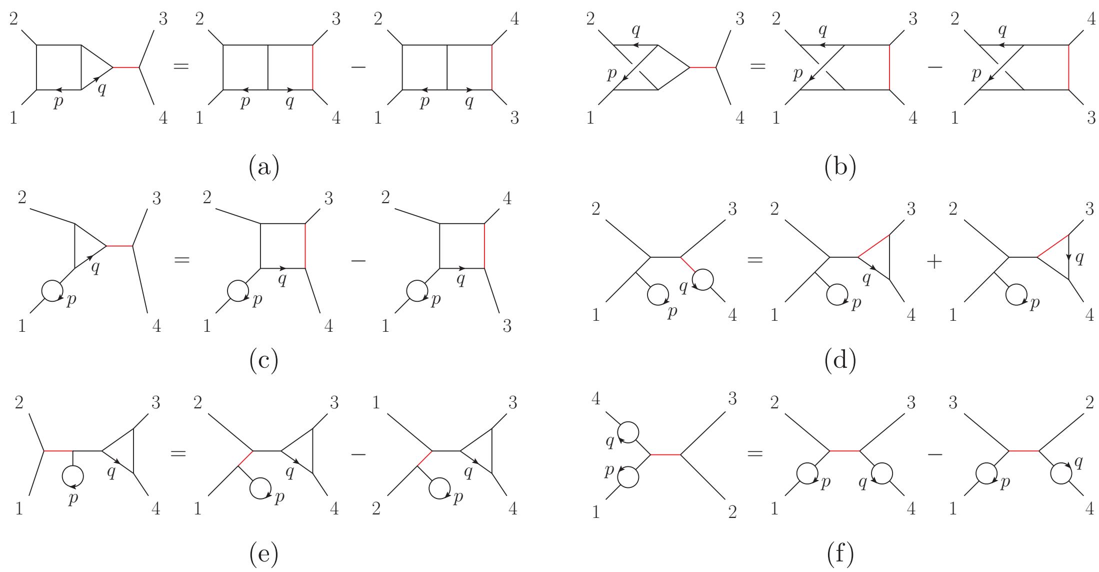

Figure 3.8: Sample duality relations (a)-(f) involving graphs with vanishing numerators. In each relation, the leftmost diagram has a vanishing numerator. The shaded (red) leg marks the central leg of the applied dual-Jacobi identity.

where each $\mathcal{P}^x$ is the numerator of an integral $\mathcal{I}_4^x[\mathcal{P}^x(\lambda_i, p, q, k_i)](s,t)$ corresponding to diagram x, depicted in Fig. 3.5. In contrast to the one-loop case, the duality-satisfying amplitudes do contain tadpole diagrams with nonvanishing numerators.

Although BCJ duality gives us a set of well defined-numerators for all diagrams, those diagrams with on-shell or vanishing intermediate propagators are ill-defined. However, all such ill-defined diagrams diagrams give vanishing contributions after integration. They also do not contribute to the standard two- and three-particle cuts. In more detail, using the numerators from Eq. (3.14), diagrams (d)–(k) in Fig. 3.5 contain scale-free integrals that vanish after integration. Note that diagrams (e), (f), and (h)–(k) are ill-defined. Diagrams (f), (h), and (j) in Fig. 3.5 contain a tadpole subdiagram. We set these to zero, just as they are set to zero in Feynman diagrams since the tadpole integral is scale free in dimensional regularization. Diagrams (e), (i), and (k) are also ill-defined for on-shell external legs because of the propagator carrying an on-shell momentum. With Feynman diagrams, this is normally dealt with by taking the legs off shell; in principle, we can also define an off-shell continuation, although it is nontrivial to do so consistently in our case. However, such ill-defined bubble-on-external-leg contributions again vanish in dimensional regularization, since the integrals are also scale free. In the gauge-theory case, although vanishing, these integrals can potentially contain ultraviolet divergences that cancel completely against infrared divergences. However, in the gravity case, which we are interested in here, the integrals are suppressed by an additional power of the on-shell invariant $k_i^2 = 0$ and therefore lead to ultraviolet divergences with zero coefficient. Diagrams (d) and (g) in Fig. 3.5 may appear to have nonvanishing cut contributions, but inverse propagators in the numerator cancel propagators, again leaving scale-free integrals that vanish.

In summary, the two-loop four-point all-plus-helicity pure Yang-Mills amplitude in a duality-satisfying representation is given by

$$\mathcal{A}_4^{(2)}(1^+, 2^+, 3^+, 4^+) = g^6 \sum_{\mathcal{S}_4} \sum_{x \in \{\text{diagrams}\}} \frac{1}{S^x} C^x_{1234} A^x_{1234} \,, \tag{3.15}$$

where x labels diagrams in Fig. 3.5 with nonvanishing numerators. S^x is the symmetry factor of diagram x, while C^x_{1234} is the color factor. The partial amplitudes are given by

$$A^x_{1234} = i\mathcal{T}\,\mathcal{I}_4^x[\mathcal{P}^x(\lambda_i, p, q, k_i)](s,t)\,, \tag{3.16}$$

where all diagrams except for those in Fig. 3.5(a), (b), (c) integrate to zero in gauge theory. In Section 3.3.2, we will use the double-copy relation (2.5) on these numerators to study the two-loop ultraviolet behavior of gravity coupled to a dilaton and an antisymmetric tensor.

3.3 Ultraviolet Properties of Gravity

We now turn to the ultraviolet properties of the gravity double-copy theory consisting of a graviton, dilaton, and antisymmetric tensor, from the perspective of the double-copy formalism. The theory generated by taking the double copy of pure Yang-Mills corresponds to the low-energy effective Lagrangian of the bosonic part of string theory [66],

$$\mathcal{L} = \sqrt{-g}\left(-\frac{2}{\kappa^2}R + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{6}e^{-2\kappa\phi/\sqrt{D-2}}H_{\mu\nu\rho}H^{\mu\nu\rho}\right), \tag{3.17}$$

where $H_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu}$, and $A_{\mu\nu} = -A_{\nu\mu}$ is the rank-two antisymmetric tensor field.

Pure Einstein gravity is one-loop finite in four dimensions [56]. However, when coupled to a scalar (dilaton) [56] or to a rank-two antisymmetric tensor [58], the theory is divergent. We find that the double-copy theory coupled to both a dilaton and an antisymmetric tensor is also divergent, although for all these theories the four-point amplitudes with at least one external graviton are finite, as expected from simple counterterm arguments. We will show that the cancellation no longer holds at two loops, and the theory has an R^3 counterterm, in much the same way as it does for pure Einstein gravity [67]. In six dimensions, pure Einstein gravity is ultraviolet divergent at one loop [59]. We find the same to be true in our double-copy theory, and we find a divergence in eight dimensions as well. We will give the explicit form of the divergences for these cases. In carrying out these computations we use the four-dimensional helicity scheme [62, 68]. It would be interesting to compare our results to ones obtained using the standard dimensional-regularization scheme, used in, for example, Ref. [67].

3.3.1 One Loop

3.3.1.1 Four Dimensions

In four dimensions, there is no one-loop four-point divergence when one external leg is a graviton [56, 58] because the potential independent counterterms for such divergences vanish on shell or can be eliminated by the equations of motion. Using the double-copy formula (2.5), we have explicitly confirmed finiteness in one-loop four-point amplitudes containing at least one external graviton, with the remaining legs either gravitons, dilatons, or antisymmetric tensors. We obtain the gravity numerator from the double-copy formula (2.5) by taking the two Yang-Mills numerators, $\tilde{n}_i$ and n_i, to be equal to the BCJ form of the Yang-Mills numerator (3.5). As an interesting cross check, we have obtained an asymmetric representation of the gravity amplitudes by taking the $\tilde{n}_i$ to be the numerators that satisfy BCJ duality and the n_i to be numerators obtained by gauge-theory Feynman rules in Feynman gauge, similar to the procedure used recently for half-maximal supergravity [38, 31]. By generalized gauge invariance [11, 15], this should be equivalent to the symmetric construction. Indeed, we find identical results for the ultraviolet divergences.

To evaluate the ultraviolet divergences, we expand in small external momenta to reduce to logarithmically divergent integrals [64]. We then simplify tensor integrals composed of loop momenta in the numerators by using Lorentz invariance, which implies that the integrals must be linear combinations of products of metric tensors $\eta^{\mu\nu}$. (See Ref. [20] for a recent discussion of evaluating tensor vacuum integrals.) With the insertion of a massive infrared regulator, we finally integrate simple one-loop integrals to find the potential ultraviolet divergence. Due to our auxiliary conditions, contributions from bubbles on external legs vanish, as they would for ordinary gravity Feynman diagrams. We therefore obtain our entire result from box, triangle, and bubble-on-internal-leg diagrams.

For completeness we have also computed the divergences directly in dimensional regularization without introducing a mass regulator, using techniques similar to those for two loops in Appendix A. After subtracting the infrared divergence as computed in Appendix B, we find complete agreement with our result found using vacuum integrals.

We obtain an expression for the divergence in terms of formal polarization vectors. By taking linear combinations of the product of polarization vectors from each copy of Yang-Mills, we can project onto the graviton, dilaton, and antisymmetric tensor states. In $D = 4$ this is conveniently implemented by using spinor helicity [69]. Graviton polarization tensors correspond to the 'left' and 'right' copies of Yang-Mills according to $\varepsilon^{h+}_{\mu\nu} \to \varepsilon^{+}_{\mathrm{L}\mu}\varepsilon^{+}_{\mathrm{R}\nu}$ and $\varepsilon^{h-}_{\mu\nu} \to \varepsilon^{-}_{\mathrm{L}\mu}\varepsilon^{-}_{\mathrm{R}\nu}$. For the dilaton and antisymmetric tensor, we symmetrize and antisymmetrize in opposite-helicity configurations according to $\varepsilon^{\phi}_{\mu\nu} \to \frac{1}{\sqrt{2}}(\varepsilon^{+}_{\mathrm{L}\mu}\varepsilon^{-}_{\mathrm{R}\nu} + \varepsilon^{-}_{\mathrm{L}\mu}\varepsilon^{+}_{\mathrm{R}\nu})$ and

$\varepsilon^A_{\mu\nu} \to \frac{1}{\sqrt{2}}(\varepsilon^+_{\text{L}\mu}\varepsilon^-_{\text{R}\nu} - \varepsilon^-_{\text{L}\mu}\varepsilon^+_{\text{R}\nu})$. By substituting the explicit polarizations, we find that all configurations where at least a single leg is a graviton are free of ultraviolet divergences,

$$\mathcal{M}^{(1)}(1^h, 2, 3, 4)\Big|_{\text{div.}} = 0\,, \tag{3.18}$$

where leg 1 is either a positive- or negative-helicity graviton, and the other three states are unspecified.

We however find divergences for the cases with no external gravitons. For the four-dilaton amplitude, we find

$$\mathcal{M}^{(1)}(1^\phi, 2^\phi, 3^\phi, 4^\phi)\Big|_{\text{div.}} = \frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{i}{(4\pi)^2}\frac{1132 - 92D_s + 3D_s^2}{120}\left(s^2 + t^2 + u^2\right), \tag{3.19}$$

corresponding to the operator,

$$\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{1}{(4\pi)^4}\frac{1132 - 92D_s + 3D_s^2}{240}(D_\mu\phi D^\mu\phi)^2\,. \tag{3.20}$$

This result is similar to the one obtained long ago by 't Hooft and Veltman [56]. However, in our case we have an antisymmetric tensor which can circulate in the loop, altering the numerical coefficient. We note that the operator in Ref. [56] looks different than above, but it can be written in a similar way through use of the field equations of motion.

The amplitude with four antisymmetric tensors is also one-loop divergent in four dimensions. In four dimensions, the antisymmetric tensor is dual to a scalar field, so we expect the divergence to be the same as that for dilatons. Indeed, the divergence in the four-antisymmetric-tensor amplitude for a theory with an antisymmetric tensor coupled to gravity is equal to that of the four-dilaton amplitude in a theory of a dilaton coupled to gravity [58]. In congruence, we find the divergence for four external antisymmetric tensors to also be given by the same expression as the four-dilaton divergence (3.19),

$$\mathcal{M}^{(1)}(1^A, 2^A, 3^A, 4^A)\Big|_{\text{div.}} = \mathcal{M}^{(1)}(1^\phi, 2^\phi, 3^\phi, 4^\phi)\Big|_{\text{div.}}\,. \tag{3.21}$$

In terms of the antisymmetric tensor fields, the divergence is generated by the operator,

$$\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{1}{(4\pi)^4}\frac{1132-92D_s+3D_s^2}{2160}\left(H_{\mu\nu\rho}H^{\mu\nu\rho}\right)^2. \tag{3.22}$$

The counterterm that cancels the divergence is given by the negative of this operator.

In addition to the above divergences, there is also a divergence in the $D=4$, $\phi\phi AA$ amplitude. This divergence is given by

$$\begin{aligned}\mathcal{M}^{(1)}(1^\phi,2^\phi,3^A,4^A)\Big|_{\text{div.}} \;=\; & \frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{i}{(4\pi)^4}\Bigg(\frac{1116-76D_s-D_s^2}{120}\,s^2 \\ & + \frac{-1124+84D_s-D_s^2}{120}\,(t^2+u^2)\Bigg),\end{aligned} \tag{3.23}$$

which corresponds to the operator,

$$\begin{aligned}\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^4 \frac{1}{(4\pi)^4}\Bigg(&\frac{1124-84D_s+D_s^2}{60}\,H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}D^\mu\phi D^\nu\phi \\ & - \frac{1132-92D_s+3D_s^2}{360}\,H_{\mu\nu\rho}H^{\mu\nu\rho}D_\sigma\phi D^\sigma\phi\Bigg).\end{aligned} \tag{3.24}$$

3.3.1.2 Six Dimensions

In six dimensions for external gravitons, the only independent invariant operator at one loop [70] is

$$R_{\alpha\beta\mu\nu}R^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}. \tag{3.25}$$

This corresponds to the known $D=6$ one-loop divergence of pure Einstein gravity given in Ref. [59]. We have computed the coefficient of the $D=6$ divergence for the double-copy theory of a graviton coupled to a dilaton and an antisymmetric tensor. In this case, the divergence is given by the operator,

$$-\frac{1}{\epsilon}\frac{1}{(4\pi)^3}\frac{(D_s-2)^2}{30240}R_{\alpha\beta\mu\nu}R^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}. \tag{3.26}$$

Appropriate powers of the coupling are generated by expanding the metric around flat space, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Although we do not include the explicit forms of the counterterms here, we have also found divergences for the following amplitudes (as well as their permutations and parity conjugates) involving external dilatons and antisymmetric tensors, where we restrict the external states to four dimensions:

$$\mathcal{M}^{(1)}(1^\phi, 2^+, 3^+, 4^+)\,,\ \mathcal{M}^{(1)}(1^\phi, 2^\phi, 3^+, 4^+)\,,\ \mathcal{M}^{(1)}(1^\phi, 2^\phi, 3^\phi, 4^+)\,,\ \mathcal{M}^{(1)}(1^\phi, 2^\phi, 3^\phi, 4^\phi)\,,$$
$$\mathcal{M}^{(1)}(1^A, 2^A, 3^+, 4^+)\,,\ \mathcal{M}^{(1)}(1^A, 2^A, 3^\phi, 4^+)\,,\ \mathcal{M}^{(1)}(1^A, 2^A, 3^\phi, 4^\phi)\,. \tag{3.27}$$

3.3.1.3 Eight Dimensions

In eight dimensions, there are seven linearly independent R^4 operators [71]:

$$\begin{aligned}
T_1 &= (R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2\,,\\
T_2 &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\lambda}R_{\gamma\delta\kappa}{}^{\sigma}R^{\gamma\delta\kappa\lambda}\,,\\
T_3 &= R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\gamma}R^{\lambda\gamma}{}_{\delta\kappa}R^{\rho\sigma\delta\kappa}\,,\\
T_4 &= R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\gamma}R^{\rho\lambda}{}_{\delta\kappa}R^{\sigma\gamma\delta\kappa}\,,\\
T_5 &= R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\lambda\gamma}R^{\rho}{}_{\delta}{}^{\lambda}{}_{\kappa}R^{\sigma\delta\gamma\kappa}\,,\\
T_6 &= R_{\mu\nu\rho\sigma}R^{\mu}{}_{\lambda}{}^{\rho}{}_{\gamma}R^{\lambda}{}_{\delta}{}^{\gamma}{}_{\kappa}R^{\nu\delta\sigma\kappa}\,,\\
T_7 &= R_{\mu\nu\rho\sigma}R^{\mu}{}_{\lambda}{}^{\rho}{}_{\gamma}R^{\lambda}{}_{\delta}{}^{\nu}{}_{\kappa}R^{\gamma\delta\sigma\kappa}\,.
\end{aligned} \tag{3.28}$$

On shell, the combination,

$$U = -\frac{T_1}{16} + T_2 - \frac{T_3}{8} - T_4 + 2T_5 - T_6 + 2T_7\,, \tag{3.29}$$

is a total derivative, so only six of the T_i are independent on shell. In terms of these operators, the divergence for gravity coupled to a dilaton and an antisymmetric tensor at one loop in

$D = 8$ is

$$\frac{1}{\epsilon}\frac{1}{(4\pi)^4}\frac{1}{1814400}\Big[(4274 - 899D_s + 11D_s^2)T_1 - 40(466 - 103D_s - 2D_s^2)T_2 \\ - 2(1886 + 319D_s - D_s^2)T_3 - 180(1034 + D_s)T_4 \\ + 16(1196 + 34D_s - D_s^2)T_6 + 64(12454 + 71D_s + D_s^2)T_7 + c\,U\Big]\,, \tag{3.30}$$

where c is a free parameter multiplying the total derivative (3.29) .

We have also found that the following four-point amplitudes involving dilatons and antisymmetric tensors diverge in $D = 8$:

$$\mathcal{M}^{(1)}(1^\phi, 2^\phi, 3^+, 4^+)\,, \quad \mathcal{M}^{(1)}(1^\phi, 2^\phi, 3^+, 4^-)\,, \quad \mathcal{M}^{(1)}(1^\phi, 2^\phi, 3^\phi, 4^\phi)\,, \quad \mathcal{M}^{(1)}(1^A, 2^A, 3^+, 4^+)\,, \\ \mathcal{M}^{(1)}(1^A, 2^A, 3^+, 4^-)\,, \quad \mathcal{M}^{(1)}(1^A, 2^A, 3^\phi, 4^+)\,, \quad \mathcal{M}^{(1)}(1^A, 2^A, 3^\phi, 4^\phi)\,, \quad \mathcal{M}^{(1)}(1^A, 2^A, 3^A, 4^A)\,, \tag{3.31}$$

where we have again chosen the external states to be four dimensional. The other configurations are finite.

3.3.2 Two Loops in Four Dimensions

Pure Einstein gravity in $D = 4$ is one-loop finite, but it does diverge at two loops [67]. This suggests that the two-loop four-graviton amplitude, including also the dilaton and antisymmetric tensor, should diverge as well. For external gravitons, the only independent operator is the same R^3 operator for one loop in six dimensions (3.25). Our aim is to find its coefficient.

The R^3 operator generates a nonvanishing four-point amplitude for identical-helicity gravitons, illustrated in Fig. 3.9. This means that we can determine the coefficient of this operator by computing the four-graviton all-plus-helicity amplitude. Fortunately, as we discussed in Section 3.2, we have the BCJ form of the required all-plus-helicity Yang-Mills amplitude. Applying the double-copy formula (2.5) to the Yang-Mills amplitude in Eq. (3.15)

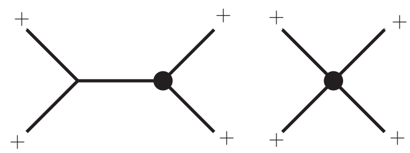

Figure 3.9: The R^3 operator diagrams that contribute to the all-plus-helicity four-graviton amplitude. The solid dot represents vertices generated by the R^3 operator.

immediately gives us the corresponding gravity integrand, simply by squaring the numerators. Diagrams (d)-(k) in Fig. 3.5 integrate to zero in gravity just as they did in Yang-Mills. In addition, as was mentioned in Section 3.2.2, the second term of the double-triangle in Eq. (3.11) also integrates to zero; in fact, due to the simple identity,

$$-(p-q)\cdot(k_1+k_2)+s=\frac{1}{2}(p-k_1-k_2)^2+\frac{1}{2}(q+k_1+k_2)^2-\frac{1}{2}p^2-\frac{1}{2}q^2\,, \tag{3.32}$$

all such terms will integrate to zero because the inverse propagators lead to scale-free integrals. Thus, the four-graviton all-plus-helicity amplitude is given by

$$\mathcal{M}^{(2)}(1^+,2^+,3^+,4^+)=\left(\frac{\kappa}{2}\right)^6\sum_{\mathcal{S}_4}\left[\frac{1}{4}M^{\rm P}_{1234}+\frac{1}{4}M^{\rm NP}_{12;34}+\frac{1}{8}M^{\rm DT}_{1234}\right]\,, \tag{3.33}$$

where

$$
\begin{aligned}
M^{\mathrm{P}}_{1234} &= i\,\mathcal{T}^2\,\mathcal{I}^{\mathrm{P}}_4\Bigg[\bigg(\frac{(D_s-2)^2}{2}(p+q)^2\lambda_p^2\lambda_q^2 + 16s\left((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2\right)\\
&\qquad + (D_s-2)\Big(s\left(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2\right)+4(p+q)^2(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)\Big)\bigg)^2\Bigg](s,t)\\
&= i\,\mathcal{T}^2\Bigg\{\mathcal{I}^{\mathrm{P}}_4\Big[\Big((D_s-2)s\left(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2\right)+16s\left((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2\right)\Big)^2\Big](s,t)\\
&\quad + \mathcal{I}^{\text{bow-tie}}_4\Bigg[2\bigg(4(D_s-2)(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)+\frac{(D_s-2)^2}{2}\lambda_p^2\lambda_q^2\bigg)\\
&\qquad\qquad \times\left((D_s-2)s\left(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2\right)+16s\left((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2\right)\right)\Bigg](s,t)\\
&\quad + \mathcal{I}^{\text{bow-tie}}_4\Bigg[(p+q)^2\bigg(\frac{(D_s-2)^2}{2}\lambda_p^2\lambda_q^2+4(D_s-2)(\lambda_p^2+\lambda_q^2)(\lambda_p\cdot\lambda_q)\bigg)^2\Bigg](s,t)\Bigg\}\,,\\
M^{\mathrm{NP}}_{12;34} &= i\,\mathcal{T}^2 s^2\,\mathcal{I}^{\mathrm{NP}}_4\Big[\Big((D_s-2)\left(\lambda_p^2\lambda_q^2+\lambda_p^2\lambda_{p+q}^2+\lambda_q^2\lambda_{p+q}^2\right)+16\left((\lambda_p\cdot\lambda_q)^2-\lambda_p^2\lambda_q^2\right)\Big)^2\Big](s,t)\,,\\
M^{\mathrm{DT}}_{1234} &= i\,\mathcal{T}^2\,\mathcal{I}^{\mathrm{DT}}_4\Bigg[\bigg(\frac{(D_s-2)^2}{2}\left(4p\cdot q+2(p-q)\cdot(k_1+k_2)-s\right)\lambda_p^2\lambda_q^2\bigg)^2\Bigg](s,t)\\
&= i\,\mathcal{T}^2\frac{1}{s}\mathcal{I}^{\text{bow-tie}}_4\Bigg[\bigg(\frac{(D_s-2)^2}{2}\left(4p\cdot q+2(p-q)\cdot(k_1+k_2)-s\right)\lambda_p^2\lambda_q^2\bigg)^2\Bigg](s,t)\,. \qquad (3.34)
\end{aligned}
$$

We have explicitly confirmed that s-, t-, and u-channel unitarity cuts are satisfied. We did so numerically keeping the internal states in integer dimensions $D=6$ and $D=8$.

To obtain the ultraviolet divergences, we integrate the amplitudes in dimensional regularization. We carry out the extraction of the ultraviolet divergences in two ways. In the first approach we simply use dimensional regularization and then subtract the known infrared divergences, leaving only the ultraviolet ones. In the second approach we introduce a mass regulator to separate the ultraviolet singularities from the infrared divergences, as carried out in Appendix C. Either method yields the same result. In fact, the second method also shows that the vanishing integrals that we dropped, including diagrams (d)-(k) in Fig. 3.5 and the second term of the double-triangle in Eq. (3.11), are not ultraviolet divergent.

The dimensionally regularized integrals are performed in Appendix A.[2] Eq. (A.10) gives

[2]We thank L. Dixon for cross-checking our integrals.

the planar double-box integrals; Eq. (A.17) gives the nonplanar double-box integrals; and Eq. (A.21) gives the bow-tie integrals. The infrared divergence from Appendix B is

$$\begin{aligned}\mathcal{M}^{(2)}(1^+,2^+,3^+,4^+)\big|_{\text{IR div.}} = & -\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\mathcal{T}^2\,\frac{(D_s-2)^2}{120}\left(s^2+t^2+u^2\right) \\ & \times\left[s\log\left(\frac{-s}{\mu^2}\right)+t\log\left(\frac{-t}{\mu^2}\right)+u\log\left(\frac{-u}{\mu^2}\right)\right]. \end{aligned} \tag{3.35}$$

We insert the divergent parts of the integrals evaluated using dimensional regularization into Eq. (3.34), then insert these results into Eq. (3.33) and perform the permutation sum. Finally we subtract the infrared divergence and arrive at the two-loop ultraviolet divergence of gravity coupled to a dilaton and an antisymmetric tensor for four external positive-helicity gravitons:

$$\begin{aligned}\mathcal{M}^{(2)}(1^+,2^+,3^+,4^+)\big|_{\text{UV div.}} = & \frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\mathcal{T}^2 \\ & \times\frac{(2D_s^4-136D_s^3+2883D_s^2-35164D_s+103052)stu}{10800}. \end{aligned} \tag{3.36}$$

For our second method, we evaluate the ultraviolet divergences of the required integrals by going to vacuum integrals and using a massive infrared regulator, sidestepping the need to subtract the infrared divergence. The ultraviolet divergences of the individual integrals are calculated in Appendix C. After permutations, the contributions of the planar double-box, nonplanar double-box, and double-triangle components are

$$\begin{aligned}\mathcal{M}^{\text{P}}(1^+,2^+,3^+,4^+)\big|_{\text{UV div.}} &= -\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\mathcal{T}^2\,\frac{(2D_s^3-63D_s^2+588D_s-1420)stu}{180}, \\ \mathcal{M}^{\text{NP}}(1^+,2^+,3^+,4^+)\big|_{\text{UV div.}} &= -\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\mathcal{T}^2\,\frac{(21D_s^2-4D_s-396)stu}{240}, \\ \mathcal{M}^{\text{DT}}(1^+,2^+,3^+,4^+)\big|_{\text{UV div.}} &= \frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\mathcal{T}^2\,\frac{(D_s-2)^4 stu}{5400}. \end{aligned} \tag{3.37}$$

Summing these contributions, we find complete agreement with Eq. (3.36).

We can re-express the two-loop divergence in terms of the operator that generates it. By matching the amplitude generated by the diagrams with an R^3 vertex shown in Fig. 3.9 to the divergence in Eq. (3.36), we find that the operator,

$$-\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^4}\frac{2D_s^4 - 136D_s^3 + 2883D_s^2 - 35164D_s + 103052}{648000} R_{\alpha\beta\mu\nu}R^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}, \qquad (3.38)$$

generates the two-loop divergence for gravity coupled to a dilaton and an antisymmetric tensor.

3.4 Conclusions

In this chapter, we constructed a representation of the one-loop four-point amplitude of pure Yang-Mills theory explicitly exhibiting the duality between color and kinematics. This construction is the first nonsupersymmetric example at loop level valid in any dimension with no restriction on the external states. The cost of this generality is relatively complicated expressions in terms of formal polarization vectors.

The duality between color and kinematics and its associated gravity double-copy structure has proven useful for unraveling ultraviolet properties in various dimensions [11, 20, 38, 31, 46]. Using the one-loop four-point pure Yang-Mills amplitude with the duality manifest, we obtained the integrand for the corresponding amplitude in a theory of a graviton, dilaton, and antisymmetric tensor. In $D = 4$, we found that one-loop four-point amplitudes with one or more external gravitons are ultraviolet finite, while amplitudes involving only external dilatons or antisymmetric tensor fields diverge. This result is similar to those of earlier studies involving gravity coupled either to a scalar, an antisymmetric tensor, or other matter and is in line with simple counterterm arguments [56, 57, 58]. We gave the explicit form, including numerical coefficients, for all four-point divergences in this theory. Since our construction is valid in any dimension, we also investigated the ultraviolet properties of the double-copy theory in higher dimensions. In particular, we showed that in $D = 6, 8$ the one-loop four-graviton amplitudes diverge, as expected, and gave the explicit form of these divergences including their numerical coefficients.

In order to investigate whether the observed $D = 4$ ultraviolet finiteness of the amplitudes with one or more external gravitons continues beyond one loop, we also computed the coefficient of the potential two-loop R^3 divergences. This was greatly simplified by the observation that the coefficient of the divergence can be determined from the identical-helicity four-graviton configuration. The required gravity amplitude was then easily constructed via the double-copy property, by first finding a representation of the pure Yang-Mills amplitude that satisfies the duality. The existence of such a representation has already been noted in Ref. [11]. Here we provided the explicit representation, including diagrams that integrate to zero not present in the original form of the two-loop identical-helicity amplitude given in Ref. [55]. We found that the two-loop amplitude with external gravitons is indeed divergent and that the R^3 counterterm has nonzero coefficient. This is not surprising given that pure Einstein gravity diverges at two loops [67]. This chapter definitively shows that, as one might have expected, the double-copy property by itself cannot render a gravity theory ultraviolet finite. For ultraviolet finiteness, an additional mechanism such as supersymmetry is needed. Further progress in clarifying the ultraviolet structure of gravity theories will undoubtedly rely on new multiloop calculations to guide theoretical developments. We expect that the duality between color and kinematics will continue to play an important role in this.

We note that after the work of this chapter was completed, a subsequent analysis has revealed that at two loops there are evanescent counterterms not accounted for in our two-loop analysis. The appearance of such counterterms is unexpected because there are no corresponding one-loop divergences in $D = 4$ that can act as subdivergences. The net effect is that the numerical coefficient of the divergence in Eq. (3.38) should be interpreted as a bare result without counterterm or subdivergence subtractions. Consequently, the coefficient will be modified, although the conclusion that there is a divergence is unaltered. This surprising phenomenon, as well as the counterterm subtraction terms, will be described in Chapter 8.

CHAPTER 4

Color-Kinematics Duality in One-Loop Four-Gluon Amplitudes with Matter

4.1 Introduction

At loop level, the duality between color and kinematics has been extensively studied for supersymmetric cases but less so for the nonsupersymmetric case. In particular, four-point one-loop amplitudes in nonsupersymmetric ($\mathcal{N} = 0$) Yang-Mills (YM) theory that satisfy the BCJ duality have been constructed in Chapter 3. They are valid in arbitrary dimensions and written in terms of formal polarization vectors—i.e., the external states are not in a helicity basis. The $\mathcal{N} = 0$ YM theory of Chapter 3 contains only gluons. Here, we extend that work by constructing duality-satisfying amplitudes that are valid for any adjoint matter content circulating in the loop, still with external gluons. The construction closely follows that of Chapter 3. We begin by building an ansatz for the amplitude in D dimensions. We use formal polarization vectors instead of dimension-specific helicity states. The ansatz is then constrained to satisfy the BCJ duality. Furthermore, we demand that the kinematic numerator factors of the ansatz obey the same relabeling symmetries as their corresponding diagrams. Using a D-dimensional variant [33, 34] of the unitarity method [35], we enforce that the ansatz obey the same unitarity cuts as the amplitude under consideration. These D-dimensional unitarity cuts completely determine the integrated amplitude.

Because we have both gluon- and fermion-loop contributions, we can compare our results to previously obtained one-loop duality-satisfying amplitudes with four-dimensional external helicity states. Namely, we look at the $\mathcal{N} = 4$ super-Yang-Mills (sYM) amplitude of Ref. [72], the maximally-helicity-violating (MHV) $\mathcal{N} = 1$ (chiral) sYM amplitude of Ref. [25], and the all-plus-helicity $\mathcal{N} = 0$ YM amplitude of Ref. [33]. We compare our results to the earlier work by restricting the external states to four dimensions and putting the formal polarization vectors into a helicity basis. While going to a helicity basis considerably simplifies our MHV $\mathcal{N} = 1$ (chiral) BCJ numerators, the all-plus-helicity $\mathcal{N} = 0$ numerators are not particularly simplified, because they contain complicated terms that integrate to zero.

We organize this chapter as follows. In Section 4.2, we present the BCJ numerators with adjoint matter content circulating in the loop. These individual contributions are then combined to construct BCJ numerators for the theories of $\mathcal{N} = 4$ sYM, $\mathcal{N} = 1$ (chiral) sYM, and $\mathcal{N} = 0$ YM. We show the simplification in combining our formal-polarization BCJ numerators into the four-dimensional supersymmetric theories of $\mathcal{N} = 4$ sYM and $\mathcal{N} = 1$ (chiral) sYM in Appendix D. In Section 4.3, we review one technique for putting formal polarization vectors into a helicity basis, with a slight digression found in Appendix E. In Section 4.4, we compare our BCJ numerators with the existing literature. Finally, in Section 4.5, we present our conclusions.

4.2 Formal-Polarization BCJ Numerators

In this section, we find the BCJ numerators for adjoint fermions and adjoint scalars circulating in the four-point one-loop box diagram—Fig. 4.1(a)—with external gluons. For completeness, we also provide the expression for a gluon in the loop. The box numerators that we give are for the external-leg ordering $(1, 2, 3, 4)$ and with the loop momentum labeling convention $p_1 \equiv p$, where p_1 is shown in Fig. 4.1(a). The other BCJ numerators, such as those displayed in Figs. 4.1(b-g), are found by solving the numerator Jacobi relations of Eq. (2.2). Figs. 4.2 and 4.3 show the Jacobi relations diagrammatically. We note that the right-hand sides of Figs. 4.2 and 4.3 can be written solely in terms of boxes. In these functional numerator relations, we encounter box numerators with different external-leg orderings and loop-momentum labels. However, we demand that these numerators are simply

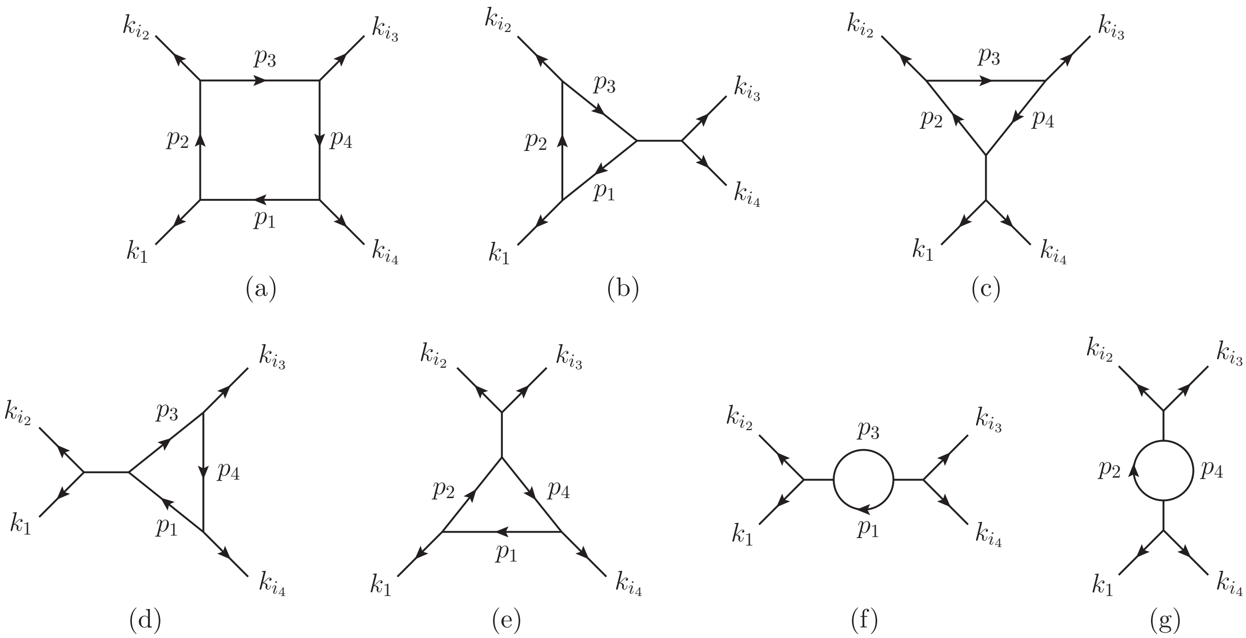

Figure 4.1: The labeling convention that we employ both for numerators with formal polarization vectors and for color-ordered amplitudes. The external legs have the ordering $(1, i_2, i_3, i_4)$, with outgoing momenta k_1, k_{i_2}, k_{i_3}, k_{i_4}. The loop momentum is denoted by $p_1 \equiv p$, while p_2, p_3, and p_4 are given by momentum conservation.

relabelings of the box numerator that we give. (In this procedure, the polarization vectors must of course be relabeled in addition to the external momenta and the loop momentum.) We also demand that the box numerator is unchanged under the three rotation relabelings and four reflection relabelings—the automorphisms of the box diagram. The other numerators have analogous relabeling properties, which follow from the color-kinematics duality.

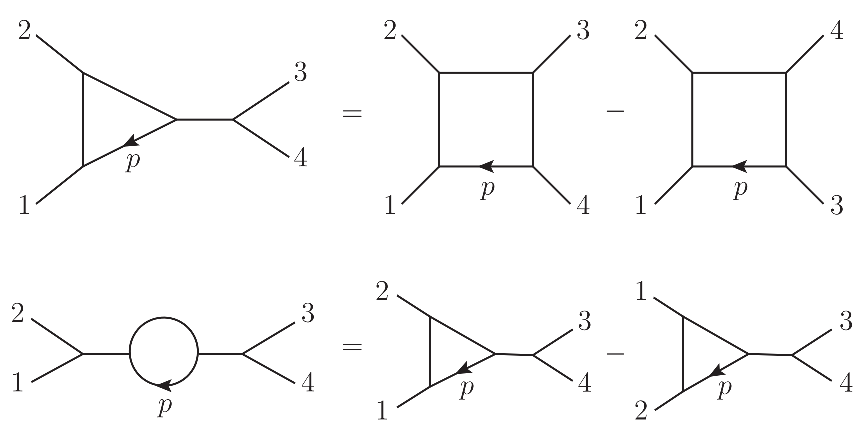

Figure 4.2: The Jacobi relations determining either color or kinematic numerators of the four-point diagrams containing either a triangle or internal bubble.

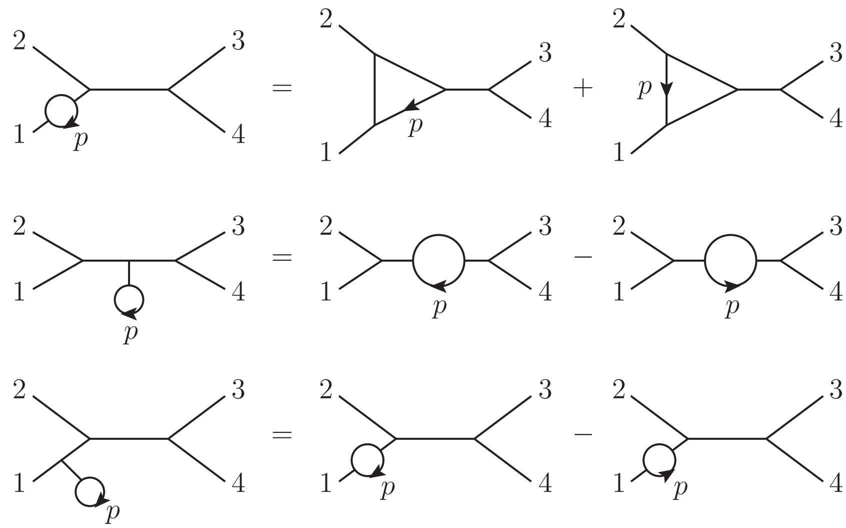

Figure 4.3: The color or kinematic Jacobi relations involving a bubble on an external leg or a tadpole. These diagrams have vanishing contribution to the integrated amplitude.

The construction of these BCJ numerators closely follows that of Chapter 3. As we will discuss, we generalize the constraints of Chapter 3 to accommodate matter in the loop, and we also make additional constraints on internal bubble numerators (Figs. 4.1(f,g)) with supersymmetry in mind. First, we build an ansatz for the box numerator with external-leg ordering $(1, 2, 3, 4)$. The ansatz is a sum of all (*468*) possible terms, each with an undetermined coefficient. Next, we impose the color-kinematics duality and the relabeling properties mentioned above. This allows us to generate the other numerators needed to construct the color-ordered amplitudes of Eq. (2.6). Then, we enforce that these amplitudes obey the appropriate two-particle D-dimensional unitarity cuts of Fig. 4.4. Fig. 4.1 displays the seven diagrams that contribute to at least one of the two two-particle unitarity cuts of the color-ordered amplitude $A_4^{(1)}(1, i_2, i_3, i_4)$. Because of our relabeling properties, we need only to consider one of the color-ordered amplitudes, say $A_4^{(1)}(1, 2, 3, 4)$. Imposing the duality, relabeling, and cut conditions fixes *447* of the 468

To clean up the expression, we the fix *12* of the remaining 21 coefficients by demanding that all tadpole numerators vanish prior to integration. That is, the left-hand side of the

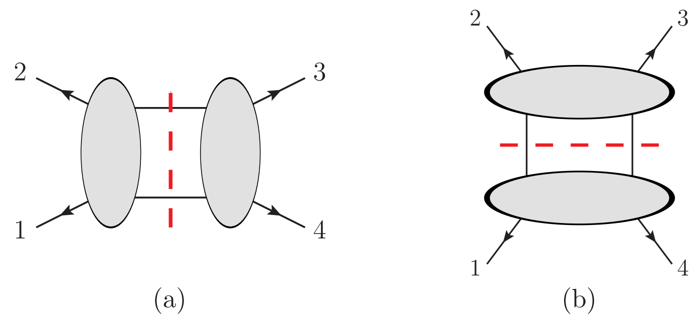

Figure 4.4: The two two-particle unitarity cuts in which the exposed internal propagators are put on shell. The one-loop contributions to cut (a) come from Figs. 4.1(a,b,d,f), and the one-loop contributions to cut (b) come from Figs. 4.1(a,c,e,g). Diagrams with a bubble on an external leg and diagrams that contain a tadpole do not contribute to either cut.

bottom two equations of Fig. 4.3 are set to zero. (In fact, solving just the bottom equation in the figure is sufficient.) Because the tadpole integrals are scale-free in dimensional regularization, they vanish regardless of the coefficient choice (see Ref. [73]). An important benefit of imposing that the tadpole numerators vanish *prior* to integration is that the maximum power of loop momentum in each BCJ numerator is p^V, where V is the number of vertices in the loop. When supersymmetry is present, the maximum power is reduced to no more than p^{V-2}, with $V-2 \geq 0$. (At one loop, this well-known improved power counting can be seen by using the second-order formalism for the fermion loop [74] and the background-field gauge for the gluon loop.)

We now fix *four* additional coefficients so that the integrals arising from the diagram with a bubble on external-leg 1—the first diagram of Fig. 4.3—are well-defined. (Our relabeling properties ensure that the integrals from bubbles on different external legs are also be well-defined.) In general, in the on-shell limit the intermediate propagator, $1/k_1^2 \sim 1/0$, can cause the the integrals to be ill-defined. Feynman diagrams avoid this because each term in the bubble-on-external-leg-1 Feynman kinematic numerator contains at least one of the following scalar products

$$p \cdot \varepsilon_1, \qquad p \cdot k_1, \qquad p^2. \tag{4.1}$$

This constraint along with the associated current conservation of the vacuum polarization ensures that the power of k_1^2 after integration is no lower than $(k_1^2)^{(D-4)/2}$. (In these integrals, we use the prescription where k_1^2 is not put on shell until the end of the calculation.) With no powers of k_1^2 in the denominator for $D \geq 4$, the expression is well-defined. Thus, we demand that each term in the bubble-on-external-leg-1 BCJ kinematic numerator contains at least one of these scalar products, following the structure found with ordinary Feynman-gauge Feynman diagrams. We now expound on this subtle restriction (see also Chapter 3).

Fig. 4.3 shows that there are no terms with an odd power of loop momentum in the bubble-on-external-leg-1 numerator due to the vanishing-tadpole condition. We choose coefficients so as to eliminate terms with no loop momentum. Now, only terms quadratic in the loop momentum remain, and they in fact contain at least one of the scalar products of Eq. (4.1). By Lorentz invariance, we have

$$\int \frac{d^D p}{(2\pi)^D} \frac{p^\mu \, p^\nu}{k_1^2 \, p^2 \, (p-k_1)^2} = \frac{1}{k_1^2} \left(g^{\mu\nu} k_1^2 \, A + k_1^\mu k_1^\nu \, B \right) , \tag{4.2}$$

where A and B are scalar integrals. If one of the loop momentum vectors of Eq. (4.2) is contracted with ε_1, then $k_1 \cdot \varepsilon_1$ appears in the prefactor of integral B. This vanishes immediately, so these terms cause no problem. Aside from $k_1 \cdot \varepsilon_1$, simple power counting (and noting that k_1^2 is the only scale in the integral) reveals that all other terms after integration are at least proportional to $(k_1^2)^{(D-4)/2}$, as mentioned above. So, the integrals clearly vanish for $D > 4$. In $D = 4$, the integrals are now well-defined and vanish in dimensional regularization. We do note that the integral

$$\int \frac{d^D p}{(2\pi)^D} \frac{k_1^2}{k_1^2 \, p^2 \, (p-k_1)^2} = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 \, (p-k_1)^2} \tag{4.3}$$

vanishes through a cancellation of UV and collinear singularities (see Ref. [73]). Thus, these integrals need to be included when calculating UV divergences in four-dimensional Yang-Mills theory. It is interesting to note that in the corresponding gravity numerator of Eq. (2.5), we ensure that there is an extra scalar product from Eq. (4.1) in each term from multiplying two Yang-Mills numerators. Thus, in the gravity case, the integrals either vanish due to

	N_g	N_f	N_s(real)
$\mathcal{N}=4$	1	4	6
$\mathcal{N}=2$	1	2	2
$\mathcal{N}=1$ (vector)	1	1	0
$\mathcal{N}=1$ (chiral)	0	1	2

Table 4.1: Four-dimensional supersymmetric field content.

$k_1 \cdot \varepsilon_1$ or go as $(k_1^2)^{(D-2)/2}$, which also vanishes with no contribution to the UV divergence.

These tadpole and bubble-on-external-leg constraints generalize those of Chapter 3 to deal with matter content in the loop. Unlike Chapter 3 where the remaining *five* coefficients are simply set to zero, here we add the additional simplifying constraint that the terms without loop momentum vanish in the bubble numerator of Fig. 4.1(f). Internal bubble and bubble-on-external-leg numerators now have no $\mathcal{O}(p^0)$ terms, so they vanish in supersymmetric theories due to the reduced maximum power of the loop momentum. The internal bubble and bubble-on-external-leg conditions are also necessary for the further-improved loop-momentum power counting in maximally-supersymmetric Yang-Mills theory (e.g., $\mathcal{N}=4$ in $D=4$ or $\mathcal{N}=1$ in $D=10$). Namely, the maximum power is p^{V-4}, where $V-4 \geq 0$. This means that triangle and bubble numerators vanish identically and that box numerators have no powers of loop momentum. Henceforth, we fix all 468 coefficients using the above constraints.

Because the Jacobi relations are linear, the linear combinations of BCJ box numerators also obey color-kinematics duality. Thus, we decompose the BCJ box numerator as follows:

$$n_{1234;p} = N_g\, n_{1234;p}^{(\text{gluon})} + N_f\, n_{1234;p}^{(\text{fermion})} + N_s\, n_{1234;p}^{(\text{scalar})}\,, \tag{4.4}$$

where 1234 refers to the external-leg ordering and p is the loop momentum. $n_{1234;p}^{(\text{gluon})}$, $n_{1234;p}^{(\text{fermion})}$, and $n_{1234;p}^{(\text{scalar})}$ are the BCJ box numerators corresponding to individual field contributions in the loop, which we provide below. The prefactors N_g, N_f, and N_s are the number of gluons, fermions, and real scalars, respectively, circulating in the loop. For instance, the allowed field content for supersymmetric theories in four dimensions is given in Table 4.1.

Using the shorthand notation,

$$\begin{gathered} p_1 = p, \qquad p_2 = p - k_1, \qquad p_3 = p - k_1 - k_2, \qquad p_4 = p + k_4, \\ \mathcal{E}_{ij} = \varepsilon_i \cdot \varepsilon_j, \qquad \mathcal{P}_{ij} = p_i \cdot \varepsilon_j, \qquad \mathcal{K}_{ij} = k_i \cdot \varepsilon_j, \end{gathered} \tag{4.5}$$

and using the labeling convention of Fig. 4.1(a), the contribution from a real scalar field circulating in the loop is as follows:

$$\begin{aligned} n^{(\text{scalar})}_{1234;p} &= -i\Big[-\tfrac{1}{24}\,\mathcal{E}_{12}\,\mathcal{E}_{34}\,p_1^2\,p_3^2 + \tfrac{1}{24}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,p_1^2\,p_3^2 + \tfrac{1}{8}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_1^2\,p_3^2 + \tfrac{1}{6}\,\mathcal{E}_{12}\,\mathcal{P}_{33}\,\mathcal{P}_{44}\,p_1^2 \\
&- \tfrac{1}{6}\,\mathcal{E}_{13}\,\mathcal{P}_{22}\,\mathcal{P}_{44}\,p_1^2 - \mathcal{E}_{14}\,\mathcal{P}_{22}\,\mathcal{P}_{33}\,p_1^2 - \tfrac{1}{6}\,\mathcal{E}_{24}\,\mathcal{P}_{11}\,\mathcal{P}_{33}\,p_1^2 + \tfrac{1}{6}\,\mathcal{E}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22}\,p_1^2 \\
&+ \mathcal{P}_{11}\,\mathcal{P}_{22}\,\mathcal{P}_{33}\,\mathcal{P}_{44} - \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{42}\,\mathcal{P}_{44}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{41}\,\mathcal{P}_{44}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{P}_{44}\,p_1^2 \\
&- \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{P}_{44}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{14}\,\mathcal{K}_{42}\,\mathcal{P}_{33}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{24}\,\mathcal{K}_{41}\,\mathcal{P}_{33}\,p_1^2 + \tfrac{1}{4}\,\mathcal{E}_{12}\,\mathcal{K}_{34}\,\mathcal{P}_{33}\,p_1^2 \\
&- \tfrac{1}{6}\,\mathcal{E}_{24}\,\mathcal{K}_{31}\,\mathcal{P}_{33}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{12}\,\mathcal{K}_{24}\,\mathcal{P}_{33}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{P}_{33}\,p_1^2 + \tfrac{1}{4}\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,p_1^2 \\
&- \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{34}\,\mathcal{P}_{22}\,p_1^2 + \tfrac{1}{6}\,\mathcal{E}_{34}\,\mathcal{K}_{31}\,\mathcal{P}_{22}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{22}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{P}_{22}\,p_1^2 \\
&- \tfrac{1}{6}\,\mathcal{E}_{14}\,\mathcal{K}_{13}\,\mathcal{P}_{22}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{34}\,\mathcal{K}_{42}\,\mathcal{P}_{11}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11}\,p_1^2 \\
&- \tfrac{1}{12}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{P}_{11}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{P}_{11}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{K}_{42}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{34}\,\mathcal{K}_{42}\,p_1^2 \\
&+ \tfrac{1}{12}\,\mathcal{E}_{34}\,\mathcal{K}_{31}\,\mathcal{K}_{42}\,p_1^2 - \tfrac{1}{6}\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{41}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{41}\,p_1^2 \\
&+ \tfrac{1}{12}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{K}_{41}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{31}\,\mathcal{K}_{34}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{34}\,p_1^2 \\
&- \tfrac{1}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{31}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{K}_{31}\,p_1^2\Big] + \text{cyclic}. \end{aligned} \tag{4.6}$$

The notation '+ cyclic' indicates a sum over the three additional cyclic permutations of indices, giving a total of four permutations $(1,2,3,4)$, $(2,3,4,1)$, $(3,4,1,2)$, and $(4,1,2,3)$ of the possible variables $\varepsilon_i, k_i, p_i, s \equiv (k_1+k_2)^2$, and $t \equiv (k_2+k_3)^2$.

The contribution from the gluon is the sum of a piece proportional to the scalar contribution and extra terms, denoted $n^{(\text{extra})}_{1234;p}$. Explicitly,

$$n^{(\text{gluon})}_{1234;p} = \mathfrak{D}_g\, n^{(\text{scalar})}_{1234;p} + n^{(\text{extra})}_{1234;p}\,, \tag{4.7}$$

where the proportionality factor, $\mathfrak{D}_g \equiv D-2$, is the number of gluonic states—i.e., on-shell degrees of freedom. The extra terms contribute the following:

$$
\begin{aligned}
n^{(\text{extra})}_{1234;p} \; = -i\Big[&\mathcal{E}_{12}\,\mathcal{E}_{34}\,p_1^2\,p_3^2 - \mathcal{E}_{13}\,\mathcal{E}_{24}\,p_1^2\,p_3^2 - \mathcal{E}_{12}\,\mathcal{E}_{34}\,p_1^2\,p_2^2 + \mathcal{E}_{13}\,\mathcal{E}_{24}\,p_1^2\,p_2^2 - \mathcal{E}_{14}\,\mathcal{E}_{23}\,p_1^2\,p_2^2\\
&+ \mathcal{E}_{14}\,\mathcal{E}_{23}\left(p_1^2\right)^2 + 4\,\mathcal{E}_{23}\,\mathcal{K}_{41}\,\mathcal{P}_{44}\,p_1^2 - 4\,\mathcal{E}_{12}\,\mathcal{K}_{34}\,\mathcal{P}_{33}\,p_1^2 + 4\,\mathcal{E}_{24}\,\mathcal{K}_{31}\,\mathcal{P}_{33}\,p_1^2 - 4\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,p_1^2\\
&- 4\,\mathcal{E}_{34}\,\mathcal{K}_{31}\,\mathcal{P}_{22}\,p_1^2 - 4\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{22}\,p_1^2 + 4\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,p_1^2 + 4\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11}\,p_1^2 + \mathcal{E}_{12}\,\mathcal{E}_{34}\,p_2^2\,s\\
&- \mathcal{E}_{13}\,\mathcal{E}_{24}\,p_2^2\,s + \mathcal{E}_{14}\,\mathcal{E}_{23}\,p_2^2\,s - \mathcal{E}_{14}\,\mathcal{E}_{23}\,p_1^2\,s + 2\,\mathcal{E}_{13}\,\mathcal{P}_{22}\,\mathcal{P}_{44}\,s + 2\,\mathcal{E}_{24}\,\mathcal{P}_{11}\,\mathcal{P}_{33}\,s\\
&- 4\,\mathcal{E}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22}\,s - 2\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{K}_{42}\,p_1^2 - 2\,\mathcal{E}_{34}\,\mathcal{K}_{31}\,\mathcal{K}_{42}\,p_1^2 - 2\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,p_1^2 + 2\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{K}_{42}\,p_1^2\\
&+ 6\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{41}\,p_1^2 + 4\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,p_1^2 - 2\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{K}_{41}\,p_1^2 + 2\,\mathcal{E}_{23}\,\mathcal{K}_{31}\,\mathcal{K}_{34}\,p_1^2\\
&- 2\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,p_1^2 + 2\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,p_1^2 + 2\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{31}\,p_1^2 - 2\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{K}_{31}\,p_1^2\\
&- 2\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{24}\,p_1^2 + 2\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,p_1^2 + 4\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,\mathcal{P}_{11}\,\mathcal{P}_{33} - 8\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22}\\
&- 8\,\mathcal{K}_{13}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22} - 2\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,s + 2\,\mathcal{E}_{13}\,\mathcal{K}_{34}\,\mathcal{P}_{22}\,s + 2\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{22}\,s - 2\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{P}_{22}\,s\\
&- 2\,\mathcal{E}_{34}\,\mathcal{K}_{42}\,\mathcal{P}_{11}\,s - 2\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,s - 2\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11}\,s + 2\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{P}_{11}\,s - 2\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{P}_{11}\,s\\
&- 4\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{42}\,\mathcal{P}_{11} + 4\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,\mathcal{P}_{11} + 4\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,\mathcal{P}_{11} + 4\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11}\\
&+ 4\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{11} + \tfrac{1}{2}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,s^2 - 2\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,s - 2\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,s - 2\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,s\\
&- 2\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,s - 4\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,s - 2\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,s + \mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,\mathcal{K}_{42} + 2\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{41}\\
&+ 4\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{31}\,\mathcal{K}_{34} + 2\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{31}\,\mathcal{K}_{34} + 4\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31} + 4\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\Big] + \text{cyclic.}
\end{aligned}
\tag{4.8}
$$

Finally, we give the contribution from the fermion loop:

$$
\begin{aligned}
n^{(\text{fermion})}_{1234;p} &= -i\,\mathfrak{D}_f \Big[-\tfrac{1}{12}\,\mathcal{E}_{12}\,\mathcal{E}_{34}\,p_1^2\,p_3^2 + \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,p_1^2\,p_3^2 - \tfrac{1}{8}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_1^2\,p_3^2 + \tfrac{1}{8}\,\mathcal{E}_{12}\,\mathcal{E}_{34}\,p_1^2\,p_2^2 \\
&- \tfrac{1}{8}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,p_1^2\,p_2^2 + \tfrac{1}{8}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_1^2\,p_2^2 - \tfrac{1}{8}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,\left(p_1^2\right)^2 - \tfrac{1}{6}\,\mathcal{E}_{12}\,\mathcal{P}_{33}\,\mathcal{P}_{44}\,p_1^2 + \tfrac{1}{6}\,\mathcal{E}_{13}\,\mathcal{P}_{22}\,\mathcal{P}_{44}\,p_1^2 \\
&+ \mathcal{E}_{14}\,\mathcal{P}_{22}\,\mathcal{P}_{33}\,p_1^2 + \tfrac{1}{6}\,\mathcal{E}_{24}\,\mathcal{P}_{11}\,\mathcal{P}_{33}\,p_1^2 - \tfrac{1}{6}\,\mathcal{E}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22}\,p_1^2 - \mathcal{P}_{11}\,\mathcal{P}_{22}\,\mathcal{P}_{33}\,\mathcal{P}_{44} + \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{42}\,\mathcal{P}_{44}\,p_1^2 \\
&- \tfrac{5}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{41}\,\mathcal{P}_{44}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{P}_{44}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{P}_{44}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{14}\,\mathcal{K}_{42}\,\mathcal{P}_{33}\,p_1^2 \\
&+ \tfrac{1}{12}\,\mathcal{E}_{24}\,\mathcal{K}_{41}\,\mathcal{P}_{33}\,p_1^2 + \tfrac{1}{4}\,\mathcal{E}_{12}\,\mathcal{K}_{34}\,\mathcal{P}_{33}\,p_1^2 - \tfrac{1}{3}\,\mathcal{E}_{24}\,\mathcal{K}_{31}\,\mathcal{P}_{33}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{12}\,\mathcal{K}_{24}\,\mathcal{P}_{33}\,p_1^2 \\
&+ \tfrac{1}{12}\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{P}_{33}\,p_1^2 + \tfrac{1}{4}\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{34}\,\mathcal{P}_{22}\,p_1^2 + \tfrac{1}{3}\,\mathcal{E}_{34}\,\mathcal{K}_{31}\,\mathcal{P}_{22}\,p_1^2 \\
&+ \tfrac{5}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{22}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{P}_{22}\,p_1^2 + \tfrac{1}{6}\,\mathcal{E}_{14}\,\mathcal{K}_{13}\,\mathcal{P}_{22}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{34}\,\mathcal{K}_{42}\,\mathcal{P}_{11}\,p_1^2 \\
&- \tfrac{5}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,p_1^2 - \tfrac{5}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{P}_{11}\,p_1^2 - \tfrac{1}{12}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{P}_{11}\,p_1^2 \\
&- \tfrac{1}{8}\,\mathcal{E}_{12}\,\mathcal{E}_{34}\,p_2^2\,s + \tfrac{1}{8}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,p_2^2\,s - \tfrac{1}{8}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_2^2\,s + \tfrac{1}{8}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_1^2\,s - \tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{P}_{22}\,\mathcal{P}_{44}\,s \\
&- \tfrac{1}{4}\,\mathcal{E}_{24}\,\mathcal{P}_{11}\,\mathcal{P}_{33}\,s + \tfrac{1}{2}\,\mathcal{E}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22}\,s + \tfrac{1}{6}\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{K}_{42}\,p_1^2 + \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{34}\,\mathcal{K}_{42}\,p_1^2 + \tfrac{1}{6}\,\mathcal{E}_{34}\,\mathcal{K}_{31}\,\mathcal{K}_{42}\,p_1^2 \\
&+ \tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,p_1^2 - \tfrac{1}{4}\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{K}_{42}\,p_1^2 - \tfrac{7}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{41}\,p_1^2 - \tfrac{5}{12}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,p_1^2 \\
&+ \tfrac{1}{12}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{41}\,p_1^2 + \tfrac{1}{6}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{K}_{41}\,p_1^2 - \tfrac{1}{6}\,\mathcal{E}_{23}\,\mathcal{K}_{31}\,\mathcal{K}_{34}\,p_1^2 + \tfrac{1}{6}\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,p_1^2 \\
&+ \tfrac{1}{12}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{34}\,p_1^2 - \tfrac{1}{6}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,p_1^2 - \tfrac{1}{6}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{31}\,p_1^2 + \tfrac{1}{6}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{K}_{31}\,p_1^2 \\
&+ \tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{24}\,p_1^2 - \tfrac{1}{4}\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,p_1^2 - \tfrac{1}{2}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,\mathcal{P}_{11}\,\mathcal{P}_{33} + \mathcal{K}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22} \\
&+ \mathcal{K}_{13}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22} + \tfrac{1}{4}\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,s - \tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{K}_{34}\,\mathcal{P}_{22}\,s - \tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{22}\,s + \tfrac{1}{4}\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{P}_{22}\,s \\
&+ \tfrac{1}{4}\,\mathcal{E}_{34}\,\mathcal{K}_{42}\,\mathcal{P}_{11}\,s + \tfrac{1}{4}\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,s + \tfrac{1}{4}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11}\,s - \tfrac{1}{4}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{P}_{11}\,s + \tfrac{1}{4}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{P}_{11}\,s \\
&+ \tfrac{1}{2}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{42}\,\mathcal{P}_{11} - \tfrac{1}{2}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,\mathcal{P}_{11} - \tfrac{1}{2}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,\mathcal{P}_{11} - \tfrac{1}{2}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11} \\
&- \tfrac{1}{2}\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{11} - \tfrac{1}{32}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,s\,t + \tfrac{1}{16}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,s\,t - \tfrac{1}{8}\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,s + \tfrac{1}{4}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,s \\
&- \tfrac{1}{8}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{41}\,s - \tfrac{1}{8}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{34}\,s - \tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{24}\,s + \tfrac{1}{4}\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,s - \tfrac{1}{8}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,\mathcal{K}_{42} \\
&- \tfrac{1}{4}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{41} - \tfrac{1}{2}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{31}\,\mathcal{K}_{34} - \tfrac{1}{4}\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{31}\,\mathcal{K}_{34} - \tfrac{1}{2}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31} \\
&- \tfrac{1}{2}\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{31} \Big] + \text{cyclic.} \qquad (4.9)
\end{aligned}
$$

Here, there is also a well-known proportionality factor, $\mathfrak{D}_f$, that denotes the number of states—i.e., on-shell degrees of freedom—of each fermion. The minimal spinor type corresponding to each spacetime dimension is provided in Table 4.2 along with its number of states (see Ref. [75]).

It is now simple to obtain BCJ numerators with four-dimensional external states that we

Dimension (D)	Minimal Spinor Type	# of States ($\mathfrak{D}_f$)
3	Majorana	1
4	Majorana	2
5	Dirac	4
6	Weyl	4
7	Dirac	8
8	Pseudo-Majorana	8
9	Pseudo-Majorana	8
10	Pseudo-Majorana and Weyl	8
11	Majorana	16

Table 4.2: The number of states in minimal spinors, dependent on dimension. We note that in $D = 5, 6$, and 7, a symplectic Majorana condition can be applied among an even number of spinors. We ignore this condition here.

use to compare to earlier work in Section 4.4:

$$n^{\mathcal{N}=4}_{1234;p} = n^{(\text{gluon})}_{1234;p}\Big|_{\mathfrak{D}_g=2} + 4\, n^{(\text{fermion})}_{1234;p}\Big|_{\mathfrak{D}_f=2} + 6\, n^{(\text{scalar})}_{1234;p}, \tag{4.10a}$$

$$n^{\mathcal{N}=1(\text{chiral})}_{1234;p} = n^{(\text{fermion})}_{1234;p}\Big|_{\mathfrak{D}_f=2} + 2\, n^{(\text{scalar})}_{1234;p}, \tag{4.10b}$$

$$n^{\mathcal{N}=0}_{1234;p} = n^{(\text{gluon})}_{1234;p}\Big|_{\mathfrak{D}_g=2}. \tag{4.10c}$$

To explicitly see the simplification due to supersymmetry, we provide the $\mathcal{N} = 4$ and $\mathcal{N} = 1$ (chiral) box numerators in Appendix D. In Section 4.4, we compare the numerators of Eqs. (4.10) to the $\mathcal{N} = 4$ numerators of Ref. [72], the $\mathcal{N} = 1$ (chiral) MHV numerators of Ref. [25], and the $\mathcal{N} = 0$ all-plus-helicity numerators of Ref. [33]. But first, we discuss how to put these formal-polarization expressions into a helicity basis in the next section.

Before we proceed, we clarify a few points related to our inclusion of matter. First, we reiterate that using nonsupersymmetric field content in Eq. (4.4) still yields valid BCJ numerators, but numerators with internal bubbles or bubbles on external legs no longer vanish and the loop momentum power counting is not improved. Second, our $\mathcal{N} = 0$ numerators—generated by the box numerator of Eq. (4.7)—differ from those presented in Chapter 3 because of our restrictions on the internal bubble numerators. Third, we emphasize that we are only considering numerators with external gluons. Amplitudes with arbitrary field content on the external legs do not always straightforwardly allow a BCJ representation. In particular, consider two different flavors of scalars minimally coupled to nonsupersymmetric YM theory. We notice at tree level that four external scalars of two different flavors can only scatter in one channel. Thus, we cannot satisfy color-kinematics duality because the numerator Jacobi relations relate three different channels. A remedy in this situation is to add a four-point contact interaction that mixes the different flavors [32, 76, 77]. This is the interaction that arises when nonsupersymmetric pure YM theory is dimensionally reduced from six dimensions to four dimensions. This has been studied in some detail in the context of multi-Regge kinematics in Ref. [76]. We expect similar properties for external fermions. Namely, we expect that the duality works straightforwardly with only one flavor[1]; however, for multiple flavors, we anticipate the need for flavor-mixing Yukawa interactions. Such interactions are seen in $\mathcal{N} = 2$ sYM theory in four dimensions. ($\mathcal{N} = 2$ sYM theory in four dimensions can be constructed by dimensionally reducing six-dimensional $\mathcal{N} = 1$ (vector) sYM theory, which has only one fermion flavor.) There are no such issues for our one-loop amplitudes with external gluons and multiple flavors of matter in the loop; each diagram can only have one flavor circulating in the loop at a time. However, this is not the case for higher loop orders.

Finally, we mention that our BCJ numerators can be used in Eq. (2.5) to calculate amplitudes in gravity theories. For our amplitudes, the external states can consist of gravitons, antisymmetric tensors, and dilatons, as discussed in Chapter 3. As an example of field content in the loop, we note that the product of our gluon box numerator and fermion box numerator, $n^{(\text{gluon})}_{1234;p} \times n^{(\text{fermion})}_{1234;p}$, gives the box numerator for a gravitino and fermion circulating in the loop. This agrees with simple state counting. The tensor product of the gluon and fermion states yields $\mathfrak{D}_g \times \mathfrak{D}_f$ states. Likewise, the total number of states of a gravitino and a fermion is $(\mathfrak{D}_g - 1)\mathfrak{D}_f + \mathfrak{D}_f = \mathfrak{D}_g \times \mathfrak{D}_f$.

[1]We note the restrictions of Ref. [78]: The four-fermion tree amplitude, $A_4^{\text{tree}}(1^\psi, 2^\psi, 3^\psi, 4^\psi)$, can only satisfy color-kinematics duality in $D = 3, 4, 6, 10$. We thank Radu Roiban for bringing this to our attention.

Plain-text, computer-readable versions of the full expressions for the numerators can be found online [79].

4.3 Polarization Vectors in a Momentum Basis

To compare our results to existing literature, we consider our numerators in specific four-dimensional helicity configurations. (Because we use dimensional regularization, the loop momentum is in $D = 4 - 2\varepsilon$.) We do this by putting the formal polarization vectors into a momentum basis, as in Ref. [63]. Because we are dealing with four-point amplitudes in four dimensions, the momentum basis consists of three independent external momentum vectors and an orthogonal dual vector. The dual vector is formed by contracting three independent external momentum vectors with the four-dimensional Levi-Civita symbol:

$$v^\mu \equiv \epsilon(\mu, k_1, k_2, k_3) \equiv \epsilon^{\mu\alpha\beta\gamma} k_{1\alpha} k_{2\beta} k_{3\gamma}. \tag{4.11}$$

We take care to preserve the phase factors associated with the polarization vectors. Phase factors arise naturally in the spinor-helicity formalism via

$$\langle ij \rangle = \sqrt{|s_{ij}|} e^{i\phi_{ij}}, \qquad [ij] = \sqrt{|s_{ij}|} e^{-i(\phi_{ij}+\pi)}, \tag{4.12}$$

where particles i and j have positive energy, i.e., $k_i^0 > 0$ and $k_j^0 > 0$ (cf. Ref. [80]). Also, we define $s_{ij} \equiv (k_i + k_j)^2$, so $s \equiv s_{12} = s_{34}$, $t \equiv s_{23} = s_{14}$, and $u \equiv s_{13} = s_{24} = -s - t$ are the standard Mandelstam variables. By antisymmetry of the spinor products and momentum conservation,

$$\phi_{ji} = \phi_{ij} + \pi, \qquad \phi_{24} = -\phi_{13} + \phi_{14} + \phi_{23} + \pi, \qquad \phi_{34} = -\phi_{12} + \phi_{14} + \phi_{23}. \tag{4.13}$$

The combination of phase factors that appear in our calculations are immediately identified with spinor-helicity expressions. For reference, we list the four-point combinations that arise:

$$\begin{aligned} e^{-2i(\phi_{14}+\phi_{23})} &= \frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle} && \sim \ \text{helicity}: ++++, \\ e^{2i(\phi_{12}+\phi_{13}-\phi_{14}-2\phi_{23})} &= -\frac{st}{u}\frac{[24]^2}{[12]\,\langle 23\rangle\,\langle 34\rangle\,[41]} && \sim \ \text{helicity}: -+++, \\ e^{2i(2\phi_{12}-\phi_{14}-\phi_{23})} &= -\frac{t}{s}\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle} && \sim \ \text{helicity}: --++. \end{aligned} \tag{4.14}$$

Of course, some care is needed when using dimensional regularization. The external momenta, k_i^μ; formal polarization vectors, ε_i^μ; and dual vector, v^μ, are still four-dimensional objects. However, the loop momentum, call it $\tilde{l}^\mu$, is in $4-2\varepsilon$ dimensions. We denote the four-dimensional components of the loop momentum as l^μ and the spacelike (-2ε)-component as $l^\mu_{[-2\varepsilon]}$. The -2ε dimensions only affect the $\tilde{l}^2$ inner product. Specifically,

$$\tilde{l}\cdot k_i = l\cdot k_i, \qquad \tilde{l}\cdot v = l\cdot v, \qquad \tilde{l}^2 = l^2 - l^2_{[-2\varepsilon]} \equiv l^2-\mu^2, \tag{4.15}$$

noting that we use the mostly-minus metric convention. The $\mu^2 \equiv l^2_{[-2\varepsilon]}$ can be treated as an effective mass of the loop momentum. (For the importance of the μ^2 in loop calculations, we direct the reader to Ref. [81].)

Using identities of the Levi-Civita symbol, we find the following properties of the dual vector:

$$v^2 = -\tfrac{1}{4}stu, \tag{4.16}$$

$$(l\cdot v)^2 = -\tfrac{1}{4}\left[t\tau_{51}(t\tau_{51}-2u\tau_{52}) + (\text{cyclic permutations of 1,2,3}) + stu(\tilde{\tau}_{55}+\mu^2)\right], \tag{4.17}$$

where we define

$$\tau_{5i}\equiv l\cdot k_i, \qquad \tilde{\tau}_{55}\equiv \tilde{l}^2. \tag{4.18}$$

Because the propagators in our amplitudes contain the $(4-2\varepsilon)$-dimensional $\tilde{l}^2$'s, we write the l^2 term in $(l\cdot v)^2$ as $\tilde{l}^2+\mu^2$. This is the only vehicle through which μ^2 terms arise in our expressions.

Now, we put the polarization vectors into a momentum basis. We define a four-dimensional representation of the polarization vector corresponding to an external leg with momentum k_i by

$$\varepsilon^{\mu}_{h_i}(i;j_1,j_2) \equiv \mathcal{P}_{h_i}(i;j_1,j_2)\sqrt{\frac{2}{s_{ij_1}s_{ij_2}s_{j_1j_2}}}\times \\ \Big[(k_{j_1}\cdot k_{j_2})\,k_i^{\mu}+(k_i\cdot k_{j_2})\,k_{j_1}^{\mu}-(k_i\cdot k_{j_1})\,k_{j_2}^{\mu}+i\,h_i\,\epsilon(\mu,k_i,k_{j_1},k_{j_2})\Big]\,. \qquad (4.19)$$

The arguments i, j_1, j_2 correspond to the external momenta k_i, k_{j_1}, k_{j_2}, where k_{j_1} and k_{j_2} are the reference momenta. μ is a free Lorentz index and $h_i=\pm 1$ defines the helicity state, which we sometimes simply denote as $h_i=\pm$. $\mathcal{P}_{h_i}(i;j_1,j_2)$ is a phase factor that we determine in Appendix E to be

$$\mathcal{P}_{h_i}(i;j_1,j_2) = -e^{-ih_i\left(\phi_{ij_2}-\phi_{j_1j_2}+\phi_{ij_1}\right)}. \qquad (4.20)$$

The coefficients of the basis vectors were determined by demanding the following:

$$k_i\cdot\varepsilon_{h_i}(i;j_1,j_2)=0, \qquad k_{j_1}\cdot\varepsilon_{h_i}(i;j_1,j_2)=0, \qquad \varepsilon^{*}_{\pm}(i;j_1,j_2)=\varepsilon_{\mp}(i;j_1,j_2), \\ \varepsilon_{h_i}(i;j_1,j_2)\cdot\varepsilon_{h_i}(i;j_1,j_2)=0, \qquad \varepsilon_{h_i}(i;j_1,j_2)\cdot\varepsilon^{*}_{h_i}(i;j_1,j_2)=-1. \qquad (4.21)$$

Note that $i\neq j_1\neq j_2$; otherwise, $\epsilon(\mu,k_i,k_{j_1},k_{j_2})=0$, and Eqs. (4.21) cannot be satisfied. This implies that $s_{ij_1}s_{ij_2}s_{j_1j_2}=stu$.

Choosing different reference momenta in Eq. (4.19) changes the expression by at most a

$\varepsilon_i \cdot \varepsilon_j$	ε_1	ε_2	ε_3	ε_4
ε_1	0	$-\frac{1}{2}(1 - h_1 h_2)$	$\frac{1}{2}(1 + h_1 h_3)$	$\frac{1}{2}(1 - h_1 h_4)$
ε_2	$-\frac{1}{2}(1 - h_1 h_2)$	0	$\frac{1}{2}(1 + h_2 h_3)$	$\frac{1}{2}(1 - h_2 h_4)$
ε_3	$\frac{1}{2}(1 + h_1 h_3)$	$\frac{1}{2}(1 + h_2 h_3)$	0	$-\frac{1}{2}(1 + h_3 h_4)$
ε_4	$\frac{1}{2}(1 - h_1 h_4)$	$\frac{1}{2}(1 - h_2 h_4)$	$-\frac{1}{2}(1 + h_3 h_4)$	0

Table 4.3: The inner product $\varepsilon_i \cdot \varepsilon_j$ in the representation given by Eqs. (4.23) with phase factors suppressed. $h_i = \pm 1$ corresponds to the helicity of leg i.

gauge shift. For example,

$$\varepsilon^\mu_{h_1}(1;3,4) = \varepsilon^\mu_{h_1}(1;2,3) - \left(\mathcal{P}_{h_1}(1;2,3)\sqrt{\tfrac{2}{stu}}\, t\right) k_1^\mu, \tag{4.22a}$$

$$\varepsilon^\mu_{h_3}(3;4,1) = \varepsilon^\mu_{h_3}(3;1,2) + \left(\mathcal{P}_{h_3}(3;1,2)\sqrt{\tfrac{2}{stu}}\, t\right) k_3^\mu, \tag{4.22b}$$

where we enforce momentum conservation. We choose the following reference momenta that simplify our $\mathcal{N} = 1$ (chiral) result, which we present later:

$$\varepsilon_1^\mu \to \varepsilon^\mu_{h_1}(1;3,4), \tag{4.23a}$$

$$\varepsilon_2^\mu \to \varepsilon^\mu_{h_2}(2;3,1), \tag{4.23b}$$

$$\varepsilon_3^\mu \to \varepsilon^\mu_{h_3}(3;4,1), \tag{4.23c}$$

$$\varepsilon_4^\mu \to \varepsilon^\mu_{h_4}(4;3,1). \tag{4.23d}$$

For consistency, we use this choice throughout the remainder of the chapter. Other gauge choices do not affect the $\mathcal{N} = 4$ result, and the $\mathcal{N} = 0$ expression will be equally lengthy regardless of reference momenta choices. With this representation, we tabulate the inner products $\varepsilon_i \cdot \varepsilon_j$ in Table 4.3, where we suppress the phase factors, and $\varepsilon_i \cdot \varepsilon_j^*$ in Table 4.4. Also, we suppress the phase factor along with $\sqrt{2/(stu)}$ and list $\varepsilon_i \cdot k_j$ in Table 4.5.

We draw attention to the fact that putting the polarization vectors into a momentum basis using the prescription above introduces a degree of non-locality. Each expression in a

$\varepsilon_i \cdot \varepsilon_j^*$	ε_1^*	ε_2^*	ε_3^*	ε_4^*
ε_1	-1	$-\frac{1}{2}(1+h_1h_2)$	$\frac{1}{2}(1-h_1h_3)$	$\frac{1}{2}(1+h_1h_4)$
ε_2	$-\frac{1}{2}(1+h_1h_2)$	-1	$\frac{1}{2}(1-h_2h_3)$	$\frac{1}{2}(1+h_2h_4)$
ε_3	$\frac{1}{2}(1-h_1h_3)$	$\frac{1}{2}(1-h_2h_3)$	-1	$-\frac{1}{2}(1-h_3h_4)$
ε_4	$\frac{1}{2}(1+h_1h_4)$	$\frac{1}{2}(1+h_2h_4)$	$-\frac{1}{2}(1-h_3h_4)$	-1

Table 4.4: The inner product $\varepsilon_i \cdot \varepsilon_j^*$ in the representation given by Eqs. (4.23). $h_i = \pm 1$ corresponds to the helicity of leg i

$\varepsilon_i \cdot k_j$	k_1	k_2	k_3	k_4
ε_1	0	$-\frac{1}{2}st$	0	$\frac{1}{2}st$
ε_2	$\frac{1}{2}su$	0	0	$-\frac{1}{2}su$
ε_3	$\frac{1}{2}tu$	$-\frac{1}{2}tu$	0	0
ε_4	$\frac{1}{2}tu$	$-\frac{1}{2}su$	0	0

Table 4.5: The inner product $\varepsilon_i \cdot k_j$ in the representation given by Eqs. (4.23), suppressing the phase factor and $\sqrt{2/(stu)}$.

four-point numerator will have the non-local factor $4/(stu)^2$ since ε_1, ε_2, ε_3, and ε_4 are present in each term. This is the same degree of non-locality that is present in the $\mathcal{N} = 1$ (chiral) numerators of Ref. [25]. In addition, the relabeling symmetries of the formal-polarization numerators in Section 4.2 are, in general, lost once the polarization vectors are put into a momentum basis. Thus, instead of one box numerator, there are now three unrelated by relabeling.

4.4 BCJ Numerator Comparisons

Here, we compare our BCJ numerators to existing representations in literature. Even if two sets of BCJ numerators satisfy the color-kinematics duality and obey the same unitarity cuts, we do not expect exact agreement because of the freedom of generalized gauge invariance [15]. Regardless, we show that the discrepancy in the amplitudes vanish upon integration.

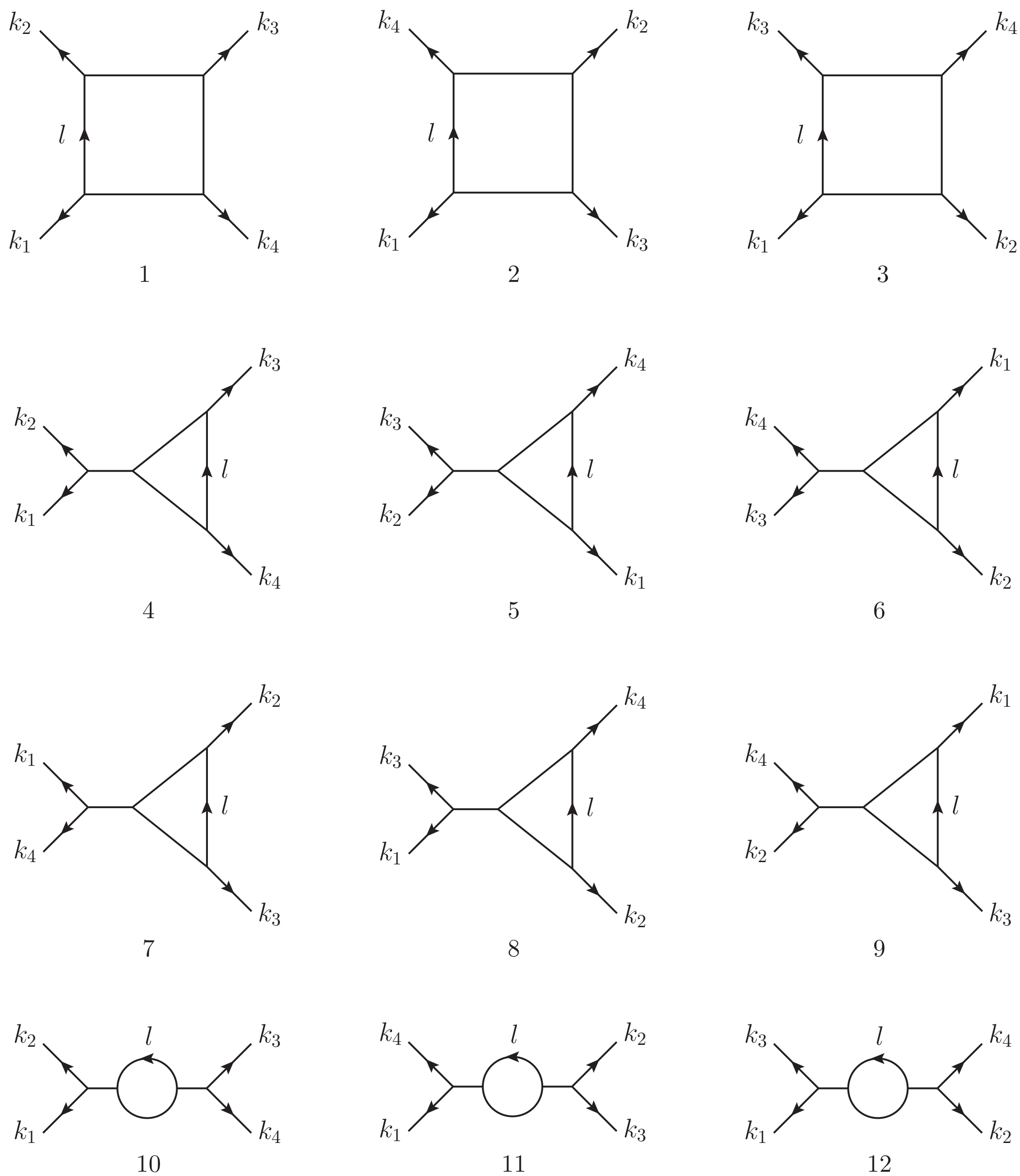

Figure 4.5: The labeling convention for numerators with polarization vectors put into a momentum basis. The labeling is identical to that of Ref. [25].

4.4.1 $\mathcal{N} = 4$ Super-Yang-Mills BCJ Numerators

As a warm-up exercise, we find duality-satisfying kinematic numerators in a helicity basis for $\mathcal{N} = 4$ sYM theory. We do not immediately exploit the simplicity of the one-loop $\mathcal{N} = 4$ box numerator, namely that it is proportional to the tree amplitude. Thus, the procedure that we outline here can be used in the more complicated cases described in the following subsections.

In general, we lose the relabeling properties mentioned in Section 4.2 when we convert formal polarization vectors to a helicity basis. So, we first relabel Eq. (D.2) to get the three independent box numerators with external-leg orderings $(1,2,3,4)$, $(1,4,2,3)$, and $(1,3,4,2)$. (The box numerator with external-leg ordering $(1,4,3,2)$, for example, is the same as the box numerator with external-leg ordering $(1,2,3,4)$ up to a relabeling of the loop momentum.) Then, we make the polarization-vector substitutions of Eqs. (4.23). Solving the numerator Jacobi relations, we generate all other independent numerators. (Alternatively, we could find all of the numerators with formal polarization vectors first, then apply Eqs. (4.23).) In this and the following subsections, we refer to Fig. 4.5 for numerator labeling conventions. We use n_i to denote the kinematic numerator corresponding to Fig. 4.5(i).

For the all-plus- and single-minus-helicity configurations,

$$h_1 = \pm, \quad h_2 = +, \quad h_3 = +, \quad h_4 = +, \tag{4.24}$$

our $\mathcal{N} = 4$ numerators vanish identically,

$$n_{1\text{-}12} = 0. \tag{4.25}$$

For the MHV configuration,

$$h_1 = -, \quad h_2 = -, \quad h_3 = +, \quad h_4 = +, \tag{4.26}$$

our numerators are

$$n_{1\text{-}3} = -is^2 e^{2i(2\phi_{12}-\phi_{14}-\phi_{23})}$$
$$= stA_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+), \quad (4.27)$$

$$n_{4\text{-}12} = 0, \quad (4.28)$$

where A_4^{tree} is the color-ordered four-point tree amplitude. These results are expected since Eq. (D.2) was already identified as stA_4^{tree} in Appendix D even before external helicities were specified. It is well known that the tree amplitude is nonvanishing only for the MHV configuration. Also, the crossing symmetry of $stA_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$ assures us that all three box numerators should be identical. Because the box numerators are identical and there is no loop momentum present—as discussed in Section 4.2—that might need to be relabeled, the numerator Jacobi equations show that the non-box numerators vanish. All of these results agree with Ref. [72].

4.4.2 $\mathcal{N} = 1$ (chiral) Super-Yang-Mills MHV BCJ Numerators

Using the same procedure as Section 4.4.1, we now construct $\mathcal{N} = 1$ (chiral) numerators in the MHV configuration, similar to those in Ref. [25]. We again use the labeling convention of Fig. 4.5, which is identical to the convention of Ref. [25]. Also, we extract a factor of $stA_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)$ from our numerators [37, 20, 25]. We define the quantity N_i by

$$n_i = stA_4^{\text{tree}}(1^-, 2^-, 3^+, 4^+)N_i. \quad (4.29)$$

The resulting N_i's are then,

$$N_1 = \tfrac{1}{2s^2}\left(4\tau_{51}\tau_{53} + 4\tau_{52}\tau_{53} - 2t\tau_{51} + 2u\tau_{52} + s\tilde{\tau}_{55}\right) + \tfrac{\mu^2}{s}, \tag{4.30a}$$

$$N_2 = \tfrac{1}{2s^2}\left(4\tau_{51}\tau_{53} + 4\tau_{52}\tau_{53} + 2s\tau_{53} + s\tilde{\tau}_{55}\right) + \tfrac{2i}{s^2}\left(l \cdot v\right) + \tfrac{\mu^2}{s}, \tag{4.30b}$$

$$N_3 = \tfrac{1}{2s^2}\left(4\tau_{51}\tau_{53} + 4\tau_{52}\tau_{53} + 2s\tau_{53} + s\tilde{\tau}_{55}\right) + \tfrac{\mu^2}{s}, \tag{4.30c}$$

$$N_5 = -\tfrac{2i}{s^2}\left(l \cdot v\right), \tag{4.30d}$$

$$N_7 = \tfrac{1}{s^2}\left(t\tau_{51} - u\tau_{52} - s\tau_{53}\right) + \tfrac{2i}{s^2}\left(l \cdot v\right), \tag{4.30e}$$

$$N_8 = \tfrac{2i}{s^2}\left(l \cdot v\right), \tag{4.30f}$$

$$N_9 = \tfrac{1}{s^2}\left(t\tau_{51} - u\tau_{52} + s\tau_{53}\right) + \tfrac{2i}{s^2}\left(l \cdot v\right), \tag{4.30g}$$

$$N_4 = N_6 = N_{10} = N_{11} = N_{12} = 0, \tag{4.30h}$$

where v, μ^2, and τ_{5i} are as defined in Eqs. (4.11), (4.15), and (4.18), respectively. These numerators have the same simplicity as those of Ref. [25]—which we denote $\tilde{N}_i$—but they do not match exactly. (In this section and the next, we use a tilde to denote the results constructed from existing literature, Ref. [25] in this case.) Furthermore, the exact numerators of Ref. [25] cannot be obtained simply by choosing different reference momenta in Eq. (4.23). The differences between our box numerators and theirs are

$$\Delta N_1 = \frac{i}{2s^2tu}\left[stu(\tilde{\tau}_{55} + 2\mu^2) - 4s^2\tau_{51}\tau_{52} - 4u^2\tau_{51}\tau_{53} - 4t^2\tau_{52}\tau_{53} + 2tu^2\tau_{51} - 2t^2u\tau_{52}\right], \tag{4.31a}$$

$$\Delta N_2 = \Delta N_1, \tag{4.31b}$$

$$\Delta N_3 = \frac{i}{2s^2tu}\left[stu(\tilde{\tau}_{55} + 2\mu^2) - 4s^2\tau_{51}\tau_{52} - 4u^2\tau_{51}\tau_{53} - 4t^2\tau_{52}\tau_{53} + 2s^2t\tau_{51} - 2st^2\tau_{53}\right]. \tag{4.31c}$$

(We use Δ to denote our result minus the result from literature, eg. $\Delta N_i \equiv N_i - \tilde{N}_i$.)

To verify that our numerators of Eq. (4.30) produce the same physical result as those in Ref. [25], we show that the three independent color-ordered amplitudes $A_4^{(1)}(1,2,3,4)$, $A_4^{(1)}(1,4,2,3)$, and $A_4^{(1)}(1,3,4,2)$ of Eq. (2.6) give the same integrated result. The differences in the integrands $I_4(1,2,3,4)$, $I_4(1,4,2,3)$, and $I_4(1,3,4,2)$, are

$$\Delta I_4(1^-,2^-,3^+,4^+) = \frac{istA_4^{\text{tree}}(1^-,2^-,3^+,4^+)}{2s(s\,t)\prod_{i=1}^4 p_i^2}\left[s\,p_1^2\,p_2^2 + t\,p_3^2\,p_4^2 + u\,p_4^2\,p_1^2 + 2\,s\,t\,\mu^2\right], \quad (4.32a)$$

$$\Delta I_4(1^-,4^+,2^-,3^+) = \frac{istA_4^{\text{tree}}(1^-,2^-,3^+,4^+)}{2s(t\,u)\prod_{i=1}^4 p_i^2}\left[u\,p_1^2\,p_2^2 + s\,p_2^2\,p_3^2 + t\,p_3^2\,p_4^2 + 2\,t\,u\,\mu^2\right], \quad (4.32b)$$

$$\Delta I_4(1^-,3^+,4^+,2^-) = \frac{istA_4^{\text{tree}}(1^-,2^-,3^+,4^+)}{2s(s\,u)\prod_{i=1}^4 p_i^2}\left[u\,p_2^2\,p_3^2 + t\,p_3^2\,p_4^2 + s\,p_4^2\,p_1^2 + 2\,s\,u\,\mu^2\right], \quad (4.32c)$$

where we use the labeling convention of Fig. 4.1 so that two-particle cut-free terms may be readily identified (see Fig. 4.4). We notice that, aside from the μ^2 term, this difference does not survive either of the two-particle cuts. Integrals of this type—bubble-on-external-leg integrals sans the on-shell intermediate propagator, as in Eq. (4.3)—vanish in dimensional regularization (see Ref. [73]). Furthermore, the μ^2 box integral does not contribute because it is $\mathcal{O}(\varepsilon)$ [33]. Hence, our integrated color-ordered amplitudes agree with those of Ref. [25].

4.4.3 $\mathcal{N}=0$ Yang-Mills All-Plus-Helicity BCJ Numerators

The contribution from a real scalar in the loop of a four-point one-loop amplitude has been computed in Ref. [33], where the external gluons are in the all-plus-helicity configuration. In the all-plus-helicity sector, the color-ordered amplitude for a gluon in the loop is simply twice that of a massless scalar,

$$\tilde{A}_4^{(1)\,\text{gluon}}\left(1^+,2^+,3^+,4^+\right) = \frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}\int \frac{d^4p}{(2\pi)^4}\frac{d^{-2\varepsilon}\mu}{(2\pi)^{-2\varepsilon}}\frac{2\,\mu^4}{\prod_{i=1}^4 p_i^2}, \quad (4.33)$$

where again we use a tilde to denote the results from existing literature. Because the spinor-helicity prefactor and μ^4 are invariant under relabelings of the external momenta and the loop momentum, we can immediately read off the BCJ numerators:

$$\tilde{n}_{1\text{-}3} = \frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}\,2\,\mu^4, \tag{4.34a}$$

$$\tilde{n}_{4\text{-}12} = 0, \tag{4.34b}$$

where we again use the labeling conventions of Fig. 4.5. (We identify $\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}2\mu^4$ as a box numerator because the four propagators present are those of the box diagram.)

Now, we construct all-plus-helicity $\mathcal{N} = 0$ BCJ numerators in the same way as Sections 4.4.1 and 4.4.2. However, we find that each box numerator in the all-plus-helicity sector is just as long as the formal-polarization expression; there is no simplification like we observed in Section 4.4.2. Furthermore, the non-box numerators do not vanish as in Eq. (4.34b). The color-ordered amplitude integrands, too, are more complicated than Eq. (4.33). For instance,

$$\Delta A_4^{(1)\,\mathrm{gluon}}\left(1^+,2^+,3^+,4^+\right) = \frac{[12]\,[34]}{\langle 12\rangle\,\langle 34\rangle}\int \frac{d^4p}{(2\pi)^4}\frac{d^{-2\varepsilon}\mu}{(2\pi)^{-2\varepsilon}}\frac{1}{\left(\prod_{i=1}^4 p_i^2\right)6s^2\,t^2\,(s+t)}\times$$
$$\Big[s\left(-3s^2-2ts+t^2\right)p_3^2\,p_4^6 + st(t-5s)\,p_1^2\,p_4^6 + \left(3s^3+ts^2+t^2s+3t^3\right)p_3^4\,p_4^4$$
$$+ s\left(6s^2+11ts+9t^2\right)p_2^2\,p_3^2\,p_4^4 - 2t\left(-3s^2+4ts+3t^2\right)p_1^2\,p_3^2\,p_4^4 + st(9s+17t)\,p_1^2\,p_2^2\,p_4^4$$
$$+ 3t^2(t-3s)\,p_1^4\,p_4^4 + t\left(s^2-2ts-3t^2\right)p_3^6\,p_4^2 - 2s\left(3s^2+4ts+3t^2\right)p_2^2\,p_3^4\,p_4^2$$
$$- st(3s+t)\,p_1^2\,p_3^4\,p_4^2 - s^2(3s+7t)\,p_2^4\,p_3^2\,p_4^2 - 2st(7s+13t)\,p_1^2\,p_2^2\,p_3^2\,p_4^2 + t^2(17s+9t)\,p_1^4\,p_3^2\,p_4^2$$
$$- 2s^2t\,p_1^2\,p_2^4\,p_4^2 - 8st^2\,p_1^4\,p_2^2\,p_4^2 - 6t^3\,p_1^6\,p_4^2 + st(s+t)\,p_2^2\,p_3^6 + 3s^2(s+t)\,p_2^4\,p_3^4$$
$$+ st(5s+9t)\,p_1^2\,p_2^2\,p_3^4 + 4s^2t\,p_1^2\,p_2^4\,p_3^2 + 4st^2\,p_1^4\,p_2^2\,p_3^2 - 12i\left(s^2-t^2\right)(l\cdot v)\,p_3^2\,p_4^4$$
$$+ 12i(s-t)t(l\cdot v)\,p_1^2\,p_4^4 + 12i\left(s^2-t^2\right)(l\cdot v)\,p_3^4\,p_4^2 + 4is(3s+7t)(l\cdot v)\,p_2^2\,p_3^2\,p_4^2$$
$$- 4it(5s+3t)(l\cdot v)\,p_1^2\,p_3^2\,p_4^2 + 8ist(l\cdot v)\,p_1^2\,p_2^2\,p_4^2 + 24it^2(l\cdot v)\,p_1^4\,p_4^2 - 12is(s+t)(l\cdot v)\,p_2^2\,p_3^4$$
$$- 16ist(l\cdot v)\,p_1^2\,p_2^2\,p_3^2 + st\left(7s^2+6ts-t^2\right)p_3^2\,p_4^4 + s(11s-t)t^2\,p_1^2\,p_4^4$$
$$+ st\left(-s^2+6ts+7t^2\right)p_3^4\,p_4^2 - s^2t(5s+t)\,p_2^2\,p_3^2\,p_4^2 - st^2(13s+5t)\,p_1^2\,p_3^2\,p_4^2 + 8s^2t^2\,p_1^2\,p_2^2\,p_4^2$$
$$+ 12st^3\,p_1^4\,p_4^2 - s^2t(s+t)\,p_2^2\,p_3^4 - 4s^2t^2\,p_1^2\,p_2^2\,p_3^2 + 24ist(s+t)(l\cdot v)\,p_3^2\,p_4^2$$
$$- 24ist^2(l\cdot v)\,p_1^2\,p_4^2 - 4s^2t^2(s+t)\,p_3^2\,p_4^2 + 6st^2\left(2s\mu^2+2t\mu^2-st\right)p_1^2\,p_4^2\Big], \tag{4.35}$$

where we again use the labeling convention of Fig. 4.1 so that two-particle cut-free terms may be readily identified. (We do not include contributions from bubble-on-external-leg diagrams since we demanded that they integrate to zero, as discussed in Section 4.2.) There are no terms that survive either two-particle cut of Fig. 4.4. We then argue, as we did in Section 4.4.2, that this difference vanishes after integration. The tensor integrals involving $p\cdot v$ present no additional complications. By Lorentz invariance, the only objects that can contract with the dual vector, v^μ, after integration are external momenta. These scalar products vanish. The other color-ordered amplitudes agree after integration, as well. Even though our amplitudes agree with literature, converting the formal-polarization numerators into a helicity basis using the method of Section 4.3 yields rather complicated terms that then integrate to zero. Of course, these terms can be dropped immediately upon encountering them because they contain no s- or t-channel cuts.

4.5 Conclusions

In Chapter 3, a representation for the one-loop four-point amplitude of pure Yang-Mills theory was constructed with the duality between color and kinematics manifest. In this chapter, we extended the discussion by finding BCJ representations with fermions and scalars circulating in the loop. The presented expressions are valid in arbitrary dimensions and are written in terms of formal polarization vectors. Knowing the contributions from matter in the loop allowed us to construct supersymmetric BCJ amplitudes with external gluons. Furthermore, we found representations with improved loop-momentum power counting when supersymmetric field content is present.

We then compared a subset of our results to three amplitudes in literature that obey color-kinematics duality: the $\mathcal{N}=4$ sYM amplitude of Ref. [72], the $\mathcal{N}=1$ (chiral) MHV sYM amplitude of Ref. [25], and the $\mathcal{N}=0$ all-plus-helicity YM amplitude of Ref. [33]. These amplitudes were expressed in a four-dimensional helicity basis. Our formal-polarization BCJ numerators are lengthy in comparison to previously obtained BCJ numerators given in four-dimensional helicity representations, so it was interesting to see what simplifications occur with helicity bases.

The $\mathcal{N}=4$ formal-polarization numerators were identified as stA_4^{tree}, so we immediately obtained the well-known results. Putting our $\mathcal{N}=1$ (chiral) numerators into a four-dimensional MHV configuration revealed a simplification on par with Ref. [25]. While these numerators are not in exact agreement with Ref. [25], we showed that both sets of BCJ numerators produced the same amplitudes after integration. Likewise, our $\mathcal{N}=0$ numerators in the all-plus-helicity configuration produced the same integrated amplitude as Ref. [33]. However, our all-plus-helicity numerators contained reasonably complicated terms that vanish on the unitarity cuts. For the generic case of nonsupersymmetric amplitudes there does not appear to be any such simplification.

In summary, we provided further examples showing that BCJ duality appears to extend to loop level even without supersymmetry. It would be important, not only to construct further loop-level examples, but to find a systematic means of constructing loop integrands in a form compatible with BCJ duality without relying on an ansatz.

Part II

Subleading Soft Theorems in Yang-Mills Theory and Gravity

CHAPTER 5

Introduction to Subleading Soft Limits

Interest in the soft behavior of gravitons and gluons has recently been renewed by a proposal from Strominger and collaborators [82, 83] showing that soft-graviton behavior follows from Ward identities of extended Bondi, van der Burg, Metzner and Sachs (BMS) symmetry [84, 85]. This has stimulated a variety of studies of the subleading soft behavior of gravitons and gluons. In four spacetime dimensions, Cachazo and Strominger [83] showed that tree-level graviton amplitudes have a universal behavior through second subleading order in the soft-graviton momentum. In Ref. [86] an analogous description of tree-level soft behavior for gluons at first subleading order was given. Interestingly, these universal behaviors hold in D dimensions as well [87]. In four dimensions, there is an interesting connection between the subleading soft behavior in gauge theory and conformal invariance [88, 89]. There are also recent constructions of twistor-related theories with the desired soft properties [90]. Soft behavior in string theory and for higher-dimension operators has also been discussed [91, 89].

Soft theorems have a long history and were recognized in the 1950s and 1960s to be an important consequence of local on-shell gauge invariance [92, 93, 94, 95, 96]. (For a discussion of the low-energy theorem for photons see Chapter 3 of Ref. [97].) For photons, Low's theorem [93] determines the amplitudes with a soft photon from the corresponding amplitudes without a photon, through $\mathcal{O}(q^0)$, where q is the soft-photon momentum.

The universal leading soft-graviton behavior was first discussed by Weinberg [94, 95]. The leading behavior is uncorrected to all loop orders [98]. Using dispersion relations, Gross and Jackiw analyzed the particular example of Compton scattering of gravitons on massive scalar particles [99]. They showed that, for fixed angle, the Born contributions have no corrections up to, but not including, fourth order in the soft momentum. Jackiw then applied gauge-invariance arguments similar to those of Low to reanalyze this case [100]. However, for our purposes this case is too special because the degenerate kinematics of $2 \to 2$ scattering leads to extra suppression not only at tree level, but at loop level as well. In particular, the soft limits are finite at fixed angle. This may be contrasted with the behavior for larger numbers of legs, where the amplitudes at all loop orders diverge as a graviton becomes soft, matching the tree behavior. Thus, the results of Refs. [99, 100] cannot be directly applied to our discussion of n-point behavior. A more recent discussion of the generic subleading behavior of soft gluons and gravitons is given in Refs. [101, 102].

CHAPTER 6

On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons

6.1 Introduction

One might hope that at least the first subleading soft behavior discussed in Chapter 5 is a theorem valid to all loop orders, as suggested by its link to BMS symmetry [83]. However, symmetries at loop level are delicate because of the need to regularize ultraviolet and infrared divergences. The required regularization can modify Ward identities derived from symmetries. In this chapter, we demonstrate in a simple way that graviton infrared singularities imply that there are loop corrections to the subleading behavior of scattering amplitudes as external gravitons become soft, when we use the standard definition of such limits. These corrections are effectively a quantum breaking of the symmetry responsible for the tree-level behavior.

In order to understand the loop-level behavior of soft gravitons, it is useful to first look at the well-studied case of loop corrections to soft gluons in quantum chromodynamics (QCD) [103, 104]. The subleading soft-gluon behavior was already discussed using the eikonal approach [101]. A simple proof of the universal subleading soft behavior of gluons at tree level was recently given [86], following the corresponding proof for gravitons [83]. The connection between the two theories is not surprising. Gravity scattering amplitudes are closely related to gauge-theory ones and can even be constructed directly from them [105, 106, 36, 98, 10, 11].

At one loop, the modifications to the leading soft-gluon behavior are directly tied to the infrared singularities, and can be used to deduce the complete correction including finite parts [103]. When a gluon becomes soft, there is a mismatch between the infrared singularities at n points and at $n-1$ points, so loop corrections to the soft function are required to absorb this mismatch. Following the gauge-theory case, we use the infrared singularities of gravity loop amplitudes [94, 95, 107, 108] to deduce the existence of loop corrections to the subleading soft-graviton behavior. As in QCD, discontinuities in the infrared singularities arise as one goes from n points to $n-1$ points by taking a soft limit in the standard way. In gravity, the leading soft-graviton behavior is smooth because the dimensionful coupling ensures that any discontinuity is suppressed by at least one additional factor of the soft momentum [98]. However, since there is less suppression in subleading soft pieces, loop corrections survive. This allows us to demonstrate in a simple way that the subleading behavior of gravitons indeed has loop corrections similar to the loop corrections that appear in QCD. As the loop order increases, the suppression increases. Hence, the first subleading behavior is protected against corrections starting at two loops and the second subleading behavior is protected against corrections starting at three loops.

This chapter is organized as follows. In Section 6.2, we give preliminaries on the tree-level behavior of soft gluons and gravitons. In Section 6.3, we turn to the main subject of this chapter: the behavior of the subleading contributions at loop level, showing that there are nontrivial one-loop corrections to subleading soft-graviton behavior. In Section 6.4, we discuss the all-loop behavior. In Section 6.5, we briefly discuss our definition of dimensionally-regularized soft limits. We give our conclusions in Section 6.6.

6.2 Preliminaries

In this section, we summarize the soft behavior of gravitons and gluons at tree level, including their subleading behavior.

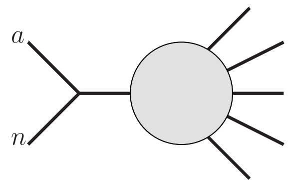

Figure 6.1: The diagrams where leading and subleading contributions to the tree soft factor arise. Leg n is the soft leg.

6.2.1 Soft gravitons

At tree level, consider the soft scaling of momentum k_n of an n-point amplitude,

$$k_n^{\alpha\dot{\alpha}} \to \delta k_n^{\alpha\dot{\alpha}}\,, \qquad \lambda_n^{\alpha} \to \sqrt{\delta}\lambda_n^{\alpha}\,, \qquad \tilde{\lambda}_n^{\dot{\alpha}} \to \sqrt{\delta}\tilde{\lambda}_n^{\dot{\alpha}}\,, \tag{6.1}$$

where $k_n^{\alpha\dot{\alpha}} = \lambda_n^{\alpha}\tilde{\lambda}_n^{\dot{\alpha}}$ is the standard decomposition of a massless momentum in terms of spinors. (See e.g. Ref. [80] for the spinor-helicity formalism used for scattering amplitudes.) In the limit (6.1), an n-point graviton tree amplitude behaves as [83]

$$M_n^{\text{tree}} \to \Big(\frac{1}{\delta}\,S_n^{(0)} + S_n^{(1)} + \delta\,S_n^{(2)}\Big)M_{n-1}^{\text{tree}} + \mathcal{O}(\delta^2)\,, \tag{6.2}$$

where δ is taken to be a small parameter. The soft operators are

$$\begin{aligned}
S_n^{(0)} &= \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^{\mu} k_i^{\nu}}{k_n \cdot k_i}\,, \\
S_n^{(1)} &= -i\sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^{\mu} k_{n\rho} J_i^{\nu\rho}}{k_n \cdot k_i}\,, \\
S_n^{(2)} &= -\frac{1}{2}\sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_{n\rho} J_i^{\mu\rho} k_{n\sigma} J_i^{\nu\sigma}}{k_n \cdot k_i}\,,
\end{aligned} \tag{6.3}$$

where $\varepsilon_{\mu\nu}$ is the graviton polarization tensor of the soft leg n and $J_i^{\mu\nu}$ is the angular momentum operator for particle i. $S_n^{(0)}$ is the leading term found long ago by Weinberg [94, 95]. For simplicity, we suppress powers of the gravitational coupling $\kappa/2$ here and in the remaining part of the chapter. In a helicity basis with a plus-helicity soft graviton, the explicit forms of the operators are

$$\begin{aligned} S_n^{(0)} &= -\sum_{i=1}^{n-1} \frac{[n\,i]\,\langle x\,i\rangle\,\langle y\,i\rangle}{\langle n\,i\rangle\,\langle x\,n\rangle\,\langle y\,n\rangle}\,, \\ S_n^{(1)} &= -\frac{1}{2}\sum_{i=1}^{n-1} \frac{[n\,i]}{\langle n\,i\rangle}\left(\frac{\langle x\,i\rangle}{\langle x\,n\rangle}+\frac{\langle y\,i\rangle}{\langle y\,n\rangle}\right)\tilde\lambda_n^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\,, \\ S_n^{(2)} &= -\frac{1}{2}\sum_{i=1}^{n-1} \frac{[n\,i]}{\langle n\,i\rangle}\tilde\lambda_n^{\dot\alpha}\tilde\lambda_n^{\dot\beta}\frac{\partial^2}{\partial\tilde\lambda_i^{\dot\alpha}\partial\tilde\lambda_i^{\dot\beta}}\,, \end{aligned} \tag{6.4}$$

where λ_x and λ_y are arbitrary massless reference spinors, which reflect gauge invariance. We follow the standard conventions of $s_{ab} = \langle a\,b\rangle\,[b\,a]$. The case of a minus-helicity soft graviton follows from parity conjugation. The first subleading behavior was discussed first in Ref. [102].

It is convenient to present the subleading behavior in terms of a holomorphic scaling of the spinors [83]. An advantage is that it makes the factorization channels clearer because the universal subleading behavior appears as poles in the scattering amplitudes. Taking leg n of an n-point amplitude to be a soft plus-helicity graviton, we scale the spinors as

$$k_n^\mu \to \delta k_n^\mu\,, \qquad \lambda_n^\alpha \to \delta\lambda_n^\alpha\,, \qquad \tilde\lambda_n^{\dot\alpha} \to \tilde\lambda_n^{\dot\alpha}\,. \tag{6.5}$$

Under this rescaling, tree-level graviton amplitudes behave as [83]

$$M_n^{\rm tree} \to \left(\frac{1}{\delta^3}S_n^{(0)} + \frac{1}{\delta^2}S_n^{(1)} + \frac{1}{\delta}S_n^{(2)}\right)M_{n-1}^{\rm tree} + \mathcal{O}(\delta^0)\,, \tag{6.6}$$

where $M_n^{\rm tree}$ is the n-point amplitude and $M_{n-1}^{\rm tree}$ is the $(n-1)$-point amplitude obtained by removing the soft leg n. The connection of the two scalings is through little-group scaling. The proof of universality [83] of the subleading soft behavior (6.3) relies on all contributions arising from factorizations on $1/(k_a + k_n)^2$ propagators in the soft kinematics (6.5), as illustrated in Fig. 6.1.

Some care is needed to interpret the soft behavior in Eq. (6.6) because the n-point kinematics of the amplitude on the left-hand side of the equation is not the same as the $(n-1)$-point kinematics normally used to define the amplitude on the right-hand side of the equation. This becomes an issue for the subleading soft terms because of feed down from leading terms to subleading ones, depending on the precise prescription. The prescription chosen by Cachazo and Strominger is to explicitly impose n-point momentum conservation on the amplitude on the left-hand side and $(n-1)$-point momentum conservation on the amplitude on the right-hand side. This constraint is conveniently implemented via

$$\tilde{\lambda}_1 = -\sum_{i=3}^{m} \frac{\langle 2\, i\rangle}{\langle 2\, 1\rangle}\tilde{\lambda}_i\,, \qquad\qquad \tilde{\lambda}_2 = -\sum_{i=3}^{m} \frac{\langle 1\, i\rangle}{\langle 1\, 2\rangle}\tilde{\lambda}_i\,, \tag{6.7}$$

so that $\sum_{i=1}^{m} \lambda_i \tilde{\lambda}_i = 0$. This constraint is imposed on the amplitudes on the left-hand side of Eq. (6.6) with $m = n$ and on the right-hand side with $m = n-1$.

For our loop-level study, we use a different prescription. We interpret the expressions on both sides of Eq. (6.6) as carrying the *same* n-point kinematics, without needing to apply any additional constraints on the kinematics. The advantage is that this prevents complicated terms from feeding down from higher- to lower-order terms in the soft expansion, which would obscure the structure at loop level. This change in prescription effectively shifts contributions between different orders in the expansion.[1]

6.2.2 Soft gluons

Following the same derivation as for gravitons, tree-level Yang-Mills amplitudes also have a universal subleading soft behavior [86]. If we scale $\lambda_n \to \delta\lambda_n$, the color-ordered amplitude behaves as

$$A_n^{\rm tree} \to \Big(\frac{1}{\delta^2} S^{(0)}_{n\,\rm YM} + \frac{1}{\delta} S^{(1)}_{n\,\rm YM}\Big) A_{n-1}^{\rm tree}\,, \tag{6.8}$$

[1]We numerically confirmed in many examples that the two prescriptions give identical results through $\mathcal{O}(\delta)$ in Eq. (6.2).

where the leading soft factor is

$$S^{(0)}_{n\,\mathrm{YM}} = \frac{k_1 \cdot \varepsilon_n}{\sqrt{2}\,k_1 \cdot k_n} - \frac{k_{n-1} \cdot \varepsilon_n}{\sqrt{2}\,k_{n-1} \cdot k_n}\,. \tag{6.9}$$

The subleading one is

$$S^{(1)}_{n\,\mathrm{YM}} = -i\varepsilon_{n\mu} k_{n\nu} \left(\frac{J_1^{\mu\nu}}{\sqrt{2}\,k_1 \cdot k_n} - \frac{J_{n-1}^{\mu\nu}}{\sqrt{2}\,k_{n-1} \cdot k_n} \right)\,. \tag{6.10}$$

Again we have suppressed the coupling constants. Using spinor-helicity, the plus-helicity gluon leading soft factor is

$$S^{(0)}_{n\,\mathrm{YM}} = \frac{\langle (n-1)\,1\rangle}{\langle (n-1)\,n\rangle \langle n\,1\rangle}\,, \tag{6.11}$$

while the subleading operator is

$$S^{(1)}_{n\,\mathrm{YM}} = \frac{1}{\langle (n-1)\,n\rangle} \tilde{\lambda}_n^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{n-1}^{\dot{\alpha}}} - \frac{1}{\langle 1\,n\rangle} \tilde{\lambda}_n^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_1^{\dot{\alpha}}}\,. \tag{6.12}$$

An earlier description was given in Ref. [101].

6.3 One-loop corrections to subleading soft behavior

As shown by Weinberg [94, 95], the leading soft-graviton behavior has no higher-loop corrections. In Ref. [83], Cachazo and Strominger demonstrated that their proposed theorem for subleading soft-graviton behavior holds at tree level.

Here, we demonstrate that there are nontrivial loop corrections for the subleading soft-graviton behavior analogous to the ones that appear in QCD for the leading soft terms, using the standard definition of soft limits in dimensional regularization. As in QCD, loop corrections linked to infrared divergences necessarily appear because of mismatches in the logarithms of the infrared singularities at n and $n-1$ points. Divergences require a regulator which can break symmetries at the quantum level. In this sense, we can think of the loop corrections as due to an anomaly in the underlying symmetry. Its origin is similar to the

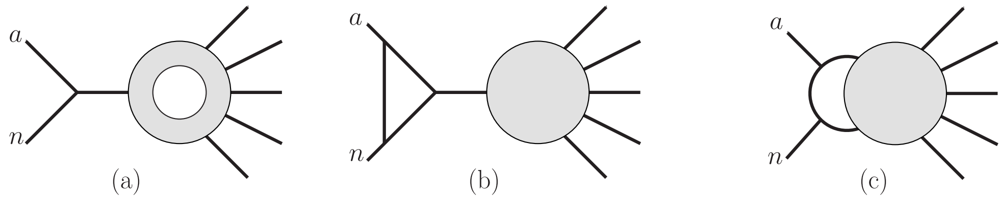

Figure 6.2: At one loop, the simple tree-level soft behavior (a) is corrected by factorizing (b) and nonfactorizing (c) contributions [103]. In gravity, the corrections are suppressed by factors of the soft momentum k_n, but they affect the subleading behavior.

twistor-space holomorphic anomaly [109], where extra contributions arise in regions of loop integration that are singular.

In general, the structure of the loop corrections to soft behavior is entangled with the infrared divergences. This phenomenon is familiar in QCD [110, 103], so we discuss this case first before turning to gravity. Besides corrections that arise from infrared singularities, we will find that there are other loop corrections due to nontrivial factorization properties [111, 112, 113], even for infrared-finite one-loop amplitudes.

6.3.1 One-loop corrections to soft-gluon behavior

In general, loop-level factorization properties of gauge theories are surprisingly nontrivial, in part, because of their entanglement with infrared singularities [110]. This causes naive notions of factorization in soft and other kinematic limits to break down; in massless gauge theories, one can obtain kinematic poles also from the loop integration. However, because the infrared singularities have a universal behavior, they offer a simple means for studying soft limits of loop amplitudes with an arbitrary number of external legs.

Fig. 6.2 shows the types of contributions to the one-loop soft behavior when the amplitude is represented in terms of the standard covariant basis of integrals. These consist of "factorizing" contributions, illustrated in Fig. 6.2(b), and "nonfactorizing" contributions, illustrated in Fig. 6.2(c).[2] The nonfactorizing contributions arise from poles in the S-matrix

[2]In light-cone gauge or the unitarity approach, by introducing light-cone denominators containing a

coming from loop integration and not directly from propagators, as illustrated in Fig. 6.2(c).

As a simple example, consider the single-external-mass box integral, displayed in Fig. 6.3. This is one of the basis integrals for one-loop amplitudes. The infrared-divergent terms of this integral are [115]

$$I_4^{1\mathrm{m}} = \frac{2i\, c_\Gamma}{s_{n1}s_{12}} \left[\frac{1}{\epsilon^2} \left(\left(\frac{\mu^2}{-s_{n1}} \right)^\epsilon + \left(\frac{\mu^2}{-s_{12}} \right)^\epsilon - \left(\frac{\mu^2}{-s_{n12}} \right)^\epsilon \right) + \text{finite} \right], \tag{6.13}$$

where the labels correspond to those in Fig. 6.3. We also have

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}, \qquad s_{i_1 i_2 \cdots i_j} = (k_{i_1} + k_{i_2} + \cdots + k_{i_j})^2 . \tag{6.14}$$

When leg n goes soft, the integral has a $1/s_{n1}$ kinematic pole from the prefactor. While one might expect such poles to cancel out of amplitudes, they, in fact, remain due to their entanglement with infrared singularities. However, this link ensures that they have a regular pattern. In general, these nonfactorizing contributions need to be accounted for in loop-level soft behavior and other factorization limits in gauge theories. The same holds for the subleading soft behavior of gravity amplitudes.

A one-loop n-gluon amplitude in QCD has ultraviolet and infrared singularities given by [116, 110]

$$A_n^{\text{1-loop}}(1, 2, \cdots, n)\Big|_{\text{div.}} = -\frac{1}{\epsilon^2} A_n^{\text{tree}}(1, 2, \cdots, n)\sigma_n^{\text{YM}}, \tag{6.15}$$

where

$$\sigma_n^{\text{YM}} = c_\Gamma \left[\sum_{j=1}^{n} \left(\frac{\mu^2}{-s_{j,j+1}} \right)^\epsilon + 2\epsilon \left(\frac{11}{6} - \frac{1}{3}\frac{n_f}{N_c} - \frac{1}{6}\frac{n_s}{N_c} \right) \right]. \tag{6.16}$$

In this expression, n_f is the number of quark flavors, n_s is the number of scalar flavors (zero in QCD) and N_c is the number of colors. Here, $\epsilon = (4-D)/2$ is the dimensional-regularization parameter, and μ^2 is the usual dimensional-regularization scale. It turns out that it is best to work with unrenormalized amplitudes containing also ultraviolet divergences because the mismatch in the number of coupling constants at n and $n-1$ points causes an additional

reference momentum, one can push all contributions into factorizing diagrams [114, 104].

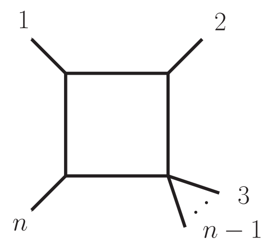

Figure 6.3: An example of an integral that has a "nonfactorizing" kinematic pole that contributes to the soft behavior.

(trivial) discontinuity in the soft behavior. By working with unrenormalized amplitudes, we avoid this. A key property of Eq. (6.16) is that the terms depending on the number of quark and scalar flavors is independent of the number of external gluons. The terms in the summation arise from soft-gluon singularities in the loop integration. In general, the expression in Eq. (6.16) should be interpreted as being series expanded in ϵ, since terms beyond $\mathcal{O}(\epsilon^0)$ that are usually not computed can mix nontrivially with these.

Consider the soft limit of the singular parts of the gauge-theory amplitude (6.15). The tree prefactor obeys the simple soft behavior given in Eq. (6.8). The infrared singularities, however, have a mismatch between n points and $n-1$ points:

$$\sigma_n^{\rm YM} = \sigma_{n-1}^{\rm YM} + \sigma_n^{\prime\rm YM} + \mathcal{O}(\epsilon^2)\,, \tag{6.17}$$

where

$$\sigma_n^{\prime\rm YM} = c_\Gamma\left(1 + \epsilon\log\left(\frac{-\mu^2 s_{(n-1)1}}{s_{(n-1)n}s_{n1}}\right)\right). \tag{6.18}$$

It turns out that this mismatch can be used to deduce the complete one-loop corrections to the leading soft factor by matching the infrared discontinuities in the basis integrals to the infrared discontinuities in the amplitude [103].

The leading soft behavior of an n-gluon amplitude with any matter content for $\lambda_n \to \delta\lambda_n$ is then [103, 104]

$$A_n^{\text{1-loop}} \to S_{n\,\rm YM}^{(0)} A_{n-1}^{\text{1-loop}} + S_{n\,\rm YM}^{(0)\text{1-loop}} A_{n-1}^{\rm tree}\,, \tag{6.19}$$

where the leading one-loop soft correction function is

$$
\begin{aligned}
S^{(0)\,\text{1-loop}}_{n\,\text{YM}} &= -S^{(0)}_{n\,\text{YM}} \frac{c_\Gamma}{\epsilon^2} \left(\frac{-\mu^2 s_{(n-1)1}}{s_{(n-1)n} s_{n1}} \right)^{\epsilon} \frac{\pi\epsilon}{\sin(\pi\epsilon)} \\
&= -S^{(0)}_{n\,\text{YM}} c_\Gamma \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log\left(\frac{-\mu^2 s_{(n-1)1}}{\delta^2 s_{(n-1)n} s_{n1}} \right) + \frac{1}{2} \log^2\left(\frac{-\mu^2 s_{(n-1)1}}{\delta^2 s_{(n-1)n} s_{n1}} \right) + \frac{\pi^2}{6} \right) \\
&\qquad + \mathcal{O}(\epsilon)\,.
\end{aligned} \tag{6.20}
$$

The form on the first line is valid to all orders in ϵ. In applying this equation, it is important to first expand in ϵ prior taking the soft limit.

Now consider the subleading soft terms. Taking the divergent part of the one-loop amplitude to have a soft limit of the form,

$$
A^{\text{1-loop}}_n \Big|_{\text{div.}} \to \left(\frac{1}{\delta^2} S^{(0)}_{n\,\text{YM}} + \frac{1}{\delta} S^{(1)}_{n\,\text{YM}} \right) A^{\text{1-loop}}_{n-1} \Big|_{\text{div.}} + \left(\frac{1}{\delta^2} S^{(0)\,\text{1-loop}}_{n\,\text{YM}} + \frac{1}{\delta} S^{(1)\,\text{1-loop}}_{n\,\text{YM}} \right) A^{\text{tree}}_{n-1} \Big|_{\text{div.}}\,, \tag{6.21}
$$

we then solve for the divergent parts of the one-loop corrections to the soft operators, denoted by $S^{(i)\,\text{1-loop}}_{n\,\text{YM}}$. We do so by comparing the soft expansion of the left-hand side of Eq. (6.21) to the terms on the right-hand side. Applying $S^{(1)}_{n\,\text{YM}}$ to the infrared singularity of the $(n-1)$-point amplitude gives

$$
\begin{aligned}
S^{(1)}_{n\,\text{YM}} \sigma^{\text{YM}}_{n-1} &= -c_\Gamma \epsilon \Big(\frac{[1\,n]}{[1\,(n-1)]\,\langle (n-1)\,n \rangle} - \frac{[(n-1)\,n]}{[(n-1)\,1]\,\langle 1\,n \rangle} \\
&\qquad + \frac{[(n-2)\,n]}{[(n-2)\,(n-1)]\,\langle (n-1)\,n \rangle} - \frac{[2\,n]}{[2\,1]\,\langle 1\,n \rangle} \Big)\,,
\end{aligned} \tag{6.22}
$$

where we use the form of σ^{YM}_{n-1} exactly as it appears in Eq. (6.16) without any additional momentum-conservation relations imposed. Taking the one-loop correction to the subleading soft function to be

$$
S^{(1)\,\text{1-loop}}_{n\,\text{YM}} = -\frac{1}{\epsilon^2} \left[\sigma'^{\text{YM}}_n S^{(1)}_{n\,\text{YM}} - \left(S^{(1)}_{n\,\text{YM}} \sigma^{\text{YM}}_{n-1} \right) \right] + \mathcal{O}(\epsilon^0)\,, \tag{6.23}
$$

we find that Eq. (6.21) holds. The simple form of the correction relies on using the specific form for $S^{(1)}\sigma_{n-1}$ in Eq. (6.22). We also interpret both sides of Eq. (6.21) as having the same n-point kinematics.

It would be important to understand the infrared-finite terms as well. These also have nontrivial corrections. For the case of the infrared-finite identical-helicity one-loop amplitudes [117], numerical analysis through 30 points shows that the amplitudes behave exactly as tree-level amplitudes with no nontrivial corrections. However, the one-loop amplitudes with a single minus helicity [111] have nontrivial subleading soft behavior. As an example, consider the one-loop five-gluon amplitude [118, 111],

$$A_5^{\text{1-loop}}(1^-,2^+,3^+,4^+,5^+) = \frac{i}{48\pi^2}\frac{1}{\langle 3\,4\rangle^2}\left[-\frac{[2\,5]^3}{[1\,2]\,[5\,1]} + \frac{\langle 1\,4\rangle^3\,[4\,5]\,\langle 3\,5\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle^2} - \frac{\langle 1\,3\rangle^3\,[3\,2]\,\langle 4\,2\rangle}{\langle 1\,5\rangle\,\langle 5\,4\rangle\,\langle 3\,2\rangle^2}\right], \tag{6.24}$$

as the momentum of leg 5 becomes soft. The four-point one-loop single-minus-helicity amplitude is [33, 62]

$$A_4^{\text{1-loop}}(1^-,2^+,3^+,4^+) = \frac{i}{48\pi^2}\frac{\langle 2\,4\rangle\,[2\,4]^3}{[1\,2]\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,[4\,1]}\,. \tag{6.25}$$

Applying the tree-level operators to the four-point amplitude, as in Eq. (6.8), yields

$$\begin{aligned}\Big(\frac{1}{\delta^2}S^{(0)}_{n\,\text{YM}} + \frac{1}{\delta}S^{(1)}_{n\,\text{YM}}\Big)&A_4^{\text{1-loop}}(1^-,2^+,3^+,4^+)\\ &= \frac{i}{48\pi^2}\frac{\langle 1\,3\rangle^3\,\langle 2\,4\rangle\,[1\,2]}{\langle 2\,3\rangle^2\,\langle 3\,4\rangle^3}\left(\frac{1}{\delta^2}\frac{\langle 4\,1\rangle}{\langle 4\,5\rangle\,\langle 5\,1\rangle} + \frac{1}{\delta}\frac{[5\,2]}{\langle 5\,1\rangle\,[1\,2]}\right).\end{aligned} \tag{6.26}$$

After applying the operators, we applied five-point momentum conservation to remove the anti-holomorphic spinors $\tilde{\lambda}_3$, $\tilde{\lambda}_4$.[3] This facilitates comparison with the soft limit of the

[3] We note that the momentum-conservation prescription of Ref. [83] gives the same conclusion.

five-point amplitude (6.24). With the same constraints applied, this is given by

$$A_5^{\text{1-loop}}(1^-,2^+,3^+,4^+,5^+) \to \frac{i}{48\pi^2}\Bigg[\frac{\langle 1\,3\rangle^3\langle 2\,4\rangle[1\,2]}{\langle 2\,3\rangle^2\langle 3\,4\rangle^3}\left(\frac{1}{\delta^2}\frac{\langle 4\,1\rangle}{\langle 4\,5\rangle\langle 5\,1\rangle}+\frac{1}{\delta}\frac{[5\,2]}{\langle 5\,1\rangle[1\,2]}\right) + \frac{1}{\delta}\frac{\langle 1\,4\rangle^3\langle 3\,5\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle^3\langle 4\,5\rangle^2}(\langle 1\,3\rangle[1\,5]+\langle 2\,3\rangle[2\,5])\Bigg]. \tag{6.27}$$

While the leading order pieces are identical, the subleading pieces differ in Eqs. (6.26) and (6.27).

The nontrivial behavior of the single-minus-helicity amplitudes is not surprising given that they contain nontrivial complex poles that cannot be interpreted as a straightforward factorization. In general, nonsupersymmetric gauge-theory loop amplitudes contain such nontrivial poles. This phenomenon complicates the construction of gauge and gravity loop amplitudes from their poles and has been described in some detail in Refs. [112, 113]. We leave the discussion of such infrared-finite contributions to the future.

6.3.2 One-loop corrections to soft-graviton behavior

Applying a similar analysis, it is straightforward to see that one-loop corrections to the subleading soft-graviton behavior do not vanish because of mismatched logarithms in the infrared singularities. At one loop, the n-graviton amplitude contains the dimensionally-regularized infrared-singular terms [119, 107, 108],

$$M_n^{\text{1-loop}}\Big|_{\text{div.}} = \frac{\sigma_n}{\epsilon}M_n^{\text{tree}}\,, \tag{6.28}$$

where M_n^{tree} is the n-graviton tree amplitude, and

$$\sigma_n = -c_\Gamma\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} s_{ij}\log\left(\frac{\mu^2}{-s_{ij}}\right), \tag{6.29}$$

where c_Γ is defined in Eq. (6.14). As in QCD, the logarithms that appear at n points are not identical to the ones appearing at $(n-1)$ points. The logarithms in the infrared singularity that differ between an n- and $(n-1)$-graviton amplitude are

$$\sigma'_n = -c_\Gamma \sum_{i=1}^{n-1} s_{in} \log\Big(\frac{\mu^2}{-s_{in}}\Big) . \qquad (6.30)$$

While this mismatch does not affect the leading soft behavior because of the suppression from the s_{in} factors, it does affect subleading terms.

By absorbing the mismatches into corrections to the subleading soft operator, we find that in the soft limit $\lambda_n \to \delta\lambda_n$, the infrared singular terms behave as

$$M_n^{\text{1-loop}}\Big|_{\text{div.}} \to \left(\frac{S_n^{(0)}}{\delta^3} + \frac{S_n^{(1)}}{\delta^2} + \frac{S_n^{(2)}}{\delta}\right) M_{n-1}^{\text{1-loop}}\Big|_{\text{div.}} + \left(\frac{S_n^{(1)\,\text{1-loop}}}{\delta^2} + \frac{S_n^{(2)\,\text{1-loop}}}{\delta}\right) M_{n-1}^{\text{tree}}\Big|_{\text{div.}} , \qquad (6.31)$$

where

$$\begin{aligned}
S_n^{(0)\,\text{1-loop}}\Big|_{\text{div.}} &= 0 \,, \\
S_n^{(1)\,\text{1-loop}}\Big|_{\text{div.}} &= \frac{1}{\epsilon}\Big[\sigma'_n S_n^{(0)} - \Big(S_n^{(1)}\sigma_{n-1}\Big)\Big] \,, \\
S_n^{(2)\,\text{1-loop}}\Big|_{\text{div.}} &= \frac{1}{\epsilon}\Big[\sigma'_n S_n^{(1)} - \Big(S_n^{(2)}\sigma_{n-1}\Big) + \sum_{i=1}^{n-1} \frac{[n\,i]}{\langle n\,i\rangle}\Big(\tilde\lambda_n^{\dot\alpha}\frac{\partial\sigma_{n-1}}{\partial\tilde\lambda_i^{\dot\alpha}}\Big)\tilde\lambda_n^{\dot\beta}\frac{\partial}{\partial\tilde\lambda_i^{\dot\beta}}\Big] .
\end{aligned} \qquad (6.32)$$

Similar to the gauge-theory case, the simple form of these corrections to the subleading soft operators relies on using the form of σ_{n-1} obtained from Eq. (6.29) with no additional momentum-conservation relations imposed. We again also interpret both sides of Eq. (6.31) as having the same n-point kinematics. As in QCD, it is important to follow the standard procedure of first series expanding the amplitude in ϵ prior to taking soft limits.

We have checked numerically through 10 points that the infrared-finite identical-helicity graviton amplitudes [120] satisfy the same subleading soft behavior as the tree amplitudes. However, more generally we expect a more complicated behavior due to the nontrivial factorization properties of loop amplitudes [111, 112]. Such nontrivial factorization properties have

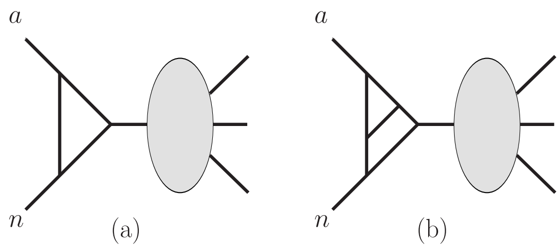

Figure 6.4: Sample factorizing (a) one- and (b) two-loop contributions to the soft behavior.

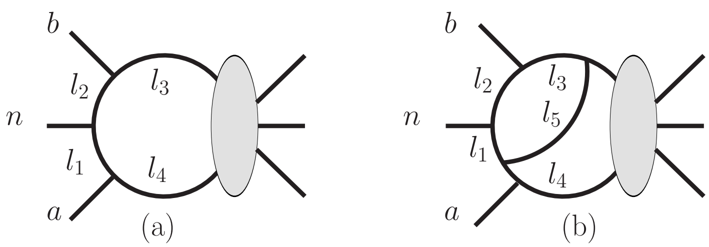

Figure 6.5: Sample nonfactorizing (a) one- and (b) two-loop contributions to the soft behavior.

been discussed for gravity theories in Refs. [121, 113]. Indeed, by numerically analyzing the infrared-finite one-loop five-graviton amplitude with a single minus helicity from Ref. [121] and the one-loop four-graviton amplitude with a single minus helicity from Ref. [122], we find that the second subleading soft behavior has nontrivial corrections. We leave a discussion of the infrared-finite corrections to the graviton soft behavior to the future.

6.4 All loop order behavior of soft gravitons

As we demonstrated in the previous section, the subleading soft behavior has loop corrections. In this section, we argue that the first subleading soft behavior has no corrections beyond one loop and that the second subleading behavior has no corrections beyond two loops.

6.4.1 General considerations

The all-loop leading soft-graviton behavior has been discussed in some detail in Section 5.2 of Ref. [98]. Here we follow this discussion for the subleading behavior. As already noted for gauge theory, potential contributions to the soft behavior can be divided into "factorizing" and "nonfactorizing" contributions [110] when the amplitude is expressed in terms of covariant Feynman integrals. We consider these types of contributions in turn.

The factorizing contributions of the type displayed in Fig. 6.4 depend on the soft momentum k_n and one additional momentum k_a. After the Lorentz indices of polarization tensors are contracted, no other Lorentz invariants are present other than s_{na}. By dimensional analysis, the L-loop correction contains an additional factor κ^{2L} of the gravitational coupling relative to the tree-level contribution in Fig. 6.1, and therefore must contain relative factors of s_{na}^L. This gives a suppression of one soft momentum k_n for each additional loop.

The nonfactorizing contributions displayed in Fig. 6.5 have a similar suppression. The nonfactorizing contributions arise in regions where loop momenta become soft in addition to the external soft leg. For example, in the one-loop case displayed in Fig. 6.5(a), as $k_n \to 0$, we must also have the loop momentum go as $l_1 \to 0$ in order to obtain a nonfactorizing contribution to the soft behavior; otherwise, there would be no large contribution for $k_n \to 0$, or equivalently for $\lambda_n \to 0$. In this region, $l_2 = l_1 - k_n$, $l_3 = l_1 - k_n - k_b$ and $l_4 = l_1 + k_a$ all go on shell. After integration, this leads to potential kinematic poles in s_{an} or s_{bn}, or equivalently in λ_n. However, because gravity has an extra power of soft momentum, either k_n or l_1 in the vertex attaching leg n to the loop will suppress the pole. Similarly, at two loops, illustrated in Fig. 6.5(b), potential contributions arise when additional loop momenta become soft, in this case l_5. Once again, the dimensionful coupling ensures that there will be additional factors of soft momenta in the numerator. More generally, after integration, we get an additional L factors of s_{jn} compared to the gauge-theory case, where j can be any momentum in the amplitude.

The net effect is that there are no loop corrections to the leading soft behavior, no corrections beyond one loop for the first subleading soft behavior, and no corrections beyond two loops for the second subleading soft behavior. We therefore expect the general form of the L-loop behavior for a plus-helicity graviton with $\lambda_n \to \delta\lambda_n$ to have no loop corrections beyond two loops.

6.4.2 All loop behavior of leading infrared singularities

Since there should be no corrections beyond two loops, we expect that the L-loop leading infrared-divergent terms should behave in the soft limit as

$$\begin{aligned} M_n^{L\text{-loop}}\Big|_{\text{lead. div.}} \to \quad & \left(\frac{S_n^{(0)}}{\delta^3} + \frac{S_n^{(1)}}{\delta^2} + \frac{S_n^{(2)}}{\delta}\right) M_{n-1}^{L\text{-loop}}\Big|_{\text{lead. div.}} \\ & + \left(\frac{S_n^{(1)\,1\text{-loop}}}{\delta^2} + \frac{S_n^{(2)\,1\text{-loop}}}{\delta}\right) M_{n-1}^{(L-1)\text{-loop}}\Big|_{\text{lead. div.}} \\ & + \frac{S_n^{(2)\,2\text{-loop}}}{\delta} M_{n-1}^{(L-2)\text{-loop}}\Big|_{\text{lead. div.}} . \end{aligned} \tag{6.33}$$

We check this using the known all-loop-order form of infrared singularities in gravity theories [94, 95, 107, 108]. The infrared singularities of gravity amplitudes are given by

$$M_n = \mathcal{S}_n \mathcal{H}_n \,, \tag{6.34}$$

where M_n is a gravity amplitude valid to all loop orders and $\mathcal{H}_n$ is the infrared-finite hard function. The all-loop infrared singularity function is a simple exponentiation of the one-loop function (6.28):

$$\mathcal{S}_n = \exp\left(\frac{\sigma_n}{\epsilon}\right) . \tag{6.35}$$

From this equation, we see that the leading infrared singularity at L loops is simply given in terms of the tree amplitude:

$$M_n^{L\text{-loop}}\Big|_{\text{lead. div.}} = \frac{1}{L!}\left(\frac{\sigma_n}{\epsilon}\right)^L M_n^{\text{tree}} . \tag{6.36}$$

This gives us a simple means for testing Eq. (6.33) and also for finding the leading infrared-singular part of the two-loop operator, $S_n^{(2)\,\text{2-loop}}$. We do so by taking the difference of the soft expansion on both sides of Eq. (6.33) and using the previously determined operators in Eq. (6.32). We need the soft expansion of the leading infrared-singular part of $M_n^{L\text{-loop}}$, given by

$$\frac{\sigma_n^L}{L!} M_n^{\text{tree}} \to \frac{(\sigma_{n-1} + \delta\sigma'_n)^L}{L!} \left(\frac{S_n^{(0)}}{\delta^3} + \frac{S_n^{(1)}}{\delta^2} + \frac{S_n^{(2)}}{\delta} \right) M_{n-1}^{\text{tree}} \,, \tag{6.37}$$

where σ'_n is defined in Eq. (6.30). We also need the results of acting on $(\sigma_{n-1}^L/L!)M_{n-1}^{\text{tree}}$ with the tree-level soft operators,

$$\left(\frac{S_n^{(0)}}{\delta^3} + \frac{S_n^{(1)}}{\delta^2} + \frac{S_n^{(2)}}{\delta} \right) \frac{\sigma_{n-1}^L}{L!} M_{n-1}^{\text{tree}} \,. \tag{6.38}$$

Evaluating these, we deduce the leading infrared-divergent contribution to the two-loop soft operator to be

$$S_n^{(2)\,\text{2-loop}}\Big|_{\text{lead. div.}} = \frac{1}{\epsilon^2} \left[\frac{1}{2} \left(\sigma'_n\right)^2 S_n^{(0)} - \sigma'_n \left(S_n^{(1)} \sigma_{n-1} \right) - \left(\frac{1}{2} \sum_{i=1}^{n-1} \frac{[n\, i]}{\langle n\, i \rangle} \left(\tilde{\lambda}_n^{\dot{\alpha}} \frac{\partial \sigma_{n-1}}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \right)^2 \right) \right] . \tag{6.39}$$

The lack of higher-loop corrections to the soft operators is a consequence of the fact that they are suppressed by additional powers of the soft momentum. As before, the form of σ_{n-1} in the correction must be specifically as given in Eq. (6.29).

6.5 A Note On Dimensionally-Regularized Soft Limits

In this chapter we have used the standard definition of dimensionally-regularized soft limits where one first series expands in the dimensional-regularization parameter before taking the soft limit. We do so because it matches the one needed for scattering amplitudes and associated physical processes as they are normally computed. After the appearance of the first version of this work, a new paper appeared [123] showing that in some simple supersymmetric examples, loop corrections to the soft operators can be removed by altering the long-standing standard definition of soft limits. This alteration involves keeping the dimensional-regularization parameter finite before taking the soft limit.

The lack of loop corrections found in the examples of Ref. [123] is not surprising and is a simple consequence of the lack of discontinuities [110, 103] with the reordered limits. This is connected to the well-known fact that with a finite dimensional-regularization parameter $\epsilon < 0$, or equivalently $D > 4$, there are no infrared singularities. One can also view the prescription as equivalent to taking soft limits on integrands instead of the integrated expressions because one can push limits through the integral when they are smooth. (One can apply soft limits directly at the integrand level, but that is a distinct problem from the one for integrated amplitudes.) As an example, we immediately see from the first line of Eq. (6.20) that one-loop corrections to the leading soft function in QCD vanish for $k_n \to 0$ if we hold $\epsilon < 0$ fixed.

However, there are a number of reasons why it is important to use the standard dimensional-regularization procedure of series expanding in ϵ prior to taking soft [103, 104] or other limits. To be useful for obtaining cross sections, soft limits must be compatible with cancellations of infrared singularities between real-emission and virtual contributions. One might imagine keeping ϵ finite in both contributions in an attempt to treat them on an equal footing. However, the use of four-dimensional helicity states on external legs makes this problematic. Even in the well-understood standard definition of soft limits, one must be careful not to violate unitarity because of the incompatible treatment of real-emission and virtual contributions. (See for example Ref. [124].) Moreover, in QCD the modified prescription disrupts the cancellation of leading infrared singularities when $\epsilon \to 0$ because it alters the real-emission sigularities without changing corresponding virtual ones.

Even if there were a way to avoid difficulties with real-emission contributions, keeping ϵ finite in virtual contributions would lead to serious complications as well. In general, loop amplitudes are computed only through a fixed order in ϵ because the higher order contributions are rather complicated, except in simple supersymmetric cases, and do not carry useful physical information for the problem at hand. (For an example of the typical forms that loop amplitudes take, see Ref. [118].)

The single-minus helicity infrared-finite amplitudes are a good example of why it is best to series expand in ϵ. As noted in Sections 6.3.1 and 6.3.2, these amplitudes have another type of loop correction to soft behavior coming from nontrivial complex factorization channels and not from infrared discontinuities. (Since the first version of our work appeared, He, Huang and Wen thoroughly investigated the single-minus helicity amplitudes [125], among other topics, confirming our finding of nontrivial loop corrections.) In general, such amplitudes are known only for $\epsilon = 0$ [111, 121]. It would be highly nontrivial to obtain the higher order in ϵ contributions for the purpose of attempting to prevent renormalization of the soft operators. Furthermore, we note that loop corrections to soft behavior are, in fact, quite useful for understanding the analytic structure of amplitudes and their associated physical properties. More generally, experience shows that it is overwhelmingly simpler to absorb complications associated with dimensional regularization into loop corrections of soft limits rather than to deal with higher order in ϵ terms in amplitudes.

Consequently, while it may be tempting to change the standard definitions of dimensional regularization and soft limits in order to remove loop corrections to soft operators associated with infrared singularities, we greatly prefer the standard definitions because of their well-understood consistency, simplicity and applicability to problems of physical and theoretical interest.

6.6 Conclusions

Recently a generalization of Weinberg's soft-graviton theorem for the subleading behavior was proposed [126, 83]. (See also previous work from White [102].) Here we showed that, unlike the leading soft-graviton behavior, the subleading soft behavior requires loop corrections. In QCD, loop corrections to the leading soft functions make up for mismatches in the infrared singularities of n-point and $(n-1)$-point amplitudes. Applying this observation to gravity, we obtained the leading infrared-singular loop contributions to the subleading soft-graviton operators valid to all loop orders. This proves in a simple way that there necessarily are non-vanishing loop corrections to soft-graviton behavior. In addition, in the simple example of a five-graviton amplitude with a single minus helicity, we found additional corrections to the second subleading behavior, not linked to infrared singularities. These come from the non-trivial complex factorization properties of generic loop amplitudes [110, 111, 112, 121, 113].

Following the discussion for the leading soft-graviton behavior [94, 95, 98], we argued that there are no loop corrections to the first subleading soft behavior beyond one loop and no new corrections to the second subleading behavior beyond two loops. This is connected to the dimensionful coupling of gravity. In the regions contributing to the soft limit, an extra power of the soft momentum is obtained for each additional loop, suppressing the contributions. By the third loop order, there are a sufficient number of powers of the soft momentum to suppress further corrections to the soft operators.

We also discussed the form of subleading corrections to the soft behavior in gauge theory as a warm-up for the gravity case. It is interesting to note that the subleading soft behavior in QCD might be useful for improved soft-gluon approximations.

An important remaining task is to determine the loop corrections to the general subleading soft behavior of the infrared-finite terms in both gauge and gravity theories. While this is simple in special cases, such as for identical-helicity amplitudes [117, 120], in general, the task is complicated by the nontrivial complex factorization properties of loop amplitudes [110, 111, 112, 121, 113], on top of well-understood feed downs from infrared singularities. We leave studies of the soft behavior of infrared-finite terms in gauge and gravity amplitudes to future work.

CHAPTER 7

Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance

7.1 Introduction

Extended BMS symmetry gives us a remarkable new understanding for the behavior of soft gravitons in four spacetime dimensions [82]. However, given that universal soft behavior holds also in D dimensions as well as for gluons, we expect that there is a more general explanation not tied to four dimensions. In this chapter, we show that, just as for photons [93], on-shell gauge invariance can be used to fully determine subleading behavior. We show that in nonabelian gauge theory, on-shell gauge invariance dictates that at tree level the first subleading term is universal and controlled by the amplitude with the soft gluon removed. Similarly, in gravity the first two subleading terms at tree level are universal. Our proof is valid in D dimensions because it uses only on-shell gauge invariance together with D-dimensional three-point vertices.

We shall also explain how loop corrections arise in this context. In nonabelian gauge theory and gravity, there are "factorizing" loop corrections to the three-vertex controlling the soft behavior. However, in gravity, generically the dimensionful nature of the coupling implies that there are no loop corrections to the leading behavior [98], no corrections beyond one loop to the first subleading behavior, and no corrections beyond two loops to the second subleading behavior (see Chapter 6).

As shown long ago, in gauge theory the factorizing contributions are suppressed: In gauge theory they vanish at leading order in the soft limit [103, 104], but are nontrivial at the first subleading order (see Ref. [125] and Chapter 6). Similarly, we prove that for the case of a scalar circulating in the loop, the factorizing loop corrections to the soft-graviton behavior vanish not only for the leading order but for the first subleading order as well. This case is particularly transparent because there are no infrared singularities [107, 108] or contributions to the soft operators arising from them. We expect that for all other particles circulating in the loop, only contributions associated with infrared singularities will appear at the first subleading soft order. Indeed, this suppression has been observed in the explicit examples of infrared-finite amplitudes studied in Ref. [125] and Chapter 6. These results suggest that, up to issues associated with infrared singularities, the soft Ward identities of BMS symmetry [82] are not anomalous. We note that while there are loop corrections to the first subleading soft-graviton behavior linked with infrared singularities, they come from a well-understood source and therefore should not be too disruptive when studying the connection to BMS symmetry.

This chapter is organized as follows. In Section 7.2, we review Low's theorem for the case of a soft photon coupled to n scalars, showing how gauge invariance determines the first subleading behavior. In Section 7.3, we repeat the analysis for a soft graviton. Next, in Section 7.4, we study the case of a soft gluon where all external particles are gluons and discuss spin contributions in some detail. The analysis for a soft graviton is extended to the case where all external particles are gravitons in Section 7.5. In Section 7.6, we explain how loop corrections to the soft operators arise from the perspective of on-shell gauge invariance and show that there are no corrections to the first subleading soft-graviton behavior for scalars in the loop. We give our conclusions in Section 7.7.

We note that while the work of this chapter was being finalized, a paper appeared constraining soft behavior using Poincaré and gauge invariance, as well as from a condition arising from the distributional nature of scattering amplitudes [127]. In this way, the authors determine the form of the subleading soft differential operators up to a numerical constant for every leg.

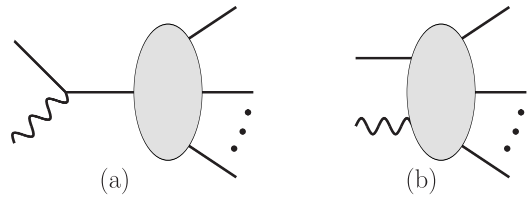

Figure 7.1: Diagrams of the form (a) give universal leading soft behavior. The subleading behavior comes from both diagrams types (a) and (b).

7.2 Photon soft limit with n scalar particles

In this section, we review the classic theorem due to Low [93] on the subleading soft behavior of photons, for simplicity focusing on the case of a single photon coupled to n scalars. As explained by Low in 1958, gauge invariance enforces the universality of the first subleading behavior, allowing us to fully determine it in terms of the amplitude without the soft photon. In subsequent sections, we will apply a similar analysis to cases with gravitons and gluons.

As illustrated in Fig. 7.1, the scattering amplitude of a single photon and n scalar particles arises from (a) contributions with a pole in the soft momentum q and (b) contributions with no pole:

$$A_n^\mu(q;k_1,\ldots,k_n) = \sum_{i=1}^{n} e_i \frac{k_i^\mu}{k_i\cdot q} T_n(k_1,\ldots,k_i+q,\ldots,k_n) + N_n^\mu(q;k_1,\ldots,k_n)\,. \tag{7.1}$$

For our purposes, it is convenient to not include the polarization vectors until the end of the discussion. The full amplitude is obtained by contracting A_n^μ with the physical photon polarization $\varepsilon_{q\mu}$. The first term in Eq. (7.1) corresponds to the emission of the photon from one of the scalar external lines as illustrated in Fig. 7.1(a) and is divergent in the soft-photon limit, while the second term, illustrated in Fig. 7.1(b), is finite in the soft-photon limit. The electric charge of particle i is e_i.

On-shell gauge invariance implies

$$
\begin{aligned}
0 &= q_\mu A_n^\mu(q; k_1, \ldots, k_n) \\
&= \sum_{i=1}^{n} e_i T_n(k_1, \ldots, k_i + q, \ldots, k_n) + q_\mu N_n^\mu(q; k_1, \ldots, k_n)\,,
\end{aligned}
\tag{7.2}
$$

valid for any value of q. Expanding around $q = 0$, we have

$$
\begin{aligned}
0 = \sum_{i=1}^{n} e_i \, &\left[T_n(k_1, \ldots, k_i, \ldots, k_n) + q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_i, \ldots, k_n) \right] \\
&+ q_\mu N_n^\mu(q = 0; k_1, \ldots, k_n) + \mathcal{O}(q^2)\,.
\end{aligned}
\tag{7.3}
$$

At leading order, this equation is

$$
\sum_{i=1}^{n} e_i = 0\,, \tag{7.4}
$$

which is simply a statement of charge conservation [94, 95]. At the next order, we have

$$
q_\mu N_n^\mu(0; k_1, \ldots, k_n) = -\sum_{i=1}^{n} e_i q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n)\,. \tag{7.5}
$$

This equation tells us that $N_n^\mu(0; k_1, \ldots, k_n)$ is entirely determined up to potential pieces that are separately gauge invariant. However, it is easy to see that the only expressions local in q that vanish under the gauge-invariance condition $q_\mu E^\mu = 0$ are of the form,

$$
E^\mu = (B_1 \cdot q) B_2^\mu - (B_2 \cdot q) B_1^\mu\,, \tag{7.6}
$$

where B_1^μ and B_2^μ are arbitrary vectors that are local in q and constructed with the momenta of the scalar particles. The explicit factor of the soft momentum q in each term means that they are suppressed in the soft limit and do not contribute to $N_n^\mu(0; k_1, \ldots, k_n)$. We can therefore remove the q_μ from Eq. (7.5), leaving

$$N_n^\mu(0; k_1, \ldots, k_n) = -\sum_{i=1}^{n} e_i \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n)\,, \tag{7.7}$$

thereby determining $N_n^\mu(0; k_1, \ldots, k_n)$ as a function of the amplitude without the photon. Inserting this into Eq. (7.1) yields

$$A_n^\mu(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[k_i^\mu - i q_\nu J_i^{\mu\nu}\right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q)\,, \tag{7.8}$$

where

$$J_i^{\mu\nu} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right)\,, \tag{7.9}$$

is the orbital angular-momentum operator and $T_n(k_1, \ldots, k_n)$ is the scattering amplitude involving n scalar particles. Eq. (7.8) is Low's theorem for the case of one photon and n scalars.

Low's theorem is unchanged at loop level for the simple reason that even at loop level, all diagrams containing a pole in the soft momentum are of the form shown in Fig. 7.1(a), with loops appearing only in the blob and not correcting the external vertex. If the scalars are massive, the integrals will not have infrared discontinuities that could lead to loop corrections of the type described in Chapter 6.

It is also interesting to see if there is any further information at higher orders in the soft expansion. If we go one order further in the expansion, we find the extra condition,

$$\frac{1}{2} \sum_{i=1}^{n} e_i q_\mu q_\nu \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\nu}} T_n(k_1, \ldots, k_n) + q_\mu q_\nu \frac{\partial N_n^\mu}{\partial q_\nu}(0; k_1, \ldots, k_n) = 0\,. \tag{7.10}$$

This implies

$$\sum_{i=1}^{n} e_i \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\nu}} T_n(k_1, \ldots, k_n) + \left[\frac{\partial N_n^\mu}{\partial q_\nu} + \frac{\partial N_n^\nu}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) = 0 \,, \tag{7.11}$$

where the final set of arguments belongs to both terms in the bracket. Gauge invariance determines only the symmetric part of the quantity $\frac{\partial N_n^\nu}{\partial q_\mu}(0; k_1, \ldots, k_n)$. The antisymmetric part is not fixed by gauge invariance; indeed, this corresponds exactly to terms of the type in Eq. (7.6). Then, up to this order, we have

$$\begin{aligned} A_n^\mu(q; k_1, \ldots, k_n) = & \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[k_i^\mu - i q_\nu J_i^{\mu\nu} \left(1 + \frac{1}{2} q_\rho \frac{\partial}{\partial k_{i\rho}} \right) \right] T_n(k_1, \ldots, k_n) \\ & + \frac{1}{2} q_\nu \left[\frac{\partial N_n^\mu}{\partial q_\nu} - \frac{\partial N_n^\nu}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) + O(q^2) \,. \end{aligned} \tag{7.12}$$

It is straightforward to see that one gets zero by saturating the previous expression with q_μ.

In order to write our universal expression in terms of the amplitude, we contract $A_n^\mu(q; k_1, \ldots, k_n)$ with the photon polarization $\varepsilon_{q\mu}$. From Eq. (7.8), we have the soft-photon limit of the single-photon, n-scalar amplitude:

$$A_n(q; k_1, \ldots, k_n) \to \left[S^{(0)} + S^{(1)} \right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q) \,, \tag{7.13}$$

where

$$\begin{aligned} S^{(0)} &\equiv \sum_{i=1}^{n} e_i \frac{k_i \cdot \varepsilon_q}{k_i \cdot q} \,, \\ S^{(1)} &\equiv -i \sum_{i=1}^{n} e_i \frac{\varepsilon_{q\mu} q_\nu J_i^{\mu\nu}}{k_i \cdot q} \,, \end{aligned} \tag{7.14}$$

and $J_i^{\mu\nu}$ is given in Eq. (7.9).

7.3 Graviton soft limit with n scalar particles

We now turn to the case of gravitons coupled to n scalars. We shall see that in the graviton case, gauge invariance can be used to fully determine the first two subleading orders in the soft-graviton momentum q. Together with the subsequent sections, this shows that the tree behavior through second subleading soft order uncovered in Ref. [83] can be understood as a consequence of on-shell gauge invariance.

In the case of a graviton scattering on n scalar particles, Eq. (7.1) becomes

$$M_n^{\mu\nu}(q;k_1,\dots,k_n) = \sum_{i=1}^{n} \frac{k_i^\mu k_i^\nu}{k_i\cdot q} T_n(k_1,\dots,k_i+q,\dots,k_n) + N_n^{\mu\nu}(q;k_1,\dots,k_n)\,, \tag{7.15}$$

where $N_n^{\mu\nu}(q;k_1,\dots,k_n)$ is symmetric under the exchange of μ and ν. For simplicity, we have set the gravitational coupling constant to unity. Similar to the gauge-theory case, we contract with the graviton polarization tensor $\varepsilon_{q\mu\nu}$ at the end. On-shell gauge invariance of the soft leg requires that the amplitude be invariant under the shift in the polarization tensor,

$$\varepsilon_{q\mu\nu} \to \varepsilon_{q\mu\nu} + q_\mu \varepsilon_{q\nu} f(q,k_i)\,, \tag{7.16}$$

where $\varepsilon_{q\nu}$ satisfies $\varepsilon_{q\nu}\cdot q = 0$ and $f(q,k_i)$ is an arbitrary function of the momenta. This implies that

$$\begin{aligned} 0 &= q_\mu M_n^{\mu\nu}(q;k_1,\dots,k_n) \\ &= \sum_{i=1}^{n} k_i^\nu T_n(k_1,\dots,k_i+q,\dots,k_n) + q_\mu N_n^{\mu\nu}(q;k_1,\dots,k_n)\,. \end{aligned} \tag{7.17}$$

Strictly speaking, Eq. (7.17) is true only after contracting the ν index with either $\varepsilon_{q\nu}$ or a conserved current. Since we contract with polarizations at the end, we can use Eq. (7.17).

At leading order in q, we then have

$$\sum_{i=1}^{n} k_i^\mu = 0\,, \tag{7.18}$$

which is satisfied due to momentum conservation. (As noted by Weinberg [94, 95], had there been different couplings to the different particles, it would have prevented this from vanishing in general; this shows that gravitons have universal coupling.)

At first order in q, Eq. (7.17) implies

$$\sum_{i=1}^{n} k_i^\nu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n) + N_n^{\mu\nu}(0; k_1, \ldots, k_n) = 0\,, \tag{7.19}$$

while at second order in q, it gives

$$\sum_{i=1}^{n} k_i^\nu \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\rho}} T_n(k_1, \ldots, k_n) + \left[\frac{\partial N_n^{\mu\nu}}{\partial q_\rho} + \frac{\partial N_n^{\rho\nu}}{\partial q_\mu}\right](0; k_1, \ldots, k_n) = 0\,. \tag{7.20}$$

As in the case of the photon, this is true up to gauge-invariant contributions to $N_n^{\mu\nu}$. However, the requirement of locality prevents us from writing any expression that is local in q yet not sufficiently suppressed in q. In fact, the most general local expression that obeys the gauge-invariance condition $q_\mu E^{\mu\nu} = q_\nu E^{\mu\nu} = 0$ is of the form,

$$E^{\mu\nu} = \Big((B_1 \cdot q) B_2^\mu - (B_2 \cdot q) B_1^\mu\Big)\Big((B_3 \cdot q) B_4^\nu - (B_4 \cdot q) B_3^\nu\Big)\,, \tag{7.21}$$

where the B_i^μ are local in q and constructed in terms of the momenta of the scalar particles. In the amplitude, $E^{\mu\nu}$ will be contracted against the symmetric traceless graviton-polarization tensor $\varepsilon_{q\mu\nu}$, so there is no need to include potential terms proportional to q^μ, q^ν or $\eta^{\mu\nu}$. The two powers of q in Eq. (7.21) mean that such terms do not contribute to the soft expansion at the orders in which we are interested.

Using Eqs. (7.19) and (7.20) in Eq. (7.15), we write the expression for a soft graviton as

$$M_n^{\mu\nu}(q;k_1\dots k_n) = \sum_{i=1}^n \frac{k_i^\nu}{k_i\cdot q}\left[k_i^\mu - iq_\rho J_i^{\mu\rho}\left(1+\frac{1}{2}q_\sigma\frac{\partial}{\partial k_{i\sigma}}\right)\right]T_n(k_1,\dots,k_n)$$
$$+\frac{1}{2}q_\rho\left[\frac{\partial N_n^{\mu\nu}}{\partial q_\rho}-\frac{\partial N_n^{\rho\nu}}{\partial q_\mu}\right](0;k_1,\dots,k_n)+\mathcal{O}(q^2)\,. \tag{7.22}$$

This is essentially the same as Eq. (7.12) for the photon except that there is a second Lorentz index in the graviton case. We will show that, unlike the case of the photon, the antisymmetric quantity in the second line of the previous equation can also be determined from the amplitude $T_n(k_1,\dots,k_n)$ without the graviton.

But, before we proceed further, let us check gauge invariance. Saturating the previous expression with q_μ, we see that the first term is vanishing because of momentum conservation, while all other terms are vanishing because $q_\mu q_\rho$ is saturated with terms that are antisymmetric in μ and ρ. If, instead, we saturate the amplitude with q_ν, the first term is vanishing as before due to momentum conservation, while the first term depending on angular momentum is vanishing because of angular-momentum conservation. The remaining terms are

$$q_\nu M_n^{\mu\nu}(q;k_1,\dots,k_n) = \frac{1}{2}q_\rho q_\sigma\Bigg\{\sum_{i=1}^n\left(k_i^\mu\frac{\partial}{\partial k_{i\rho}}-k_i^\rho\frac{\partial}{\partial k_{i\mu}}\right)\frac{\partial}{\partial k_{i\sigma}}T_n(k_1,\dots,k_n)$$
$$+\left[\frac{\partial N_n^{\mu\sigma}}{\partial q_\rho}-\frac{\partial N_n^{\rho\sigma}}{\partial q_\mu}\right](0;k_1,\dots,k_n)\Bigg\}$$
$$=0\,, \tag{7.23}$$

where the vanishing follows from Eq. (7.20), remembering that $N_n^{\mu\nu}$ is a symmetric matrix. Therefore the amplitude in Eq. (7.22) is gauge invariant. Actually, Eq. (7.20) allows us to write the relation,

$$-i\sum_{i=1}^n J_i^{\mu\rho}\frac{\partial}{\partial k_{i\sigma}}T_n(k_1,\dots,k_n) = \left[\frac{\partial N_n^{\rho\sigma}}{\partial q_\mu}-\frac{\partial N_n^{\mu\sigma}}{\partial q_\rho}\right](0;k_1,\dots,k_n)\,, \tag{7.24}$$

which fixes the antisymmetric part of the derivative of $N_n^{\mu\nu}$ in terms of the amplitude $T_n(k_1, \dots, k_n)$ without the graviton. Inserting this into Eq. (7.22), we can then rewrite the terms of $\mathcal{O}(q)$ as follows:

$$\begin{aligned}
M_n^{\mu\nu}(q; k_1, \dots, k_n)\big|_{\mathcal{O}(q)} &= -\frac{i}{2}\sum_{i=1}^{n} \frac{q_\rho q_\sigma}{k_i \cdot q}\left[k_i^\nu J_i^{\mu\rho}\frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma J_i^{\mu\rho}\frac{\partial}{\partial k_{i\nu}}\right] T_n(k_1, \dots, k_n) \\
&= -\frac{i}{2}\sum_{i=1}^{n} \frac{q_\rho q_\sigma}{k_i \cdot q}\left[J_i^{\mu\rho}k_i^\nu \frac{\partial}{\partial k_{i\sigma}} - \left(J_i^{\mu\rho}k_{i\nu}\right)\frac{\partial}{\partial k_{i\sigma}}\right. \\
&\qquad\qquad \left. - J_i^{\mu\rho}k_i^\sigma \frac{\partial}{\partial k_{i\nu}} + \left(J_i^{\mu\rho}k_i^\sigma\right)\frac{\partial}{\partial k_{i\nu}}\right] T_n(k_1, \dots, k_n) \\
&= \frac{1}{2}\sum_{i=1}^{n} \frac{1}{k_i \cdot q}\left[\Big((k_i \cdot q)(\eta^{\mu\nu}q^\sigma - q^\mu \eta^{\nu\sigma}) - k_i^\mu q^\nu q^\sigma\Big)\frac{\partial}{\partial k_i^\sigma}\right. \\
&\qquad\qquad \left. - q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}\right] T_n(k_1, \dots, k_n)\,. \qquad (7.25)
\end{aligned}$$

Finally, we wish to write our soft-limit expression in terms of the amplitude, so we contract with the physical polarization tensor of the soft graviton, $\varepsilon_{q\mu\nu}$. We see that the physical-state conditions set to zero the terms in Eq. (7.25) that are proportional to $\eta^{\mu\nu}$, q^μ and q^ν. We are then left with the following expression for the graviton soft limit of a single-graviton, n-scalar amplitude:

$$M_n(q; k_1, \dots, k_n) \to \left[S^{(0)} + S^{(1)} + S^{(2)}\right] T_n(k_1, \dots, k_n) + \mathcal{O}(q^2)\,, \qquad (7.26)$$

where

$$\begin{aligned}
S^{(0)} &\equiv \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q}\,, \\
S^{(1)} &\equiv -i\sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q}\,, \\
S^{(2)} &\equiv -\frac{1}{2}\sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q}\,. \qquad (7.27)
\end{aligned}$$

These soft factors follow from gauge invariance and agree with those computed in Ref. [83].

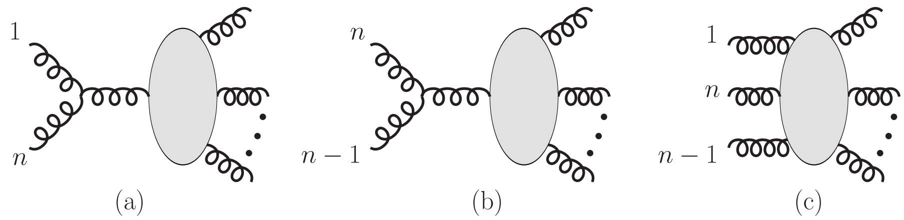

Figure 7.2: Diagrams (a) and (b) give leading universal soft-gluon behavior. The first subleading behavior of the amplitude contained in the non-pole diagram (c) can be determined via on-shell gauge invariance.

We have also looked at higher-order terms and found that gauge invariance does not fully determine them in terms of derivatives acting on $T_n(k_1, \ldots, k_n)$.

7.4 Soft limit of n-gluon amplitudes

7.4.1 Behavior of gluon tree amplitudes

In this section, we generalize the procedure of Section 7.2 to the case of n-gluon tree amplitudes prior to turning to the case of n gravitons in the next section. As we shall discuss in Section 7.6, the soft-gluon behavior has loop corrections.

We consider a tree-level color-ordered amplitude (see e.g. Ref. [80]) where gluon n becomes soft, where we define $q \equiv k_n$. As before, we find it convenient to contract the expression with polarization vectors only at the end to obtain the full amplitude. In the case of n gluons, we have two pole terms: one where the soft gluon is attached to leg 1 (see Fig. 7.2(a)) and the other where the soft gluon is attached to leg $n-1$ (see Fig. 7.2(b)). In addition to the contributions containing a pole in the soft momentum, we have the usual term $N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q; k_1, \ldots, k_{n-1})$ that is regular in the soft limit (see Fig. 7.2(c)). Together, the contributions in Fig. 7.2 give

$$
\begin{aligned}
A_n^{\mu;\mu_1\cdots\mu_{n-1}}&(q;k_1,\ldots,k_{n-1})\\
&= \frac{\delta_\rho^{\mu_1}k_1^\mu + \eta^{\mu\mu_1}q_\rho - \delta_\rho^\mu q^{\mu_1}}{\sqrt{2}(k_1\cdot q)}\,A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1+q,k_2,\ldots,k_{n-1})\\
&\quad - \frac{\delta_\rho^{\mu_{n-1}}k_{n-1}^\mu + \eta^{\mu_{n-1}\mu}q_\rho - \delta_\rho^\mu q^{\mu_{n-1}}}{\sqrt{2}(k_{n-1}\cdot q)}\,A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-2},k_{n-1}+q)\\
&\quad + N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\ldots,k_{n-1})\,. \qquad (7.28)
\end{aligned}
$$

We have dropped terms from the three-gluon vertex that vanish when saturated with the external-gluon polarization vectors in addition to using the current-conservation conditions,

$$
\begin{aligned}
&(k_1+q)_\rho\,A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1+q,k_2,\ldots,k_{n-1}) = 0\,,\\
&(k_{n-1}+q)_\rho\,A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-2},k_{n-1}+q) = 0\,, \qquad (7.29)
\end{aligned}
$$

which are valid once we contract with the polarization vectors carrying the μ_j indices. By introducing the spin-one angular-momentum operator,

$$
(\Sigma_i^{\mu\sigma})^{\mu_i\rho} \equiv i\left(\eta^{\mu\mu_i}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\mu_i\sigma}\right)\,, \qquad (7.30)
$$

we can write Eq. (7.28) as

$$
\begin{aligned}
A_n^{\mu;\mu_1\cdots\mu_{n-1}}&(q;k_1,\ldots,k_{n-1})\\
&= \frac{\delta_\rho^{\mu_1}k_1^\mu - iq_\sigma(\Sigma_1^{\mu\sigma})^{\mu_1}{}_\rho}{\sqrt{2}(k_1\cdot q)}\,A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1+q,k_2,\ldots,k_{n-1})\\
&\quad - \frac{\delta_\rho^{\mu_{n-1}}k_{n-1}^\mu - iq_\sigma(\Sigma_{n-1}^{\mu\sigma})^{\mu_{n-1}}{}_\rho}{\sqrt{2}(k_{n-1}\cdot q)}\,A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\ldots,k_{n-2},k_{n-1}+q)\\
&\quad + N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\ldots,k_{n-1})\,. \qquad (7.31)
\end{aligned}
$$

Notice that the spin-one terms independently vanish when contracted with q_μ.

The on-shell gauge invariance of Eq. (7.31) requires

$$
\begin{aligned}
0 &= q_\mu A_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\dots,k_{n-1}) \\
&= \frac{1}{\sqrt{2}} A_{n-1}^{\mu_1\mu_2\cdots\mu_{n-1}}(k_1+q,k_2,\dots,k_{n-1}) - \frac{1}{\sqrt{2}} A_{n-1}^{\mu_1\cdots\mu_{n-2}\mu_{n-1}}(k_1,\dots,k_{n-2},k_{n-1}+q) \\
&\quad + q_\mu N_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1,\dots,k_{n-1})\,. \qquad (7.32)
\end{aligned}
$$

For $q=0$, this is automatically satisfied. At the next order in q, we obtain

$$
-\frac{1}{\sqrt{2}}\left[\frac{\partial}{\partial k_{1\mu}} - \frac{\partial}{\partial k_{n-1\mu}}\right] A_{n-1}^{\mu_1\cdots\mu_{n-1}}(k_1,k_2\dots k_{n-1}) = N_n^{\mu;\mu_1\cdots\mu_{n-1}}(0;k_1,\dots,k_{n-1})\,. \qquad (7.33)
$$

Similar to the photon case, we ignore local gauge-invariant terms in $N_n^{\mu;\mu_1\cdots\mu_{n-1}}$ because they are necessarily of a higher order in q. Thus, $N_n^{\mu;\mu_1\cdots\mu_{n-1}}(0;k_1,\dots,k_{n-1})$ is determined in terms of an expression without the soft gluon. With this, the total expression in Eq. (7.31) becomes

$$
\begin{aligned}
A_n^{\mu;\mu_1\cdots\mu_{n-1}}(q;k_1\dots k_{n-1}) &= \left(\frac{k_1^\mu}{\sqrt{2}(k_1\cdot q)} - \frac{k_{n-1}^\mu}{\sqrt{2}(k_{n-1}\cdot q)}\right) A_{n-1}^{\mu_1\cdots\mu_{n-1}}(k_1,\dots,k_{n-1}) \\
&\quad - i\frac{q_\sigma (J_1^{\mu\sigma})^{\mu_1}{}_\rho}{\sqrt{2}(k_1\cdot q)} A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1,\dots,k_{n-1}) \\
&\quad + i\frac{q_\sigma (J_{n-1}^{\mu\sigma})^{\mu_{n-1}}{}_\rho}{\sqrt{2}(k_{n-1}\cdot q)} A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1,\dots,k_{n-1}) + \mathcal{O}(q)\,, \qquad (7.34)
\end{aligned}
$$

where

$$
(J_i^{\mu\sigma})^{\mu_i\rho} \equiv L_i^{\mu\sigma}\eta^{\mu_i\rho} + (\Sigma_i^{\mu\sigma})^{\mu_i\rho}, \qquad (7.35)
$$

the spin-one angular-momentum operator is given in Eq. (7.30), and the orbital angular-momentum operator is

$$
L_i^{\mu\sigma} \equiv i\left(k_i^\mu \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma \frac{\partial}{\partial k_{i\mu}}\right)\,. \qquad (7.36)
$$

Both angular-momentum operators satisfy the same commutation relations,

$$[L_i^{\mu\nu}, L_i^{\rho\sigma}] = i\left(\eta^{\nu\rho} L_i^{\mu\sigma} + \eta^{\mu\rho} L_i^{\sigma\nu} + \eta^{\mu\sigma} L_i^{\nu\rho} + \eta^{\nu\sigma} L_i^{\rho\mu}\right) ,$$
$$[\Sigma_i^{\mu\nu}, \Sigma_i^{\rho\sigma}] = i\left(\eta^{\nu\rho} \Sigma_i^{\mu\sigma} + \eta^{\mu\rho} \Sigma_i^{\sigma\nu} + \eta^{\mu\sigma} \Sigma_i^{\nu\rho} + \eta^{\nu\sigma} \Sigma_i^{\rho\mu}\right) , \tag{7.37}$$

where the suppressed indices on $\Sigma_i^{\mu\nu}$ should be treated as matrix indices.

In order to write the final result in terms of full amplitudes, we contract with external polarization vectors. On the right-hand side of Eq. (7.34), we must pass polarization vectors $\varepsilon_{1\mu_1}$ and $\varepsilon_{n-1\mu_{n-1}}$ through the spin-one angular-momentum operator such that they will contract with the ρ index of, respectively, $A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1, \ldots, k_{n-1})$ and $A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1, \ldots, k_{n-1})$. It is convenient write the spin angular-momentum operator as

$$\varepsilon_{i\mu_i} (\Sigma_i^{\mu\sigma})^{\mu_i}{}_{\rho} A^\rho = i\left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}}\right) \varepsilon_{i\rho} A^\rho \,. \tag{7.38}$$

We may therefore write

$$A_n(q; k_1, \ldots, k_{n-1}) \to \left[S_n^{(0)} + S_n^{(1)}\right] A_{n-1}(k_1, \ldots, k_{n-1}) + \mathcal{O}(q)\,, \tag{7.39}$$

where

$$S_n^{(0)} \equiv \frac{k_1 \cdot \varepsilon_n}{\sqrt{2}\,(k_1 \cdot q)} - \frac{k_{n-1} \cdot \varepsilon_n}{\sqrt{2}\,(k_{n-1} \cdot q)} \,,$$
$$S_n^{(1)} \equiv -i \varepsilon_{n\mu} q_\sigma \left(\frac{J_1^{\mu\sigma}}{\sqrt{2}\,(k_1 \cdot q)} - \frac{J_{n-1}^{\mu\sigma}}{\sqrt{2}\,(k_{n-1} \cdot q)}\right) . \tag{7.40}$$

Here

$$J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + \Sigma_i^{\mu\sigma} \,, \tag{7.41}$$

where

$$\Sigma_i^{\mu\sigma} \equiv i\left(\varepsilon_i^{\mu}\frac{\partial}{\partial\varepsilon_{i\sigma}} - \varepsilon_i^{\sigma}\frac{\partial}{\partial\varepsilon_{i\mu}}\right) . \tag{7.42}$$

In using Eq. (7.39), one must interpret $L_i^{\mu\sigma}$ as not acting on explicit polarization vectors, i.e., $L_i^{\mu\sigma}\varepsilon_i = 0$. If one instead interprets polarization vectors as functions of momenta (see e.g. Sect. 5.9 of Ref. [128]) and returns a nonzero value for $L_i^{\mu\sigma}\varepsilon_i$, then one should not include the spin term (7.42). To be concrete, the action of the total angular-momentum operator on momenta and polarizations is given by

$$\begin{aligned} J_i^{\mu\sigma}k_i^{\rho} &= i\left(\eta^{\sigma\rho}k_i^{\mu} - \eta^{\mu\rho}k_i^{\sigma}\right) , \\ J_i^{\mu\sigma}\varepsilon_i^{\rho} &= i\left(\eta^{\sigma\rho}\varepsilon_i^{\mu} - \eta^{\mu\rho}\varepsilon_i^{\sigma}\right) . \end{aligned} \tag{7.43}$$

We comment more on the action of the operator on polarization vectors in Section 7.4.2.

In conclusion, the first two leading terms in the soft-gluon expansion of an n-gluon amplitude are given directly in terms of the amplitude without the soft gluon. This derivation is valid in D dimensions. We have explicitly checked the soft-gluon formula (7.39) using explicit four-, five- and six-gluon tree amplitudes of gauge theory in terms of formal polarization vectors.

7.4.2 Connection to spinor helicity

To connect with the spinor-helicity formalism used in e.g. Refs. [83, 125] and Chapter 6, we show that, up to a gauge transformation, the action of the above subleading soft operators on polarization vectors expressed in terms of spinor helicity is identical to the ones defined as differential operators acting on spinors. In the spinor-helicity formalism, polarization vectors are expressed directly in terms of spinors depending on the momenta:

$$\varepsilon_i^{+\,\rho}(k_i,k_r) = \frac{\langle r|\,\gamma^{\rho}\,|i]}{\sqrt{2}\,\langle r\,i\rangle} , \qquad \varepsilon_i^{-\,\rho}(k_i,k_r) = -\frac{\langle i|\,\gamma^{\rho}\,|r]}{\sqrt{2}\,[r\,i]} , \tag{7.44}$$

where k_i is the momentum of gluon i and k_r is a null reference momentum. Henceforth, we will leave the k_i argument implicit and only display the reference momentum. The spinors are standard Weyl spinors. We follow the conventions of Ref. [80] aside from our use of angle and square brackets instead of the $\pm$ angle-bracket convention. In our convention, we have

$$\langle i| = \langle i^-|\,, \qquad [i| = \langle i^+|\,, \qquad |i\rangle = |i^+\rangle\,, \qquad |i] = |i^-\rangle\,. \tag{7.45}$$

In terms of spinors, the subleading soft factor for a tree-level gauge-theory amplitude is [86]

$$S_n^{(1)\lambda} = \frac{1}{\langle (n-1)\,n\rangle}\tilde{\lambda}_n^{\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}_{n-1}^{\dot\alpha}} - \frac{1}{\langle 1\,n\rangle}\tilde{\lambda}_n^{\dot\alpha}\frac{\partial}{\partial\tilde{\lambda}_1^{\dot\alpha}}\,, \tag{7.46}$$

where $\lambda^\alpha \equiv |i^+\rangle^\alpha$ and $\tilde{\lambda}^{\dot\alpha} \equiv |i^-\rangle^{\dot\alpha}$. We consider the explicit action of $S_n^{(1)\lambda}$ in Eq. (7.46) and $S_n^{(1)}$ in Eq. (7.40) on $\varepsilon_1^{\pm\,\rho}(k_{r_1})$ to show equivalence after contraction with the polarization-stripped amplitude. The action on $\varepsilon_{n-1}^{\pm\,\rho}(k_{r_{n-1}})$ follows similarly. We act with Eq. (7.46) on the vectors in Eq. (7.44)—with $i \to 1$ and $k_r \to k_{r_1}$—in turn:

$$S_n^{(1)\lambda}\,\varepsilon_1^{+\,\rho}(k_{r_1}) = -\frac{1}{\langle 1\,n\rangle}\frac{\langle r_1|\,\gamma^\rho\,|n]}{\sqrt{2}\,\langle r_1\,1\rangle} = -\frac{\langle r_1\,n\rangle}{\langle r_1\,1\rangle\,\langle 1\,n\rangle}\varepsilon_n^{+\,\rho}(k_{r_1})\,, \tag{7.47}$$

and

$$\begin{aligned} S_n^{(1)\lambda}\,\varepsilon_1^{-\,\rho}(k_{r_1}) &= -\frac{1}{\langle 1\,n\rangle}\left(-\frac{\langle 1|\,\gamma^\rho\,|r_1]}{\sqrt{2}}\right)\left(-\frac{[r_1\,n]}{[r_1\,1]^2}\right) \\ &= \frac{[r_1\,n]}{[r_1\,1]\,\langle 1\,n\rangle}\varepsilon_1^{-\,\rho}(k_{r_1}) \\ &= \frac{[r_1\,n]}{[r_1\,1]\,\langle 1\,n\rangle}\left[\varepsilon_1^{-\,\rho}(k_n) + \frac{\sqrt{2}\,[r_1\,n]}{[r_1\,1]\,[n\,1]}\,k_1^\rho\right] \\ &= \frac{[r_1\,n]}{[r_1\,1]\,[1\,n]}\left[\varepsilon_n^{+\,\rho}(k_1) - \frac{\sqrt{2}\,[r_1\,n]}{[r_1\,1]\,\langle 1\,n\rangle}\,k_1^\rho\right], \end{aligned} \tag{7.48}$$

where we used

$$\varepsilon_i^{-\rho}(k_r) = \varepsilon_i^{-\rho}(k_{\tilde{r}}) + \frac{\sqrt{2}\,[r\,\tilde{r}]}{[r\,i]\,[\tilde{r}\,i]}\,k_i^\rho\,, \tag{7.49}$$

in the second-to-last line. The last line of Eq. (7.48) follows from

$$\varepsilon_j^{+\,\rho}(k_i) = \frac{[i\,j]}{\langle i\,j\rangle}\,\varepsilon_i^{-\,\rho}(k_j)\,. \tag{7.50}$$

We can write Eq. (7.48) more simply as

$$S_n^{(1)\lambda}\,\varepsilon_1^{-\,\rho}(k_{r_1}) \cong \frac{[r_1\,n]}{[r_1\,1]\,[1\,n]}\,\varepsilon_n^{+\,\rho}(k_1)\,, \tag{7.51}$$

where the symbol $\cong$ denotes equivalence up to a term proportional to k_1^ρ. Such terms will vanish when contracted with the polarization-stripped $(n-1)$-point amplitude, so we are free to drop them. Similar spinor-helicity algebra reveals that the action of $S_n^{(1)}$ from Eq. (7.40) on $\varepsilon_1^{\pm\,\rho}(k_{r_1})$ yields

$$\begin{aligned} S_n^{(1)}\,\varepsilon_1^{+\,\rho}(k_{r_1}) &= -i\varepsilon^+_{n\,\mu}(k_{r_n})k_{n\,\sigma}\frac{\Sigma_1^{\mu\sigma}}{\sqrt{2}\,(k_1\cdot k_n)}\varepsilon_1^{+\,\rho}(k_{r_1}) \\ &= -\frac{\langle r_1\,n\rangle}{\langle r_1\,1\rangle\,\langle 1\,n\rangle}\,\varepsilon_n^{+\,\rho}(k_{r_1})\,, \end{aligned} \tag{7.52}$$

and

$$S_n^{(1)}\,\varepsilon_1^{-\,\rho}(k_{r_1}) = \frac{[r_1\,n]}{[r_1\,1]\,[1\,n]}\,\varepsilon_n^{+\,\rho}(k_1)\,. \tag{7.53}$$

We can summarize the action of the operators as

$$S_n^{(1)\lambda}\,\varepsilon_1^{\pm\,\rho}(k_{r_1}) \cong S_n^{(1)}\,\varepsilon_1^{\pm\,\rho}(k_{r_1}) = -\left(\frac{\varepsilon_1^{\pm}(k_{r_1})\cdot p_n}{\sqrt{2}(p_1\cdot p_n)}\right)\times\begin{cases}\varepsilon_n^{+\,\rho}(k_{r_1}), & \text{for } +\,,\\ \varepsilon_n^{+\,\rho}(k_1), & \text{for } -\,.\end{cases} \tag{7.54}$$

We see that, up to terms proportional to k_1^ρ, the action of $S_n^{(1)\lambda}$ and $S_n^{(1)}$ on the polarization vectors yield completely equivalent expressions as expected.

7.5 Soft limit of n-graviton amplitudes

In this section, we generalize what has been done for the case of n gluons to the case of n gravitons. As before, we write the amplitude as a sum of two pieces: the first contains terms where the soft graviton is attached to one of the other $n-1$ external gravitons, giving a contribution divergent as $1/q$ for $q \to 0$, while in the second the soft graviton attaches to one of the internal graviton lines and is of $\mathcal{O}(q^0)$ in the same limit. Leaving the expression uncontracted with polarization tensors for now, we write

$$\begin{aligned} M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}&(q;k_1,\ldots,k_{n-1}) \\ &= \sum_{i=1}^{n-1} \frac{1}{k_i\cdot q}\left[k_i^\mu \eta^{\mu_i\alpha} - iq_\rho(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\left[k_i^\nu \eta^{\nu_i\beta} - iq_\sigma(\Sigma_i^{\mu\sigma})^{\nu_i\beta}\right] \\ &\qquad\qquad \times M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i+q,\ldots,k_{n-1}) \\ &\quad + N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1})\,, \end{aligned} \tag{7.55}$$

where

$$(\Sigma_i^{\mu\rho})^{\mu_i\alpha} \equiv i\left(\eta^{\mu\mu_i}\eta^{\alpha\rho} - \eta^{\mu\alpha}\eta^{\mu_i\rho}\right)\,. \tag{7.56}$$

The simple form of the three vertex used in Eq. (7.55) can be obtained from the standard de Donder gauge one, using current conservation and tracelessness properties of external polarization tensors and M_{n-1}, as well as assigning terms to N_n where the $i/k_i\cdot q$ propagator cancels. We note that it is important to keep the lowered indices of M_{n-1} in their appropriate slots. On-shell gauge invariance implies

$$
\begin{aligned}
0 &= q_\mu M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1}) \\
&= \sum_{i=1}^{n-1}\left[k_i^\nu \eta^{\nu_i\beta} - iq_\rho(\Sigma_i^{\nu\rho})^{\nu_i\beta}\right] M_{n-1}^{\mu_1\nu_1\cdots\mu_i}{}_{\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i+q,\ldots,k_{n-1}) \\
&\quad + q_\mu N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1})\,, \qquad (7.57)
\end{aligned}
$$

provided that as usual we will contract all free indices of M_n with polarization tensors at the end. This includes contracting the ν index with a polarization vector ε_n^μ satisfying $\varepsilon_n \cdot q = 0$. Expanding the previous expression for small q, we find that the leading term vanishes because of momentum conservation, while the next-to-leading term gives two conditions by taking the symmetric and antisymmetric parts:

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n-1}\eta^{\mu_i\alpha}\eta^{\nu_i\beta}\left(k_i^\mu\frac{\partial}{\partial k_{i\nu}} + k_i^\nu\frac{\partial}{\partial k_{i\mu}}\right) M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1}) \\
&\qquad\qquad = -N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(0;k_1,\ldots,k_{n-1})\,, \qquad (7.58)
\end{aligned}
$$

and

$$
\sum_{i=1}^{n-1}\left[L_i^{\nu\rho}\eta^{\nu_i\beta} + 2(\Sigma_i^{\nu\rho})^{\nu_i\beta}\right] M_{n-1}^{\mu_1\nu_1\cdots\mu_i}{}_{\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1}) = 0\,. \qquad (7.59)
$$

As in the earlier cases, we can ignore potential terms that are local in q and vanish when dotted into q^μ since they will not contribute to the desired order. The first condition determines $N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(0;k_1,\ldots,k_{n-1})$ in terms of the amplitude without the soft graviton, while the second one reflects conservation of total angular momentum. The factor of 2 in front of the spin term in Eq. (7.59) reflects the fact that the graviton has spin 2.

Finally, the terms of order q^2 in Eq. (7.57) imply the following condition:

$$\sum_{i=1}^{n-1} q_\rho \left[k_i^\nu \eta^{\nu_i\beta} \frac{\partial^2}{\partial k_{i\rho} \partial k_{i\mu}} - 2i(\Sigma_i^{\nu\rho})^{\nu_i\beta} \frac{\partial}{\partial k_{i\mu}} \right] M_{n-1}^{\mu_1\nu_1\cdots\mu_i}{}_{\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1, \ldots, k_i, \ldots, k_{n-1})$$
$$= -q_\rho \left[\frac{\partial N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\rho} + \frac{\partial N_n^{\rho\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\mu} \right] (0; k_1, \ldots, k_{n-1})\,. \quad (7.60)$$

Using the previous results, for a soft graviton of momentum q, we have

$$\begin{aligned}
&M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q; k_1, \ldots, k_{n-1}) \\
&\quad = \sum_{i=1}^{n-1} \frac{1}{k_i \cdot q} \Bigg\{ k_i^\mu k_i^\nu \eta^{\mu_i\alpha} \eta^{\nu_i\beta} \\
&\qquad - \frac{i}{2} q_\rho \Big[k_i^\mu \eta^{\mu_i\alpha} \left[L_i^{\nu\rho} \eta^{\nu_i\beta} + 2(\Sigma_i^{\nu\rho})^{\nu_i\beta} \right] + k_i^\nu \eta^{\nu_i\beta} \left[L_i^{\mu\rho} \eta^{\mu_i\alpha} + 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha} \right] \Big] \\
&\qquad - \frac{i}{2} q_\rho q_\sigma \Bigg[k_i^\nu \eta^{\mu_i\alpha} \eta^{\nu_i\beta} L_i^{\mu\rho} \frac{\partial}{\partial k_{i\sigma}} - 2i(\Sigma_i^{\mu\rho})^{\mu_i\alpha} (\Sigma_i^{\nu\sigma})^{\nu_i\beta} - 2k_i^\sigma \eta^{\nu_i\beta} (\Sigma_i^{\nu\rho})^{\nu_i\beta} \frac{\partial}{\partial k_{i\mu}} \\
&\qquad\qquad + 2 \left[\eta^{\mu_i\alpha} k_i^\mu (\Sigma_i^{\nu\rho})^{\nu_i\beta} + \eta^{\nu_i\beta} k_i^\nu (\Sigma_i^{\mu\rho})^{\mu_i\alpha} \right] \frac{\partial}{\partial k_{i\sigma}} \Bigg] \Bigg\} \\
&\qquad\qquad \times M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1, \ldots, k_i, \ldots, k_{n-1}) \\
&\qquad + \frac{1}{2} q_\rho \left[\frac{\partial N_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\rho} - \frac{\partial N_n^{\rho\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\mu} \right] (0; k_1, \ldots, k_{n-1}) \\
&\qquad + \mathcal{O}(q^2)\,.
\end{aligned} \quad (7.61)$$

As in the case of gluon scattering, it may seem that we cannot determine the order q contributions in terms of M_{n-1} because the antisymmetric part of the matrix N_n is still present in Eq. (7.61). However, it turns out that there is additional information from on-shell gauge invariance. When we saturate it with q_μ, we get of course zero because this is the way that Eq. (7.61) is constructed. When we saturate it with q_ν, however, we obtain the extra condition:

$$
\begin{aligned}
0 &= q_\nu M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1}) \\
&= q_\rho q_\sigma \Bigg\{ \sum_{i=1}^{n-1} [L_i^{\mu\rho}\eta^{\mu_i\alpha} + 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}] \frac{\partial}{\partial k_{i\sigma}} M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha}{}^{\nu_i\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1}) \\
&\qquad + i\left[\frac{\partial N_n^{\mu\sigma;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\rho} - \frac{\partial N_n^{\rho\sigma;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\mu}\right](0;k_1,\ldots,k_{n-1})\Bigg\}, \qquad (7.62)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\sum_{i=1}^{n-1} q_\rho [L_i^{\mu\rho}\eta^{\mu_i\alpha} + 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}] \frac{\partial}{\partial k_{i\sigma}} M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha}{}^{\nu_i\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1}) \\
&\qquad = -iq_\rho \left[\frac{\partial N_n^{\mu\sigma;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\rho} - \frac{\partial N_n^{\rho\sigma;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}}{\partial q_\mu}\right](0;k_1,\ldots,k_{n-1}). \qquad (7.63)
\end{aligned}
$$

We can now use it in Eq. (7.61) to obtain our final expression giving the soft limit entirely in terms of the $(n-1)$-point amplitude:

$$
\begin{aligned}
&M_n^{\mu\nu;\mu_1\nu_1\cdots\mu_{n-1}\nu_{n-1}}(q;k_1,\ldots,k_{n-1}) \\
&\quad = \sum_{i=1}^{n-1} \frac{1}{k_i\cdot q}\Bigg\{ k_i^\mu k_i^\nu \eta^{\mu_i\alpha}\eta^{\nu_i\beta} \\
&\qquad - \frac{i}{2} q_\rho \Big[k_i^\mu \eta^{\mu_i\alpha}\left[L_i^{\nu\rho}\eta^{\nu_i\beta} + 2(\Sigma_i^{\nu\rho})^{\nu_i\beta}\right] + k_i^\nu \eta^{\nu_i\beta}\left[L_i^{\mu\rho}\eta^{\mu_i\alpha} + 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}\right]\Big] \\
&\qquad - \frac{1}{2} q_\rho q_\sigma \Big[[L_i^{\mu\rho}\eta^{\mu_i\alpha} + 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}]\left[L_i^{\nu\sigma}\eta^{\nu_i\beta} + 2(\Sigma_i^{\nu\sigma})^{\nu_i\beta}\right] - 2(\Sigma_i^{\mu\rho})^{\mu_i\alpha}(\Sigma_i^{\nu\sigma})^{\nu_i\beta}\Big]\Bigg\} \\
&\qquad\qquad \times M_{n-1}^{\mu_1\nu_1\cdots}{}_{\alpha\beta}{}^{\cdots\mu_{n-1}\nu_{n-1}}(k_1,\ldots,k_i,\ldots,k_{n-1}) + \mathcal{O}(q^2). \qquad (7.64)
\end{aligned}
$$

In order to write our expression in terms of amplitudes, we saturate with graviton polarization tensors using $\varepsilon_{\mu\nu} \to \varepsilon_\mu \varepsilon_\nu$ where ε_μ are spin-one polarization vectors. As we did for the case with gluons, we must pass the polarization vectors through the spin-one operators. We are then left with

$$M_n(q; k_1, \ldots, k_{n-1}) = \left[S_n^{(0)} + S_n^{(1)} + S_n^{(2)}\right] M_{n-1}(k_1, \ldots, k_{n-1}) + \mathcal{O}(q^2)\,, \tag{7.65}$$

where

$$\begin{aligned} S_n^{(0)} &\equiv \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q}\,, \\ S_n^{(1)} &\equiv -i \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^\mu q_\rho J_i^{\nu\rho}}{k_i \cdot q}\,, \\ S_n^{(2)} &\equiv -\frac{1}{2} \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q}\,. \end{aligned} \tag{7.66}$$

Here

$$J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + \Sigma_i^{\mu\sigma}\,, \tag{7.67}$$

with

$$L_i^{\mu\sigma} \equiv i \left(k_i^\mu \frac{\partial}{\partial k_{i\sigma}} - k_i^\sigma \frac{\partial}{\partial k_{i\mu}} \right), \qquad \Sigma_i^{\mu\sigma} \equiv i \left(\varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_{i\sigma}} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_{i\mu}} \right). \tag{7.68}$$

Since the graviton polarization tensor is quadratic in spin-one polarization vectors ε_i^μ, the differential operator in Eq. (7.68) picks up factors of 2 as required for Eq. (7.65) to be compatible with Eq. (7.64).

In conclusion, in the case of a soft graviton, on-shell gauge invariance completely determines the first two subleading contributions. Using the Kawai-Lewellen-Tye relations [105] we have generated graviton amplitudes with formal polarization tensors up to six points. Using these we analytically confirmed Eq. (7.65) through five points and numerically through six points.

7.6 Comments on Loop Corrections

In gauge and gravity theories in four dimensions, the operators describing the soft behavior have nontrivial loop corrections (see Ref. [125] and Chapter 6). Indeed, in QCD loop corrections linked to infrared singularities are present already at leading order in the soft limit [103, 104]. One may wonder how loop corrections to the soft operators arise from the perspective of the constraints imposed by on-shell gauge invariance. In this section we explain this. We first describe the case of gauge theory before turning to gravity.

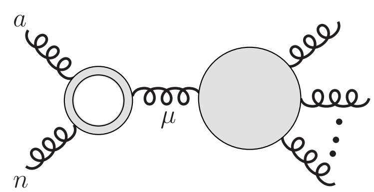

Figure 7.3: The potential factorizing contributions to the one-loop corrections to the leading soft function which then cancel. Leg n is the soft leg which carries momentum q. At subleading order there are additional contributions.

As explained in Ref. [103], we can separate the contributions into two distinct sources. The first source of potential corrections is the "factorizing" one that, for leading order, arises from loop corrections of the form displayed in Fig. 7.3 (see Refs. [103, 125] and Chapter 6). The second type of contributions are "nonfactorizing" infrared-divergent pieces that can come from discontinuities in the amplitudes associated with infrared divergences [110]. (Alternatively these nonfactorizing contributions can be pushed into factorizing contributions that have light-cone denominators coming from a careful application of unitarity [104].)

Here we will focus on the factorizing pieces. In gauge theory we will explain why they do not enter in the leading soft behavior [103, 104]. For the case of scalars in the loops, which is an especially clean case since there are no infrared singularities even for massless scalars, we show that there are no factorizing loop corrections at the leading and first subleading orders of the soft-graviton expansion. This suppression was discussed earlier in explicit examples of soft limits of one-loop infrared-finite gravity amplitudes in Ref. [125] and Chapter 6.

7.6.1 Gauge Theory

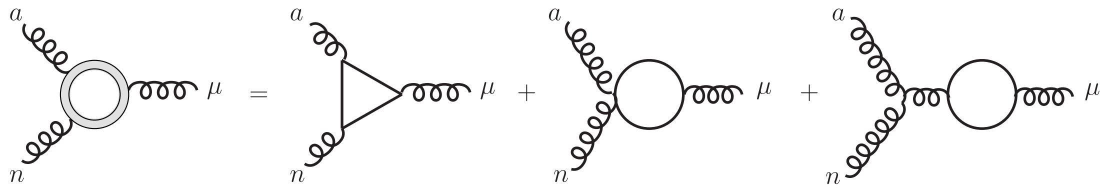

Figure 7.4: The diagrams with potential factorizing contributions to the one-loop soft function. At subleading order there are additional contributions.

As a warm up to the gravity case, we first discuss the well-understood gauge-theory case. The explicit forms of the factorizing one-loop corrections to the soft behavior have been described in some detail in Refs. [103, 104] for QCD at leading order in the soft (and collinear) limits.

For the case of external gluons, the potential factorizing contributions to one-loop modifications of the leading soft behavior are shown in Fig. 7.3. We can expand these corrections into triangle and bubble diagrams as shown in Fig. 7.4. As derived in Ref. [103], these diagrams evaluate to

$$D^{\mu,\text{fact}} = \frac{i}{\sqrt{2}}\frac{1}{3}\frac{1}{(4\pi)^2}\Big(1 - \frac{n_f}{N_c} + \frac{n_s}{N_c}\Big)(q - k_a)^\mu\Big[(\varepsilon_n \cdot \varepsilon_a) - \frac{(q\cdot\varepsilon_a)(k_a\cdot\varepsilon_n)}{(k_a\cdot q)}\Big]\,, \tag{7.69}$$

where n_f is the number of fundamental representation fermions, n_s the number of fundamental representation complex scalars (using the normalization conventions of Ref. [103]), and N_c is the number of colors. As usual we take the soft momentum of leg n to be q. After integration this result is both ultraviolet- and infrared-finite, so we have taken $\epsilon = 0$ in the final integrated result. The all orders in ϵ form of Eq. (7.69) is given in Refs. [103, 104].

The result (7.69) has a few surprising features that explain why we cannot use it to obtain the full subleading soft correction via gauge invariance. The first is that the correction to the three-vertex is nonlocal because of the pole in $q \cdot k_a$ that arises from the loop integration. Indeed, after we include the intermediate propagator $-i/(k_a + q)^2$, there is a double pole[1]

[1]While this might seem to violate basic factorization properties of field theories, in fact it does not,

in $q \cdot k_a$. A second curious feature is that Eq. (7.69) is gauge invariant by itself: It vanishes when ε_n^μ is replaced by $q^\mu \equiv k_n^\mu$ for any value of the intermediate off-shell momentum. The nonlocal nature of the result is what allows us to write such a gauge-invariant term with the correct dimensions.

A third feature is that, in fact, there is no contribution from Eq. (7.69) to the leading one-loop correction to the soft function, as noted in Refs. [103, 104]. To see this, we sew Eq. (7.69) onto the rest of the amplitude across the factorization channel:

$$D_\mu^{\text{fact}} \frac{-i}{2q \cdot k_a} \mathcal{J}^\mu \,, \tag{7.70}$$

as illustrated in Fig. 7.3. We observe that $\mathcal{J}^\mu$ is a conserved current:

$$(q + k_a)_\mu \mathcal{J}^\mu = 0 \,, \tag{7.71}$$

assuming that all the remaining legs are contracted with on-shell polarizations. This immediately implies

$$D_\mu^{\text{fact}} \frac{-i}{2q \cdot k_a} \mathcal{J}^\mu = \mathcal{O}(q^0) \,, \tag{7.72}$$

because D_μ^{fact} is proportional to $(q - k_a)_\mu$ which is equivalent to $2q_\mu$ when dotted into a conserved current. This reproduces the fact that there is no leading $\mathcal{O}(1/q)$ factorizing contribution to the one-loop soft function [103, 104].

Unfortunately, the $\mathcal{O}(q^0)$ terms in the full factorizing contributions are not under control via gauge invariance. Once we allow for an extra $1/(q \cdot k_a)$ nonlocality arising from the loop integration, we lose control over the subleading piece. This cannot happen at tree level because there is no source of a second factor of $1/(q \cdot k_a)$. In fact, Eq. (7.69) is incomplete for capturing all subleading contributions. The additional contributions have already been described in some detail at one loop on a case-by-case basis in Refs. [111, 112, 113]. Unfortunately, no universal factorization formula is known for these types of

---

because for real momenta the double pole is reduced to a single pole. See Refs. [111, 112] for a detailed discussion of this phenomenon.

corrections, although case-by-case their forms appear to be relatively simple. An example of this type of nontrivial factorization can be found in Eq. (61) of Ref. [111] or Eq. (3.9) of Ref. [125]; the precise form of the correction depends on the helicities of other legs.

Interestingly, these contributions resemble an anomaly that seemingly vanishes if we take the loop integrand strictly in four dimensions. This arises from the integration where a $1/\epsilon$ ultraviolet pole cancels a factor of ϵ from numerator algebra, leaving terms of $\mathcal{O}(1)$. This is reminiscent of the way the chiral anomaly arises from triangle diagrams in dimensional regularization. Indeed, for the single minus-helicity case discussed in Ref. [125] and Chapter 6, not only does this contribution vanish but the entire amplitude would vanish if we were not careful to keep in the integrand in $D = 4 - 2\epsilon$ instead of four dimensions. It is interesting that these types of contributions do not appear in supersymmetric theories.

Besides the loop contributions described above, there is a second type of loop correction to the soft operators (7.40) arising from non-smoothness in the amplitude due to infrared singularities [110]. In QED the integrals are smooth because the electron mass acts as an infrared cutoff, but in QCD or gravity there is no such physical cutoff on gluons or gravitons. It is therefore much more difficult to consistently introduce a mass regulator without breaking gauge symmetry or altering the number of propagating degrees of freedom. As is standard practice, it is far simpler to use dimensional regularization. As discussed in some detail in Refs. [110, 103] and Chapter 6, as gluons become soft or collinear, the matrix elements develop discontinuities that are absorbed into modifications of the loop splitting or soft operators. Alternatively, by using light-cone gauge or carefully applying unitarity, one introduces light-cone denominators containing a reference momentum, and one can push all contributions into factorizing diagrams [114, 104]. Either way, the conclusion is the same: There are nontrivial contributions due to infrared singularities not accounted for in the naive tree-level soft limit.

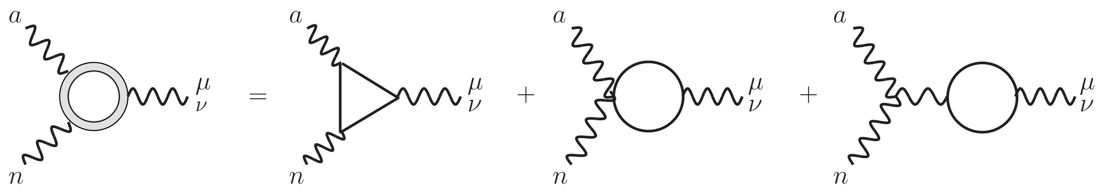

Figure 7.5: The diagrams with potential factorizing contributions to the one-loop soft behavior in gravity with a scalar in the loop. This captures all such potential leading and first subleading contributions, but it is incomplete at second subleading order.

7.6.2 Gravity

We now show that the situation in gravity is similar. Here the dimensionful coupling ensures that there are no loop corrections at leading order [98], only one-loop corrections at the first subleading order, and only up to two-loop corrections at second subleading order, as shown in Chapter 6. Thus, we need only analyze one loop to show that the factorizing contributions do not modify the soft operator at first subleading order.

We focus on the case of a scalar in the loop. This case is particularly transparent because there are no infrared singularities associated with scalars circulating in a loop [107, 108]. This allows us to study the soft behavior without being entangled with the issue of infrared divergences. We can determine the behavior through the first subleading soft order simply by computing the diagrams in Fig. 7.5.

We have carried out the analogous computation to the one performed in Ref. [103] for gluons, but for gravity with a real scalar in the loop. The result of this computation is

$$\mathcal{D}^{\mu\nu,\text{fact,s}} = -\frac{i}{(4\pi)^2}\left(\frac{\kappa}{2}\right)^3 \frac{1}{30q\cdot k_a}\Big((\varepsilon_n\cdot\varepsilon_a)(q\cdot k_a) - (q\cdot\varepsilon_a)(k_a\cdot\varepsilon_n)\Big)^2 k_a^\mu k_a^\nu + \mathcal{O}(q^2)\,, \quad (7.73)$$

where we have kept all terms involving no more than one overall power of the soft momentum $q \equiv k_n$. Such terms naively appear to contribute at the first subleading order in the correction to the amplitude. However, as in the gauge-theory case, the diagrams $\mathcal{D}^{\mu\nu,\text{fact,s}}$ contract into a current $\mathcal{J}_{\mu\nu}$ which results in a suppression of an extra factor of the soft momentum q. In the gravity case we find

$$(k_a + q)^\mu \mathcal{J}_{\mu\nu} = f(k_i, \varepsilon_i)(k_a + q)_\nu \,, \tag{7.74}$$

where f is some function of the momenta and polarizations of both the hard and soft legs. With $k_a^\mu k_a^\nu$ contracting with $\mathcal{J}_{\mu\nu}$, we then have

$$\begin{aligned} k_a^\mu k_a^\nu \mathcal{J}_{\mu\nu} &= (k_a + q)^\mu (k_a + q)^\nu \mathcal{J}_{\mu\nu} + \mathcal{O}(q) \\ &= f(k_i, \varepsilon_i)(k_a + q)^2 + \mathcal{O}(q) \\ &= 2f(k_i, \varepsilon_i) q \cdot k_a + \mathcal{O}(q) \\ &= \mathcal{O}(q) \,. \end{aligned} \tag{7.75}$$

Therefore as far the correction to the amplitude is concerned, we can effectively view $\mathcal{D}^{\mu\nu,\text{fact,s}}$ as being of order q^2. We then finally have

$$\mathcal{D}^{\mu\nu,\text{fact,s}} \frac{i}{2q \cdot k_a} \mathcal{J}_{\mu\nu} = \mathcal{O}(q) \,. \tag{7.76}$$

After including the $1/q$ from the intermediate propagator, we find the potential correction to the soft operator is of $\mathcal{O}(q)$ and therefore does not modify the first subleading soft behavior. Unfortunately, for the second subleading soft behavior we lose control, in much the same way that we did for the first subleading behavior of gauge theory. Indeed, nontrivial contributions are found in explicit examples (see Ref. [125] and Chapter 6.

As in the QCD case (7.69), we expect the cases with other particles circulating in the loop to be similar and that factorizing contributions not linked to infrared singularities should appear starting only at the second subleading order in the soft expansion. In addition, the explicit gravity examples studied in Ref. [125] and Chapter 6 are exactly in line with this expectation. We leave a discussion of cases with infrared singularities to future work.

7.7 Conclusions

In this chapter we extended Low's proof of the universality of subleading behavior of photons to nonabelian gauge theory and to gravity. In particular, we showed that in gauge theory, on-shell gauge invariance can be used to fully determine the first subleading soft-gluon behavior at tree level. In gravity the first two subleading terms in the soft expansion found in Ref. [83] can also be fully determined from on-shell gauge invariance. Our discussion is similar to the ones given by Low [93] for photons and by Jackiw [100] for gravitons coupled to a scalar at four points. We focused mainly on n-gluon and n-graviton amplitudes, but also discussed simpler cases with scalars.

A motivation for studying soft-graviton theorems is to understand their relation to the extended BMS symmetry. It will, of course, be very important to understand how BMS symmetry relates to the proof of soft properties in n-graviton amplitudes given here.

Unlike the case of photons, for gluons there are loop corrections to the soft operators starting at leading order. In gauge theory, leading-order corrections are linked to infrared singularities, while subleading-order corrections can also arise from contributions not linked to infrared singularities. Gravity also has loop corrections but not at leading order. In this chapter we prove that for the case of a scalar circulating in the loop, there is no modification to the soft behavior of graviton amplitudes until the second subleading order. We expect this to hold in general for contributions not linked to infrared singularities. On the other hand, graviton loop contributions that are infrared divergent give corrections to the soft operators starting at the first subleading order, as shown in Chapter 6, using the standard definition of dimensional regularization. Since infrared singularities are well-understood, we do not expect this to be too disruptive for studying the consequences of extended BMS symmetry at loop level. We will describe loop level in more detail elsewhere.

Part III

Evanescent Effects in Gravity at Two Loops

CHAPTER 8

Evanescent Effects Can Alter Ultraviolet Divergences in Quantum Gravity without Physical Consequences

8.1 Introduction

Although theories of quantum gravity have been studied for many decades, basic questions about their ultraviolet (UV) structure persist. One subtlety is the conformal anomaly[1], also known as the Weyl or trace anomaly [129]. At one loop, the conformal anomaly provides the coefficient of the Gauss-Bonnet (GB) term. The physical significance of this relationship has not been settled, however. In particular, Duff and van Nieuwenhuizen showed that the conformal anomaly changes under duality transformations of p-form fields, suggesting that theories related through such transformations are quantum-mechanically inequivalent [130]. In response, Siegel argued that this effect is a gauge artifact and therefore not physical [131]; Fradkin and Tseytlin and Grisaru et al. have also argued that duality should hold at the quantum level [132]. Furthermore, for $D = 4$ external states, one-loop divergences in gravity theories coupled to two-form antisymmetric tensors are unchanged under a duality transformation relating two-forms to zero-form scalars [58, 133]. However, as we shall see, intuition based on one-loop analyses can be deceptive.

As established in the seminal work of 't Hooft and Veltman [56], pure gravity is finite at one loop because the only available counterterm is the GB term, which integrates to zero in a topologically trivial background. While amplitudes with external matter fields diverge at one loop, amplitudes with only external gravitons remain finite. At two loops, however,

[1] Einstein gravity is not conformally invariant, so this is not an anomaly in the traditional sense.

pure gravity diverges, as demonstrated explicitly by Goroff and Sagnotti [67] and confirmed by van de Ven [134].

In this chapter, we investigate the UV properties of the two-loop amplitude for scattering of four identical-helicity gravitons, including the effect of p-form duality transformations. We use dimensional regularization, which forces us to consider the effects of evanescent operators like the GB term, which are legitimate operators in D dimensions but vanish (or are total derivatives) in four dimensions. We show that the GB counterterm is required to cancel subdivergences and reproduce the two-loop counterterm coefficient found previously [67, 134].

Evanescent operators are well-studied in gauge theory (see e.g. Ref. [135]), where they can modify subleading corrections. In contrast, we find that evanescent effects can alter the leading UV divergence in gravity.[2] Despite this change in the UV divergence, the physical dependence of the renormalized amplitude on the renormalization scale remains unchanged. This break in the link between the UV divergence and the renormalization-scale dependence is unlike familiar one-loop examples. We arrive at a similar conclusion when comparing the divergences and renormalization-scale dependences in gravity coupled to scalars versus antisymmetric-tensor fields.

8.2 Setup

Pure gravity is defined by the Einstein-Hilbert Lagrangian,

$$\mathcal{L}_{\text{EH}} = -\frac{2}{\kappa^2}\sqrt{-g}R\,, \tag{8.1}$$

where $\kappa^2 = 32\pi G_N = 32\pi/M_P^2$ and the metric signature is $(+---)$. We also augment $\mathcal{L}_{\text{EH}}$ by matter Lagrangians for one of the following: n_0 scalars, n_2 two-form fields (antisymmetric

[2]Effects of the GB term have also been studied in renormalizable, but non-unitary, R^2 gravity [136].

tensors) or n_3 three-form fields:

$$\mathcal{L}_0 = \frac{1}{2}\sqrt{-g}\sum_{j=1}^{n_0} \partial_\mu \phi_j \partial^\mu \phi_j \,,$$
$$\mathcal{L}_2 = \frac{1}{6}\sqrt{-g}\sum_{j=1}^{n_2} H_{j\,\mu\nu\rho} H_j^{\mu\nu\rho} \,,$$
$$\mathcal{L}_3 = -\frac{1}{8}\sqrt{-g}\sum_{j=1}^{n_3} H_{j\,\mu\nu\rho\sigma} H_j^{\mu\nu\rho\sigma} \,. \tag{8.2}$$

Here ϕ_j is a scalar field and $H_{j\,\mu\nu\rho}$ and $H_{j\,\mu\nu\rho\sigma}$ are the field-strengths of the two- and three-form antisymmetric-tensor fields $A_{j\,\mu\nu}$ and $A_{j\,\mu\nu\rho}$. The index j labels distinct fields. Standard gauge-fixing for the two- and three-form actions, as well as for $\mathcal{L}_{\rm EH}$, leads to a nontrivial ghost structure. We avoid such complications by using the generalized unitarity method [36, 35, 6, 33, 55], which directly imposes appropriate D-dimensional physical-state projectors on the on-shell states crossing unitarity cuts.

Under a duality transformation, in four dimensions the two-form field is equivalent to a scalar:

$$H_{j\,\mu\nu\rho} \;\leftrightarrow\; \frac{i}{\sqrt{2}}\,\varepsilon_{\mu\nu\rho\alpha}\,\partial^\alpha \phi_j \,, \tag{8.3}$$

and the three-form field is equivalent to a cosmological-constant contribution via

$$H_{j\,\mu\nu\rho\sigma} \;\leftrightarrow\; \frac{2}{\sqrt{3}}\,\varepsilon_{\mu\nu\rho\sigma}\,\frac{\sqrt{\Lambda_j}}{\kappa} \,. \tag{8.4}$$

As usual, we expand the graviton field around a flat-space background: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Similarly, we expand the scalar, two-form field and three-form field around trivial background values. It is interesting to note that the three-form field has been proposed as a means for neutralizing the cosmological constant [137].

8.3 One Loop

For a theory with n_0 scalars, n_2 two-forms and n_3 three-forms coupled to gravity, the one-loop UV divergence takes the form of the GB term [129, 67, 130],

$$\begin{aligned}\mathcal{L}_{\rm GB} &= \frac{1}{(4\pi)^2}\frac{1}{\epsilon}\Big(\frac{53}{90}+\frac{n_0}{360}+\frac{91n_2}{360}-\frac{n_3}{2}\Big)\\ &\quad\times\sqrt{-g}(R^2-4R_{\mu\nu}^2+R_{\mu\nu\rho\sigma}^2)\,,\end{aligned}\tag{8.5}$$

which is proportional to the conformal anomaly. The calculations of the conformal anomaly and of the UV divergence are essentially the same, except that we replace a graviton polarization tensor with a trace over indices. Contracting Eq. (8.5) with four on-shell $D = 4$ graviton polarization tensors gives zero. This is because the GB combination is evanescent in $D = 4$: It is a total derivative and vanishes when integrated over a topologically trivial space; hence pure Einstein gravity is finite at one loop [56]. In a topologically nontrivial space, the integral over the GB term gives the Euler characteristic. When matter is added to the theory, the four-graviton amplitude is still UV finite at one loop, although divergences appear in amplitudes with external matter states.

Using the unitarity method, we verified Eq. (8.5) by considering the one-loop four-graviton amplitude with external states in arbitrary dimensions and internal ones in $D = 4-2\epsilon$ dimensions. On-shell scattering amplitudes are sensitive only to the coefficient of the $R_{\mu\nu\rho\sigma}^2$ operator, because the R^2 and $R_{\mu\nu}^2$ operators can be eliminated by field redefinitions at leading order in the derivative expansion. The GB combination is especially simple to work with in dimensional regularization since there are no propagator corrections in any dimension [138].

For the case of antisymmetric tensors coupled to gravity, another relevant one-loop four-point divergence is that of two gravitons and two antisymmetric tensors, generated by the operator,

$$\mathcal{L}_{RHH} = \Big(\frac{\kappa}{2}\Big)^2\frac{1}{(4\pi)^2}\frac{1}{\epsilon}\sqrt{-g}\sum_{j=1}^{n_3}R^{\mu\nu}{}_{\rho\sigma}H_{j\,\mu\nu\alpha}H_j^{\alpha\rho\sigma}\,.\tag{8.6}$$

Like the GB term, this operator is evanescent. In particular, in $D = 4$, we can dualize the antisymmetric tensors to scalars, which collapses the Riemann tensor into the Ricci scalar and tensor. Under field redefinitions, they can be eliminated in favor of the dualized scalars, removing the one-loop divergence in two-graviton two-antisymmetric-tensor amplitudes with $D = 4$ external states. The four-scalar amplitude does diverge.

The change in Eq. (8.5) under duality transformations is central to the claim by Duff and van Nieuwenhuizen of quantum inequivalence under such transformations [130]. Here we analyze their effects on the two-loop amplitude. First let us note that our unitarity-based evaluation of Eq. (8.5) sews together physical, gauge-invariant tree amplitudes. This explicitly demonstrates that the numerical coefficient of the $R^2_{\mu\nu\rho\sigma}$ term in Eq. (8.5) is gauge invariant, in contrast to implications of Ref. [131]. This gauge invariance suggests that by two loops, Eq. (8.5) could lead to duality-violating contributions to non-evanescent operators. To see if this happens, we must account for subdivergences and renormalization.

8.4 Two Loops

At two loops, pure gravity diverges in $D = 4$. The coefficient of this divergence was determined by Goroff and Sagnotti [67] from a three-point computation in the standard $\overline{\text{MS}}$ regularization scheme and later confirmed by van de Ven [134]:

$$\mathcal{L}_{R^3} = -\frac{209}{1440}\left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^4}\frac{1}{\epsilon}\sqrt{-g}\, R^{\alpha\beta}{}_{\gamma\delta} R^{\gamma\delta}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta}\,, \tag{8.7}$$

where we account for the fact that Refs. [67, 134] define $\epsilon = 4-D$ instead of our $\epsilon = (4-D)/2$. The divergence in Eq. (8.7) uses four-dimensional identities to simplify it.

In order to reproduce the Goroff and Sagnotti result, we evaluate the identical-helicity four-graviton amplitude. This is the simplest amplitude containing the two-loop divergence (8.7). While a four-point amplitude may seem to be unnecessarily complicated with respect to a three-point function, there are several advantages to considering an amplitude for a

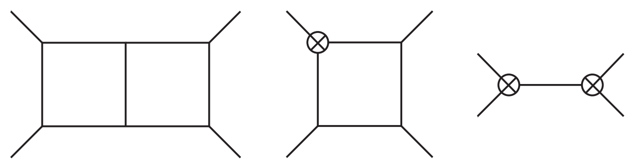

Figure 8.1: Representative diagrams of the (a) bare, (b) single-counterterm and (c) double-counterterm insertions.

physical process with real momenta. The first is that we can use the unitarity method to obtain a compact integrand [36, 35]. This method is particularly efficient for identical-helicity particles, having been used to obtain compact integrands for the gauge-theory case [55]. More importantly, the question of quantum equivalence under duality transformations can only be properly answered in the context of physical observables, such as renormalized and infrared-subtracted $2 \to 2$ scattering amplitudes entering physical cross-sections.

To facilitate comparisons to the two-loop four-point amplitude, we need the R^3 divergence (8.7) inserted into the four-plus-helicity tree amplitude:

$$A_{R^3} = \frac{209}{24}\frac{\mathcal{K}}{\epsilon}\,, \tag{8.8}$$

where

$$\mathcal{K} \equiv \left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4} stu \left(\frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\right)^2, \tag{8.9}$$

and $s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$ and $u = (k_1 + k_3)^2$ are the usual Mandelstam invariants. The last factor is a pure phase constructed from the spinor products $\langle ab\rangle$ and $[ab]$ defined in, for example, Ref. [139].

Fig. 8.1 shows that there are three types of contributions to consider: (a) the bare two-loop contribution, (b) the one-loop single-counterterm subtraction and (c) the double-counterterm subtraction. One might expect the net subdivergence subtractions (b) and (c) each to be zero because there are no corresponding $D = 4$ one-loop divergences. However, this is not correct. A careful analysis of the two-loop integrands [140] reveals subdivergences associated with the GB term (8.5). For the case of two-forms, a subdivergence corresponding to $\mathcal{L}_{RHH}$ in Eq. (8.6) must also be subtracted. In principle, when three-forms are present, there might have been subdivergences due to operators containing three-forms, but these do not appear. It is somewhat surprising that there are subdivergences at two loops without any corresponding one-loop divergences in $D = 4$. However, because some legs external to the subdivergence are in D dimensions, the cancellations that are specific to $D = 4$ do not occur.

While Goroff and Sagnotti also subtracted subdivergences, they did so integral by integral, rather than tracking the operator origin of the subdivergences as we do. Here we use dimensional regularization for both infrared and UV divergences; we subtract the well-known infrared singularities [95, 107, 108, 141] from the final result.

We evaluate the bare and single-subtraction contributions via the unitarity method. We take the external legs to be identical-helicity gravitons and each internal leg to be D-dimensional. The bare integrand obtained in this way is similar to integrands found earlier for gauge theory [6, 33, 55] and for the "double-copy" theory containing a graviton, an antisymmetric tensor and a dilaton (see Chapter 3). A key property of these integrands is that they vanish when the loop momenta are taken to reside in $D = 4$, yet the amplitudes are still nonvanishing. This phenomenon is related to the observation by Bardeen and Cangemi [142] that the nonvanishing of identical-helicity amplitudes is connected to an anomaly in the self-dual sector.

We follow the same regularization prescriptions used in Ref. [55], where algebraic manipulations on the integrand are performed with $\epsilon < 0$. We use the 't Hooft-Veltman variant: We place the external momenta and polarizations in $D = 4$ and take the loop momenta and internal states to reside in $D = 4 - 2\epsilon$ dimensions. Here we focus on the UV divergences and defer presentation of the integrands and finite terms in the amplitudes to Ref. [140].

We integrate over the loop momenta with the same techniques used to obtain two-loop four-point helicity amplitudes in QCD, including their finite parts [143, 144]. As a cross check, we also directly extract the UV divergences using masses to regulate the infrared. (See the procedures in Chapter 3 and the relevant appendices.)

8.4.1 Coupling Three-Form Fields to Gravity

	$1/\epsilon$	$\ln\mu^2$
bare	$-\frac{3431}{5400} - \frac{199n_3}{30} + 6n_3^2$	$-\frac{3431}{2700} - \frac{199n_3}{15} + 12n_3^2$
GB	$\frac{4\cdot 53-180n_3}{360} \cdot \frac{2\cdot(13+180n_3)}{15}$	$\frac{689}{675} + \frac{199n_3}{15} - 12n_3^2$
GB^2	$24\left(\frac{4\cdot 53-180n_3}{360}\right)^2$	0
total	$\frac{209}{24} - \frac{15}{2}n_3$	$-\frac{1}{4}$

Table 8.1: Coefficients of the $1/\epsilon$ UV pole and of $\ln\mu^2$ in the identical-helicity four-graviton two-loop amplitude for pure gravity coupled to n_3 three forms. We omit the overall factor of $\mathcal{K}$ defined in Eq. (8.9). The first row gives the bare two-loop contribution, the second row the single GB-counterterm insertion at one loop, and the third row that of a double GB insertion at tree level. The final row gives the total.

Consider first the case of n_3 three-forms coupled to gravity. In Table 8.1, we give both the divergence and renormalization-scale dependence of each of the three components illustrated in Fig. 8.1. In the bare and one-loop single-insertion components, the $\ln\mu^2$ dependence, where μ is the renormalization scale, is proportional to the UV divergence. For the bare two-loop part, the $\ln\mu^2$ coefficient is twice the coefficient of the $1/\epsilon$ divergence. For the single counterterm, it is equal to the $1/\epsilon$ coefficient, and for the double-insertion tree contribution, it vanishes. This follows from dimensional analysis of the loop integrals, with measure $\int d^{4-2\epsilon}\ell$ per loop, requiring an overall factor of $\mu^{2L\epsilon}$ at L loops. The counterterm subtractions are pure poles that do not carry such factors. In the sum over terms, there is no simple relation between the $1/\epsilon$ and the $\ln\mu^2$ coefficients, in contrast to many textbook examples at one loop.

As seen from the last line of Table 8.1, with no three-form fields we match exactly the Goroff and Sagnotti divergence (8.8). The addition of n_3 three-form fields shifts the divergence from the pure gravity result. One might think that this shift would lead to a physical change in the scattering amplitudes through a different dependence on μ. However, the $\ln\mu^2$ column of Table 8.1 shows that the n_3-dependence of the bare and single-counterterm contributions precisely cancels in the sum. The scale dependence is therefore unaffected by three-form fields. The differences in the divergent parts can be removed by adjusting the coefficient of the $1/\epsilon$ R^3 counterterm. We have also obtained the amplitude's finite parts [140]. Their form allows for a finite R^3 subtraction that completely eliminates the effects of three-form fields in the two-loop renormalized identical-helicity amplitude.

8.4.2 Coupling Two- or Zero-Form Fields to Gravity

	$1/\epsilon$	$\ln\mu^2$
bare	$-\frac{3431}{5400}-\frac{277n_0}{10800}+\frac{n_0^2}{5400}$	$-\frac{3431}{2700}-\frac{277n_0}{5400}+\frac{n_0^2}{2700}$
GB	$\frac{4\cdot 53+n_0}{360}\cdot\frac{2\cdot(13-n_0)}{15}$	$\frac{689}{675}-\frac{199n_0}{2700}-\frac{n_0^2}{2700}$
GB^2	$24\left(\frac{4\cdot 53+n_0}{360}\right)^2$	0
total	$\frac{209}{24}-\frac{1}{48}n_0$	$-\frac{2+n_0}{8}$

Table 8.2: Coefficients of the $1/\epsilon$ UV pole and of $\ln\mu^2$ in the four-graviton amplitude for gravity coupled to n_0 scalars. The table follows the same format as Table 8.1.

	$1/\epsilon$	$\ln\mu^2$
bare	$-\frac{3431}{5400}+\frac{8543n_2}{10800}+\frac{8281n_2^2}{5400}$	$-\frac{3431}{2700}+\frac{8543n_2}{5400}+\frac{8281n_2^2}{2700}$
GB	$\frac{4\cdot 53+91n_2}{360}\cdot\frac{2\cdot(13-91n_2)}{15}$	$\frac{689}{675}-\frac{18109n_2}{2700}-\frac{8281n_2^2}{2700}$
GB^2	$24\left(\frac{4\cdot 53+91n_2}{360}\right)^2$	0
RHH	$5n_2$	$5n_2$
total	$\frac{209}{24}+\frac{299}{48}n_2$	$-\frac{2+n_2}{8}$

Table 8.3: Coefficients of the $1/\epsilon$ UV pole and of $\ln\mu^2$ in the two-loop four-graviton amplitude for gravity coupled to n_2 antisymmetric-tensor fields. The table follows the same format as Table 8.1. The second-to-last row gives the contribution of the RHH counterterm inserted into the one-loop amplitude.

We now turn to the case of duality transformations between antisymmetric-tensor fields and scalars. In Tables 8.2 and 8.3, the coefficients of $1/\epsilon$ and $\ln\mu^2$ terms are collected. The tables show that, while the individual components are quite different and the final $1/\epsilon$ divergence changes under duality transformations, scalars and two-forms have exactly the same renormalization-scale dependence. As for the case of three-forms, we find that the UV divergence does depend on the field representations, but the renormalization-scale dependence does not. Again, finite subtractions can be found to make the dual pair of renormalized amplitudes identical [140].

8.4.3 General Results

From Tables 8.1–8.3, we find that in all cases, the scale dependence in the identical-helicity four-graviton amplitude follows a simple behavior:

$$\mathcal{M}_4^{\text{2-loop}}\Big|_{\ln\mu^2} = -\mathcal{K}\,\frac{N_b - N_f}{8}\,\ln\mu^2\,, \tag{8.10}$$

where N_b (N_f) is the number of bosonic (fermionic) four-dimensional states in the theory. We only computed Eq. (8.10) explicitly for $N_f = 0$, but the identical-helicity graviton amplitude vanishes in supersymmetric theories, forcing Eq. (8.10) to be proportional to $N_b - N_f$.

The $\ln\mu^2$ dependence is clearly a more appropriate quantity for deciding whether a theory should be thought of as nonrenormalizable. If the coefficient of the $\ln\mu^2$ is nonvanishing, as is the case for pure gravity, the coefficient will run, and we consider such a theory to be nonrenormalizable. Our result shows that instead of focusing on the divergences, one should study the $\ln\mu^2$ coefficient to see if there is a principle that can be applied to set it to zero. One obvious useful principle is that renormalization schemes should be chosen that maintain the equality of theories related by duality transformations.

In this light, one might wonder if the recently-computed four-loop divergence of pure $\mathcal{N} = 4$ supergravity [39] is an artifact of the particular $SU(4)$ formulation of the theory that was used. However, with the uniform mass infrared regulator used in that calculation, extensive checks reveal that all subdivergences cancel. Therefore the coefficient of $\ln\mu^2$ is proportional to that of the $1/\epsilon$ divergence. When matter multiplets are added there are one-loop subdivergences, but those are not evanescent. In other formulations, it is possible that the divergences will change, but we do not expect the $\ln\mu^2$ coefficients to change.

8.5 Conclusions

In summary, our investigation of the ultraviolet divergences of nonsupersymmetric gravity reveals a number of striking features. The first is the nontrivial role of the conformal anomaly and the associated evanescent Gauss-Bonnet term entering subdivergences. It is remarkable that a term that vanishes in four dimensions can contribute directly to the leading divergence of a graviton amplitude. Another important feature is that the integrand of the identical-helicity amplitude vanishes if the loop momenta are taken to be four-dimensional; this feature of identical-helicity amplitudes, which follows straightforwardly from unitarity, is also tied to anomalous behavior [142]. Similar connections to anomalous behavior [145, 77] were noted in the four-loop divergence of $\mathcal{N} = 4$ pure supergravity [39].

A key lesson is that under duality transformations, the values of two-loop divergences can change, contrary to the situation at one loop [58, 133]. However, the difference in these divergences are unphysical, in the sense that they can be absorbed into a redefinition of the coefficient of a local operator. In other words, our results for scattering amplitudes are consistent with quantum equivalence under duality transformations when that equivalence allows for the adjustment of coefficients of higher-dimension operators. The dependence on the renormalization scale does not change under duality transformations in the examples we studied; it is a more appropriate measure of the UV properties of the theory. It would be quite interesting to establish this property beyond two loops. Together with recent examples of ultraviolet finiteness in supergravity amplitudes, despite the existence of seemingly valid counterterms [38, 31, 146], the results summarized in this chapter show that much more remains to be learned about both duality at the quantum level and the ultraviolet structure of gravity theories.

APPENDIX A

Two-Loop Dimensionally Regularized Integrals

In this appendix, we explicitly compute the divergent parts of dimensionally regularized two-loop integrals in $D = 4 - 2\epsilon$, appearing in Section 3.3.2. In general, both ultraviolet and infrared divergences appear as poles in ϵ so we must subtract the infrared ones in order to obtain the ultraviolet ones.

We start with the planar double-box integral, displayed in Fig. 3.5(a), following the discussion in Section 4 of Ref. [144],

$$\begin{aligned} &\mathcal{I}_4^{\rm P}[\mathcal{P}(\lambda_i, p, q, k_i)](s,t) \\ &\qquad \equiv \int \frac{d^D p}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{\mathcal{P}(\lambda_i, p, q, k_i)}{p^2 q^2 (p+q)^2 (p-k_1)^2 (p-k_1-k_2)^2 (q-k_4)^2 (q-k_3-k_4)^2} \,. \end{aligned} \tag{A.1}$$

Using Schwinger parameters, we rewrite the planar double-box integral with constant numerator as

$$\mathcal{I}_4^{\rm P}[1](s,t) = \frac{1}{(4\pi)^D} \prod_{i=1}^{7} \int_0^\infty dt_i \, [\Delta_{\rm P}(T)]^{-\frac{D}{2}} \exp\left[-\frac{Q_{\rm P}(s,t,t_i)}{\Delta_{\rm P}(T)} \right] , \tag{A.2}$$

where

$$\Delta_{\rm P}(T) = (T_p T_q + T_p T_{pq} + T_q T_{pq}) \,, \tag{A.3}$$

and

$$T_p = t_3 + t_4 + t_5 \,, \quad T_q = t_1 + t_2 + t_7 \,, \quad T_{pq} = t_6 \,. \tag{A.4}$$

T_p, T_q, and T_{pq} are sums of Schwinger parameters corresponding to propagators with loop momenta p, q, and $p+q$, respectively. We also have

$$Q_{\rm P}(s,t,t_i) = -s\left(t_1t_2T_p + t_3t_4T_q + t_6(t_1+t_3)(t_2+t_4)\right) - t\,t_5t_6t_7\,. \qquad \text{(A.5)}$$

To account for factors of λ_p^2, λ_q^2, and λ_{p+q}^2 in the numerator, we take derivatives on the (-2ϵ)-dimensional part of the (Wick-rotated) integral:

$$\int d\lambda_p^{-2\epsilon} d\lambda_q^{-2\epsilon} \exp\left[-T_p\lambda_p^2 - T_q\lambda_q^2 - T_{pq}\lambda_{p+q}^2\right] \propto \left[\Delta_{\rm P}(T)\right]^{\epsilon}\,, \qquad \text{(A.6)}$$

with respect to T_p, T_q, and T_{pq}. This introduces additional factors to be inserted in the integrand in Eq. (A.2). For example,

$$\begin{aligned}
(\lambda_p^2)^4 &\to -\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)\left(\frac{T_q+T_{pq}}{\Delta_{\rm P}(T)}\right)^4, \\
(\lambda_p^2)^3\lambda_q^2 &\to \epsilon^2(1-\epsilon)(2-\epsilon)\frac{(T_q+T_{pq})^2}{\Delta_{\rm P}(T)^3} - \epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)\frac{(T_q+T_{pq})^2T_{pq}^2}{\Delta_{\rm P}(T)^4}\,, \\
(\lambda_p^2)^2\lambda_q^2\lambda_{p+q}^2 &\to \epsilon^2(1-\epsilon)^2\frac{1}{\Delta_{\rm P}(T)^2} + \epsilon(1-\epsilon)(2-\epsilon)\frac{\epsilon(T_q^2+T_{pq}^2)+2T_qT_{pq}}{\Delta_{\rm P}(T)^3} \\
&\qquad - \epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)\frac{T_q^2T_{pq}^2}{\Delta_{\rm P}(T)^4}\,.
\end{aligned} \qquad \text{(A.7)}$$

We account for extra factors of $\Delta_{\rm P}^a(T)$ by shifting the dimension $D \to D-2a$. Following Smirnov [147], we change six of the seven Schwinger parameters to Feynman parameters with the delta-function constraint $\sum_{i\neq 6}\alpha_i = 1$:

$$\mathcal{I}_4^{\rm P}[\mathcal{P}(\lambda_p,\lambda_q)](s,t) = \frac{\Gamma[7-D+\gamma]}{(4\pi)^D}\int_0^\infty d\alpha_6 \prod_{i\neq 6}\int_0^1 d\alpha_i\,\delta\Big(1-\sum_{i\neq 6}\alpha_i\Big)\frac{\left[\Delta_{\rm P}(T)\right]^{7-\frac{3D}{2}+\gamma}}{\left[Q_{\rm P}(s,t,\alpha_i)\right]^{7-D+\gamma}}D(\alpha_i)\,, \qquad \text{(A.8)}$$

where $D(\alpha_i)$ represents the extra factors in one term of Eq. (A.7), with $t_i \to \alpha_i$. The parameter γ counts the factors of α_i in $D(\alpha_i)$ and can take on values 0, 2, and 4 for the integrals under consideration here. Next we perform a change of variables that imposes the delta-function constraint [147]:

$$\alpha_1 = \beta_1\xi_3\,, \quad \alpha_2 = (1-\xi_5)(1-\xi_4)\,, \quad \alpha_3 = \beta_2\xi_1\,, \quad \alpha_4 = \xi_5(1-\xi_2)\,,$$
$$\alpha_5 = \beta_2(1-\xi_1)\,, \quad \alpha_7 = \beta_1(1-\xi_3)\,, \quad \beta_1 = (1-\xi_5)\xi_4\,, \quad \beta_2 = \xi_5\xi_2\,. \qquad \text{(A.9)}$$

We then integrate these parameters to obtain a Mellin-Barnes representation, which we again integrate. Finally we arrive at the dimensionally regularized results of our required planar double-box integrals:

$$\begin{aligned}
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^4](s,t) &= \mathcal{I}'^{\rm P} - \frac{1}{(4\pi)^4}\frac{s+2t}{360\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm P}[(\lambda_{p+q}^2)^4](s,t) &= 2\mathcal{I}'^{\rm P} - \frac{1}{(4\pi)^4}\frac{29s+4t}{180\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^3\lambda_q^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{480\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^3\lambda_{p+q}^2](s,t) &= \mathcal{I}'^{\rm P} + \frac{1}{(4\pi)^4}\frac{s-t}{360\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2(\lambda_q^2)^2](s,t) &= \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2(\lambda_{p+q}^2)^2](s,t) &= \mathcal{I}'^{\rm P} - \frac{1}{(4\pi)^4}\frac{s+2t}{720\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2\lambda_q^2\lambda_{p+q}^2](s,t) &= \frac{1}{(4\pi)^4}\frac{s}{720\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm P}[\lambda_p^2\lambda_q^2(\lambda_{p+q}^2)^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{240\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm P}[\lambda_p^2(\lambda_{p+q}^2)^3](s,t) &= \mathcal{I}'^{\rm P} - \frac{1}{(4\pi)^4}\frac{5s+t}{180\epsilon} + \mathcal{O}(\epsilon^0)\,,
\end{aligned} \qquad \text{(A.10)}$$

where

$$\begin{aligned}
\mathcal{I}'^{\rm P} \;\equiv\; \frac{1}{(4\pi)^4}\Bigg[&\frac{1}{840s\epsilon^2}\left(2s^2+st+2t^2\right)(-s)^{-2\epsilon}e^{-2\epsilon\gamma_{\rm E}}\\
&+\frac{1}{88200su^4\epsilon}(4s^6+753s^5t+4306s^4t^2+9144s^3t^3\\
&\qquad\qquad -315\pi^2s^3t^3+9381s^2t^4+4813st^5+1019t^6)\\
&+\frac{t^3(11s^2+7st+2t^2)}{840su^3\epsilon}\log\Big(\frac{t}{s}\Big)-\frac{s^2t^3}{280u^4\epsilon}\log^2\Big(\frac{t}{s}\Big)\Bigg]+\mathcal{O}(\epsilon^0)\,.
\end{aligned} \qquad \text{(A.11)}$$

All integrals above are symmetric under $\lambda_p \leftrightarrow \lambda_q$.

Next we look at the nonplanar double-box integrals:

$$\mathcal{I}_4^{\rm NP}[\mathcal{P}(\lambda_i,p,q,k_i)](s,t) \equiv \int \frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{\mathcal{P}(\lambda_i,p,q,k_i)}{p^2q^2(p+q)^2(p-k_1)^2(q-k_2)^2(p+q+k_3)^2(p+q+k_3+k_4)^2}\,, \tag{A.12}$$

whose evaluation follows that of the planar double-box integrals quite closely. $\Delta_{\rm NP}(T)$ takes the same form as $\Delta_{\rm P}(T)$ in Eq. (A.3), except that

$$T_p = t_1+t_2\,,\quad T_q = t_3+t_4\,,\quad T_{pq} = t_5+t_6+t_7\,. \tag{A.13}$$

We then also have

$$Q_{\rm NP}(s,t,u,t_i) = -s\left(t_1t_3t_5+t_2t_4t_7+t_5t_7(T_p+T_q)\right) - t\,t_2t_3t_6 - u\,t_1t_4t_6\,. \tag{A.14}$$

In this case, we find it advantageous to only change the four Schwinger parameters associated with T_p and T_q to Feynman parameters, resulting in

$$\mathcal{I}_4^{\rm NP}[\mathcal{P}(\lambda_p,\lambda_q)] = \frac{\Gamma[7-D+\gamma]}{(4\pi)^D} \times \prod_{i=5}^{7}\int_0^\infty d\alpha_i \prod_{j=1}^{4}\int_0^1 d\alpha_j\,\delta\left(1-\sum_{i=1}^{4}\alpha_i\right)\frac{[\Delta_{\rm NP}(T)]^{7-\frac{3D}{2}+\gamma}}{[Q_{\rm NP}(s,t,u,\alpha_i)]^{7-D+\gamma}}D(\alpha_i)\,. \tag{A.15}$$

We impose the delta-function constraint via further redefinition:

$$\alpha_1 = \xi_3(1-\xi_1)\,,\qquad \alpha_2 = \xi_3\xi_1\,,\qquad \alpha_3 = (1-\xi_3)(1-\xi_2)\,,\qquad \alpha_4 = (1-\xi_3)\xi_2\,. \tag{A.16}$$

Once again we can straightforwardly integrate the parameters and use the Mellin-Barnes representation to evaluate our required nonplanar double-box integrals:

$$
\begin{aligned}
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^4](s,t) &= \mathcal{I}'^{\rm NP} - \frac{1}{(4\pi)^4}\frac{215s^2+342st+342t^2}{50400s\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm NP}[(\lambda_{p+q}^2)^4](s,t) &= \frac{1}{(4\pi)^4}\frac{s}{80\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^3\lambda_q^2](s,t) &= \mathcal{I}'^{\rm NP} - \frac{1}{(4\pi)^4}\frac{215s^2+342st+342t^2}{50400s\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^3\lambda_{p+q}^2](s,t) &= \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2(\lambda_q^2)^2](s,t) &= \mathcal{I}'^{\rm NP} - \frac{1}{(4\pi)^4}\frac{230s^2+171st+171t^2}{25200s\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2(\lambda_{p+q}^2)^2](s,t) &= \frac{1}{(4\pi)^4}\frac{s}{160\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2\lambda_q^2\lambda_{p+q}^2](s,t) &= \frac{1}{(4\pi)^4}\frac{s}{1440\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm NP}[\lambda_p^2\lambda_q^2(\lambda_{p+q}^2)^2](s,t) &= \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\rm NP}[\lambda_p^2(\lambda_{p+q}^2)^3](s,t) &= \frac{1}{(4\pi)^4}\frac{s}{160\epsilon} + \mathcal{O}(\epsilon^0)\,,
\end{aligned}
\tag{A.17}
$$

where

$$
\begin{aligned}
\mathcal{I}'^{\rm NP} \equiv \frac{1}{(4\pi)^4}\Bigg[&\frac{1}{840s\epsilon^2}\left(2t^2+tu+2u^2\right)(-s)^{-\epsilon}(-t)^{-\epsilon}e^{-2\epsilon\gamma_{\rm E}}\\
&+\frac{1}{352800s^5\epsilon}(5581u^6+25188u^5t+51783u^4t^2+64352u^3t^3\\
&\qquad -1260\pi^2u^3t^3+51783u^2t^4+25188ut^5+5581t^6)\\
&+\frac{u^3(11t^2+7tu+2u^2)}{840s^4\epsilon}\log\left(\frac{u}{t}\right)-\frac{t^3u^3}{280s^5\epsilon}\log^2\left(\frac{u}{t}\right)\Bigg]+\mathcal{O}(\epsilon^0)\,.
\end{aligned}
\tag{A.18}
$$

As with the planar results, the above are valid under the exchange $\lambda_p \leftrightarrow \lambda_q$.

Finally we evaluate the bow-tie integrals:

$$
\begin{aligned}
&\mathcal{I}_4^{\text{bow-tie}}[\mathcal{P}(\lambda_i,p,q,k_i)](s)\\
&\qquad\equiv\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\frac{\mathcal{P}(\lambda_i,p,q,k_i)}{p^2q^2(p-k_1)^2(p-k_1-k_2)^2(q-k_4)^2(q-k_3-k_4)^2}\,.
\end{aligned}
\tag{A.19}
$$

The bow-tie integrals are relatively simple because they are products of two one-loop integrals. Similar techniques involving Schwinger parameters and Mellin-Barnes representations can be used on each one-loop integral. Since bubbles with a massless leg vanish in dimensional regularization, the replacement $(p+q)^2 \to 2p\cdot q$ is valid in the numerator. We also use the tensor reduction $(\lambda_p \cdot \lambda_q)^2 \to \lambda_p^2 \lambda_q^2/(-2\epsilon)$. For the bow-tie integrals appearing in Eq. (3.34), this tensor reduction is the only source of an ultraviolet divergence. When evaluating the bow-tie contributions then, we expose $(\lambda_p \cdot \lambda_q)^2$ factors through the substitutions,

$$\lambda_{p+q}^2 \to \lambda_p^2 + \lambda_q^2 + 2(\lambda_p \cdot \lambda_q)\,, \qquad (p+q)^2 \to (2p_{(4)}\cdot q_{(4)}) - 2(\lambda_p\cdot\lambda_q)\,. \tag{A.20}$$

Only terms containing a $(\lambda_p \cdot \lambda_q)^2$ are ultraviolet divergent; there are no terms with $(\lambda_p \cdot \lambda_q)^4$ or higher powers of $(\lambda_p \cdot \lambda_q)$. The relevant bow-tie integrals are then given by

$$\begin{aligned}
\mathcal{I}_4^{\text{bow-tie}}[(\lambda_p^2)^2(\lambda_p\cdot\lambda_q)^2](s) &= \frac{1}{(4\pi)^4}\frac{s^2}{720\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\text{bow-tie}}[\lambda_p^2\lambda_q^2(\lambda_p\cdot\lambda_q)^2](s) &= \frac{1}{(4\pi)^4}\frac{s^2}{1152\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\text{bow-tie}}[(\lambda_p^2)^2\lambda_q^2(\lambda_p\cdot\lambda_q)^2](s) &= \frac{1}{(4\pi)^4}\frac{s^3}{8640\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\text{bow-tie}}[(\lambda_p^2)^2(\lambda_q^2)^2(\lambda_p\cdot\lambda_q)^2](s) &= \frac{1}{(4\pi)^4}\frac{s^4}{64800\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\text{bow-tie}}[(\lambda_p^2)^2(\lambda_p\cdot\lambda_q)^2(2p_{(4)}\cdot q_{(4)})](s,t) &= -\frac{1}{(4\pi)^4}\frac{s^2(10s-t)}{15120\epsilon} + \mathcal{O}(\epsilon^0)\,,\\
\mathcal{I}_4^{\text{bow-tie}}[\lambda_p^2\lambda_q^2(\lambda_p\cdot\lambda_q)^2(2p_{(4)}\cdot q_{(4)})](s,t) &= -\frac{1}{(4\pi)^4}\frac{s^2(12s-t)}{28800\epsilon} + \mathcal{O}(\epsilon^0)\,.
\end{aligned} \tag{A.21}$$

These are also symmetric under the exchange $\lambda_p \leftrightarrow \lambda_q$.

APPENDIX B

Two-Loop Infrared Divergence

In this appendix we obtain the two-loop infrared divergence for the four-point all-plus-helicity graviton amplitude in the theory of gravity coupled to a dilaton and an antisymmetric tensor using dimensional regularization in $D = 4 - 2\epsilon$. We subtract the infrared divergence from the total divergence to obtain the ultraviolet divergence. Infrared divergences in gravity can be obtained by exponentiating the divergence found at the one-loop order [95, 107, 141, 52]. In the cases where there is a divergence at one loop, the infrared singularities are 'one-loop exact'; however, in the all-plus-helicity gravitons case, the first divergence occurs at two loops. Nevertheless, the same principles apply. More specifically we are concerned with the exponentiation of the gravitational soft function, which describes the effects of soft graviton exchange between external particles.

Following the discussion of Ref. [107, 141], a gravity scattering amplitude can be written as

$$\mathcal{M}_n = S_n \cdot H_n \,, \tag{B.1}$$

where S_n is the infrared-divergent soft function and H_n is the infrared-finite hard function. Each quantity in Eq. (B.1) can be written as a loop expansion in powers of $(\kappa/2)^2(4\pi e^{-\gamma_E})^\epsilon$:

$$\mathcal{M}_n = \sum_{L=0}^{\infty} \mathcal{M}_n^{(L)}, \qquad S_n = 1 + \sum_{L=1}^{\infty} S_n^{(L)}, \qquad H_n = \sum_{L=0}^{\infty} H_n^{(L)}. \tag{B.2}$$

The soft function is given by the exponential of the lowest-order infrared divergence:

$$S_n = \exp\left[\frac{\sigma_n}{\epsilon}\right], \qquad \sigma_n = \left(\frac{\kappa}{2}\right)^2 \frac{1}{(4\pi)^{2-\epsilon}} e^{-\gamma_E \epsilon} \sum_{j=1}^{n} \sum_{i<j} s_{ij} \log\left(\frac{-s_{ij}}{\mu^2}\right), \qquad s_{ij} = (k_i + k_j)^2. \tag{B.3}$$

An L-loop amplitude can then be written as

$$\mathcal{M}_n^{(L)} = \sum_{l=0}^{L} \frac{1}{(L-l)!} \left[\frac{\sigma_n}{\epsilon}\right]^{L-l} H_n^{(l)}(\epsilon)\,. \tag{B.4}$$

For four-point amplitudes, we have

$$\sigma_4 = \left(\frac{\kappa}{2}\right)^2 \frac{2}{(4\pi)^{2-\epsilon}} e^{-\gamma_E \epsilon} \left[s \log\left(\frac{-s}{\mu^2}\right) + t \log\left(\frac{-t}{\mu^2}\right) + u \log\left(\frac{-u}{\mu^2}\right) \right], \tag{B.5}$$

and the one-loop infrared divergence is given by

$$\mathcal{M}_4^{(1)}\Big|_{\text{IR div.}} = \frac{\sigma_4}{\epsilon} \mathcal{M}_4^{(0)}. \tag{B.6}$$

We used this to subtract the infrared divergence from our dimensionally regularized one-loop result in Section 3.3.1.1 to isolate the ultraviolet divergence. The four-point two-loop infrared divergence is given by

$$\mathcal{M}_4^{(2)}\Big|_{\text{IR div.}} = \frac{1}{2}\left[\frac{\sigma_4}{\epsilon}\right]^2 \mathcal{M}_4^{(0)} + \frac{\sigma_4}{\epsilon} H_4^{(1)}(\epsilon)\Big|_{\text{IR div.}}. \tag{B.7}$$

For the all-plus-helicity gravitons case, the tree amplitude $\mathcal{M}_4^{(0)}$ vanishes. The one-loop amplitude is therefore infrared finite and equal to the one-loop infrared-finite hard function. The one-loop amplitude can be computed using the double-copy procedure in Section 3.3.1.1 and is given by [119]

$$\mathcal{M}^{(1)}(1^+, 2^+, 3^+, 4^+) = -\left(\frac{\kappa}{2}\right)^4 \frac{i}{(4\pi)^2} \left(\frac{[1\,2][3\,4]}{\langle 1\,2\rangle\langle 3\,4\rangle}\right)^2 \frac{(D_s-2)^2}{240} \left(s^2 + t^2 + u^2\right). \tag{B.8}$$

The two-loop infrared divergence is then

$$\mathcal{M}^{(2)}(1^+,2^+,3^+,4^+)\Big|_{\text{IR div.}} = -\frac{1}{\epsilon}\left(\frac{\kappa}{2}\right)^6 \frac{i}{(4\pi)^4}\left(\frac{[1\,2][3\,4]}{\langle 1\,2\rangle\langle 3\,4\rangle}\right)^2 \frac{(D_s-2)^2}{120}\left(s^2+t^2+u^2\right) \times \left[s\log\left(\frac{-s}{\mu^2}\right)+t\log\left(\frac{-t}{\mu^2}\right)+u\log\left(\frac{-u}{\mu^2}\right)\right]. \qquad \text{(B.9)}$$

APPENDIX C

Two-Loop Ultraviolet Divergences from Vacuum Integrals

In this appendix we compute the ultraviolet divergences of the integrals in Section 3.3.2. The techniques are very similar to those used to study the one-loop ultraviolet properties of gravity in Section 3.3. However, before we can use them, we must deal with the (-2ϵ)-dimensional components λ_p, λ_q, and λ_{p+q} in the numerators of the integrals using the techniques in Section 4.1 of Ref. [144].

The effect of inserting factors of λ_p, λ_q, and λ_{p+q} into the planar and nonplanar double-box integrals is very similar to inserting factors of $v \cdot p$, $v \cdot q$, and $v \cdot (p+q)$, where

$$v^\mu \equiv \epsilon^\mu{}_{\nu_1\nu_2\nu_3} k_1^{\nu_1} k_2^{\nu_2} k_3^{\nu_3} \,. \tag{C.1}$$

Example parameter insertions for factors of λ_i are given in Eq. (A.7). For polynomials in $v \cdot p$ and $v \cdot q$, we have

$$\begin{aligned}
(v\cdot p)^8 \rightarrow& 105 \left(\frac{stu}{8}\right)^4 \frac{(T_q + T_{pq})^4}{\Delta^4} \,, \\
(v\cdot p)^6 (v\cdot q)^2 \rightarrow& \left(\frac{stu}{8}\right)^4 \left[15 \frac{(T_q + T_{pq})^2}{\Delta^3} + 105 \frac{(T_q + T_{pq})^2 T_{pq}^2}{\Delta^4} \right] , \\
(v\cdot p)^4 (v\cdot q)^4 \rightarrow& \left(\frac{stu}{8}\right)^4 \left[9 \frac{1}{\Delta^2} + 90 \frac{T_{pq}^2}{\Delta^3} + 105 \frac{T_{pq}^4}{\Delta^4} \right] , \\
(v\cdot p)^4 (v\cdot q)^2 (v\cdot(p+q))^2 \rightarrow& \left(\frac{stu}{8}\right)^4 \left[9 \frac{1}{\Delta^2} + 15 \frac{3T_q^2 + 3T_{pq}^2 - 2(T_q + T_{pq})^2}{\Delta^3} + 105 \frac{T_q^2 T_{pq}^2}{\Delta^4} \right] .
\end{aligned} \tag{C.2}$$

These are valid for both the planar and nonplanar double boxes provided the corresponding definitions for Δ, T_p, T_q, and T_{pq} given in Appendix A are used.

We can also relate polynomials in $v\cdot p$ and $v\cdot q$ to the λ_i. The four-dimensional component of the loop momenta p can be written as

$$p^\mu_{[4]} \equiv c^p_1 k^\mu_1 + c^p_2 k^\mu_2 + c^p_3 k^\mu_3 + c^p_v v^\mu , \tag{C.3}$$

where

$$\begin{aligned} c^p_1 =& \frac{1}{2su}\left[-t(2p\cdot k_1) + u(2p\cdot k_2) + s(2p\cdot k_3)\right], \\ c^p_2 =& \frac{1}{2st}\left[t(2p\cdot k_1) - u(2p\cdot k_2) + s(2p\cdot k_3)\right], \\ c^p_3 =& \frac{1}{2tu}\left[t(2p\cdot k_1) + u(2p\cdot k_2) - s(2p\cdot k_3)\right], \\ c^p_v =& -\frac{4}{stu}\epsilon_{\mu\nu_1\nu_2\nu_3} p^\mu k_1^{\nu_1} k_2^{\nu_2} k_3^{\nu_3} = -\frac{4}{stu} v\cdot p\,. \end{aligned} \tag{C.4}$$

We therefore have

$$p^2 + \lambda_p^2 = p_{[4]}\cdot p_{[4]} = s c^p_1 c^p_2 + t c^p_2 c^p_3 + u c^p_1 c^p_3 - \frac{1}{4} stu (c^p_v)^2\,, \tag{C.5}$$

or

$$\lambda_p^2 = -\frac{4}{stu}(v\cdot p)^2 + \hat{\mathcal{P}}_p\,, \tag{C.6}$$

where

$$\hat{\mathcal{P}}_p \equiv -p^2 + s c^p_1 c^p_2 + t c^p_2 c^p_3 + u c^p_1 c^p_3\,. \tag{C.7}$$

Similarly, we have

$$
\begin{aligned}
\lambda_q^2 &= -\frac{4}{stu}(v\cdot q)^2 + \hat{\mathcal{P}}_q\,, \\
\lambda_{p+q}^2 &= -\frac{4}{stu}(v\cdot (p+q))^2 + \hat{\mathcal{P}}_{pq}\,,
\end{aligned} \tag{C.8}
$$

where

$$
\begin{aligned}
\hat{\mathcal{P}}_q \equiv & -q^2 + s\,c_1^q c_2^q + t\,c_2^q c_3^q + u\,c_1^q c_3^q\,, \\
\hat{\mathcal{P}}_{pq} \equiv & -(p+q)^2 + s(c_1^p + c_1^q)(c_2^p + c_2^q) + t(c_2^p + c_2^q)(c_3^p + c_3^q) + u(c_1^p + c_1^q)(c_3^p + c_3^q)\,.
\end{aligned} \tag{C.9}
$$

These relations, along with the parameter replacements in Eqs. (A.7), (C.2), allow us to rewrite the integrals involving factors λ_i in terms of integrals involving tensor products between the loop momenta and the external momenta. For a general function $f(p\cdot k_i, q\cdot k_i)$, we have

$$
\begin{aligned}
\int (\lambda_p^2)^4 f &= -\frac{\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)}{105}\left(\frac{8}{stu}\right)^4 \int (v\cdot p)^8 f \\
&= -\frac{16\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)}{(1-2\epsilon)(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)} \int \hat{\mathcal{P}}_p^4 f\,, \\
\int (\lambda_p^2)^3 \lambda_q^2 f &= -\frac{16\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)}{(1-2\epsilon)(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)} \int \hat{\mathcal{P}}_p^3 \hat{\mathcal{P}}_q f \\
&\quad + \frac{12\epsilon(1-\epsilon)(2-\epsilon)}{(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)} \int \frac{\hat{\mathcal{P}}_p^2 f}{\Delta}\,, \\
\int (\lambda_p^2)^2 (\lambda_q^2)^2 f &= -\frac{16\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)}{(1-2\epsilon)(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)} \int \hat{\mathcal{P}}_p^2 \hat{\mathcal{P}}_q^2 f \\
&\quad + \frac{16\epsilon(1-\epsilon)(2-\epsilon)}{(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)} \int \frac{\hat{\mathcal{P}}_p \hat{\mathcal{P}}_q f}{\Delta} \\
&\quad - \frac{6\epsilon(1-\epsilon)}{(5-2\epsilon)(7-2\epsilon)} \int \frac{f}{\Delta^2}\,, \\
\int (\lambda_p^2)^2 \lambda_q^2 \lambda_{p+q}^2 f &= -\frac{16\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)}{(1-2\epsilon)(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)} \int \hat{\mathcal{P}}_p^2 \hat{\mathcal{P}}_q \hat{\mathcal{P}}_{pq} f \\
&\quad + \frac{4\epsilon(1-\epsilon)(2-\epsilon)}{(3-2\epsilon)(5-2\epsilon)(7-2\epsilon)} \int \frac{\hat{\mathcal{P}}_p(\hat{\mathcal{P}}_p + 2\hat{\mathcal{P}}_q + 2\hat{\mathcal{P}}_{pq}) f}{\Delta} \\
&\quad - \frac{6\epsilon(1-\epsilon)}{(5-2\epsilon)(7-2\epsilon)} \int \frac{f}{\Delta^2}\,,
\end{aligned}
\tag{C.10}
$$

where a factor $1/\Delta$ indicates that a shift in dimension of the integral should be made: $D \to D+2$, $\epsilon \to \epsilon - 1$ (ϵ's in prefactors in Eq. (C.10) should *not* be shifted, however).

Once we have integrals in a form involving tensor products between the loop momenta and external momenta, we expand in small external momenta to reduce to logarithmically divergent integrals, just as we did in the one-loop case. This gives us vacuum integrals. We then reduce the tensors involving loop momenta using Lorentz covariance and insert an infrared mass regulator. By integrating we obtain the ultraviolet divergences. Since every prefactor in Eq. (C.10) contains a factor of ϵ, to get the ultraviolet divergence, we only need the $1/\epsilon^2$ pole of the integrals on the right-hand side. These leading contributions have no dependence on the mass regulator, so we are unaffected by subdivergence issues due to the mass regulator. The ultraviolet divergences of the planar and nonplanar double-box integrals are then

$$
\begin{aligned}
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^4](s,t) &= \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm P}[(\lambda_{p+q}^2)^4](s,t) &= -\frac{1}{(4\pi)^4}\frac{14s+t}{90\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^3\lambda_q^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{480\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^3\lambda_{p+q}^2](s,t) &= \frac{1}{(4\pi)^4}\frac{2s+t}{360\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2(\lambda_q^2)^2](s,t) &= \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2(\lambda_{p+q}^2)^2](s,t) &= \frac{1}{(4\pi)^4}\frac{s+2t}{720\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm P}[(\lambda_p^2)^2\lambda_q^2\lambda_{p+q}^2](s,t) &= \frac{1}{(4\pi)^4}\frac{s}{720\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm P}[\lambda_p^2\lambda_q^2(\lambda_{p+q}^2)^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{240\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm P}[\lambda_p^2(\lambda_{p+q}^2)^3](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{40\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^4](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{80\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[(\lambda_{p+q}^2)^4](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{80\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^3\lambda_q^2](s,t) &= \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^3\lambda_{p+q}^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{80\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2(\lambda_q^2)^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{7s}{1440\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2(\lambda_{p+q}^2)^2](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{160\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[(\lambda_p^2)^2\lambda_q^2\lambda_{p+q}^2](s,t) &= \frac{1}{(4\pi)^4}\frac{s}{1440\epsilon} + \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[\lambda_p^2\lambda_q^2(\lambda_{p+q}^2)^2](s,t) &= \mathcal{O}(\epsilon^0)\,, \\
\mathcal{I}_4^{\rm NP}[\lambda_p^2(\lambda_{p+q}^2)^3](s,t) &= -\frac{1}{(4\pi)^4}\frac{s}{160\epsilon} + \mathcal{O}(\epsilon^0)\,.
\end{aligned}
\tag{C.11}
$$

The bow-tie integrals do not contain infrared divergences, and their ultraviolet divergences were computed in Appendix A. Combining all the pieces then gives us the ultraviolet divergence in Eq. (3.36).

APPENDIX D

SUSY BCJ Box Numerators

In this appendix, we provide the four-dimensional $\mathcal{N} = 4$ and $\mathcal{N} = 1$ (chiral) BCJ box numerators with formal polarization vectors. As discussed in Section 4.2, supersymmetry reduces the maximum power of loop momentum in our box numerators. Specifically, it is reduced from $\mathcal{O}(p^4)$ in $\mathcal{N} = 0$ to $\mathcal{O}(p^2)$ in $\mathcal{N} = 1$ and $\mathcal{O}(p^0)$ in $\mathcal{N} = 4$. While clearly seen in the $\mathcal{N} = 4$ expression below, this property is not explicit in the $\mathcal{N} = 1$ (chiral) expression. The improved power counting is only manifest when the inverse propagators in the numerators are expanded. For example, observing the labeling convention of Fig. 4.1 for the box numerator with external-leg ordering $(1, 2, 3, 4)$, we would have to expand p_3^2 as so,

$$p_3^2 = (p - k_1 - k_2)^2 = p^2 - 2\,(p \cdot k_1) - 2\,(p \cdot k_2) + s^2. \tag{D.1}$$

The $\mathcal{N} = 4$ BCJ box numerator in four dimensions is as follows:

$$\begin{aligned} n^{\mathcal{N}=4}_{1234;p} \;=& -i\Big[-\tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,s\,t + \tfrac{1}{2}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,s\,t + \tfrac{1}{2}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,s^2 - \mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,s \\ &- \mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{41}\,s - 2\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,s - \mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{34}\,s - 2\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,s \\ &- 2\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,s - 2\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{24}\,s - 2\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,s - 2\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,s\Big] + \text{cyclic}. \end{aligned} \tag{D.2}$$

Even with formal polarization vectors, we can identify this as $stA_4^{\text{tree}}(1, 2, 3, 4)$.

The $\mathcal{N}=1$ (chiral) BCJ box numerator in four dimensions is

$$
\begin{aligned}
n^{\mathcal{N}=1(\text{chiral})}_{1234;p} &= -i\Big[-\tfrac{1}{4}\,\mathcal{E}_{12}\,\mathcal{E}_{34}\,p_1^2\,p_3^2+\tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,p_1^2\,p_3^2+\tfrac{1}{4}\,\mathcal{E}_{12}\,\mathcal{E}_{34}\,p_1^2\,p_2^2-\tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,p_1^2\,p_2^2\\
&+\tfrac{1}{4}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_1^2\,p_2^2-\tfrac{1}{4}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\left(p_1^2\right)^2-\mathcal{E}_{23}\,\mathcal{K}_{41}\,\mathcal{P}_{44}\,p_1^2+\mathcal{E}_{12}\,\mathcal{K}_{34}\,\mathcal{P}_{33}\,p_1^2-\mathcal{E}_{24}\,\mathcal{K}_{31}\,\mathcal{P}_{33}\,p_1^2\\
&+\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,p_1^2+\mathcal{E}_{34}\,\mathcal{K}_{31}\,\mathcal{P}_{22}\,p_1^2+\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{22}\,p_1^2-\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,p_1^2-\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11}\,p_1^2\\
&-\tfrac{1}{4}\,\mathcal{E}_{12}\,\mathcal{E}_{34}\,p_2^2\,s+\tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,p_2^2\,s-\tfrac{1}{4}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_2^2\,s+\tfrac{1}{4}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,p_1^2\,s-\tfrac{1}{2}\,\mathcal{E}_{13}\,\mathcal{P}_{22}\,\mathcal{P}_{44}\,s\\
&-\tfrac{1}{2}\,\mathcal{E}_{24}\,\mathcal{P}_{11}\,\mathcal{P}_{33}\,s+\mathcal{E}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22}\,s+\tfrac{1}{2}\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{K}_{42}\,p_1^2+\tfrac{1}{2}\,\mathcal{E}_{34}\,\mathcal{K}_{31}\,\mathcal{K}_{42}\,p_1^2+\tfrac{1}{2}\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,p_1^2\\
&-\tfrac{1}{2}\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{K}_{42}\,p_1^2-\tfrac{3}{2}\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{41}\,p_1^2-\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,p_1^2+\tfrac{1}{2}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{K}_{41}\,p_1^2-\tfrac{1}{2}\,\mathcal{E}_{23}\,\mathcal{K}_{31}\,\mathcal{K}_{34}\,p_1^2\\
&+\tfrac{1}{2}\,\mathcal{E}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,p_1^2-\tfrac{1}{2}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,p_1^2-\tfrac{1}{2}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{31}\,p_1^2+\tfrac{1}{2}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{K}_{31}\,p_1^2\\
&+\tfrac{1}{2}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{24}\,p_1^2-\tfrac{1}{2}\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,p_1^2-\mathcal{K}_{24}\,\mathcal{K}_{42}\,\mathcal{P}_{11}\,\mathcal{P}_{33}+2\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22}\\
&+2\,\mathcal{K}_{13}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,\mathcal{P}_{22}+\tfrac{1}{2}\,\mathcal{E}_{34}\,\mathcal{K}_{41}\,\mathcal{P}_{22}\,s-\tfrac{1}{2}\,\mathcal{E}_{13}\,\mathcal{K}_{34}\,\mathcal{P}_{22}\,s-\tfrac{1}{2}\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{22}\,s+\tfrac{1}{2}\,\mathcal{E}_{14}\,\mathcal{K}_{23}\,\mathcal{P}_{22}\,s\\
&+\tfrac{1}{2}\,\mathcal{E}_{34}\,\mathcal{K}_{42}\,\mathcal{P}_{11}\,s+\tfrac{1}{2}\,\mathcal{E}_{23}\,\mathcal{K}_{34}\,\mathcal{P}_{11}\,s+\tfrac{1}{2}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11}\,s-\tfrac{1}{2}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{P}_{11}\,s+\tfrac{1}{2}\,\mathcal{E}_{34}\,\mathcal{K}_{12}\,\mathcal{P}_{11}\,s\\
&+\mathcal{K}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{42}\,\mathcal{P}_{11}-\mathcal{K}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,\mathcal{P}_{11}-\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,\mathcal{P}_{11}-\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,\mathcal{P}_{11}\\
&-\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{P}_{11}-\tfrac{1}{16}\,\mathcal{E}_{13}\,\mathcal{E}_{24}\,s\,t+\tfrac{1}{8}\,\mathcal{E}_{14}\,\mathcal{E}_{23}\,s\,t-\tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{42}\,s+\tfrac{1}{2}\,\mathcal{E}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{41}\,s\\
&-\tfrac{1}{4}\,\mathcal{E}_{24}\,\mathcal{K}_{23}\,\mathcal{K}_{41}\,s-\tfrac{1}{4}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{34}\,s-\tfrac{1}{2}\,\mathcal{E}_{13}\,\mathcal{K}_{12}\,\mathcal{K}_{24}\,s+\tfrac{1}{2}\,\mathcal{E}_{14}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,s-\tfrac{1}{4}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\,\mathcal{K}_{42}\\
&-\tfrac{1}{2}\,\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{34}\,\mathcal{K}_{41}-\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{31}\,\mathcal{K}_{34}-\tfrac{1}{2}\,\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{31}\,\mathcal{K}_{34}-\mathcal{K}_{12}\,\mathcal{K}_{23}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\\
&-\mathcal{K}_{12}\,\mathcal{K}_{13}\,\mathcal{K}_{24}\,\mathcal{K}_{31}\Big]+\text{cyclic.}
\end{aligned}
\tag{D.3}
$$

APPENDIX E

Determination of the Phase Factor

We determine the phase factor, $\mathcal{P}_{h_i}(i; j_1, j_2)$, through comparison with the spinor-helicity representation of the polarization vectors (cf. Ref. [80]):

$$\varepsilon^{\mu}_{\pm}(i; j_1) \equiv \pm \frac{\left\langle j_1^{\mp}\right| \gamma^{\mu} \left|i^{\mp}\right\rangle}{\sqrt{2}\left\langle j_1^{\mp}|i^{\pm}\right\rangle}. \tag{E.1}$$

Using the spin sum completeness relation in the massless limit,

$$\not{p} = \sum_{s=1,2} u_s(p)\bar{u}_s(p) = u_+(p)\bar{u}_+(p) + u_-(p)\bar{u}_-(p) = \left|p^+\right\rangle \left\langle p^+\right| + \left|p^-\right\rangle \left\langle p^-\right|, \tag{E.2}$$

and

$$\left\langle i^-|j^+\right\rangle \equiv \langle ij\rangle, \qquad \left\langle i^+|j^-\right\rangle \equiv [ij], \qquad \left\langle i^+|j^+\right\rangle = 0, \qquad \left\langle i^-|j^-\right\rangle = 0, \tag{E.3}$$

we find, for $k_{j_2} \neq k_i \neq k_{j_1}$,

$$k_{j_2} \cdot \varepsilon_{h_i}(i; j_1) = \pm \frac{\left\langle j_1^{\mp}\right| \not{k}_{j_2} \left|i^{\mp}\right\rangle}{\sqrt{2}\left\langle j_1^{\mp}|i^{\pm}\right\rangle} = \begin{cases} +\frac{\langle j_1 j_2\rangle [j_2 i]}{\sqrt{2}\langle j_1 i\rangle}, & \text{if } h_i \text{ is } + \\ -\frac{[j_1 j_2]\langle j_2 i\rangle}{\sqrt{2}[j_1 i]}, & \text{if } h_i \text{ is } - \end{cases} = -\sqrt{\frac{s_{j_1 j_2} s_{j_2 i}}{2\, s_{j_1 i}}}\, e^{-ih_i\left(\phi_{ij_2} - \phi_{j_1 j_2} + \phi_{ij_1}\right)}. \tag{E.4}$$

Without loss of generality, we compare this result to the inner product of k_{j_2} with the momentum basis representation of Eq. (4.19),

$$k_{j_2} \cdot \varepsilon_{h_i}(i; j_1, j_2) = \mathcal{P}_{h_i}(i; j_1, j_2)\sqrt{\frac{s_{j_1 j_2} s_{j_2 i}}{2 s_{j_1 i}}}, \tag{E.5}$$

to conclude that

$$\mathcal{P}_{h_i}(i; j_1, j_2) = -e^{-ih_i\left(\phi_{ij_2} - \phi_{j_1 j_2} + \phi_{ij_1}\right)}. \tag{E.6}$$

Since there are only three independent external momenta, there is nothing special about choosing k_{j_2} as both the second reference momentum in Eq. (E.5) and as the momentum to contract with the polarization vectors in Eqs. (E.4) and (E.5). The other choice leads to the same result: there is a minus sign from momentum conservation on the external legs and also a minus sign due to momentum conservation in the phase factors from Eqs. (4.13).

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