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SQUARE PRINCIPLES IN $\mathbb{P}_{\max}$ EXTENSIONS

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ABSTRACT

By forcing with $\mathbb{P}_{\max}$ over strong models of determinacy, we obtain models
where different square principles at ω_2 and ω_3 fail. In particular, we obtain
a model of $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \neg\square(\omega_2) + \neg\square(\omega_3)$.

1. Introduction

The forcing notion $\mathbb{P}_{\max}$ was introduced by W. Hugh Woodin in the early 1990s,
see Woodin [Woo10]. When applied to models of the Axiom of Determinacy, it
achieves a number of effects not known to be obtainable by forcing over models
of ZFC.

Recall that $\mathsf{AD}_\mathbb{R}$ asserts the determinacy of all length ω perfect information
two player games where the players alternate playing real numbers, and Θ
denotes the least ordinal that is not a surjective image of the reals. As usual,
MM denotes the maximal forcing axiom, Martin's Maximum. By $\mathsf{MM}(\mathfrak{c})$ we
denote its restriction to partial orders of size at most continuum. For the
strengthening $\mathsf{MM}^{++}(\mathfrak{c})$ of this latter principle, see Woodin [Woo10, Definition
2.47].

Woodin [Woo10, Theorem 9.39] shows that when $\mathbb{P}_{\max}$ is applied to a model of $\mathsf{AD}_{\mathbb{R}} +$ "Θ is regular", the resulting extension satisfies $\mathsf{MM}^{++}(\mathfrak{c})$. A natural question is to what extent one can extend this result to partial orders of size $\mathfrak{c}^+$. For many partial orders of this size, obtaining the corresponding forcing axiom from a determinacy hypothesis should greatly reduce the known upper bound for its large cardinal consistency strength. Moreover, using the Core Model Induction, a method pioneered by Woodin (see Schindler–Steel [SS14] and Sargsyan [Sar15b]), one can find lower bounds for the consistency strength of $\mathsf{MM}^{++}(\mathfrak{c}^+)$ and its consequences, which leads to the possibility of proving equiconsistencies.

In this paper we apply $\mathbb{P}_{\max}$ to theories stronger than $\mathsf{AD}_{\mathbb{R}} +$ "Θ is regular" and obtain some consequences of $\mathsf{MM}^{++}(\mathfrak{c}^+)$ on the extent of square principles, as introduced by Ronald B. Jensen [Jen72]. We recall the definitions:

Definition 1.1: Given a cardinal κ, the principle $\square_\kappa$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ such that for each $\alpha < \kappa^+$,

 (1) C_α is club in α;
 (2) for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$;
 (3) the order type of each C_α is at most κ.

Definition 1.2: Given an ordinal γ, the principle $\square(\gamma)$ asserts the existence of a sequence $\langle C_\alpha \mid \alpha < \gamma \rangle$ satisfying the following conditions:

 (1) For each $\alpha < \gamma$,
   - C_α is club in α;
   - for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$.
 (2) There is no *thread* through the sequence, that is, there is no club $E \subseteq \gamma$ such that $C_\alpha = E \cap \alpha$ for each limit point α of E.

We refer to sequences witnessing these principles as $\square_\kappa$-sequences or $\square(\gamma)$-sequences, respectively. A sequence satisfying the first part of Definition 1.2 (but possibly having a thread) is called a *coherent sequence*. Note that $\square_\kappa$ implies $\square(\kappa^+)$, and that $\square_\omega$ is true.

Remark 1.3: Suppose that κ is uncountable. A key distinction between these principles is that $\square_\kappa$ persists to outer models that agree about κ^+, while $\square(\kappa^+)$ need not. This seems to be folklore; since we could not locate an argument in the literature, we sketch one below.

For example, consider the poset $\mathbb{P}$ that attempts to add a $\square(\kappa^+)$-sequence with initial segments. Note that $\mathbb{P}$ is $(\kappa + 1)$-strategically closed.

Let G be V-generic for $\mathbb{P}$, and assume for the moment that the generic sequence added by $\mathbb{P}$ is indeed a $\square(\kappa^+)$-sequence, say $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$. Then one can thread it by further forcing over V[G] with the poset $\mathbb{Q}$ whose conditions are closed bounded subsets c of κ^+ such that $\max(c)$ is a limit point of c and, for every $\alpha \in \lim(c)$, we have $c \cap \alpha = C_\alpha$.

This threading does not collapse κ^+, because $\mathbb{Q}$ is κ^+-distributive in V[G], by a standard argument. In fact, the forcing $\mathbb{P} * \dot{\mathbb{Q}}$ has a κ^+-closed dense set consisting of conditions of the form $(p, \dot{q})$ where p decides the value $\dot{q}$ to be $p(\alpha)$, for α the largest ordinal in $\mathrm{dom}(p)$. This can also be verified by a standard density argument.

Assume now that PFA holds in V, so $\square(\kappa^+)$ fails (as does any $\square(\gamma)$ for $\mathrm{cf}(\gamma) > \omega_1$, by Stevo Todorcevic [Tod84]). Since PFA is preserved by ω_2-closed forcing, by König–Yoshinobu [KY04], it holds in the extension by $\mathbb{P} * \dot{\mathbb{Q}}$. (One could argue similarly starting from a universe where κ is indestructibly supercompact.)

It remains to argue that the sequence $\vec{C}$ added by $\mathbb{P}$ is a $\square(\kappa^+)$-sequence. The (standard) argument verifying this was suggested by James Cummings, and simplifies our original approach, where a more elaborate poset than $\mathbb{P}$ was being used. Assume instead that the generic sequence is threadable, and let $\dot{c}$ be a name for a thread. Now inductively construct a descending sequence of conditions p_n, and an increasing sequence of ordinals γ_n, for $n \in \omega$, such that, letting α_n be the length of p_n, we have:

(1) $\alpha_n < \gamma_n < \alpha_{n+1}$,
(2) $p_{n+1} \Vdash \gamma_n \in \dot{c}$, and
(3) p_{n+1} determines the value of $\dot{c} \cap \alpha_n$.

Let $\gamma = \sup_n \gamma_n = \sup_n \alpha_n$, and let p' be the union of all p_n. Then p' is not a condition, but can be made into one, call it p^*, by adding at γ as the value $p^*(\gamma)$, some cofinal subset of γ of order type ω that is distinct from $\dot{c} \cap \gamma$.

Then p^* forces that the γ-th member of $\vec{C}$ is different from $\dot{c} \cap \gamma$. But γ is forced by p^* to be a limit point of both $\dot{c}$ and the γ-th member of $\vec{C}$, and therefore p^* forces that $\dot{c}$ is not a thread through $\vec{C}$.

Viewing this as a density argument, we see that densely many conditions force that $\dot{c}$ is not a thread thorough $\vec{C}$. Thus, $\vec{C}$ is a $\square(\kappa^+)$-sequence in V[G], as we wanted.

This shows that neither $\square(\kappa^+)$, nor its negation, is upward absolute to models that agree on κ^+. See also the discussion on *terminal square* in Ernest Schimmerling [Sch07, §6].

Question 1.4: Assuming that $\square(\kappa^+)$ fails, can it be made to hold by κ^+-closed forcing? This seems unlikely, though we do not see a proof at the moment. If the answer is no, then the argument above can be simplified, as there is no need to assume PFA or any such hypothesis on the background universe.

Via work of Todorcevic [Tod84, Tod02], it is known that $\mathsf{MM}^{++}(\mathfrak{c})$ implies $2^{\aleph_1} = \aleph_2 + \neg\square(\omega_2)$, and $\mathsf{MM}^{++}(\mathfrak{c}^+)$ implies $\neg\square(\omega_3)$. Through work of Schimmerling [Sch07] and Steel (via the Core Model Induction, see Schindler–Steel [SS14]) it is known that the following statement implies that the Axiom of Determinacy holds in the inner model $L(\mathbb{R})$:

$$(1) \qquad\qquad \neg\square(\omega_2) + \neg\square_{\omega_2} + 2^{\aleph_1} = \aleph_2.$$

Theorem 3.1 below shows that if $\mathsf{AD}_{\mathbb{R}} +$ "Θ is regular" holds, and there is no $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ such that $L(\Gamma, \mathbb{R}) \models$ "Θ is Mahlo in HOD", then a weak form of $\square_{\omega_2}$ (restricted to a club subset of Θ, which becomes ω_3 in the $\mathbb{P}_{\max}$ extension) holds in HOD (the inner model of all hereditarily ordinal definable sets, see Jech [Jec03]). The argument is in essence a standard adaptation of the usual proof of square principles in fine structural models; the HOD analysis of Sargsyan [Sar15b] makes this adaptation possible. As shown in Corollary 3.5, this gives a lower bound for the hypotheses needed for the threadability results in this paper.

The axiom AD^+ is an ostensible strengthening of AD, introduced by Woodin (we refer the reader to page 611 of [Woo10] for the definition). It is an open question whether the two statements are equivalent. It is also an open question whether AD^+ follows from $\mathsf{AD}_{\mathbb{R}}$; Woodin has shown that it does follow from $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

Woodin's analysis of $\mathbb{P}_{\max}$ is carried out abstractly in the context of AD^+. Woodin has shown that the theory $\mathsf{AD}^+ + \mathsf{Uniformization}$ is equivalent to the theory $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$ (see Theorem 9.22 or Theorem 9.24 of [Woo10]); we use this theory for several of our results.[1] A hypothesis weaker than AD^+ +

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[1] Uniformization is the statement that if $A \subseteq \mathbb{R} \times \mathbb{R}$ is such that each vertical cross-section of A is nonempty, then A contains the graph of a function with domain $\mathbb{R}$. Uniformization follows easily from $\mathsf{AD}_{\mathbb{R}}$.

Uniformization + "Θ is Mahlo in HOD" suffices to make $\square_{\omega_2}$ fail in the $\mathbb{P}_{\max}$ extension; see Theorem 4.1. However, this extension will not be a model of Choice: a standard argument (see Remark 2.5) shows that if AD and Uniformization hold, then forcing with $\mathbb{P}_{\max}$ cannot well-order $\mathcal{P}(\mathbb{R})$. Theorem 4.3 gives a result on the failure of $\square(\Theta)$ in determinacy models.

The hypothesis that we use to force the negation of $\square_{\omega_2}$ in a model of Choice is in the end just slightly stronger than the assumption of Theorem 3.1 (see Theorem 5.1, and the discussion before):

AD$^+$ + Uniformization + $V = L(\mathcal{P}(\mathbb{R}))$ + "$\{\kappa \mid \kappa$ is regular in HOD, is a member of the Solovay sequence, and has cofinality $\omega_1\}$ is stationary in Θ".

We show from this hypothesis that $\square_{\omega_2}$ fails in the extension given by $\mathbb{P}_{\max}$ followed by a natural forcing $(\mathrm{Add}(\omega_3, 1))$ well-ordering the power set of the reals. That $2^{\aleph_1} = \aleph_2$ and $\neg\square(\omega_2)$ also hold follows from Woodin's work. Similarly, a stronger hypothesis allows us to conclude that even $\square(\omega_3)$ fails in the final extension; see Theorem 6.5.

The determinacy hypotheses we use are all weaker in consistency strength than a Woodin cardinal that is limit of Woodin cardinals [Sar15b, Sar15c, Sar15a, ST]. This puts them within the region suitable to be reached from current techniques by a Core Model Induction. Moreover, these hypotheses are much weaker than the previously known upper bounds on the strength of (1) and similar theories.

Prior to our work, two methods were known to show the consistency of (1): It is a consequence of PFA($\mathfrak{c}^+$), the restriction of the proper forcing axiom to partial orders of size $\mathfrak{c}^+$ (see Todorcevic [Tod84]), and it can be forced directly from the existence of a *quasicompact* cardinal. Quasicompactness was introduced by Jensen, see Cummings [Cum05] and Jensen [Jen98]; Sean Cox (unpublished), and possibly others, observed that the classical argument from James E. Baumgartner [Bau76] obtaining the consistency of "every stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$ reflects" from weak compactness adapts straightforwardly to this setting.

We expect that the HOD analysis (see Sargsyan [Sar15b]) should allow us to extend the Core Model Induction to establish the precise consistency strength of (1). The question of whether it is possible to obtain MM$^{++}(\mathfrak{c}^+)$ or even PFA($\mathfrak{c}^+$) in a $\mathbb{P}_{\max}$ extension of some determinacy model remains open.

Since forcing axioms are connected with *failures* of square principles, we want to suggest some notation to refer to these negations in a positive way, highlighting their *compactness* character, and solving the slight notational inconvenience that refers to the square principle at a cardinal successor κ^+ as $\square_\kappa$, as if it were a property of its predecessor.

Definition 1.5: Let γ be an ordinal. We say that γ **is threadable** if and only if $\square(\gamma)$ fails.

If $\gamma = \lambda^+$ is a successor cardinal, we say that γ **is square inaccessible** if and only if $\square_\lambda$ fails.

For general background on descriptive set theory, we refer to Kechris [Kec95] and Moschovakis [Mos09]; the latter is also a good reference for basic determinacy results. In addition, for determinacy and Woodin's AD^+ theory, we also refer to Woodin [Woo10], Jackson [Jac10], Caicedo–Ketchersid [CK11], Ketchersid [Ket11], and references therein. Basic knowledge of determinacy will be assumed in what follows.

We also assume some ease with $\mathbb{P}_{\max}$ arguments, although the properties of $\mathbb{P}_{\max}$ that we require could be isolated and treated as black boxes. We refer to Woodin [Woo10] and Larson [Lar10] for background.

2. From HOD to $\mathrm{HOD}_{\mathcal{P}(\mathbb{R})}$

Some of our results use hypotheses on inner models of the form $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$, where $\mathcal{P}_\kappa(\mathbb{R})$ denotes the collection of sets of reals of Wadge rank less than κ. We show in this section how some of these hypotheses follow from statements about HOD. With the possible exception of Theorem 2.6, none of the results in this section is new. The key technical tool is given by Lemma 2.2.

We refer the reader to Kechris [Kec95] for background on the Wadge hierarchy. If A is a set of reals, we let $|A|_W$ denote its Wadge rank. Recall that Θ is the least ordinal that is not a surjective image of the reals, and that the *Solovay sequence*, introduced by Robert M. Solovay [Sol78], is the unique increasing sequence of ordinals $\langle \theta_\alpha \mid \alpha \leq \gamma \rangle$ such that

- θ_0 is the least ordinal that is not the surjective image of the reals by an ordinal definable function;

- for each $\alpha < \gamma$, $\theta_{\alpha+1}$ is the least ordinal that is not the surjective image of the reals by a function definable from an ordinal and a set of reals of Wadge rank θ_α;
- for each limit ordinal $\beta \leq \gamma$, $\theta_\beta = \sup\{\theta_\alpha \mid \alpha < \beta\}$; and
- $\theta_\gamma = \Theta$.

Remark 2.1: The proof of Solovay [Sol78, Lemma 0.2] shows that, under AD, whenever γ is an ordinal, and $\phi\colon \mathbb{R} \to \gamma$ is a surjection, there exists a set of reals of Wadge rank γ definable from ϕ. From this it follows that if $|F|_W = \theta_\alpha < \Theta$, then every set of reals of Wadge rank less than $\theta_{\alpha+1}$ is definable from F, a real and an ordinal. In turn, it follows from this that there is no surjection from $\mathbb{R}$ to $\theta_{\alpha+1}$ ordinal definable from any set of reals of Wadge rank less than $\theta_{\alpha+1}$. It follows moreover that for each $\alpha \leq \gamma$ as in the definition of the Solovay sequence, $\mathcal{P}_{\theta_\alpha}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap \mathrm{HOD}_{\mathcal{P}_{\theta_\alpha}(\mathbb{R})}$ and θ_α is the Θ of $\mathrm{HOD}_{\mathcal{P}_{\theta_\alpha}(\mathbb{R})}$.

The following is due to Woodin, building on work of Petr Vopěnka (see Jech [Jec03, Theorem 15.46]):

LEMMA 2.2 (ZF): *For each ordinal ξ there exists a complete Boolean algebra $\mathbb{B}$ in HOD such that for each $E \subseteq \xi$ there is a HOD-generic filter $H \subseteq \mathbb{B}$ with*

$$\mathrm{HOD}_E = \mathrm{HOD}[H].$$

Furthermore, if AD holds, $\xi < \Theta$ and θ is the least member of the Solovay sequence greater than ξ, then $\mathbb{B}$ can be taken to have cardinality at most θ in V.

Proof. Fix an ordinal γ such that every ordinal definable subset of $\mathcal{P}(\xi)$ is ordinal definable in V_γ. Let $\mathbb{B}_0$ be the following version of the *Vopěnka algebra*: $\mathbb{B}_0$ is the Boolean algebra consisting of all sets of the form

$$A_{\phi,s} = \{x \subseteq \xi \mid V_\gamma \models \phi(x,s)\},$$

where ϕ is a formula and s is a finite subset of γ, ordered by inclusion.

The relation $A_{\phi,s} = A_{\phi',s'}$ is ordinal definable, and there is an ordinal definable well-ordering of the corresponding equivalence classes. Let η be the length of this well-ordering, let $h\colon \eta \to \mathbb{B}_0$ be the corresponding inverse rank function, and let $\mathbb{B}_1$ be the Boolean algebra with domain η induced by h. Since the relation $A_{\phi,s} \subseteq A_{\phi',s'}$ is ordinal definable, $\mathbb{B}_1$ is in HOD.

Given a filter $G \subseteq \mathbb{B}_0$, let $E(G)$ be the set of $\alpha \in \xi$ such that

$$\{E \subseteq \xi \mid \alpha \in E\} \in G.$$

Then for any $E \subseteq \xi$, $E(\{A \in \mathbb{B}_0 \mid E \in A\}) = E$. According to Vopěnka's Theorem (see Caicedo–Ketchersid [CK11]),

(1) for every $E \subseteq \xi$, $H_E = h^{-1}[\{A \in \mathbb{B}_0 \mid E \in A\}]$ is HOD-generic for $\mathbb{B}_1$;
(2) there exists a $\mathbb{B}_1$-name $\dot{E} \in \text{HOD}$ such that if $H \subseteq \mathbb{B}_1$ is HOD-generic and $G = h[H]$, then $E(G) = \dot{E}_H$.

Now suppose that $H \subseteq \mathbb{B}_1$ is HOD-generic, and let $E = \dot{E}_H$. Let us see first that $\text{HOD}_E \subseteq \text{HOD}[H]$. Suppose that A is a set of ordinals in HOD_E, and fix an ordinal δ such that A and E are both subsets of δ. Then there is an ordinal definable relation $T \subseteq \delta \times \mathcal{P}(\delta)$ such that $A = \{\zeta < \delta \mid T(\zeta, E)\}$. Define the relation T^* on $\delta \times \mathbb{B}_1$ by setting $T^*(\zeta, p)$ if and only if $T(\zeta, D)$ holds for all $D \in h(p)$. Then $T^* \in \text{HOD}$, and A is in $\text{HOD}[H]$, since A is the set of $\zeta < \sup(A)$ such that there exists a p in H for which $T^*(\zeta, p)$ holds.

For the other direction, note first that for any set $F \subseteq \xi$, H_F is in HOD_F. It follows that H_E is in $\text{HOD}[H]$. Moreover, the previous paragraph shows that there is a $\mathbb{B}_1$-name in HOD for $H_{\dot{E}_H}$. Densely many conditions in $\mathbb{B}_1$ then decide whether or not the generic filter H will be equal to $H_{\dot{E}_H}$. However, for any condition $p \in \mathbb{B}_1$, if F is any element of $h(p)$ then $\dot{E}_{H_F} = F$, which means that p cannot force the generic filter H to be different from $H_{\dot{E}_H}$. It follows then that H is in HOD_E so $\text{HOD}[H] = \text{HOD}_E$, whenever $H \subseteq \mathbb{B}_1$ is HOD-generic and $E = \dot{E}_H$.

Finally, assume that AD holds, and let θ be as in the statement of the lemma. Then θ is either θ_0 or $\theta_{\alpha+1}$ for some α. Let F be empty if the first case holds, and a set of reals of Wadge rank θ_α otherwise.

Let us see that the cardinality of η is at most θ in V. By Remark 2.1 and the Moschovakis Coding Lemma (see for instance Koellner–Woodin [KW10, Theorem 3.2] or Kanamori [Kan09, Theorem 28.15]) there is a surjection $\pi \colon \mathbb{R} \to \mathcal{P}(\xi)$ definable from ξ and F. If A is an ordinal definable subset of $\mathcal{P}(\xi)$, then $\pi^{-1}[A]$ is ordinal definable from F, which means that $|\pi^{-1}[A]|_W < \theta$, which again by Remark 2.1 implies that A is definable from F, a real and a finite subset of θ. For each fixed finite $a \subseteq \theta$, the definability order on the sets $A \subseteq \mathcal{P}(\xi)$ definable from F, a, and a real induces a pre-well-ordering of the

reals definable from F and a, which must then have order type less than θ. It follows from this that η also has cardinality at most θ.   ∎

By Theorem 9.10 of [Woo10], if AD^+ holds, then it holds in every inner model of ZF containing $\mathbb{R}$.

Remark 2.3: Under AD^+, every set of reals is ordinal definable from a set of ordinals, and in fact this set can be taken to be a bounded subset of Θ (see Woodin [Woo10, Lemma 9.5]).

Combining this fact with Lemma 2.2 gives the following folklore result.

THEOREM 2.4: *Assume that AD^+ holds. If θ is a member of the Solovay sequence and θ is regular in HOD, then θ is regular in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$. For any $S \subseteq \theta$ in HOD, if S is stationary in HOD then S is stationary in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.*

Proof. We prove the first part and leave the proof of the second, which is similar, to the reader. If $\theta = \theta_0$ or θ is a successor member of the Solovay sequence, there is a set of reals A from which Wadge-cofinally many sets of reals are ordinal definable. It follows then that θ is regular in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ without any assumption on HOD, since for each function $f \colon \mathbb{R} \to \theta$ in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ there is a surjection from $\mathbb{R}$ to $\sup(f[\mathbb{R}])$ definable from A.

Now suppose that θ is a limit in the Solovay sequence, and that $f \colon \alpha \to \theta$ is a cofinal function in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$, for some $\alpha < \theta$. Then f is ordinal definable from some set of reals in $\mathcal{P}_\theta(\mathbb{R})$ that itself is ordinal definable from a bounded subset A of θ (see Remark 2.3).

Pick $\theta_\xi < \theta$ such that A is bounded in θ_ξ. By Lemma 2.2, there is a set H, generic over HOD via a partial order of cardinality less than $\theta_{\xi+1}$ in HOD, and such that $\mathrm{HOD}_A \subseteq \mathrm{HOD}[H]$. Then $f \in \mathrm{HOD}[H]$. By cardinality considerations, the regularity of θ in HOD is preserved in $\mathrm{HOD}[H]$, giving a contradiction.   ∎

With a little more work, one gets Theorem 2.6 below.

Remark 2.5: A standard argument shows that, assuming that both AD and $\mathsf{Uniformization}$ hold, there is no set A such that every set of reals is ordinal definable from A and a real (consider the set of pairs (x, y) such that y is not ordinal definable from A and x). It follows, for instance, that $\mathsf{AD} + \mathsf{Uniformization}$

implies that there is no function from an ordinal to $\mathcal{P}(\mathbb{R})$ whose range is Wadge-cofinal, and also that the Solovay sequence has limit length (applying Remark 2.1). Similarly, since $\mathbb{P}_{\max} \subseteq H(\aleph_1)$, AD + Uniformization implies that there is no $\mathbb{P}_{\max}$-name for a well-ordering of $\mathcal{P}(\mathbb{R})$.

By Theorem 9.24 of [Woo10] (modulo the fact that $\mathsf{AD}_\mathbb{R}$ reflects to the inner model $L(\mathcal{P}(\mathbb{R}))$), if AD^+ holds and the Solovay sequence has limit length then $\mathsf{AD}_\mathbb{R}$ (and thus AD + Uniformization) holds.

Given a cardinal θ, a filter on θ is *θ-complete* if it is closed under intersections of cardinality less than θ, and *$\mathbb{R}$-complete* if it is closed under intersections indexed by $\mathbb{R}$.

THEOREM 2.6: *Assume that AD^+ holds, and that θ is a limit on the Solovay sequence. Let F be a θ-complete filter on θ in HOD, and let F' be the set of elements of $\mathcal{P}(\theta) \cap \mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ containing some member of F. Then the following hold:*

    (1) *The filter F' is $\mathbb{R}$-complete in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.*
    (2) *If F is normal in HOD, then F' is normal in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.*
    (3) *If F is an ultrafilter in HOD, then F' is an ultrafilter in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.*

Moreover, every θ-complete filter on θ in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ is $\mathbb{R}$-complete.

Proof. Again by Remark 2.3 (and the last line of Remark 2.1), every element of $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ is ordinal definable from a bounded subset of θ. By Lemma 2.2, every element of $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ exists in a generic extension of HOD by a partial order of cardinality less than θ in HOD. The second and third conclusions of the lemma follow, as well as the fact that F' is θ-complete in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$. It suffices then to prove the last part of the theorem.

Let F' be a θ-complete filter on θ, and G a function from $\mathbb{R}$ to F', both in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$. For each $\alpha < \theta$, let $B_\alpha = \{x \in \mathbb{R} \mid \alpha \in G(x)\}$. Then the sequence $\bar{B} = \langle B_\alpha \mid \alpha < \theta \rangle$ is ordinal definable from a set of reals of Wadge rank less than θ. Since AD + Uniformization holds in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ (see Remarks 2.3 and 2.5), and θ is the Θ of this model (see Remark 2.1), the Wadge ranks of the B_α cannot be cofinal in θ (by Remark 2.5). On the other hand, if unboundedly many of the B_α were to be distinct sets Wadge-below some fixed set of reals, then one could define from this situation a pre-well-ordering of length θ, which is impossible. So $\bar{B}$ must contain fewer than θ many distinct sets.

Suppose that $B \subseteq \mathbb{R}$ is such that $\{\alpha < \theta \mid B_\alpha = B\}$ is F'-positive. For each $x \in \mathbb{R}$, $G(x) \in F'$, so there is an $\alpha < \theta$ for which $x \in B_\alpha$ and $B_\alpha = B$. It follows that $B = \mathbb{R}$. Since F' is θ-complete,

$$\{\alpha \mid B_\alpha = \mathbb{R}\} \in F'.$$

Since $\bigcap_{x \in \mathbb{R}} G(x) = \{\alpha \mid B_\alpha = \mathbb{R}\}$, we are done.   ∎

3. Square in $\mathbb{P}_{\max}$ extensions of weak models of determinacy

Theorem 3.1 below shows that under a certain minimality hypothesis on our determinacy models, partial □-sequences of length Θ exist. Corollary 3.5 then shows that $\square_{\omega_2}$ holds in the $\mathbb{P}_{\max} * \mathrm{Add}(\Theta, 1)$-extension of a model of this hypothesis. In Section 4 we will see that $\mathbb{P}_{\max}$ alone does not necessarily add a $\square_{\omega_2}$-sequence in this context.

THEOREM 3.1: *Assume that* $\mathsf{AD}_\mathbb{R}$ *holds and that there is no* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *such that* $\mathrm{L}(\Gamma, \mathbb{R}) \models$ "Θ *is Mahlo in* HOD". *Then in* HOD *there exist a closed unbounded set* $C^* \subseteq \Theta$ *and a sequence* $\langle c_\tau \mid \tau \in C^* \rangle$ *such that the following hold.*

(a) *Each* c_τ *is a closed unbounded subset of* τ *and* $c_\tau \subseteq C^*$ *unless the order-type of* c_τ *is* ω.
(b) *If* $\bar{\tau}$ *is a limit point of* c_τ *then* $c_{\bar{\tau}} = c_\tau \cap \bar{\tau}$.
(c) $\mathsf{otp}(c_\tau) < \tau$.

Remark 3.2: The assumption that Θ is not Mahlo in $\mathrm{L}(\mathcal{P}(\mathbb{R}))$ is necessary for the theorem above, but the hypothesis that there is no Γ properly contained in $\mathcal{P}(\mathbb{R})$ such that $\mathrm{L}(\Gamma, \mathbb{R}) \models$ "Θ is Mahlo in HOD" may possibly be weakened. Its role is to make the HOD analysis from [Sar15b] applicable.

The heart of the construction takes place in HOD and is a straightforward combination of standard constructions of square sequences as developed in Jensen [Jen72], Schimmerling–Zeman [SZ04], and Zeman [Zem10], adapted to the context of strategic extender models as developed in Sargsyan [Sar15b]. In order to stay close to the constructions in Schimmerling–Zeman [SZ04] and Zeman [Zem10], we use fine structure notation and terminology as in Zeman [Zem02]; the rest of the notation and terminology is consistent with that in Mitchell–Steel [MS94], Steel [Ste96], and Sargsyan [Sar15b].

Proof. The key technical tool is a condensation lemma for initial segments of HOD which can be proved using the standard argument modified to the strategic extender models from Sargsyan [Sar15b]. Unlike the square constructions in Schimmerling–Zeman [SZ04] and Zeman [Zem10], our situation is specific in the sense that the initial segments of HOD used for the definition of the elements of our square sequence are never pluripotent (see Schimmerling–Zeman [SZ04] and Zeman [Zem10]), that is, they do not give rise to protomice. This makes it possible to run the construction without analysis of extender fragments, so the construction does not differ too much from that in L. The reason why this is the case is a consequence of the following corollary of our smallness assumption; see Sargsyan [Sar15b]:

(2)
   If $\theta_\alpha < \Theta$, then θ_α is not overlapped by an extender on the HOD-sequence.

We now formulate the condensation lemma. Recall that if $\theta_\alpha < \Theta$, then Σ_α is the iteration strategy for HOD$\,|\,\theta_\alpha$ in HOD.

LEMMA 3.3: *Assume $\mathcal{N}$ is an initial segment of HOD. Let $\theta_\alpha < \Theta$, and let $\mathcal{M}$ be a sound strategic premouse such that $\rho_{\mathcal{M}}^{n+1} = \theta_\alpha$. Let finally $\sigma : \mathcal{M} \to \mathcal{N}$ be a $\Sigma_0^{(n)}$-preserving map with critical point θ_α, $\sigma(\theta_\alpha) = \theta_\beta$, and $\sigma \in$ HOD. Then $\mathcal{M}$ is an initial segment of HOD.*

Proof (Sketch). Since $\sigma \restriction \theta_\alpha = \mathsf{id}$, the model $\mathcal{M}$ agrees with HOD below θ_α.

By (2), all critical points of the iteration tree on the $\mathcal{M}$-side of the comparison of $\mathcal{M}$ against HOD are strictly larger than θ_α. The preservation degree of σ guarantees that $\mathcal{M}$ is iterable when using extenders with critical points larger than θ_α, so $\mathcal{M}$ can be compared with HOD. By the theory developed in Sargsyan [Sar15b], HOD wins the comparison against $\mathcal{M}$; the assumption $\sigma \in$ HOD is used here. (We sketch the argument below, using freely notation and results form Sargsyan [Sar15b].) But then $\mathcal{M}$ is not moved in the coiteration, as it projects to θ_α, is sound, and all critical points on the $\mathcal{M}$-side are larger than θ_α. This gives the result. This is an instance of a more general result from Sargsyan [Sar15b], namely, that HOD thinks that it is *full*: Letting λ^{HOD} denote the order type of the set of Woodin cardinals in HOD and their limits, if $\alpha < \lambda^{\text{HOD}}$, and $\eta \in [\theta_\alpha, \theta_{\alpha+1})$ is a cutpoint, then any sound Σ_α-mouse $\mathcal{M}$ over HOD$\,|\,\eta$ with $\rho(\mathcal{M}) = \eta$ *is* an initial segment of HOD.

To see that HOD wins the comparison, let Σ be the strategy of $\mathcal{N}$, so Σ respects Σ_β, the iteration strategy for $\mathrm{HOD} \,|\, \theta_\beta$ in HOD. By the arguments of Sargsyan [Sar15b], all we need to check is that $\mathcal{M}$ is a Σ_α-premouse and Σ^σ respects Σ_α. This follows from hull condensation, and the former is immediate.

(3)

$$
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\ \mathcal{T}^\sigma\ } & \mathcal{N}' \\[4pt]
\sigma \uparrow & & \uparrow \sigma' \\[4pt]
\mathcal{M} & \xrightarrow{\ \mathcal{T}\ } & \mathcal{M}'
\end{array}
$$

To see this, suppose that $\mathcal{T}$ is the tree arising on the $\mathcal{M}$-side of the comparison, as in diagram (3), so $\mathcal{T}$ has critical point above θ_α. Let $\mathcal{U}$ be on the sequence of $\mathcal{M}'$. We must argue that $\mathcal{U}$ is according to Σ_α. We use σ' to produce $\sigma'(\mathcal{U})$, and note that the pointwise image $\sigma'[\mathcal{U}]$ is a hull of $\sigma'(\mathcal{U})$. But then $\mathcal{U}$ is also a hull of $\sigma'(\mathcal{U})$, and since $\sigma \in \mathrm{HOD}$, so is $\sigma' \in \mathrm{HOD}$. This allows us to invoke hull condensation, as claimed. ∎

In the following we treat the case where $\Theta = \theta_\Omega$ for some limit ordinal Ω which is not Mahlo in HOD. The remaining case, namely when Θ is of the form $\theta_{\alpha+1}$, is discussed at the end of this section.

We modify the construction in Zeman [Zem10] to the current context to obtain a global square sequence below Θ in HOD of the form $\langle C_\tau \mid \tau \in C^* \rangle$ where $C^* \in \mathrm{HOD}$ is a closed subset of Θ fixed in advance such that

- $\theta_\tau = \tau$ and τ is singular in HOD whenever $\tau \in C^*$;
- $\mathrm{cf}(\tau) = \omega$ whenever τ is a successor point of C^*.

Notice that the latter can be arranged by replacing C^* with $\lim(C^*)$.

In the rest of this section, we describe the modifications to the construction in Zeman [Zem10] that yield the sequence $\langle C_\tau \mid \tau \in C^* \rangle$. The point of our description is to separate aspects of the construction that can be accomplished by abstract fine structural considerations from those that are specific to strategic extender models. By "abstract fine structural considerations", we mean here methods within the framework of Zeman [Zem02, Chapter 1]. Our notation is consistent with that in Zeman [Zem10] with two exceptions: First, in our case Θ plays the same role the class of ordinals plays in Zeman [Zem10], and second, the sets C_τ from Zeman [Zem10] do not correspond to the sets denoted by C_τ here.

The construction in Zeman [Zem10] is carried out separately on two disjoint sets $\mathcal{S}^0, \mathcal{S}^1$. The class $\mathcal{S}^1$ consists of all those ordinals for which the singularizing structure is a protomouse. In our case, protomice do not arise in the construction due to (2), so we have $\mathcal{S}^1 = \varnothing$. This greatly simplifies the situation, as the verification that $C_\tau \subseteq \mathcal{S}^i$ whenever $\tau \in \mathcal{S}^i$ ($i = 0, 1$) and $\mathsf{otp}(C_\tau) > \omega$ involved a substantial amount of work in Zeman [Zem10]. Thus we will refer only to the portion of the construction in Zeman [Zem10] that concerns the set $\mathcal{S}^0$.

To each $\tau \in C^*$, we assign the singularizing level of HOD for τ, which we denote by $\mathcal{N}_\tau$. We then define the auxiliary objects for $\mathcal{N}_\tau$ exactly as in Zeman [Zem10]:

- By $\tilde{h}_\tau^k$, we denote the uniform $\Sigma_1^{(k-1)}$-Skolem function for $\mathcal{N}_\tau$; this is a partial function from $\omega \times \mathcal{N}_\tau$ into $\mathcal{N}_\tau$.
- We write $\tilde{h}_\tau^k(\gamma \cup \{p\})$ to denote the set of all values $\tilde{h}_\tau^k(i, \langle \xi, p \rangle)$, where $i \in \omega$ and $\xi < \gamma$.
- We let p_τ be the standard parameter of $\mathcal{N}_\tau$.
- We let n_τ be the complexity degree of a singularizing function for τ over $\mathcal{N}_\tau$, or equivalently the least n such that $\tilde{h}_\tau^{n+1}(\gamma \cup \{p_\tau\})$ is cofinal in τ for some $\gamma < \tau$.
- $\tilde{h}_\tau = \tilde{h}_\tau^{n_\tau + 1}$.
- We let α_τ be the largest $\alpha < \tau$ such that $\tilde{h}_\tau(\alpha \cup \{p_\tau\}) \cap \tau = \alpha$.

We then define the sets B_τ, which are the first approximations to C_τ, analogously as in Zeman [Zem10]. Recall that B_τ may be bounded in τ and even empty if τ is countably cofinal, but on the other hand B_τ will be "almost" coherent also at successor points.

To be precise: Recall that an ordinal ζ is in p_τ if and only if some generalized solidity witness for ζ with respect to $\mathcal{N}_\tau$ and p_τ is an element of $\mathcal{N}_\tau$, and the standard solidity witness for ζ can be reconstructed from any generalized solidity witness for ζ inside $\mathcal{N}_\tau$. Here, by the *standard* solidity witness, we mean the transitive collapse of the hull $\tilde{h}_\tau^{k+1}(\zeta \cup \{p_\tau \setminus (\zeta + 1)\})$, where $\rho_{\mathcal{N}_\tau}^{k+1} \leq \zeta < \rho_{\mathcal{N}_\tau}^k$.

Recall also that premice may be either passive or active, and active premice are of three types, depending on the set of generators of the top extenders. When we talk about embeddings between premice, we always require that the premice in question are of the same kind according to this categorization. We will say the premice are of the same type also in the case where both are passive.

Now let $\bar{\tau} \in B_\tau$ if and only if the following hold:

(1) $\mathcal{N}_{\bar{\tau}}$ is a strategic premouse of the same type as $\mathcal{N}_\tau$,

(2) $n_{\bar{\tau}} = n_\tau$, and

(3) there is a $\Sigma_0^{(n)}$-preserving embedding $\sigma : \mathcal{N}_{\bar{\tau}} \to \mathcal{N}_\tau$ such that:

- $\sigma \upharpoonright \bar{\tau} = \mathsf{id}$,
- $\sigma(\bar{\tau}) = \tau$ if $\bar{\tau} \in \mathcal{N}_{\bar{\tau}}$,
- $\sigma(p_{\bar{\tau}}) = p_\tau$, and
- for each $\xi \in p_\tau$, there is some generalized solidity witness Q for ξ with respect to $\mathcal{N}_\tau$ and p_τ, such that $Q \in \mathrm{rng}(\sigma)$,

(4) $\alpha_{\bar{\tau}} = \alpha_\tau$.

The following facts are proved by means of abstract fine structure theory, and the proofs look exactly as the corresponding proofs in Zeman [Zem10]:

(A) σ is unique, and we denote it by $\sigma_{\bar{\tau},\tau}$.

(B) $\sigma_{\bar{\tau},\tau}[\omega\rho_{\mathcal{N}_{\bar{\tau}}}^{n_{\bar{\tau}}}]$ is bounded in $\omega\rho_{N_\tau}^{n_\tau}$.

(C) If $\tau^* < \bar{\tau}$ are in B_τ, then there is a unique map $\sigma_{\tau^*,\bar{\tau}} : \mathcal{N}_{\tau^*} \to \mathcal{N}_{\bar{\tau}}$ with the list of properties (1)–(4) stipulated above, and with $(\tau^*, \bar{\tau})$ in place of $(\bar{\tau}, \tau)$.

(D) If $\tau^* < \bar{\tau} < \tau'$ are in $B_\tau \cup \{\tau\}$, then $\sigma_{\bar{\tau},\tau'} \circ \sigma_{\tau^*,\bar{\tau}} = \sigma_{\tau^*,\tau'}$.

(E) If $\bar{\tau} \in B_\tau$, then $B_\tau \cap \bar{\tau} = B_{\bar{\tau}} \setminus \min(B_\tau)$.

The proof of unboundedness of B_τ, for τ of uncountable cofinality, is similar to that in Zeman [Zem10], but uses the Condensation Lemma 3.3 where the argument from Zeman [Zem10] used condensation for $L[E]$-models:

For λ regular and sufficiently large, given $\tau' < \tau$, and working in HOD, we construct a countable elementary substructure $X \prec H_\lambda$, such that

$$\mathcal{N}_\tau, \tau', C^* \in X.$$

Letting $\tilde{\tau} = \sup(X \cap \tau)$, notice that $\tau' < \tilde{\tau} < \tau$ and $\tilde{\tau} \in C^*$, as C^* is closed. Let $\bar{\mathcal{N}}$ be the transitive collapse of $\mathcal{N}_\tau$, and let $\sigma : \bar{\mathcal{N}} \to \mathcal{N}_\tau$ be the inverse of the collapsing isomorphism. Set $\tilde{\mathcal{N}} = \mathrm{Ult}^{n_\tau}(\bar{\mathcal{N}}, \sigma \upharpoonright (\bar{\mathcal{N}} | \bar{\tau}))$, where $\sigma(\bar{\tau}) = \tau$, and let $\sigma' : \tilde{\mathcal{N}} \to \mathcal{N}_\tau$ be the factor map, that is, $\sigma' \circ \tilde{\sigma} = \sigma$, and $\sigma' \upharpoonright \tilde{\tau} = \mathsf{id}$.

Exactly as in Zeman [Zem10], one can show the following facts by means of abstract fine structural considerations:

(F) $\tilde{\sigma}$ is $\Sigma_0^{(n)}$-preserving, and maps $\omega\rho_{\bar{\mathcal{N}}}^{n_\tau}$ cofinally into $\omega\rho_{\tilde{\mathcal{N}}}^{n_\tau}$. Similarly, σ' is $\Sigma_0^{(n)}$-preserving, but maps $\omega\rho_{\tilde{\mathcal{N}}}^{n_\tau}$ boundedly into $\omega\rho_{\mathcal{N}}^{n_\tau}$.

(G) $\tilde{\mathcal{N}}$ is a sound and solid strategic premouse.

(H) $\tilde{\mathcal{N}}$ is a singularizing structure for $\tilde{\tau}$ with singularization degree n_τ.

(I) $\sigma'(\tilde{\tau}) = \tau$, $\sigma'(p_{\tilde{\mathcal{N}}}) = p_\tau$, and α_τ is the largest $\alpha < \tilde{\tau}$ satisfying

$$\tilde{h}_{\tilde{\mathcal{N}}}^{n_\tau + 1}(\alpha \cup \{p_{\tilde{\mathcal{N}}}\}) \cap \tilde{\tau} = \alpha.$$

Since the entire construction took place inside HOD, Lemma 3.3 can be applied to the map $\sigma' : \tilde{\mathcal{N}} \to \mathcal{N}_\tau$. The rest follows again by abstract fine structural considerations, literally as in Zeman [Zem10]. In particular, these considerations can be used to show that $\tilde{\mathcal{N}} = \mathcal{N}_{\tilde{\tau}}$ and $\sigma'(p_{\tilde{\mathcal{N}}}) = p_\tau$, and hence $\sigma' = \sigma_{\tilde{\tau}, \tau}$. Additionally, $\alpha_{\tilde{\tau}} = \alpha_\tau$. This shows that $\tilde{\tau} \in B_\tau$.

In Zeman [Zem10] it is only proved that B_τ is closed on a tail-end; this was again caused by the fact that one has to consider protomice. Here, we prove that B_τ is itself closed.

Given a limit point $\tilde{\tau}$ of B_τ, notice first that $\tilde{\tau} \in C^*$, let $\tilde{\mathcal{N}}$ be the direct limit of $\langle \mathcal{N}_{\bar{\tau}}, \sigma_{\tau^*, \bar{\tau}} \mid \tau^* < \bar{\tau} < \tilde{\tau} \rangle$, and let $\sigma' : \tilde{\mathcal{N}} \to \mathcal{N}_\tau$ be the direct limit map. Again, by abstract fine structural considerations that are essentially identical to those in Zeman [Zem10], we establish the conclusions analogous to (F)–(I) for the current version of $\tilde{\mathcal{N}}$ and σ', and then as above apply Lemma 3.3 to $\sigma' : \tilde{\mathcal{N}} \to \mathcal{N}_\tau$, and conclude that $\tilde{\mathcal{N}}$ is an initial segment of HOD. As above, we then conclude that $\tilde{\mathcal{N}} = \mathcal{N}_{\tilde{\tau}}$ and $\sigma' = \sigma_{\tilde{\tau}, \tau}$.

Having established closure and unboundedness of B_τ, we follow the construction from Zeman [Zem10], and obtain fully coherent sets B_τ^* by "stacking" the sets B_τ; so

$$B_\tau^* = B_{\nu_0} \cup B_{\nu_1} \cup \cdots \cup B_{\nu_{\ell_\tau}},$$

where $\nu_0 = \tau$, $\nu_{i+1} = \min(B_{\nu_i})$, and ℓ_τ is the least ℓ such that $B_{\nu_{\ell+1}} = \varnothing$. We then let C_τ^* be the set of all ordinals τ_ι, defined inductively as follows:

$$
\begin{aligned}
\tau_0 &= \min(B_\tau), \\
\xi_\iota^\tau &= \text{the least } \xi < \tau \text{ such that } \tilde{h}_\tau(\{\xi\} \cup \{p_\tau\}) \not\subseteq \mathrm{rng}(\sigma_{\tau_\iota, \tau}), \\
\tau_{\iota+1} &= \text{the least } \bar{\tau} < \tau \text{ such that } \tilde{h}_\tau(\{\xi_\iota^\tau\} \cup \{p_\tau\}) \subseteq \mathrm{rng}(\sigma_{\bar{\tau}, \tau}), \\
\tau_\iota &= \sup\{\tau_{\bar{\iota}} \mid \bar{\iota} < \iota\} \text{ for limit } \iota.
\end{aligned}
$$

The proof that the C_τ^* are fully coherent, unbounded in τ whenever τ has uncountable cofinality in HOD, and $\mathrm{otp}(C_\tau^*) < \tau$, can be carried out using abstract fine structural considerations, and is essentially the same as the argument in Zeman [Zem10].

Finally, we let $C_\tau = \lim(C_\tau^*)$ whenever $\lim(C_\tau^*)$ is unbounded in τ, and otherwise let C_τ be some randomly chosen cofinal ω-sequence in τ. (Recall again that

here our notation diverges from that in Zeman [Zem10], where C_τ denoted sets that are fully coherent but not necessarily cofinal at countably cofinal τ.) ∎

Remark 3.4: Above, we worked in the theory $\mathsf{AD}_\mathbb{R}$, as it directly relates to our negative results. However, as long as we are in the situation where there is no $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ such that $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_\mathbb{R} +$ "Θ is Mahlo in HOD", Theorem 3.1 holds assuming only AD^+.

We briefly sketch the argument in the case where $\Theta = \theta_{\Omega+1}$ for some ordinal Ω. In this case we use the following facts.

(a) Models that admit the HOD analysis are of the form $\mathsf{Lp}^\Sigma(\mathbb{R})$.

(b) Under our minimality assumption, the HOD analysis applies to our universe V.

That (a) and (b) hold follows from [SS15]. Assuming them, let G be $\mathbb{P}_{\max}$-generic over V. By the S-construction from Section 2.11 of Sargsyan [Sar15b], the model $\mathsf{Lp}^\Sigma(\mathbb{R})[G]$ can be rearranged into the form $\mathsf{Lp}^\Sigma(\mathbb{R}, G)$. Since G well-orders $\mathbb{R}$ in order type ω_2, there is $A \subseteq \omega_2$ such that $\mathsf{Lp}^\Sigma(\mathbb{R}, G) = \mathsf{Lp}(A)$. The standard construction of the canonical $\square_{\omega_2}$-sequence in $\mathsf{Lp}(A)$ thus yields a $\square_{\omega_2}$-sequence in the $\mathbb{P}_{\max}$-extension.

Suppose now that we have a sequence $\langle c_\tau \mid \tau \in C^* \rangle$ as in Theorem 3.1, and that the Axiom of Choice holds. We may assume in addition that c_τ consists of successor ordinals whenever $c_\tau \not\subseteq C^*$. It is a well-known fact that if $\alpha < \beta$ are ordinals in the interval (ω_2, ω_3), then there is a coherent sequence $\langle d_\tau \mid \tau \in \lim \cap (\alpha, \beta) \rangle$ such that each d_τ is a closed unbounded subset of τ contained in the interval (α, β) and has order-type at most ω_2. (This is proved by induction on the length of initial segments of such a sequence.) Using Choice we can pick such a sequence $\langle d_\tau \rangle_\tau$ for each interval (α, β) where $\alpha < \beta$ are two adjacent elements of C^*, and assemble them together with the sequence $\langle c_\tau \mid \tau \in C^* \rangle$ into a single coherent sequence $\langle c'_\tau \mid \tau \in \lim \cap (\omega_2, \omega_3) \rangle$. This sequence is almost a $\square_{\omega_2}$-sequence, except that it satisfies a bit more relaxed restriction on order-types, namely $\mathsf{otp}(c'_\tau) < \tau$ for every limit ordinal $\tau \in (\omega_2, \omega_3)$. For auxiliary purposes, set also $c'_\tau = \tau$ whenever $\tau \leq \omega_2$ is a limit ordinal. It is then easy to verify that the following recursive construction from Jensen [Jen72] turns the sequence $\langle c'_\tau \mid \tau \in \lim \cap (\omega_2, \omega_3) \rangle$ into a $\square_{\omega_2}$-sequence $\langle c^*_\tau \mid \tau \in \lim \cap (\omega_2, \omega_3) \rangle$. Let $\pi_\tau : \mathsf{otp}(c'_\tau) \to c'_\tau$ be the unique order isomorphism. Then set

$$c^*_\tau = \pi_\tau[c^*_{\mathsf{otp}(c'_\tau)}].$$

As the partial order $\mathbb{P}_{\max} * \text{Add}(\Theta, 1)$ forces Choice, this argument gives the following corollary of Theorem 3.1.

COROLLARY 3.5: *Assume that* $\text{AD}_{\mathbb{R}}$ *holds and that there is no* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *such that* $\text{L}(\Gamma, \mathbb{R}) \models$ *"Θ is Mahlo in HOD". Then* $\square_{\omega_2}$ *holds after forcing with* $\mathbb{P}_{\max} *$ $\text{Add}(\Theta, 1)$.

4. Choiceless extensions where square fails

In this section we present two results showing that ω_3 is square inaccessible, and even threadable (see Definition 1.5), in the $\mathbb{P}_{\max}$ extension of suitable models of determinacy. These arguments do not show that the subsequent forcing $\text{Add}(\omega_3, 1)$, which adds a Cohen subset of ω_3, and in the process well-orders $\mathcal{P}(\mathbb{R})$, does not ("accidentally") add a square sequence. In fact it can, as the hypotheses of Theorems 3.1 and 4.1 are jointly consistent relative to suitable large cardinals. Moreover, the hypothesis of Theorem 4.1 (with θ as Θ) follows from $\text{AD}^+ + \text{Uniformization} + \text{V=L}(\mathcal{P}(\mathbb{R})) + $ "Θ is Mahlo in HOD" (and is strictly weaker than it, if it is consistent). To see this, note that the Solovay sequence is closed, and, by Remark 2.5, $\text{AD}^+ + \text{Uniformization}$ implies that the Solovay sequence has limit length. Finally, an application of Theorem 2.2 gives a model of the form $L_\kappa(\mathcal{P}_{\theta_\alpha}(\mathbb{R}))$ satisfying the hypothesis of Theorem 4.1.

Given a set S of ordinals, let us say that a sequence of sets $\langle x_\xi \mid \xi \in S \rangle$ is a *singularizing sequence* for S if each x_ξ is a cofinal subset of ξ of order-type strictly smaller than ξ.

THEOREM 4.1: *Assume that* AD^+ *holds, and that* θ *is a limit element of the Solovay sequence. Let* R_θ *be the set of* $\kappa < \theta$ *which are both limit elements of the Solovay sequence and regular in HOD. Assume that* R_θ *is unbounded in* θ, *and let* $R \subseteq R_\theta$ *be an unbounded subset of* θ *in HOD. Then there does not exist a singularizing sequence for* R *in the* $\mathbb{P}_{\max}$ *extension of* $\text{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.

In particular, ω_3 *is square inaccessible in the* $\mathbb{P}_{\max}$ *extension of* $\text{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.

Proof. In $\text{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ fix a $\mathbb{P}_{\max}$-name σ for a singularizing sequence for R. Fix $\kappa \in R$ such that σ is ordinal definable from some $A \in \mathcal{P}_\kappa(\mathbb{R})$. By Theorem 2.4, κ is regular in $\text{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$. The canonical $\mathbb{P}_{\max}$-name

$$\dot{x}_\kappa = \{(p, \check{\alpha}) \mid p \Vdash \alpha \in \sigma_{\check{\kappa}}\}$$

for the member of the interpretation of σ associated with κ is definable from σ and κ, and therefore belongs to $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$.

Then x_κ, the interpretation of $\dot{x}_\kappa$, belongs to the $\mathbb{P}_{\max}$ extension of $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$. But this is a contradiction, since, on the one hand, x_κ is unbounded in κ and has order type less than κ, and, on the other hand, the regularity of κ is preserved in the $\mathbb{P}_{\max}$ extension of $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$, as $|\mathbb{P}_{\max}| = \mathfrak{c}$.

The square-inaccessibility of ω_3 in the $\mathbb{P}_{\max}$ extension of $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ follows from the fact that any square sequence witnessing $\square_{\omega_3}$ induces a singularizing sequence for R_θ. ∎

Remark 4.2: Let us emphasize that the negative result in Theorem 4.1 is a negative result on the existence of a singularizing sequence rather than a negative result on square sequences. The contradiction in the proof of the theorem is obtained not from a combinatorial situation that would block the existence of a square sequence, rather this contradiction exhibits the lack of choice in the $\mathbb{P}_{\max}$ extension: Every $\kappa \in R_\theta$ is singular, but we cannot pick a singularizing sequence. The coherence requirement does not play any role here.

An uncountable cardinal θ is *weakly compact* if for every $S \subset \mathcal{P}(\theta)$ of cardinality θ there is a θ-complete filter F on θ such that $\{A, \theta \setminus A\} \cap F \neq \emptyset$ for each $A \in S$ (this is not the usual definition, but is easily seen to be equivalent to the Extension Property in Theorem 4.5 of [Kan09]). It follows easily from this formulation (or, more directly, the Extension Property) that θ is threadable if it is weakly compact.

THEOREM 4.3: *Assume that* AD^+ *holds, and that* θ *is a limit element of the Solovay sequence, and weakly compact in* HOD. *Then* θ *remains weakly compact in the* $\mathbb{P}_{\max}$ *extension of* $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.

Proof. Let τ be a $\mathbb{P}_{\max}$ name in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ for a collection of θ many subsets of θ. Then τ is ordinal definable in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ from a set of reals that is itself definable from a bounded subset S of θ. For each $\mathbb{P}_{\max}$ condition p and each ordinal $\alpha < \theta$, let $A_{p,\alpha}$ be the set of ordinals forced by p to be in the α-th set represented by τ.

Fix an ordinal $\delta < \theta$ such that $S \subseteq \delta$, and let

$$C \colon \mathcal{P}(\omega) \times \mathcal{P}(\delta) \to \mathcal{P}(\omega + \delta)$$

be defined by letting $C(a,b) \cap \omega = a$, and, for all $\gamma < \delta$, $\omega + \gamma \in C(a,b)$ if and only if $\gamma \in b$. Fix also a coding of elements of $H(\aleph_1)$ by subsets of ω.

By Lemma 2.2, there is a partial order Q of cardinality less than θ in HOD such that for each $a \subseteq \omega$, $C(a,S)$ is HOD-generic for Q. The proof of Lemma 2.2 shows that there is a sequence $\langle \sigma_\alpha \mid \alpha < \theta \rangle$ in HOD, consisting of Q-names, such that for each $\alpha < \theta$ and each $a \subseteq \omega$ coding some $p \in \mathbb{P}_{\max}$, the realization of σ_α by $C(a,S)$ is the set $A_{p,\alpha}$. To see this, following the argument from Lemma 2.2, let T be an ordinal definable relation on $\mathcal{P}(\omega + \delta) \times \theta \times \theta$ such that for all $a \subseteq \omega$ coding $p \in \mathbb{P}_{\max}$ and all $(\alpha, \beta) \in \theta \times \theta$, we have that $\beta \in A_{p,\alpha}$ if and only if $T(C(a,S), \alpha, \beta)$. Define T^* on $\theta \times \theta \times Q$ by letting $T^*(\alpha, \beta, q)$ hold if and only if $T(D, \alpha, \beta)$ holds for all $D \in h(q)$, where h is the function from the proof of Lemma 2.2. Then T^* is (essentially) the desired sequence $\langle \sigma_\alpha \mid \alpha < \theta \rangle$.

Applying the weak compactness of θ in HOD, we can find in HOD a θ-complete filter F on θ, such that, for each $\alpha < \theta$ and each $q \in Q$, F contains either the set $\{\beta < \theta \mid q \Vdash \check{\beta} \in \sigma_\alpha\}$, or its complement. Since $|Q| < \theta$ in HOD, for each $p \in \mathbb{P}_{\max}$ and each $\alpha < \theta$, either $A_{p,\alpha}$ or its complement contains a set in F. By Theorem 2.6, the filter generated by F is $\mathbb{R}$-complete in $\text{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$. It follows then that the filter generated by F measures all the sets in the realization of τ in the $\mathbb{P}_{\max}$ extension. ∎

5. Forcing the square inaccessibility of ω_3

A *partial $\square_\kappa$-sequence* is a sequence $\langle C_\alpha \mid \alpha \in A \rangle$ (for A a subset of κ^+) satisfying the three conditions in Definition 1.1 for each $\alpha \in A$. Note that condition (2) implies that $\beta \in A$ whenever $\alpha \in A$ and β is a limit point of C_α.

The partial order $\text{Add}(\omega_3, 1)$ adds a subset of ω_3 by initial segments. When $\mathfrak{c} = \aleph_2$, as in a $\mathbb{P}_{\max}$ extension, $\text{Add}(\omega_3, 1)$ well-orders $\mathcal{P}(\mathbb{R})$ in order type ω_3.

The hypotheses of Theorem 5.1 below imply that Θ is regular, since any singularizing function would exist in HOD_A for some set of reals A, and, by Theorem 2.4, this would give a club of singular cardinals in HOD below Θ. Modulo the HOD analysis (and the assumption of $\text{AD}_\mathbb{R} + \text{V} = \text{L}(\mathcal{P}(\mathbb{R}))$), the hypothesis of Theorem 5.1 is close to the negation of the hypothesis of Theorem 3.1 (see Remark 5.6).

THEOREM 5.1: *Assume that $\text{AD}^+ + \text{Uniformization}$ holds, that $\text{V} = \text{L}(\mathcal{P}(\mathbb{R}))$, and that stationarily many elements θ of cofinality ω_1 in the Solovay sequence*

are regular in HOD. Then in the $\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)$-extension there is no partial $\square_{\omega_2}$-sequence defined on all points of cofinality at most ω_1.

Proof. Let κ be a regular cardinal which is not the surjective image of $\mathcal{P}(\mathbb{R})$, and let T be the theory of $\mathrm{L}_\kappa(\mathcal{P}(\mathbb{R}))$. Suppose that τ is a $\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)$-name in $\mathrm{L}(\mathcal{P}(\mathbb{R}))$ whose realization is forced by some condition p_0 to be such a partial $\square_{\omega_2}$-sequence. We may assume that τ is coded by a subset of $\mathcal{P}(\mathbb{R})$, and (by using the least ordinal parameter defining a counterexample to the theorem) that (τ, p_0) is definable in $\mathrm{L}_\kappa(\mathcal{P}(\mathbb{R}))$ from some $A \subseteq \mathbb{R}$. Using our hypothesis (and Theorem 2.4 for item (4)), we get $\theta < \Theta$ with A in $\mathcal{P}_\theta(\mathbb{R})$, and ordinals ξ_0 and ξ_1 such that

(1) $\theta < \xi_0 < \Theta < \xi_1 \leq \Theta^+$;

(2) $\mathrm{L}_{\xi_1}(\mathcal{P}(\mathbb{R}))$ satisfies T;

(3) θ is a limit element of the Solovay sequence of $\mathrm{L}(\mathcal{P}(\mathbb{R}))$ of cofinality ω_1;

(4) θ is regular in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$;

(5) $p_0 \in \mathrm{L}_{\xi_0}(\mathcal{P}_\theta(\mathbb{R}))$;

(6) every element of $\mathrm{L}_{\xi_0}(\mathcal{P}_\theta(\mathbb{R}))$ is definable in $\mathrm{L}_{\xi_0}(\mathcal{P}_\theta(\mathbb{R}))$ from a set of reals in $\mathcal{P}_\theta(\mathbb{R})$;

(7) in $\mathrm{L}_{\xi_1}(\mathcal{P}(\mathbb{R}))$, τ is a $\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)$-name whose realization is forced by p_0 to be a partial $\square_{\omega_2}$-sequence defined on the ordinals of cofinality at most ω_1; and

(8) there exist $\sigma \in \mathrm{L}_{\xi_0}(\mathcal{P}_\theta(\mathbb{R}))$ and an elementary embedding

$$j \colon \mathrm{L}_{\xi_0}(\mathcal{P}_\theta(\mathbb{R})) \to \mathrm{L}_{\xi_1}(\mathcal{P}(\mathbb{R}))$$

with critical point θ such that $j(\sigma) = \tau$.

To see this, let ξ_1 be the least ordinal $\xi > \Theta$ such that $\mathrm{L}_\xi(\mathcal{P}(\mathbb{R})) \models T$, and τ is definable from A in $\mathrm{L}_\xi(\mathcal{P}(\mathbb{R}))$. Then every element of $\mathrm{L}_{\xi_1}(\mathcal{P}(\mathbb{R}))$ is definable in $\mathrm{L}_{\xi_1}(\mathcal{P}(\mathbb{R}))$ from a set of reals. For each $\alpha < \Theta$, let X_α be the set of elements of $\mathrm{L}_{\xi_1}(\mathcal{P}(\mathbb{R}))$ definable from a set of reals of Wadge rank less than α. Then, by the definition of the Solovay sequence, and Remark 2.1, the order type of $X_\alpha \cap \Theta$ is always less than the least element of the Solovay sequence above α. Since Θ is regular, $X_\alpha \cap \Theta$ is bounded below Θ. Let $f(\alpha) = \sup(X_\alpha \cap \Theta)$. Then f is continuous, and we can find θ satisfying items (3) and (4) above, and such that $f(\theta) = \theta$. Let ξ_0 be the order type of $X_\theta \cap \xi_1$, so that $\mathrm{L}_{\xi_0}(\mathcal{P}_\theta(\mathbb{R}))$ is the transitive collapse of X_θ, the embedding j is simply the inverse of the collapse, and σ is the collapse of τ.

Let $M_0 = \mathrm{L}_{\xi_0}(\mathcal{P}_\theta(\mathbb{R}))$ and $M_1 = \mathrm{L}_{\xi_1}(\mathcal{P}(\mathbb{R}))$. Let G be $\mathbb{P}_{\max}$-generic over M_1, containing the first coordinate of p_0. Then j lifts to

$$j \colon M_0[G] \to M_1[G],$$

and $j(\mathrm{Add}(\theta, 1)^{M_0[G]}) = \mathrm{Add}(\Theta, 1)^{M_1[G]}$. Since $\mathbb{P}_{\max}$ is countably closed, θ has cofinality ω_1 in $M_1[G]$. It follows that each countable subset of $\mathrm{Add}(\theta, 1)^{M_0[G]}$ in $M_1[G]$ is an element of $\mathrm{L}(A, \mathbb{R})[G]$ for some $A \in \mathcal{P}(\mathbb{R})^{M_0}$, and therefore that $\mathrm{Add}(\theta, 1)^{M_0[G]}$ is ω-closed in $M_1[G]$. Let z be the second coordinate of p_0, as realized by G.

CLAIM 5.2: *If H is $M_1[G]$-generic over $\mathrm{Add}(\theta, 1)^{M_0[G]}$, with $z \in H$, then σ_{G*H} has a thread in $M_1[G][H]$.*

We first finish the proof of the theorem, assuming the existence of an H as in the claim. Since θ has uncountable cofinality in $M_1[G][H]$, there can be at most one thread through σ_{G*H} in $M_1[G][H]$. The thread, being unique, would be in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}[G][H]$ (note that $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ has the same sets of reals as M_0). This leads to a contradiction, as θ would be collapsed in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}[G][H]$, which is impossible since θ is regular in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.

It suffices then to prove the claim.

Proof of Claim 5.2. Toward a contradiction, suppose that the claim were false. Let $C \in M_1$ be club in θ, with order type ω_1. By the Coding Lemma, C is in $\mathrm{L}(B, \mathbb{R})$ for some set of reals B with $|B|_W = \theta$. If H is $M_1[G]$-generic as above, with $z \in H$, then σ_{G*H} is a coherent sequence of length θ with no thread in $M_1[G][H]$. We can fix $(\mathbb{P}_{\max} * \mathrm{Add}(\theta, 1)^{M_0[G]})$-names ρ and ψ in $\mathrm{L}(B, \mathbb{R})$ such that

- ρ_{G*H} is the tree of attempts to build a thread through σ_{G*H} along C (i.e., the relation consisting of those pairs (α, β) from C for which the β-th member of σ_{G*H} extends the α-th member), and
- ψ_{G*H} is the poset that specializes ρ_{G*H} (i.e., which consists of finite partial functions mapping C to ω in such a way that ρ_{G*H}-compatible elements of C are mapped to distinct elements of ω, ordered by inclusion).

Since σ_{G*H} is forced by p_0 to have no thread in $M_1[G][H]$, $\mathrm{Add}(\theta, 1)^{M_0[G]} * \psi_{G*H}$ is ω-closed*c.c.c., and thus proper, in $M_1[G]$.

SUBCLAIM 5.3: *In $M_1[G]$, there are H, f such that*

- H is $\mathrm{Add}(\theta, 1)^{M_0[G]}$-generic over $M_0[G]$, with $z \in H$, and
- f specializes ρ_{G*H}.

Proof. In $M_0[G]$, $\mathrm{Add}(\theta, 1)$ is $< \theta$-closed.

For each $\alpha \in C$ and each ternary formula ϕ, (recalling condition (6) from the choice of ξ_0 and θ) let $E_{\alpha,\phi}$ be the collection of sets of the form

$$\{x \mid M_0[G] \models \phi(A, G, x)\}$$

which are dense subsets of $\mathrm{Add}(\theta, 1)^{M_0[G]}$, where A is an element of $\mathcal{P}_\alpha(\mathbb{R})$. As $\mathcal{P}_\alpha(\mathbb{R})$ is a surjective image of $\mathbb{R}$ in M_0, $E_{\alpha,\phi}$ has cardinality less than θ in $M_0[G]$. It follows that there is a dense subset $D_{\alpha,\phi}$ of $\mathrm{Add}(\theta, 1)^{M_0[G]}$ refining all the members of $E_{\alpha,\phi}$.

Since $\mathsf{PFA}(\mathfrak{c})$ holds in $M_1[G]$ (by Woodin [Woo10, Theorem 9.39]), there is a filter $H * K$ on $\mathrm{Add}(\theta, 1)^{M_0[G]} * \psi$ in $M_1[G]$ such that H meets each $D_{\alpha,\phi}$, and $H * K$ meets the dense sets guaranteeing that K determines a specializing function f for ρ_{G*H}. This gives the subclaim. ∎

Let H and f be as in the subclaim. Then in $M_1[G]$, H is a condition in $\mathrm{Add}(\Theta, 1)$. We can therefore find a generic H_1 over $M_1[G]$ such that j lifts to

$$j \colon M_0[G][H] \to M_1[G][H_1].$$

But then σ_{G*H} has the thread $(\tau_{G*H_1})_\theta$. But f is in $M_1[G]$, so ω_1 was collapsed by going to $M_1[G][H_1]$, giving a contradiction. ∎

This completes the proof of Theorem 5.1. ∎

Definition 5.4: Given cardinals κ and λ, the principle $\square_{\kappa,\lambda}$ asserts the existence of a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa^+ \rangle$ of nonempty sets such that for each $\alpha < \kappa^+$,

(1) $|\mathcal{C}_\alpha| \leq \lambda$;
(2) each element of $\mathcal{C}_\alpha$ is club in α, and has order type at most κ;
(3) for each member C of $\mathcal{C}_\alpha$, and each limit point β of C, $C \cap \beta \in \mathcal{C}_\beta$.

We call a sequence witnessing the above principle a $\square_{\kappa,\lambda}$-sequence. The statement $\square_{\omega_2,\omega_2}$ follows from $2^{\aleph_1} = \aleph_2$ [Jen72], which holds in $\mathbb{P}_{\max}$ extensions.

Question 5.5: Can one improve Theorem 5.1 to obtain the failure of $\square_{\omega_2,\omega}$?

Remark 5.6: In terms of consistency strength, the hypothesis of Theorem 5.1 is below a Woodin limit of Woodin cardinals; this follows from the proof of Sargsyan [Sar15b, Theorem, 3.7.3] (see also [Sar15c, Sar15a]). Furthermore,

the hypothesis is equiconsistent with a determinacy statement that is easier to state. In fact, something stronger than mere equiconsistency holds, as we proceed to sketch.

THEOREM 5.7: *Assume* $\mathsf{AD}^+ + \mathsf{Uniformization} + \mathrm{V} = \mathrm{L}(\mathcal{P}(\mathbb{R}))$, *and that the* HOD *analysis applies. Then the following are equivalent.*

(1) HOD $\models$ "Θ *is Mahlo to measurables*".
(2) Θ *is Mahlo to measurables of* HOD.
(3) *There are stationarily many* θ *in the Solovay sequence that have cofinality* ω_1 *and are regular in* HOD.

Proof. That clause 2 implies clause 1 is immediate. Assume now that clause 1 holds. Let

$$S = \{\theta_\alpha < \Theta \mid \mathrm{HOD} \models \text{``}\theta_\alpha \text{ is measurable''}\}.$$

We have that HOD $\models$ "S is stationary". It follows from Theorem 2.4 that S is stationary in V. Hence, clause 1 implies clause 2.

Next suppose Θ is Mahlo to measurables of HOD. Let S be as above. It follows from the HOD analysis (in particular, see [Ste10, Lemma 8.25]) that every member of S has cofinality ω_1. This shows that clause 2 implies clause 3.

Finally, assume that there are stationarily many θ in the Solovay sequence that have cofinality ω_1 and are regular in HOD. Let

$$S = \{\theta_\alpha < \Theta \mid \mathrm{cf}(\theta_\alpha) = \omega_1 \text{ and } \mathrm{HOD} \models \text{``}\theta_\alpha \text{ is regular''}\}.$$

It follows from the HOD analysis that each $\theta_\alpha \in S$ is measurable in HOD. To see this, fix $\theta_\alpha \in S$ and let $(\mathcal{Q}, \Lambda) \in \mathcal{F}$ be such that $\theta_\alpha \in \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ and such that for some $\kappa \in \mathcal{Q}$, $\pi_{\mathcal{Q},\infty}^{\Lambda}(\kappa) = \theta_\alpha$. Since θ_α is regular in HOD we have that κ is regular in $\mathcal{Q}$. If κ is not measurable in $\mathcal{Q}$,, then $\pi_{\mathcal{Q},\infty}^{\Lambda}[\kappa]$ is cofinal in θ_α implying that θ_α has countable cofinality. Therefore, we must have that κ is measurable in $\mathcal{Q}$ implying that HOD $\models$ "θ_α is measurable". This shows that clause 3 implies clause 1. ∎

6. Stronger hypotheses and the threadability of ω_3

In this section we apply a hypothesis stronger than the one used in Theorem 5.1 to obtain the failure of a weakening of $\square(\omega_3)$.

Definition 6.1: Given an ordinal γ and a cardinal λ, the principle $\square(\gamma, \lambda)$ asserts the existence of a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \gamma \rangle$ satisfying the following conditions.

(1) For each $\alpha < \gamma$,
  - $0 < |\mathcal{C}_\alpha| \leq \lambda$;
  - each element of $\mathcal{C}_\alpha$ is club in α;
  - for each member C of $\mathcal{C}_\alpha$, and each limit point β of C, $C \cap \beta \in \mathcal{C}_\beta$.
(2) There is no thread through the sequence, that is, there is no club $E \subseteq \gamma$ such that $E \cap \alpha \in \mathcal{C}_\alpha$ for every limit point α of E.

Again, we refer to sequences witnessing the above principle as $\square(\gamma, \lambda)$-sequences. Notice that $\square_{\kappa, \lambda}$ implies $\square(\kappa^+, \lambda)$. The arguments of Todorcevic [Tod84, Tod02] mentioned above show that $\mathsf{MM}(\mathfrak{c})$ implies the failure of $\square(\gamma, \omega_1)$ for any ordinal γ of cofinality at least ω_2.

Before stating our result, we recall a result of Woodin that will be useful in what follows. The notion of *A-iterability* for A a set of reals, crucial in the theory of $\mathbb{P}_{\max}$, is introduced in Woodin [Woo10, Definition 3.30]. For $X \prec H(\omega_2)$, Woodin denotes by M_X its transitive collapse. Woodin [Woo10, §3.1] presents a series of covering theorems for $\mathbb{P}_{\max}$ extensions. In particular, we have:

THEOREM 6.2 (Woodin [Woo10, Theorem 3.45]): *Suppose that M is a proper class inner model that contains all the reals and satisfies* $\mathsf{AD} + \mathsf{DC}$. *Suppose that for any $A \in \mathcal{P}(\mathbb{R}) \cap M$, the set*

$$\{X \prec H(\omega_2) \mid X \text{ is countable, and } M_X \text{ is } A\text{-iterable}\}$$

is stationary. Let X in V be a bounded subset of Θ^M of size ω_1. Then there is a set $Y \in M$, of size ω_1 in M, and such that $X \subseteq Y$.

Typically, we apply this result as follows: We start with $M = \mathrm{L}(\mathcal{P}(\mathbb{R}))$, a model of $\mathsf{AD}^+ + \mathsf{DC}$, and force with $\mathbb{P}_{\max}$ to produce an extension $M[G]$. The technical stationarity assumption is then true in $M[G]$ by virtue of Woodin [Woo10, Theorem 9.32], and we can then apply Theorem 6.2 in this setting.

The following lemma will be used in the proof of Theorem 6.5. A similar fact was used in the proof of Theorem 5.1.

LEMMA 6.3: *Suppose that $M_0 \subseteq M_1$ are models of* $\mathsf{ZF} + \mathsf{AD}^+$ *such that*
  - $\mathbb{R} \subseteq M_0$;
  - $\mathcal{P}(\mathbb{R})^{M_0}$ *is a proper subset of* $\mathcal{P}(\mathbb{R})^{M_1}$;
  - $M_1 \models$ *"Θ is regular";*

- Θ^{M_0} has cofinality at least ω_2 in M_1.

Let $G \subset \mathbb{P}_{\max}$ be an M_1-generic filter. Then $\mathrm{Add}(\omega_3, 1)^{M_0[G]}$ is closed under ω_1-sequences in $M_1[G]$.

Proof. Since $M_0 \models \mathsf{AD}^+$ and G is an M_0-generic filter for $\mathbb{P}_{\max}$, $\omega_3^{M_0[G]} = \Theta^{M_0}$. We show first that $\mathrm{cf}(\Theta^{M_0}) = \omega_2$ in $M_1[G]$. Since we have assumed that $\mathrm{cf}(\Theta^{M_0}) \geq \omega_2$ in M_1, equality is a consequence of the Covering Theorem 6.2, as follows. Given any bounded subset X of Θ^{M_1} of cardinality $\aleph_1$ in $M_1[G]$, let $A \in M_1$ be a set of reals of Wadge rank at least $\sup(X)$, and apply Theorem 6.2 with M as $\mathrm{L}(A, \mathbb{R})$ and V as $L(A, \mathbb{R})[G]$. Since $\mathcal{P}(\omega_1)^{M_1[G]}$ is contained in $\mathrm{L}(\mathbb{R})[G]$ (Woodin [Woo10, Theorem 9.32]), X will be an element of $\mathrm{L}(A, \mathbb{R})[G]$, and thus a subset of an element of $\mathrm{L}(A, \mathbb{R})$ of cardinality $\aleph_1$ in $\mathrm{L}(A, \mathbb{R})$.

Let $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence in $M_1[G]$ of conditions in $\mathrm{Add}(\omega_3, 1)^{M_0[G]}$. Since $\mathrm{cf}(\Theta^{M_0}) = \omega_2$ in $M_1[G]$, we may fix a $\gamma < \Theta^{M_0}$ such that each p_α is a subset of γ. Since ω_2-DC holds in $M_1[G]$ by Woodin [Woo10, Theorem 9.36], we may find in $M_1[G]$ a sequence $\langle \tau_\alpha \mid \alpha < \omega_1 \rangle$ consisting of $\mathbb{P}_{\max}$-names in M_0 such that $\tau_{\alpha,G} = p_\alpha$ for all $\alpha < \omega_1$.

Via a pre-well-ordering R of length γ in M_0, we may assume that each τ_α is coded by R and a set of reals S_α in M_0, in such a way that the sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ is in $M_1[G]$. Letting $\eta < \Theta^{M_0}$ be a bound on the Wadge ranks of the sets S_α, we have that the sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ is coded by a single set of reals E in M_0 and an ω_1-sequence of reals in $M_1[G]$. Finally, since $\mathcal{P}(\omega_1)^{M_1[G]} \subseteq \mathrm{L}(\mathbb{R})[G] \subseteq M_0[G]$ (as mentioned in the first paragraph of this proof), we have that $\langle p_\alpha \mid \alpha < \omega_1 \rangle \in M_0[G]$. ∎

The following principle is introduced in Woodin [Woo10, §9.5]. If one assumes that I is the nonstationary ideal, then one gets Todorcevic's reflection principle $\mathsf{SRP}(\kappa)$. The principle $\mathsf{SRP}(\omega_2)$ follows easily from $\mathsf{MM}^{++}(\mathfrak{c})$.

Definition 6.4: Given a cardinal $\kappa \geq \omega_2$, $\mathsf{SRP}^*(\kappa)$ is the statement that there is a proper, normal, fine ideal $I \subseteq \mathcal{P}([\kappa]^{\aleph_0})$ such that for all stationary $T \subseteq \omega_1$,

$$\{X \in [\kappa]^{\aleph_0} \mid X \cap \omega_1 \in T\} \notin I,$$

and such that for all $S \subseteq [\kappa]^{\aleph_0}$, if S is such that for all stationary $T \subseteq \omega_1$,

$$\{X \in S \mid X \cap \omega_1 \in T\} \notin I,$$

then there is a set $Y \subseteq \kappa$ such that

- $\omega_1 \subseteq Y$;
- $|Y| = \aleph_1$;
- $\mathrm{cf}(\sup(Y)) = \omega_1$;
- $S \cap [Y]^{\aleph_0}$ contains a club in $[Y]^{\aleph_0}$.

The hypothesis of Theorem 6.5 is stronger than that of Theorem 5.1, but still below a Woodin limit of Woodin cardinals in terms of consistency strength [ST]. By Woodin [Woo10, Theorem 9.10], the hypotheses of the theorem imply that AD^+ holds in M_0.

THEOREM 6.5: *Suppose that* $M_0 \subseteq M_1$ *are models of* $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{Uniformization}$ *with the same reals such that, letting* $\Gamma_0 = \mathcal{P}(\mathbb{R}) \cap M_0$, *the following hold:*

- $M_0 = \mathrm{HOD}^{M_1}_{\Gamma_0}$;
- $M_0 \models$ *"Θ is regular";*
- $\Theta^{M_0} < \Theta^{M_1}$;
- Θ^{M_0} *has cofinality at least* ω_2 *in* M_1.

Let $G \subset \mathbb{P}_{\max}$ *be* M_1-*generic, and let* $H \subset \mathrm{Add}(\omega_3, 1)^{M_0[G]}$ *be* $M_1[G]$-*generic. Then the following hold in* $M_0[G][H]$:

- ω_3 *is threadable; in fact, we have* $\neg\square(\omega_3, \omega)$;
- $\mathsf{SRP}^*(\omega_3)$;

Proof. Suppose that τ is a $\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)$-name in M_0 for a $\square(\omega_3, \omega)$-sequence. We may assume that the realization of τ comes with an indexing of each member of the sequence in order type at most ω. In M_0, τ is ordinal definable from some set of reals S.

By Woodin [Woo10, Theorem 9.39], $\mathsf{MM}^{++}(\mathfrak{c})$ holds in $M_1[G]$. As in the first paragraph of the proof of Lemma 6.3, Θ^{M_0} has cofinality ω_2 in $M_1[G]$. Forcing with $(<\omega_2)$-directed closed partial orders of size at most $\mathfrak{c}$ preserves $\mathsf{MM}^{++}(\mathfrak{c})$ (see Larson [Lar00]). It follows then from Lemma 6.3 that $\mathsf{MM}^{++}(\mathfrak{c})$ holds in the $\mathrm{Add}(\omega_3, 1)^{M_0[G]}$-extension of $M_1[G]$, and thus that in this extension every candidate for a $\square(\Theta^{M_0}, \omega)$-sequence is threaded.

Let $\mathcal{C} = \langle \mathcal{C}_\alpha \mid \alpha < \Theta^{M_0} \rangle$ be the realization of τ in the $\mathrm{Add}(\omega_3, 1)^{M_0[G]}$-extension of $M_1[G]$. Since Θ^{M_0} has cofinality at least ω_2 in this extension (which satisfies Choice), $\mathcal{C}$ has at most ω many threads, since otherwise one could find a $\mathcal{C}_\alpha$ in the sequence with uncountably many members. Therefore, some member of some $\mathcal{C}_\alpha$ in the realization of τ will be extended by a unique thread through the sequence, and since the realization of τ indexes each $\mathcal{C}_\alpha$ in order type at

most ω, there is in M_1 a name, ordinal definable from S, for a thread through the realization of τ. This name is then a member of $M_0 = \mathrm{HOD}_{\Gamma_0}^{M_1}$.

That $\mathsf{SRP}^*(\omega_3)$ holds in $M_0[G][H]$ follows from the fact that the nonstationary ideal on $[\Theta^{M_0}]^{\aleph_0}$ as defined in $M_1[G][H]$ is an element of $M_0[G][H]$, and the facts that $|\Theta^{M_0}|^{M_1[G][H]} = \aleph_2$, and $M_1[G][H]$ satisfies $\mathsf{SRP}(\omega_2)$. ∎

Question 6.6: Can one improve Theorem 6.5 to obtain the failure of $\square(\omega_3, \omega_1)$?

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