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On Shifted-Localized Derivators

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On Shifted-Localized Derivators

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ABSTRACT OF THE DISSERTATION

On Shifted-Localized Derivators

by

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Doctor of Philosophy in Mathematics
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Professor Paul Balmer, Chair

This dissertation investigates objects known as "shifted-localized derivators" through the lens of algebraic geometry by building affine and projective space objects over an arbitrary derivator. For affine space, we give a definition of $\mathbb{A}^n$ over a derivator $\mathbb{D}$, and then show a series of results identifying it as extending the $\mathbb{A}^n$-construction in algebraic geometry, including a universal property. We then note that this construction is not specific to the choice of $\mathbb{A}^n$ but can be used for any choice of abelian, unital monoid.

In order to tackle the projective space case, first we prove some technical results on two topics: compact generation of triangulated derivators, and a Day convolution structure on symmetric monoidal derivators shifted by symmetric monoidal categories. To construct $\mathbb{P}^n$, we first shift by a symmetric monoidal category Q_n to achieve an analogue of graded modules over polynomial rings, and then localize by localizing to copies of $\mathbb{A}^n$. We prove generation and semiorthogonal decomposition results in $\mathbb{P}^n$ by using this formulation of localization, and the aforementioned technical results.

The dissertation of John Min Zhang is approved.

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2019

All that is gold does not glitter

Not all those who wander are lost

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PUBLICATIONS

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CHAPTER 1

Introduction

Since the 1960s, category theory has developed into a multifaceted, multipurpose toolbox for researchers in algebra, representation theory, algebraic geometry, topology, and myriad other mathematical fields. Triangulated categories are one primary tool in this toolbox, as discussed in [Nee01]. Triangulated categories arise in nature as derived categories of abelian or exact categories (in algebraic geometry and representation theory), as stable categories (in representation theory), or as homotopy categories of (stable) model categories or ∞-categories (in algebraic topology), just to name a few examples.

One fundamental construction in a triangulated category $\mathcal{T}$ is the cone $C(f)$ of a morphism $f : X \to Y$ in $\mathcal{T}$, viewed either as a mapping cone of complexes in homological algebra or as the mapping cone in topology. This mapping cone is well-known to be nonfunctorial, and in [Ver96] it is noted that if the cone is functorial, then the category is somewhat similar to being abelian semi-simple. As almost all structure in a triangulated category is dependent on the cone, this creates a problematic situation where many things exist up to non-unique isomorphism. However, in most triangulated categories occurring in nature, there is a model for the cone, which have led mathematicians to develop stronger structures beyond merely triangulated categories. Two main approaches include dg-categories, see [Kel94] and stable ∞-categories, see [Lur09] or [Lur17]. Derivators, introduced by Grothendieck, Heller, Franke, and others, see [Gro91], [Hel88], or [Fra96], are another attempt to clarify this issue.

Consider the construction of the mapping cone in the derived category of an abelian category. Given the abelian category $\mathcal{A}$ and some category of chain complexes denoted $\mathrm{Ch}(\mathcal{A})$, we would form the mapping cone in $\mathcal{D}(\mathcal{A})$ of a map $f : B \to C$ by choosing a representative for $f : B \to C$ in $\mathrm{Ch}(\mathcal{A})$, take the mapping cone in $\mathrm{Ch}(\mathcal{A})$, and then take a representative in $\mathcal{D}(\mathcal{A})$. The lift back to the category of chain complexes creates functoriality issues.

Based off this idea, we see that correctly speaking, the mapping cone of a map in the derived category is not so much being taken in $(\mathcal{D}(\mathcal{A}))^{[1]}$, the arrow category of $\mathcal{D}(\mathcal{A})$, but rather in the derived category $\mathcal{D}(\mathcal{A}^{[1]})$. If we knew what $\mathcal{A}$ was we would be afforded access to the category $\mathcal{D}(\mathcal{A}^{[1]})$, but this is a fundamentally un-triangulated construction, as we return to the underlying abelian category first before making another triangulated category. On another note, it is a simple exercise to see that $\mathcal{D}(\mathcal{A})^{[1]}$ is only triangulated if $\mathcal{A}$ is the zero category 0.

Derivators provide a means of addressing this issue: specifically, to every category J in a diagram category **Dia**, we assign a category $\mathbb{D}(J)$. These categories are connected by a number of adjoints: for every functor $u : I \to J$ in **Dia**, there is a corresponding functor $u^* : \mathbb{D}(J) \to \mathbb{D}(I)$, along with left and right adjoints given by $u_! \dashv u^* \dashv u_*$.

While the list of conditions may look imposing, derivators appear naturally in most categorical contexts, such as:

1. The assignment $I \mapsto \mathcal{D}(\mathcal{A}^I)$ gives a collection of derived categories building off the derived category $\mathcal{D}(\mathcal{A})$.

2. The assignment $I \mapsto \mathrm{HO}(\mathcal{M}^I)$, for a Quillen model category $\mathcal{M}$ gives a collection of homotopy categories building off the homotopy category $\mathrm{HO}(\mathcal{M})$.

3. The assignment $I \mapsto C^I$ for any bicomplete category C gives the usual collection of presheaves with values in C.

Essentially all triangulated categories appearing in nature can be turned into derivators instead, as we will see in the next section. As such, there is ample reason to investigate the full structure of a derivator instead of the single triangulated category.

In fact, derivators and triangulated derivators have been studied to great effect in algebraic topology, see [Fra96], [GPS14a], [GPS14b], representation theory, see [GS16a] and [GS16b], and algebraic geometry, see [Hor17a], [Hor17b], and K-theory, see [Pat17], [Tab08], [CT11]. Our contribution to this field is in a more algebro-geometric direction, wherein general affine and projective spaces are defined and studied. These constructions generalize results known about triangulated categories of affine and projective space, while also providing new insights that triangulated categories presumably did not have sufficient structure to conclude.

Moreover, similar to the study of stable ∞-categories, in a series of papers [Cis03], [Cis04], [Cis08] Cisinski shows that stable derivators are spectrally enhanced. In [Tab08] and [CT11] this is used as a weapon to study the homotopy theory of dg-categories.

Apart from the aforementioned issues in the theory of triangulated categories, there are precious few ways to make new triangulated categories out of old. Taking full triangulated categories is a boring tool that also rarely yields anything new, and though the theory of localization is well-developed, see [Kra10], the categories it can produce are still limited. More recently there have been two other constructions to create new triangulated categories, namely modules of a separable monad, see [Bal11], and the theory of completion with respect to a metric, see [Kra18], [Nee18], [Nee19]. These are more specialized and fragile constructions that may not always exist, or give us a category that we know how to interpret.

Derivators give us a new tool of constructing new triangulated categories. In a triangulated derivator $\mathbb{D}$, typically one category $\mathbb{D}(e)$ will be the original triangulated category we were interested in investigating. However, as a part of the axioms, every other category $\mathbb{D}(I)$ is also triangulated in a compatible manner with $\mathbb{D}(e)$. For a derivator-oriented construction, we could construct the shifted derivator $\mathbb{D}^I$, defined as $\mathbb{D}^I(J) = \mathbb{D}(I \times J)$.

As in [Bal16], taking modules over a separable monad in a triangulated category is akin to pulling back along an étale map, localization being a special case of this. Here we will describe affine space $\mathbb{A}^n$ and projective space $\mathbb{P}^n$-constructions over derivators, which combined with the above can describe a large class of "schemes over a derivator."

We saw earlier that we wanted to access the derived category $\mathcal{D}(\mathcal{A}^{[1]})$. This category can be conceived of as the base of a shifted derivator. Alternatively, under the assignment $\mathbb{D}(I) = \mathcal{D}(\mathcal{A}^I)$, the derived category $\mathcal{D}(\mathcal{A}^{[1]})$ would simply be $\mathbb{D}([1])$, a part of the derivator. Something that is difficult to access from a triangulated perspective occurs naturally in the derivator world.

Shifted derivators $\mathbb{D}^I$ or values of a derivator $\mathbb{D}(I)$ at $I \neq e$ often carry meaning that perhaps is not immediately apparent. We have seen this already with $I = [1]$. Other possible choices of I allow us to construct affine and projective space constructions, as well as naive for of G-equivariance for a group G.

In addition, there is a major philosophical reason to use derivators in lieu of triangulated categories. With the definition of triangulated category, we speak of a *chosen* self-equivalence Σ and a *chosen* choice of cones $C(f)$ for every morphism $f : X \to Y$, unique up to non-canonical isomorphism. In practice, of course we have a single choice of Σ, either given to us in the structure of a stable model category, or self-evidently a shift in degree for chain complexes, and a clear choice of $C(f)$ up to non-canonical ismorphism. The choices for the triangulated structure are self-evident with no other alternative, so one would hope for a way to have a *canonical* triangulated structure, see [Gro13] or [Gro18] for more details and philosophical motivation.

In the world of derivators, most parts of the triangulated structure are canonically constructed. Assuming that we have a pointed derivator $\mathbb{D}$, we can create a canonical suspension morphism Σ by sending an element $X \in \mathbb{D}(I)$ to an element of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \\ 0 & & \end{array}$$

located in $\mathbb{D}(I \times \ulcorner)$, and then taking the homotopy pushout to get back to $\mathbb{D}(I)$. This can all be done functorially, and the right adjoint Ω can be constructed in a similar fashion. Note here the similarity to topological constructions as a justification for calling this suspension.

If we add the condition that (Σ, Ω) are not merely an adjunction but actually equivalences, the derivator is known as being *stable*. With a lifting condition known as strongness or (Der5), values of the derivator $\mathbb{D}$ will have canonical triangulated structures. Notably, the lifting condition (Der5) is satisfied in basically all examples of derivators in nature. See [Gro13] for a deeper discussion of this axiom and precise justifications; we will simply sketch how to construct triangles and only do so on the base $\mathbb{D}(e)$.

Given a morphism $f : X \to Y$ in $\mathbb{D}(e)$, the condition (Der5) guarantees that we can lift it to an object of the form $(X \to Y)$ in $\mathbb{D}([1])$. From then on, we extend $(X \to Y)$ via two successive Kan extensions to an object of shape $[2] \times [1]$:

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z & \longrightarrow & \Sigma X \end{array}$$

Extending this in both directions gives the usual triangle

$$\cdots \Omega Z \to X \to Y \to Z \to \Sigma X \cdots .$$

From then on in [Gro13] it is shown why this gives a canonical triangulated structure on $\mathbb{D}(e)$. Using shifts $\mathbb{D}^I$ we extract each $\mathbb{D}(I)$ as the base of a triangulated derivator and endow it with a canonical triangulated structure. Moreover, the usual functors $u_!, u^*, u_*$ that appear will be triangulated functors also.

It is evident that due to the lift, the cone is also not functorial, not that we would expect it to be. However, we can speak of a "functorial cone construction" from $\mathbb{D}([1]) \to \mathbb{D}(e)$, which is sufficient for most purposes.

This poses a major philosophical difference with the usual ideas of triangulated categories; instead of specifying the triangulated structure on the category, when using derivators all of the triangulated structure appears canonically whenever the derivator satisfies certain reasonable conditions. This is in contrast to, say, the use of dg-categories, which seek to impose yet more structure on the triangulated category.

Starting from [Bal02] and [Bal05], Balmer and a succession of authors have defined and implemented a program of *tensor-triangular geometry*. Broadly speaking, Balmer takes usual triangulated categories arising in nature with a compatible symmetric monoidal (tensor) structure and creates a space that is similar to a Zariski spectrum, where the points are so-called *prime ideals* in the tensor-triangulated category. Some core motivations of this program are the classification of thick subcategories of derived categories of schemes, and the study of Morava K-theories in algebraic topology.

The guiding principle in tensor-triangular geometry is that while we cannot hope to classify all (compact) objects inside a given tensor-triangulated category, or perhaps even the thick subcategories, we can hope to do so with aid of the tensor structure. This philosophy has been successfully adapted to do computations in algebraic geometry, representation theory, algebraic topology, C^*-algebras, and so forth. In particular, as in [Bal02] it is possible to reconstruct a scheme X (not just the topological space, but also local data) from the derived category $\mathcal{D}(X)$, thereby in theory subsuming scheme-theoretic study of algebraic geometry into tensor-triangular geometry, the question was posed as to what the notion of $\mathbb{A}^1$ *of a triangulated category* meant. This presents one motivating question of our research.

While we do not make direct tensor-triangulated computations here, we can still detect strong connections to tensor-triangular geometry. As it stands it does not necessarily make sense to make an attempt on the direct computation of the spectrum. Rather, we can use other tools, for example developed in [Bal10a], to get a better grasp on how we have $\mathbb{A}^1$-objects in tensor-triangular geometry.

Our approach to constructing $\mathbb{P}^n$ also nods to tensor-triangular concerns; sheaves on $\mathbb{P}^n$ are graded modules that are equivalent if their restrictions to copies of $\mathbb{A}^n$ are the same. However, in general it is not clear how universal properties in tensor-triangular geometry and derivators relate, though the connection is undeniable.

The structure of the thesis is as follows. In Chapter 2, we give necessary definitions for the rest of our work. Chapter 3 sets the technical stage with various results about shifted derivators and monoidal structures, while Chapters 4-6 introduce the $\mathbb{A}^n$-construction, describe its useful properties, and discusses a large class of shifting diagrams. In Chapter 7 we describe a specific class of open subscheme and describe the relationship with the universal properties described in Chapters 5 and 6. Chapters 8 and 9 are devoted to discussing the $\mathbb{P}^n$ construction.

Specifically, Chapter 2 will give a brief discussion of the definition of derivators, the use of homotopy exact squares, and morphisms between derivators. Particular attention will be paid to the homotopy exact squares, which arise repeatedly whenever computations are needed. Chapter 3 concerns monoidal derivators and the interaction with the triangulated structure, questions of generation of derivators, and the development of the homotopical Day convolution as previously mentioned. Moreover, we make a brief foray into the theory of localization of derivators, which will be important in the last three chapters. A mixture of old and new technical results will be given in Chapter 3 and this chapter can be viewed as a partial contribution to the foundational theory of derivators. Again we attempt to restrict ourselves to purely necessary results. Many of these results will appear similar to their triangulated analogues and indeed should be thought as generalizations of triangulated results to the derivator world.

In Chapter 4, we define the $\mathbb{A}^1$ construction on a derivator $\mathbb{D}$ as the shifted derivator $\mathbb{D}^{\underline{\mathbb{N}}}$. In fact, the usual $\mathbb{A}^1$-monoidal structure can be computed in one swoop using a homotopical analogue of Day convolution. Moreover as we would expect, this extends naturally to $\mathbb{A}^n$. In Chapter 5, we see that there is an Eilenberg-Watts type Theorem for this $\mathbb{A}^n$-construction. However, one soon realizes that there is nothing special about $\underline{\mathbb{N}}$, and that other shifts by categories with one object gives completely analoguous results, as we explain in Chapter 6. We will explore the interactions between various shifted derivators and their universal properties in Chapter 7.

In Chapter 8 we move to the projective space construction. Starting here we will make some assumptions on the triangulated nature of our derivator for technical reasons. In the last two chapters we will prove a variety of results about the projective space construction, using the previous chapters as basis.

1.1 Notation

We fix some recurring notation throughout. The category e is the terminal category with one object and the identity morphism. For a positive integer n, the poset $0 \to 1 \to \cdots \to n$ will be denoted by $[n]$. If $(M, +)$ is a monoid with unit, $\underline{M}$ will be the category with one object and endomorphism monoid $(M, +)$.

If a is an object in a category A, by abuse of notation we will also use a to refer to the functor $e \to A$ picking out the object. For any category A, the functor $\pi : A \to e$ is the only possible functor to the terminal category.

CHAPTER 2

Technical Introduction and Definitions

2.1 Basic definitions

Derivators were simultaneously defined by a variety of authors. From an algebraic topology perspective, in [Hel88] Alex Heller arguably gave the first definition of derivators, while in [Gro91] Grothendieck defines them with an eye on the derived category of an abelian category. Independently of both authors, Franke in [Fra96] also gives an effectively equivalent definition.

There are two variance conventions on (pre)derivators. The definition by Grothendieck considers (pre)derivators as systems of diagram categories as in [Gro91] and also authors such as Moritz Groth in [Gro16] and other works. Alternatively, Heller and later Cisinski consider them as CAT-valued presheaves, with an extra contravariance in the 2-direction; see [Cis03], [Cis04]. Here we will adopt the former convention of diagram categories.

We first define the notion of prederivator.

Definition 2.1.1. A prederivator $\mathbb{D}$ is a strict 2-functor $\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$.

Here Cat is the 2-category of small categories, *i.e.* objects are categories with a set of objects and **Hom**-sets, 1-morphisms are functors, and 2-morphisms are natural transformations. Similarly, CAT is the 2-category with objects consisting of large categories, 1 morphisms being functors, and 2-morphisms being natural transformations.

The op encodes the fact that a prederivator $\mathbb{D}$ reverses the direction of the 1-morphisms (functors), that is, if we have a prederivator $\mathbb{D}$ and a functor $u : I \to J$, then

$$\mathbb{D}(u) := u^*\colon \mathbb{D}(J) \to \mathbb{D}(I).$$

For natural transformations, given two functors $u, v : I \to J$ and a natural transformation $\alpha\colon u \to v$, we have an induced natural transformation in the same direction, $\alpha^*\colon u^* \to v^*$; there is only contravariance in the 1-level.

For a functor $u\colon A \to B$, we call u^* *restriction along* u or *pullback along* u. In particular, if e denotes the terminal category with one object and the identity morphism, and $a\colon e \to A$ is the functor that sends the single object in e to $a \in A$, then we call a^* the *value at* a functor.

For $X \in \mathbb{D}(A)$ and $a \in A$, sometimes we may write X_a for a^*X. For a prederivator $\mathbb{D}$, the category $\mathbb{D}(e)$ is called the *underlying category* or *base* of $\mathbb{D}$.

Example Let $\mathcal{C}$ be any (possibly large) category. The *represented prederivator* of $\mathcal{C}$ is defined to be the 2-functor $y_{\mathcal{C}}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$ to take $I \mapsto \mathcal{C}^I$, with the usual pullbacks/natural transformations on the functor categories for functors and natural transformations.

Definition 2.1.2. Let J be a small category. The *shifted prederivator* $\mathbb{D}^J$ is defined by $\mathbb{D}^J(I)\colon = \mathbb{D}(J \times I)$.

We can call objects in $\mathbb{D}(A)$ *coherent diagrams of shape* A, to distinguish them from *incoherent* diagrams of shape A, which are objects in $\mathbb{D}(e)^A$. However, coherent and incoherent diagrams are connected by the so-called "partial underlying diagram functor":

Definition 2.1.3. Let $\mathbb{D}$ be a derivator and I be a small category. We define the partial underlying diagram functor $\mathrm{dia}_I : \mathbb{D}(I) \to \mathbb{D}(e)^I$ as follows.

Let $X \in \mathbb{D}(I)$ be an object. For any object $i \in I$, we can evaluate X at i to get the object i^*X. For a morphism $\alpha : i \to j$ in I, we have the corresponding natural transformation $\alpha^* : i^* \to j^*$, which we can evaluate at X. Stitching all this data together gives us a functor $I \to \mathbb{D}(e)$, sending $i \in I$ to i^*X. This gives a functor $\mathbb{D}(I) \to \mathbb{D}(e)^I$ that we call dia_I.

Remark 2.1.4. Replacing $\mathbb{D}$ with a shifted prederivator $\mathbb{D}^J$ gives a similar partial underlying diagram functor

$$\mathrm{dia}_I : \mathbb{D}^J(I) \to \mathbb{D}^J(e)^I,$$

which we will denote

$$\mathrm{dia}_{I,J} : \mathbb{D}(J \times I) \to \mathbb{D}(J)^I.$$

That is to say, if we have an object in $\mathbb{D}(I \times J)$, we can "remove the coherence" in one direction while preserving it in the other direction.

Remark 2.1.5. While the functor $\mathrm{dia}_{I,J}$ is useful for gaining an intuitive understanding, it is almost never faithful. For certain I the functor may be full or essentially surjective.

Next we give the definition of a derivator.

Definition 2.1.6. A derivator is a prederivator $\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$ satisfying the following conditions.

Der1: $\mathbb{D}\colon \mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$ takes finite coproducts to products, *i.e.*

$$\mathbb{D}(\coprod_i J_i) \cong \prod_i \mathbb{D}(J_i).$$

In particular, $\mathbb{D}(\varnothing)$ is the terminal category.

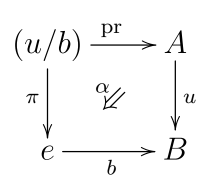
Der2: For any $A \in \mathsf{Cat}$, a morphism $f\colon X \to Y$ is an isomorphism in $\mathbb{D}(A)$ if and only if the morphisms

$$a^* f\colon a^* X \to a^* Y$$

are isomorphisms in $\mathbb{D}(e)$ for all $a \in A$.

Der3: For each functor $u\colon A \to B$, the corresponding functor $\mathbb{D}(u) := u^*\colon \mathbb{D}(B) \to \mathbb{D}(A)$ has a left adjoint $u_!$ and a right adjoint u_*. The two functors $u_!$ and u_* are also referred to as the (left/right) *homotopy Kan extensions* along u.

Der4: For any functor $u\colon A \to B$ and any object $b \in B$, let us identify b with the functor $b\colon e \to B$. We have a natural transformation given by



Here (u/b) is the slice category whose objects are pairs $(a \in A, f\colon u(a) \to b)$, and morphism

$$(a, f) \to (a', f')$$

given by a morphism $g : a \to a'$ in A such that $f' \circ u(g) = f$. The projection functor sending $(a, f) \mapsto a$ is denoted pr, and π the projection to e. Here the natural transformation α is constructed as follows: for an object $f : u(a) \to b$, the composition $u \circ \mathrm{pr}$ sends it to $u(a)$, while the composition $b \circ \pi$ sends it to b. Thus the natural transformation α on the object $(f\colon u(a) \to b)$ is given by the morphism $f\colon u(a) \to b$ in B. By definition of (u/b) it is easy to see that this patches to a natural transformation $u \circ \mathrm{pr} \to b \circ \pi$.

After applying $\mathbb{D}$, since u^* and π^* have left adjoints $u_!$ and $\pi_!$ respectively, we obtain the diagram.

$$\begin{array}{ccccc}
\mathbb{D}(e) & \xleftarrow{\pi_!} & \mathbb{D}(u/b) & \xleftarrow{\mathrm{pr}^*} & \mathbb{D}(A) \\
\uparrow{\scriptstyle \mathrm{Id}} \quad \overset{\epsilon}{\swarrow} & & \uparrow{\scriptstyle \pi^*} \quad \overset{\alpha^*}{\swarrow} & & \uparrow{\scriptstyle u^*} \quad \overset{\eta}{\swarrow} \quad \nwarrow{\scriptstyle \mathrm{Id}} \\
 & & \mathbb{D}(e) & \xleftarrow[b^*]{} & \mathbb{D}(B) \xleftarrow[u_!]{} \mathbb{D}(A)
\end{array}$$

The morphisms in the two triangles are induced by the unit/counit transformations, respectively. The combined transformation is a morphism

$$\mathrm{Hocolim}_{(u/b)} \circ \mathrm{pr}^* \to b^* \circ u_!.$$

We require this to be an isomorphism. Similarly, we have a diagram

$$\begin{array}{ccc}
(b/u) & \xrightarrow{\mathrm{pr}} & A \\
{\scriptstyle \pi}\downarrow & \nearrow_{\alpha} & \downarrow{\scriptstyle u} \\
e & \xrightarrow[b]{} & B
\end{array}$$

and a similar natural transformation

$$b^* u_* \to \mathrm{Holim}_{(b/u)} pr^*,$$

which we also require to be an isomorphism.

Remark 2.1.7. We may speak of *left derivators*, which are prederivators satisfying (Der1), (Der2), and (Der3) and (Der4) but only for homotopy left Kan extensions $u_!$.

Similarly we may speak of *right derivators* who only have adjoints u_*. A prederivator that is both a left and right derivator is simply a derivator.

In particular, from (Der1) and (Der3) we see that $\mathbb{D}(A)$ has all coproducts and products and therefore also have initial and final objects.

There is sometimes a fifth axiom appended to the above list. We will give it separately as it is orthogonal to the previous four, but it occurs frequently and is useful enough to warrant a discussion here.

Definition 2.1.8. A prederivator $\mathbb{D}$ is said to be *strong* if it satisfies the following condition:

Der5: For every $K \in \mathsf{Cat}$, the partial underlying diagram functor

$$\mathrm{dia}_{[1],K} : \mathbb{D}([1] \times K) \to \mathbb{D}(K)^{[1]}$$

is full and essentially surjective.

We can interpret this as the ability to lift arrows in $\mathbb{D}(K)$ to $\mathbb{D}([1] \times K)$. This becomes important in defining a triangulated structure.

Example The following are some examples of derivators.

1. Let $\mathcal{C}$ be a complete and cocomplete category. The represented prederivator $y(\mathcal{C})\colon J \mapsto \mathcal{C}^J$ is a derivator.

2. Let $\mathcal{M}$ be a model category and $\mathcal{W}$ the subcategory of weak equivalences. Then we have the *homotopy derivator*

$$\mathbb{HO}(\mathcal{M}, \mathcal{W})\colon I \mapsto \mathcal{M}^I[(\mathcal{W}^I)^{-1}].$$

This is a theorem of Cisinski, see [Cis03, Theorem 1].

3. Let $\mathcal{A}$ be a Grothendieck abelian category. We can associate a derivator $\mathbb{D}_{\mathcal{A}}$ to $\mathcal{A}$ by $\mathbb{D}_{\mathcal{A}}\colon I \mapsto \mathcal{D}(\mathcal{A}^I)$, where $\mathcal{D}$ denotes the derived category.

We can impose more categorical structures onto our derivators. For example, we can require that a derivator $\mathbb{D}$ be *pointed* by asking the base $\mathbb{D}(e)$ be pointed. The existence of adjoints guarantees that $\mathbb{D}(I)$ is pointed for all I.

We can also impose stronger structures, for example that the values of the derivator be triangulated as in the third example. Since for any Grothendieck abelian category $\mathcal{A}$ the category of complexes $\mathrm{Ch}(\mathcal{A})$ has a natural model structure, the case of Grothendieck abelian categories is subsumed by the general model category case. Nevertheless, for the derivators associated to Grothendieck abelian categories as in case (3), their values are triangulated categories. Such derivators whose values are triangulated are called *triangulated derivators*, and they are an important object of study in their own right. Triangulated derivators have a big advantage over usual triangulated categories, because both the suspension Σ and class of distinguished triangles are determined by the natural structure of the derivator, and this makes cones functorial. For more information, see [Gro13, §4], where given a strong stable derivator Groth constructs the natural triangulated structure.

We have the notion of pointed derivator as follows:

Definition 2.1.9. We say a derivator is $\mathbb{D}$ is pointed if $\mathbb{D}(e)$ is pointed, *i.e.* if it contains a zero object.

Proposition 2.1.10. *For a derivator $\mathbb{D}$, if $\mathbb{D}(e)$ is pointed, every value $\mathbb{D}(I)$ is pointed.*

Proof. Let $\pi : I \to e$ denote the projection functor for any category I. By definition π^* is both a left and right adjoint. Therefore, $\pi^* : \mathbb{D}(e) \to \mathbb{D}(I)$ preserves both initial and final objects, which we know that $\mathbb{D}(I)$ has. Therefore, the initial and final objects in $\mathbb{D}(I)$ coincide and $\pi^* 0_{\mathbb{D}(e)}$ is a zero object in $\mathbb{D}(I)$. □

Next we sketch the notion of stability and its consequences.

Proposition 2.1.11. *A pointed derivator $\mathbb{D}$ has canonical suspension Σ, loop Ω, and cone and fiber functors.*

We leave the definitions to [Gro13, §3].

Definition 2.1.12. A pointed derivator $\mathbb{D}$ is *stable* if the adjunction (Σ, Ω) is actually an equivalence.

We term a derivator *triangulated* if it is both strong and stable. See [Gro13, §4] for a discussion of triangulated derivators and the construction of the canonical triangulated structure. At this point it would be prudent to mention that the construction of cone, fiber, suspension, and loop functors are all done from the internal structure of the derivator, *i.e.* restriction functors and Kan extensions.

Another notion of importance is that of *monoidal derivator*, the situation where the values of the derivator have compatible monoidal structures. Thematically, this is more in line with the content of the next chapter and we reserve the definition for then. Putting together the monoidal and triangulated sides of the equations gives us so-called *tensor triangular categories*. [Bal10] gives a good summary on how we study these categories with geometric techniques.

We have seen the notion of shifted prederivator. It turns out that the shift of a derivator is a new derivator.

Proposition 2.1.13. *Let $\mathbb{D}$ be a derivator and L be a small category. Then the prederivator*

$$\mathbb{D}^L(I) := \mathbb{D}(L \times I)$$

is also a derivator.

We direct the reader to [Gro13, Theorem 1.31] for the proof of this result. This construction is of fundamental importance in the world of derivators, and is generally unavailable in other homotopy-theoretic contexts (without passing back to the model).

2.2 Homotopy exact squares

Here we are broadly interested in the interactions of various left and right Kan extensions along with restriction functors. Much of the material is drawn off of [Gro13].

Proposition 2.2.1. *[Gro13, Prop 1.26] Let $u : J \to K$ be a fully faithful functor and $\mathbb{D}$ be a derivator. Then the left and right Kan extension functors, $u_!, u_* : \mathbb{D}(J) \to \mathbb{D}(K)$ are fully faithful.*

Next we define sieves and cosieves, which are very useful for calculations.

Definition 2.2.2. Let $u\colon I \to J$ be a fully faithful functor that is injective on objects.

1. Call u a *cosieve* if whenever we have a morphism $u(i) \to j$, then j lies in the image of u.

2. Call u a *sieve* if whenever we have a morphism $k \to u(i)$, then k lies in the image of u.

Homotopy left Kan extensions along cosieves and homotopy right Kan extensions along sieves are simple:

Proposition 2.2.3. *[Gro13, Prop 1.29] Let $\mathbb{D}$ be a derivator.*

1. *Let $u\colon I \to J$ be a cosieve. Then the homotopy left Kan extension $u_!$ is fully faithful, and $X \in \mathbb{D}(J)$ lies in the essential image of $u_!$ if and only if $X_j \cong \varnothing$ for all $j \in J - u(I)$.*

2. *Let $u\colon I \to J$ be a sieve. Then the homotopy right Kan extension u_* is fully faithful, and $X \in \mathbb{D}(J)$ lies in the essential image of u_* if and only if $X_j \cong *$ for all $j \in J - u(I)$.*

In particular, if $\mathbb{D}$ is pointed intuitively these Kan extensions are "extensions by 0." Moreover, there is an interesting alternative formulation of pointedness.

Definition 2.2.4. [Gro13, Definition 3.4] A derivator $\mathbb{D}$ is *strongly pointed* if it has the following two properties:

1. For every sieve $j : J \to K$, the homotopy right Kan extension j_* has a further right adjoint $j^!$. That is to say, we have a chain of four adjoints $j_! \dashv j^* \dashv j_* \dashv j^!$ for any sieve j.

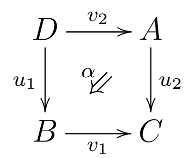
2. For every cosieve $i : J \to K$, the homotopy left Kan extension $i_!$ has a further left adjoint $j^?$. That is to say, we have a chain of four adjoints $i^? \dashv i_! \dashv i^* \dashv i_*$ for any cosieve i.

Every strongly pointed derivator is pointed, and vice versa.

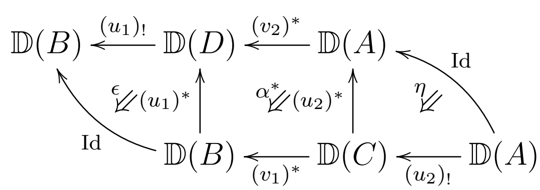
This alternative formulation of strongly pointed in lieu of pointedness will be useful for us when we examine $\mathbb{P}^n$.

Next we must define the notion of a *homotopy exact square*. The squares we produced in (Der4) are one large class of examples, and they are useful in many computational contexts.

Definition 2.2.5. Consider a square with natural transformation:

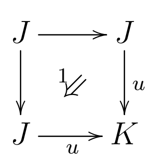


It is said to be homotopy exact if for every derivator $\mathbb{D}$, the natural transformation below is an isomorphism



That is to say, we have a natural isomorphism $(u_1)_!(v_2)^* \cong (v_1)^*(u_2)_!$ given by the whiskering of the three natural transformations in the above diagram.

Example Given that Kan extensions along fully faithful functors are fully faithful in any derivator, we know that if $u : J \to K$ is a fully faithful functor, the following square is homotopy exact:



[Gro13, §1.2] and [GPS14a, §3] discuss some specific classes of homotopy exact squares. We mention one technical theorem that we will employ repeatedly.

Definition 2.2.6. Let A be a small category. Call A *homotopy contractible* if the counit

$$(\pi_A)_!(\pi_A)^* \to \mathrm{Id}_{\mathbb{D}(e)}$$

is an isomorphism for all derivators $\mathbb{D}$, where π_A is the projection $A \to e$.

Equivalently, A is homotopy contractible if the square

$$\begin{array}{ccc} A & \longrightarrow & e \\ \big\downarrow & \overset{1}{\swarrow} & \big\downarrow \\ e & \longrightarrow & e \end{array}$$

is homotopy exact.

Example 1. If A can be connected to e via a zigzag of adjunctions, then A is homotopy contractible. For example, the category $(\mathbb{Z}, <)$ is homotopy contractible, as the inclusion of $(\mathbb{N}, <)$ is a right adjoint, and then $(\mathbb{N}, <)$ has an initial element, so the inclusion of 0 is a left adjoint.

2. For regular limits and colimits, a constant colimit of shape A evaluates to the common element if A is connected. However, connectedness is not sufficient for homotopy contractible, though it is a necessary condition.

Definition 2.2.7. Consider a homotopy exact square as in Definition 2.2.5. Let γ be a morphism in C, $a \in A$ and $b \in B$ be objects. Define the category $(a/D/b)_\gamma$ to have objects triples $(d, f\colon a \to u_1(d), g\colon u_2(d) \to b)$ such that $v_1(g) \circ \alpha(d) \circ v_2(f) = \gamma$.

The morphisms between two triples

$$(d, f\colon a \to u_1(d), g\colon u_2(d) \to b) \to (d', f'\colon a \to u_1(d'), g'\colon u_2(d') \to b)$$

are morphisms $h\colon d \to d'$ in D such that $u_1(h) \circ f = f'$ and $g' \circ u_2(h) = g$.

Theorem 2.2.8. *[GPS14a, Theorem 3.8] Consider a homotopy exact square*

$$\begin{array}{ccc} D & \xrightarrow{u_1} & A \\ {\scriptstyle u_2}\big\downarrow & \overset{\alpha}{\swarrow} & \big\downarrow{\scriptstyle v_2} \\ B & \xrightarrow[v_1]{} & C \end{array}$$

as in Definition 2.2.5. The square is homotopy exact if and only if for all morphisms $\gamma \in C$ and objects $a \in A$ and $b \in B$, the category $(a/D/b)_\gamma$ is homotopy contractible.

We direct the reader to [GPS14a] for the proof. We will frequently appeal to this theorem to check that squares are homotopy exact.

Next we define homotopy final functors.

Definition 2.2.9. [GPS14a, Definition 3.14] A functor $f : A \to B$ is *homotopy final* if the following square is homotopy exact:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & {}^{1}\swarrow & \downarrow \\ e & \longrightarrow & e \end{array}$$

Practically, this means that $(\pi_B)_! \cong (\pi_A)_! f^*$, *i.e.* we can compute the homotopy colimit on B by restricting first to A and then computing the homotopy colimit.

Proposition 2.2.10. *[GPS14a, Prop 3.15] A functor $f : A \to B$ is homotopy final if and only if for each $b \in B$, the category (b/f) is homotopy contractible.*

Example Every right adjoint $f : A \to B$ is homotopy final. Let L be the left adjoint to f, then $f^* \cong L_!$ and the definition is clearly satisfied.

We may employ homotopy final functors when we may not be interested in a specific colimit computation but wish to identify its similarity with other colimits we do know.

2.3 Morphisms of derivators

Here we define morphisms of derivators. The category of prederivators forms a 2-category.

Definition 2.3.1. Let $\underline{PDER}$ denote the 2-category of prederivators. That is to say, the objects in this category are prederivators, the 1-morphisms are pseudonatural transformations and the 2-morphisms are modifications.

We clarify the meaning of pseudonatural transformation:

Definition 2.3.2. A *morphism of prederivators* $F\colon \mathbb{D} \to \mathbb{E}$ is a pseudonatural transformation of 2-functors. This means that for each $I \in \mathsf{Cat}$, we have a functor $F_I\colon \mathbb{D}(I) \to \mathbb{E}(I)$, and for every $u\colon A \to B$ in Cat, we have a *chosen* natural isomorphism

$$\gamma_u^F : u^* F_B \to F_A u^*,$$

encoded in the following diagram with the usual coherence data.

$$\begin{array}{ccc} \mathbb{D}(B) & \xrightarrow{F_B} & \mathbb{E}(B) \\ {\scriptstyle u^*}\downarrow & \overset{\gamma_u^F}{\swarrow} & \downarrow{\scriptstyle u^*} \\ \mathbb{D}(A) & \xrightarrow[F_A]{} & \mathbb{E}(A) \end{array}$$

Here, both the functors F_I and the natural transformations γ_u^F are part of the data of the morphism of prederivators.

This is what we would normally call a *strong morphism of prederivators*. There are similar notions of *lax morphism* and *strict morphism* of prederivators with γ_u^F merely being natural transformations or identities, respectively. Some authors prefer to restrict their attention to strict morphisms, but the inclusion of strong morphisms for this discussion is absolutely essential. While it may be intuitive to look only at strict morphisms, the world of strict morphisms is not rich enough and many examples are only strong rather than strict. For us, a *morphism* of (pre) derivators will always refer to a strong morphism.

Definition 2.3.3. A morphism of derivators is simply a morphism of prederivators, except that the source and target are derivators instead of merely prederivators.

Important classes of morphisms of derivators are given by the following:

Example For $\mathbb{D}$ a derivator, $A, B \in \mathsf{Cat}$ and a functor $u\colon A \to B$, there is a *strict* morphism of derivators

$$u^*\colon \mathbb{D}^B \to \mathbb{D}^A,$$

which is defined at $I \in Cat$ by

$$(u \times 1_I)^*\colon \mathbb{D}^B(I) = \mathbb{D}(B \times I) \to \mathbb{D}(A \times I) = \mathbb{D}^A(I).$$

These morphisms are strict by the 2-functoriality of the derivator $\mathbb{D}$.

Of course, if $\mathbb{D}$ is a derivator, then for a functor $u\colon A \to B$ in Cat we also have left and right Kan extensions $u_!, u_*$. We can similarly define morphisms of derivators $u_!, u_*\colon \mathbb{D}^A \to \mathbb{D}^B$. These are not strict, and this is already one justification of why we should attempt to utilize strong morphisms instead of strict morphisms.

The natural transformation γ_u^F also induces natural transformations

$$\gamma_{u_!}^F\colon u_! F_A \to F_B u_!, \gamma_{u_*}^F\colon F_B u_* \to u_* F_A$$

by composing with the unit and counit transformations as needed for the definitions of morphisms $u_!$, u_*. These are typically refered to as the mates of γ_u^F.

Definition 2.3.4. Let $F : \mathbb{D} \to \mathbb{E}$ be a morphism of derivators, $u : A \to B$ is a functor. The morphism F *preserves homotopy left Kan extensions along* u if the natural transformation $\gamma_{u_!}^F$ is an isomorphism, and F *preserves homotopy right Kan extensions along* u if the natural transformation $\gamma_{u_*}^F$ is an isomorphism.

Similarly, F is *cocontinuous* if it preserves all homotopy left Kan extensions and F is *continuous* if it preserves all homotopy right Kan extensions.

Example For a functor $u\colon A \to B$, the morphism of derivators $u^*\colon \mathbb{D}^B \to \mathbb{D}^A$ is a strict morphism of derivators that is both continuous and cocontinuous.

The morphism of derivators $u_!\colon \mathbb{D}^A \to \mathbb{D}^B$ is a cocontinuous strong morphism of derivators, while $u_*\colon \mathbb{D}^A \to \mathbb{D}^B$ is a continuous strong morphism of derivators.

Proposition 2.3.5. *A morphism of derivators preserves homotopy left Kan extensions if and only if it preserves homotopy colimits, and preserves homotopy right Kan extensions if and only if it preserves homotopy limits.*

This result essentially follows from (Der2), (Der4), and the fact that morphisms of prederivators interact well with restrictions.

Recall that a morphism of derivators is simply a morphism of their underlying prederivators. In this 2-categorical world, we also have a notion of adjoints.

Proposition 2.3.6. *[Gro13, Lemma 2.10] Suppose $L : \mathbb{D} \to \mathbb{E}$ is a morphism of prederivators and that $L_K : \mathbb{D}(K) \to \mathbb{E}(K)$ is a left adjoint, with right adjoint R_K for each $K \in \mathsf{Cat}$. Then there is a unique way to extend the collection $\{R_K\}$ to a lax morphism of prederivators, $R : \mathbb{E} \to \mathbb{D}$ such that the following diagram commutes for all $u : J \to K$, $X \in \mathbb{D}(K)$, $Y \in \mathbb{E}(K)$.*

$$\begin{array}{ccc} Hom_{\mathbb{E}(K)}(LX, Y) & \longrightarrow & Hom_{\mathbb{D}(K)}(X, RY) \\ {\scriptstyle u^*}\downarrow & & {\scriptstyle u^*}\downarrow \\ Hom_{\mathbb{E}(J)}(u^*LX, u^*Y) & \longrightarrow & Hom_{\mathbb{D}(J)}(u^*X, u^*RY) \\ {\scriptstyle \gamma^L}\downarrow & & {\scriptstyle \gamma^R}\downarrow \\ Hom_{\mathbb{E}(J)}(Lu^*X, u^*Y) & \longrightarrow & Hom_{\mathbb{D}(J)}(u^*X, Ru^*Y) \end{array}$$

Moreover, if $\mathbb{D}$ and $\mathbb{E}$ were derivators to begin with, we have the following:

Proposition 2.3.7. *[Gro13, Prop 2.11] Let $L : \mathbb{D} \to \mathbb{E}$ is a morphism of derivators with levelwise right adjoints, and let R be the lax morphism constructed above. Then L is a left adjoint morphism of derivators if and only if L preserves homotopy left Kan extensions if and only if R is a morphism of derivators.*

In particular, a morphism of derivators is an equivalence if and only if it is a levelwise equivalence of categories.

However, in tracing the proofs of these two assertions in fact only homotopy left Kan extensions are needed in $\mathbb{D}$ and $\mathbb{E}$. This will become relevant later.

Example Let $u : J \to K$ be a functor and $\mathbb{D}$ be a derivator. Then $(v_!, v^*)$ and (v^*, v_*) are two adjunctions of derivators between $\mathbb{D}^J$ and $\mathbb{D}^K$.

We may occasionally use $\mathrm{PDER}_!(\mathbb{D}, \mathbb{E})$ to denote the 1-category of cocontinuous morphisms and all modifications between two derivators, and similarly for PDER_*. We rarely consider the 2-category of all derivators, so we do not give a specific notation.

CHAPTER 3

Results on Shifted Derivators

We split this chapter into several thematic sections: monoidal derivators and closed monoidal derivators, generation results about derivators and a derivator Brown representability theorem, and some basic localization theory of derivators.

3.1 Monoidal Derivators

Definition 3.1.1. Let $\mathbb{D}$ be a prederivator. We say that $(\mathbb{D}, \otimes)$ is a monoidal prederivator if it is a pseudomonoid object in the category of prederivators, *i.e.* there exists a (strong) morphism of prederivators

$$\otimes : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$$

along with unit morphism $\mathbb{1} : y_e \to \mathbb{D}$ (recall that for any (possible large) category $\mathcal{C}$, $y_{\mathcal{C}}$ is the represented prederivator associated to $\mathcal{C}$).

Remark 3.1.2. The morphism of prederivators means that for each $I \in \mathsf{Cat}$, $\mathbb{D}(I)$ is equipped with a monoidal structure $\otimes_I$ arising from the two morphisms described above, and that the monoidal structures are compatible via restriction functors u^*.

An alternative presentation of the monoidal structure is the *external product*. Take $A \in \mathbb{D}(I)$, $B \in \mathbb{D}(J)$: we define the external product

$$A \boxtimes B := (1 \times \pi_J)^* A \otimes_{I \times J} (\pi_I \times 1)^* B.$$

That is to say, we lift A and B back to $\mathbb{D}(I \times J)$ and then take the tensor product in $\mathbb{D}(I \times J)$.

Then if objects X, Y are both in $\mathbb{D}(I)$, we can recover their tensor product from the above-defined external product by

$$X \otimes Y = \Delta_I^*(X \boxtimes Y),$$

where $\Delta_I : I \to I \times I$ is the diagonal functor $\Delta_I(i) = (i, i)$ for all $i \in I$. Thus the external product and the tensor products on various $\mathbb{D}(I)$ contain the same information, and we will opt to use the former idea throughout.

Moreover, the usual identities needed to make $\mathbb{D}(I)$ a monoidal category in this case can be checked using the fact that $\otimes$ is a morphism of prederivators, see [GPS14a, §3] for details.

One might expect that a monoidal derivator is a monoidal prederivator that also happens to be a monoidal derivator. This is close, we just require an additional condition.

Definition 3.1.3. Let $(\mathbb{D}, \otimes)$ be a derivator that is also a monoidal prederivator. We say that $(\mathbb{D}, \otimes)$ is a monoidal derivator if in addition the external product is cocontinuous in each variable, *i.e.*

$$(u_! X \boxtimes Y) \cong (u \times 1)_!(X \boxtimes Y)$$

for all $X \in \mathbb{D}(A)$, $Y \in \mathbb{D}(B)$, and functors $u : A \to C$, and similarly for $X \boxtimes v_! Y$.

Remark 3.1.4. We may call $\mathbb{D}$ braided or symmetric monoidal if the individual tensor products $\otimes_I$ induce braided or symmetric monoidal structures.

Remark 3.1.5. We can also consider partially external and partially internal tensor products: for categories A, B, C and objects $X \in \mathbb{D}(A \times C)$, $Y \in \mathbb{D}(B \times C)$, we can consider the mixed product

$$\mathbb{D}(A \times C) \times \mathbb{D}(B \times C) \to \mathbb{D}(A \times B \times C \times C) \to \mathbb{D}(A \times B \times C),$$

where the first arrow is the external product, and the second arrow is induced via the pullback along the diagonal functor $\Delta_C : C \to C \times C$.

It is easy to see that if $\mathbb{D}$ is a monoidal (pre)derivator, since Δ^* is a cocontinuous morphism of derivators, this defines a monoidal structure on any shifted (pre)derivator $\mathbb{D}^C$. This is sometimes referred to as the *pointwise monoidal structure* on $\mathbb{D}^C$, as opposed to the Day convolution structure that we define below.

We will see soon that this is a naive solution. If $\mathbb{D}$ is a symmetric monoidal derivator and C can be given the structure of a symmetric monoidal category, then there is a Day convolution model structure. We will demonstrate how this is more likely to be one that arises "in reality."

Proposition 3.1.6. *Let $(\mathcal{C}, \otimes_C)$ be a symmetric monoidal category and let $\mathbb{D}$ be a symmetric monoidal derivator. We can define a new monoidal structure on $\mathbb{D}^{\mathcal{C}}$ via the composition morphism:*

$$\mathbb{D}^{\mathcal{C}} \times \mathbb{D}^{\mathcal{C}} \xrightarrow{\boxtimes_{\mathcal{C}}} \mathbb{D}^{\mathcal{C}\times\mathcal{C}} \xrightarrow{(\otimes_{\mathcal{C}})_!} \mathbb{D}^{\mathcal{C}}$$

Proof. This is clearly a morphism of derivators as each part of the composition is. By construction, the morphisms $\boxtimes_{\mathcal{C}}$ and $(\otimes_{\mathcal{C}})_!$ are both cocontinuous. Since both $\boxtimes_{\mathcal{C}}$ and $(\otimes_{\mathcal{C}})_!$ are invariant under the transposition τ_{12}, we have a symmetric monoidal derivator structure. $\square$

In this symmetric monoidal derivator structure, the external product is given by

$$(X \boxtimes_{\mathcal{C}} Y) := (\otimes_{\mathcal{C}})_!(X \boxtimes Y).$$

We call it the *Day monoidal structure* because it is effectively just Day convolution.

Proposition 3.1.7. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two symmetric monoidal categories and let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a monoidal functor. Then $F_! : \mathbb{D}^{\mathcal{C}_1} \to \mathbb{D}^{\mathcal{C}_2}$ is a monoidal morphism of derivators with the respect to the Day monoidal structure on the two shifted derivators.*

Proof. The statement that $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a monoidal functor can be conveyed by the commutative square:

$$\begin{array}{ccc} \mathcal{C}_1 \times \mathcal{C}_1 & \xrightarrow{F\times F} & \mathcal{C}_2 \times \mathcal{C}_2 \\ {\scriptstyle \otimes_{\mathcal{C}_1}}\downarrow & & {\scriptstyle \otimes_{\mathcal{C}_2}}\downarrow \\ \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 \end{array}$$

Let $X \in \mathbb{D}^{\mathcal{C}_1}(I)$, $Y \in \mathbb{D}^{\mathcal{C}_1}(J)$. Assigning left Kan extensions and remembering that the external product $\boxtimes$ for $\mathbb{D}$ is cocontinuous in both variables, we have

$$
\begin{aligned}
F_!(X \boxtimes_{\mathcal{C}_1} Y) &= F_!(\otimes_{\mathcal{C}_1})_!(X \boxtimes Y) \\
&= (\otimes_{\mathcal{C}_2})_!(F \times F)_!(X \boxtimes Y) \\
&= (\otimes_{\mathcal{C}_2})_!(F_! X \boxtimes F_! Y) \\
&= (F_! X \boxtimes_{\mathcal{C}_2} F_! Y)
\end{aligned}
$$

Hence $F_!$ is a monoidal morphism of derivators. □

After discussing generation conditions on triangulated derivators, we will come back to a perspective about why this Day convolution structure is useful from a tensor-triangular geometry perspective.

3.2 Generation of derivators

Proposition 3.2.1. *From [Nee01] let us recall the notion of perfectly generated triangulated category (resp. compactly generated, well generated). Let $\mathbb{D}$ be a triangulated derivator. If $\mathbb{D}(e)$ is a perfectly generated (resp. compactly generated, well generated) triangulated category, then $\mathbb{D}(I)$ is perfectly generated (resp. compactly generated, well generated) triangulated category for all $I \in \mathsf{Cat}$.*

See [Nee01], [Kra10] for a discussion of various types of generation on triangulated categories and the implications for the localization theory thereof.

Proof. We prove the case of compact generation. The other two cases follow similarly. Suppose $\mathbb{D}(e)$ is a compactly generated triangulated category with generating set $\{S_\lambda : \lambda \in \Lambda\}$ and let $I \in \mathsf{Cat}$. Recall that a triangulated category $\mathcal{T}$ has a compact generating set $\mathcal{G}$ if the following two conditions are satisfied:

1. If X is an object such that $[\Sigma^n G, X] = 0$ for all $G \in \mathcal{G}$ and all $n \in \mathbb{Z}$, then $X = 0$.

2. For each object $G \in \mathcal{G}$ and any collection $\{X_\alpha : \alpha \in A\}$, the canonical morphism of Hom-groups

$$\coprod_{\alpha \in A} [G, X_\alpha] \to [G, \coprod_{\alpha \in A} X_\alpha]$$

is an isomorphism.

Under this setup, $\{i_! S_\lambda : i \in I, \lambda \in \Lambda\}$ is a compact generating set for $\mathbb{D}(I)$. Recall via the theory of triangulated derivators that left and right Kan extension functors are exact, so they commute with the suspension Σ and loop Ω.

1. Let $X \in \mathbb{D}(I)$ be an object such that $[\Sigma^n(i_! S_\lambda), X] = 0$ for all $i \in I$, $\lambda \in \Lambda$. Fix an $i \in I$ for consideration. We have that

$$\begin{aligned} 0 &= [\Sigma^n(i_! S_\lambda), X] \\ &= [i_! \Sigma^n S_\lambda, X] \\ &= [\Sigma^n S_\lambda, i^* X] \end{aligned}$$

And as $\{S_\lambda : \lambda \in \Lambda\}$ is a generating set, this means $i^* X = 0$ for all $i \in I$. Hence $X = 0$ in $\mathbb{D}(I)$, as there is a natural morphism $X \to 0$ that is pointwise $0 \to 0$, and so the assertion follows by (Der2) .

2. Now let's consider a coproduct $\coprod_{\alpha \in A} X_\alpha$ along with an object of the form $i_! S_\lambda$. We have the following chain of isomorphisms

$$\begin{aligned} \coprod_{\alpha \in A} [i_! S_\lambda, X_\alpha] &= \coprod_{\alpha \in A} [S_\lambda, i^* X_\alpha] \\ &= [S_\lambda, \coprod_{\alpha \in A} i^* X_\alpha] \\ &= [S_\lambda, i^* \coprod_{\alpha \in A} X_\alpha] \\ &= [i_! S_\lambda, \coprod_{\alpha \in A} X_\alpha] \end{aligned}$$

Therefore, $\mathbb{D}(I)$ is compactly generated for all $I \in \mathsf{Cat}$ and moreover a generating set on $\mathbb{D}(I)$ can be constructed from a generating set on $\mathbb{D}(e)$. The cases of perfect generation and well generation are similar, and we leave them to the reader. In each case, if $\{S_\lambda : \lambda \in \Lambda\}$ is a generating set for $\mathbb{D}(e)$, then $\{i_! S_\lambda : \lambda \in \Lambda, i \in I\}$ is a generating set for $\mathbb{D}(I)$. □

Definition 3.2.2. Let $\mathbb{D}$ be a triangulated derivator. Call $\mathbb{D}$ a compactly generated (resp. perfectly generated, resp. well generated) triangulated derivator if $\mathbb{D}(e)$ is a compactly generated (resp. perfectly generated, resp. well generated) triangulated category. In this case each $\mathbb{D}(I)$ is compactly generated (resp. perfectly generated, resp. well generated) and the generating sets for $\mathbb{D}(I)$ can be constructed by taking a generating for $\mathbb{D}(e)$ and taking a possible left Kan extensions $i_!$ over objects in I.

Proposition 3.2.3. *Let $\mathbb{D}$ be a compactly generated triangulated derivator and $\mathbb{E}$ be another triangulated derivator, and $F : \mathbb{D} \to \mathbb{E}$ be an exact morphism commuting with all coproducts. Then F is a left adjoint morphism of derivators.*

Proof. Since each $\mathbb{D}(I)$ is a compactly generated triangulated category, F_I is a left adjoint, with right adjoints R_I. We need to patch the R_I to a strong morphism of derivators. From [Gro13, Prop 2.10] we know that the R_I can be patched to a lax morphism.

Accordingly, it will be sufficient to prove that F commutes with all homotopy left Kan extensions. As F commutes with cones, it must also commute with homotopy pushouts per [GPS14b, Theorem 6.1]. Since F also preserves coproducts, F will preserve all homotopy colimits, or equivalentlly all homotopy left Kan extensions.

Therefore, F is a left adjoint morphism of derivators, and we call the patching of $\{R_I\}$ its right adjoint R. □

Remark 3.2.4. This may appear to be a facile restating of Brown representabiliy à la [Nee01], but it will be useful to have the exact statement for use.

3.3 Closed monoidal derivators

Our main goal in this section is to show that for a symmetric monoidal derivator $\mathbb{D}$ and symmetric monoidal category $(\mathcal{C}, \otimes_{\mathcal{C}})$, the shifted derivator $\mathbb{D}^{\mathcal{C}}$ with Day monoidal structure is closed and further that when applicable, compact and rigid objects in the respective tensor-triangulated categories coincide.

Definition 3.3.1. A monoidal derivator $\mathbb{D}$ closed if the tensor product

$$\otimes \colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$$

is a two-variable left adjoint.

For a general discussion of two-variable left adjoints, we defer to [GPS14b, Section 8].

Theorem 3.3.2. *If* $(\mathbb{D}, \boxtimes)$ *is a closed symmetric monoidal derivator and* $(\mathcal{C}, \otimes_{\mathcal{C}})$ *is a symmetric monoidal category, then* $\mathbb{D}^{\mathcal{C}}$ *is a closed symmetric monoidal derivator with the Day convolution structure.*

Proof. We have already proven that $\mathbb{D}^{\mathcal{C}}$ admits the structure of a symmetric monoidal derivator. It remains to proved closed-ness.

Recall that the external product associated to this new derivator is $\boxtimes_{\mathcal{C}} : \mathbb{D}^{\mathcal{C}}(I) \times \mathbb{D}^{\mathcal{C}}(J) \to \mathbb{D}^{\mathcal{C}}(I \times J)$ defined as

$$\mathbb{D}(\mathcal{C} \times I) \times \mathbb{D}(\mathcal{C} \times J) \xrightarrow{\boxtimes_{\mathbb{D}}} \mathbb{D}(\mathcal{C} \times I \times \mathcal{C} \times J) \xrightarrow{(\otimes_{\mathcal{C}} \times 1)_!} \mathbb{D}(\mathcal{C} \times I \times J)$$

To show this is a two-variable left adjoint, we see that

1. One two-variable adjoint is $\rhd_{[\mathcal{C},J]} \colon \mathbb{D}^{\mathcal{C}}(J)^{op} \times \mathbb{D}^{\mathcal{C}}(I \times J) \to \mathbb{D}^{\mathcal{C}}(I)$, given by the composition

   $$\rhd_{[\mathcal{C},J]} \circ (\mathrm{Id} \times (\otimes_{\mathcal{C}} \times 1_{I \times J})^*),$$

   which we obtain by taking the right adjoints of $\boxtimes_{\mathbb{D}}$ and $(\otimes_{\mathcal{C}} \times 1)_!$. By abuse of notation, we use $\rhd[\mathcal{C}, J]$ in the composition expression to also denote the two-variable adjoint of $\boxtimes_{\mathbb{D}}$.

2. The other two-variable adjoint is $\lhd_{[\mathcal{C},I]}\colon \mathbb{D}^{\mathcal{C}}(I \times J) \times \mathbb{D}^{\mathcal{C}}(I)^{op} \to \mathbb{D}^{\mathcal{C}}(I)$, given by the composition

$$\lhd_{[\mathcal{C}\times I]} \circ ((\otimes_{\mathcal{C}} \times 1_{I\times J})^{*} \times \mathrm{Id}),$$

again obtained by taking right adjoints of $\boxtimes_{\mathbb{D}}$ and $(\otimes_{\mathcal{C}} \times 1)_!$. Similar to the above, we use $\lhd_{[\mathcal{C},I]}$ in the composition to denote the two-variable adjoint of $\boxtimes_{\mathbb{D}}$.

We should then have, for any $X \in \mathbb{D}^{\mathcal{C}}(I)$, $Y \in \mathbb{D}^{\mathcal{C}}(J)$, and $Z \in \mathbb{D}^{\mathcal{C}}(I \times J)$, natural isomorphisms

$$\mathbb{D}^{\mathcal{C}}(I \times J)(X \boxtimes_{\mathcal{C}} Y, Z) \cong \mathbb{D}^{\mathcal{C}}(I)(X, Y \rhd_{[\mathcal{C},J]} Z) \cong \mathbb{D}^{\mathcal{C}}_{\mathbb{D}}(J)(Y, Z \lhd_{[\mathcal{C},I]} X)$$

The first isomorphism above is given by the composition

$$\begin{aligned}
\mathbb{D}^{\mathcal{C}}(I \times J)(X \boxtimes_{\mathcal{C}} Y, Z) &\cong \mathbb{D}(\mathcal{C} \times I \times J)((\otimes_{\mathcal{C}} \times 1)_!(X \boxtimes_{\mathbb{D}} Y), Z) \\
&\cong \mathbb{D}(\mathcal{C} \times \mathcal{C} \times I \times J)(X \boxtimes_{\mathbb{D}} Y, (\otimes_{\mathcal{C}} \times 1)^{*} Z) \\
&\cong \mathbb{D}(\mathcal{C} \times I)(X, Y \rhd_{[J]} (\otimes_{\mathcal{C}} \times 1)^{*} Z) \\
&\cong \mathbb{D}(\mathcal{C} \times I)(X, Y \rhd_{[\mathcal{C},J]} Z) \\
&\cong \mathbb{D}^{\mathcal{C}}(I)(X, Y \rhd_{[\mathcal{C},J]} Z)
\end{aligned}$$

Here each step uses either the $(\otimes_!, \otimes^{*})$-adjunction or $\boxtimes_{\mathbb{D}}$ being an adjunction of two variables. The second-to-last isomorphism is via abuse of notation, transitioning between two-variable left adjoints for the $\mathbb{D}$-monoidal structure and the shifted monoidal structure on $\mathbb{D}^{\mathcal{C}}$.

Meanwhile the isomorphism $\mathbb{D}^{\mathcal{C}}(I \times J)(X \boxtimes_{\mathcal{C}} Y, Z) \cong \mathbb{D}^{\mathcal{C}}(J)(Y, Z \lhd_{[\mathcal{C},I]} X)$ is similarly given by

$$\begin{aligned}
\mathbb{D}^{\mathcal{C}}(I \times J)(X \boxtimes_{\mathcal{C}} Y, Z) &\cong \mathbb{D}(\mathcal{C} \times I \times J)((\otimes_{\mathcal{C}} \times 1)_!(X \boxtimes_{\mathbb{D}} Y), Z) \\
&\cong \mathbb{D}(\mathcal{C} \times \mathcal{C} \times I \times J)(X \boxtimes_{\mathbb{D}} Y, (\otimes_{\mathcal{C}} \times 1)^{*} Z)
\end{aligned}$$

$$
\begin{aligned}
&\cong \mathbb{D}(\mathcal{C}\times J)(Y, (\otimes_{\mathcal{C}}\times 1)^* Z \lhd_{[J]} X)\\
&\cong \mathbb{D}(\mathcal{C}\times J)(Y, Z \lhd_{[\mathcal{C},J]} X)\\
&\cong \mathbb{D}^{\mathcal{C}}(J)(Y, Z \lhd_{[\mathcal{C},I]} X)
\end{aligned}
$$

Here again each isomorphism is either induced from the adjunction $(\otimes_!, \otimes^*)$ or the two-variable adjunction for $\mathbb{D}$, and is therefore natural.

For the canonical mates, fix $Y \in \mathbb{D}^{\mathcal{C}}(J)$, $Z \in \mathbb{D}^{\mathcal{C}}(I\times J)$, and let $u\colon I' \to I$ be any functor. Then we have

$$
\begin{aligned}
u^*(Y \rhd_{[\mathcal{C},J]} Z) &\cong u^*(Y \rhd_{[\mathcal{C}\times J]} \otimes_{\mathcal{C}}^* Z)\\
&\cong (Y \rhd_{[\times J]} (u\times 1)^* \otimes_{\mathcal{C}}^* Z)\\
&\cong (Y \rhd_{[\mathcal{C}\times J]} (+^*(u\times 1)^* Z)\\
&\cong (Y \rhd_{[\mathcal{C},J]} (u\times 1)^* Z)
\end{aligned}
$$

where the isomorphisms come from naturality and the known natural isomorphisms for $\rhd_{[J]}$. Similarly for the other mate we fix $X \in \mathbb{D}^{\mathcal{C}}(I)$ and $Z \in \mathbb{D}^{\mathcal{C}}(J)$, and a functor $v\colon J' \to J$. Then we have

$$
\begin{aligned}
v^*(Z \lhd_{[\mathcal{C},J]} X) &\cong v^*(\otimes_{\mathcal{C}}^* Z \lhd_{[\mathcal{C}\times J]} X)\\
&\cong ((1\times v)^* \otimes_{\mathcal{C}}^* Z \lhd_{[\mathcal{C}\times J]} X)\\
&\cong (\otimes_{\mathcal{C}}^*(1\times v)^* Z \lhd_{[\mathcal{C}\times J]} X)\\
&\cong ((1\times v)^* Z \lhd_{[\mathcal{C},J]} X)
\end{aligned}
$$

This completes the verification the new tensor product is a two-variable left adjoint and that the shifted derivator $\mathbb{D}^{\mathcal{C}}$ with this tensor product is a closed symmetric monoidal derivator. □

If $(\mathbb{D}, \boxtimes, \mathbb{1})$ is a monoidal derivator, it comes equipped with a so-called "projection morphism": for objects $A \in \mathbb{D}(I)$, $B \in \mathbb{D}(J)$, and functor $u\colon I \to J$, we have a morphism

$u_!(A \otimes_{\mathbb{D}(I)} u^*B) \to u_!A \otimes_{\mathbb{D}(J)} B$ defined to be the adjoint of the following morphism under the $u_! \dashv u^*$ adjunction.

$$A \otimes_{\mathbb{D}(I)} u^*B \longrightarrow u^*u_!A \otimes_{\mathbb{D}(I)} u^*B \cong u^*(u_!A \otimes_{\mathbb{D}(J)} B)$$

The projection morphism is moreover an isomorphism if $(\mathbb{D}, \boxtimes, \mathbb{1})$ is a closed monoidal derivator, see [Gal14, Lemma 2]. Thus, $\mathbb{D}^{\mathcal{C}}$ is equipped with a projection isomorphism whenever $\mathbb{D}$ is closed.

Recall the definition of rigid object in a closed symmetric monoidal category, see for example [BDS16]. From [Bal10], recall the definition of compactly generated tensor triangulated category:

Definition 3.3.3. [Bal10, Definition 44] Let $\mathcal{T}$ be a tensor-triangulated category with arbitrary coproducts. The subcategory $\mathcal{T}^c$ of compact objects is triangulated but not closed under coproducts. We say that $\mathcal{T}$ is a *compactly generated tensor triangulated category* if

1. $\mathcal{T}^c$ generates $\mathcal{T}$, that is, $\mathcal{T} = \mathrm{Loc}(\mathcal{T}^c)$ is the smallest localizing triangulated subcategory of $\mathcal{T}$ that contains $\mathcal{T}^c$.

2. $\mathcal{T}^c$ is essentially small, consists of strongly dualizable elements, and the unit $\mathbb{1}$ is compact.

In this case, an object is compact if and only if it is strongly dualizable, and one should apply the tensor-triangular geometric machine to the category $\mathcal{T}^c$.

Remark 3.3.4. If $\mathcal{T} = \mathbb{D}(e)$ is the base of a symmetric monoidal triangulated derivator, then $\mathcal{T}$ is a tensor-triangulated category. If $\mathcal{C}$ is a symmetric monoidal category, then $\mathbb{D}(\mathcal{C})$ has two possible tensor products: the pointwise tensor product induced from $\mathbb{D}$, and the Day convolution structure induced by thinking of $\mathbb{D}(\mathcal{C})$ as the base of $\mathbb{D}^{\mathcal{C}}$.

Operating under the added assumption that $\mathcal{C}$ is closed, an ideal monoidal structure on $\mathbb{D}(\mathcal{C})$ is one such that if $\mathbb{D}(e)$ is compactly generated tensor triangulated, then so is $\mathbb{D}(\mathcal{C})$ with this monoidal structure.

Theorem 3.3.5. *If $\mathbb{D}(e)$ is a compactly generated tensor triangulated category and $\mathcal{C}$ is closed symmetric monoidal, then $\mathbb{D}(\mathcal{C})$ with the Day convolution monoidal structure is also a compactly generated tensor triangulated category.*

Proof. The compact objects evidently are defined independently of the monoidal structure. Suppose $\{X_\lambda : \lambda \in \Lambda\}$ is a set of rigid-compact generators for $\mathbb{D}(e)$. Then we know that $\{c_! X_\lambda : \lambda \in \Lambda, c \in C\}$ is a set of *compact* generators for $\mathbb{D}(\mathcal{C})$. We endeavor to prove each compact generator is also rigid. For an object $c \in C$, let $\hat{c}$ denote its dual, and similarly let $\hat{X}_\lambda$ denote the dual of X_λ in $\mathbb{D}(e)$.

I claim that $c_! X_\lambda$ has dual $(\hat{c})_!(\hat{X}_\lambda)$. This at least intuitively makes sense: we can construct the evaluation $c_! X_\lambda \otimes_{\mathcal{C}} (\hat{c})_!(\hat{X}_\lambda) \to \mathbb{1}$ by noting the isomorphism

$$c_! X_\lambda \otimes_{\mathcal{C}} (\hat{c})_!(\hat{X}_\lambda) \cong (c \otimes \hat{c})_!(X_\lambda \otimes_{\mathbb{D}} \hat{X}_\lambda)$$

and then evaluating twice, once on $(c \otimes \hat{c})$ and once on $X_\lambda \otimes \hat{X}_\lambda$. The coevaluation maps are defined similarly, and one can check the triangle identities.

Thus it stands that the set of compact generators is also a collection of rigid generators. As such, every compact object in $\mathbb{D}(\mathcal{C})$ is rigid with respect to this monoidal structure. □

Remark 3.3.6. We will not trace the other monoidal structure. Instead, it is clear that the compact objects do not depend on the monoidal structure, whereas the rigid objects do. Thus given another monoidal structure, we would not expect the compact and rigid objects to coincide on $\mathbb{D}(\mathcal{C})$.

3.4 Localization

Here we will collect some results on the localization theory of derivators. There are a variety of standard tools at our disposal, many of which are motivated by localization theory of categories, for example see [GZ67] or [Kra10].

We will begin with results that make limited assumptions on our underlying derivators.

Lemma 3.4.1. *[Cis08, Lemme 4.2] Let $\mathbb{D}$ be a derivator. A prederivator $\mathbb{E}$ is called a full subprederivator of $\mathbb{D}$ if there is a morphism $\iota : \mathbb{E} \to \mathbb{D}$ such that $(\iota)_I$ is fully faithful for all small categories I.*

Let $\mathbb{E} \to \mathbb{D}$ be a full subprederivator of $\mathbb{D}$. If the inclusion morphism ι admits either a left or right adjoint, then $\mathbb{E}$ is also a derivator.

A proof is also given in [Col19, Lemma 3.3]. If ι admits a left adjoint we call $\mathbb{E}$ a *localization* and if ι admits a right adjoint we call $\mathbb{E}$ a *colocalization*.

We also utilize the following notion of localization functor:

Proposition 3.4.2. *[Kra10, Proposition 2.4.1] Let $L : \mathcal{C} \to \mathcal{C}$ be a functor, $\eta : 1_{\mathcal{C}} \to L$ be a natural transformation. Then the following are equivalent.*

1. *$\eta_L : L \to L^2$ and $L\eta : L \to L^2$ are natural isomorphisms.*
2. *There exists a functor $F : \mathcal{C} \to \mathcal{D}$ with fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$ such that $L = GF$ and $\eta : 1_C \to GF$ is the unit of the adjunction.*

Then we call L a localization functor*.*

By naming abuse if the functor F has a fully faithful right adjoint G as above, we may sometimes also refer to F as the localization functor.

We have introduced the notion of a shifted derivator, and so we would like to combine that with localization-theoretic techniques. In [Col19] a large class of potential examples is provided.

Definition 3.4.3. Let $\mathbb{D}$ be a pointed derivator, B a category, $i : A \to B$ a full subcategory. Define $\mathbb{D}(B, A)$ to be the full subcategory of $\mathbb{D}(B)$ *vanishing at* A, *i.e.*

$$\mathbb{D}(B, A) = \{X \in \mathbb{D}(B) : i^*X = 0 \in \mathbb{D}(A)\}.$$

In [Col19, Lemma 4.8, Theorem 4.10] Coley shows that this induces a localization and colocalization of derivators.

We now turn to the triangulated world for additional examples. Recall that we have proven a derivator version of Brown representability.

Theorem 3.4.4. *Let $\mathbb{D}$ be a compactly generated triangulated derivator and $C \subset \mathbb{D}(A)$ be a localizing subcategory closed under homotopy colimits. By abuse of notation, let $C_I \subset \mathbb{D}(A \times I)$ be the subcategory whose objects are pointwise (in the I-direction) in C and let $\mathcal{C}$ denote this prederivator.*

By design, $\mathcal{C}$ is a left derivator that is also levelwise triangulated. Denote the resulting localized prederivator as $\mathbb{D}^A/\mathcal{C}$, where

$$(\mathbb{D}^A/\mathcal{C})(I) = \mathbb{D}(A \times I)/\mathcal{C}_I$$

This is another triangulated derivator.

Proof. We prove statements in the following order:

1. The localization is a well-defined left derivator because $\mathcal{C}$ is a left derivator.

2. The localization is a stable derivator by Brown representability.

So first, let's show that $\mathbb{D}^A/\mathcal{C}$ is a left derivator. The conditions (Der1) and (Der2) are satisfied easily. We prove that the left Kan extensions in $\mathbb{D}^A/\mathcal{C}$ are induced by those in $\mathbb{D}^A$. Let $f : I \to J$ be a functor, and $X \in \mathbb{D}^A/\mathcal{C}(I)$ be an object; define $f_!X$ by picking a representative of X in $\mathbb{D}(A \times I)$, applying $f_!$, and localizing at $\mathcal{C}$.

We need to check that this is well-defined: let $\bar{X}, \bar{X}'$ be two different representatives for X. This means that they are connected by a zigzag of morphisms in $\mathrm{Mor}_{\mathcal{C}}$. Let's suppose that $\bar{X}$ and $\bar{X}'$ are connected by a morphism $\alpha : \bar{X} \to \bar{X}'$ with $C\alpha \in \mathcal{C}_I$. Then $f_!\bar{X}$ and $f_!\bar{X}'$ have isomorphic images in the localization, as $f_!$ is an exact functor and $\mathcal{C}$ is a left derivator, telling us that $C(f_!\alpha) \cong f_!C(\alpha) \cong 0$. Therefore, $f_!\bar{X}$ and $f_!\bar{X}'$ have the same value in the localization.

(Der4) follows easily for left Kan extensions as the computation can be done in $\mathbb{D}^A$ prior to localization by choosing representatives. Therefore, $\mathbb{D}^A/\mathcal{C}$ is a left derivator. Moreover, since the left Kan extensions are computed by choosing representatives in $\mathbb{D}^A$ the localization morphism is cocontinuous.

Now in particular, the localization morphism levelwise preserves coproducts. By Brown representability, we have a levelwise right adjoint which therefore must be a levelwise inclusion. The right adjoints are levelwise fully faithful inclusions, hence by [Cis08, Lemme 4.2], the quotient is a derivator. □

Remark 3.4.5. In this situation the left Kan extensions of the localized derivator can be computed from the left Kan extensions of the shifted (but not localized) $\mathbb{D}^A$. However, we do not know what the right Kan extensions are, nor do we expect the localization morphism to preserve right Kan extensions.

Remark 3.4.6. We term new derivators obtained from a known derivator with a shift and localization a shift-loc derivator, as it involves these two important operations.

Example As we will see, the construction of $\mathbb{P}^n$ on a (triangulated) derivator $\mathbb{D}$ is an example of a shift-loc derivator.

Remark 3.4.7. Now let's examine the 'composition' of two shift-loc derivators. Let $\mathbb{D}$ be a compactly generated triangulated derivator, $\mathbb{D}^A/\mathcal{C}_1$ a shift-loc derivator. Now let's examine another shift-loc derivator:

$$(\mathbb{D}^A/\mathcal{C}_1)^B/\mathcal{C}_2.$$

Now, a priori the shift-loc derivator $\mathbb{D}^A/\mathcal{C}_1$ is not compactly generated, which might prevent the composite shift-loc from being a derivator. However, this worry turns out to be unfounded.

Let $L_1 : \mathbb{D}^A \to \mathbb{D}^A/\mathcal{C}_1$ be the localization morphism. Recall by construction that L_1 is cocontinuous. Then

$$(\mathbb{D}^A/\mathcal{C}_1)^B/\mathcal{C}_2 \cong \mathbb{D}^{A\times B}/(L_1^{-1}(\mathcal{C}_2)).$$

Here $L_1^{-1}(\mathcal{C}_2)$ is the obvious object to localize at: the preimage of $\mathcal{C}_2$ in $\mathbb{D}^{A\times B}$ prior to localization. We need only show that $L_1^{-1}(\mathcal{C}_2)$ is a left derivator; in which case similar logic will tell us that the composite shift-loc is another stable derivator.

Proof. There are two things to prove here; first that the two localizations are equivalent and then that $L_1^{-1}(\mathcal{C}_2)$ is a left derivator.

For the equivalence of the localizations we can do this on the base and extend to the entire derivator by placing the shifting diagram inside $\mathbb{D}$. The category $(\mathbb{D}^A/\mathcal{C}_1)^B/\mathcal{C}_2(e)$ is the universal with respect to morphisms out of $(\mathbb{D}^A/\mathcal{C}_1)(B)$ sending $\mathcal{C}_2$ to 0, that is to say, universal with respect to morphisms out of $\mathbb{D}(A \times B)$ sending anything in the preimage of $\mathcal{C}_2$ with respect to the localization to 0. That preimage is precisely $L_1^{-1}(\mathcal{C}_2)$.

Therefore, the two localizations satisfy the same universal property and hence are equivalent.

Next I show that $L_1^{-1}(\mathcal{C}_2)$ is a left derivator. Recall that $\mathcal{C}_2$ is a left derivator and L_1 is a cocontinuous morphism of (left) derivators. By those two conditions it's clear that (Der1) and (Der2) are satisfied. For (Der3), $L_1^{-1}(\mathcal{C}_2)$ is closed under left Kan extensions as $\mathcal{C}_2$ is a left derivator and L_1 is cocontinuous. Then (Der4) is obviously satisfied as we are in a full subderivator of $\mathbb{D}^{A\times B}$. Therefore, this is an appropriate situation for our shift-loc derivator. □

Example Let X be a scheme, $\mathbb{D}_X$ be the associated derivator. We will see that $\mathbb{D}_{\mathbb{A}^1_X}$ is the shifted derivator $\mathbb{D}_X^{\underline{\mathbb{N}}}$. Let $\mathcal{C}$ be the subderivator of complexes with cohomology supported on $\{0\} \times X$. Then $\mathbb{D}^{\underline{\mathbb{N}}}/\mathcal{C}$ is the derivator associated to $\mathbb{G}_{m,X}$, which is also the shifted derivator $\mathbb{D}^{\underline{\mathbb{Z}}}$.

In diagrammatic form, the localization is the left Kan extension along the inclusion $i_{\mathbb{N}} : \underline{\mathbb{N}} \to \underline{\mathbb{Z}}$ while its right adjoint is $i_{\mathbb{N}}^*$.

Theorem 3.4.8. *Let $\mathbb{D}$ be a compactly generated derivator, $R \subset \mathbb{D}^A$ be a triangulated right subderivator. Then the quotient $\mathbb{D}^A/R$ is also a derivator.*

Proof. This is symmetric with the left situation, and we use the dual statement of Brown representability as prove in the previous chapter. □

CHAPTER 4

Affine lines and spaces over derivators

4.1 The definitions of $\mathbb{A}^1$ and $\mathbb{A}^n$

We briefly recall the definition of $\mathbb{A}^1$ and $\mathbb{A}^n$ over a derivator, as drawn from [BZ17].

We want to follow the intuition that a module over a polynomial ring $R[T]$ is just an R-module together with a chosen R-linear endomorphism corresponding to the action of T. Of course, once T acts on an R-module then T^n also acts for all $n \in \mathbb{N}$. Consider then the *loop* category

$$\underline{\mathbb{N}} = \bullet \circlearrowright \mathbb{N} \tag{4.1.1}$$

which has a single object with endomorphism monoid $\mathbb{N} = (\{0, 1, 2, 3, \ldots\}, +)$. Similarly, we let $\underline{\mathbb{N}}^n$ denote the n-fold product of $\underline{\mathbb{N}}$ with itself.

Definition 4.1.2. Let $\mathbb{D}$ be a derivator. We define:

1. $\mathbb{A}^1_{\mathbb{D}}$ to be the shifted derivator $\mathbb{D}^{\underline{\mathbb{N}}}$.

2. For any natural number $n \in \mathbb{N}$, $\mathbb{A}^n_{\mathbb{D}}$ to be the shifted derivator $\mathbb{D}^{\underline{\mathbb{N}}^n}$.

From this definition, some facts are clear about the $\mathbb{A}^1$-construction.

Proposition 4.1.3. *1. The $\mathbb{A}^n$-construction on a derivator $\mathbb{D}$ is just the $\mathbb{A}^1$-construction on $\mathbb{D}$ iterated n times.*

2. If $\mathbb{D}$ is pointed or stable, so is $\mathbb{A}^n_{\mathbb{D}}$ for all $n \in \mathbb{N}$.

3. If $\mathbb{D}$ is a (symmetric) monoidal derivator, then $\mathbb{A}^n_{\mathbb{D}}$ inherits a (symmetric) monoidal structure from the pointwise tensor product on $\mathbb{D}$.

Proof. 1. This is clear from the definition of $\mathbb{A}^n$ and shifted derivators.

2. Pointedness is equivalent to asking that each value of the derivator has a zero object, while stability asks for the Σ and Ω functors on each value to be an equivalence. Since each value of the shifted derivator $\mathbb{A}^n_{\mathbb{D}}$ is also a value for $\mathbb{D}$, pointedness and stability are also preserved.

3. This is clear from the definition of (symmetric) monoidal structure on a derivator, see Chapter 3.

□

We invite the reader to peruse [BZ17] for a justification of why $\mathbb{A}^1_{\mathbb{D}}$ mirrors the usual $\mathbb{A}^1$ construction in algebraic geometry. Subsequently, we will discuss more parallels between the derivator and algebraic geometric versions of $\mathbb{A}^1$.

4.2 Canonical morphisms between the base and the affine line

Let R be a commutative ring with unit. Remember then that $\mathbb{A}^1_{\text{Spec } R}$ is just Spec $R[t]$. We have three "natural" morphisms of rings and their Zariski spectra given by:

1. $R \hookrightarrow R[t]$, inducing the *structure morphism*

$$\mathbb{A}^1_{\text{Spec } R} \to \text{Spec } R$$

2. $R[t] \to R[t]/(t) \cong R$, inducing the *evaluation at 0 morphism*

$$\text{Spec } R \to \mathbb{A}^1_{\text{Spec } R}$$

3. $R[t] \to R[t]/(t-1) \cong R$, inducing the *evaluation at 1 morphism*

$$\text{Spec } R \to \mathbb{A}^1_{\text{Spec } R}$$

Precisely, we are applying the Spec functor to the ring homomorphisms above to get the requisite morphisms of affine schemes. We note that for a quasi-compact, quasi-separated scheme X with affine cover $X = \cup_i \text{Spec } A_i$, we can take the corresponding cover for $\mathbb{A}^1_X$ as $\cup_i \text{Spec } A_i[t]$.

Definition 4.2.1. Let R be a ring. The prederivator $\mathbb{D}_R \colon \mathsf{Cat} \longrightarrow \mathsf{CAT}$ taking $I \mapsto D(R\text{-Mod}^I)$ is a derivator, where we take derived categories. Its base $\mathbb{D}_R(e)$ is the derived category of R, and we call it the *derivator extending the derived category of* R.

Let X be a scheme. The prederivator $\mathbb{D}_X \colon \mathsf{Cat} \longrightarrow \mathsf{CAT}$ taking $I \mapsto D(QCoh(X)^I)$ is a derivator. Its base is the derived category of quasi-coherent sheaves on X, and we call the *derivator extending the derived category of* X.

If X is further *separated*, recall that the derived category of quasi-coherent sheaves, $D(\text{QCoh}(X))$, is equivalent to the usual derived category with quasi-coherent cohomology, $D_{\text{QCoh}}(X)$, see [BN93, Corollary 5.5]. So with the mild additional condition of being separated, we recover the "usual" derived categories of our scheme X.

Here we have equivalences of derivators between $\mathbb{D}_R$ and $\mathbb{D}_{\text{Spec R}}$ for a ring R, owing to the isomorphism between QCoh(Spec$R$) and R-Mod.

Moreover, a ring homomorphism $f : R \to S$ induces morphisms between the corresponding derivators of R and S via derived extension of scalars along f and restriction of scalars along f. Similarly, given a morphism $g : X \to Y$ of schemes, there are morphisms of derivators induced by the derived direct and inverse image functors along g. With R and $R[t]$, we can describe some of these functors in a diagrammatic manner.

The above ring homomorphisms between R and $R[t]$ generate morphisms between the categories R-Mod and $R[t]$-Mod via extension of scalars, which extend to morphisms of derivators. These will be our models for the evaluation at 0, evaluation at 1, and structure morphisms.

We first define the structure morphism.

Definition 4.2.2. The structure morphism of a derivator $\mathbb{D}$ and its affine line $\mathbb{A}^1_{\mathbb{D}}$ is given by the left Kan extension morphism $i_!$ of derivators

$$i_! \colon \mathbb{D} \to \mathbb{D}^{\mathbb{N}},$$

where $i : e \to \underline{\mathbb{N}}$ is the assignment of the single object in e to the single object in $\underline{\mathbb{N}}$, with identities mapping to identities.

On affine schemes, the structure morphism is the map $f\colon \mathbb{A}^1_R \to \mathrm{Spec}R$ induced by the inclusion $R \hookrightarrow R[t]$. Then i^* is just the direct image f_*, while $i_!$ is the inverse image functor f^*, so in the case of derivators associated to affine schemes, our definition of structure morphism extends the usual definition of structure morphism.

In particular, we should keep in mind the following:

$$\begin{array}{ccc} \mathbb{D}_R & \xrightarrow{i_!} & \mathbb{A}^1_{\mathbb{D}_R} \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathbb{D}_R & \xrightarrow[\mathrm{Spec}f^*]{} & \mathbb{D}_{R[t]} \end{array}$$

which explains that the choice of $i_!$ is indeed appropriate.

Next, we have the evaluation at 1 morphism. It is also a homotopy left Kan extension.

Definition 4.2.3. The "evaluation at 1" morphism relating a derivator $\mathbb{D}$ and its affine line $\mathbb{A}^1_{\mathbb{D}}$ is given by the homotopy left Kan extension morphism of derivators

$$p_!\colon \mathbb{D}^{\underline{\mathbb{N}}} \to \mathbb{D}.$$

We can see this in case of an affine scheme $\mathrm{Spec}R$, when we have the evaluation at 1 map $g\colon \mathrm{Spec}\ R \to \mathbb{A}^1_R$ induced by the ring homomorphism

$$R[t] \to R[t]/(t-1) \cong R$$

The map $p^*\colon R\text{-Mod} \to R\text{-Mod}^{\underline{\mathbb{N}}} = R[t]\text{-Mod}$ is the direct image functor g_* and $p_!$ is the inverse image functor g^*. For an affine scheme, our "evaluation at 1" map is induced from the ring homomorphism $R[t] \to R[t]/(t-1) \cong R$.

Again, we have the following picture:

$$\begin{array}{ccc} \mathbb{A}^1_{\mathbb{D}_R} & \xrightarrow{i_!} & \mathbb{D}_R \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathbb{D}_{R[t]} & \xrightarrow[\mathrm{Spec}f^*]{} & \mathbb{D}_R \end{array}$$

Using the machinery of monoidal derivators, we will be able to give a unified definition of evaluation at 0 and 1 along with other "coherent endomorphisms." For now, we will have to stick with a somewhat unwieldy definition for evaluation at 0. Let us assume that the derivator $\mathbb{D}$ is now pointed, so that the right Kan extension $i_{[1]*}$ is just extension by zero.

Let $u : [1] \to \underline{\mathbb{N}}$ be the functor sending $0 \to 1$ to $1 \in \mathbb{N}$. Given a derivator $\mathbb{D}$ and $X \in \mathbb{D}(\underline{\mathbb{N}})$ we can restrict via the functor $u\colon [1] \to \underline{\mathbb{N}}$ to obtain $u^*(X) \in \mathbb{D}([1])$.

From there, we include $i_{[1]}\colon [1] \to \ulcorner$ and $i_{\ulcorner}\colon \ulcorner \to \square$. In terms of underlying diagrams, the composition $i_{\ulcorner !} i_{[1]*} u^*$ takes an element (M, f) in $\mathbb{D}^{\underline{\mathbb{N}}}$ to the (coherent) homotopy pushout square (i.e. element of $\mathbb{D}(\square)$) below. As $i_{[1]}$ is a sieve, the bottom left corner of the coherent diagram is the 0 object. Left Kan extension along $i_{\ulcorner}$ constructs a (coherent) homotopy pushout square. Identifying the bottom right corner gives us the proposed evaluation at zero map$\mathrm{ev}_0 = (1,1)^* i_{\ulcorner !} i_{[1]*} u^*$. Here the underlying diagram looks like

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & M/f(M) \end{array}$$

ev_0 is a morphism of derivators as each operation in the composition is a morphism of derivators. We may simplify this one step further. The composition $(1,1)^* i_{\ulcorner !}$ is actually just the homotopy colimit of the $\ulcorner$-shaped diagram, and we can write it as $\pi_{\ulcorner !}$.

Proposition 4.2.4. *The evaluation at 0 morphism is* $ev_0 = (\pi_{\ulcorner})_! i_{[1]*} u^*$.

We see also that this fits the "evaluation at 0" morphism for affine schemes, in that case being the map $\mathrm{Spec} f\colon \mathrm{Spec} R \to \mathbb{A}^1_R$, induced by the homomorphism $f\colon R[t] \to R[t]/(t)$. We see directly that the construction emulates

$$M \mapsto M \otimes_{R[t]} R[t]/(t) \cong M/tM$$

for an $R[t]$-module M.

Specifically we are referring to the following diagram that commutes up to natural isomorphism: here $\mathbb{D}_R$ is the derivator associated to the ring R as usual;

$$\begin{array}{ccc} \mathbb{A}^1_{\mathbb{D}_R} & \xrightarrow{i_!} & \mathbb{D}_R \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \cong} \\ \mathbb{D}_{R[t]} & \xrightarrow[\mathrm{Spec} f^*]{} & \mathbb{D}_R \end{array}$$

4.3 Compatibility of canonical morphisms

In this section, we want to define a general *evaluation at* α morphism. As mentioned in the introduction, our model for this is the extension by scalars via the homomorphism $R[t] \to R[t]/(t-r) \cong R$, *i.e.*

$$- \otimes_{R[t]} R[t]/(t-r) \colon R[t]\text{-Mod} \to R\text{-Mod}\,.$$

For rings, "scalars" are simply elements of R, but in the derivator there is no obvious analogue of elements of the monoidal unit in $\mathbb{D}$. However, it we examine $R[t]$-modules with underlying R-module R, then the possibilities for the t-actions correspond precisely to elements of R.

Therefore, our intuition should be: scalars are endomorphisms of the unit.

In the derivator case, these *evaluation at alpha* morphisms ought to be strong monoidal. Furthermore, the composition

$$R \hookrightarrow R[t] \to R[t]/(t-r) \cong R$$

is the identity. The structure morphism is a model for extension of scalars along $R \hookrightarrow R[t]$, so the evaluation at α should form a section of the structure morphism.

First, we show that the structure morphism $i_! \colon \mathbb{D} \to \mathbb{A}^1_{\mathbb{D}}$ is a strong monoidal morphism under the $\mathbb{A}^1_{\mathbb{D}}$-monoidal structure as defined in the previous section.

Proposition 4.3.1. *The structure morphism is strong monoidal, i.e.*

$$i_!(X \boxtimes_{\mathbb{D}} Y) \cong (i_! X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} i_! Y)$$

Proof. This is a straightforward computation, we have

$$(i_! X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} i_! Y) \quad = \quad +_!(i_! X \boxtimes_{\mathbb{D}} i_! Y)$$

$$\cong +_!(i \times 1)_!(X \boxtimes_{\mathbb{D}} i_! Y)$$

$$\cong +_!(i \times 1)_!(1 \times i)_!(X \boxtimes_{\mathbb{D}} Y)$$

$$\cong (+ \circ (i \times 1) \circ (1 \times i))_!(X \boxtimes_{\mathbb{D}} Y)$$

$$\cong i_!(X \boxtimes_{\mathbb{D}} Y)$$

□

The construction of the *evaluation at* α morphism strongly mirrors the ring case.

Definition 4.3.2. Let $\mathbb{D}$ be a derivator, and X be an object of $\mathbb{D}(\underline{\mathbb{N}}) = \mathbb{A}^1_{\mathbb{D}}(e)$, such that $i^*X = \mathbb{1}_{\mathbb{D}}$. Then call X an *coherent endomorphism of the unit*. We can write $\mathrm{dia}_{\underline{\mathbb{N}},e}(X) = (\mathbb{1}, \alpha)$. In this case we call X the *coherent α endomorphism*.

The term *coherent α endomorphism* can be a bit deceptive, as there may be more than one object with the same underlying diagram. Now we can define what the *evaluation at α* morphism means.

Let $\mathbb{D}$ be a symmetric monoidal derivator and equip $\mathbb{D}^{\underline{\mathbb{N}}}$ with the $\mathbb{A}^1_{\mathbb{D}}$-monoidal structure. We would take a coherent endomorphism of the unit, i.e. an object in $\mathbb{D}(\underline{\mathbb{N}})$ whose underlying diagram is $(\mathbb{1}, \alpha)$, take its $\mathbb{A}^1_{\mathbb{D}}$-external product with any object in some $\mathbb{A}^1_{\mathbb{D}}(I)$, and then forget the endomorphism part originating in the $(\mathbb{1}, \alpha)$-direction. That is to say, for some $X \in \mathbb{D}^{\underline{\mathbb{N}}}(I)$ and letting ev_α denote our desired evaluation at α map, our formula for ev_α is

$$\mathrm{ev}_\alpha(X) = i^* +_! (X \boxtimes_{\mathbb{D}} (\mathbb{1}, \alpha)).$$

We call $p^*\mathbb{1}$ the *coherent identity morphism*, and evaluation at 1 means evaluating at this particular element of $\mathbb{D}(\underline{\mathbb{N}})$.

Definition 4.3.3. Let $\mathbb{D}$ be a symmetric monoidal derivator, and consider $\mathbb{A}^1_{\mathbb{D}}$ with the $\mathbb{A}^1_{\mathbb{D}}$-monoidal structure. The *evaluation at (coherent) α* for any $Y \in \mathbb{A}^1_{\mathbb{D}}(I)$ is

$$\mathrm{ev}_\alpha(Y) = i^*(Y \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (\mathbb{1}_{\mathbb{D}}, \alpha)).$$

We will show in due course that this is a strong monoidal morphism. First we show that it coincides with our previous notion of evaluation at 1.

Lemma 4.3.4. *The definition of "evaluation at 1" given by* $p_!$ *and the new definition of evaluation at 1 coincide.*

Proof. Using general definition of evaluation at 1, our coherent endomorphism is $p^*\mathbb{1}_{\mathbb{D}}$. Thus we have

$$\begin{aligned} \mathrm{ev}_1(X) &= i^* +_! (X \boxtimes_{\mathbb{D}} p^*\mathbb{1}) \\ &= i^* +_! (1 \times p)^*(X) \end{aligned}$$

So the task at hand is now simply to prove the isomorphism $i^* +_! (1 \times p)^* \cong p_!$. We will prove a related isomorphism, namely $+_!(1 \times p)^* \cong p^* p_!$. The required isomorphism now follows from this one since $i^*p^* \cong \mathrm{Id}$, and so post-composing both sides of $+_!(1 \times p)^* \cong p^*p_!$ gives precisely $i^* +_! (1 \times p)^* \cong p_!$. Thus, we would like to show the following (commutative) square is homotopy exact.

$$\begin{array}{ccc} \underline{\mathbb{N}} \times \underline{\mathbb{N}} & \xrightarrow{1\times p} & \underline{\mathbb{N}} \\ {\scriptstyle +}\downarrow & \overset{\mathrm{Id}}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle p} \\ \underline{\mathbb{N}} & \xrightarrow[p]{} & e \end{array}$$

Here we check this directly via [GPS14a, Theorem 3.8]. In our good fortune, because we have the terminal category in the lower right corner, we need only check that a single category is homotopy contractible. The category $(\bullet/\underline{\mathbb{N}} \times \underline{\mathbb{N}}/\bullet)_{\mathrm{id}}$ as stated in the theorem, where both objects $\bullet$ are the sole objects in the two copies of $\underline{\mathbb{N}}$, has objects triples

$$(m \in \mathbb{N} = \mathrm{Hom}_{\underline{\mathbb{N}}}(\bullet, \bullet), \bullet \in \underline{\mathbb{N}} \times \underline{\mathbb{N}}, n \in \mathbb{N} = \mathrm{Hom}_{\underline{\mathbb{N}}}(\bullet, \bullet)),$$

which we view as merely a pair of natural numbers (m, n).

The morphisms in this category are morphisms $(a_1, a_2) \in \underline{\mathbb{N}} \times \underline{\mathbb{N}}$ such that $(1 \times p)(a_1, a_2) + m = m'$, and $+(a_1, a_2) + n' = n$, *i.e.* we have a morphism $(a_1, a_2) : (m, n) \to (m', n')$ if

$a_1 + m = m'$ and $a_1 + a_2 + n' = n$. Between any two objects of this category $(\bullet/\underline{\mathbb{N}} \times \underline{\mathbb{N}}/\bullet)_{\mathrm{id}}$, there is at most only a single morphism. There is a morphism $(m, n) \to (m', n')$, if $m \leq m'$ and $n - n' \geq m' - m$, and in particular we must have $m \leq m'$ and $n \geq n'$.

Thus, we may view $(\bullet/\underline{\mathbb{N}} \times \underline{\mathbb{N}}/\bullet)_{\mathrm{id}}$ as a subcategory of $(\mathbb{N}, \leq) \times (\mathbb{N}, \geq)$ containing all objects but not all morphisms, where we have a morphism $(m, n) \to (m', n')$ if and only if $m \leq m'$ and $n \geq n'$, and $m + n \leq m' + n'$. However, we may view this as a subcategory of $(\mathbb{N}, \leq) \times (\mathbb{N}, \geq)$ by taking $(m, n) \mapsto (m + n, n)$. Being a fully faithful functor, it is an equivalence onto its image, and this second category is a full subcategory of $(\mathbb{N}, \leq) \times (\mathbb{N}, \geq)$ consisting of objects (k, l) with $k \geq l$. Call this subcategory $L \subset (\mathbb{N}, \leq) \times (\mathbb{N}, \geq)$.

There is an adjunction connecting L and $(\mathbb{N}, \leq) \times (\mathbb{N}, \geq)$. The right adjoint is the inclusion, and the left adjoint takes $(m, n) \in (\mathbb{N}, \leq) \times (\mathbb{N}, \geq)$ to $(m, n) \in L$ if $m \geq n$ and $(n, n) \in L$ if $m < n$. Then $(\mathbb{N}, \leq) \times (\mathbb{N}, \geq)$ is a product of two homotopy contractible categories, since $(\mathbb{N}, \leq)$ has an initial object while $(\mathbb{N}, \geq)$ has a final object, and hence homotopy contractible.

This category L is then connected via adjunction to the terminal category, hence it is homotopy contractible. Therefore, our commutative square is homotopy exact and the two definitions of evaluation at 1 coincide. □

Lemma 4.3.5. *Let $Y = (\mathbb{1}, \alpha)$ be a coherent endomorphism in $\mathbb{D}(\underline{\mathbb{N}})$ and let $M, N \in \mathbb{D}(e)$. Then $(M \boxtimes_{\mathbb{D}} Y) \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (N \boxtimes_{\mathbb{D}} Y) \cong (M \boxtimes_{\mathbb{D}} N) \boxtimes_{\mathbb{D}} Y$.*

Proof. This can be proven by showing that $+_! +^* \cong \mathrm{Id}$, or equivalently that the square

$$\begin{array}{ccc} \underline{\mathbb{N}} \times \underline{\mathbb{N}} & \xrightarrow{1} & \underline{\mathbb{N}} \times \underline{\mathbb{N}} \\ \downarrow{\scriptstyle 1} & \overset{\mathrm{Id}}{\swarrow} & \downarrow{\scriptstyle +} \\ \underline{\mathbb{N}} \times \underline{\mathbb{N}} & \xrightarrow{+} & \underline{\mathbb{N}} \end{array}$$

is homotopy exact. However, this statement is certainly true if $\mathbb{D}$ is a model category, so by [GPS14a, Theorem 3.16], it holds for all derivators $\mathbb{D}$. □

Proposition 4.3.6. *The general "evaluation at α" morphism is a strong monoidal morphism of derivators for any coherent endomorphism $(\mathbb{1}, \alpha) \in \mathbb{D}^{\mathbb{N}}(e)$.*

Proof. Again we simply restrict to examining the evaluation at α morphism on $\mathbb{A}^1_{\mathbb{D}}(e)$. Consider objects X and Y in $\mathbb{D}(\underline{\mathbb{N}})$-we wish to show

$$\mathrm{ev}_\alpha(X) \boxtimes_{\mathbb{D}} \mathrm{ev}_\alpha(Y) \cong \mathrm{ev}_\alpha(X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} Y).$$

Let us use the formulation of $\mathrm{ev}_\alpha(-) = i^* +_! (- \boxtimes_{\mathbb{D}} (1, \alpha))$. Then we have

$$\begin{aligned}
\mathrm{ev}_\alpha(X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} Y) &= i^* +_! ((X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} Y) \boxtimes_{\mathbb{D}} (\mathbb{1}, \alpha)) \\
&= i^*((X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} Y) \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (\mathbb{1}, \alpha)) \\
&\cong i^*((X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} Y) \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} ((\mathbb{1}, \alpha) \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (\mathbb{1}, \alpha))) \\
&\cong i^*(X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (\mathbb{1}, \alpha)) \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (Y \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (\mathbb{1}, \alpha)) \\
&\cong i^*(X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (\mathbb{1}, \alpha)) \boxtimes_{\mathbb{D}} i^*(Y \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (\mathbb{1}, \alpha)) \\
&= (i^* +_!)(X \boxtimes_{\mathbb{D}} (\mathbb{1}, \alpha)) \boxtimes_{\mathbb{D}} (i^* +_!)(Y \boxtimes_{\mathbb{D}} (\mathbb{1}, \alpha)) \\
&= \mathrm{ev}_\alpha(X) \boxtimes_{\mathbb{D}} \mathrm{ev}_\alpha(Y)
\end{aligned}$$

Here the last isomorphism comes from the preceding lemma. □

Therefore, the monoidal structures on $\mathbb{A}^1_{\mathbb{D}}$ and $\mathbb{D}$ are compatible, in that the structure and evaluation morphisms are all strong monoidal.

Proposition 4.3.7. *The evaluation at α morphisms are cocontinuous.*

Proof. To be precise, we would like to show that for any functor $f \colon A \to B$, the diagram below commutes.

$$\begin{array}{ccc}
\mathbb{A}^1_{\mathbb{D}}(A) & \xrightarrow{f_!} & \mathbb{A}^1_{\mathbb{D}}(B) \\
{\scriptstyle \mathrm{ev}_\alpha}\downarrow & \overset{\cong}{\swarrow} & \downarrow{\scriptstyle \mathrm{ev}_\alpha} \\
\mathbb{D}(A) & \xrightarrow[f_!]{} & \mathbb{D}(B)
\end{array}$$

Remember that $\mathrm{ev}_\alpha(X)$ for any $X \in \mathbb{A}^1_{\mathbb{D}}(I)$ is the composition $i^* +_! (X \boxtimes_{\mathbb{D}} (\mathbb{1}, \alpha))$. Taking an external product with $(\mathbb{1}, \alpha)$ commutes with $f_!$ since external products are cocontinuous. The homotopy left Kan extensions $+_!$ and $f_!$ commute as they occur in different variables, and similarly i^* and $f_!$. □

This last result will be of importance when we discuss the “universal property” of the affine line.

CHAPTER 5

The universal property of $\mathbb{A}^n$

Recall that our definition for the affine line over a derivator rested upon the intuition that if R is a commutative ring, an $R[t]$-module is nothing more than an R-module with an R-module endomorphism, allowing us to define $\mathbb{A}^1_{\mathbb{D}} = \mathbb{D}^{\underline{\mathbb{N}}}$ for any derivator $\mathbb{D}$. For rings, a homomorphism $R[t] \to S$ can be broken down into a two simple parts, the "underlying morphism" $R \to S$ and the value of $t \in R[t]$ under the homomorphism, while conversely the combination of a ring homomorphism $R \to S$ and a value $s \in S$ for the assignment of t is sufficient to determine a ring homomorphism $R[t] \to S$. We expect a similar bijective compatibility for (symmetric monoidal) derivators, that a (cocontinuous) strong monoidal morphism $F\colon \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$ for symmetric monoidal derivators $\mathbb{D}$, $\mathbb{E}$ can be determined by an "underlying morphism" $\mathbb{D} \to \mathbb{E}$ and the value of an object in $\mathbb{E}(\underline{\mathbb{N}})$.

We note that in the setting of noncommutative rings there is no expectation of a similarly clean bijection, and indeed our proof relies significantly upon heavily on the monoidal structures that we have previously defined.

5.1 Two technical lemmas

We will begin by proving two results that will be frequently used.

Lemma 5.1.1. *Let $F : \mathbb{D} \to \mathbb{E}$ be a cocontinuous strong monoidal morphism between monoidal derivators. Then $F^{\underline{\mathbb{N}}}\colon \mathbb{A}^1_{\mathbb{D}} \to \mathbb{A}^1_{\mathbb{E}}$ is again strong monoidal for the $\mathbb{A}^1_{\mathbb{D}}$, $\mathbb{A}^1_{\mathbb{E}}$-monoidal structures on $\mathbb{D}^{\underline{\mathbb{N}}}$, $\mathbb{E}^{\underline{\mathbb{N}}}$, respectively.*

Proof. The following diagram commutes up to natural isomorphism:

$$
\begin{array}{ccc}
\mathbb{D}^{\underline{\mathbb{N}}} \times \mathbb{D}^{\underline{\mathbb{N}}} & \xrightarrow{F^{\underline{\mathbb{N}}} \times F^{\underline{\mathbb{N}}}} & \mathbb{E}^{\underline{\mathbb{N}}} \times \mathbb{E}^{\underline{\mathbb{N}}} \\
{\scriptstyle \boxtimes_{\mathbb{D}}} \downarrow & \overset{\cong}{\swarrow\!\!\swarrow} & \downarrow {\scriptstyle \boxtimes_{\mathbb{E}}} \\
\mathbb{D}^{\underline{\mathbb{N}} \times \underline{\mathbb{N}}} & \xrightarrow[F^{\underline{\mathbb{N}} \times \underline{\mathbb{N}}}]{} & \mathbb{E}^{\underline{\mathbb{N}} \times \underline{\mathbb{N}}} \\
{\scriptstyle +_!} \downarrow & \underset{\cong}{\swarrow\!\!\swarrow} & {\scriptstyle +_!} \downarrow \\
\mathbb{D}^{\underline{\mathbb{N}}} & \xrightarrow[F^{\underline{\mathbb{N}}}]{} & \mathbb{E}^{\underline{\mathbb{N}}}
\end{array}
$$

Here the commutativity of the top square up to natural isomorphism expresses that F is strong monoidal on the $\mathbb{A}^n$ monoidal structures, while the commutativity of the bottom square is a consequence of F being cocontinuous. □

Note that in this case $F^{\underline{\mathbb{N}}}$ is again a cocontinuous strong monoidal functor, so we can iterate this construction.

Now we take some steps towards the decomposition discussed above. If we are given a morphism of derivators $F\colon \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$, composing with $i_!\colon \mathbb{D} \to \mathbb{D}^{\underline{\mathbb{N}}}$ gives us the "underlying" morphism of derivator $F_0\colon \mathbb{D} \to \mathbb{E}$. Indeed this is what we would expect if $\mathbb{D}$ and $\mathbb{E}$ are derivators associated to rings.

We define two pieces of terminology.

Definition 5.1.2. Let $\mathbb{D}$, $\mathbb{E}$ be two symmetric monoidal derivators, and $F : \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$ be a monoidal morphism of derivators.

1. Call the composition $F_0 := F \circ i_! : \mathbb{D} \to \mathbb{E}$ the *base* of the morphism F.

2. The image of the object $+^* i_! \mathbb{1}_{\mathbb{D}}$ in $\mathbb{D}(\underline{\mathbb{N}} \times \underline{\mathbb{N}}) = \mathbb{A}^1_{\mathbb{D}}(\underline{\mathbb{N}})$ under F evaluated at $\underline{\mathbb{N}}$, *i.e.* $F_{\underline{\mathbb{N}}}(+^* i_! \mathbb{1}_{\mathbb{D}})$ is the *type* of F.

   Later we will show that the type of F is a coherent endomorphism of the unit.

Lemma 5.1.3. *Let $F_0\colon \mathbb{D} \to \mathbb{E}$ be a cocontinuous strong monoidal morphism of derivators and $(\mathbb{1}, \alpha) \in \mathbb{E}(\underline{\mathbb{N}})$ be any coherent endomorphism. Then the composition*

$$ev_\alpha \circ F_0^{\underline{\mathbb{N}}}\colon \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$$

is also a cocontinuous, strong monoidal morphism.

Proof. This is clear, since both ev_α and $F_0^{\mathbb{N}}$ were known to be strong monoidal and cocontinuous. □

Moreover, such a morphism $\mathrm{ev}_\alpha \circ F_0^{\mathbb{N}}$ has underlying morphism F_0, since

$$\mathrm{ev}_\alpha \circ F_0^{\mathbb{N}} \circ i_! \cong \mathrm{ev}_\alpha \circ i_! \circ F_0 \cong F_0$$

by co-continuity of F_0, and then by noting that $\mathrm{ev}_\alpha \circ i_!$ is just the identity morphism.

5.2 Main Theorem and proof

The main theorem regarding the $\mathbb{A}^1$-universal property is given below.

Theorem 5.2.1. *Let $\mathbb{D}$, $\mathbb{E}$ be two monoidal derivators, and let $F\colon \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$ be a cocontinuous monoidal morphism. The only cocontinuous monoidal morphism with base F_0 and type α is*

$$ev_\alpha \circ F_0^{\mathbb{N}}$$

That is to say, every cocontinuous monoidal morphism of derivators

$$F : \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$$

can be obtained as a composition $ev_\alpha \circ F_0^{\mathbb{N}}$ for some coherent endomorphism $(\mathbb{1}_{\mathbb{E}}, \alpha)$ and some strong monoidal morphism

$$F \circ i_! = F_0 : \mathbb{D} \to \mathbb{E}.$$

First we see where the information of the coherent endomorphism α can be obtained.

Lemma 5.2.2. *The following diagram commutes up to isomorphism:*

$$\begin{array}{ccc} \mathbb{D}^{\mathbb{N}\times\mathbb{N}} & \xrightarrow{F^{\mathbb{N}}} & \mathbb{E}^{\mathbb{N}} \\ {\scriptstyle +^*}\uparrow & \overset{\cong}{\swarrow} & \downarrow{\scriptstyle i^*} \\ \mathbb{D}^{\mathbb{N}} & \xrightarrow[F]{} & \mathbb{E} \end{array}$$

Proof. Since the composition

$$\underline{\mathbb{N}} \xrightarrow{1\times i} \underline{\mathbb{N}} \times \underline{\mathbb{N}} \xrightarrow{+} \underline{\mathbb{N}}$$

is the identity, so too is the composition $(1 \times i)^* +^*$. Therefore, the below diagram commutes (without isomorphism):

$$\begin{array}{ccc} \mathbb{D}^{\underline{\mathbb{N}}\times\underline{\mathbb{N}}} & \xrightarrow{\mathrm{Id}} & \mathbb{D}^{\underline{\mathbb{N}}\times\underline{\mathbb{N}}} \\ {\scriptstyle +^*}\uparrow & & \downarrow{\scriptstyle (1\times i)^*} \\ \mathbb{D}^{\underline{\mathbb{N}}} & \xrightarrow{\mathrm{Id}} & \mathbb{D}^{\underline{\mathbb{N}}} \end{array}$$

Then, for any morphism $\mathbb{D}^{\underline{\mathbb{N}}} \to \mathbb{E}$, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{D}^{\underline{\mathbb{N}}\times\underline{\mathbb{N}}} & \xrightarrow{F^{\underline{\mathbb{N}}}} & \mathbb{E}^{\underline{\mathbb{N}}} \\ {\scriptstyle (1\times i)^*}\downarrow & \overset{\cong}{\swarrow} & \downarrow{\scriptstyle i^*} \\ \mathbb{D}^{\underline{\mathbb{N}}} & \xrightarrow[F]{} & \mathbb{E} \end{array}$$

Pasting the two squares horizontally gives precisely our desired diagram. □

In the specific case of $F^{\underline{\mathbb{N}}}(+^*\mathbb{1}_{\mathbb{A}^1_{\mathbb{D}}})$, since F is strong monoidal we know that $i^*F^{\underline{\mathbb{N}}}(+^*\mathbb{1}_{\mathbb{A}^1_{\mathbb{D}}})$ is just $\mathbb{1}_{\mathbb{E}}$. Therefore, $F^{\underline{\mathbb{N}}}(+^*\mathbb{1}_{\mathbb{A}^1_{\mathbb{D}}})$ is equal to $(\mathbb{1}_{\mathbb{E}}, \alpha) \in \mathbb{E}(\underline{\mathbb{N}})$ for some α.

So let $F\colon \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$ be a morphism of derivators. We wish to determine the image of $F^{\underline{\mathbb{N}}}(+^*X)$ for any $X \in \mathbb{D}^{\underline{\mathbb{N}}}$. First we give a decomposition of $+^*X$ as the $\mathbb{A}^1_{\mathbb{A}^1_{\mathbb{D}}}$-tensor product.

Proposition 5.2.3. $+^*X$ *is the* $\mathbb{A}^1_{\mathbb{A}^1_{\mathbb{D}}}$*-tensor product of* $(1 \times i)_!X$ *and* $+^*i_!\mathbb{1}_{\mathbb{D}}$.

Proof. Recall the definition of the $\mathbb{A}^1$-monoidal structure. Here we end up taking a $\mathbb{A}^1_{\mathbb{A}^1_{\mathbb{D}}}$-tensor product, *i.e.* doing the Day convolution product on $\mathbb{D}^{\underline{\mathbb{N}}\times\underline{\mathbb{N}}}$.

Consider $\mathbb{1}_{\mathbb{D}} \in \mathbb{D}(e)$ and $X \in \mathbb{A}^1_{\mathbb{D}}(I) = \mathbb{D}(\underline{\mathbb{N}} \times I)$.

First we re-write $+^*X = +^*(X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} i_!\mathbb{1}_{\mathbb{D}})$. Here we can draw upon the following diagram, expressing the definition of the $\mathbb{A}^1$-monoidal structure.

$$\begin{array}{ccc}
\mathbb{D}(\underline{\mathbb{N}}) \times \mathbb{D}(\underline{\mathbb{N}}) & \xrightarrow{\boxtimes_{\mathbb{D}}} & \mathbb{D}(\underline{\mathbb{N}} \times \underline{\mathbb{N}}) \\
\Big\downarrow \mathrm{Id} & \overset{\mathrm{Id}}{\swarrow} & \Big\downarrow +_! \\
\mathbb{A}^1_{\mathbb{D}}(e) \times \mathbb{A}^1_{\mathbb{D}}(e) & \xrightarrow[\boxtimes_{\mathbb{A}^1_{\mathbb{D}}}]{} & \mathbb{D}(\underline{\mathbb{N}}) \cong \mathbb{A}^1_{\mathbb{D}}(e)
\end{array}$$

So we can write $+^*(X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} i_! \mathbb{1}_{\mathbb{D}}) = +^* +_! (X \boxtimes_{\mathbb{D}} i_! \mathbb{1}_{\mathbb{D}})$. Now, we also have

$$\begin{aligned}
(1 \times i)_! X \boxtimes_{\mathbb{A}^1_{\mathbb{A}^1_{\mathbb{D}}}} +^* i_! \mathbb{1}_{\mathbb{D}} &= +_!((1 \times i)_! X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} +^* i_! \mathbb{1}_{\mathbb{D}}) \\
&\cong +_!(1 \times i)_!(X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} +^* i_! \mathbb{1}_{\mathbb{D}}) \\
&\cong X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (+^* i_! \mathbb{1}_{\mathbb{D}}) \\
&= (+ \times 1)_!(X \boxtimes +^* i_! \mathbb{1}_{\mathbb{D}}) \\
&\cong (+ \times 1)_!(1 \times +)^*(X \boxtimes i_! \mathbb{1}_{\mathbb{D}})
\end{aligned}$$

The individual isomorphisms are as follows. The first equality is just the definition of the $\mathbb{A}^1_{\mathbb{A}^1_{\mathbb{D}}}$-external product relative to the $\mathbb{A}^1_{\mathbb{D}}$-external product. The second isomorphism is from co-continuity of the external product. Therefore, $((1 \times i)_! X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} +^* i_! \mathbb{1}_{\mathbb{D}}) \cong (i \times 1)_!(X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} +^* i_! \mathbb{1}_{\mathbb{D}})$,

For the third isomorphism, by naturality we know that $+_!(1 \times i)_! \cong (+ \circ (1 \times i))_!$, but $+ \circ (1 \times i)$ is just the identity functor on $\underline{\mathbb{N}}$. Thus, $+_!(1 \times i)_! \cong 1_!$, and $1_! \cong \mathrm{Id}$. Hence we can simply remove $+_!(1 \times i)_!$ for the third isomorphism.

The fourth equality is once again a definition of the $\mathbb{A}^1_{\mathbb{D}}$-external product relative to the $\mathbb{D}$-external product, while the fifth isomorphism is a reflection of the fact that taking the external product with any object is a morphism of derivators.

Thus, we have $(1 \times i)_! X \boxtimes_{\mathbb{A}^1_{\mathbb{A}^1_{\mathbb{D}}}} +^* i_! \mathbb{1}_{\mathbb{D}} = (+ \times 1)_!(1 \times +)^*(X \boxtimes_{\mathbb{D}} i_! \mathbb{1}_{\mathbb{D}})$. We would like to show it to be isomorphic to $+^*(M, f) = +^* +_! (i_! \mathbb{1}_{\mathbb{D}} \boxtimes_{\mathbb{D}} (M, f))$. A sufficient statement would be simply that $(+ \times 1)_!(1 \times +)^* \cong +^* +_!$. $\square$

Lemma 5.2.4. *The following square is homotopy exact:*

$$
\begin{array}{ccc}
\underline{\mathbb{N}} \times \underline{\mathbb{N}} \times \underline{\mathbb{N}} & \xrightarrow{1\times+} & \underline{\mathbb{N}} \times \underline{\mathbb{N}} \\
{\scriptstyle 1\times+}\downarrow & \overset{Id}{\swarrow} & \downarrow{\scriptstyle +} \\
\underline{\mathbb{N}} \times \underline{\mathbb{N}} & \xrightarrow[+]{} & \underline{\mathbb{N}}
\end{array}
\tag{5.2.5}
$$

Here the natural transformation in the middle is just the identity, as both compositions are just

$$\underline{\mathbb{N}} \times \underline{\mathbb{N}} \times \underline{\mathbb{N}} \to \underline{\mathbb{N}},$$

taking a map $(a, b, c) \mapsto a + b + c$.

Proof. Using (Der4), we know that the square

$$
\begin{array}{ccc}
(+ \times 1)/e & \xrightarrow{pr} & \underline{\mathbb{N}} \times \underline{\mathbb{N}} \times \underline{\mathbb{N}} \\
{\scriptstyle \pi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle +\times 1} \\
e & \xrightarrow[i]{} & \underline{\mathbb{N}} \times \underline{\mathbb{N}}
\end{array}
\tag{5.2.6}
$$

is homotopy exact. Our original square is homotopy exact if and only if its pasting with the above is homotopy exact, as homotopy exactness can be checked pointwise and $\underline{\mathbb{N}} \times \underline{\mathbb{N}}$ has precisely one object.

The category $(+ \times 1/e)$, by definition has objects $(\bullet \in \underline{\mathbb{N}} \times \underline{\mathbb{N}}, (a, b) \colon \bullet \to \bullet)$, of which the information we can just condense to $(a, b) \in \mathbb{N} \times \mathbb{N}$. A morphism $(a, b) \to (c, d)$ will be a morphism (j, k, l) in $\underline{\mathbb{N}} \times \underline{\mathbb{N}} \times \underline{\mathbb{N}}$ such that $(+ \times 1)(j, k, l) = (a - c, b - d)$-i.e. that $j + k + c = a$ and $l + d = b$.

Hence, if the pasting of (5.2.5) and (5.2.6) can be shown to be homotopy exact, as (5.2.6) we see that (5.2.5) is homotopy exact. This pasting of (5.2.5) and (5.2.6) looks like

$$\begin{array}{ccccc}
(+\times 1)/e & \xrightarrow{pr} & \underline{\mathbb{N}}\times\underline{\mathbb{N}}\times\underline{\mathbb{N}} & \xrightarrow{1\times +} & \underline{\mathbb{N}}\times\underline{\mathbb{N}} \\
{\scriptstyle\pi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle +\times 1} & \overset{\mathrm{Id}}{\swarrow} & \downarrow{\scriptstyle +} \\
e & \xrightarrow[i]{} & \underline{\mathbb{N}}\times\underline{\mathbb{N}} & \xrightarrow{+} & \underline{\mathbb{N}}
\end{array} \tag{5.2.7}$$

Then we can whisker the natural transformations, to make this a single square with natural transformation. Below, we take the functor p to be the composition of the top line in (5.2.7):

$$(+\times 1)/e \to \underline{\mathbb{N}}\times\underline{\mathbb{N}}\times\underline{\mathbb{N}} \to \underline{\mathbb{N}}\times\underline{\mathbb{N}}$$

$$\begin{array}{ccc}
(+\times 1)/e & \xrightarrow{p} & \underline{\mathbb{N}}\times\underline{\mathbb{N}} \\
{\scriptstyle\pi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle +} \\
e & \xrightarrow[i]{} & \underline{\mathbb{N}}
\end{array} \tag{5.2.8}$$

It is not clear why this square would be homotopy exact given its current description. From (Der4) we consider the square

$$\begin{array}{ccc}
(+/e) & \xrightarrow{pr} & \underline{\mathbb{N}}\times\underline{\mathbb{N}} \\
{\scriptstyle\pi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle +} \\
e & \xrightarrow[i]{} & \underline{\mathbb{N}}
\end{array} \tag{5.2.9}$$

which we know to be homotopy exact. Here an object of the category $(+/e)$ has objects $(\bullet \in \underline{\mathbb{N}}, m\colon \bullet \to \bullet)$, information that we can just condense to $m \in \mathbb{N}$. A morphism $(m) \to (n)$ is a morphism (i,j) in $\underline{\mathbb{N}}\times\underline{\mathbb{N}}$ such that $+(i,j)) = m - n$. We would like to write (5.2.8) as a pasting of (5.2.9) with another square, i.e. obtain a pasting of the form

$$
\begin{array}{ccccc}
(+\times 1)/e & \xrightarrow{G} & (+/e) & \xrightarrow{pr} & \underline{\mathbb{N}}\times\underline{\mathbb{N}} \\
\Big\downarrow{\scriptstyle \pi} & \overset{\alpha}{\swarrow} & \Big\downarrow{\scriptstyle \pi} & \overset{\mathrm{Id}}{\swarrow} & \Big\downarrow{\scriptstyle +} \\
e & \xrightarrow[Id]{} & e & \xrightarrow[i]{} & \underline{\mathbb{N}}
\end{array}
\tag{5.2.10}
$$

such that $\mathrm{pr}_{(+/e)} \circ G = (1\times+) \circ \mathrm{pr}_{(+\times 1)/e}$, *i.e.* we want to find a functor $G : (+\times 1/e) \to (+/e)$ such that the below square commutes:

$$
\begin{array}{ccc}
(+\times 1/e) & \xrightarrow{G} & (+/e) \\
{\scriptstyle pr}\Big\downarrow & & {\scriptstyle pr}\Big\downarrow \\
\underline{\mathbb{N}}\times\underline{\mathbb{N}}\times\underline{\mathbb{N}} & \xrightarrow{1\times+} & \underline{\mathbb{N}}\times\underline{\mathbb{N}}
\end{array}
$$

Upon further examination, such a functor will be induced by $(1\times+)$ in the following way; taking an object (a, b) in $(+\times 1/e)$ to $(a+b) \in (+/e)$, and a morphism $(j, k, l)\colon (a, b) \to (c, d)$ to $(j, k+l)\colon (a+b) \to (c+d)$. Checking that the above commutes tells us that G is precisely what is required. Thus, we can re-write (5.2.8) in the guise of (5.2.10).

The right-hand square of this pasting (5.2.10) is homotopy exact by (Der4), so it simply suffices to prove the left-hand square is homotopy exact. This most obvious step would be to check that the the functor G is a right adjoint, by [Gro13, Proposition 1.24], but this fails. Instead, let us denote $(+\times 1/e) = \mathcal{C}$, $(+/e) = \mathcal{D}$, and let $\mathcal{C}_0 \subset \mathcal{C}$ be the full subcategory with objects $(a, 0)$. We form the pasted square

$$
\begin{array}{ccccc}
\mathcal{C}_0 & \xrightarrow{i_0} & \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\
{\scriptstyle \pi}\Big\downarrow & \overset{\mathrm{Id}}{\swarrow} & \Big\downarrow{\scriptstyle \pi} & \overset{\mathrm{Id}}{\swarrow} & \Big\downarrow{\scriptstyle \pi} \\
e & \xrightarrow[1]{} & e & \xrightarrow[1]{} & e
\end{array}
\tag{5.2.11}
$$

Here, I claim that both the inclusion $i_0 : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ and $G \circ i_0$ are right adjoints. This will prove that both the left-hand square and the pasting are homotopy exact squares, and hence that the right-hand square is. We detail the respective adjunctions.

The left adjoint L to $i_0 : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ takes $(a,b) \in \mathcal{C}$ to $(a,0) \in \mathcal{C}_0$ and a map $(j,k,l)\colon (a,b) \to (c,d)$ to $(j,k,0) : (a,0) \to (c,0)$. So we take $(a,b) \in \mathcal{C}$ and $(c,0) \in \mathcal{C}_0$, then $\mathrm{Hom}_{\mathcal{C}_0}((a,0),(c,0)) \cong \mathrm{Hom}_{\mathcal{C}}((a,b),(c,0))$. The former consists of maps of the form $(j,k,0)$ with $j+k+c=a$, while the latter consists of maps of the form (j,k,b) with $j+k+c=a$, rendering an obvious bijection.

The left adjoint F to $\mathcal{C}_0 \hookrightarrow \mathcal{C} \to \mathcal{D}$ takes $(m) \in \mathcal{D}$ to $(m,0) \in \mathcal{C}_0$ and a map $(a,b) : (m) \to (n)$ to $(a,b,0) : (a,0) \to (c,0)$. Take $(a,0) \in \mathcal{C}_0$ and $(n) \in \mathcal{D}$. Then $\mathrm{Hom}_{\mathcal{C}_0}((n,0),(a,0)) \cong \mathrm{Hom}_{\mathcal{D}}((n),(a))$, as the former consists of maps $(j,k,0)$ where $j+k+a=n$, while the latter consists of maps (j,k) where $j+k+a=n$, with the obvious bijection. Therefore, both $i_0 : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ and $G \circ i_0$ are both right adjoints.

Therefore, the square

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{G} & \mathcal{D} \\
{\scriptstyle \pi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle \pi} \\
e & \xrightarrow[1]{} & e
\end{array}
\tag{5.2.12}
$$

is homotopy exact. This implies that (5.2.10) is homotopy exact, which is the same square as (5.2.8). Therefore, our original square (5.2.5) is homotopy exact, and we rewrite it below:

$$
\begin{array}{ccc}
\underline{\mathbb{N}} \times \underline{\mathbb{N}} \times \underline{\mathbb{N}} & \xrightarrow{1\times +} & \underline{\mathbb{N}} \times \underline{\mathbb{N}} \\
{\scriptstyle +\times 1}\downarrow & \overset{\mathrm{Id}}{\swarrow} & \downarrow{\scriptstyle +} \\
\underline{\mathbb{N}} \times \underline{\mathbb{N}} & \xrightarrow[+]{} & \underline{\mathbb{N}}
\end{array}
\tag{5.2.13}
$$

Taking the respective adjoints, we have an isomorphism

$$(+ \times 1)_!(1 \times +)^* \cong +^* +_! \,.$$

□

Now we can complete the proof of the proposition.

Proof. From the previous lemma we have

$$\begin{aligned} +^*X &= +^*(i_!\mathbb{1}_{\mathbb{D}} \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} X) \\ &= +^* +_! (i_!\mathbb{1}_{\mathbb{D}} \boxtimes_{\mathbb{D}} X) \end{aligned}$$

As $+^*+_! \cong (+\times 1)_!(1\times +)^*$, we have

$$+^* +_! (i_!\mathbb{1}_{\mathbb{D}} \boxtimes_{\mathbb{D}} X) \cong (+\times 1)_!(1\times +)^*(i_!\mathbb{1}_{\mathbb{D}} \boxtimes_{\mathbb{D}} X).$$

We also have that

$$(1\times i)_!X \boxtimes_{\mathbb{A}^1_{\mathbb{A}^1_{\mathbb{D}}}} +^*i_!\mathbb{1}_{\mathbb{D}} = (+\times 1)_!(1\times +)^*(X \boxtimes_{\mathbb{D}} i_!\mathbb{1}_{\mathbb{D}}).$$

The proof of the theorem will rely on the two isomorphic representations of $+^*X$ that we have produced. ☐

Lastly, we can tackle the proof of the Theorem.

Proof. We know that $F^{\underline{\mathbb{N}}}((1\times i)_!X) = F_0^{\underline{\mathbb{N}}}X$, while $F^{\underline{\mathbb{N}}}(+^*i_!\mathbb{1}_{\mathbb{D}}) = (\mathbb{1}_{\mathbb{E}}, \alpha)$ by characterization of having type α. Therefore, $F^{\underline{\mathbb{N}}}(+^*X) = F_0^{\underline{\mathbb{N}}}X \boxtimes_{\mathbb{A}^1_{\mathbb{E}}} (\mathbb{1}_{\mathbb{E}}, \alpha)$. Thus,

$$\begin{aligned} F(X) &\cong i^*F^{\underline{\mathbb{N}}}(+^*X) \\ &\cong i^*F^{\underline{\mathbb{N}}}(i_!X \boxtimes_{\mathbb{A}^1_{\mathbb{A}^1_{\mathbb{D}}}} (+^*(i_!\mathbb{1}_{\mathbb{D}}))) \\ &\cong i^*(F^{\underline{\mathbb{N}}}(i_!X) \boxtimes_{\mathbb{A}^1_{\mathbb{E}}} F^{\underline{\mathbb{N}}}(+^*\mathbb{1}_{\mathbb{A}^1_{\mathbb{D}}})) \\ &= i^*(F_0^{\underline{\mathbb{N}}}X \boxtimes_{\mathbb{A}^1{}_{\mathbb{E}}} (\mathbb{1}_{\mathbb{E}}, \alpha)) \\ &= \mathrm{ev}_\alpha F_0^{\underline{\mathbb{N}}}X \end{aligned}$$

Above, the first isomorphism $F(X) \cong i^*F^{\underline{\mathbb{N}}}(+^*X)$ is since F is a morphism of derivators. The second isomorphism is the decomposition

$$+^*X = +^*i_!\mathbb{1}_{\mathbb{D}} \boxtimes_{\mathbb{A}^1_{\mathbb{A}^1_{\mathbb{D}}}} (1\times i)_!X,$$

and the third just follows by co-continuity of F.

Our very last equality is precisely the definition of the evaluation at α morphism.

Therefore, cocontinuous monoidal morphisms of derivators $F : \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$ can be determined simply by their base F_0 and their type α, in that a morphism F with designated base F_0 and type α is simply $\mathrm{ev}_\alpha F_0^{\mathbb{N}}$. □

Definition 5.2.14. Let $\mathbb{D}$, $\mathbb{E}$ be two derivators. Let $PDER(\mathbb{D}, \mathbb{E})$ denote the category whose objects are strong morphisms of prederivators $\mathbb{D} \to \mathbb{E}$ and morphisms are modifications.

Let $PDER_!(\mathbb{D}, \mathbb{E})$ denote the full category whose objects are cocontinuous morphisms of derivators $\mathbb{D} \to \mathbb{E}$. Similarly, if $\mathbb{D}$ and $\mathbb{E}$ are monoidal derivators, then we can let $PDER_\otimes(\mathbb{D}, \mathbb{E})$ denote the category of monoidal morphisms and pseudonatural transformations. Finally, if we are looking at $PDER(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E})$ we can let $PDER_{\otimes,!,\mathbb{D}}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E})$ denote the cocontinuous, monoidal morphisms $F : \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$ such that $F \circ i_!$ is a given morphism $\mathbb{D} \to \mathbb{E}$.

Theorem 5.2.15. *We describe* (5.2.1) *via a more global perspective.*

1. *There exists a functor*

$$PDER_{\otimes,!}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E}) \to \mathbb{E}(\underline{\mathbb{N}})$$

that sends a cocontinuous, monoidal morphism $F\colon \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$ *to* $F(+^* i_! \mathbb{1}_{\mathbb{D}})$.

2. *Fix a cocontinuous monoidal morphism of derivators* $F_0\colon \mathbb{D} \to \mathbb{E}$. *If we restrict to cocontinuous monoidal morphisms with base* F_0, *this induces an equivalence of categories*

$$PDER_{\otimes,!,\mathbb{D}}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E}) \cong \{X : i^* X = \mathbb{1}_{\mathbb{E}}\} \subset \mathbb{A}^1_{\mathbb{E}}(e).$$

Call the latter category $\mathbb{E}_{\mathbb{1}}$, *which we can also think of as all the coherent endomorphisms of the identity in* $\mathbb{E}$. *Here* $PDER_{\otimes,!,\mathbb{D}}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E})$ *consists of cocontinuous, strong monoidal morphisms* F *of derivators between* $\mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$ *with* $Fi_!$ *equal to some fixed* F_0. *In the above equivalence, the forward direction functor*

$$PDER_{\otimes,!,\mathbb{D}}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E}) \to \mathbb{E}_{\mathbb{1}}$$

is the functor in the first part. Its inverse takes $(\mathbb{1}, \alpha)$ *to* $ev_\alpha \circ F_0^{\mathbb{N}}$.

3. *Alternatively, we have an equivalence of categories*

$$PDER_{\otimes,!}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E}) \cong \mathbb{E}_{\mathbb{1}} \times PDER_{\otimes,!}(\mathbb{D}, \mathbb{E}).$$

The forward direction functor splits F into the information of its type α and its base F_0. Its inverse takes a coherent endomorphism $(\mathbb{1}_{\mathbb{E}}, \alpha)$ plus a cocontinuous, monoidal morphism $F_0 \colon \mathbb{D} \to \mathbb{E}$ to $ev_\alpha F_0^{\underline{\mathbb{N}}}$.

Proof. 1. The first part is clear: every pseudonatural transformation between two morphisms $F, G : \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$ gives a morphism in $\mathbb{E}(\underline{\mathbb{N}})$,

$$F(+^*i_!\mathbb{1}_{\mathbb{D}}) \to G(+^*i_!\mathbb{1}_{\mathbb{D}}).$$

2. It clear by (5.2.1) that the functor in part (1) can have its codomain restricted to to $\mathbb{E}_{\mathbb{1}}$.

   The two functors between $PDER_{\otimes,!,\mathbb{D}}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E})$ and $\mathbb{E}_{\mathbb{1}}$ are as follows. Given $F : \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$, we send it to $F_{\underline{\mathbb{N}}}(+^*i_!\mathbb{1}_{\mathbb{D}}) \in \mathbb{E}_{\mathbb{1}}$. For a monoidal natural transformation $F \to F'$, we send it to the morphism

$$F^{\underline{\mathbb{N}}}(+^*i_!\mathbb{1}_{\mathbb{D}}) \to F'^{\underline{\mathbb{N}}}(+^*i_!\mathbb{1}_{\mathbb{D}}).$$

   Conversely, given $(\mathbb{1}, \alpha) \in \mathbb{E}_{\mathbb{1}}$ we send it to $\mathrm{ev}_\alpha \circ F_0^{\underline{\mathbb{N}}} \colon \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}$, which we know to be in $PDER_{\otimes,!,\mathbb{D}}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E})$. Given a morphism $(\mathbb{1}, \alpha) \to (\mathbb{1}, \beta) \in \mathbb{E}_{\mathbb{1}}$, there is the corresponding monoidal natural transformation $\mathrm{ev}_\alpha \to \mathrm{ev}_\beta$. Recall the definition of the evaluation at α morphism, (4.3.3); given a morphism $g \colon (\mathbb{1}, \alpha) \to (\mathbb{1}, \beta)$ there is a corresponding morphism, $i^*(\mathrm{Id}_X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} g)$ from

$$i^*(X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (\mathbb{1}, \alpha)) \to i^*(X \boxtimes_{\mathbb{A}^1_{\mathbb{D}}} (\mathbb{1}, \beta))$$

   and these paste to become a monoidal natural transformation $\mathrm{ev}_\alpha \to \mathrm{ev}_\beta$. Hence we also obtain a monoidal natural transformation $\mathrm{ev}_\alpha F_0^{\underline{\mathbb{N}}} \to \mathrm{ev}_\beta F_0^{\underline{\mathbb{N}}}$.

   The compositions

$$PDER_{\otimes,!,\mathbb{D}}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E}) \to \mathbb{E}_{\mathbb{1}} \to PDER_{\otimes,!,\mathbb{D}}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E})$$

$$\mathbb{E}_{\mathbb{1}} \to PDER_{\otimes,!,\mathbb{D}}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E}) \to \mathbb{E}_{\mathbb{1}}$$

are both seen to be isomorphisms, showing that there is actually an equivalence between the two categories.

3. As in the previous part, given $F \in PDER_{\otimes,!}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E})$ we map it to

$$(F(+^*i_!\mathbb{1}_{\mathbb{D}}), F \circ i_!).$$

For a modification $F \to G$ in $PDER_{\otimes,!}(\mathbb{A}^1_{\mathbb{D}}, \mathbb{E})$, wesend it to the morphisms $F(+^*i_!\mathbb{1}_{\mathbb{D}}) \to G(+^*i_!\mathbb{1}_{\mathbb{D}})$ and $F \circ i_! \to G \circ i_!$ in the categories $\mathbb{E}_{\mathbb{1}}$ and $PDER_{\otimes,!}(\mathbb{D}, \mathbb{E})$ respecitvely.

In the opposite direction, we take a pair $(\mathbb{1}_{\mathbb{E}}, \alpha)$ and $F_0\colon \mathbb{D} \to \mathbb{E}$ and send it to

$$\mathrm{ev}_\alpha \circ F_0^{\mathbb{N}}\colon \mathbb{A}^1_{\mathbb{D}} \to \mathbb{E}.$$

Given a pair of morphisms

$$(\mathbb{1}_{\mathbb{E}}, \alpha) \to (\mathbb{1}_{\mathbb{E}}, \beta), F_0 \to G_0,$$

we have a transformation

$$\mathrm{ev}_\alpha F_0^{\mathbb{N}} \to \mathrm{ev}_\alpha G_0^{\mathbb{N}} \to \mathrm{ev}_\beta G_0^{\mathbb{N}}$$

where the first arrow is given by the transformation $F_0 \to G_0$ and the second induced by the transformation $\mathrm{ev}_\alpha \to \mathrm{ev}_\beta$ as described in the previous part. From (5.2.1), we know that these two functors are essential inverses to each other.

□

We note one special case below, when the base is the identity.

Corollary 5.2.16. *The only monoidal morphisms $F\colon \mathbb{A}^1_{\mathbb{D}} \to \mathbb{D}$ that $F \circ i_!$ is the identity are the* evaluation at α *morphisms ev_α.*

Indeed it's clear by the definition of the evaluation at α morphism that each one is a section to the structure morphism $i_!$. Now we have seen that these are the only sections.

5.3 Universal property of affine space $\mathbb{A}^n$

Now we aim to extend this result to $\mathbb{A}^n_{\mathbb{D}}$ in a natural way. Let us fix the following notation.

1. Let $i_n\colon \underline{\mathbb{N}}^{n-1} \to \underline{\mathbb{N}}^n$ denote the functor $1_{\underline{\mathbb{N}}^{n-1}} \times i$

2. Let $0 \leq m < n$ be two integers. Let $i_{m,n}$ denote the composition

$$i_n \circ i_{n-1} \circ \cdots \circ i_{m+1}\colon \underline{\mathbb{N}}^m \to \underline{\mathbb{N}}^n$$

3. Let $F\colon \mathbb{A}^n_{\mathbb{D}} \to \mathbb{E}$ be a cocontinuous monoidal morphism of derivators. Let $F_{m,n}$ denote the morphism

$$F \circ (i_{m,n})_!\colon \mathbb{A}^m_{\mathbb{D}} \to \mathbb{E}$$

for $0 \leq m < n$.

We generate the universal property for $\mathbb{A}^n$ essentially by iterating the universal property for $\mathbb{A}^1$ along the inclusions i_k. Note that if we think of $\mathbb{A}^n_{\mathbb{D}}$ as $\mathbb{A}^1_{\mathbb{A}^{n-1}_{\mathbb{D}}}$, then with the above notation $F_{n-1,n}$ is the base of the morphism of derivators $F\colon \mathbb{A}^n_{\mathbb{D}} \to \mathbb{E}$, and more generally $F_{m-1,n}$ is the base of the morphism

$$F_{m,n}\colon \mathbb{A}^m_{\mathbb{D}} \to \mathbb{E}.$$

So we can use the universal property of $\mathbb{A}^1$ ((5.2.1), (5.2.15)) to bootleg up to $\mathbb{A}^n$.

Corollary 5.3.1. *We have the following analogues of* (5.2.15)*.*

1. *There is a functor*

$$PDER_{\otimes,!}(\mathbb{A}^n_{\mathbb{D}}, \mathbb{E}) \to \Pi_n \mathbb{E}(\underline{\mathbb{N}}),$$

sending a cocontinuous monoidal morphism $F \cong \mathbb{A}^n_{\mathbb{D}} \to \mathbb{E}$ to the product of the types of $F_{m,n}$ for each $1 \leq m \leq n$, from the universal property of $\mathbb{A}^1$.

2. *Fix a cocontinuous, monoidal morphism of derivators $G\colon \mathbb{D} \to \mathbb{E}$. Consider the subcategory*

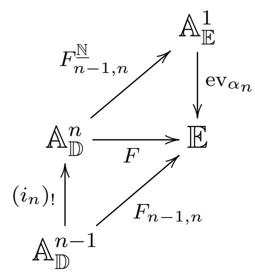
$$PDER_{\otimes,!,\mathbb{D}}(\mathbb{A}^n_{\mathbb{D}}, \mathbb{E}) \subset PDER_{\otimes,!}(\mathbb{A}^n_{\mathbb{D}}, \mathbb{E})$$

consisting of morphisms F with $F_{0,n} = G$. The functor in part (1) induces an equivalence of categories with $\Pi_n \mathbb{E}_1$.

3. *From (2) we have an equivalence of categories $PDER_{\otimes,1}(\mathbb{A}^n_{\mathbb{D}}, \mathbb{E})$ with the category*

$$PDER_{\otimes,!}(\mathbb{D}, \mathbb{E}) \times \Pi_n \mathbb{E}_1.$$

Proof. First we will describe how to generate the product $\Pi_n \mathbb{E}_1$ by induction. The main theorem (5.2.15) is the case for $n = 1$. Inductively, note that we have the following commutative diagram of two commuting triangles:



The ev_{α_n} is obtained from the universal property of $\mathbb{A}^1$ on for the derivator $\mathbb{A}^{n-1}_{\mathbb{D}}$, which gives the commutativity of the top triangle. So for $\mathbb{A}^n$ we get the information of n evaluation at α morphisms giving us the requisite functor for (1).

For (2) it's clear that one can restrict the codomain of the functor in (1) to $\Pi_n \mathbb{E}_1$. Note that suppose we are given a cocontinuous monoidal morphism $F \colon \mathbb{A}^n_{\mathbb{D}} \to \mathbb{E}$ with $F_{0,n} = G$. Then $F_{n-1,n} \colon \mathbb{A}^{n-1}_{\mathbb{D}} \to \mathbb{E}$ again has base G and we can write $F = \mathrm{ev}_{\alpha_n} \circ F^{\mathbb{N}}_{n-1,n}$. Then one can write $F_{n-1,n} = \mathrm{ev}_{\alpha_{n-1}} \circ F^{\mathbb{N}}_{n-2,n}$, and so forth.

So given a product $\Pi^n_{i=1}(\mathbb{1}_{\mathbb{E}}, \alpha_i)$ and a morphism G which should be equal to $F_{0,n}$ for some $F \colon \mathbb{A}^n_{\mathbb{D}} \to \mathbb{E}$, we can recursively define $F_{k+1,n}$ as $\mathrm{ev}_k \circ (F_{k,n})^{\mathbb{N}}$, until we get to $F_{n,n}$ which is simply our desired morphism of derivators.

This gives the inverse to our stated equivalence in part (2), while (3) is simply a re-writing of (2). □

For us, this is the most telling signal that the definition of $\mathbb{A}^1_{\mathbb{D}}$ and more generally $\mathbb{A}^n_{\mathbb{D}}$ is a reasonable one. It mirrors behavior that we would expect polynomial algebras over a commutative ring or the affine spaces over a reasonable scheme to have.

CHAPTER 6

Derivators shifted by other monoids

Here we consider the general case where M is a cancellative, abelian monoid with unit. We can consider the associated category $\underline{M}$, which has one object $\bullet$ and endomorphism monoid M. For a derivator $\mathbb{D}$, we can form the derivator $\mathbb{D}^{\underline{M}}$ and ask whether it has similar properties as outlined for $\mathbb{A}^1_{\mathbb{D}}$. Generally speaking, the answer is yes, and below we outline four main aspects of the relationship between $\mathbb{D}^{\underline{M}}$ and $\mathbb{D}$:

1. Agreement: i.e. if $\mathbb{D}$ is the derivator $\mathbb{D}_R$ associated to a commutative ring R, then $\mathbb{D}^{\underline{M}}$ is the derivator $\mathbb{D}_{R[M]}$ associated to the (commutative) monoid ring $R[M]$

2. Canonical morphisms: that $(i_M)_!$ and $(p_M)_!$ where i_M and p_M are the canonical monoid maps from a point into M and M to a point, correspond to extension of scalars along the ring homomorphisms $R \hookrightarrow R[M]$ and $R[M] \to R$, where the latter map sends every element of M to 1

3. Monoidal structure: the verification that if $\mathbb{D}$ is a symmetric monoidal derivator, then the Day convolution formula gives a symmetric monoidal structure on $\mathbb{D}^{\underline{M}}$ such that if $\mathbb{D}$ is the derivator associated to a commutative ring R and its monoidal product is induced by $- \otimes_R -$, then the monoidal structure on $\mathbb{D}^{\underline{M}}$ is induced by $- \otimes_{R[M]} -$

4. Universal property: the verification that a cocontinuous, monoidal morphism of derivators

$$F : \mathbb{D}^{\underline{M}} \to \mathbb{E}$$

can be determined by its pre-composition with $(i_M)_!$ and an element in $X \in \mathbb{E}(\underline{M})$ such that $(i_M)^* X = \mathbb{1}_{\mathbb{E}}$.

In the following subsections we will outline why each aspect is true. All proofs are very similar to the proofs for $\mathbb{A}^1_{\mathbb{D}}$ or $M = \mathbb{N}$ and can be obtained by simply substituting M for $\mathbb{N}$ in the relevant steps, so we do not repeat them for the sake of brevity.

6.1 Agreement

As with the case of $\mathbb{A}^1$, we have an isomorphism on categories of modules,

$$R\text{-Mod}^{\underline{M}} \cong R[M]\text{-Mod}\,.$$

Therefore, it follows also that we have an isomorphism on the chain complexes,

$$\text{Ch}(R\text{-Mod}^{\underline{M}}) \cong \text{Ch}(R[M]\text{-Mod}).$$

Moreover, the quasi-isomorphisms in both categories are the same. We recall the proofs in [BZ17], specifically [BZ17, Theorem 5] with $\mathbb{N}$ replaced by M and the general [**?**, Lemma 8], touching on the first and secon assertions above.

Example We give some basic examples of shifts by monoids that produce meaningful results.

1. If $M = \mathbb{N}^n$, then we recover affine space constructions of the previous two chapters.

2. If M is an abelian group G, then $R\text{-Mod}^{\underline{M}} \cong R[G]\text{-Mod}$, with potential representation-theoretic value. Again, here the Day convolution product is the usual tensor of $R[G]$-modules, while the usual shifted tensor product gives the tensor product of underlying R-modules with the diagonal G-action.

3. Again if M is an abelian group G, and Top denotes a model for topological spaces, then Top^G consists of G-equivariant topological spaces. Top can be replaced with pointed spaces, a model for spectra, etc, and in the case of spectra we obtain naive G-equivariant spectra.

6.2 Canonical morphisms

Here we note merely that the structure morphism and the evaluation at 1 morphism are constructed in a similar fashion.

Proposition 6.2.1. *The structure morphism for* $\mathbb{D}^{\underline{M}}$ *is*

$$(i_M)_! : \mathbb{D} \to \mathbb{D}^{\underline{M}}.$$

The adjoint of this functor, $(i_M)^*$, is the morphism of derivators induced by the forgetful functor

$$R[M]\text{-Mod} \to R\text{-Mod}$$

that takes an $R[M]$-module and forgets all structure apart from the R-module. Equivalently it is the restriction of scalars functor along $R \hookrightarrow R[M]$. Hence $(i_M)_!$ is the extension of scalars functor along $R \hookrightarrow R[M]$.

Proposition 6.2.2. *The "evaluation at 1" morphism for* $\mathbb{D}^{\underline{M}}$ *is*

$$(p_M)_! : \mathbb{D}^{\underline{M}} \to \mathbb{D}.$$

Again, the right adjoint $(p_M)^*$ is the morphism of derivators induced by restriction of scalars along the ring homomorphism $R[M] \to R$, sending each $m \in M$ to 1 and is the identity on R. So its left adjoint is the extension of scalars along the same homomorphism.

6.3 Monoidal structures

We first define the monoidal structure on $\mathbb{D}^{\underline{M}}$. Recall that we can think of $\underline{M}$ as a symmetric monoidal category; it has only one object, and the tensor product of maps $m_1, m_2 \in M$ is the composition $m_1 + m_2 = m_2 + m_1 \in M$. Then we can use a construction akin to Day convolution:

Proposition 6.3.1. *Let* $(\mathbb{D}, \boxtimes)$ *be a symmetric monoidal derivator. Then* $(\mathbb{D}^{\underline{M}}, \boxtimes_{\mathbb{D}^{\underline{M}}})$ *is a symmetric monoidal derivator where*

$$(X \boxtimes_{\mathbb{D}^{\underline{M}}} Y) = (+_M)_!(X \boxtimes_{\mathbb{D}} Y).$$

The proof that this defines a symmetric monoidal derivator is precisely analoguous to the case of $M = \mathbb{N}$. As for why this is the "correct" construction, we have as usual that if $\mathbb{D}$ is the monoidal derivator associated to a ring R then $\mathbb{D}^{\underline{M}}$ should be the monoidal derivator associated to $R[M]$.

The external product of two elements of $R\text{-Mod}^{\underline{M}}$ gives an object in $R\text{-Mod}^{\underline{M}\times\underline{M}}$, i.e.an $R[M \times M]$-module. The functor $+^*$ induced by

$$+ : M \times M \to M$$

takes an $R[M]$-module and turns it into an $R[M \times M]$-module by pulling back the actions of elements of M along $+$ and hence $+_!$, as the left adjoint, is the correct option.

Definition 6.3.2. We define a coherent M-endomorphism of the unit to be an element X of $\mathbb{D}(\underline{M})$ such that $(i_M)^*X = \mathbb{1}_{\mathbb{D}}$. In terms of the underlying diagram functor these look like

$$(\mathbb{1}_{\mathbb{D}}, \alpha_m : m \in M)$$

such that $\alpha_m \circ \alpha_n = \alpha_{m+n}$.

For each coherent M-endomorphism $(\mathbb{1}_{\mathbb{D}}, \alpha_m : m \in M)$ we can define an evaluation at $\{\alpha_m : m \in M\}$ as follows:

Definition 6.3.3. Define

$$\text{ev}_{\{\alpha_m : m \in M\}} : \mathbb{D}^{\underline{M}} \to \mathbb{D}$$

to take

$$X \mapsto (i_M)^*(+_M)_!(X \boxtimes (\mathbb{1}, \alpha_m : m \in M))$$

The two main properties we need about these morphisms are as follows:

1. ev_{α_m} is a section to $(i_M)_!$
2. ev_{α_m} is a strong monoidal morphism of derivators, when utilizing the Day convolution monoidal structure on $\mathbb{D}^{\underline{M}}$ and the monoidal structure on $\mathbb{D}$.

As usual, the proofs are analoguous to the case for $M = \mathbb{N}$.

6.4 Universal properties

We enumerate the universal property of $\mathbb{D}^{\underline{M}}$ with respect to $\mathbb{D}$ when $\mathbb{D}$ is a symmetric monoidal derivator.

Theorem 6.4.1. *The main result is as follows. Let $\mathbb{D}$, $\mathbb{E}$ be two symmetric monoidal derivators;*

$$F : \mathbb{D}^{\underline{M}} \to \mathbb{E}$$

be a cocontinuous, monoidal functor.

As in the $\mathbb{A}^1$-case, the object $F((+_M)^* \mathbb{1}_{\mathbb{D}^{\underline{M}}})$ is a coherent M-endomorphism of the unit. Let $F_0 = F \circ (i_M)_!$. Then

$$F \cong ev_{F((+_M)^* \mathbb{1}_{\mathbb{D}^{\underline{M}}})} \circ F_0^{\underline{M}}.$$

An equivalent assertion would be that any cocontinuous, monoidal morphism from $\mathbb{D}^{\underline{M}}$ to $\mathbb{E}$ can be characterized by its composition with $(i_M)_!$ and the image of a certain element, namely $(+_M)^* \mathbb{1}_{\mathbb{D}^{\underline{M}}}$.

We omit the proof of this assertion: the discussion with homotopy exact squares for the product of $\mathbb{A}^1$ indicates that having an abelian, cancellative monoid M with unit will be sufficient for the proof; there was nothing special about $\mathbb{N}$ or $\underline{\mathbb{N}}^n$.

CHAPTER 7

Group Completion and Localization

Given a commutative monoid M, it is natural to consider the group completion which we denote as $g(M)$. In our algebro-geometric intuition, going from $\mathbb{D}^{\underline{M}}$ to $\mathbb{D}^{\underline{g(M)}}$ is akin to taking an open subscheme by removing the origin.

We will tackle some general results about the localization theory of derivators, and then restrict to our specific case.

7.1 Localization Theory of Derivators

We will give some more specialized methods for localization of derivators, in relation to the shifted derivators we have discussed in the previous three chapters.

Here we give an attempt to summarize the theory of *localizations of derivators*. For a discussion of localizations of categories, see [GZ67]. In CAT, the mechanism is simple: we have a class of morphisms $\mathcal{W} \subset C^{[1]}$ satisfying reasonable properties for some category C, and we ask whether there is a category $C[\mathcal{W}^{-1}]$ along with a functor $L : C \to C[\mathcal{W}^{-1}]$ inverting all morphisms in $\mathcal{W}$ and admitting a fully faithful right adjoint.

In this case both the functor L and the category $C[\mathcal{W}^{-1}]$ are unique up to equivalence as they satisfy a universal property in CAT. While more special localization theories exist, for example with model categories or triangulated categories, we will stick to a base model of localization that is applicable to more derivators. Later we will make some generalizations from the case of triangulated categories to triangulated derivators.

Proposition 7.1.1. *[Col19, Proposition 3.5] Let $F : \mathbb{D} \to \mathbb{E}$ be a localization of derivators,*

$\mathcal{W}_F \subset \mathbb{D}(e)^{[1]}$ *be the collection of morphisms in* $\mathbb{D}(e)$ *such that* $F(f)$ *is an isomorphism. Let* $\mathbb{T}$ *be another derivator. Then precomposition by* F*,*

$$F^* : PDER(\mathbb{E}, \mathbb{T}) \to PDER(\mathbb{D}, \mathbb{T})$$

is fully faithful with essential image consisting of morphisms $\xi : \mathbb{D} \to \mathbb{T}$ *such that* ξ_e *sends every morphism in* $\mathcal{W}_F$ *to an isomorphism.*

Remark 7.1.2. As usual let $\mathrm{PDER}_!(\mathbb{D}, \mathbb{E}) \subset \mathrm{PDER}(\mathbb{D}, \mathbb{E})$ denote the full subcategory consisting of cocontinuous morphisms. A subscript $\mathcal{W}_F$ still denotes the subcategory of morphisms inverting $\mathcal{W}_F$ on the base.

Proposition 7.1.3. *[Col19, Proposition 3.7] In the situation of the above Proposition, if we restrict to the category of cocontinuous morphisms we obtain an equivalence*

$$F^* : PDER_!(\mathbb{E}, \mathbb{T}) \simeq PDER_{!,\mathcal{W}_F}(\mathbb{D}, \mathbb{T}).$$

Moreover, we can give the inverse.

Lemma 7.1.4. *If* $F : \mathbb{D} \to \mathbb{E}$ *is a localization of derivators with fully faithful right adjoint* G*, then*

$$G^* : PDER_{\mathcal{W}_F}(\mathbb{D}, \mathbb{T}) \to PDER(\mathbb{E}, \mathbb{T})$$

is a quasi-inverse to F^**. This equivalence also restricts to the collection of cocontinuous morphisms.*

Moreover it is also clear that we have categorical localizations:

Proposition 7.1.5. *If* $F : \mathbb{D} \to \mathbb{E}$ *is a localization of derivators by the collection* $\mathcal{W}_F$ *in* $\mathbb{D}(e)^{[1]}$*, then* $F_I : \mathbb{D}(I) \to \mathbb{E}(I)$ *is a localization of categories by the collection* $\mathcal{W}_F(I)$*.*

Normally it may not be necessary to know what the collection of inverted morphisms is. In the future we hope to give in the case of $\mathbb{D}$ a stable derivator what the subcategory being quotiented out is on the base. The associated subderivator will be the analogue of a "subcategory supported on a closed subset."

7.2 $\mathbb{A}^1$ and $\mathbb{G}_m$

Recall that we have defined $\mathbb{A}^1_{\mathbb{D}}$ of a derivator as $\mathbb{D}^{\underline{\mathbb{N}}}$, using the intuition of the scheme case. Similarly, we may define $\mathbb{G}_m$ of a derivator as follows.

Definition 7.2.1. The punctured affine line $\mathbb{G}_m$ of a derivator $\mathbb{D}$ is $\mathbb{D}^{\underline{\mathbb{Z}}}$, where $\underline{\mathbb{Z}}$ is the category with one object $\bullet$ and $\text{Hom}(\bullet, \bullet) = (\mathbb{Z}, +)$.

The intuition for this definition is like the definition for the affine line: given a ring R, $\mathbb{G}_{m,R}$ is defined as Spec$R[t, t^{-1}]$. We note that $R[t, t^{-1}]$-Mod $\cong R$-Mod$^{\mathbb{Z}}$, indicating that a shift by $\mathbb{Z}$ is the correct intuition for $(\mathbb{G}_m)_{\mathbb{D}}$.

Let $g_{\mathbb{N}}$ denote the categorification of the usual monoid map $\mathbb{N} \hookrightarrow \mathbb{Z}$. Note that

$$(g_{\mathbb{N}})^* : R[t, t^{-1}]\text{-Mod} \to R[t]\text{-Mod}$$

is the restriction of scalars functor along the usual inclusion $R[t] \hookrightarrow R[t, t^{-1}]$. Thus the left adjoint $(g_{\mathbb{N}})_!$ presents the extension of scalars functor along $R[t] \hookrightarrow R[t, t^{-1}]$. As usual, we note that this is a monoidal functor and that it extends to a cocontinuous monoidal morphism of derivators

$$(g_{\mathbb{N}})_! : \mathbb{D}_{R[t]} \to \mathbb{D}_{R[t,t^{-1}]}.$$

Proposition 7.2.2. *The morphism of derivators $(g_{\mathbb{N}})_!$ is a localization of derivators.*

Proof. A result of Cisinski indicates that if one has a left adjoint morphism of derivators $L : \mathbb{D} \to \mathbb{D}'$ and that $R : \mathbb{D}' \to \mathbb{D}$ is a fully faithful right adjoint, then L is a localization. In our case, $(g_{\mathbb{N}})_!$ has right adjoint $(g_{\mathbb{N}})^*$, so we merely need to show that it is fully faithful.

This is equivalent to showing that $(g_{\mathbb{N}})_!(g_{\mathbb{N}})^* \cong Id$. Equivalently, we show that

$$i^*(g_{\mathbb{N}})_!(g_{\mathbb{N}})^* \cong i^*.$$

But we can understand $i^*(g_{\mathbb{N}})_!$ via a (Der4) square as follows:

$$\begin{array}{ccc} (g_{\mathbb{N}}/\bullet) & \xrightarrow{\text{pr}} & \underline{\mathbb{N}} \\ {\scriptstyle\pi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle g_{\mathbb{N}}} \\ e & \xrightarrow[i]{} & \underline{\mathbb{Z}} \end{array}$$

telling us that $i^*(g_{\mathbb{N}})_! \cong (\pi_{g_{\underline{\mathbb{N}}}/\bullet})_! \circ \mathrm{pr}^*$. Thus

$$i^*(g_{\mathbb{N}})_!(g_{\mathbb{N}})^* \cong (\pi_{g_{\underline{\mathbb{N}}}/\bullet})_! \mathrm{pr}^* \circ (g_{\mathbb{N}})^*.$$

However, let us actually consider $\mathrm{pr}^*(g_{\mathbb{N}})^*(X)$ for some $X \in \mathbb{D}(\underline{\mathbb{Z}})$. First let us construct $(g_{\mathbb{N}}/\bullet)$: this category has objects indexed by $\mathbb{Z}$, corresponding to the maps $\bullet \to \bullet$. We see that there is a map $(m : \bullet \to \bullet) \longrightarrow (n : \bullet \to \bullet)$ if and only if $m \geq n$. Thus $(g_{\mathbb{N}}/\bullet)$ is actually just the poset $(\mathbb{Z}, \geq)$. □

Lemma 7.2.3. *The poset* $(\mathbb{Z}, \geq)$ *has contractible nerve.*

Proof. First we consider the subcategory $(\mathbb{N}, \geq) \subset (\mathbb{Z}, \geq)$. This full subcategory evidently has contractible nerve, as 0 is a terminal object. I claim that the inclusion here is a left adjoint, with right adjoint

$$R : (\mathbb{Z}, \geq) \to (\mathbb{N}, \geq)$$

defined as follows: $R(n) = n$ if $n \geq 0$ and $R(n) = 0$ if $n < 0$.

One can check directly that this constitutes an adjunction. Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. Then $\mathrm{Hom}(inc(m), n)$ consists of a single map if $m \geq n$ and of no maps if $m < n$. Similarly $\mathrm{Hom}(m, R(n))$ consists of a single map if $n < 0$ or if $m \geq n$, and of no maps if $m < n$. Therefore, we have a natural isomorphism

$$\mathrm{Hom}_{(\mathbb{Z},\geq)}(inc(m), n) \cong \mathrm{Hom}_{(\mathbb{N},\geq)}(m, R(n))$$

for all $m \in (\mathbb{N}, \geq)$ and $n \in (\mathbb{Z}, \geq)$.

As mentioned above $(\mathbb{N}, \geq)$ is connected to e via inclusion to the terminal object and the projection to e. Thus, $(\mathbb{Z}, \geq)$ is connected via a series of adjunctions to the terminal category.

This proves that $(\mathbb{Z}, \geq)$ is contractible. □

Proof. Now we continue the proof that our morphism of derivators is a localization. Recall that an object in $\mathbb{D}(\underline{\mathbb{Z}})$ is an underlying object X with a collection of morphisms ω_n for each

$n \in \mathbb{Z}$, such that ω_0 is the identity and $\omega_n \circ \omega_m = \omega_{n+m}$. This implies that each ω_n is an isomorphism. We denote such an object then as $\{X, \omega_n : n \in \mathbb{Z}\}$. Applying $\mathrm{pr}^*(g_{\mathbb{N}})^*$ to this object gives a diagram of the form

$$... \xrightarrow{\omega_1} X \xrightarrow{\omega_1} X \xrightarrow{\omega_1} X \xrightarrow{\omega_1} ...$$

where recall that each ω_1 is an isomorphism. However, this is pointwise equivalent to $\pi^* X$. Recall that $\pi^* X$ is a diagram of the form

$$... \xrightarrow{id} X \xrightarrow{id} X \xrightarrow{id} X \xrightarrow{id} X...$$

The pointwise isomorphism is given as follows: for $n \in \mathbb{Z}$ we have the isomorphism

$$\omega_n : (\pi^* X)_n \to \mathrm{pr}^*(g_{\mathbb{N}})^* \{X, \omega_n : n \in \mathbb{Z}\}$$

One checks that this is a pointwise isomorphism, and hence we can replace $\mathrm{pr}^*(g_{\mathbb{N}})^* \{X, \omega_n : n \in \mathbb{Z}\}$ by $\pi^* X$ in future computations.

Our task then reduces to proving that $\pi_! \pi^* X \cong X$. This is precisely the case if the category $(\mathbb{Z}, \geq)$ is contractible, which was proven above. Therefore, π^* is a fully faithful functor.

Then the same [Cis08, Lemme 4.2] indicates that $\mathbb{D}^{\underline{\mathbb{Z}}}$ will be a localization of $\mathbb{D}^{\underline{\mathbb{N}}}$. □

7.3 The general case

Here we let M be a finitely generated, cancellative, abelian, unital monoid, where we denote the monoid operation as $+$. Let $\{x_1, \cdots, x_k\}$ be a collection of generators (i.e. each element of M can be written as an $\mathbb{N}$-linear combination of $\{x_1, \cdots, x_l\}$), and let $g(M)$ denote the Grothendieck group of M. Then because M is cancellative, the canonical morphism $M \to g(M)$ is an inclusion. Let $g_M : \underline{M} \to \underline{g(M)}$ be the corresponding functor on the categories.

Let $\mathbb{D}$ be a derivator, then we have a cocontinuous morphism of derivators

$$(g_M)_! : \mathbb{D}^{\underline{M}} \to \mathbb{D}^{\underline{g(M)}}.$$

Theorem 7.3.1. *This morphism of derivators $(g_M)_!$ is a localization of derivators.*

The proof of this result is akin to that of the previous result. We split it up into two steps, first by making an approximation of the $(g_M)_!(g_M)^*$ via a (Der4) square and then replaing it with an isomorphic object, as in the case of $M = \mathbb{N}$. This part is no different from the previous portion. The second step is to show that the category $(g_M/\bullet)$ is has contractible nerve. This procedure is a bit different. We did not find a connection between $(g_M/\bullet)$ and the terminal category via adjunctions, but we will write it as a colimit of contractible categories.

Proposition 7.3.2. *The category $(g_M/\bullet)$ has contractible nerve.*

Proof. The category $(g_M/\bullet)$ is defined as follows: its objects are indexed by elements of the abelian group $g(M)$, and if m_1, m_2 are elements of $g(M)$ or equivalently $(g_M/\bullet)$, there is a map $m_1 \to m_2$ if and only if the difference $m_1 - m_2$ lies in $M \subset g(M)$.

We first define the subcategory $C_0 \subset (g_M/\bullet)$ as the full subcategory consisting of all objects in M. This category has a terminal object, namely 0 as clearly each object of C_0 admits a unique map to 0. Thus its nerve is contractible.

Now for each $n \geq 0$, define C_n to be the full subcategory consists of objects m such that $m + n(x_1 + \cdots + x_k) \in M$. We have an increasing union

$$C_0 \subset C_1 \subset \cdots \subset C_n \subset \cdots (g_M/\bullet),$$

as $\{x_1, \cdots, x_k\}$ also generate $g(M)$ as a group (since each element of $g(M)$ is a $\mathbb{Z}$-linear combination of $\{x_1, \cdots, x_k\}$).

Thus, $(g_M/\bullet)$ is the colimit of the C_n along inclusions $C_n \hookrightarrow C_{n+1}$ for all $n \in \mathbb{N}$. We see that the colimit diagram is the shape $(\mathbb{N}, \leq)$ and hence this is a directed colimit. As nerves preserve directed colimits, we obtain that

$$N(g_m/\bullet) \simeq \mathrm{colim}_{(\mathbb{N},\leq)} N(C_n)$$

Now, each C_n has contractible nerve as C_n has a terminal object, namely $-n(x_1+\cdots+x_k)$. Indeed it is clear that C_0 and C_n are isomorphic for each n, via functors $C_0 \to C_n$ sending $m \mapsto m - n(x_1 + \cdots + x_k)$ and $C_n \to C_0$.

Since our colimit is in sSet using the Quillen model structure, we know that homotopy colimits and colimits are equivalent. Hence

$$N(g_M/\bullet) = \mathrm{colim}_{(\mathbb{N},\leq)} N(C_n) \simeq \mathrm{hocolim}_{(\mathbb{N},\leq)} N(C_n) \simeq \mathrm{hocolim}_{(\mathbb{N},\leq)} * = *.$$

This proves that $(g_M/\bullet)$ has contractible nerve. ∎

Now we proceed to the proof of the theorem.

Proof. As usual we wish to determine that $(g_M)_!(g_M)^* \cong Id$. This can be accomplished by proving that $i^*(g_M)_!(g_M)^* \cong i^*$. We can switch out $i^*(g_M)_!$ via a (Der4) square:

$$\begin{array}{ccc} (g_M/\bullet) & \xrightarrow{\mathrm{pr}} & \underline{M} \\ {\scriptstyle\pi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle g_M} \\ e & \xrightarrow[i]{} & g(M) \end{array} \tag{7.3.3}$$

From (Der4), we see that $i^*(g_M)_! \cong \pi_! pr^*$. Thus, $i^*(g_M)_!(g_M)^* \cong \pi_! pr^*(g_M)^*$. Let us examine this second composition $\pi_! pr^*(g_M)^*$.

As usual, consider an object in $\mathbb{D}(\underline{g(M)})$. It consists of an underlying object X along with a collection of coherent maps ω_m indexed by elements of the group $g(M)$, satisfying the condition that $\omega_0 = \mathrm{id}$ and $\omega_n \circ \omega_m = \omega_{n+m}$. We write it as $\{X, \omega_m : m \in g(M)\}$.

I claim that $pr^*(g_M)^*\{X, \omega_m : m \in g(M)\}$ is isomorphic to just $\pi^* X$. The proof here mirrors the case of $M = \mathbb{N}$. The pointwise isomorphism is given at the point $m \in g(M)$ by $\omega_m : (\pi^* X)_m \to (pr^*(g_M)^*\{X, \omega_m : m \in M\})$. Hence again we may replace $pr^*(g_M)^*\{X, \omega_m : m \in g(M)\}$ with $\pi^* X$.

The proof now boils down to showing that $\pi_! \pi^* X \cong X$, which is the case as the category $(g_M/\bullet)$ has contractible nerve. ∎

We remark that an alternate formulation of this is that the classifying spaces of $\underline{\mathbb{N}}$ and $\underline{\mathbb{Z}}$ are homotopy equivalent, and that the map of classifying spaces induced by the inclusion

$$i_{\underline{\mathbb{N}}} : \underline{\mathbb{N}} \to \underline{\mathbb{Z}}$$

is a homotopy equivalence.

But in fact more is true: both conditions of finite generation and cancellative can be removed.

7.4 In relation to the universal property

Let us now return to the case that M is a cancellative, abelian monoid. Recall that in this case, if $\mathbb{D}$ is a symmetric monoidal derivator, then $\mathbb{D}^{\underline{M}}$ is governed by a universal property, roughly stating that (cocontinuous, monoidal) morphisms from $\mathbb{D}^{\underline{M}}$ to $\mathbb{E}$ are governed by a morphism $\mathbb{D} \to \mathbb{E}$ and the image of a specific element.

Given the morphism

$$(g_M)_! : \mathbb{D}^{\underline{M}} \to \mathbb{D}^{\underline{g(M)}}$$

for a symmetric monoidal derivator $\mathbb{D}$ and another symmetric monoidal derivator $\mathbb{E}$, we have the induced morphism on Hom-categories:

$$((g_M)_!)^* : \mathrm{PDER}_{\otimes,!}(\mathbb{D}^{\underline{g(M)}}, \mathbb{E}) \to \mathrm{PDER}_{\otimes,!}(\mathbb{D}^{\underline{M}}, \mathbb{E}).$$

Due to the universal property in question, the induced functor can instead be thought of as a functor

$$((g_M)_!)^* : \mathrm{PDER}_{\otimes,!}(\mathbb{D}, \mathbb{E}) \times \mathbb{E}(\underline{g(M)})_{\mathbb{1}} \to \mathrm{PDER}_{\otimes,!}(\mathbb{D}, \mathbb{E}) \times \mathbb{E}(\underline{M})_{\mathbb{1}}.$$

We can ask whether it admits a simple characterization; and the answer is that

$$((g_M)_!)^* = \mathrm{Id}_{\mathrm{PDER}_{\otimes,!}(\mathbb{D},\mathbb{E})} \times (g_M)^*.$$

Proof. We first note that the functor onto the first component is indeed the identity. This is obvious, as $(g_M) \circ (i_M) = i_{g(M)}$. The functor onto the second component is more interesting:

let $F : \mathbb{D}^{\underline{g(M)}} \to \mathbb{E}$ be a cocontinuous, monoidal morphism of derivators (i.e. one under which the universal property holds). Recall the element we care about is

$$F_{\underline{g(M)}}(+^*_{g(M)}\mathbb{1}_{\mathbb{D}^{\underline{g(M)}}}) \in \mathbb{E}(\underline{g(M)})$$

and that this particular element gets mapped to

$$(F \circ (g_M)_!)_{\underline{M}}(+^*_M\mathbb{1}_{\mathbb{D}^{\underline{M}}}) \in \mathbb{E}(\underline{M}).$$

Furthermore, recall that when we merely examine their underlying objects they are both $\mathbb{1}_{\mathbb{E}}$, hence we have a functor between the described categories.

We aim to show that it is indeed just $(g_M)^*$.

First, we can get from $(+^*_M\mathbb{1}_{\mathbb{D}^{\underline{M}}})$ to $(+^*_{g(M)}\mathbb{1}_{\mathbb{D}^{\underline{g(M)}}})$ via the application of the functor $((+_{g(M)})^*(g_M)_!(1\times i_M)^*$. So knowing this, it would suffice to show that $(1\times g_M)^* +^*_{g(M)} \mathbb{1}_{\mathbb{D}^{\underline{g(M)}}}$ and $(g_M \times 1)_!((+^*_M\mathbb{1}_{\mathbb{D}^{\underline{M}}})$ are isomorphic.

Let us parse these two objects: the first object $(1 \times g_M)^* +^*_{g(M)} \mathbb{1}_{\mathbb{D}^{\underline{g(M)}}}$ can be expanded as

$$\begin{aligned}(1 \times g_M)^* +^*_{g(M)} \mathbb{1}_{\mathbb{D}^{\underline{g(M)}}} &= (+_{g(M),M})^*(i_{g(M)})_!\mathbb{1}_{\mathbb{D}} \\ &= (+_{g(M),M})^*(g_M)_!(i_M)_!\mathbb{1}_{\mathbb{D}}\end{aligned}$$

While the second object $(g_M \times 1)_!((+^*_M\mathbb{1}_{\mathbb{D}^{\underline{M}}})$ can also be expanded as

$$(g_M \times 1)_!((+^*_M\mathbb{1}_{\mathbb{D}^{\underline{M}}}) = (g_M \times 1)_!(+_M)^*(i_M)_!\mathbb{1}_{\mathbb{D}}$$

As such, it would be sufficient to show that

$$(+_{g(M),M})^*(g_M)_! \cong (g_M \times 1)_!(+_M)^*$$

for all commutative, cancellative monoids M. We define the functor

$$+_{g(M),M} : \underline{g(M)} \times \underline{M} \to \underline{g(M)}$$

to be the restriction of the functor $+_{g(M)}$ to the subcategory

$$\underline{g(M)} \times \underline{M} \subset \underline{g(M)} \times \underline{g(M)}.$$

We note that this is equivalent to asking the square below

$$\begin{array}{ccc} \underline{M} \times \underline{M} & \xrightarrow{+_M} & \underline{M} \\ {\scriptstyle g_M \times 1}\downarrow & \overset{\text{id}}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle g_M} \\ \underline{g(M)} \times \underline{M} & \xrightarrow[+]{} & \underline{g(M)} \end{array} \tag{7.4.1}$$

to be homotopy exact. Here the $+$ at the bottom is addition of elements of $g(M)$ and $M \subset g(M)$ as M is cancellative.

As usual, this can be done via a direct verification using Theorem 3.8 of [GPS14a]. As all categories in this diagram only have a single object, we need to verify that categories of the following form are contractible: for each $g \in g(M)$, the objects are a pair $(g_1, m_1) \in g(M) \times M$ along with $m_2 \in M$, such that $g_1 + m_1 + m_2 = g$, and a morphism $(g_1, m_1, m_2) \to (g_1', m_1', m_2')$ is indexed by a pair (k, l) both in M, where $g_1 + k = g_1'$ and $m_1 + l = m_1'$ (the third coordinate is resolved by the fact that $g_1 + m_1 + m_2 = g_1' + m_1' + m_2'$). There is at most only a single map from one object to another, as M is cancellative.

This means that all such categories we need to check are directed sets, and we know that directed sets have contractible nerve. □

CHAPTER 8

Definition of projective space

Next we move on to constructing projective space. There is an intuitive model of projective space $\mathbb{P}^n$ as being the gluing of $n+1$ different copies of $\mathbb{A}^n$ along various copies of $\mathbb{A}^{n-1} \times \mathbb{G}_m$ with cocycle conditions. In theory, since we know how to construct all of the objects that gluing needs to be taken along, we could attempt to define $\mathbb{P}^n$ in the above fashion.

However, in light of how we attacked the $\mathbb{A}^n$-problem, we should consider an approach based on quasicoherent sheaves on $\mathbb{P}^n$. Again, we will begin with the case of a projective space over a ring, and then we generalize to projective space over a scheme. Recall that a quasicoherent sheaf on $\mathbb{P}^n_R$ can be given by a graded module over $R[t_0, \cdots, t_n]$ with two graded modules representing the same sheaf under some localizing conditions. This will form the starting point of our inquiry.

8.1 A step to constructing projective space

As discussed above, we begin by constructing the category of $\mathbb{N}$-graded modules over $R[t_0, \cdots, t_n]$, where each t_n has degree 1.

Definition 8.1.1. Let $n \in \mathbb{N}$ be a natural number. Define Q_n to be the category with:

1. Objects $0, 1, \cdots$ indexed by $\mathbb{N}$
2. Maps $i \to i+1$ given by $x_0, x_1, \cdots x_n$, with the commutativity relation $x_j x_k = x_k x_j$ for composable maps where the composition has the same domain and codomain.

Due to this commutativity relation, when we write a composition of maps in Q_n we simply do so as a monomial. Composition in Q_n is given by multiplication of monomials, and so we see that $\mathrm{Hom}(i, i+k)$ is given by the set of monomials in $n+1$ variables with total degree k.

Further, because of the commutativity relation as long as the morphisms are composable the superscripts are irrelevant.

Proposition 8.1.2. *Let $\mathbb{D}$ be a derivator associated to R-*Mod *(represented or derived), then $\mathbb{D}^{Q_n}$ is the derivator associated to the category of $\mathbb{N}$-graded $R[t_0, \cdots, t_n]$-modules.*

Proof. Let us first litigate the content of the category of graded modules. By an $\mathbb{N}$-graded $R[t_0, \cdots, t_n]$-module I mean to treat $R[t_0, \cdots, t_n]$ as a graded ring, with each t_i graded in degree 1 and R in degree 0. Recall that an $\mathbb{N}$-graded module over a $\mathbb{N}$-graded ring $\oplus_{i \in \mathbb{N}} R_i$ is a module $\oplus_{i \in \mathbb{N}} S_i$ where $R_i S_j \subset S_{i+j}$.

Of course, our specific graded ring tells us that a graded module $\oplus_{i \in \mathbb{N}} S_i$ should have $RS_i \subset S_i$, so that each S_i is an R-module. Furthermore, due to additivity inherent in looking at categories of modules we can simply consider the generators t_i, and note that we have maps $t_i : S_j \to S_{j+1}$ for all $j \in \mathbb{N}$, $0 \leq i \leq n$ such that the maps $t_i t_k$ and $t_k t_i$ are equal due to commutativity.

So this gives for us a diagrammatic shape we desire: for each $i \in \mathbb{N}$ we have an R-module S_i, and between each S_i and S_{i+1} there is a map $t_j : S_i \to S_{i+1}$ for each $0 \leq j \leq n$. Conversely, given the additive structure on module maps it is clear that the information of R-modules S_i for each $n \in \mathbb{N}$ along with the maps $t_j : S_i \to S_{i+1}$ is sufficient to determine the structure of the $R[t_0, \cdots, t_n]$-module $\oplus_{i \in \mathbb{N}} S_i$.

The desired Homs are degree 0 maps of graded modules, so given two $R[t_0, \cdots, t_n]$-modules $\oplus_{i \in \mathbb{N}} S_i$ and $\oplus_{i \in \mathbb{N}} S'_i$, a morphism of graded modules consists of maps $f_i : S_i \to S'_i$ such that $t_j f_i \cong f_{i+1} t_j$ for all $i \in \mathbb{N}$, $0 \leq j \leq n$. This is precisely a morphism between two Q_n-shaped diagrams of R-modules.

Hence we have an isomorphism of categories $R\text{-Mod}^{Q_n}$ and $R[t_0, \cdots, t_n] - \mathrm{grMod}$. This means that their represented derivators are equivalent.

R-Mod is a Grothendieck abelian category, hence also $R\text{-Mod}^{Q_n}$. In addition, the isomorphism above that we have described $R[t_0, \cdots, t_n] - \text{grMod} \to R\text{-Mod}^{Q_n}$ is a left and right exact functor between abelian categories.

Hence the isomorphism on abelian categories extends to a Quillen equivalence on model categories of unbounded chain complexes that also happens to be an isomorphism, and so they induce equivalent derivators of unbounded derived categories. □

8.2 The monoidal structure

In fact, the above equivalence is a monoidal equivalence. First we describe the monoidal structure on $R\text{-Mod}^{Q_n}$, which will be done via Day convolution. Then we will see that this is the same as the tensor product of graded modules.

Definition 8.2.1. Q_n is a symmetric monoidal category, with tensor product

$$\otimes : Q_n \times Q_n \to Q_n$$

given by $(m, n) \mapsto m + n$ and tensor product of morphisms given by multiplication of monomials.

The unit in the category is the object 0. The various coherence conditions for the symmetric monoidal structure are clear.

We note of course that the monoidal product is symmetric due to the aforementioned commtuativity relation.

Therefore, we can use Day convolution to define another symmetric monoidal structure on $R\text{-Mod}^{Q_n}$ and a corresponding shifted monoidal derivator $\mathbb{D}^{Q_n}$.

We note that the monoidal structure induced via Day is the same as the graded tensor product.

Proposition 8.2.2. *The induced monoidal structure on $R\text{-Mod}^{Q_n}$ is the same as the tensor product of $\mathbb{N}$-graded $R[t_0, \cdots, t_n]$-modules, where the R-part of the the diagram corresponds to the k-part of the graded module.*

Proof. Recall first the monoidal structure on $\mathbb{N}$-graded $R[t_0, \cdots, t_n]$-modules. Let $\oplus_{i\in\mathbb{N}} M_i$ and $\oplus_{j\in\mathbb{N}} N_j$ be two graded $R[t_0, \cdots, t_n]$-modules. The R-graded part of $\oplus_{i\in\mathbb{N}} M_i \otimes \oplus_{j\in\mathbb{N}} N_j$ is given by $\oplus_{i+j=k} M_i \otimes N_j$ modulo the relation $m \otimes rn = rm \otimes n$, for the degrees of r, m, n adding up to k.

Now let's consider the Day monoidal structure on R-Mod^{Q_n}. According to $(Der4)$, we have the following square for a pointwise computation of the colimit:

$$\begin{array}{ccc} (\otimes/k) & \xrightarrow{\mathrm{pr}} & Q_n \times Q_n \\ {\scriptstyle\pi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle\otimes} \\ e & \xrightarrow[k]{} & Q_n \end{array}$$

The category $(\otimes/k)$ can be described as follows: Its objects are pairs (i, j) with $i+j \leq k$, along with a map $\prod x_k^{t_k} : (i+j) \to k$. A map $\{(i, j) : (i+j) \to k\} \to \{(i', j') : (i'+j') \to k\}$ is given by a pair of maps $i \to i'$ and $j \to j'$ such that the maps under $\otimes$ commute. Taking $\oplus_{i\in\mathbb{N}} M_i$ and $\oplus_{j\in\mathbb{N}} N_j$ as our inputs, the object at $\{(i, j) : i+j \to k\}$ in the colimit is $M_i \otimes N_j$.

The maximal elements of this set are at the points $\{(i, j) : i+j = k \to k\}$, i.e. $M_i \otimes N_{k-i}$. However, given another $\{(i, j) : \prod {t_m}^{n_m} : i+j \to k\}$, there are maps

$$\begin{aligned} (\prod t_m^{n_m}, id) : \{(i, j), \prod {t_m}^{n_m} : i+j \to k\} &\to \{(i, k-i), id : k \to k\} \\ (id, \prod t_m^{n_m}) : \{(i, j), \prod {t_m}^{n_m} : i+j \to k\} &\to \{(k-j, j), id : k \to k\} \end{aligned}$$

which need to commute under the canonical map to the colimit. This gives precisely the relation $m \otimes rn = rm \otimes n$. Therefore, this gives precisely the same monoidal structure as the tensor product of $\mathbb{N}$-graded $R[t_0, \cdots, t_n]$-modules.

We also compute the structure of the monoidal unit, which will be $0_! \mathbb{1}$. In fact, we compute $0_! X$ for any $X \in R$-Mod using a (Der4) computation: the relevant square is

$$\begin{array}{ccc} (0/k) & \xrightarrow{\mathrm{pr}} & e \\ {\scriptstyle\pi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle 0} \\ e & \xrightarrow[k]{} & Q_n \end{array}$$

The objects of $(0/k)$ are just the maps from 0 to k in Q_n. As 0 has no endomorphisms we can view this category as merely a set. Morphisms $0 \to k$ correspond precisely to monomials of degree k in $n+1$ variables. Therefore, the degree k portion of $i_! X$ is a coproduct of $\binom{n+k}{k}$ X's.

In particular, if X is the base ring, then the degree k part is the degree k part of the graded polynomial ring, showing that $i_! \mathbb{1}$ is the monoidal unit here. □

Therefore, we can see that $0_! \mathbb{1}$ plays the role of $\mathcal{O}$ in general, as it is the monoidal unit for the shifted derivator $\mathbb{D}^{Q_n}$ with Day convolution monoidal structure, .

Remark 8.2.3. One can use both $\mathbb{N}$-graded modules or $\mathbb{Z}$-graded modules for part of the $\mathbb{P}^n$-construction, since the localization will not be able to tell the difference. For doing everything with a $\mathbb{Z}$-grading, we utilize the category $Q_n^{\mathbb{Z}}$, which has objects indexed over $\mathbb{Z}$ and morphisms defined the same way. It also has a symmetric monoidal structure, and there is no particular reason why one setup is superior to the other purely for the definition of $\mathbb{P}^n$.

We will stick with $\mathbb{N}$-indexing as that was our original intuition, and because it simplifies some auxiliary computations.

The twisting sheaves/modules $\mathcal{O}(i)$ come from shifting the module degree by i. We can produce the twisting in a diagrammatic manner: let $\tau : Q_n \to Q_n$ be the map sending $k \mapsto k+1$ and $x_i \mapsto x_i$.

Proposition 8.2.4. *$\tau : Q_n \to Q_n$ can also be considered as the inclusion of the full subcategory on objects $\{1, 2, 3, \cdots, \}$.*

Similarly, τ^k is the inclusion of the full subcategory on objects $\geq k$.

Proposition 8.2.5. *The pointwise computations for $(\tau^k)_! X$ are as follows: if $l < k$, $l^*(\tau^k)_! X \cong 0$, and if $l \geq k$, $l^*(\tau_k)_! X \cong X_{k-l}$.*

Proof. This is a simple exercise in (Der4). So we see that $(\tau^k)_!$ just shifts the object k degrees upward and inserts k zeroes at $\{0, 1, \cdots, k-1\}$. □

This is therefore akin to "twisting" the module by k degrees or by $\mathcal{O}(-k)$, as we would usually denote it. We would expect its adjoint τ^* to twist the module in the other direction.

However, because our indexing is on $\mathbb{N}$, the statement does not quite work the way we want. Rather, obtaining the desired equivalence from $(\tau_!, \tau^*)$ will occur after we localize to $\mathbb{P}^n$.

Moreover, the twist $\tau_!$ respects tensor products in a certain sense. By this, I mean that for two objects $X, Y \in \mathbb{D}^{Q_n}$, $\tau_! X \boxtimes_{Q_n} Y \cong \tau_!(X \boxtimes_{Q_n} Y)$. We can see this through the below computation.

$$\begin{aligned}
(\tau_! X \boxtimes_{Q_n} Y) &\cong (\otimes_{Q_n})_!(\tau_! X \boxtimes Y) \\
&\cong (\otimes_{Q_n})_!(\tau \times 1)_!(X \boxtimes Y) \\
&\cong \tau_!(\otimes_{Q_n})_!(X \boxtimes Y) \\
&\cong \tau_!(X \boxtimes_{Q_n} Y)
\end{aligned}$$

Putting $\tau_!$ on the other side is the same by symmetry. This also fits with our usual intuition about twisting sheaves.

8.3 Projective space as a localization of $\mathbb{D}^{Q_n}$

Now, recall that every quasi-coherent $\mathcal{O}_{\mathbb{P}^n}$-module over $\mathbb{P}^n_R$ comes from a $\mathbb{N}$-graded $R[t_0, \cdots, t_n]$-module. In addition, as $\mathbb{P}^n$ is covered by $n+1$ copies of $\mathbb{A}^n$, a sheaf on $\mathbb{P}^n$ can be determined by its local presence on the n+1 $\mathbb{A}^n$'s that cover it.

Proposition 8.3.1. *The restriction from a sheaf on $\mathbb{P}^n$ to one of the component $\mathbb{A}^n$'s that cover it is the left Kan extension along a specific localization functor.*

Proof. Suppose we want to examine the component of $\mathbb{A}^n$ with $x_k = 1$. The coordinates on $\mathbb{A}^n$ are $(\frac{x_0}{x_k}, \frac{x_1}{x_k}, \cdots, \hat{\frac{x_k}{x_k}}, \cdots, \frac{x_n}{x_k})$, i.e. $\frac{x_k}{x_k}$ is omitted.

Now recall how the restriction on the sheaf level works: let M be a $\mathbb{N}$-graded $k[x_0, \cdots, x_n]$-module and $\tilde{M}$ be the associated sheaf on $\mathbb{P}^n$. The restriction on $\mathbb{A}^n$ is the sheaf associated to $(M_{(x_k)})_0$, which we can just think of as the module $(M_{(x_k)})_0$.

This can be constructed in two steps: first given the category Q_n, we invert all maps of the form x_k and call the resulting localization $Q_n[x_k^{-1}]$.

Let's examine the structure of $Q_n[x_k^{-1}]$. First we note that all the objects in this new category are isomorphic, as the maps x_k have become isomorphisms. Secondly, in Q_n two composable x_i, x_j commute, so $x_i x_k^{-1} = x_k^{-1} x_i$ as long as they are composable in $Q_n[x_k^{-1}]$, i.e. left or right fractions don't matter.

Now let's look at a morphism $a \to b$ in $Q_n[x_k^{-1}]$. For ease of notation, we will describe the case $k = 0$ but naturally the case for $1 \leq k \leq n$ are exactly similar.

Each morphism $x_0, \cdots, x_n$ increases the object number by 1, and x_0^{-1} decreases it by 1. Therefore, a map $a \to b$ can be written as a fraction $\frac{\prod_{i=0}^n x_i^{t_i}}{x_0^m}$, where $\sum_{i=0}^n t_i - m = b - a$.

Equivalently, this can be determined by some (nonnegative) powers of $x_1, x_2, \cdots, x_n$, and the remaining degrees must be made up by positive or negative powers of the isomorphism x_0. Thus we easily see that $\mathrm{Hom}_{Q_n[x_k^{-1}]}(a, b) = (\mathbb{N}^n, +)$. Indeed, e_i in $\mathbb{N}^n$ corresponds to $\frac{x_i}{x_0}$.

Now we have an equivalence of categories $\underline{\mathbb{N}}^n \cong Q_n[x_k^{-1}]$. The functor $F : Q_n[x_k^{-1}] \to \underline{\mathbb{N}}^n$ sending each object to $\bullet \in \underline{\mathbb{N}}^n$ and x_i to $e_i \in \mathbb{N}^n$, x_0 to the identity map, is an obvious equivalence, while any functor $\underline{\mathbb{N}}^n \to Q_n[x_k^{-1}]$ sending $\bullet$ to any $i \in Q_n$ with $\mathbb{N}^n$ mapping appropriately to $\mathrm{Hom}(i, i)$ in a manner described above will be a quasi-inverse.

The above description we can check gives us a functor $Q_n[x_k^{-1}] \to \underline{\mathbb{N}}^n$, sending $k \in Q_n$ to $\bullet \in \underline{\mathbb{N}}^n$. This functor is surjective and fully faithful according to the above description, hence an equivalence.

Conversely, one quasi-inverse $\underline{\mathbb{N}}^n \to Q_n[x_k^{-1}]$ sends $\bullet$ to 0 and e_i to $\frac{x_i}{x_0}$. We can check that this is in fact a quasi-inverse.

Let's now check the left Kan extension functor along $(x_k)^{-1} : Q_n \to Q_n[x_k^{-1}]$, or rather just the part at index 0. Once again, consider the relevant (Der4) square:

$$\begin{array}{ccc} ((x_k)^{-1}/0) & \xrightarrow{\mathrm{pr}} & Q_n \\ {\scriptstyle \pi}\downarrow & \overset{\alpha}{\swArrow} & \downarrow{\scriptstyle (x_k)^{-1}} \\ e & \xrightarrow[0]{} & Q_n[x_k^{-1}] \end{array}$$

Let's examine the category $((x_k)^{-1}/0)$. The objects in this category are $i \in \mathbb{N}$, along with a map $i \to 0$ in $Q_n[x_k^{-1}]$. Again for simplicity we assume $k = 0$, noting the symmetry here. Then a map $i \to 0$ can be written as a monomial $\frac{\prod_{j=1}^n t_j^{m_j}}{(x_0)^m}$, where $\sum m_j - m = -i$, so an object in our category is $i \in \mathbb{N}$ along with such a monomial.

This category is rather complicated; however, we note that we have a homotopy final subcategory, which is the full subcategory consisting of the objects $\{m, (x_0)^{-m}\}$, i.e. a copy of $(\mathbb{N}, <)$. Indeed, given an object $(i, \frac{\prod_{j=1}^n t_j^{m_j}}{(x_0)^m})$, there is a unique map to $(i+\sum_{j=1}^n m_j, (x_0)^{-m})$ induced by $\prod_{j=1}^n t_j^{m_j}$.

We can check that this is an adjunction, with the inclusion from $(\mathbb{N}, <)$ to $((x_0^{-1}/0)$ the right adjoint and the functor mapping $(i, \frac{\prod_{j=1}^n t_j^{m_j}}{(x_0)^m}) \mapsto (i + \sum_{j=1}^n m_j)$ the left adjoint.

Viewing $(\mathbb{N}, \leq)$ as a subcategory that is homotopy final, we find that the colimit along $((x_0)^{-1}/0)$ is isomorphic to the sequential colimit along the given subcategory.

Nonetheless, this is just a sequential colimit. Now, suppose we are given a graded $k[t_0, \cdots, t_n]$-module, $\oplus_{i \in \mathbb{N}} M_i$. The relevant colimit is just

$$M_0 \xrightarrow{t_0} M_1 \xrightarrow{t_0} \cdots \xrightarrow{t_0}$$

and if we compute via elements, we see that this is nothing more that $(M_{t_0})_0$, as $a \in M_i$ and $t_0^m a \in M_{m+i}$ are identified, so if we identify $a \in M_i$ with $\frac{a}{(t_0)^i}$ then $a \in M_i$ and $t_0^m a \in M_{m+i}$ are equivalent. □

Proposition 8.3.2. *The passage from a sheaf on $\mathbb{P}^n$ (thought of as a module) to a sheaf on $\mathbb{A}^n$ (also thought of as a module) is induced via the left Kan extension given above.*

Let's just remember that the associated sheaf on $\mathbb{A}^n$ (localizing at x_0) has associated maps $x_0^{-1}x_i$ for $1 \leq i \leq n$, although we may also consider the entire category $Q_n[x_0^{-1}]$ which is equivalent.

Remark 8.3.3. A morphism between $\mathbb{N}$-graded $k[t_0 \cdots, t_n]$-modules is an isomorphism of sheaves on $\mathbb{P}^n$ if they induce isomorphic sheaves (modules) on the $n + 1$ copies of $\mathbb{A}^n$.

Moreover, let's consider not just localizing at one x_k, but two coordinates, say x_0, x_1. Then indeed, we can either first invert x_0 or then x_1, or vice versa. If we view the localization at x_0 as just $\underline{\mathbb{N}^n}$, then subsequent localization at x_1 is equivalent to the localization of the map $x_0^{-1}x_1$, or equivalently $e_1 \in \mathbb{N}^n$, to induce a copy of $\underline{\mathbb{Z}} \times \underline{\mathbb{N}}^{n-1}$.

Needless to say, inverting some collection of $\{x_i\}$ and looking at inverting them one at a time gives me a collection of gluing data on the copies of $\mathbb{A}^n$ and their double/triple intersections, etc. Specifically, the $n+1$ copies of $\mathbb{A}^n$ referred to here are the shifted derivators $\mathbb{D}^{Q_n[x_i^{-1}]}$ with $Q_n[x_i^{-1}]$ equivalent to $\underline{\mathbb{N}^n}$. Inverting x_i and x_j gives $Q_n[x_i^{-1}, x_j^{-1}]$, which is equivalent to $\underline{\mathbb{N}^{n-1} \times \mathbb{Z}}$, and $\mathbb{D}^{\underline{\mathbb{N}^{n-1}\times\mathbb{Z}}}$ is nothing more than $\mathbb{G}_m \times \mathbb{A}^{n-1}$.

A sheaf on $\mathbb{P}^n$ is zero if and only if its restriction to each $\mathbb{A}^n$ is zero.

Definition 8.3.4. Let $n \in \mathbb{N}$. Call an object in $\mathbb{D}^{Q_n}$ nilpotent if $(x_i)_!^{-1}X = 0$ for all $0 \leq i \leq n$ and denote the full subderivator of pointwise nilpotent objects $Nilp(\mathbb{D})(I)$. By definition, we obtain a full subprederivator of $\mathbb{D}$ from this, which we term $Nilp(\mathbb{D})$.

Call a morphism $f : X \to Y$ in $\mathbb{D}^{Q_n}$ an $\mathbb{P}^n$-equivalence if $(x_i^{-1})_!f$ is an isomorphism for all $0 \leq i \leq n$.

Now suppose in addition that $\mathbb{D}$ is a compactly generated triangulated derivator. Then

Proposition 8.3.5. *The full subderivator $Nilp(\mathbb{D})$ evaluated at e, $Nilp(\mathbb{D})(e)$ is a localizing triangulated subcategory of $\mathbb{D}^{Q_n}(e)$.*

Proof. Clearly $0 \in Nilp(\mathbb{D})(e)$. We need only prove that $Nilp(\mathbb{D})(e)$ is closed under taking cones. However, this is obvious as all Kan extensions here viewed as morphisms are exact morphisms. So we need only take a triangle

$$X \to Y \to Z \to \cdots$$

in $Nilp(\mathbb{D})(e)$ and apply $(x_i^{-1})_!$ to it, which gives us the triangle

$$0 \to 0 \to 0 \to (x_i^{-1})_!Z \to \cdots$$

hence naturally $(x_i)_!^{-1}Z = 0$.

The category contains all coproducts as $(x_i^{-1})_!$ is cocontinuous. It is also thick. Suppose $X = X_1 \oplus X_2$ is in $Nilp(\mathbb{D})(e)$. Because left Kan extensions commute with coproducts, indeed

$$0 = (x_i)_!^{-1} X = (x_i)_!^{-1} X_1 \oplus (x_i)_!^{-1} X_2,$$

hence both $(x_i)_!^{-1} X_1$ and $(x_i)_!^{-1} X_2$ are also 0. □

By a simple shifting argument, this means we can take the levelwise Verdier localization $\mathbb{D}(I)/\mathrm{Nilp}(\mathbb{D})(I)$ for every $I \in \mathsf{Cat}$. We would like to show that $\mathbb{P}^n_{\mathbb{D}}$ defines a derivator.

Proposition 8.3.6. *Let $\mathbb{D}$ be a compactly generated triangulated derivator. The assignment*

$$I \mapsto \mathbb{D}^{Q_n}(I)/Nilp(\mathbb{D})(I)$$

defines a triangulated derivator. We call this new derivator $\mathbb{P}^n_{\mathbb{D}}$.

Proof. With the additional assumption of compact generation, the statement is nearly trivial. We can apply our derivator version of the Brown reprsentability theorem, and then an application of [Cis08, Lemme 4.2] using the fully faithful right adjoint will guarantee that we have a derivator.

Without the compact generation of $\mathbb{D}$, we would still obtain that our derivator is a *left derivator* with triangulated values. We will prove this result separately, as it is indicative of how we would approach such issues without applying a cudgel such as Brown representability.

□

Remark 8.3.7. However, we note in fact that this localization will already have all finite homotopy right Kan extensions also. Since our categories are additive, finite products and coproducts coincide. Furthermore, one condition of being a triangulated derivator is that it is *regular*, i.e. that sequential colimits and pullbacks (since we are additive, equivalently, finite limits) commute.

Therefore, to obtain that all right Kan extensions exist for this derivator, it would be sufficient to prove that the $(x_i^{-1})_!$ commute with all products. Generally speaking we probably should not expect for sequential colimits to commute with all products. Luckily for us, the assumption of compact generation (perfect generation seems to be sufficient) circumvents this problem as discussed above.

For the general approach without assuming Brown representability, we take the following steps.

1. $\mathbb{P}^n_{\mathbb{D}}$ is a left derivator, the left Kan extensions come from left Kan extensions of $\mathbb{D}^{Q_n}$.

2. The localization functor preserves left Kan extensions and homotopy finite right Kan extensions.

3. Prove that $\mathbb{P}^n_{\mathbb{D}}$ is strong.

Lemma 8.3.8. *As long as $\mathbb{D}$ is triangulated, $\mathbb{P}^n_{\mathbb{D}}$ is a left derivator.*

Proof. It is clear that $\mathbb{D}$ is a prederivator as the localizing subcategories are defined pointwise.

We need to check (Der1), (Der2), (Der3) for left Kan extensions, and (Der4) for left Kan extensions.

(Der1): This is clear as,

$$\mathbb{P}^n_{\mathbb{D}}(\coprod_{\lambda\in\Lambda} I_\lambda) := \mathbb{D}^{Q_n}(\coprod_{\lambda\in\Lambda} I_\lambda)/Nilp(\mathbb{D}(\coprod_{\lambda\in\Lambda} I_\lambda)).$$

However, due to the presence of (Der1) for the derivator $\mathbb{D}^{Q_n}$, we see that

$$\mathbb{D}^{Q_n}(\coprod_{\lambda\in\Lambda} I_\lambda)/Nilp(\mathbb{D}(\coprod_{\lambda\in\Lambda} I_\lambda)) \cong \prod_{\lambda\in\Lambda} \mathbb{D}^{Q_n}(I_\lambda)/(\prod_{\lambda\in\Lambda} Nilp(\mathbb{D}(I_\lambda))) \cong \prod_{\lambda\in\Lambda} \mathbb{P}^n_{\mathbb{D}}(I_\lambda).$$

(Der2): Suppose $f : X \to Y$ is a morphism in $\mathbb{P}^n_{\mathbb{D}}(A)$ such that a^*f was an isomorphism in $\mathbb{P}^n_{\mathbb{D}}(e)$ for all $a \in A$. If so, this means that $(x_i^{-1})_!(a^*X)$ and $(x_i^{-1})_!(a^*Y)$ are isomorphic for all $0 \leq i \leq n$. However, a^* is a bicontinuous morphism of derivators for all $a \in A$, so in fact

$$(x_i^{-1})_!(a^*X) \cong a^*(x_i^{-1})_!X, (x_i^{-1})_!(a^*Y) \cong a^*(x_i^{-1})_!Y,$$

so in fact we have we have an isomorphism $a^*(x_i^{-1})_! X$ and $a^*(x_i^{-1})_! Y$ for all $a \in A$ and all $0 \leq i \leq n$, hence an isomorphism of $(x_i^{-1})_! X$ and $(x_i^{-1})_! Y$ for all $0 \leq i \leq n$, by (Der2). Therefore, the f we started off with is an isomorphism.

(Der3L): Here we only attempt to prove the existence of left adjoints. Let $u : J \to K$ be any functor. I claim that the left Kan extension $u_! : \mathbb{D}^{Q_n}(J) \to \mathbb{D}^{Q_n}(K)$ induces a left Kan extension $u_! : \mathbb{P}^n_{\mathbb{D}}(J) \to \mathbb{P}^n_{\mathbb{D}}(K)$.

Firstly, as $u_!$ is a cocontinuous morphism of derivators it commutes with $(x_i^{-1})_!$ for all $0 \leq i \leq n$. So naturally $u_!$ (simply as a functor) from $\mathbb{D}^{Q_n}(J) \to \mathbb{D}^{Q_n}(K)$ to $\mathbb{P}^n_{\mathbb{D}}(J) \to \mathbb{P}^n_{\mathbb{D}}(K)$.

Now I do now know whether this functor is a left adjoint to u^* or not. However, u^* for $\mathbb{P}^n$ is also induced by restrictions u^* from $\mathbb{D}^{Q_n}$. Since adjunctions can be defined via the triangle identities, those descend naturally from $\mathbb{D}^{Q_n}$ to $\mathbb{P}^n_{\mathbb{D}}$, hence this $u_!$ we have constructed is an actual left adjoint to u^* on $\mathbb{P}^n_{\mathbb{D}}$.

(Der4L): Since the homotopy left Kan extensions $u_!$ were defined by lifting them to $\mathbb{D}^{Q_n}$, computing there, and then localizing, (Der4L) holds for this definition of $u_!$ on $\mathbb{P}^n_{\mathbb{D}}$.

Collectively, we have managed to show that $\mathbb{P}^n_{\mathbb{D}}$ is a left derivator. □

Corollary 8.3.9. *The localization morphism $L : \mathbb{D}^{Q_n} \to \mathbb{P}^n_{\mathbb{D}}$ is a cocontinuous morphism of (left) derivators.*

Proof. This is obvious by the construction of left Kan extensions in $\mathbb{P}^n_{\mathbb{D}}$ as coming from left Kan extensions in $\mathbb{D}^{Q_n}$. □

Proposition 8.3.10. *Suppose now that $\mathbb{D}$ is a compactly generated derivator. Then the localization morphism $L : \mathbb{D}^{Q_n} \to \mathbb{P}^n_{\mathbb{D}}$ has a fully faithful right adjoint.*

Proof. We already know that $\mathbb{D}^{Q_n}(I)$ is compactly generated for all small categories I. In the previous corollary we have noted that the localization morphism L is cocontinuous, so in particular it preserves all coproducts. Therefore, by Brown representability L_I has an adjoint R_I for each $I \in \mathsf{Cat}$.

By [Gro13, Lemma 2.10], we know that in this case, the R_I patch to a *lax* morphism of prederivators. However, since in this the morphism L is cocontinuous essentially by definition, the lax morphism of derivators is actually a *strong* morphism. This is the right adjoint in question.

Moreover, since L_I are localization functors, their right adjoints R_I are fully faithful inclusions. $\square$

Corollary 8.3.11. *$\mathbb{P}^n_{\mathbb{D}}$ is a derivator when $\mathbb{D}$ is a compactly generated triangulated derivator.*

Proof. By [Cis03, Lemme 4.2], any full subderivator of a derivator that is either reflective or coreflective is a derivator. We have indicated above that $\mathbb{P}^n_{\mathbb{D}}$ is a full subderivator of $\mathbb{D}^{Q_n}$, and the inclusion via R_I is a right adjoint, hence it is reflective. Therefore, $\mathbb{P}^n_{\mathbb{D}}$ is a derivator. $\square$

Proposition 8.3.12. *$\mathbb{P}^n_{\mathbb{D}}$ is a* strong stable *derivator. The triangulation on $\mathbb{P}^n_{\mathbb{D}}$ that we have constructed via the localization of $\mathbb{D}^{Q_n}$ coincides with the triangulation arises on $\mathbb{P}^n_{\mathbb{D}}$ due to its structure as a strong stable derivator.*

Proof. Since the localization $L : \mathbb{D}^{Q_n} \to \mathbb{P}^n_{\mathbb{D}}$ is a Bousfield localization, (Der5) follows immediately, by simply including the morphism $f : X \to Y$ in $\mathbb{P}^n_{\mathbb{D}}(I)$ via the inclusion R_I, noting (Der5) from $\mathbb{D}^{Q_n}$, and localizing back down to $\mathbb{P}^n_{\mathbb{D}}$.

The construction of Σ and Ω morphisms on $\mathbb{P}^n_{\mathbb{D}}$ are precisely induced by the localization functor. Since the L_I are all triangulated, by definition the R_I are also all triangulated, i.e. the triangles in $\mathbb{P}^n_{\mathbb{D}}$ as well as the (Σ, Ω) adjunction are precisely those that arise in $\mathbb{D}^{Q_n}$. $\square$

Now we examine monoidal structures. We show that the monoidal structure on $\mathbb{D}^{Q_n}$ descends to $\mathbb{P}^n_{\mathbb{D}}$ and that this models the tensor product of shaves on $\mathbb{P}^n_A$ if $\mathbb{D}$ is the derivator associated to a ring A.

Definition 8.3.13. Recall the monoidal structure on Q_n as described above. Let $\mathbb{D}$ be a symmetric monoidal derivator. Then there is a symmetric monoidal structure on $\mathbb{D}^{Q_n}$ given by

$$X \boxtimes_{Q_n} Y := (\otimes_{Q_n})_!(X \boxtimes Y).$$

We remember that this is a standard construction mimicking the Day convolution. Recall that this makes each value $D^{Q_n}(I)$ a symmetric monoidal category. This part of the construction does not depend on stability.

From now on, assume $\mathbb{D}$ is a compactly generated symmetric monoidal triangulated derivator.

Lemma 8.3.14. *With the monoidal structure defined above, $Nilp(\mathbb{D}) \subset \mathbb{D}^{Q_n}(e)$ is a localizing tensor-ideal.*

Proof. We have already proven above that the category is a thick triangulated subcategory. Therefore, it just remains to check that if $X \in Nilp(\mathbb{D})$ and $Y \in \mathbb{D}^{Q_n}(e)$, then $X \otimes Y \in Nilp(\mathbb{D})$.

First, let us note that $Q_n[x_i^{-1}]$ is still a symmetric monoidal category, with tensor product induced by addition on elements and multiplication on maps. We note that the localization functor $(x_i)^{-1} : Q_n \to Q_n[x_i^{-1}]$ is a monoidal functor. Therefore, the associated left Kan extension

$$(x_i)_!^{-1} : \mathbb{D}^{Q_n} \to \mathbb{D}^{Q_n[x_i^{-1}]}$$

is a monoidal functor.

Hence in fact

$$\begin{aligned} (x_i^{-1})_!(X \otimes Y) &= (x_i^{-1})_! X \otimes (x_i^{-1})_! Y \\ &= 0 \otimes (x_i^{-1})_! Y \\ &= 0 \end{aligned}$$

□

Therefore, since we take the quotient of a thick tensor-ideal, the tensor product descends to the quotient.

Remark 8.3.15. There is an "alternative" definition of taking projective space to be $\mathbb{D}^{Q_n}$ localized at the collection of $\mathbb{P}^n$-equivalences. This may be a vehicle for defining $\mathbb{P}^n_{\mathbb{D}}$ in non-triangulated settings, and it has the same intuitive basis that two modules are $\mathbb{P}^n$-equivalent precisely when they are isomorphic on each open $\mathbb{A}^n$-component.

There is a potential size issue here if $\mathbb{D}$ is not triangulated, not to mention that the two localizations may not be equal in this case, but certainly if $\mathbb{D}$ is triangulated then it is an equivalent definition.

8.4 Twisting in $\mathbb{P}^n_{\mathbb{D}}$

Now we explore the twisting functors $\tau_!$ and τ^*. Recall that in the case of sheaves on $\mathbb{P}^n$, what we call twisting up or down is a tensor product by $\mathcal{O}(k)$ for an integer k.

There are two main things that ought to be true for our derivator analogue of twisting. First, the twisting should respect the tensor product in a certain way, i.e. $(\tau_! \mathbb{1} \otimes X)$ and $\tau_! X$ ought to be isomorphic, and this we saw in the previous section. Secondly, the twisting $\tau_!$ should be an equivalence, and this statement was not true for $\mathbb{D}^{Q_n}$, partly because there is no strong analogue of $\mathcal{O}(1)$ in $\mathbb{D}^{Q_n}$ in the sense of an object in $\mathbb{D}^{Q_n}$ that is a tensor-inverse to $\mathcal{O}(-1)$ or $\tau_! \mathbb{1}$.

Therefore, we have an adjunction $(\tau_!, \tau^*)$, and the left adjoint is fully faithful, but τ^* is not an equivalence. We hope that the adjunction descends to $\mathbb{P}^n_{\mathbb{D}}$, which is sufficient for our needs.

We first give some preparatory results for these two points. Recall that the underlying object after applying $(x_i)_!^{-1}$ is actually computed via a sequential colimit in the x_i-direction. Note that τ restricts to this $(\mathbb{N}, <)$-shaped subcategory, and is simply $i \mapsto i+1$ on $(\mathbb{N}, <)$.

Lemma 8.4.1. $\tau : (\mathbb{N}, <) \to (\mathbb{N}, <)$ *induces an isomorphism* $(\pi_{(\mathbb{N},<)})_! \tau^* \cong (\pi_{(\mathbb{N},<)})_!$.

Proof. We verify that the square below is homotopy exact.

$$\begin{array}{ccc} (\mathbb{N},<) & \xrightarrow{\tau} & (\mathbb{N},<) \\ {\scriptstyle\pi}\downarrow & \overset{\alpha}{\swarrow} & \downarrow{\scriptstyle\pi} \\ e & \xrightarrow[id]{} & e \end{array}$$

Fix an $n \in \mathbb{N}$, we need to verify simply that the category with objects x equipped with a morphism $n \to \tau(x)$ with morphisms induced by those in $(\mathbb{N},<)$ is contractible. However, if $n \neq 0$ there is an obvious initial object for these categories given by $x = n-1$ and at $n = 0$ the category is simply $(\mathbb{N},<)$ which is also contractible. Therefore, by [GPS14b, Theorem 3.8], the square above is homotopy exact, and checking the definition yields the appropriate isomorphism in the lemma. □

This makes good intuitive sense; putting an extra 0 at the beginning of a homotopy colimit should not change the homotopy colimit in any way.

Therefore, if $X \in Nilp(\mathbb{D})$, then $\tau^*(X) \in Nilp(\mathbb{D})$. Hence the morphism τ^* on $\mathbb{D}^{Q_n}$ descends to a similar morphism that we also call τ^* on $\mathbb{P}^n_{\mathbb{D}}$.

We have explored how τ^* on $\mathbb{D}^{Q_n}$ is analoguous to the twist by degree 1, and so τ^* is also a twist by degree 1, hence tensoring with $\mathcal{O}(1)$.

Obviously, τ^* has left and right adjoints $\tau_!$ and τ_*. We specifically examine $\tau_!$, and claim that it too descends to $\mathbb{P}^n_{\mathbb{D}}$ and that actually the adjunction $(\tau_!, \tau^*)$ are equivalences on $\mathbb{P}^n_{\mathbb{D}}$.

Lemma 8.4.2. *$\tau : Q_n \to Q_n$ is a cosieve.*

Proof. τ is evidently fully faithful, and we can easily verify that it is a cosieve as there are no maps $n \to 0$ if $n > 0$. □

Equivalently, we can think of τ being the inclusion of the full subcategory of elements except from 0 in Q_n. The left Kan extension along τ is fully faithful, and an extension of zero. Therefore, we know already that $\tau^*\tau_!$ is the identity on $\mathbb{D}^{Q_n}$. We seek to prove that $\tau_!\tau^*$ is also isomorphic to the identity on $\mathbb{P}^n_{\mathbb{D}}$.

Proposition 8.4.3. *$\tau_!\tau^*$ is isomorphic to the identity on $\mathbb{P}^n_{\mathbb{D}}$.*

Proof. Recall that $\tau_!$ is an extension by 0. The composition$\tau_!\tau^*$ takes an object in $\mathbb{D}^{Q_n}$, which pointwise is $\{X_n, n \in \mathbb{N}\}$ with maps $x_i : X_n \to X_{n+1}$, and sends it to an object in $\mathbb{D}^{Q_n}$, which is equal to X_n for all $n \geq 1$ along with associated maps $x_i : X_n \to X_{n+1}$, but sets $X_0 = 0$, due to $\tau_!$ being an extension by 0.

Let's consider the counit $\tau_!\tau^*X \to X$, and extend this to a distinguished triangle

$$\tau_!\tau^*X \to X \to Z \to \Sigma(\tau_!\tau^*X) \to \cdots .$$

We note that applying τ^* to the counit is an isomorphism (as the unit of the adjunction is an isomorphism), so applying n^* for each $n \geq 1$, we in fact have isomorphisms there.

Furthermore, recall that triangles in $\mathbb{D}(Q_n)$ are built pointwise out of triangles in $\mathbb{D}(e)$. At $n \geq 1$, our triangles look like

$$X_n \xrightarrow{\text{iso}} X_n \xrightarrow{0} 0 \longrightarrow \Sigma X_n \cdots$$

While at $n = 0$ the triangle is

$$0 \longrightarrow X_0 \xrightarrow{\text{id}} X_0 \longrightarrow 0 \cdots$$

Therefore, reconstructing the object Z we see that it is only supported in degree 0 where $0^*Z_0 = X_0$. Note that $\tau^*(x_i)^*Z = 0$, hence $(\pi_{\mathbb{N}})_!Z \cong (\pi_{\mathbb{N}})_!\tau^*Z \cong (\pi_{\mathbb{N}})_!0 = 0$.

What we have proven then is that the cone of any counit map is in fact in the localizing subcategory, so in fact in $\mathbb{P}^n_{\mathbb{D}}$ all the counit maps for the $(\tau_!, \tau^*)$ adjunction are isomorphisms. This proves that descending to $\mathbb{P}^n_{\mathbb{D}}$, $\tau_!$ and τ^* form an adjoint equivalence pair. □

Corollary 8.4.4. *Similarly, τ_* descends to $\mathbb{P}^n_{\mathbb{D}}$ and is isomorphic to $\tau_!$.*

Proof. One can check that τ_* also descends to $\mathbb{P}^n_{\mathbb{D}}$ via a similar argument to $\tau_!$.

τ^* is an equivalence on $\mathbb{P}^n_{\mathbb{D}}$, hence its quasi-inverse $\tau_!$ is both a left and right adjoint. The right adjoint τ_* will then be equivalent to the quasi-inverse $\tau_!$. □

Now we verify that the twisting respects the monoidal structure.

Proposition 8.4.5. *In* $\mathbb{P}^n_{\mathbb{D}}$, $\tau_!(X) \cong X \boxtimes \mathcal{O}(-1)$.

Proof. This is clear, as τ^* models the twist by $\mathcal{O}(1)$ and $\tau_!$ is a quasi-inverse to τ^*. □

Proposition 8.4.6. τ^* *respects the monoidal structure on* $\mathbb{P}^n_{\mathbb{D}}$*, in the sense that* $(\tau^* X \otimes Y) \cong \tau^*(X \otimes Y)$.

Proof. We verify that $\tau_!$ respects the monoidal structure on $\mathbb{P}^n_{\mathbb{D}}$. This is akin to the verification for $\mathbb{D}^{Q_n}$, as it follows from verifying the commutativity of the square

$$\begin{array}{ccc} Q_n \times Q_n & \xrightarrow{\tau \times 1} & Q_n \times Q_n \\ {\scriptstyle\otimes}\downarrow & & {\scriptstyle\otimes}\downarrow \\ Q_n & \xrightarrow{\tau} & Q_n \end{array}$$

The requisite assertion on $\tau_!$ follows, noting that in $\mathbb{D}^{Q_n}$,

$$\begin{aligned} \tau_!(X \otimes Y) &= \tau_!(\otimes)_!(X \boxtimes Y) \\ &\cong (\otimes)_!(\tau \times 1)_!(X \boxtimes Y) \\ &= (\otimes)_!(\tau_! X \boxtimes Y) \\ &= (\tau_! X \otimes Y) \end{aligned}$$

Therefore this also descends to $\mathbb{P}^n_{\mathbb{D}}$. τ^* is a quasi-inverse to $\tau_!$, so indeed we also have $(\tau^* X \otimes Y) \cong \tau^*(X \otimes Y)$. □

Recall that we had a collection of derivator morphisms

$$(x_i^{-1})_! : \mathbb{D}^{Q_n} \to \mathbb{A}^n_{\mathbb{D}}$$

for $0 \leq i \leq n$. Note that if $X \in Nilp(\mathbb{D})$, then by definition $(x_i^{-1})_! X = 0$, therefore these morphisms factor through $\mathbb{P}^n_{\mathbb{D}}$.

Proposition 8.4.7. $(x_i^{-1})_! : \mathbb{P}^n_{\mathbb{D}} \to \mathbb{A}^n_{\mathbb{D}}$ *models the restriction of a sheaf from* $\mathbb{P}^n$ *to the canonical open set* $D(x_i) \cong \mathbb{A}^n$.

Proof. The above discussion indicates that $(x_i^{-1})_!$ factors through $\mathbb{P}^n_{\mathbb{D}}$. On the level of graded modules we proved above that this precisely takes a graded module, views it as a sheaf, and constructs the restriction on $D(x_i) \cong \mathbb{A}^n$. □

Moreover, as above we have gluing data that $(x_i)^{-1}(x_j)^{-1} \cong (x_i x_j)^{-1} \cong (x_j)^{-1}(x_i)^{-1}$ (the functors are actually equal). We can take left Kan extensions of these (equal) functors and they are naturally equal. But what does this actually mean?

Proposition 8.4.8. *The localization $Q_n[(x_i x_j)^{-1}]$ is equivalent to $\underline{\mathbb{N}^{n-1} \times \mathbb{Z}}$.*

Proof. This localization is just $Q_n[(x_i)^{-1}][x_j^{-1}]$. The canonical identification we will use of $Q_n[x_i^{-1}]$ with $\underline{\mathbb{N}^n}$ is going to be the object at 0 and with the map e_j given by $x_i^{-1}x_j$, with $0 \leq j \leq n$ and $i \neq j$.

To take an additional x_j^{-1} means I'm inverting the map e_j in $\mathbb{N}^n$, giving us the category $\underline{\mathbb{N}^{n-1} \times \mathbb{Z}_j}$. We write $\mathbb{Z}_j$ to remind us that the $\mathbb{Z}$ sits in the j-th coordinate. □

Remember that derivator shifted by $\underline{\mathbb{N}^n}$ represents affine n-space, while the derivator shifted by $\underline{\mathbb{N}^{n-1} \times \mathbb{Z}_j}$ represents the open subset of affine n-space with j-th coordinate nonzero, i.e. an appropriate copy of $\mathbb{A}^{n-1} \times \mathbb{G}_m$, which is naturally the intersection of opens $D(x_i)$ and $D(x_j)$.

Proposition 8.4.9. *There is a canonical morphism of derivators going from $\mathbb{P}^n_{\mathbb{D}}$ to the gluing of $n+1$ copies of $\mathbb{A}^n_{\mathbb{D}}$ with the gluing between the i-th and j-th copy along a copy of $\mathbb{A}^{n-1} \times \mathbb{G}_m$ in the correct way that gives a gluing of $\mathbb{P}^n$ if one thinks of these as simply their underlying topological spaces.*

Proposition 8.4.10. *The restriction $(x_i^{-1})_! : \mathbb{D}^{Q_n} \to \mathbb{A}^n_{\mathbb{D}}$ is a Bousfield localization of derivators.*

Proof. By a Bousfield localization we mean that the functor $(x_i^{-1}))_!$ has a fully faithful right adjoint. We will show the case $i = 0$, enabling simplicity in notation, and note that the cases $1 \leq i \leq n$ are all equivalent.

Recall that the equivalence between $Q_n[x_0^{-1}]$ and $\underline{\mathbb{N}}^n$ is given by sending all $n \in \mathbb{N}$ to the single object $\bullet$ in $\underline{\mathbb{N}}^n$, while $x_1, \cdots, x_n$ are sent to $e_1, \cdots, e_n \in \mathbb{N}^n$, respectively, while the isomorphisms x_0 and x_0^{-1} are sent to the identity. A quasi-inverse going in the other direction can embed the single object $\bullet \in \underline{\mathbb{N}}^n$ as any $n \in \mathbb{N}$, but we will choose to embed at 0, and send the maps e_i to $x_0^{-1}x_i$. It can be easily verified that these two maps are mutual quasi-inverses. Of course, we have also that $i \cong j$ for any $i, j \in Q_n[x_0^{-1}]$, with isomorphisms given by the correct powers of x_0 in each direction.

So we aim to compute $(x_0^{-1})_!(x_0^{-1})^*$. Let $X \in \mathbb{A}^n_{\mathbb{D}}$ be an object, which we write as an underlying object X_0 along with n compatible maps $f_i, 1 \leq i \leq n$. The object $(x_0^{-1})^*X$ has at each level n the object X_0, with map x_0 the identity, and x_i^j the maps f_i for all $j \in \mathbb{N}$.

Now we give the calculation of $(x_0^{-1})_!$ at the object 0, via a (Der4) calculation. The relevant square is:

$$\begin{array}{ccc} (x_0^{-1}/0) & \xrightarrow{\ \mathrm{pr}\ } & Q_n \\ {\scriptstyle\pi}\downarrow & \overset{\alpha}{\swArrow} & \downarrow{\scriptstyle x_0^{-1}} \\ e & \xrightarrow[\ 0\ ]{} & Q_n[x_0^{-1}] \end{array}$$

Recall the definition of slice category; the objects of $(x_0^{-1}/0)$ are given by an integer $j \in \mathbb{N}$ along with a map $j \to 0$ in $Q_n[x_0^{-1}]$, i.e. a monomial $\prod_{0 \leq i \leq n} x_i^{k_i}$ of total degree $-j$, where $k_i \geq 0$ for $1 \leq i \leq n$ and k_0 being any integer. It is easily seen that the maximal possible k_0 in such an object will be $-j$. A morphism $(j, \prod_{0 \leq i \leq n} x_i^{k_i}) \to (j', \prod_{0 \leq i \leq n} x_i^{k'_i})$ is induced by a morphism in Q_n $\prod_{0 \leq i \leq n} x_i^{k''_i}$, such that the diagram below commutes:

$$\begin{array}{ccc} j & & \\ & \searrow^{\prod x_i^{k_i}} & \\ {\scriptstyle \prod x_i^{k''_i}}\downarrow & & 0 \\ & \nearrow_{\prod x_i^{k'_i}} & \\ j' & & \end{array}$$

Now let us consider the full $(\mathbb{N}, \leq)$-shaped subcategory of $(x_0^{-1}/0)$ consisting of the objects (j, x_0^{-j}). We can easily see that any object $(j, \prod x_i^{k_i})$ has a canonical map induced by $\prod_{i=1}^n x_i^{k_i}$ to $(k_0, x_0^{-k_0})$.

Furthermore, such an assignment is functorial. So this induces a functor from $(x_0^{-1}/0)$ to

$(\mathbb{N}, \leq)$ by viewing the designated subcategory of (j, x_0^{-j})-s as a copy of $(\mathbb{N}, \leq)$. It is easily checked that this functor is a left adjoint and that the inclusion i_0 of $(\mathbb{N}, \leq)$ by sending $j \mapsto (j, x_0^{-j})$ is the right adjoint.

Therefore, i_0 is a "homotopy final" functor, i.e. the square below is homotopy exact.

$$\begin{array}{ccc} (\mathbb{N}, \leq) & \xrightarrow{i_0} & (x_0^{-1}/0) \\ {\scriptstyle\pi}\big\downarrow & \overset{id}{\swarrow} & \big\downarrow{\scriptstyle\pi} \\ e & \xrightarrow[1]{} & e \end{array}$$

The implication of this is that for $Y \in \mathbb{D}^{(x_0^{-1}/0)}$, $\pi_! Y \cong \pi_!(i_0)^* Y$. However, if Y is an object of the form $\mathrm{pr}^*(x_0^{-1})^* X$, then in fact $(i_0)^* Y$ is in fact just a constant diagram: $(\pi_{(\mathbb{N},\leq)})^* X_0$. However, $(\mathbb{N}, \leq)$ obviously has contractible nerve as it contains an initial object 0. Therefore, this colimit is just X_0.

This gives the underlying object for applying $(x_0^{-1})_!(x_0^{-1})^*$ to $X \in \mathbb{A}^n_{\mathbb{D}}$. The various coherent morphisms in the diagram $\mathbb{N}^n$ can be induced by on the colimit. Utilizing the equivalence $\underline{\mathbb{N}}^n$ with $Q_n[x_0^{-1}]$, the action of maps $e_i \in \mathbb{N}^n$ are induced by the maps $\pi^*_{(\mathbb{N},\leq)} f_i$ on the diagram $\pi^*_{(\mathbb{N},\leq)}(X_0)$, i.e. simply f_i. Therefore, we see that $(x_0^{-1})_!(x_0^{-1})^* X \cong X$ for all $X \in \mathbb{A}^n_{\mathbb{D}}$. □

Proposition 8.4.11. *The induced morphism $(x_i)_!^{-1} : \mathbb{P}^n_{\mathbb{D}} \to \mathbb{A}^n_{\mathbb{D}}$ is also a Bousfield localization.*

Proof. While we can't use Brown representability, we can precompose with the localization from $\mathbb{D}^{Q_n}$ to $\mathbb{P}^n_{\mathbb{D}}$. The assertion is then clear. □

Proposition 8.4.12. *The shifting diagram over $\mathbb{Z}$ or over $\mathbb{N}$ give equivalent derivators/constructions for $\mathbb{P}^n$.*

Proof. Consider the embedding $i_{\mathbb{N}} : Q_n^{\mathbb{N}} \hookrightarrow Q_n^{\mathbb{Z}}$. The left Kan extension along this is fully faithful and an extension by 0. One can check that the embedding $(i_{\mathbb{N}})_!$ induces an equivalence after localization. □

8.5 Compatibility

Naturally, after discussion of various forms of $\mathbb{P}^n$-like activity, it is incumbent upon us to prove that we have actually recovered the usual $\mathbb{P}^n$.

Proposition 8.5.1. *This model gives a construction for the (left) derivator of projective space over any additive derivator. By this we simply mean to take the localization of* $\mathbb{D}^{Q_n}$ *at the collection of morphisms* f *such that* $(x_i^{-1})_! f$ *is an isomorphism. In the proof above we would have that this is a left derivator as the left Kan extensions in* $\mathbb{P}^n_{\mathbb{D}}$ *can be taken in* $\mathbb{D}^{Q_n}$ *prior to localization.*

Proof. We realize that sheaves on $\mathbb{P}^n_R$ are precisely $\mathbb{N}$-graded modules over $R[t_0, \cdots, t_n]$ with the relation that the restrictions of the modules to the $n+1$ copies of $\mathbb{A}^n_R$ covering $\mathbb{P}^n_R$ are isomorphic. As described in the construction section, if a morphism f in $\mathbb{D}^{Q_n}$ has $(x_i^{-1})_! f$ an isomorphism, this means that f restricted to the i-th copy of $\mathbb{A}^n_R$ is an isomorphism.

It is now apparent that the localization $\mathbb{D}^{Q_n}/\mathrm{Nilp}(\mathbb{D})$ represents the derivator of projective space if $\mathbb{D}$ is the represented derivator of R-modules. Taking the derived categories gives the analoguous result for triangulated derivators. □

8.6 Embeddings of Projective Space

Recall from basic algebraic geometry the following two embeddings:

1. The *Segre embedding*

$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$

   via $\sigma([X],[Y]) = [\{X_i Y_j\}]$.

2. The *degree d Veronese embedding*

$$\sigma : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$$

   via $[X_0 : X_1 : \cdots : X_n]$ to all monomials of total degree d.

Owing to the "uniform" nature of these functions, each of these embeddings can be realized in the world of derivators. Specifically, we will construct inverse image morphisms on the shifted derivators $\mathbb{D}^{Q_n}$ while verifying that the kernels of the localization to projective space map to kernels.

Let us begin with the Segre embedding. We should construct a monoidal morphism $\mathbb{P}_{\mathbb{D}}^{(n+1)(m+1)-1} \to (\mathbb{P}^n \times \mathbb{P}^m)_{\mathbb{D}}$ by constructing a morphism

$$\mathbb{D}^{Q_{(n+1)(m+1)-1}} \to \mathbb{D}^{Q_m \times Q_n}$$

and then checking that it descends.

First we describe what the product of projective spaces would be.

Theorem 8.6.1. *The derivator $(\mathbb{P}^n \times \mathbb{P}^m)_{\mathbb{D}}$ is the shifted derivator $\mathbb{D}^{Q_n \times Q_m}$ localized at the following set of morphisms f such that $(x_i \times x_j)_!^{-1}(f)$ is an isomorphism for all $0 \leq i \leq n$, $0 \leq j \leq m$.*

Proof. Beginning with just Q_n is the original proof of $\mathbb{P}^n$; we can view this construction as $\mathbb{P}^m$-construction on $\mathbb{P}^n_{\mathbb{D}}$, and hence this represents the correct construction. □

Definition 8.6.2. Define a functor $S_{n,m} : Q_{(n+1)(m+1)-1} \to Q_m \times Q_n$ as follows: for each k in the domain we send it to $(k, k) \in Q_m \times Q_n$, and for a morphism of the form $x_i : k \to k+1$, we make an assignment of a morphism $(x_{l_1}, x_{l_2}) : (k, k) \to (k+1, k+1)$, where $0 \leq l_1 \leq m$ and $0 \leq l_2 \leq n$.

It is easy to see that this is sufficient to ensure functoriality on compositions in $Q_{(n+1)(m+1)-1}$, and the commutativity of x_i and x_j ensures that it is well-defined.

Proposition 8.6.3. *The above functor is monoidal and fully faithful.*

Proof. By construction we see that the functor is an equivalence onto the subcategory of $Q_m \times Q_n$ with objects (k, k). Moreover, the functor is clearly monoidal given the product monoidal structure on $Q_m \times Q_n$. □

Remark 8.6.4. We can think of an object of $\mathbb{D}^{Q_m \times Q_n}$ as a bigraded module in $R[x_0, \cdots, x_m, y_0, \cdots, y_n]$ where each x_i has bidegree $(1, 0)$ and each y_j has bidegree $(0, 1)$.

Now I claim that left Kan extensions on the above two functors, after localization to $\mathbb{P}^n_{\mathbb{D}}$, give models of Segre and Veronese embeddings. Let's begin with the Segre embedding.

Proposition 8.6.5. *The descended morphism of derivators* $\overline{S_{n,m!}}$ *from* $\mathbb{P}_{\mathbb{D}}^{(n+1)(m+1)-1} \to (\mathbb{P}^n \times \mathbb{P}^m)_{\mathbb{D}}$ *represents the inverse image of the Segre embedding.*

Proof. As with some of our $\mathbb{A}^1$-related proofs, we examine the right adjoint $\overline{S_{n,m}}^*$. Recall that this takes an object in $\mathbb{D}^{Q_n \times Q_m}$ and restricts it to the full subcategory on the diagonal component, *i.e.* objects of the form (k,k). □

Now we examine the Veronese embedding.

Similarly, we have a monoidal functor that will realize the Veronese embedding.

Definition 8.6.6. For each $n, d > 0$, we have a functor

$$V_{n,d} : Q_{\binom{n+d}{d-1}} \to Q_n,$$

sending $k \mapsto dk$ and assigning each morphism x_i in $Q_{\binom{n+d}{d}-1}$ to a degree d monomial in Q_n.

Proposition 8.6.7. *The above functor is monoidal.*

Proof. This is clear from the definition on both objects and morphisms. □

Proposition 8.6.8. *The descended morphism of derivators* $\overline{V_{n,d!}}$ *from* $\mathbb{P}^{\binom{n+d}{d}-1} \to \mathbb{P}^n_{\mathbb{D}}$ *represents the inverse image of the Segre embedding.*

Proof. Again, the right adjoint $\overline{V^*_{n,d}}$ takes an object X in $\mathbb{D}^{Q_n}$ to $V^*_{n,d}(X)$ in $\mathbb{D}^{Q_{\binom{n+d}{d}-1}}$, where $V^*_{n,d}(X)_k$ is X_{dk}, and the degree 1 morphisms on $V^*_{n,d}(X)$ are precisely degree d monomials, which is precisely what we'd expect for the direct image.

Therefore, its left adjoint $\overline{V_{n,d!}}$ is indeed a model for the inverse image. □

CHAPTER 9

Generation of projective space

One extremely interesting result about the derived category of $\mathbb{P}^n$ is that it is generated by $n+1$ consecutive twisting sheaves. Since our $\mathbb{P}^n$ is defined as a localization, we don't have a great grasp on the morphisms inside $\mathbb{P}^n_{\mathbb{D}}(e)$ (unless we better understood the embedding back to $\mathbb{D}^{Q_n}$, which is an issue in and of itself).

Rather, our strategy is to obtain a skeleton of this result in the unlocalized $\mathbb{D}^{Q_n}$, and then to push the generators down via the localization to $\mathbb{P}^n_{\mathbb{D}}$. Since ultimately our interest is more in objects of $\mathbb{P}^n_{\mathbb{D}}$, there is no substantive worry of having an incomplete understanding of morphisms in $\mathbb{P}^n_{\mathbb{D}}$.

We begin with a discussion of the case where $\mathbb{D}$ has a symmetric monoidal structure. There is no strong obligation for this assumption, but the construction of the Koszul complex is more intuitive and generalizable in the monoidal case. However, the assumption of compact generation is paramount here, since this is what allows us to transition between two notions of generation (the Hom-set notion versus the localizing category notion).

After we see how $\mathbb{P}^n_{\mathbb{D}}$ is generated when $\mathbb{D}(e)$ is compactly generated and $\mathbb{D}$ is a symmetric monoidal triangulated derivator, we may drop the symmetric monoidal assumption. This is due to the fact that by definition, taking the external product with an object in a symmetric monoidal triangulated derivator is a triangulated functor, certain constructions that we made in the symmetric monoidal case can be replaced by cofibers when a monoidal structure is lacking.

9.1 The monoidal case

As said above, here we assume that $\mathbb{D}$ admits a monoidal structure compatible with its tensor-triangulated structure.

Remark 9.1.1. From now on let $\mathbb{D}$ be a tensor-triangulated derivator that is generated by the monoidal unit $\mathbb{1}$. As we mentioned, there is no reason for this apart from ease of presentation.

However, everything done with $\mathbb{1}$ can be replaced by twisted versions of X, as everything in this section can simply be done by tensoring with X. After the next proposition we can explain why this simplification is sufficient.

Proposition 9.1.2. *$\mathbb{D}^{Q_n}(e)$ is generated by $\{k_!\mathbb{1}_{\mathbb{D}} : k \in \mathbb{N}\}$.*

Proof. We made the simplifying assumption that $\mathbb{D}(e)$ is generated by $\mathbb{1}$. Therefore, from 3.2.1 it is clear that a generating set for $\mathbb{D}(Q_n)$ is given by $\{k_!\mathbb{1} : k \in \mathbb{N}\}$. The presentation of $\mathbb{D}^{Q_n}(e)$ is immaterial here. □

The reasoning for the assumption with $\mathbb{1}$ is as follows. Suppose $\mathbb{D}(e)$ had a generating set consisting of $\{X_\lambda : \lambda \in \Lambda\}$, then we know $\mathbb{D}^{Q_n}(e)$ has a generating set consisting of $\{k_!X_\lambda : \lambda \in \Lambda, k \in \mathbb{N}\}$. For $\mathbb{P}^n_{\mathbb{D}}$ we are interested in truncating this generating set for each $\lambda \in \Lambda$, *i.e.* by only taking $\{k_!X_\lambda : 0 \le k \le n\}$ is sufficient to obtain $\{k_!X_\lambda : k < n\}$. It would be sufficient to look at $X_\lambda = \mathbb{1}$, and then tensor with X_λ, knowing that tensoring respects left Kan extensions.

Proposition 9.1.3. *The category $\mathbb{P}^n_{\mathbb{D}}(e)$ is generated by the same collection in the image $\{k_!\mathbb{1}_{\mathbb{D}} : k \in \mathbb{N}\}$.*

Proof. The shifted derivator $\mathbb{D}^{Q_n^{\mathbb{N}}}$ is generated by that set $\{\mathcal{O}(-k) : k \in \mathbb{N}\}$. So the localization is also generated by the same set. □

Theorem 9.1.4. *In fact, it is sufficient to use the equivalent of $\mathcal{O}, \cdots, \mathcal{O}(-n)$, as $\mathcal{O}(-n-1) \in \langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-n)\rangle$ in $\mathbb{P}^n$. Equivalently, in $\mathbb{P}^n_{\mathbb{D}}(e)$, $\mathcal{O}(-n-k-1) \in \langle \mathcal{O}(-k), \cdots, \mathcal{O}(-n-$*

$k)\rangle$. Here when we use $\langle X, \cdots, \rangle$ we mean the thick subcategory generated by the collection of objects.

Therefore, $\mathbb{P}^n_{\mathbb{D}}$ is generated by $\{\mathcal{O}(-k) : 0 \leq k \leq n\}$.

We will attack this by building $\mathcal{O}(-n-1)$ out of $\{\mathcal{O}(-k), 0 \leq k \leq n\}$ and an element in $\mathrm{Nilp}(\mathbb{D})$.

Recall the functor $\tau : Q_n \to Q_n$. We proved that $\tau_!$ is fully faithful, *i.e.* $\tau^* \tau_! \cong \mathrm{Id}$.

There are natural transformations $\mathrm{Id} \to \tau$ given by mapping along x_i for a specified $0 \leq i \leq n$. Therefore, for any $X \in \mathbb{D}^{Q_n}$ and any $0 \leq i \leq n$, there is a natural transformation

$$x_i^* : \tau_! X \to \tau^* \tau_! X \cong X.$$

Let $\mathrm{Kos}_X(x_i)$ denote the cone of this morphism. If $X = \mathbb{1}$ we drop the X label.

Remark 9.1.5. Note that we can also make sense of the natural transformation x_i in $Q_n[x_i^{-1}]$, where it is a natural isomorphism for the corresponding i.

Proposition 9.1.6. *For any $0 \leq i \leq n$ we have that $(x_i^{-1})_! Kos(x_i) = 0$ in $\mathbb{D}^{Q_n^{\mathbb{Z}}}[x_i^{-1}] \cong \mathbb{A}^n_{\mathbb{D},i}$.*

Proof. The map $x_i : \mathcal{O}(-1) \to \mathcal{O}$ is an isomorphism after applying the functor $(x_i)^{-1})_!$, and the left Kan extension is exact, so the cone of $(x_i^{-1})_! x_i$ is $(x_i^{-1})_! Kos(x_i)$. □

Proposition 9.1.7. *In $\mathbb{P}^n_{\mathbb{D}}$, the object $\otimes_{i=0}^n Kos(x_i)$ is zero.*

Proof. Each $(x_i)_!^{-1}$ is a monoidal functor. Furthermore, $(x_i)^{-1})_! Kos(x_i)$ is a zero object, hence applying the functor $(x_i^{-1})_!$ to $\otimes_{i=0}^n Kos(x_i)$ is zero. Therefore, $\otimes_{i=0}^n Kos(x_i)$ is a zero object in $\mathbb{P}^n_{\mathbb{D}}$. □

Now we go back to proving Theorem 9.1.4 by giving a precise reformulation.

Theorem 9.1.8. *In $\mathbb{D}^{Q_n}(e)$, the object $\mathcal{O}(-n-1)$ lies in $\langle \mathcal{O}, \cdots, \mathcal{O}(-n), \otimes_{i=0}^n Kos(x_i) \rangle$.*

Proof. Let us fix an $n \in \mathbb{N}$ and proceed by induction on k ($k \leq n$). The base case $k = 0$ is clear, as the triangle

$$\cdots \mathcal{O}(-1) \xrightarrow{x_0} \mathcal{O} \longrightarrow Kos(x_0) \cdots$$

indicates that $\mathcal{O}(-1)$ lies in $\langle \mathcal{O}, Kos(x_0) \rangle$.

Next, let us suppose that $\mathcal{O}(-k) \in \langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-k+1), \otimes_{i=0}^{k-1} Kos(x_i) \rangle$. Then we will show that $\mathcal{O}(-k-1) \in \langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-k), \otimes_{i=0}^{k} Kos(x_i) \rangle$.

Let us first note that given the triangle

$$\cdots \mathcal{O}(-1) \xrightarrow{x_k} \mathcal{O} \longrightarrow Kos(x_k) \cdots$$

we can twist everything by $\mathcal{O}(-i)$ to obtain the triangle

$$\cdots \mathcal{O}(-i-1) \xrightarrow{x_t} \mathcal{O}(-i) \longrightarrow Kos(x_k)(-i) \cdots$$

Therefore, if $0 \leq i \leq k-1$, we see that $Kos(x_k)(-i) \in \langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-k) \rangle$. So we see that the objects $Kos(x_k) \otimes \mathcal{O}, Kos(x_k) \otimes \mathcal{O}(-1) \cdots, Kos(x_k) \otimes \mathcal{O}(-k+1), Kos(x_k) \otimes (\otimes_{i=0}^{k-1} Kos(x_i))$ all lie in $\langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-k), \otimes_{i=0}^{k} Kos(x_i) \rangle$.

However, the collection of objects we've picked up are just $Kos(x_k)$ tensored with $\mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-k+1), \otimes_{i=0}^{k-1} Kos(x_i)$. By the inductive hypothesis, we know that $\mathcal{O}(-k)$ is in the triangulated subcategory generated by those objects, hence $Kos(x_k)(-k)$ lies in the triangulated subcategory $\langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-k), \otimes_{i=0}^{k} Kos(x_i) \rangle$. Now note that when we twist the triangle defining $Kos(x_k)$ by $\mathcal{O}(-k)$, we get

$$\cdots \mathcal{O}(-k-1) \xrightarrow{x_k} \mathcal{O}(-k) \longrightarrow Kos(x_k)(-k) \cdots$$

So we see that $\mathcal{O}(-k-1)$ is generated by $\mathcal{O}(-k)$ along with $Kos(x_k)(-k)$, hence by $\langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-k), \otimes_{i=0}^{k} Kos(x_i) \rangle$. By induction, this proves the proposition. □

This gets us the original reformulation. Now we turn to proving the original statement of theorem 9.1.4.

Proof. First note that after localizing, the claim in the theorem reduces to the statement that $\mathcal{O}(-n-1) \in \langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-n) \rangle$ in $\mathbb{P}^n_{\mathbb{D}}(e)$, as $\otimes_{i=0}^{n} Kos(x_i)$ has been sent to 0. Therefore,

$\mathcal{O}(-m)$ for all $m > n$ lies in $\langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-n) \rangle$ simply by taking the triangles to generate $\mathcal{O}(-n-1)$ and begin twisting down by $\mathcal{O}(-1)$. Therefore, $\mathcal{O}, \cdots, \mathcal{O}(-n)$ together generate $\mathbb{P}^n_{\mathbb{D}}$. □

Proposition 9.1.9. *Theorem 9.1.4 is a mechanism to "twist down" in $\mathbb{P}^n_{\mathbb{D}}(e)$. We can also move in the reverse direction in $\mathbb{P}^n_{\mathbb{D}}(e)$. In particular, $\mathcal{O} \in \langle \mathcal{O}(-1), \cdots, \mathcal{O}(-k-1), \otimes_{i=0}^{k} Kos(x_i) \rangle$.*

Proof. Again we prove the statement by induction. $k = 0$ is obvious by definition of $Kos(x_0)$.

The induction follows easily again. If the inductive hypothesis is that $\mathcal{O} \in \langle \mathcal{O}(-1), \cdots, \mathcal{O}(-k-1), \otimes_{i=0}^{k} \mathrm{Kos}(x_i) \rangle$, then changing k to $k+1$ we ought to look at the category $\langle \mathcal{O}(-1), \cdots, \mathcal{O}(-k-2), \otimes_{i=0}^{k+1} \mathrm{Kos}(x_i) \rangle$. We first note that $\mathrm{Kos}(x_{k+1}) \otimes \mathcal{O}(-1), \cdots, \mathrm{Kos}(x_{k+1}) \otimes \mathcal{O}(-k-1)$ lie in the thick subcategory generated by those elements.

It now follows that $\mathrm{Kos}(x_{k+1})$ lies in this thick subcategory, and hence $\mathcal{O}$ can be built out of that and $\mathcal{O}(-1)$. □

Corollary 9.1.10. *Let k be any integer. Then $\mathcal{O}(k), \cdots, \mathcal{O}(k+n)$ also generate $\mathbb{P}^n_{\mathbb{D}}$.*

Proof. If $k \geq -n$ we simply begin twisting down to obtain $\mathcal{O}(k-1)$, etc until we obtain $\mathcal{O}(-n), \cdots, \mathcal{O}$, which we know does generate $\mathbb{P}^n_{\mathbb{D}}$.

Otherwise we start twisting up until we obtain $(\mathcal{O}(-n), \cdots, \mathcal{O})$ by similar logic. □

Now if $\mathbb{D}$ is not generated by the unit $\mathbb{1}$ but instead by some collection of generators $\mathcal{G} = \{G_\lambda : \lambda \in \Lambda\}$. Then $\mathbb{D}^{Q_n^{\mathbb{N}}}$ is generated by $\{k_! G_\lambda : \lambda \in \Lambda, k \in \mathbb{N}\}$. But in fact again we can restrict the k to $0, 1, \cdots, n$ again by the above proof. The proof proceeds as follows:

Proof. We have that for $\mathbb{P}^n_{\mathbb{D}}$, $\mathcal{O}(-m) \in \langle \mathcal{O}, \cdots, \mathcal{O}(-n) \rangle$ for all $m \geq 0$. Tensoring with any object G, the object $G \boxtimes \mathcal{O}(-m) \in \langle G \boxtimes \mathcal{O}, \cdots, G \boxtimes \mathcal{O}(-n) \rangle$. Therefore, we take G to range over a set of generators for $\mathbb{D}$ to ensure that the set $\{G \boxtimes \mathcal{O}(-k) : 0 \leq k \leq n\}$ forms a generating set for $\mathbb{P}^n_{\mathbb{D}}$. □

9.2 An adaptation for the non-monoidal case

In fact, everything can be done without the use of a monoidal structure. The question is how to transpose the analogues of the tensor products of the Koszul objects.

Proposition 9.2.1. *$\otimes_{i=0}^{k} Kos(x_i)$ can be constructed by the formula $C^{k+1}\gamma_k^*(\tau^{k+1})_!(\mathbb{1})$ (essentially, by non-monoidal means).*

Proof. First we explain what the composition $C^{k+1}\gamma_k^*(\tau^{k+1})_!$ is.

Let γ_k be the functor from $[1]^{k+1} \times Q_n^{\mathbb{N}} \to Q_n^{\mathbb{N}}$ sending $(e_0, \cdots, e_k), l \mapsto \sum e_i + l$ on objects, and it sends $(0 \to 1)$ in coordinate i to the map x_i between two neighboring objects in $Q_n^{\mathbb{N}}$. Let $G \in \mathbb{D}^{Q_n^{\mathbb{N}}}$ be an object, and consider the object $\gamma_k^*(\tau^{k+1})_!G$.

For an object $(e_0, \cdots, e_k) \in [1]^{k+1}$, the $Q_n^{\mathbb{N}}$-shaped object (i.e. with $(e_0, \cdots, e_k)^*$ evaluated at the result) at this point is $(\tau^{\sum_{i=0}^{k} e_i})^*(\tau^{k+1})_!G$.

However, that object is precisely $(\tau^{k+1-\sum_{i=0}^{k} e_i})_!G$ as each e_i is either 0 or 1. According to the definition of the functor $(0 \to 1)$ in coordinate i corresponds to the map x_i as previously mentioned. To wit, taking the case $k = 1$ and applying $\gamma_1^* \circ (\tau^2)_!$ to an object G we get a coherent diagram of the form

$$\begin{array}{ccc} \tau_!^2 G & \xrightarrow{x_0} & \tau_! G \\ {\scriptstyle x_1}\downarrow & & {\scriptstyle x_1}\downarrow \\ \tau_! G & \xrightarrow{x_0} & G \end{array}$$

How would we obtain such a tensor product? First recall that x_i is a canonical morphism that exists for any object X, there exists a morphism $x_i : \tau_! X \to X$ induced by the isomorphism $X \cong \tau^*\tau_! X$, so $Kos(x_i)$ is the cone of that morphism applied at $X = \mathcal{O}$. For any X we have a similar object $Kos_{x_i}(X)$ via this construction and $Kos(x_i)$ is just $Kos_{x_i}(\mathbb{1})$.

Now let's examine $Kos(x_0) \otimes Kos(x_1)$. We have a triangle

$$\cdots Kos(x_0) \otimes \mathcal{O}(-1) \xrightarrow{1 \otimes x_1} Kos(x_0) \otimes \mathcal{O} \longrightarrow Kos(x_0) \otimes Kos(x_1) \cdots$$

But remembering of course that $Kos(x_0)$ itself was the cone of $x_0 : \mathcal{O}(-1) \to \mathcal{O}$, so in fact we have the following 3×3 diagram where every'second arrow' is the cofiber morphism of the first and vice versa:

$$\begin{array}{ccccc}
\mathcal{O}(-1) \otimes \mathcal{O}(-1) & \xrightarrow{x_0 \otimes 1} & \mathcal{O} \otimes \mathcal{O}(-1) & \longrightarrow & Kos(x_0) \otimes \mathcal{O}(-1) \\
{\scriptstyle 1\otimes x_1}\downarrow & & {\scriptstyle 1\otimes x_1}\downarrow & & {\scriptstyle 1\otimes x_1}\downarrow \\
\mathcal{O}(-1) \otimes \mathcal{O} & \xrightarrow{x_0 \otimes 1} & \mathcal{O} \otimes \mathcal{O} & \longrightarrow & Kos(x_0) \otimes \mathcal{O} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{O}(-1) \otimes Kos(x_1) & \xrightarrow{x_0 \otimes 1} & \mathcal{O} \otimes Kos(x_1) & \longrightarrow & Kos(x_0) \otimes Kos(x_1)
\end{array}$$

That is to say, given a coherent diagram

$$\begin{array}{ccc}
\mathcal{O}(-2) & \xrightarrow{x_1} & \mathcal{O}(-1) \\
{\scriptstyle x_0}\downarrow & & {\scriptstyle x_0}\downarrow \\
\mathcal{O}(-1) & \xrightarrow{x_1} & \mathcal{O}
\end{array}$$

we can first take cones in the x_1-direction and then in the x_0-direction, or vice versa to obtain the object $Kos(x_0) \otimes Kos(x_1)$. This provides a natural means to obtain the object $Kos(x_0) \otimes Kos(x_1)$ without needing any monoidal structure.

We can tensor the entire diagram with G to construct the object $Kos(x_0) \otimes Kos(x_1) \otimes G$, for instance. To achieve the tensor product $\otimes_{i=0}^{k} Kos(x_i)$, we would have to construct a $k+1$-dimensional cube where each object was of the form $\tau_!^j G$ and each map of the form x_i, before taking the cone simultaneously along all $n+1$ coordinates.

Therefore, in the presence of the monoidal structure, $\otimes_{i=0}^{k} Kos(x_i)$ is $C^{k+1}\gamma_k^*(\tau^{k+1})_!(\mathbb{1})$ and $X \otimes (\otimes_{i=0}^{k} Kos(x_i))$ is $C^{k+1}\gamma_k^*(\tau^{k+1})_!(X)$. However, this is obviously defined without the use of any monoidal structure. □

Moreover, above we have verified that $(x_i^{-1})_! Kos(x_i) = 0$, and indeed $(x_i^{-1})_! Kos_{x_i}(G)$ is 0 by similar logic. Now we can verify that

Proposition 9.2.2. *In $\mathbb{P}^n_{\mathbb{D}}$ the object $C^{n+1}\gamma_n^*(\tau^{n+1})_!(X)$ is zero for any X.*

Proof. This is equivalent to checking that $(x_i^{-1})_! C^{n+1}\gamma_n^*(\tau^{n+1})_!(X)$ is zero for any X and all $0 \leq i \leq n$.

First it's clear that the cone morphism, γ_n^*, and $(\tau^{n+1})_!$ are all cocontinuous since they are a composition of left adjoint morphisms. But after applying $(x_i^{-1})_!$ we see that the maps x_i are all isomorphisms, hence applying only the cone in the i-direction we get all 0s. This proves the requisite statement. $\square$

Therefore, the proofs in the previous section requiring a monoidal structure can be done without a monoidal structure. We will not repeat them here as no new insight would be gained by it.

Proposition 9.2.3. *With the assumptions above,*

$$\mathbb{D}^{Q_n^{\mathbb{N}}} = Loc\langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-n), (\tau^k)_! \otimes_{i=0}^n Kos(x_i)\rangle$$

for $k \in \mathbb{N}$.

Proof. This is evident as $\mathbb{D}^{Q_n^{\mathbb{N}}} = \mathrm{Loc}\langle \mathcal{O}(-k), k \in \mathbb{N}\rangle$ and we have indicated how to build $\mathcal{O}(-n-1)$ out of $\{\mathcal{O}, \cdots, \mathcal{O}(-n), \otimes_{i=0}^n Kos(x_i)\}$ and then we can obtain higher $\mathcal{O}(-n-k-1)$ by twisting the construction for $\mathcal{O}(-n-1)$ by $\mathcal{O}(-k)$. $\square$

We also have a $\mathbb{Z}$-version.

Proposition 9.2.4. *With the assumptions above,*

$$\mathbb{D}^{Q_n^{\mathbb{Z}}} = Loc\langle \mathcal{O}, \mathcal{O}(-1), \cdots, \mathcal{O}(-n), (\tau^k)_! \otimes_{i=0}^n Kos(x_i)\rangle$$

for $k \in \mathbb{Z}$.

The proof is similar, since the inclusion left Kan extension $\mathrm{inc}_! : \mathbb{D}^{Q_n^{\mathbb{N}}} \to \mathbb{D}^{Q_n^{\mathbb{Z}}}$ is a fully faithful morphism as are $\tau_!$ on both $\mathbb{D}^{Q_n^{\mathbb{N}}}$ and $\mathbb{D}^{Q_n^{\mathbb{Z}}}$. τ is an equivalence on $Q_n^{\mathbb{Z}}$.

We'd like to gain an understanding of the object $\otimes_{i=0}^n \mathrm{Kos}(x_i)$. In the ring-theoretic world, we know that the only homology of the total Koszul complex is the base ring R concentrated in degree 0. This turns out to be a purely diagrammatic fact, as we demonstrate below.

Proposition 9.2.5. $\otimes_{i=0}^{n} Kos(x_i) \cong 0_* \mathbb{1}$.

We attempt to compute the pointwise values $l^* \otimes_{i=0}^{n} \mathrm{Kos}(x_i)$ for $l \in \mathbb{N}$ and find that they are 0 except when $l = 0$, and there the value is $\mathbb{1}$.

Proof. Recall that an alternate characterization of this Koszul object is the total cofiber of the diagram $\Theta_n^*(n+1)_! \mathbb{1}_{\mathbb{D}}$, where $\Theta_n : Q_n \times [1]^{n+1} \to Q_n$ is the functor that sends $(K, e_0, e_1, \cdots, e_n)$ to $K + \sum_0^n e_i$ on objects and $(\prod_{i=0}^n x_i^{a_i}, c_0, \cdots, c_n)$ to $\prod_{i=0}^n x_i^{a_i + c_i}$ on morphisms.

For example, in $\mathbb{D}^{Q_1}$, $\Theta_1^* 2_! \mathbb{1}_{\mathbb{D}}$ is an object in $\mathbb{D}^{Q_n}([1]^2)$ with the following underlying diagram:

$$\begin{array}{ccc} 2_! \mathbb{1} & \xrightarrow{x_1^*} & 1_! \mathbb{1} \\ {\scriptstyle x_0^*}\downarrow & & {\scriptstyle x_0^*}\downarrow \\ 1_! \mathbb{1} & \xrightarrow{x_1^*} & 0_! \mathbb{1} \end{array}$$

Total cofibers can be computed pointwise: if we first apply 0^* to any diagram ${\Theta_n}^*(n+1)_! X$, then only the $(1, \cdots, 1)$-entry of the $(n+1)$-cube is X, and hence the total cofiber is the cone of $0 \to X$, or X.

Now apply i^* to the diagram for $i > 0$: we know that $i^* k_! X = 0$ if $i < k$, and that we can simplify $i^* k_! X = (i-k)^* 0_! X$ owing to the full faithfulness of $\tau_!$. We know that $l^* 0_! X = \coprod_{\sum_0^n a_j = l, a_j \geq 0} X_{x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}}$. A map x_j^* is the map $l^* 0_! X \to (l+1)^* 0_! X$ obtained by multiplying by x_j on all X-subscripts.

Therefore, when $i > 0$ the (n+1)-cube can be decomposed thusly: (this is not technically proper, but the cofiber is also the composition of iterated cones, which can be taken incoherently). Examine $m^* 0_! X = \coprod_{\sum_0^n a_j = i, a_j \geq 0} X_{x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}}$ and take a single $X_{x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}}$ therein.

There is a subset S of $\{0, \cdots, n\}$ where the power of x_i is 0, consider the $(n+1)$-cube which has objects 0 if any of the S-coordinates are 0, and X otherwise. The total $(n+1)$-cube is the direct sum of all such $(n+1)$-sub-cubes, taken over all monomials $x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}$ of total degree m. Each such cube is cocartesian (take first the iterated cones in the S^c-direction to get 0, then cones in the S-direction, which are subsequently trivial.

Therefore, $l^*\Theta_n^*(n+1)_!X$ is 0 if $l > 0$, and equal to X if $l = 0$.

This admits a map to 0_*X, i.e. the adjoint of the identity $0^*\Theta_n^*(n+1)_!X \cong X$. The adjoint is the identity after applying 0^* and must be an isomorphism for $l > 0$ since objects on both sides are 0. By (Der2), the total cofiber is isomorphic to 0_*X. □

Example We give an example computation in the case $n = 1$. Recall that we were computing pointwise the total cofiber of

$$\begin{array}{ccc} 2_!X & \xrightarrow{x_1^*} & 1_!X \\ {\scriptstyle x_0^*}\downarrow & & \downarrow{\scriptstyle x_0^*} \\ 1_!X & \xrightarrow{x_1^*} & 0_!X \end{array}$$

Applying 0^* to the diagram gives:

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array}$$

which obviously has total cofiber X. Applying 1^* gives:

$$\begin{array}{ccc} 0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle x_0} \\ X_1 & \xrightarrow{x_1} & X_{x_0} \coprod X_{x_1} \end{array}$$

and this has total cofiber 0.

Applying 2^* gives:

$$\begin{array}{ccc} X_1 & \xrightarrow{x_1} & X_{x_0} \coprod X_{x_1} \\ {\scriptstyle x_0}\downarrow & & \downarrow{\scriptstyle x_0} \\ X_{x_0} \coprod X_{x_1} & \xrightarrow{x-1} & X_{x_0^2} \coprod X_{x_0 x_1} \coprod X_{x_1}^2 \end{array}$$

and this decomposes into three squares (one constant, one zero in the left-hand column, one zero in the top row) that are each cocartesian and thus have total cofiber 0. Further computations are similar.

Proposition 9.2.6. *For any $X \in \mathbb{D}(e)$, $Hom((\tau^n)_! 0_* X, 0_* X) = 0$ in $\mathbb{D}^{Q_n}(e)$.*

Proof. This is evident by switching τ^n to the other side and noting that $(\tau^n)^* 0_* X = 0$, since $0_* X$ is an object that pointwise is X at $0 \in Q_n$ and 0 otherwise. Applying any number of operations of τ^* thus gives 0. □

9.3 The localization for projective space

Starting from a compactly generated triangulated derivator $\mathbb{D}$, we have created a triangulated derivator $\mathbb{P}^n_{\mathbb{D}}$ and exhibited a set of generators. Moreover, we have demonstrated twisting phenomena.

Theorem 9.3.1. *The canonical generating set $\{0_!\mathbb{D}, 1_!\mathbb{D}, \cdots, n_!\mathbb{D}, (\tau^k)_! 0_* \mathbb{D} : k \in \mathbb{N}\}$ splits $\mathbb{D}^{Q_n}$ into two components:*

$$Loc\{(\tau^k)_! 0_* \mathbb{D}\}^{\perp} = Loc(i_! \mathbb{D} : 0 \leq i \leq n\}$$

and vice versa,

$$Loc\{(\tau^k)_! 0_* \mathbb{D}\} =^{\perp} Loc(i_! \mathbb{D} : 0 \leq i \leq n\}$$

We are effectively tasked with the following: for each $k \in \mathbb{N}$ and each $0 \leq i \leq n$, to show that $\mathrm{Hom}((\tau^k)_! 0_* X, i_! Y) = 0$. This admits a slight reduction once we allow us to set either k or i to be 0 by noting that $\tau_!$ is a fully faithful functor. In each case, we utilize a (Der4) calculation to obtain the desired orthogonality result.

Lemma 9.3.2. *$Hom((\tau^k)_! 0_* X, 0_! Y) = 0$ for every $k \in \mathbb{N}$, $X, Y \in \mathbb{D}$.*

Proof. Here we can move the τ^k over to the right side, whence the Hom becomes $\prod_{\binom{n+k}{n}} \mathrm{Hom}(0_* X, 0_! Y)$.

The key is to remember that 0_*X is equivalently $C^{n+1}\Theta_n^*(n+1)_!X$. Therefore,

$$\mathrm{Hom}_{\mathbb{D}(Q_n)}(0_*X, 0_!Y) = \mathrm{Hom}_{\mathbb{D}(e)}(X, (n+1)^*(\Theta_n)_*(1^{n+1})_!(\tau^k)^*0_!Y).$$

The outer two morphisms we can compute via (Der4). We have the following (Der4) square:

$$\begin{array}{ccc}
((n+1)/\Theta_n) & \xrightarrow{\mathrm{pr}} & Q_n \times [1]^{n+1} \\
{\scriptstyle \pi}\downarrow & \Nearrow_{\alpha} & \downarrow{\scriptstyle \Theta_n} \\
e & \xrightarrow[n+1]{} & Q_n
\end{array}$$

Consider the structure of the slice category: it is a subcategory of $\mathbb{N}^{n+1} \times [1]^{n+1}$ consisting of points whose summed coordinates is at least $n+1$ (moving in the $\mathbb{N}^{n+1}$ direction looks at morphisms in Q_n, and moving in $[1]^{n+1}$-direction are naturally morphisms in $[1]^{n+1}$), and a homotopy initial subcategory of the slice category consists of the minimal join of elements whose coordinates actually sum to $n+1$, as the slice category is a poset.

The homotopy initial subcategory is a subcategory of $[1]^{n+1} \times [1]^{n+1}$, and we can compute the limit by taking partial fibers. Everything off of the $(1, \cdots, 1) \times [1]^{n+1}$ is naturally 0 and we need not worry about it. The information of the $(1, \cdots, 1) \times [1]^{n+1}$ is homotopy cartesian, see [BG18, §7] for a discussion of the notion. Therefore, all total fibers in the second $[1]^{n+1}$ are 0, and therefore the limit is 0.

This proves that the Hom-sets in question are in fact 0. □

Lemma 9.3.3. *Hom*$(0_*X, i_!Y) = 0$ *for every* $0 \leq i \leq n$.

Proof. We use the same tactic of redefining 0_*. The discussion of the slice categories is the exact same as the previous lemma and we do not reproduce it.

For this limit, it is a short combinatorial exercise to see that the $(1, \cdots, 1) \times [1]^{n+1}$-slice is also homotopy cartesian as long as $0 \leq i \leq n$ (also when $i > n+1$, as then the entire n+1-cube consists of zero entries). □

Remark 9.3.4. We give an illustration of the case $n = 0$, as this is the only case that is amenable to diagram-drawing. In this case, Q_n is just $\mathbb{N}$ viewed as a poset. We need to prove the following assertions for all $X, Y \in \mathbb{D}(e)$:

1. $\mathrm{Hom}((\tau^k)_! 0_* X, 0_! Y) = 0$ for every $k \in \mathbb{N}$
2. $\mathrm{Hom}(0_* X, 0_! Y) = 0$

We can comfort ourselves in knowing these statements are true without (Der4), as $0_! \cong \pi^*$ in this case and then we may move π^* over the left as a homotopy colimit, and all left-hand expressions above have homotopy colimit equal to 0. However, we will use (Der4) to show them anyways.

Θ_0 is the functor $\mathbb{N} \times [1] \to \mathbb{N}$ taking $(n, k) \mapsto n + k$ with the only choice available on objects. We compute $(1/\Theta_0)$: its objects are morphisms $(1 \to \Theta_0(n, k))$ and this is the full subcategory of $\mathbb{N} \times [1]$ with (0,0) removed. A homotopy initial subcategory is given by the full subcategory with objects $(0, 1), (1, 0), (1, 1)$.

Accordingly, the relevant homotopy limit will have shape

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ & & \uparrow \\ & & X \end{array}$$

and naturally the homotopy limit is 0. The entire homotopy limit has shape

$$\begin{array}{ccc} \cdots & \longrightarrow & \cdots \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & X \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & X \\ & & \uparrow \\ & & X \end{array}$$

where every map in the diagram is 0 or an identity of X, and we see that the truncated portion is obviously sufficient to compute the homotopy limit.

We can also see in this example that since $n = 0$, a natural triangle that arises is

$$\cdots \to 1_!X \to 0_!X \to 0_*X \to \Sigma 1_!X \cdots ,$$

and since $1_!X \to 0_!X$ is not a split monomorphism, the map $0_*!X \to \Sigma 1_!X$ is not zero, thereby providing an example to the obstacle at $n+1$ that was presented.

This gives rise to the following corollaries.

Corollary 9.3.5. *The localization $\mathbb{D}^{Q_n}/Loc\{(\tau^k)_! 0_* \mathbb{D}\}$ is $\mathbb{P}^n_{\mathbb{D}}$. Moreover, $\mathbb{P}^n_{\mathbb{D}}$ can be visualized as the subderivator generated by $\{0_!\mathbb{D}, \cdots, n_!\mathbb{D}\}$ in $\mathbb{D}^{Q_n}$.*

Corollary 9.3.6. *The above localization is finite.*

Proof. We are localizing the localizing subcategory generated by a set of compact objects. □

Corollary 9.3.7 (Derivator semiorthogonal decomposition)**.** *There exists the usual semiorthogonal decomposition to $\mathbb{P}^n_{\mathbb{D}}$. By this we mean that we have a semiorthogonal decomposition*

$$Loc\langle 0_!\mathbb{D}\rangle \subset Loc\langle 0_!\mathbb{D}, 1_!\mathbb{D}, \rangle \cdots \subset Loc\langle 0_!\mathbb{D}, \cdots n_!\mathbb{D}\rangle = \mathbb{P}^n_{\mathbb{D}}.$$

Equivalently, looking on an object level this just replicates the result that $Hom_{\mathbb{D}^{Q_n}}(k_!X, l_!Y) = 0$ if $k < l$, and then using Lemma 9.3.2 and Lemma 9.3.3 to push this down to the localization.

This is intended to mirror the existence of a strong full exceptional sequence in $\mathbb{P}^n_{\mathbb{D}}$, however the actual definition of strong full exceptional sequence makes some assumptions about generation and how morphisms work in $\mathbb{D}(e)$ that we would like to avoid making, so we have this marginally weaker version.

Proof. There is a semiorthogonal decomposition $0_!X \dashv 1_!X \dashv \cdots \dashv (n-1)_!X \dashv n_!X$ in Loc$\langle n_!X, \cdots, 0_!X\rangle$ for any $X \in \mathbb{D}(e)$. By construction of $\mathbb{P}^n_{\mathbb{D}}$ the statement is clear. □

This gives rise to a Projective Bundle Formula for K_0 on projective space; recall that we need to restrict to the compact part for Eilenberg swindle reasons.

Corollary 9.3.8 (Projective Bundle Formula). *Let $\mathbb{D}$ be a compactly generated triangulated derivator. Then from the semiorthogonal decomposition of $\mathbb{P}^n_{\mathbb{D}}$ we know that*

$$K_0(\mathbb{P}^n_{\mathbb{D}}(e)^c) = \oplus_{n+1} K_0(\mathbb{D}(e)^c)$$

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