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Statistical Mechanics and Black Hole Entropy

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Abstract

I review a new (and still tentative) approach to black hole thermodynamics that seeks to explain black hole entropy in terms of microscopic quantum gravitational boundary states induced on the black hole horizon. (Talk given at CAM '95, joint meeting of the Canadian Association of Physicists, the American Physical Society, and the Mexican Physical Society, Quebec City, Canada.)

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It has been over twenty years since we first learned from Bekenstein [1] and Hawking [2] that black holes are thermodynamic systems, characterized by temperatures and entropies. But despite considerable progress in the field, we still lack a convincing “statistical mechanical” picture of black hole thermodynamics. Indeed, black hole entropy remains rather fundamentally paradoxical. On the one hand, given the macroscopic parameters of mass, charge, and angular momentum, a black hole configuration has the highest obtainable entropy, implying at least naively that the black hole has a large number of macroscopically indistinguishable microscopic states. On the other hand, a black hole has no hair: given the same macroscopic parameters, there is, in fact, only one classical black hole state.

A number of attempts have been made to resolve this paradox (see [3] for a review), but none is yet generally accepted. In this paper, I would like to advocate a new, still rather tentative approach, which seeks to explain black hole entropy in terms of microscopic quantum gravitational states on the horizon. I do not yet know how to apply this picture to realistic (3+1)-dimensional black holes, but at least in the simpler case of 2+1 dimensions, it has been shown (modulo some reasonable assumptions about quantization) to lead to the correct entropy [4].

My starting point is the observation that any quantum mechanical statement about black holes is necessarily a statement about *conditional* probabilities: for instance, “If spacetime contains an event horizon of a certain size, then we should see Hawking radiation with a certain spectrum.” So the first question we must ask is how to impose such a condition—a restriction on the geometry of a spacetime hypersurface—in a quantum mechanical computation. One obvious answer is to start with a path integral formalism, split spacetime M along a hypersurface Σ into two pieces, say M_1 and M_2 , and perform separate path integrals over M_1 and M_2 with suitable boundary conditions on Σ (figure 1). We are therefore naturally led to consider the problem of “sewing” path integrals.

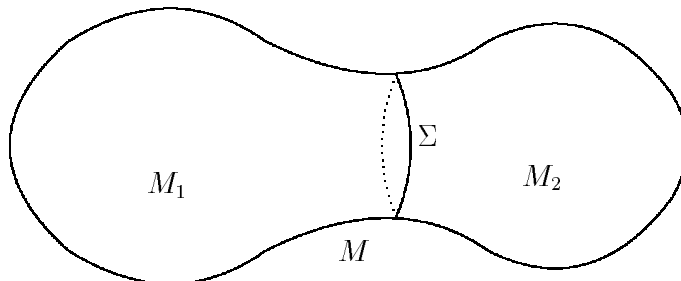


Figure 1: The manifold M is formed by “sewing” M_1 and M_2 along Σ .

In the remainder of this paper, I will deal with three successively more complicated models: a scalar field, an abelian Chern-Simons theory, and (2+1)-dimensional gravity. I will conclude with a discussion of possible generalization to realistic (3+1)-dimensional gravity and implications for black hole thermodynamics.

1. Sewing Scalar Fields

Let us begin with a simple example, which nevertheless can teach us some important lessons. Consider a scalar field ϕ on M , with an action

$$I_M[\phi] = \frac{1}{2} \int_M d^n x \sqrt{-g} \phi \Delta \phi \quad (1.1)$$

and a partition function

$$Z[M] = \int [d\phi] e^{iI_M[\phi]} = \det^{-1/2} \Delta_M. \quad (1.2)$$

Splitting M along Σ , one might hope to find a relationship of the form

$$Z[M] = Z[M_1] Z[M_2], \quad (1.3)$$

which would require that

$$\det \Delta_M = \det \Delta_{M_1} \det \Delta_{M_2}. \quad (1.4)$$

Strictly speaking, equation (1.4) doesn't quite make sense, since determinants on M_1 and M_2 require boundary conditions for their definition. But in fact, (1.4) does not hold for any choice of boundary conditions.

It is clear where we have gone wrong: the action (1.1) is simply not correct for a manifold with boundary. In fact, $I_{M_1}[\phi]$ has no classical extrema for generic boundary values: its variation is

$$\delta I_{M_1}[\phi] = \int_{M_1} d^n x \sqrt{-g} \delta \phi \Delta \phi + \frac{1}{2} \int_{\Sigma} d^{n-1} x \sqrt{h} (\phi n^\mu \nabla_\mu \delta \phi - \delta \phi n^\mu \nabla_\mu \phi), \quad (1.5)$$

and the surface term does not vanish for either Dirichlet or Neumann boundary conditions. (Here h is the induced metric on Σ , and n^μ is the unit normal.) The cure is obvious, however—if we choose boundary conditions in which ϕ is fixed at the boundary, for instance, we must add to the action a surface term

$$I_\Sigma[\phi] = -\frac{1}{2} \int_{\Sigma} d^{n-1} x \sqrt{h} \phi n^\mu \nabla_\mu \phi \quad (1.6)$$

to obtain a total action

$$I'_{M_1}[\phi] = I_{M_1}[\phi] + I_\Sigma[\phi] = -\frac{1}{2} \int_M d^n x \sqrt{-g} \nabla^\mu \phi \nabla_\mu \phi. \quad (1.7)$$

(If we choose instead to fix the normal derivative of ϕ at Σ , we should take as our action $I''_{M_1}[\phi] = I_{M_1}[\phi] - I_\Sigma[\phi]$.)

The computation of $Z[M_1]$ is now standard. If we denote the boundary value of ϕ by ϕ_0 , and let $\bar{\phi}$ be the classical solution with boundary value ϕ_0 , then

$$Z[M_1][\phi_0] = \int [d\phi] \exp \{iI'_{M_1}[\phi]\} = \exp \{iI'_{M_1}[\bar{\phi}]\} \det^{-1/2} \Delta_{M_1}, \quad (1.8)$$

where the determinant is taken with Dirichlet boundary conditions. Note that the classical action $I'_{M_1}[\bar{\phi}]$ may be written as a bilinear functional of ϕ_0 ,

$$I'_{M_1}[\bar{\phi}] = \int_\Sigma d^{n-1}x \sqrt{h(x)} \int_\Sigma d^{n-1}x' \sqrt{h(x')} \phi_0(x) K_{M_1}(x, x') \phi_0(x'). \quad (1.9)$$

The kernel $K(x, x')$ is known as the Poisson kernel [5]; it may be written in terms of normal derivatives of the Greens function for the Laplacian Δ_{M_1} , and has a straightforward generalization to other field theories.

We can now “sew” M_1 and M_2 by integrating over boundary values ϕ_0 :

$$Z[M] = \int [d\phi_0] Z[M_1][\phi_0] Z[M_2][\phi_0]. \quad (1.10)$$

The integral over ϕ_0 is again Gaussian, and can be performed exactly; we find that (1.4) must be replaced by

$$\det \Delta_M = \det \Delta_{M_1} \det \Delta_{M_2} \det(K_{M_1} + K_{M_2}). \quad (1.11)$$

This relationship among determinants, and its generalization to a wide variety of free field theories, has been rigorously demonstrated to be true [6].

None of this is surprising, of course, but it does demonstrate two crucial lessons:

1. An action appropriate for a closed manifold may not be suitable for a manifold with boundary;
2. The precise form of the required boundary term depends on the choice of boundary conditions, and can be found by demanding the existence of a classical extremum of the action.

2. Gauge Invariance and Chern-Simons Theory

The problem of sewing path integrals becomes more interesting when one starts with a theory with gauge invariance. The simple archetype is an abelian Chern-Simons theory on a three-manifold. Let A_μ be an abelian gauge potential (i.e., a connection on a $U(1)$ bundle over M), and consider the action

$$I_M[A] = \frac{k}{2\pi} \int_M d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \quad (2.1)$$

This action is invariant under gauge transformations

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad (2.2)$$

and leads to Euler-Lagrange equations

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 0. \quad (2.3)$$

The space of classical solutions is thus the space of flat connections modulo gauge transformations. The corresponding quantum theory is fairly simple, and it may be shown to have a finite-dimensional Hilbert space (see, for example, [7]).

We next split M into two pieces along Σ and repeat the analysis of the preceding section. The analogue of the surface term in equation (1.5) is

$$\delta I_{M_1}[A] = \dots - \frac{k}{2\pi} \int_\Sigma d^2x n_\rho \epsilon^{\rho\mu\nu} A_\mu \delta A_\nu, \quad (2.4)$$

and we must choose boundary conditions and a surface term I_Σ to cancel this variation. A standard approach is to choose a complex structure on Σ and to fix the component A_z , which is canonically conjugate to $A_{\bar{z}}$. The variation (2.4) can then be cancelled by the variation of a boundary action

$$I_\Sigma[A] = \frac{k}{2\pi} \int_\Sigma d^2x A_z A_{\bar{z}}. \quad (2.5)$$

Observe now that the action

$$I'_{M_1}[A] = I_{M_1}[A] + I_\Sigma[A] \quad (2.6)$$

is no longer invariant under gauge transformations (2.2) unless Λ vanishes at the boundary. We can make this noninvariance explicit by decomposing A_μ as

$$A_\mu = \bar{A}_\mu + \partial_\mu \Lambda, \quad (2.7)$$

where \bar{A}_μ is a gauge-fixed potential; then

$$I'_{M_1}[A] = I'_{M_1}[\bar{A}] + \frac{k}{2\pi} \int_{\Sigma} d^2x \left(\partial_z \Lambda \partial_{\bar{z}} \Lambda + 2\bar{A}_z \partial_{\bar{z}} \Lambda \right). \quad (2.8)$$

The “would-be gauge transformation” Λ has thus become a dynamical field on Σ , with an action that can be recognized as a chiral Wess-Zumino-Witten action. The partition function correspondingly factorizes,

$$Z[M_1][\bar{A}_z] = Z_{\text{bulk}}[\bar{A}_z] Z_{\text{WZW}}[\bar{A}_z]. \quad (2.9)$$

This is a dramatic result: we have gone from a Chern-Simons quantum theory with a finite-dimensional Hilbert space to a theory that includes an infinite-dimensional Hilbert space describing boundary degrees of freedom.

An analogous process occurs in the nonabelian case. Let $A = A_\mu^a T_a dx^\mu$ denote a connection one-form for a nonabelian gauge group G with generators T_a . Then the Chern-Simons action

$$I'_{M_1}[A] = \frac{k}{4\pi} \int_{M_1} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \frac{k}{4\pi} \int_{\Sigma} \text{Tr} A_z A_{\bar{z}} \quad (2.10)$$

appropriate for fixing A_z at Σ again splits into two pieces; under the decomposition

$$A = g^{-1} dg + g^{-1} \bar{A} g, \quad (2.11)$$

the action becomes [8, 9]

$$I'_{M_1}[\bar{A}, g] = I'_{M_1}[\bar{A}] + k I_{\text{WZW}}[g, \bar{A}_z]. \quad (2.12)$$

The term $I_{\text{WZW}}[g, \bar{A}_z]$ is now the action of a nonabelian WZW model at the boundary Σ , and a decomposition of the partition function of the form (2.9) again holds.

As in the case of a scalar field, the factor in $Z[M_1]$ coming from the boundary term is necessary to ensure a sewing relation analogous to (1.10). Witten has shown that when the WZW action is included, Chern-Simons theory does indeed sew properly at boundaries [10].

Once again, none of this is new. But we can add two more lessons to the two at the end of the preceding section:

3. In a gauge theory, the presence of a boundary term can break the gauge invariance, leading to new boundary degrees of freedom;
4. The new, dynamical “would-be gauge” degrees of freedom can drastically alter the Hilbert space of the corresponding quantum theory.

This last observation is the key to the proposed explanation of black hole entropy.

3. (2+1)-Dimensional Gravity

As our third example, let us examine quantum gravity in three spacetime dimensions. This model has the beautiful feature, first noticed by Achúcarro and Townsend [11] and later developed by Witten [12], that it can be rewritten as a Chern-Simons theory. In particular, suppose that a negative cosmological constant $\Lambda = -1/\ell^2$ is present, as is appropriate for the black hole solution of Bañados, Teitelboim, and Zanelli [13]. Define the two $SO(2, 1)$ gauge fields

$$A^a = \omega^a + \frac{1}{\ell}e^a, \quad \tilde{A}^a = \omega^a - \frac{1}{\ell}e^a, \quad (3.1)$$

where $e^a = e_\mu^a dx^\mu$ is a triad ($g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ is the metric) and $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}dx^\mu$ is a spin connection. The first-order form of the standard Einstein-Hilbert action for general relativity may then be written as

$$I_{\text{grav}}[e, \omega] = I[A] - I[\tilde{A}], \quad (3.2)$$

where $I[A]$ is the Chern-Simons action (2.10) with

$$k = \frac{\ell\sqrt{2}}{8G}. \quad (3.3)$$

(See reference [4] for my conventions.) In this formulation, the diffeomorphisms and local Lorentz transformations of ordinary general relativity are transmuted into $SO(2, 1) \times SO(2, 1)$ gauge transformations, parametrized by the group elements g as in (2.11).

Based on what we have seen of Chern-Simons theory, we might expect a boundary Wess-Zumino-Witten action to again be induced on Σ . In particular, if Σ is the horizon of a black hole, the boundary fields found above are candidates for the microscopic degrees of freedom responsible for the black hole entropy. The boundary data A_z and \tilde{A}_z of the preceding section are not quite suitable for this situation, but the appropriate data—which specify that Σ is an apparent horizon of a fixed circumference $2\pi r_+$ —may be shown to again give rise to an $SO(2, 1) \times SO(2, 1)$ WZW action [4].

Now, WZW models for noncompact groups such as $SO(2, 1)$ are not yet completely understood. In the large k (or small Λ) limit, however, the $SO(2, 1) \times SO(2, 1)$ action may be approximated by a system of six independent bosonic string oscillators. Such a system has an infinite number of states, but most of these are eliminated by a remaining gauge symmetry—a remnant of the Wheeler-DeWitt equation—that expresses invariance under shifts of the angular coordinate ϕ . Given reasonable assumptions about the Hilbert space upon which this symmetry acts, it is shown in

reference [4] that the number of boundary states is

$$n \sim \exp \left\{ \frac{2\pi r_+}{4G} \right\}. \quad (3.4)$$

The logarithm of this expression gives the correct Bekenstein-Hawking entropy [13],

$$S = \frac{2\pi r_+}{4G}, \quad (3.5)$$

for the (2+1)-dimensional black hole.

4. (3+1)-Dimensional Black Holes

Unfortunately, these results in 2+1 dimensions cannot be translated directly to realistic (3+1)-dimensional gravity. In particular, the Chern-Simons formulation of (2+1)-dimensional gravity allows a clean separation of “physical” and “gauge” degrees of freedom, and no (3+1)-dimensional analogue is known. There are, however, some tantalizing hints of a similar mechanism:

1. The obvious (3+1)-dimensional analogues of the dynamical “would-be gauge degrees of freedom” in Chern-Simons theory are the “would-be diffeomorphisms” that do not preserve the location of the boundary. Just as the boundary term (2.5) breaks the gauge invariance of Chern-Simons theory, the boundary terms*

$$\int_{\Sigma} d^3x \sqrt{h} K \quad \text{and} \quad \int_{\Sigma} d^3x \sqrt{\sigma} n^{\mu} \nabla_{\mu} N$$

break the naive diffeomorphism invariance of general relativity. As Marolf has noted [15], one must be careful about how the boundary is specified, but it may be possible to obtain a WZW-like induced boundary action.

2. A standard method for obtaining the physical degrees of freedom in general relativity [16] is to split fluctuations h_{ij} of the spatial metric as

$$h_{ij} = h_{ij}^{TT} + (L\xi)_{ij} + \frac{1}{3}g_{ij}h,$$

$$\text{with } (L\xi)_{ij} = D_i\xi_j + D_j\xi_i - \frac{2}{3}g_{ij}D_k\xi^k, \quad D^j h_{ij}^{TT} = 0,$$

*The first of these terms is standard, although I am not sure of the right generalization to null surfaces; the second occurs at the bifurcation two-sphere of a black hole [14].

where D_i denotes the spatial covariant derivative. One then argues that once the gauge is fixed and the constraints are imposed, only the transverse traceless components h_{ij}^{TT} remain as physical degrees of freedom. In the presence of a boundary, however, this decomposition must be altered: one must impose boundary conditions on ξ_i and h_{ij}^{TT} in order to make the operator L self-adjoint. As far as I know, the decomposition of the metric on a spatial slice with boundary has not been analyzed, but it is at least suggestive that the splitting into physical and gauge degrees of freedom must change.

3. As Balachandran, Chandar, and Momen have emphasized [17], the smeared generators of diffeomorphisms in canonical general relativity,

$$\mathcal{D}_\xi = -2 \int d^3x \xi_i D_j \pi^{ij},$$

are functionally differentiable—and thus represent genuine invariances—only when the vector ξ^i vanishes at the boundary. When $\xi^i \neq 0$ at the boundary, the would-be constraints instead give rise to an algebra of edge observables, which should presumably have a representation on a Hilbert space of edge states. An analysis of similar edge observables in Chern-Simons theory can, indeed, reproduce the WZW states described in section 2.

4. Preliminary investigations of (3+1)-dimensional quantum gravity in the loop variable approach have indicated the appearance of boundary states [18,19]. In particular, Smolin has found that in Euclidean quantum gravity with certain self-dual boundary conditions, the would-be diffeomorphisms of the boundary become dynamical observables, which act on states that may be described in terms of an induced topological field theory at the boundary.

These results remain scattered, and do not yet give a full, coherent picture of the origin of black hole entropy. I believe, however, that such a picture is emerging, and that we have cause for optimism.

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