University of California
Los Angeles

Some variable-coefficient nonlinear Schrödinger equations at critical regularity

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Casey Jao

2016

Abstract of the Dissertation

Some variable-coefficient nonlinear Schrödinger equations at critical regularity

by

Casey Jao

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2016

Professor Rowan Brett Killip, Co-Chair

Professor Monica Visan, Co-Chair

We present several large-data results for nontranslation-invariant analogues of the energy-critical and mass-critical nonlinear Schrödinger equations, obtained by introducing external potentials or non-Euclidean geometries.

The dissertation of Casey Jao is approved.

Eric D'Hoker

Monica Visan, Committee Co-Chair

Rowan Brett Killip, Committee Co-Chair

University of California, Los Angeles

2016

Table of Contents

Acknowledgments

I will forever be grateful for the remarkable resourcefulness, encouragement, and friendship of my mentors and tireless advocates Rowan Killip and Monica Visan.

Among my many fantastic teachers, Mr. Roger Demaree bears the most responsibility for starting me on this road.

Finally, none of this would have been possible without the unwavering support of my family over the last twenty-six years.

Vita

2011	B. A. in Mathematics, California Institute of Technology
2012-2013	Teaching Assistant
2014-2015	Department of Mathematics University of California, Los Angeles
2015-2016	UCLA Dissertation Year Fellowship

Publications

Jao, C. "The energy-critical quantum harmonic oscillator". Comm. Partial Differential Equations 41 (2016), no. 1, 79-133

CHAPTER 1

Introduction

This thesis investigates the long-time behavior of solutions to variable-coefficient semilinear Schrödinger equations. A typical nonlinear Schrödinger equation (NLS) takes the form

$$i\partial_t u = -\frac{1}{2}\Delta u + Vu + \mu|u|^p u,\ u(0,x) = \phi(x), \tag{1.1}$$

where $u : \mathbf{R}_t \times \mathbf{R}^d_x \to \mathbf{C}$ is a complex scalar field on spacetime with prescribed initial data ϕ, $V(t,x)$ is a real-valued potential, and the nonlinear term $\mu|u|^p u$ for fixed constants μ and p describes a self-interaction. Such equations arise when modeling Bose-Einstein condensates, water waves, and the propagation of light in fiber optics. One recovers the linear Schrödinger equation from quantum mechanics by setting $\mu = 0$.

To place our results in context, we first review the main features of the equation (1.1) in the case $V = 0$, which was the first to be thoroughly analyzed.

1.1 Background

Most rigorous studies of NLS have considered equations of the form

$$i\partial_t u = -\frac{1}{2}\Delta u + \mu|u|^p u,\ \ u(0,\cdot) = \phi \in H^s(\mathbf{R}^d), \tag{1.2}$$

where $p > 0$, $\mu = \pm 1$, and $H^s = (1-\Delta)^{-s/2}L^2(\mathbf{R}^d)$ is the L^2 Sobolev space. The equation is *defocusing* if $\mu = 1$ and *focusing* if $\mu = -1$. There are two known conserved quantities

$$M[u] = \int_{\mathbf{R}^d} |u|^2\,dx \tag{1.3}$$

$$E[u] = \int_{\mathbf{R}^d} \frac{1}{2}|\nabla u|^2 + \frac{2\mu}{p+2}|u|^{p+2}\,dx, \tag{1.4}$$

called the *mass* and *energy*, respectively. Note, however, that these are only defined for sufficiently regular initial data. For instance, functions merely in L^2 do not have finite energy.

A major advantage of considering the constant-coefficient equation (1.2) is that its linear part (that is, its linearization at 0) is diagonalized by the Fourier transform and may be profitably analyzed with technology from harmonic analysis, such as stationary phase asymptotics, Littlewood-Paley decompositions, and Fourier restriction estimates. These tools yield detailed insights into the concentration and decay of linear solutions, basic stepping stones to the nonlinear analysis.

Equation (1.2) enjoys a large group of symmetries. It is clearly invariant under spacetime translations $u_{t_0,x_0} = u(t-t_0, x-x_0)$. Also, the scaling $u_\lambda = \lambda^{-2/p} u(\lambda^{-2}t, \lambda^{-1}x)$ preserves the class of solutions. For each p there is a corresponding scale-invariant homogeneous Sobolev norm

$$\|u\|_{\dot{H}^{s_c}} := \|(-\Delta)^{\frac{s_c}{2}} u\|_{L^2},$$

where $s_c = \frac{d}{2} - \frac{2}{p}$ denotes the *critical* regularity; indeed, we have $\|u_\lambda(0)\|_{\dot{H}^s} = \lambda^{s_c - s}\|u(0)\|_{\dot{H}^s}$ and

$$M[u_\lambda] = \lambda^{2s_c} M[u], \quad E[u_\lambda] = \lambda^{2(s_c-1)} E[u].$$

The problem is H^s-*critical* (resp. subcritical, supercritical) if $u(0) \in H^s$, $s = s_c$ (resp. $s > s_c$, $s < s_c$).

s_c is the minimum regularity for which wellposedness is expected. When $s \geq s_c$, the equation (1.2) is locally wellposed; that is, for any initial data $\phi \in H^s$, there is a unique local-in-time solution which also depends continuously on ϕ. Very little is known about the case $s < s_c$, but heuristically one expects illposedness. We shall not discuss supercritical problems in the sequel.

One crucial difference between the subcritical and critical problems is that the guaranteed lifespan of local solutions depends merely on the norm of the initial data in the former but also on the profile in the latter. This distinction has dramatic consequences for the long-time analysis of solutions, particularly at conserved regularity. For example, for defocusing H^1-subcritical equations, mass and energy conservation imply a uniform bound in H^1, which when combined with the local theory immediately implies global existence and uniqueness of solutions. For the H^1-critical equation, however, the conservation laws provide no long-time control as $\dot{H}^1$ norm is by definition insensitive to the scaling of the equation. Thus, solutions could conceivably concentrate at a point and cease to exist after finite time while remaining bounded in H^1.

Thanks in part to sophisticated techniques and insights from harmonic analysis, the last twenty years have witnessed substantial progress toward understanding the solutions to NLS at critical regularity. The strongest conclusions have been obtained for the *mass-critical* and *energy-critical* equations

$$i\partial_t u = -\frac{1}{2}\Delta u + \mu|u|^{\frac{4}{d}}u, \qquad u(0) \in L^2(\mathbf{R}^d) \tag{1.5}$$

$$i\partial_t u = -\frac{1}{2}\Delta u + \mu|u|^{\frac{4}{d-2}}u, \quad u(0) \in \dot{H}^1(\mathbf{R}^d),\ d \geq 3, \tag{1.6}$$

where the conservation laws control the critical Sobolev norm. Broadly speaking, all solutions are known to not only exist globally but also scatter, at least when $\mu = 1$. The case $\mu = -1$, where the solution carves out a potential well that exerts a self-focusing effect, is more subtle due to the possibility of solitons or blowup. We will state these results more precisely in the next chapter.

1.2 Variable-coefficient generalizations

In this thesis, we consider several variable-coefficient analogues of the mass-critical and energy-critical NLS by introducing external potentials $V \neq 0$ or non-Euclidean geometries. Such modifications are quite natural since most real-world systems are not spatially homogeneous. Our aim is to generalize results concerning global wellposedness and asymptotic behavior for the constant coefficient equations to the variable-coefficient setting.

A key property of all the equations we shall study is that although they lack scaling symmetry, highly concentrated solutions evolve approximately according to the corresponding scaling-invariant equation (that is, equations (1.6) or (1.5)). This claim is intuitively plausible since the coefficients in the equation are nearly constant over very small length scales. It can in fact be justified rigorously. As a consequence, the large-data theory for these equations faces the same difficulty as before in that the conservation laws by themselves provide no control over the long-time behavior of solutions.

To study the long-time behavior of large-data solutions, we apply the Kenig-Merle concentration compactness and rigidity strategy which will be described in some detail in Chapter 2. While the general strategy has proved to be quite versatile, it is nontrivial to implement in the absence of various symmetries. In particular, many existing arguments for both the linear and nonlinear constant coefficient equations rely on the Fourier transform, which is ill-adapted to nontranslation-invariant systems. To compensate for this and in particular to prove the required profile decompositions, we need microlocal techniques.

Chapters 2 and 3 investigate global wellposedness for the energy-critical NLS in the presence of external potentials. One can regard sufficiently small potentials as perturbations to the constant-coefficient equation (1.6), which is by now understood. But the potentials we shall mainly consider are much too large for naive perturbative arguments.

We begin by studying the energy-critical quantum harmonic oscillator

$$i\partial_t u = \Big(-\frac{1}{2}\Delta + \frac{1}{2}|x|^2\Big)u + \mu|u|^{\frac{4}{d-2}}u,\ u(0) \in \Sigma(\mathbf{R}^d),$$

where Σ is a weighted version of H^1 adapted to the conserved energy

$$E(u) = \int \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|x|^2|u|^2 + \Big(1 - \frac{2}{d}\Big)\mu|u|^{\frac{2d}{d-2}}\,dx.$$

As we shall see, highly concentrated solutions will evolve approximately according to equation(1.6), which is by now well understood. However, these are not the only kinds of solutions one must account for when proving that arbitrary finite-energy initial data lead to globally defined solutions. This chapter essentially follows the paper [Jao16].

In Chapter 3, we generalize the results for the harmonic oscillator to a class of potentials that obey similar estimates as the exact quadratic potential. Our main point is that the results for the harmonic oscillator are in no way wedded to any algebraic miracles. Rather, we ultimately exploit the fact that the bicharacteristics for the symbol

$$h(x, \xi) = \tfrac{1}{2}|\xi|^2 + V(x)$$

are nearly straight lines in the relevant region of phase space.

In Chapter 4, we study the defocusing energy-critical NLS (1.6) in three spatial dimensions with the Laplacian Δ replaced by the Laplace-Beltrami operator Δ_g for a Riemannian metric g on $\mathbf{R}^3$. We show that if g is a sufficiently mild deformation of the Euclidean metric (in a sense to be made precise later), then all finite-energy solutions not only exist globally and also scatter to linear Euclidean solutions. This situation is considerably more delicate compared to the case of an external potential because the equation is perturbed in the highest order terms. Indeed, even small perturbations of the metric may cause the bicharacteristics of the principal symbol $h(x, \xi) = g^{jk}(x)\xi_j\xi_k$ to converge at multiple points. At the level of the Schrödinger equation, this refocusing manifests in the failure of a fundamental linear decay estimate, and an important part of our analysis will be to prove a weakened form that still suffices for our purposes.

The final part of the thesis investigates reverse Strichartz theorems in connection with mass-critical NLS. Such inverse theorems are essential for studying how solutions concentrate in the long-time large-data theory. In the Euclidean setting, these have been obtained with the aid of Fourier restriction estimates which are very much tied to spacetime translation-invariance. In Chapter 5, we discuss an alternate approach to these inverse theorems in one space dimension which generalizes to a class of Schrödinger operators that includes the harmonic oscillator.

CHAPTER 2

The energy-critical quantum harmonic oscillator

2.1 Introduction

We study the initial value problem for the energy-critical nonlinear Schrödinger equation on $\mathbf{R}^d,\ d \geq 3$, with a harmonic oscillator potential:

$$\begin{cases} i\partial_t u = (-\frac{1}{2}\Delta + \frac{1}{2}|x|^2)u + \mu|u|^{\frac{4}{d-2}}u, \quad \mu = \pm 1, \\ u(0) = u_0 \in \Sigma(\mathbf{R}^d). \end{cases} \tag{2.1}$$

The equation is *defocusing* if $\mu = 1$ and *focusing* if $\mu = -1$. Solutions to this PDE conserve energy, defined as

$$E(u(t)) = \int_{\mathbf{R}^d} \left[\tfrac{1}{2}|\nabla u(t)|^2 + \tfrac{1}{2}|x|^2|u(t)|^2 + \tfrac{d-2}{d}\mu|u(t)|^{\frac{2d}{d-2}}\right] dx = E(u(0)). \tag{2.2}$$

The term "energy-critical" refers to the fact that if we ignore the $|x|^2$ term in the equation and the energy, the scaling

$$u(t,x) \mapsto u^\lambda(t,x) := \lambda^{-\frac{2}{d-2}}u(\lambda^{-2}t, \lambda^{-1}x) \tag{2.3}$$

preserves both the equation and the energy. We take initial data in the weighted Sobolev space Σ, which is the natural space of functions associated with the energy functional. This space is equipped with the norm

$$\|f\|_\Sigma^2 = \|\nabla f\|_{L^2}^2 + \|xf\|_{L^2}^2 = \|f\|_{\dot{H}^1}^2 + \|f\|_{L^2(|x|^2\,dx)}^2 \tag{2.4}$$

We will frequently employ the notation

$$H = -\tfrac{1}{2}\Delta + \tfrac{1}{2}|x|^2, \quad F(z) = \mu|z|^{\frac{4}{d-2}}z.$$

Definition. A (strong) *solution* to (2.1) is a function $u : I \times \mathbf{R}^d \to \mathbf{C}$ that belongs to $C_t^0(K;\Sigma)$ for every compact interval $K \subset I$, and that satisfies the Duhamel formula

$$u(t) = e^{-itH}u(0) - i\int_0^t e^{-i(t-s)H}F(u(s))\,ds \quad \text{for all} \quad t \in I. \tag{2.5}$$

The hypothesis on u implies that $F(u) \in C^0_{t,loc}L_x^{\frac{2d}{d+2}}(I \times \mathbf{R}^d)$. Consequently, the right side above is well-defined, at least as a weak integral of tempered distributions.

Equation (2.1) and its variants

$$i\partial_t u = (-\tfrac{1}{2}\Delta + V)u + F(u), \quad V = \pm\tfrac{1}{2}|x|^2, \quad F(u) = \pm|u|^p u, \quad p > 0$$

have received considerable attention, especially in the energy-subcritical regime $p < 4/(d-2)$. The equation with a confining potential $V = |x|^2/2$ has been used to model Bose-Einstein condensates in a trap (see [Zha00], for example). Let us briefly review the mathematical literature.

Carles [Car02], [Car03] proved global wellposedness for a defocusing nonlinearity $F(u) = |u|^p u,\ p < 4/(d-2)$ when the potential $V(x) = |x|^2/2$ is either confining or repulsive, and obtained various wellposedness and blowup results for a focusing nonlinearity $F(u) = -|u|^p u$. In [Car05], he also studied the case of an anisotropic harmonic oscillator with $V(x) = \sum_j \delta_j x_j^2/2,\ \delta_j \in \{1, 0, -1\}$.

There has also been interest in more general potentials. The paper [Oh89] proves long-time existence in the presence of a focusing, mass-subcritical nonlinearity $F(u) = -|u|^p u,\ p < 4/d$ when $V(x)$ is merely assumed to grow subquadratically (by which we mean $\partial^\alpha V \in L^\infty$ for all $|\alpha| \geq 2$). More recently, Carles [Car11] considered *time-dependent* subquadratic potentials $V(t,x)$. Taking initial data in Σ, he established global existence and uniqueness when $4/d \leq p < 4/(d-2)$ for the defocusing nonlinearity and $0 < p < 4/d$ in the focusing case.

We are concerned with the energy-critical problem $p = 4/(d-2)$. While the critical equation still admits a local theory, the duration of local existence obtained by the usual fixed-point argument depends on the profile and not merely on the norm of the initial data u_0. Therefore, one cannot pass directly from local wellposedness to global wellposedness using conservation laws as in the subcritical case. This issue is most evident if we temporarily discard the potential and consider the equation

$$i\partial_t u = -\tfrac{1}{2}\Delta u + \mu|u|^{\frac{4}{d-2}}u, \quad u(0) = u_0 \in \dot{H}^1(\mathbf{R}^d), \quad d \geq 3, \tag{2.6}$$

which has the Hamiltonian

$$E_\Delta(u) = \int \tfrac{1}{2}|\nabla u|^2 + \mu\tfrac{d-2}{d}|u|^{\frac{2d}{d-2}}\,dx.$$

We shall refer to this equation in the sequel as the "potential-free", "translation-invariant", or "scale-invariant" problem. Since the spacetime scaling (2.3) preserves both the equation and the $\dot{H}^1$ norm of the initial data, the lifespan guaranteed by the local wellposedness theory cannot depend merely on $\|u_0\|_{\dot{H}^1}$. One cannot iterate the local existence argument to obtain global existence because with each iteration the solution could conceivably become more concentrated in space while remaining bounded in $\dot{H}^1$; the lifespans might therefore shrink to zero too quickly to cover all of $\mathbf{R}$. The scale invariance makes the analysis of (2.6) highly nontrivial.

We mention equation (2.6) because the original equation increasingly resembles (2.6) as the initial data concentrates at a point; see sections 2.4.2 and 2.5 for more precise statements concerning this limit. Hence, one would expect the essential difficulties in the energy-critical NLS to also manifest themselves in the energy-critical harmonic oscillator. Understanding the scale-invariant problem is therefore an important step toward understanding the harmonic oscillator. The last fifteen years have witnessed intensive study of the former, and the following conjecture has been verified in all but a few cases:

Conjecture 2.1.1. *When $\mu = 1$, solutions to* (2.6) *exist globally and scatter. That is, for any $u_0 \in \dot{H}^1(\mathbf{R}^d)$, there exists a unique global solution $u : \mathbf{R} \times \mathbf{R}^d \to \mathbf{C}$ to* (2.6) *with $u(0) = u_0$, and this solution satisfies a spacetime bound*

$$S_{\mathbf{R}}(u) := \int_{\mathbf{R}}\int_{\mathbf{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\,dx\,dt \leq C(E_\Delta(u_0)) < \infty. \tag{2.7}$$

Moreover, there exist functions $u_\pm \in \dot H^1(\mathbf{R}^d)$ *such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{\pm\frac{it\Delta}{2}} u_\pm\|_{\dot H^1} = 0,$$

and the correspondences $u_0 \mapsto u_\pm(u_0)$ *are homeomorphisms of* $\dot H^1$.

When $\mu = -1$*, one also has global wellposedness and scattering provided that*

$$E_\Delta(u_0) < E_\Delta(W), \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2},$$

where the ground state

$$W(x) = \big(1 + \frac{2|x|^2}{d(d-2)}\big)^{-\frac{d-2}{2}} \in \dot H^1(\mathbf{R}^d)$$

solves the elliptic equation $\frac{1}{2}\Delta + |W|^{\frac{4}{d-2}} W = 0$.

Theorem 2.1.1. *Conjecture 2.1.1 holds for the defocusing equation. For the focusing equation, the conjecture holds for radial initial data when* $d \geq 3$*, and for all initial data when* $d \geq 5$.

Proof. See [Bou99, CKS08, RV07, Vis07] for the defocusing case and [KM06, KV10] for the focusing case. □

One can formulate a similar conjecture for (2.1); however, as the linear propagator is periodic in time, one only expects uniform local-in-time spacetime bounds.

Conjecture 2.1.2. *When* $\mu = 1$*, equation* (2.1) *is globally wellposed. That is, for each* $u_0 \in \Sigma$ *there is a unique global solution* $u : \mathbf{R} \times \mathbf{R}^d \to \mathbf{C}$ *with* $u(0) = u_0$*. This solution obeys the spacetime bound*

$$S_I(u) := \int_I \int_{\mathbf{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}} \, dx \, dt \leq C(|I|, \|u_0\|_\Sigma) \tag{2.8}$$

for any compact interval $I \subset \mathbf{R}$.

If $\mu = -1$*, then the same is true provided also that*

$$E(u_0) < E_\Delta(W) \quad \text{and} \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}.$$

In [KVZ09], Killip-Visan-Zhang verified this conjecture for $\mu = 1$ and spherically symmetric initial data. By adapting an argument of Bourgain-Tao for the equation without potential (2.6), they proved that the defocusing problem (2.1) is globally wellposed, and also obtained scattering for the repulsive potential. We consider only the confining potential. In this chapter, we remove the assumption of spherical symmetry for the defocusing harmonic oscillator. In addition, we establish global wellposedness for the focusing problem under the assumption that Conjecture 2.1.1 holds for all dimensions.

Theorem 2.1.2. *Assume that Conjecture 2.1.1 holds. Then Conjecture 2.1.2 holds.*

By Theorem 2.1.1, this result is conditional only in the focusing situation for nonradial data in dimensions 3 and 4. Moreover, in the focusing case we have essentially the same blowup result as for the potential-free NLS with the same proof as in that case; see [KV10]. We recall the argument in Section 2.7.

Theorem 2.1.3 (Blowup)**.** *Suppose* $\mu = -1$ *and* $d \geq 3$*. If* $u_0 \in \Sigma$ *satisfies* $E(u_0) < E_\Delta(W)$ *and* $\|\nabla u_0\|_2 > \|\nabla W\|_2$*, then the solution to* (2.1) *blows up in finite time.*

Remark. By Lemma 2.7.1, $E(u_0) < E_\Delta(W)$ implies that either $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$ or $\|\nabla u\|_{L^2} > \|\nabla W\|_{L^2}$.

Mathematically, the energy-critical NLS with quadratic potential has several interesting properties. On one hand, it is a nontrivial variant of the potential-free equation. If the quadratic potential is replaced by a weaker potential, the proof of global wellposedness can sometimes ride on the coat tails of Theorem 2.1.1. For example, we show in Section 2.8 that for smooth, bounded potentials with bounded derivative, one obtains global wellposedness by treating the potential as a perturbation to (2.6). Further, the Avron-Herbst formula given in [Car11] reduces the problem with a linear potential $V(x) = Ex$ to (2.6). On the other hand, the linear propagator e^{-itH} for the harmonic oscillator does admit an explicit formula. In view of the preceding remarks, we believe that (2.1) is the most accessible generalization of (2.6) which does not come for free.

Proof outline. The local theory for (2.1) shows that global existence is equivalent to the uniform *a priori* spacetime bound (2.8). To prove this bound for all solutions, we apply the general strategy of induction on energy pioneered by Bourgain [Bou99] and refined over the years by Colliander-Keel-Staffilani-Takaoka-Tao [CKS08], Keraani [Ker06], Kenig-Merle [KM06], and others. These arguments proceed roughly as follows.

(1) Show that the failure of Theorem 2.1.2 would imply the existence of a minimal-energy counterexample.

(2) Show that the counterexample cannot actually exist.

By the local theory, uniform spacetime bounds hold for all solutions with sufficiently small energy $E(u)$. Assuming that Theorem 2.1.2 fails, we obtain a positive threshold $0 < E_c < \infty$ such that (2.8) holds whenever $E(u) < E_c$ and fails when $E(u) > E_c$.

As the spacetime estimates of interest are local-in-time, it suffices to prevent the blowup of spacetime norm on unit-length time intervals. This will be achieved by a Palais-Smale compactness theorem (Proposition 2.6.1), from which one deduces that failure of Theorem 2.1.2 would imply the existence of a solution u_c with $E(u_c) = E_c$, which blows up on a unit time interval, and which also has an impossibly strong compactness property (namely, its orbit $\{u_c(t)\}$ must be precompact in Σ). Put differently, we shall discover that the only scenario where blowup could possibly occur is when the solution is highly concentrated at a point and behaves like a solution to the potential-free equation (2.6); but that equation is already known to be wellposed.

This paradigm of recovering the potential-free NLS in certain limiting regimes has been applied to various other equations. See [KKS12, KSV12, IPS12, IP12, KVZb] for adaptations to gKdV, Klein-Gordon, and NLS in various domains and manifolds. While the particulars are unique to each case, a common key step is to prove an appropriate compactness theorem in the style of Proposition 2.6.1. As in the previous work, our proof of that proposition uses three main ingredients.

The first prerequisite is a local wellposedness theory that gives local existence and uniqueness as well as stability of solutions with respect to perturbations of the initial data or the equation itself. In our case, local wellposedness will follow from familiar arguments employing the dispersive estimate satisfied by the linear propagator e^{-itH}, as well the fractional product and chain rules for the operators $H^{\gamma},\ \gamma \geq 0$. We review the relevant results in Section 2.3.

We also need a linear profile decomposition for the Strichartz inequality

$$\|e^{-itH} f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \lesssim \|H^{\frac{1}{2}} f\|_{L_x^2}. \tag{2.9}$$

Such a decomposition in the context of energy-critical Schrödinger equations was first proved by Keraani [Ker01] in the translation-invariant setting for the free particle Hamiltonian $H = -\Delta$, and quantifies the manner in which a sequence of functions f_n with $\|H^{1/2} f_n\|_{L^2}$ bounded may fail to produce a subsequence of $e^{-itH} f_n$ converging in the spacetime norm. The defect of compactness arises in Keraani's case from a noncompact group of symmetries of the inequality (2.9), which includes spatial translations and scaling. In our setting, there are no obvious symmetries of (2.9); nonetheless, compactness can fail and in Section 2.4 we formulate a profile decomposition for (2.9) when H is the Hamiltonian of the harmonic oscillator.

Finally, we need to study (2.1) when the initial data is highly concentrated in space, corresponding to a single profile in the linear profile decomposition just discussed. In Section 2.5, we show that blowup cannot occur in this regime. The basic idea is that while the solution to (2.1) remains highly localized in space, it can be well-approximated up to a phase factor by the corresponding solution to the scale-invariant energy-critical NLS

$$(i\partial_t + \tfrac{1}{2}\Delta) u = \pm |u|^{\frac{4}{d-2}} u. \tag{2.10}$$

By the time this approximation breaks down, the solution to the original equation will have dispersed and can instead be approximated by a solution to the linear equation $(i\partial_t - H)u = 0$. We use as a black box the nontrivial fact (which is still a conjecture in a few cases) that solutions to (2.6) obey global spacetime bounds. By stability theory, the spacetime bounds for the approximations will be transferred to the solution for the original equation and will therefore preclude blowup.

While this discussion considers the potential $V(x) = \frac{1}{2}|x|^2$, the argument can be adapted to a wider class of subquadratic potentials defined by the following hypotheses:

- $\partial^k V \in L^\infty$ for all $k \geq 2$.
- $V(x) \geq \delta |x|^2$ for some $\delta > 0$.

Under these assumptions, Fujiwara [Fuj80] constructed a Fourier integral operator representation for the propagator, which can be used as a substitute for the Mehler formula (2.12). We focus on the harmonic oscillator because this concrete case already contains the main ideas. In the second part of this chapter we describe the modifications required to treat the more general case.

Acknowledgements

The author is indebted to his advisors Rowan Killip and Monica Visan for their helpful discussions, particularly regarding the energy trapping arguments for the focusing case, as well as their feedback on the manuscript. This work was supported in part by NSF grants DMS-0838680 (RTG), DMS-1265868 (PI R. Killip), DMS-0901166, and DMS-1161396 (both PI M. Visan).

2.2 Preliminaries

2.2.1 Notation and basic estimates

We write $X \lesssim Y$ to mean $X \le CY$ for some constant C, and $X \sim Y$ if both $X \lesssim Y$ and $Y \lesssim X$. If $I \subset \mathbf{R}$ is an interval, the mixed Lebesgue norms on $I \times \mathbf{R}^d$ are defined by

$$\|f\|_{L_t^q L_x^r(I\times\mathbf{R}^d)} = \left(\int_I \left(\int_{\mathbf{R}^d} |f(t,x)|^r\,dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}} = \|f(t)\|_{L_t^q(I;L_x^r(\mathbf{R}^d))},$$

The operator $H = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2$ is positive on $L^2(\mathbf{R}^d)$. Its associated heat kernel is given by Mehler's formula

$$e^{-tH}(x,y) = e^{\tilde{\gamma}(t)(x^2+y^2)} e^{\frac{\sinh(t)\Delta}{2}}(x,y), \tag{2.11}$$

where

$$\tilde{\gamma}(t) = \frac{1-\cosh t}{2\sinh t} = -\frac{t}{4} + O(t^3) \quad \text{as} \quad t \to 0.$$

By analytic continuation, the associated one-parameter unitary group has the integral kernel

$$e^{-itH} f(x) = \frac{1}{(2\pi i \sin t)^{\frac{d}{2}}} \int e^{\frac{i}{\sin t}\left(\frac{x^2+y^2}{2}\cos t - xy\right)} f(y)\,dy. \tag{2.12}$$

Comparing this to the well-known free propagator

$$e^{\frac{it\Delta}{2}} f(x) = \frac{1}{(2\pi i t)^{\frac{d}{2}}} \int e^{\frac{i|x-y|^2}{2t}} f(y)\,dy, \tag{2.13}$$

we obtain the relation

$$e^{-itH} f = e^{i\gamma(t)|x|^2} e^{\frac{i\sin(t)\Delta}{2}} (e^{i\gamma(t)|x|^2} f), \tag{2.14}$$

where

$$\gamma(t) = \frac{\cos t - 1}{2\sin t} = -\frac{t}{4} + O(t^3) \quad \text{as} \quad t \to 0.$$

Mehler's formula immediately implies the local-in-time dispersive estimate

$$\|e^{-itH} f\|_{L_x^\infty} \lesssim |\sin t|^{-\frac{d}{2}} \|f\|_{L^1}. \tag{2.15}$$

For $d \ge 3$, call a pair of exponents (q,r) *admissible* if $q \ge 2$ and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$. Write

$$\|f\|_{S(I)} = \|f\|_{L_t^\infty L_x^2} + \|f\|_{L_t^2 L_x^{\frac{2d}{d-2}}}$$

with all norms taken over the spacetime slab $I \times \mathbf{R}^d$. By interpolation, this norm controls the $L_t^q L_x^r$ norm for all other admissible pairs. Let

$$\|F\|_{N(I)} = \inf\{\|F_1\|_{L_t^{q_1'} L_x^{r_1'}} + \|F_2\|_{L_t^{q_2'} L_x^{r_2'}} : (q_k, r_k) \text{ admissible}, \ F = F_1 + F_2\},$$

where (q_k', r_k') is the Hölder dual to (q_k, r_k).

Lemma 2.2.1 (Strichartz estimates)**.** *Let I be a compact time interval containing t_0, and let $u : I \times \mathbf{R}^d \to \mathbf{C}$ be a solution to the inhomogeneous Schrödinger equation*

$$(i\partial_t - H)u = F.$$

Then there is a constant $C = C(|I|)$, depending only on the length of the interval, such that

$$\|u\|_{S(I)} \leq C(\|u(t_0)\|_{L^2} + \|F\|_{N(I)}).$$

Proof. This follows from the dispersive estimate (2.15), the unitarity of e^{-itH} on L^2, and general considerations; see [KT98]. By partitioning time into unit intervals, we see that the constant C grows at worst like $|I|^{\frac{1}{2}}$ (which corresponds to the time exponent $q = 2$). $\square$

We use the fractional powers H^γ of the operator H, defined via the Borel functional calculus, as a substitute for the usual derivative $(-\Delta)^\gamma$. The former has the advantage of commuting with the linear propagator e^{-itH}. Trivially

$$\|H^{\frac{1}{2}} f\|_{L^2} \sim \|(-\Delta)^{\frac{1}{2}} f\|_{L^2} + \||x| f\|_{L^2} \sim \|f\|_\Sigma.$$

Using complex interpolation, Killip, Visan, and Zhang extended this equivalence to other L^p norms and other powers of H.

Lemma 2.2.2 ([KVZ09, Lemma 2.7])**.** *For $0 \leq \gamma \leq 1$ and $1 < p < \infty$, one has*

$$\|H^\gamma f\|_{L^p(\mathbf{R}^d)} \sim \|(-\Delta)^\gamma f\|_{L^p(\mathbf{R}^d)} + \||x|^{2\gamma} f\|_{L^p(\mathbf{R}^d)}.$$

As a consequence, H^γ inherits many properties of $(-\Delta)^\gamma$, including Sobolev embedding:

Lemma 2.2.3 ([KVZ09, Lemma 2.8]). *Suppose* $\gamma \in [0,1]$ *and* $1 < p < \frac{d}{2\gamma}$, *and define* p^* *by* $\frac{1}{p^*} = \frac{1}{p} - \frac{2\gamma}{d}$. *Then*

$$\|f\|_{L^{p^*}(\mathbf{R}^d)} \lesssim \|H^\gamma f\|_{L^p(\mathbf{R}^d)}.$$

Similarly, the fractional chain and product rules carry over to the current setting:

Corollary 2.2.4 ([KVZ09, Proposition 2.10]). *Let* $F(z) = |z|^{\frac{4}{d-2}} z$. *For any* $0 \le \gamma \le \frac{1}{2}$ *and* $1 < p < \infty$,

$$\|H^\gamma F(u)\|_{L^p(\mathbf{R}^d)} \lesssim \|F'(u)\|_{L^{p_0}(\mathbf{R}^d)} \|H^\gamma f\|_{L^{p_1}(\mathbf{R}^d)}$$

for all $p_0, p_1 \in (1,\infty)$ *with* $p^{-1} = p_0^{-1} + p_1^{-1}$.

Using Lemma 2.2.2 and the Christ-Weinstein fractional product rule for $(-\Delta)^\gamma$ (e.g. [Tay00]), we obtain

Corollary 2.2.5. *For* $\gamma \in (0,1]$, $r, p_i, q_i \in (1,\infty)$ *with* $r^{-1} = p_i^{-1} + q_i^{-1}$, $i = 1,2$, *we have*

$$\|H^\gamma (fg)\|_r \lesssim \|H^\gamma f\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \|H^\gamma g\|_{q_2}.$$

The exponent $\gamma = \frac{1}{2}$ is particularly relevant to us, and it will be convenient to use the notation

$$\|f\|_{L_t^q \Sigma_x^r (I \times \mathbf{R}^d)} = \|H^{\frac{1}{2}} f\|_{L_t^q L_x^r (I \times \mathbf{R}^d)}.$$

The superscript of Σ is assumed to be 2 if omitted. We shall need the following refinement of Fatou's Lemma due to Brézis and Lieb:

Lemma 2.2.6 (Refined Fatou [BL83]). *Fix* $1 \le p < \infty$, *and suppose* f_n *is a sequence of functions in* $L^p(\mathbf{R}^d)$ *such that* $\sup_n \|f_n\|_p < \infty$ *and* $f_n \to f$ *pointwise. Then*

$$\lim_{n \to \infty} \int_{\mathbf{R}^d} \big| |f_n|^p - |f_n - f|^p - |f|^p \big| \, dx = 0.$$

Finally, we record a Mikhlin-type spectral multiplier theorem.

Theorem 2.2.7 (Hebisch [Heb90]). *If* $F : \mathbf{R} \to \mathbf{C}$ *is a bounded function which obeys the derivative estimates*

$$|\partial^k F(\lambda)| \lesssim_k |\lambda|^{-k} \quad \textit{for all} \quad 0 \le k \le \tfrac{d}{2} + 1,$$

then the operator $F(H)$, defined initially on L^2 via the Borel functional calculus, is bounded on L^p for all $1 < p < \infty$.

2.2.2 Littlewood-Paley theory

Using Theorem 2.2.7 as a substitute for the Mikhlin multiplier theorem, we obtain a Littlewood-Paley theory adapted to H by mimicking the classical development for Fourier multipliers. We define Littlewood-Paley projections using both compactly supported bump functions and also the heat kernel of H. The parabolic maximum principle implies that

$$0 \le e^{-tH}(x,y) \le e^{\frac{t\Delta}{2}}(x,y) = \tfrac{1}{(2\pi t)^{d/2}} e^{-\frac{|x-y|^2}{2t}}. \tag{2.16}$$

Fix a smooth function φ supported in $|\lambda| \le 2$ with $\varphi(\lambda) = 1$ for $|\lambda| \le 1$, and let $\psi(\lambda) = \varphi(\lambda) - \varphi(2\lambda)$. For each dyadic number $N \in 2^{\mathbf{Z}}$, which we will often refer to as "frequency," define

$$\begin{aligned} P^H_{\le N} &= \varphi(\sqrt{H/N^2}), \quad & P^H_N &= \psi(\sqrt{H/N^2}), \\ \tilde{P}^H_{\le N} &= e^{-H/N^2}, & \tilde{P}^H_N &= e^{-H/N^2} - e^{-4H/N^2}. \end{aligned}$$

The associated operators $P^H_{<N}, P^H_{>N}$, etc. are defined in the usual manner.

Remark. As the spectrum of H is bounded away from 0, by choosing φ appropriately we can arrange for $P_{<1} = 0$; thus we will only consider frequencies $N \ge 1$.

Later on we shall need the classical Littlewood-Paley projectors

$$P^\Delta_{\le N} = \varphi(\sqrt{-\Delta/N^2}) \quad P^\Delta_N = \psi(\sqrt{-\Delta/N^2}), \tag{2.17}$$

$$\tilde{P}^\Delta_{\le N} = e^{\Delta/2N^2} \qquad \tilde{P}^\Delta_N = e^{\Delta/2N^2} - e^{2\Delta/N^2}. \tag{2.18}$$

The maximum principle implies the pointwise bound

$$|\tilde{P}^H_N f(x)| + |\tilde{P}^H_{\le N} f(x)| \lesssim \tilde{P}^\Delta_{\le N}|f|(x) + \tilde{P}^\Delta_{\le N/2}|f|(x). \tag{2.19}$$

To reduce clutter we usually suppress the superscripts H and Δ unless both types of projectors arise in the same context. For the rest of this section, $P_{\le N}$ and P_N denote $P^H_{\le N}$ and P^H_N, respectively.

Lemma 2.2.8 (Bernstein estimates)**.** *For $f \in C_c^\infty(\mathbf{R}^d)$, $1 < p \le q < \infty$, $s \ge 0$, one has the Bernstein inequalities*

$$\|P_{\le N} f\|_p \lesssim \|\tilde{P}_{\le N} f\|_p, \quad \|P_N f\|_p \lesssim \|\tilde{P}_N f\|_p \tag{2.20}$$

$$\|P_{\le N} f\|_p + \|P_N f\|_p + \|\tilde{P}_{\le N} f\|_p + \|\tilde{P}_N f\|_p \lesssim \|f\|_p \tag{2.21}$$

$$\|P_{\le N} f\|_q + \|P_N f\|_q + \|\tilde{P}_{\le N} f\|_q + \|\tilde{P}_N f\|_q \lesssim N^{\frac{d}{p} - \frac{d}{q}} \|f\|_p \tag{2.22}$$

$$N^{2s} \|P_N f\|_p \sim \|H^s P_N f\|_p \tag{2.23}$$

$$\|P_{>N} f\|_p \lesssim N^{-2s} \|H^s P_{>N} f\|_p. \tag{2.24}$$

In (2.22)*, the estimates for $\tilde{P}_{\le N} f$ and $\tilde{P}_N f$ also hold when $p = 1$, $q = \infty$. Further,*

$$f = \sum_N P_N f = \sum_N \tilde{P}_N f \tag{2.25}$$

where the series converge in L^p, $1 < p < \infty$. Finally, we have the square function estimate

$$\|f\|_p \sim \|(\sum_N |P_N f|^2)^{1/2}\|_p. \tag{2.26}$$

Proof. The estimates (2.20) follow immediately from Theorem 2.2.7. To see (2.21), observe that the functions $\varphi(\sqrt{\cdot/N^2})$, $e^{-\cdot/N^2}$ satisfy the hypotheses of Theorem 2.2.7 uniformly in N. Next use (2.16) together with Young's convolution inequality to get

$$\|\tilde{P}_{\le N} f\|_q + \|\tilde{P}_N f\|_q \lesssim N^{\frac{d}{q} - \frac{d}{p}} \|f\|_p \quad \text{for } 1 \le p \le q \le \infty. \tag{2.27}$$

From (2.20) we obtain the rest of (2.22). Now consider (2.23). Let $\tilde{\psi}$ be a fattened version of ψ so that $\tilde{\psi} = 1$ on the support of ψ. Put $F(\lambda) = \lambda^s \tilde{\psi}(\sqrt{\lambda})$. By Theorem 2.2.7, the relation $\psi = \tilde{\psi}\psi$, and the functional calculus,

$$\|N^{-2s} H^s P_N f\|_p = \|F(H/N^2) P_N f\|_p \lesssim \|P_N f\|_p.$$

The reverse inequality follows by considering $F(x) = \lambda^{-s} \tilde{\psi}(\lambda)$.

We turn to (2.25). The equality holds in L^2 by the functional calculus and the fact that the spectrum of H is bounded away from 0. For $p \ne 2$, choose q and $0 < \theta < 1$ so that

$p^{-1} = 2^{-1}(1-\theta) + q^{-1}\theta$. By (2.21), the partial sum operators

$$S_{N_0,N_1} = \sum_{N_0 < N \le N_1} P_N, \qquad \tilde{S}_{N_0,N_1} = \sum_{N_0 < N \le N_1} \tilde{P}_N$$

are bounded on every L^p, $1 < p < \infty$, uniformly in N_0, N_1. Thus by Hölder's inequality,

$$\|f - S_{N_0,N_1} f\|_p \le \|f - S_{N_0,N_1} f\|_2^{1-\theta} \|f - S_{N_0,N_1} f\|_q^{\theta} \to 0 \text{ as } N_0 \to 0,\ N_1 \to \infty,$$

and similarly for the partial sums $\tilde{S}_{N_0,N_1} f$. The estimate (2.24) follows from (2.21), (2.23), and the decomposition $P_{>N} f = \sum_{M>N} P_M f$.

To prove the square function estimate, run the usual Khintchine's inequality argument using Theorem 2.2.7 in place of the Mikhlin multiplier theorem. □

2.2.3 Local smoothing

The following local smoothing lemma and its corollary will be needed when proving properties of the nonlinear profile decomposition in Section 2.6.

Lemma 2.2.9. *If $u = e^{-itH}\phi$, $\phi \in \Sigma(\mathbf{R}^d)$, then*

$$\int_I \int_{\mathbf{R}^d} |\nabla u(x)|^2 \langle R^{-1}(x-z)\rangle^{-3} \, dx \, dt \lesssim R(1+|I|) \|u\|_{L_t^\infty L_x^2} \|H^{1/2} u\|_{L_t^\infty L_x^2}.$$

with the constant independent of $z \in \mathbf{R}^d$ and $R > 0$.

Proof. We recall the Morawetz identity. Let a be a sufficiently smooth function of x; then for any u satisfying the linear equation $i\partial_t u = (-\frac{1}{2}\Delta + V)u$, one has

$$\begin{aligned} \partial_t \int \nabla a \cdot \operatorname{Im}(\overline{u}\nabla u)\, dx = \int a_{jk} \operatorname{Re}(u_j \overline{u}_k)\, dx - \tfrac{1}{4} \int |u|^2 a_{jjkk}\, dx \\ - \tfrac{1}{2} \int |u|^2 \nabla a \cdot \nabla V \, dx \end{aligned} \tag{2.28}$$

We use this identity with $a(x) = \langle R^{-1}(x-z)\rangle$ and $V = \frac{1}{2}|x|^2$, and compute

$$\begin{aligned} a_j(x) &= \frac{R^{-2}(x_j - z_j)}{\langle R^{-1}(x-z)\rangle},\ a_{jk}(x) = R^{-2}\left[\frac{\delta_{jk}}{\langle R^{-1}(x-z)\rangle} - \frac{R^{-2}(x_j-z_j)(x_k-z_k)}{\langle R^{-1}(x-z)\rangle^3}\right] \\ \Delta^2 a(x) &\le -\frac{15R^{-4}}{\langle R^{-1}(x-z)\rangle^7}. \end{aligned}$$

As $\Delta^2 a \le 0$, the right side of (2.28) is bounded below by

$$R^{-2}\int \langle R^{-1}(x-z)\rangle^{-1}\Big[|\nabla u|^2 - \big|\tfrac{R^{-1}(x-z)}{\langle R^{-1}(x-z)\rangle}\cdot \nabla u\big|^2\Big]\,dx - \tfrac{1}{2R}\int |u|^2 \tfrac{R^{-1}(x-z)}{\langle R^{-1}(x-z)\rangle}\cdot x\,dx$$
$$\ge R^{-2}\int |\nabla u(x)|^2 \langle R^{-1}(x-z)\rangle^{-3}\,dx - \tfrac{R^{-1}}{2}\int |u|^2|x|\,dx.$$

Integrating in time and applying Cauchy-Schwarz, we get

$$\begin{aligned} R^{-2}\int_I\int_{\mathbf{R}^d} &\langle R^{-1}(x-z)\rangle^{-3}|\nabla u(t,x)|^2\,dxdt \\ &\lesssim \sup_{t\in I} R^{-1}\int \tfrac{R^{-1}(x-z)}{\langle R^{-1}(x-z)\rangle}|u(t,x)||\nabla u(t,x)|\,dx + \tfrac{1}{2R}\int_I\int_{\mathbf{R}^d}|x||u|^2\,dxdt \\ &\lesssim R^{-1}(1+|I|)\|u\|_{L^\infty_t L^2_x}\|H^{1/2}u\|_{L^\infty_t L^2_x}. \end{aligned}$$

This completes the proof of the lemma. □

Corollary 2.2.10. *Fix $\phi \in \Sigma(\mathbf{R}^d)$. Then for all $T, R \le 1$, we have*

$$\|\nabla e^{-itH}\phi\|_{L^2_{t,x}(|t-t_0|\le T,\ |x-x_0|\le R)} \lesssim T^{\frac{2}{3(d+2)}} R^{\frac{3d+2}{3(d+2)}} \|\phi\|_\Sigma^{\frac{2}{3}} \|e^{-itH}\phi\|^{\frac{1}{3}}_{L^{\frac{2(d+2)}{d-2}}_{t,x}}.$$

When $d = 3$, we also have

$$\|\nabla e^{-itH}\phi\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}(|t-t_0|\le T,\ |x-x_0|\le R)} \lesssim T^{\frac{23}{180}} R^{\frac{11}{45}} \|e^{-itH}\phi\|^{\frac{5}{48}}_{L^{10}_{t,x}} \|\phi\|_\Sigma^{\frac{43}{48}}$$

Proof. The proofs are fairly standard (see [Vis14] or [KVZb]); we present the details for the second claim, which is slightly more involved. Let E the region $\{|t-t_0| \le T,\ |x-x_0| \le R\}$. Norms which do not specify the region of integration are taken over the spacetime slab $\{|t-t_0| \le T\} \times \mathbf{R}^3$. By Hölder,

$$\|\nabla e^{-itH}\phi\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}(E)} \le \|\nabla e^{-itH}\phi\|^{\frac{1}{3}}_{L^2_{t,x}(E)} \|\nabla e^{-itH}\phi\|^{\frac{2}{3}}_{L^5_t L_x^{\frac{20}{9}}(E)}.$$

By Hölder and Strichartz,

$$\|\nabla e^{-itH}\phi\|_{L^5_t L_x^{\frac{20}{9}}(E)} \lesssim T^{\frac{1}{8}} \|\nabla e^{-itH}\phi\|_{L_t^{\frac{40}{3}} L_x^{\frac{20}{9}}} \lesssim T^{\frac{1}{8}}\|\phi\|_\Sigma. \tag{2.29}$$

We now estimate $\|\nabla e^{-itH}\phi\|_{L^2_{t,x}}$. Let $N \in 2^{\mathbf{N}}$ be a dyadic number to be chosen later, and decompose

$$\|\nabla e^{-itH}\phi\|_{L^2_{t,x}(E)} \le \|\nabla e^{-itH}P^H_{\le N}\phi\|_{L^2_{t,x}(E)} + \|\nabla e^{-itH}P^H_{>N}\phi\|_{L^2_{t,x}(E)}.$$

For the low frequency piece, apply Hölder and the Bernstein inequalities to obtain

$$\|\nabla e^{-itH} P^H_{\leq N}\phi\|_{L^2_{t,x}} \lesssim T^{\frac{2}{5}} R^{\frac{6}{5}} \|\nabla e^{-itH} P^H_{\leq N}\phi\|_{L^{10}_{t,x}} \lesssim T^{\frac{2}{5}} R^{\frac{6}{5}} N \|e^{-itH}\phi\|_{L^{10}_{t,x}}.$$

For the high-frequency piece, apply local smoothing and Bernstein:

$$\|\nabla e^{-itH} P^H_{>N}\phi\|_{L^2_{t,x}} \lesssim R^{\frac{1}{2}} \|P^H_{>N}\phi\|^{\frac{1}{2}}_{L^2} \|H^{\frac{1}{2}}\phi\|^{\frac{1}{2}}_{\Sigma} \lesssim R^{\frac{1}{2}} N^{-\frac{1}{2}} \|\phi\|_{\Sigma}.$$

Optimizing in N, we obtain

$$\|\nabla e^{-itH}\phi\|_{L^2_{t,x}} \lesssim T^{\frac{2}{15}} R^{\frac{11}{15}} \|e^{-itH}\phi\|^{\frac{1}{3}}_{L^{10}_{t,x}} \|\phi\|^{\frac{2}{3}}_{\Sigma}.$$

Combining this estimate with (2.29) yields the conclusion of the corollary. □

2.3 Local theory

We record some standard results concerning local-wellposedness for (2.1). These are direct analogues of the theory for the scale-invariant equation. By Lemma 2.2.3 and Corollaries 2.2.4 and 2.2.5, we can use essentially the same proofs as in that case. The reader should consult [KV13] for those proofs.

Proposition 2.3.1 (Local wellposedness). *Let $u_0 \in \Sigma(\mathbf{R}^d)$ and fix a compact time interval $0 \in I \subset \mathbf{R}$. Then there exists a constant $\eta_0 = \eta_0(d, |I|)$ such that whenever $\eta < \eta_0$ and*

$$\|H^{\frac{1}{2}} e^{-itH} u_0\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}} (I \times \mathbf{R}^d)} \leq \eta,$$

there exists a unique solution $u : I \times \mathbf{R}^d \to \mathbf{C}$ to (2.1) which satisfies the bounds

$$\|H^{\frac{1}{2}} u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}} (I \times \mathbf{R}^d)} \leq 2\eta \quad \text{and} \quad \|H^{\frac{1}{2}} u\|_{S(I)} \lesssim \|u_0\|_{\Sigma} + \eta^{\frac{d+2}{d-2}}.$$

Corollary 2.3.2 (Blowup criterion). *Suppose $u : (T_{min}, T_{max}) \times \mathbf{R}^d \to \mathbf{C}$ is a maximal lifespan solution to (2.1), and fix $T_{min} < t_0 < T_{max}$. If $T_{max} < \infty$, then*

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}} ([t_0, T_{max}))} = \infty.$$

If $T_{min} > -\infty$, then

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}} ((T_{min}, t_0])} = \infty.$$

Proposition 2.3.3 (Stability)**.** *Fix $t_0 \in I \subset \mathbf{R}$ an interval of unit length and let $\tilde{u} : I \times \mathbf{R}^d \to \mathbf{C}$ be an approximate solution to* (2.1) *in the sense that*

$$i\partial_t \tilde{u} = H\tilde{u} \pm |\tilde{u}|^{\frac{4}{d-2}}\tilde{u} + e$$

for some function e. Assume that

$$\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \leq L, \quad \|H^{\frac{1}{2}}\tilde{u}\|_{L_t^\infty L_x^2} \leq E, \tag{2.30}$$

and that for some $0 < \varepsilon < \varepsilon_0(E, L)$ one has

$$\|\tilde{u}(t_0) - u_0\|_\Sigma + \|H^{\frac{1}{2}}e\|_{N(I)} \leq \varepsilon, \tag{2.31}$$

Then there exists a unique solution $u : I \times \mathbf{R}^d \to \mathbf{C}$ to (2.1) *with $u(t_0) = u_0$ and which further satisfies the estimates*

$$\|\tilde{u} - u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|H^{\frac{1}{2}}(\tilde{u} - u)\|_{S(I)} \leq C(E, L)\varepsilon^c \tag{2.32}$$

where $0 < c = c(d) < 1$ and $C(E, L)$ is a function which is nondecreasing in each variable.

2.4 Concentration compactness

The purpose of this section is to prove a linear profile decomposition for the Strichartz inequality

$$\|e^{-itH}f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbf{R}^d)} \leq C(|I|, d)\|f\|_\Sigma.$$

Our decomposition resembles that of Keraani [Ker01] in the sense that each profile has a characteristic length scale and location in spacetime. But since the space Σ lacks both translation and scaling symmetry, the precise definitions of our profiles are more complicated.

Keraani considered the analogous Strichartz estimate

$$\|e^{it\Delta}f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(\mathbf{R}\times\mathbf{R}^d)} \lesssim \|f\|_{\dot{H}^1(\mathbf{R}^d)}.$$

Recall that in that situation, if f_n is a bounded sequence in $\dot{H}^1$ with nontrivial linear evolution, then one has a decomposition $f_n = \phi_n + r_n$ where $\phi_n = e^{it_n\Delta}G_n\phi$, G_n are certain unitary scaling and translation operators on $\dot{H}^1$ (defined as in (2.33)), and ϕ is a weak limit of $G_n^{-1}e^{-it_n\Delta}f_n$ in $\dot{H}^1$. The "bubble" ϕ_n is nontrivial and decouples from the remainder r_n in various norms. By applying this decomposition inductively to the remainder r_n, one obtains the full collection of profiles constituting f_n.

We follow the general presentation in [KV13, Vis14]. Let $f_n \in \Sigma$ be a bounded sequence. Using a variant of Keraani's argument, we seek an $\dot{H}^1$-weak limit ϕ in terms of f_n and write $f_n = \phi_n + r_n$ where ϕ_n is defined analogously as before by "moving the operators onto f_n." However, two main issues arise.

The first is that while f_n belong to Σ, an $\dot{H}^1$ weak limit of a sequence like $G_n^{-1}e^{it_nH}f_n$ need only belong to $\dot{H}^1$. Indeed, the $\dot{H}^1$ isometries G_n^{-1} will in general have unbounded norm as operators on Σ because of the $|x|^2$ weight. To define ϕ_n, we need to introduce spatial cutoffs to obtain functions in Σ.

Secondly, to establish the various orthogonality assertions one must understand how the linear propagator e^{-itH} interacts with the $\dot{H}^1$ symmetries of translation and scaling in certain limits. This interaction is studied in Section 2.4.2. In particular, the convergence results obtained there serve as a substitute for the scaling relation

$$e^{it\Delta}G_n = G_n e^{iN_n^2 t\Delta} \quad \text{where} \quad G_n\phi = N_n^{\frac{d-2}{2}}\phi(N_n(\cdot - x_n)).$$

They can also be regarded as a precise form of the heuristic that as the initial data concentrates at a point x_0, the potential $V(x) = |x|^2/2$ can be regarded over short time intervals as essentially equal to the constant potential $V(x_0)$; hence for short times the linear propagator e^{-itH} can be approximated up to a phase factor by the free particle propagator. Section 2.5 addresses a nonlinear version of this statement.

2.4.1 An inverse Strichartz inequality

Unless indicated otherwise, $0 \in I$ in this section will denote a fixed interval of length at most 1, and all spacetime norms will be taken over $I \times \mathbf{R}^d$.

Suppose f_n is a sequence of functions in Σ with nontrivial linear evolution $e^{-itH}f_n$. The following refined Strichartz estimate shows that there must be a "frequency" N_n which makes a nontrivial contribution to the evolution.

Proposition 2.4.1 (Refined Strichartz)**.**

$$\|e^{-itH}f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \lesssim \|f\|_{\Sigma}^{\frac{4}{d+2}} \sup_N \|e^{-itH}P_N f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{d-2}{d+2}}$$

Proof. Using the Littlewood-Paley theory, we may quote essentially verbatim the proof of refined Strichartz for the free particle propagator ([Vis14] Lemma 3.1). Write f_N for $P_N f$, where $P_N = P_N^H$ unless indicated otherwise. When $d \geq 6$, apply the square function estimate (2.26), Hölder, Bernstein, and Strichartz to get

$$\begin{aligned}
&\|e^{-itH}f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} \sim \Big\|(\sum_N |e^{-itH}f_N|^2)^{1/2}\Big\|_{\frac{2(d+2)}{d-2}}^{\frac{2(d+2)}{d-2}} = \iint (\sum_N |e^{-itH}f_N|^2)^{\frac{d+2}{d-2}}\, dx\, dt \\
&\lesssim \sum_{M \leq N} \iint |e^{-itH}f_M|^{\frac{d+2}{d-2}} |e^{-itH}f_N|^{\frac{d+2}{d-2}}\, dx\, dt \\
&\lesssim \sum_{M \leq N} \|e^{-itH}f_M\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|e^{-itH}f_M\|_{L_{t,x}^{\frac{2(d+2)}{d-4}}} \|e^{-itH}f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|e^{-itH}f_N\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\
&\lesssim \sup_N \|e^{-itH}f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{8}{d-2}} \sum_{M \leq N} M^2 \|e^{-itH}f_M\|_{L_t^{\frac{2(d+2)}{d-4}} L_x^{\frac{2d(d+2)}{d^2+8}}} \|f_N\|_{L^2} \\
&\lesssim \sup_N \|e^{-itH}f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{8}{d-2}} \sum_{M \leq N} M^2 \|f_M\|_{L_x^2} \|f_N\|_{L_x^2} \\
&\lesssim \sup_N \|e^{-itH}f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{8}{d-2}} \sum_{M \leq N} \frac{M}{N} \|H^{1/2}f_M\|_{L^2} \|H^{1/2}f_N\|_{L_x^2} \\
&\lesssim \sup_N \|e^{-itH}f_N\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{8}{d-2}} \|f\|_{\Sigma}^2.
\end{aligned}$$

The cases $d = 3, 4, 5$ are handled similarly with some minor modifications in the applications of Hölder's inequality. □

The next proposition goes one step further and asserts that the sequence $e^{-itH}f_n$ with nontrivial spacetime norm must in fact contain a bubble centered at some (t_n, x_n) with spatial scale N_n^{-1}. First we introduce some vocabulary and notation which are common to concentration compactness arguments.

Definition 2.4.1. A *frame* is a sequence $(t_n, x_n, N_n) \in I \times \mathbf{R}^d \times 2^{\mathbf{N}}$ conforming to one of the following scenarios:

1. $N_n \equiv 1,\ t_n \equiv 0$, and $x_n \equiv 0$.

2. $N_n \to \infty$ and $N_n^{-1}|x_n| \to r_\infty \in [0, \infty)$.

The parameters $t_n,\ x_n,\ N_n$ will specify the temporal center, spatial center, and inverse length scale of a function. The condition that $|x_n| \lesssim N_n$ reflects the fact that we only consider functions obeying some uniform bound in Σ, and such functions cannot be centered arbitrarily far from the origin. We need to augment the frame $\{(t_n, x_n, N_n)\}$ by an auxiliary parameter N_n', which corresponds to a sequence of spatial cutoffs adapted to the frame.

Definition 2.4.2. An *augmented frame* is a sequence $(t_n, x_n, N_n, N_n') \in I \times \mathbf{R}^d \times 2^{\mathbf{N}} \times \mathbf{R}$ belonging to one of the following types:

1. $N_n \equiv 1,\ t_n \equiv 0,\ x_n \equiv 0,\ N_n' \equiv 1$.

2. $N_n \to \infty,\ N_n^{-1}|x_n| \to r_\infty \in [0, \infty)$, and either

   (a) $N_n' \equiv 1$ if $r_\infty > 0$, or

   (b) $N_n^{1/2} \le N_n' \le N_n,\ N_n^{-1}|x_n|(\frac{N_n}{N_n'}) \to 0$, and $\frac{N_n}{N_n'} \to \infty$ if $r_\infty = 0$.

Associated to an augmented frame (t_n, x_n, N_n, N_n') is a family of scaling and translation operators

$$\begin{aligned} (G_n\phi)(x) &= N_n^{\frac{d-2}{2}} \phi(N_n(x - x_n)) \\ (\tilde{G}_n f)(t, x) &= N_n^{\frac{d-2}{2}} f(N_n^2(t - t_n), N_n(x - x_n)), \end{aligned} \tag{2.33}$$

as well as spatial cutoff operators

$$S_n\phi = \begin{cases} \phi, & \text{for frames of type 1} \quad (\text{i.e. } N_n \equiv 1), \\ \chi(\frac{N_n}{N_n'}\cdot)\phi, & \text{for frames of type 2} \quad (\text{i.e. } N_n \to \infty), \end{cases} \tag{2.34}$$

where χ is a smooth compactly supported function equal to 1 on the ball $\{|x| \le 1\}$. An easy computation yields the following mapping properties:

$$\begin{aligned} &\lim_{n\to\infty} S_n = I \text{ strongly in } \dot{H}^1 \text{and in } \Sigma, \\ &\limsup_{n\to\infty} \|G_n\|_{\Sigma\to\Sigma} < \infty. \end{aligned} \tag{2.35}$$

For future reference, we record a technical lemma that, as a special case, asserts that the Σ norm is controlled almost entirely by the $\dot{H}^1$ norm for functions concentrating near the origin.

Lemma 2.4.2 (Approximation)**.** *Let (q, r) be an admissible pair of exponents with $2 \le r < d$, and let $\mathcal{F} = \{(t_n, x_n, N_n, N_n')\}$ be an augmented frame of type 2.*

1. *Suppose $\mathcal{F}$ is of type 2a in Definition 2.4.2. Then for $\{f_n\} \subseteq L_t^q H_x^{1,r}(\mathbf{R}\times\mathbf{R}^d)$, we have*

$$\limsup_n \|\tilde{G}_n S_n f_n\|_{L_t^q \Sigma_x^r} \lesssim \limsup_n \|f_n\|_{L_t^q H_x^{1,r}}.$$

2. *Suppose $\mathcal{F}$ is of type 2b and $f_n \in L_t^q \dot{H}_x^{1,r}(\mathbf{R}\times\mathbf{R}^d)$. Then*

$$\limsup_n \|\tilde{G}_n S_n f_n\|_{L_t^q \Sigma_x^r} \lesssim \limsup_n \|f_n\|_{L_t^q \dot{H}_x^{1,r}}.$$

Here $H^{1,r}(\mathbf{R}^d)$ and $\dot{H}^{1,r}(\mathbf{R}^d)$ denote the Sobolev spaces equipped with the norms

$$\|f\|_{H^{1,r}} = \|\langle\nabla\rangle\|_{L^r(\mathbf{R}^d)}, \quad \|f\|_{\dot{H}^{1,r}} = \||\nabla| f\|_{L^r(\mathbf{R}^d)}.$$

Proof. By time translation invariance we may assume $t_n \equiv 0$. By Lemma 2.2.2, it suffices to separately bound $\|\nabla \tilde{G}_n S_n f_n\|_{L_t^q L_x^r}$ and $\||x|\tilde{G}_n S_n f_n\|_{L_t^q L_x^r}$. Using a change of variables, the admissibility of (q, r), Hölder, and Sobolev embedding (hence the restriction $r < d$), we have

$$\begin{aligned} &\|\nabla \tilde{G}_n S_n f_n\|_{L_t^q L_x^r} = \|\nabla[N_n^{\frac{d-2}{2}} f_n(N_n^2 t, N_n(x-x_n))\chi(N_n'(x-x_n))]\|_{L_t^q L_x^r} \\ &\lesssim \|(\nabla f_n)(t,x)\|_{L_t^q L_x^r} + \tfrac{N_n'}{N_n}\|f_n(t,x)\|_{L_t^q L_x^r(\mathbf{R}\times\{|x|\sim \frac{N_n}{N_n'}\})} \\ &\lesssim \|\nabla f_n\|_{L_t^q \dot{H}_x^{1,r}}. \end{aligned}$$

To estimate $\| |x| \tilde{G}_n S_n f_n \|_{L_t^q L_x^r}$ we distinguish the two cases. Consider first the case where $f_n \in L_t^q H_x^{1,r}$. Using the bound $|x_n| \lesssim N_n$ and a change of variables, we obtain

$$\| |x| \tilde{G}_n S_n f_n \|_{L_t^q L^r} \lesssim N_n^{\frac{d}{2}} \| f_n(N_n^2 t, N_n(x - x_n)) \|_{L^r} \lesssim \| f_n \|_{L_t^q L^r} \lesssim \| f_n \|_{L_t^q H_x^{1,r}}.$$

Next, consider the case where f_n are merely assumed to lie in $L_t^q \dot{H}_x^{1,r}$. For each t, we apply Hölder and Sobolev embedding to get

$$\begin{aligned} \| |x| \tilde{G}_n S_n f_n \|_{L_x^r}^r &= N_n^{\frac{dr}{2} - d - r} \int_{|x| \lesssim \frac{N_n}{N_n'}} |x_n + N_n^{-1} x|^r |f_n(N_n^2 t, x)|^r dx \\ &\lesssim N_n^{\frac{dr}{2} - d} \left[N_n^{-r} |x_n|^r + N_n^{-2r} (\tfrac{N_n}{N_n'})^r \right] \int_{|x| \lesssim \frac{N_n}{N_n'}} |f_n(N_n^2 t, x)|^r dx \\ &\lesssim N_n^{\frac{dr}{2} - d} \left[N_n^{-r} |x_n|^r (\tfrac{N_n}{N_n'})^r + (N_n')^{-2r} \right] \| \nabla f_n(N_n^2 t) \|_{L_x^r}^r. \end{aligned}$$

By the hypotheses on the parameter N_n' in Definition 2.4.2, the expression inside the brackets goes to 0 as $n \to \infty$. After integrating in t and changing variables, we conclude

$$\| |x| \tilde{G}_n S_n f_n \|_{L_t^q L_x^r} \lesssim c_n \| f_n \|_{L_t^q \dot{H}_x^{1,r}}$$

where $c_n = o(1)$ as $n \to \infty$. This completes the proof of the lemma. □

Proposition 2.4.3 (Inverse Strichartz). *Let I be a compact interval containing 0 of length at most 1, and suppose f_n is a sequence of functions in $\Sigma(\mathbf{R}^d)$ satisfying*

$$0 < \varepsilon \le \| e^{-itH} f_n \|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I \times \mathbf{R}^d)} \lesssim \| f_n \|_\Sigma \le A < \infty.$$

Then, after passing to a subsequence, there exists an augmented frame

$$\mathcal{F} = \{(t_n, x_n, N_n, N_n')\}$$

and a sequence of functions $\phi_n \in \Sigma$ such that one of the following holds:

1. *$\mathcal{F}$ is of type 1 (i.e. $N_n \equiv 1$) and $\phi_n = \phi$ where $\phi \in \Sigma$ is a weak limit of f_n in Σ.*

2. *$\mathcal{F}$ is of type 2, either $t_n \equiv 0$ or $N_n^2 t_n \to \pm\infty$, and $\phi_n = e^{it_n H} G_n S_n \phi$ where $\phi \in \dot{H}^1(\mathbf{R}^d)$ is a weak limit of $G_n^{-1} e^{-it_n H} f_n$ in $\dot{H}^1$. Moreover, if $\mathcal{F}$ is of type 2a, then ϕ also belongs to $L^2(\mathbf{R}^d)$.*

The functions ϕ_n have the following properties:

$$\liminf_{n} \|\phi_n\|_\Sigma \gtrsim A \left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}, \tag{2.36}$$

$$\lim_{n\to\infty} \|f_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|f_n - \phi_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|\phi_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} = 0, \tag{2.37}$$

$$\lim_{n\to\infty} \|f_n\|_\Sigma^2 - \|f_n - \phi_n\|_\Sigma^2 - \|\phi_n\|_\Sigma^2 = 0. \tag{2.38}$$

Proof. Our plan is as follows. First we identify the parameters t_n, x_n, N_n, which define the location of the bubble ϕ_n and its characteristic size, and dispose of the case where $N_n \equiv 1$.

The case where $N_n \to \infty$ is more involved. First we define the profile ϕ_n and verify the assertions (2.36) and (2.38). Passing to a subsequence, we may assume that the sequence $N_n^2 t_n$ converges in $[-\infty, \infty]$. If the limit is infinite, decoupling (2.37) in the $L^{\frac{2d}{d-2}}$ norm will also follow. If instead $N_n^2 t_n$ has a finite limit, we show that in fact the time parameter t_n can actually be redefined to be identically zero after making a negligible correction to the profile ϕ_n, and verify that the modified profile (with $t_n = 0$ now) satisfies property (2.37) in addition to (2.36) and (2.38). We shall see along the way that in this regime of short time scales and initial data concentrated near the origin, the potential may be essentially regarded as constant.

By Proposition 2.4.1, there exist frequencies N_n such that

$$\|P_{N_n} e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \gtrsim \varepsilon^{\frac{d+2}{4}} A^{-\frac{d-2}{4}}.$$

The comparison of Littlewood-Paley projectors (2.20) implies

$$\|\tilde{P}_{N_n} e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \gtrsim \varepsilon^{\frac{d+2}{4}} A^{-\frac{d-2}{4}},$$

where $\tilde{P}_N = e^{-H/N^2} - e^{-4H/N^2}$ denote the projections based on the heat kernel. By Hölder, Strichartz, and Bernstein,

$$\begin{aligned}\varepsilon^{\frac{d+2}{4}} A^{-\frac{d-2}{4}} &\lesssim \|\tilde{P}_{N_n} e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \lesssim \|\tilde{P}_{N_n} e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-2}{d}} \|\tilde{P}_{N_n} e^{-itH} f_n\|_{L_{t,x}^\infty}^{\frac{2}{d}} \\ &\lesssim (N_n^{-1} A)^{\frac{d-2}{d}} \|\tilde{P}_{N_n} e^{-itH} f_n\|_{L_{t,x}^\infty}^{\frac{2}{d}}.\end{aligned}$$

Therefore, there exist $(t_n, x_n) \in I \times \mathbf{R}^d$ such that

$$|e^{-it_n H} \tilde{P}_{N_n} f_n(x_n)| \gtrsim N_n^{\frac{d-2}{2}} A(\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{8}}. \tag{2.39}$$

The parameters t_n, x_n, N_n will determine the center and width of a bubble.

We observe first that the boundedness of f_n in Σ limits how far the bubble can live from the spatial origin.

Lemma 2.4.4. *We have*

$$|x_n| \le C_{A,\varepsilon} N_n.$$

Proof. Put $g_n = |e^{-it_n H} f_n|$. By the kernel bound (2.19),

$$N_n^{\frac{d-2}{2}} A(\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{8}} \lesssim |\tilde{P}_{N_n} e^{-it_n H} f_n(x_n)| \lesssim \tilde{P}^{\Delta}_{\le N_n} g_n(x_n) + \tilde{P}^{\Delta}_{\le N_n/2} g_n(x_n).$$

Thus one of the terms on the right side is at least half as large as the left side, and it suffices to consider the case when

$$\tilde{P}^{\Delta}_{\le N_n} g_n(x_n) \gtrsim N_n^{\frac{d-2}{2}} A(\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{8}}$$

since the argument with N_n replaced by $N_n/2$ differs only cosmetically. Informally, $\tilde{P}^{\Delta}_{\le N_n} g_n$ is essentially constant over length scales of order N_n^{-1}, so if it is large at a point x_n then it is large on the ball $|x - x_n| \le N_n^{-1}$. More precisely, when $|x - x_n| \le N_n^{-1}$ we have

$$\begin{aligned}
\tilde{P}^{\Delta}_{\le N_n/2} g_n(x) &= \frac{N_n^d}{2^d (2\pi)^{\frac{d}{2}}} \int g_n(x-y) e^{-\frac{N_n^2 |y|^2}{8}}\, dy \\
&= \frac{N_n^d}{2^d (4\pi)^{\frac{d}{2}}} \int g_n(x_n - y) e^{-\frac{N_n^2 |y+x-x_n|^2}{8}}\, dy \\
&\ge e^{-1} \frac{N_n^d}{2^d (4\pi)^{\frac{d}{2}}} \int g_n(x_n - y) e^{-\frac{N_n^2 |y|^2}{2}}\, dy = e^{-1} 2^{-d} \tilde{P}^{\Delta}_{\le N_n} g_n(x_n) \\
&\gtrsim N_n^{\frac{d-2}{2}} A(\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{8}}.
\end{aligned}$$

On the other hand, the mapping properties of the heat kernel imply that

$$\|\tilde{P}^{\Delta}_{\le N_n/2} g_n\|_{\Sigma} \lesssim (1 + N_n^{-2}) A.$$

Thus,

$$A \gtrsim \|\tilde{P}^{\Delta}_{\le N_n/2} g_n\|_{\Sigma} \gtrsim \|x \tilde{P}^{\Delta}_{\le N_n/2} g_n\|_{L^2(|x-x_n| \le N_n^{-1})} \gtrsim |x_n| N_n^{-\frac{d}{2}} N_n^{\frac{d-2}{2}} A(\tfrac{\varepsilon}{A})^{\frac{d(d+2)}{8}},$$

which yields the claim. □

Case 1. Suppose the N_n have a bounded subsequence, so that (passing to a subsequence) $N_n \equiv N_\infty$. The x_n's stay bounded by Lemma 2.4.4, so after passing to a subsequence we may assume $x_n \to x_\infty$. We may also assume $t_n \to t_\infty$ since the interval I is compact. The functions f_n are bounded in Σ, hence (after passing to a subsequence) converge weakly in Σ to some ϕ.

To see that ϕ is nontrivial in Σ, we have

$$\begin{aligned}\langle \phi, e^{it_\infty H} \tilde{P}_{N_\infty} \delta_{x_\infty} \rangle &= \lim_n \langle f_n, e^{it_\infty H} \tilde{P}_{N_\infty} \delta_{x_\infty} \rangle \\ &= \lim_{n\to\infty} [e^{-it_n H} \tilde{P}_{N_\infty} f_n(x_n) + \langle f_n, (e^{it_\infty H} - e^{it_n H}) \tilde{P}_{N_\infty} \delta_{x_n} \rangle \\ &+ \langle f_n, e^{it_\infty H} \tilde{P}_{N_n} (\delta_{x_\infty} - \delta_{x_n}) \rangle].\end{aligned}$$

Using the heat kernel bounds (2.19) and the fact that, by the compactness of the embedding $\Sigma \subset L^2$, the sequence f_n converges to ϕ in L^2, one verifies easily that the second and third terms on the right side vanish. So

$$|\langle \phi, e^{it_\infty H} \tilde{P}_{N_\infty} \delta_{x_\infty} \rangle| = \lim_{n\to\infty} |e^{-it_n H} \tilde{P}_{N_\infty} f_n(x_n)| \gtrsim N_\infty^{\frac{d-2}{2}} \varepsilon^{\frac{d(d+2)}{8}} A^{-\frac{(d-2)(d+4)}{8}}.$$

On the other hand, by Hölder and (2.19),

$$\begin{aligned}|\langle \phi, e^{it_\infty H} \tilde{P}_{N_\infty} \delta_{x_\infty} \rangle| &\le \|e^{-it_\infty H} \phi\|_{L^{\frac{2d}{d-2}}} \|\tilde{P}_{N_\infty} \delta_{x_\infty}\|_{L^{\frac{2d}{d+2}}} \\ &\lesssim \|\phi\|_\Sigma N_\infty^{\frac{d-2}{2}}.\end{aligned}$$

Therefore

$$\|\phi\|_\Sigma \gtrsim \varepsilon^{\frac{d(d+2)}{8}} A^{-\frac{(d-2)(d+4)}{8}}.$$

Set

$$\phi_n \equiv \phi,$$

and define the augmented frame $(t_n, x_n, N_n, N_n') \equiv (0, 0, 1, 1)$. The decoupling in Σ (2.38) can be proved as in Case 2 below, and we refer the reader to the argument detailed there. It remains to establish decoupling in $L^{\frac{2d}{d-2}}$. As the embedding $\Sigma \subset L^2$ is compact, the sequence f_n, which converges weakly to $\phi \in \Sigma$, converges to ϕ strongly in L^2. After passing to a subsequence we obtain convergence pointwise a.e. The decoupling (2.37) now follows from Lemma 2.2.6. This completes the case where N_n have a bounded subsequence.

Case 2. Now we address the case where $N_n \to \infty$. The main nuisance is that the weak limits ϕ will usually be merely in $\dot{H}^1(\mathbf{R}^d)$, not in Σ, so defining the profiles ϕ_n will require spatial cutoffs.

As the functions $N_n^{-(d-2)/2}(e^{-it_nH} f_n)(N_n^{-1} \cdot + x_n)$ are bounded in $\dot{H}^1(\mathbf{R}^d)$, the sequence has a weak subsequential limit

$$N_n^{-\frac{d-2}{2}}(e^{-it_nH} f_n)(N_n^{-1} \cdot + x_n) \rightharpoonup \phi \text{ in } \dot{H}^1(\mathbf{R}^d). \tag{2.40}$$

By Lemma 2.4.4, after passing to a further subsequence we may assume

$$\lim_{n\to\infty} N_n^{-1}|x_n| = r_\infty < \infty \quad \text{and} \quad \lim_{n\to\infty} N_n^2 t_n = t_\infty \in [-\infty, \infty]. \tag{2.41}$$

It will be necessary to distinguish the cases $r_\infty > 0$ and $r_\infty = 0$, corresponding to whether the frame $\{(t_n, x_n, N_n)\}$ is type 2a or 2b, respectively.

Lemma 2.4.5. *If $r_\infty > 0$, the function ϕ defined in* (2.40) *also belongs to L^2.*

Proof. By (2.40) and the Rellich-Kondrashov compactness theorem, for each $R \geq 1$ we have

$$N_n^{-\frac{d-2}{2}}(e^{-it_nH} f_n)(N_n^{-1} \cdot + x_n) \to \phi \text{ in } L^2(\{|x| \leq R\}).$$

By a change of variables,

$$\begin{aligned} N_n^{-\frac{d-2}{2}}(e^{-it_nH} f_n)(N_n^{-1} \cdot + x_n)\|_{L^2(|x|\leq R)} &= N_n \|e^{-it_nH} f_n\|_{L^2(|x-x_n|\leq RN_n^{-1})} \\ &\lesssim \|xe^{-it_nH} f_n\|_{L^2} \end{aligned}$$

whenever $|x_n| \geq \frac{N_n r_\infty}{2}$ and $RN_n^{-1} \leq \frac{r_\infty}{10}$, so we have uniformly in $R \geq 1$ that

$$\limsup_n \|N_n^{-\frac{d-2}{2}}(e^{-it_nH} f_n)(N_n^{-1} \cdot + x_n)\|_{L^2(|x|\leq R)} \lesssim \sup_n \|e^{-it_nH} f_n\|_\Sigma \lesssim 1.$$

Therefore $\|\phi\|_{L^2} = \lim_{R\to\infty} \|\phi\|_{L^2(|x|\leq R)} \lesssim 1$. □

Remark. The claim fails if $r_\infty = 0$. Indeed, if $\phi \in \dot{H}^1(\mathbf{R}^d) \backslash L^2(\mathbf{R}^d)$, then $f_n = N_n^{(d-2)/2} \phi(N_n \cdot)\chi(\cdot)$ are bounded in Σ, and $N_n^{-(d-2)/2} f_n(N_n^{-1} \cdot) = \phi(\cdot)\chi(N_n^{-1} \cdot)$ converges strongly in $\dot{H}^1$ to ϕ.

Next we prove that ϕ is nontrivial in $\dot{H}^1$.

Lemma 2.4.6. $\|\phi\|_{\dot{H}^1} \gtrsim A\left(\frac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}$.

Proof. From (2.19) and (2.39),

$$N_n^{\frac{d-2}{2}} A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}} \lesssim \tilde{P}^{\Delta}_{\leq N_n}|e^{-it_nH}f_n|(x_n) + \tilde{P}^{\Delta}_{\leq N_n/2}|e^{-it_nH}f_n|(x_n),$$

so one of the terms on the right is at least half the left side. Suppose first that

$$\tilde{P}^{\Delta}_{\leq N_n}|e^{-it_nH}f_n|(x_n) \gtrsim N_n^{\frac{d-2}{2}} A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}.$$

Put $\check{\psi} = \tilde{P}^{\Delta}_{\leq 1}\delta_0 = e^{\Delta}\delta_0$. Since $\check{\psi}$ is Schwartz,

$$|\langle|\phi|, \check{\psi}\rangle_{L^2}| \leq \|\phi\|_{\dot{H}^1}\|\check{\psi}\|_{\dot{H}^{-1}} \lesssim \|\phi\|_{\dot{H}^1}.$$

On the other hand, as the absolute values $N_n^{-\frac{d-2}{2}}|e^{-it_nH}f_n|(N_n^{-1}\cdot + x_n)$ converge weakly in $\dot{H}^1$ to $|\phi|$,

$$\begin{aligned}\langle|\phi|, \check{\psi}\rangle_{L^2} &= \lim_n \langle N_n^{-\frac{d-2}{2}}|e^{-it_nH}f_n|(N_n^{-1}\cdot + x_n), \check{\psi}\rangle_{L^2} \\ &= \lim_n \tilde{P}^{\Delta}_{\leq N_n}|e^{-it_nH}f_n|(x_n) \gtrsim A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}.\end{aligned}$$

from which the claim follows. Similarly if

$$\tilde{P}^{\Delta}_{\leq N_n/2}|e^{-it_nH}f_n|(x_n) \gtrsim N_n^{\frac{d-2}{2}} A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}},$$

then we obtain $\|\phi\|_{\dot{H}^1} \sim \|\phi(2\cdot)\|_{\dot{H}^1} \gtrsim N_n^{\frac{d-2}{2}} A\left(\frac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}$. □

Having extracted a nontrivial bubble ϕ, we are ready to define the ϕ_n. The basic idea is to undo the operations applied to f_n in the definition (2.40) of ϕ. However, we need to first apply a spatial cutoff to embed ϕ in Σ.

With the frame $\{(t_n, x_n, N_n)\}$ defined according to (2.39), form the augmented frame $\{(t_n, x_n, N_n, N_n')\}$ with the cutoff parameter N_n' chosen according to the second case in Definition 2.4.2. Let G_n, S_n be the $\dot{H}^1$ isometries and spatial cutoff operators associated to $\{(t_n, x_n, N_n, N_n')\}$. Set

$$\phi_n = e^{it_nH}G_nS_n\phi = e^{it_nH}[N_n^{\frac{d-2}{2}}\phi(N_n(\cdot - x_n))\chi(N_n'(\cdot - x_n))]. \tag{2.42}$$

Let us check that ϕ_n satisfies the various properties asserted in the proposition.

Lemma 2.4.7. $A\left(\frac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}} \lesssim \liminf_{n\to\infty} \|\phi_n\|_\Sigma \leq \limsup_{n\to\infty} \|\phi_n\|_\Sigma \lesssim 1.$

Proof. By the definition of the Σ norm and a change of variables,

$$\|\phi_n\|_\Sigma = \|G_n S_n\|_\Sigma \geq \|S_n \phi\|_{\dot{H}^1}.$$

Lemma 2.4.6 and the remarks following Definition 2.4.2 together imply the lower bound

$$\liminf_n \|\phi_n\|_\Sigma \gtrsim A\left(\frac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}}.$$

The upper bound follows immediately from the case $(q,r) = (\infty, 2)$ in Lemma 2.4.2. □

Next we verify the decoupling assertion (2.38). By the Pythagorean theorem,

$$\begin{aligned} \|f_n\|_\Sigma^2 - \|f_n - \phi_n\|_\Sigma^2 - \|\phi_n\|_\Sigma^2 &= 2\operatorname{Re}\langle f_n - \phi_n, \phi_n\rangle_\Sigma) \\ &= 2\operatorname{Re}\langle e^{-it_n H} f_n - G_n S_n \phi, G_n S_n \phi\rangle_\Sigma \\ &= 2\operatorname{Re}\langle w_n, G_n S_n \phi\rangle_\Sigma. \end{aligned}$$

where $w_n = e^{-it_n H} f_n - G_n S_n \phi$. By definition,

$$\langle w_n, G_n S_n \phi\rangle_\Sigma = \langle w_n, G_n S_n \phi\rangle_{\dot{H}^1} + \langle x w_n, x G_n S_n \phi\rangle_{L^2}.$$

From (2.35) and the definition (2.40) of ϕ, it follows that

$$G_n^{-1} w_n \to 0 \quad \text{weakly in } \dot{H}^1 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} \langle w_n, G_n S_n \phi\rangle_{\dot{H}^1} = \lim_{n\to\infty} \langle G_n^{-1} w_n, S_n \phi\rangle_{\dot{H}^1} = \lim_{n\to\infty} \langle G_n^{-1} w_n, \phi\rangle_{\dot{H}^1} = 0.$$

We turn to the second component of the inner product. Fix $R > 0$, and estimate

$$\begin{aligned} &|\langle x w_n, x G_n S_n \phi\rangle_{L^2}| \\ &\leq \int_{\{|x - x_n| \leq R N_n^{-1}\}} |x w_n||x G_n S_n \phi|\, dx + \int_{\{|x - x_n| > R N_n^{-1}\}} |x w_n||x G_n S_n \phi|\, dx \\ &= (I) + (II) \end{aligned}$$

Use a change of variable and the bound $|x_n| \lesssim N_n$ to obtain

$$(I) \lesssim \int_{|x|\leq R} |G_n^{-1} w_n||\phi|\,dx \to 0 \quad \text{as} \quad n \to \infty.$$

Next, apply Cauchy-Schwartz and the upper bound of Lemma 2.4.7 to see that

$$\begin{aligned}(II)^2 &\lesssim \int_{\{|x-x_n|>RN_n^{-1}\}} |xG_nS_n\phi|^2\,dx \\ &\lesssim N_n^{-2} \int_{R\leq|x|\lesssim \frac{N_n}{N_n'}} |x_n + N_n^{-1}x|^2|\phi(x)|^2 dx \\ &\lesssim (N_n^{-2}|x_n|^2 + N_n^{-2}(N_n')^{-2}) \int_{R\leq|x|\lesssim \frac{N_n}{N_n'}} |\phi(x)|^2\,dx.\end{aligned}$$

Suppose that the frame $\{(t_n, x_n, N_n)\}$ is of type 2a, so that $\lim_n N_n^{-1}|x_n| > 0$. By Lemma 2.4.5 and dominated convergence, the right side above is bounded by

$$\int_{R\leq|x|} |\phi(x)|^2\,dx \to 0 \quad \text{as} \quad R \to \infty,$$

uniformly in n. If instead $\{(t_n, x_n, N_n)\}$ is of type 2b, use Hölder to see that the right side is bounded by

$$(N_n^{-2}|x_n|(\tfrac{N_n}{N_n'})^2 + (N_n')^{-4})\|\phi\|_{L^{\frac{2d}{d-2}}}.$$

By Sobolev embedding and the construction of the parameter N_n' in Definition 2.4.2, the above vanishes as $n \to \infty$. In either case, we obtain

$$\lim_{R\to\infty} \limsup_{n\to\infty} (II) = 0.$$

Combining the two estimates and choosing R arbitrarily large, we conclude as required that

$$\lim_{n\to\infty} |\langle xw_n, xG_nS_n\phi\rangle_{L^2}| = 0.$$

To close this subsection, we verify the $L^{\frac{2d}{d-2}}$ decoupling property (2.37) when $N_n^2 t_n \to \pm\infty$. Assume first that the ϕ appearing in the definition (2.42) of ϕ_n has compact support. By the dispersive estimate (2.15) and a change of variables,

$$\lim_{n\to\infty} \|\phi_n\|_{L^{\frac{2d}{d-2}}} \lesssim |t_n|^{-1}\|G_n\phi\|_{L^{\frac{2d}{d+2}}} \lesssim (N_n^2|t_n|)^{-1}\|\phi\|_{L^{\frac{2d}{d+2}}} = 0.$$

The claimed decoupling follows immediately.

For general ϕ in H^1 or $\dot{H}^1$ (depending on whether $\lim_n N_n^{-1}|x_n|$ is positive or zero), select $\psi^\varepsilon \in C_c^\infty$ converging to ϕ in the appropriate norm as $\varepsilon \to 0$. Then for all n large enough, we have

$$\|\phi_n\|_{L^{\frac{2d}{d-2}}} \leq \|e^{it_nH} G_n S_n[\phi - \psi^\varepsilon]\|_{L^{\frac{2d}{d-2}}} + \|e^{it_nH} G_n S_n \psi^\varepsilon\|_{L^{\frac{2d}{d-2}}},$$

and decoupling follows from Lemmas 2.2.3 and 2.4.2 and the special case just proved. □

2.4.2 Convergence of linear propagators

To complete the proof of Proposition 2.4.3, we need a more detailed understanding of how the linear propagator e^{-itH} interacts with the $\dot{H}^1$-symmetries G_n associated to a frame in certain limits. This section is inspired by the discussion surrounding [KSV12, Lemma 5.2], which proves analogous results relating the linear propagators of the 2D Schrödinger equation and the complexified Klein-Gordon equation $-iv_t + \langle \nabla \rangle v = 0$.

Definition 2.4.3. We say two frames $\mathcal{F}^1 = \{(t_n^1, x_n^1, N_n^1)\}$ and $\mathcal{F}^2 = \{(t_n^2, x_n^2, N_n^2)\}$ (where the superscripts are indices, not exponents) are *equivalent* if

$$\tfrac{N_n^1}{N_n^2} \to R_\infty \in (0, \infty),\ N_n^1(x_n^2 - x_n^1) \to x_\infty \in \mathbf{R}^d,\ (N_n^1)^2(t_n^1 - t_n^2) \to t_\infty \in \mathbf{R}.$$

The frames are *orthogonal* should any of the above statements fail. Note that replacing the N_n^1 in the second and third expressions above by N_n^2 yields an equivalent definition of orthogonality.

Remark. If $\mathcal{F}^1$ and $\mathcal{F}^2$ are equivalent, it follows from the above definition that they must be of the same type in Definition 2.4.1, and that $\lim_n (N_n^1)^{-1}|x_n^1|$ and $\lim_n (N_n^2)^{-1}|x_n^2|$ are either both zero or both positive.

The following lemma and its corollary make precise the heuristic that when acting on functions concentrated at a point, e^{-itH} can be approximated for small t by regarding the $|x|^2/2$ potential as essentially constant on the support of the initial data; thus one obtains a modulated free particle propagator $e^{-it|x_0|^2/2}e^{it\Delta/2}$, where x_0 is the spatial center of the data.

Lemma 2.4.8 (Strong convergence)**.** *Suppose*

$$\mathcal{F}^M = (t_n^M, x_n, M_n), \quad \mathcal{F}^N = (t_n^N, y_n, N_n)$$

are equivalent frames. Define

$$\begin{aligned} &R_\infty = \lim_{n\to\infty} \tfrac{M_n}{N_n},\ t_\infty = \lim_{n\to\infty} M_n^2(t_n^M - t_n^N),\ x_\infty = \lim_{n\to\infty} M_n(y_n - x_n),\\ &r_\infty = \lim_n M_n^{-1}|x_n| = \lim_n M_n^{-1}|y_n|. \end{aligned}$$

Let G_n^M, G_n^N be the scaling and translation operators attached to the frames $\mathcal{F}^M$ and $\mathcal{F}^N$ respectively. Then $(e^{-it_n^N H}G_n^N)^{-1}e^{-it_n^M H}G_n^M$ converges strongly as operators on Σ to the operator U_∞ defined by

$$U_\infty \phi = e^{-\frac{it_\infty (r_\infty)^2}{2}} R_\infty^{\frac{d-2}{2}} [e^{\frac{it_\infty \Delta}{2}} \phi](R_\infty \cdot + x_\infty).$$

Proof. If $M_n \equiv 1$, then by the definition of a frame we must have $\mathcal{F}^M = \mathcal{F}^N = \{(1,0,0)\}$, so the claim is trivial. Thus we may assume that $M_n \to \infty$. Put $t_n = t_n^M - t_n^N$. Using Mehler's formula (2.14), we write

$$\begin{aligned} &(e^{-it_n^N H}G_n^N)^{-1}e^{-it_n^M H}G_n^M \phi(x) = (G_n^N)^{-1} e^{-it_n H} G_n^M \phi(x)\\ &= (\tfrac{M_n}{N_n})^{\frac{d-2}{2}} e^{i\gamma(t_n)|y_n + N_n^{-1}x|^2} e^{\frac{iM_n^2 \sin(t_n)\Delta}{2}} [e^{i\gamma(t_n)|x_n + M_n^{-1}\cdot|^2}\phi](\tfrac{M_n}{N_n}x + M_n(y_n - x_n)). \end{aligned}$$

where

$$\gamma(t) = \tfrac{\cos t - 1}{2\sin t} = -\tfrac{t}{4} + O(t^3).$$

Observe that

$$e^{i\gamma(t_n)|x_n + M_n^{-1}\cdot|^2}\phi \to e^{-\frac{it_\infty (r_\infty)^2}{4}}\phi \quad \text{in } \Sigma.$$

Indeed,

$$\begin{aligned} &\|\nabla[e^{i\gamma(t_n)|x_n + M_n^{-1}\cdot|^2}\phi - e^{i\gamma(t_n)|x_n|^2}\phi]\|_{L^2} = \|\nabla[(e^{i\gamma(t_n)[M_n^{-2}|x|^2 + 2M_n^{-1}x_n\cdot x]} - 1)\phi]\|_{L^2}\\ &\lesssim \|t_n(M_n^{-2}x + 2M_n^{-1}x_n)\phi\|_{L^2} + \|(e^{i\gamma(t_n)[M_n^{-2}|x|^2 + 2M_n^{-1}x_n\cdot x]} - 1)\nabla\phi\|_{L^2}\\ &\lesssim |t_n| M_n^{-2}\|x\phi\|_{L^2} + |t_n||x_n| M_n^{-1}\|\phi\|_{L^2} + \|(e^{i\gamma(t_n)[M_n^{-2}|x|^2 2M_n^{-1}x_n\cdot x]} - 1)\nabla\phi\|_{L^2}. \end{aligned}$$

As $n \to \infty$, the first two terms vanish because $\|x\phi\|_2 + \|\phi\|_2 \lesssim \|\phi\|_\Sigma$, while the third term vanishes by dominated convergence. Dominated convergence also implies that

$$\|x[e^{i\gamma(t_n)|x_n+M_n^{-1}x|^2}\phi - e^{i\gamma(t_n)|x_n|^2}\phi]\|_{L^2} \to 0 \text{ as } n \to \infty.$$

On the other hand, since

$$\gamma(t_n)|x_n|^2 = -\tfrac{M_n^2 t_n M_n^{-2}|x_n|^2}{4} + O(M_n^{-4}) \to -\frac{t_\infty(r_\infty)^2}{4},$$

it follows that

$$\|e^{i\gamma(t_n)|x_n+M_n^{-1}\cdot|^2}\phi - e^{-\frac{it_\infty(r_\infty)^2}{4}}\phi\|_\Sigma \to 0$$

as claimed. As $e^{\frac{iM_n^2\sin(t_n)\Delta}{2}} \to e^{\frac{it_\infty\Delta}{2}}$ strongly, we obtain

$$e^{\frac{iM_n^2\sin(t_n)\Delta}{2}}[e^{i\gamma(t_n)|x_n+M_n^{-1}\cdot|^2}\phi] \to e^{-\frac{it_\infty(r_\infty)^2}{4}} e^{\frac{it_\infty\Delta}{2}}\phi \text{ in } \Sigma,$$

and the conclusion quickly follows. □

Corollary 2.4.9. *Let* $\{(t_n^M, x_n, M_n, M_n')\}$ *and* $\{(t_n^N, y_n, N_n, N_n')\}$ *be equivalent frames, and* S_n^M, S_n^N *be the associated spatial cutoff operators. Then*

$$\lim_{n\to\infty} \|e^{-it_n^M H} G_n^M S_n^M \phi - e^{-it_n^N H} G_n^N S_n^N U_\infty \phi\|_\Sigma = 0 \tag{2.43}$$

and

$$\lim_{n\to\infty} \|e^{-it_n^M H} G_n^M S_n^M \phi - e^{-it_n^N H} G_n^N U_\infty S_n^N \phi\|_\Sigma = 0 \tag{2.44}$$

whenever $\phi \in H^1$ *if the frames conform to case 2a and* $\phi \in \dot{H}^1$ *if they conform to case 2b in Definition 2.4.2.*

Proof. As before, the result is immediate if $M_n \equiv 1$ since all operators in sight are trivial. Thus we may assume $M_n \to \infty$. Suppose first that $\phi \in C_c^\infty$. Using the unitarity of e^{-itH} on Σ, the operator bounds (2.35), and the fact that $S_n^M \phi = \phi$ for all n sufficiently large, we write the left side of (2.43) as

$$\begin{aligned}
&\|G_n^N[(G_n^N)^{-1} e^{-i(t_n^M - t_n^N)H} G_n^M \phi - S_n^N U_\infty \phi]\|_\Sigma \\
&\lesssim \|(G_n^N)^{-1} e^{-i(t_n^M - t_n^N)H} G_n^M \phi - S_n^N U_\infty \phi\|_\Sigma \\
&\lesssim \|(G_n^N)^{-1} e^{-i(t_n^M - t_n^N)H} G_n^M \phi - U_\infty \phi\|_\Sigma + \|(1 - S_n^N) U_\infty \phi\|_\Sigma
\end{aligned}$$

which goes to zero by Lemma 2.4.8 and dominated convergence. This proves (2.43) under the additional hypothesis that $\phi \in C_c^\infty$.

We now remove this crutch and take $\phi \in H^1$ or $\dot{H}^1$ depending on whether the frames are of type 2a or 2b in Definition 2.4.2, respectively. For each $\varepsilon > 0$, choose $\phi^\varepsilon \in C_c^\infty$ such that $\|\phi - \phi^\varepsilon\|_{H^1} < \varepsilon$ or $\|\phi - \phi^\varepsilon\|_{\dot{H}^1} < \varepsilon$, respectively. Then

$$\|e^{-it_n^M H} G_n^M S_n^M \phi - e^{-it_n^N H} G_n^N S_n^N U_\infty \phi\|_\Sigma \leq \|e^{-it_n^M H} G_n^M S_n^M (\phi - \phi^\varepsilon)\|_\Sigma$$
$$+ \|e^{-it_n H} G_n^M S_n^M \phi^\varepsilon - e^{-it_n^N H} G_n^N S_n^N U_\infty \phi^\varepsilon\|_\Sigma + \|e^{-it_n^N H} G_n^N S_n^N U_\infty (\phi - \phi^\varepsilon)\|_\Sigma$$

In the limit as $n \to \infty$, the middle term vanishes and we are left with a quantity at most a constant times

$$\limsup_{n\to\infty} \|G_n^M S_n^M (\phi - \phi^\varepsilon)\|_\Sigma + \limsup_{n\to\infty} \|G_n^N S_n^N U_\infty (\phi - \phi^\varepsilon)\|_\Sigma.$$

Applying Lemma 2.4.2 and using the mapping properties of U_∞ on $\dot{H}^1$ and H^1, we see that

$$\limsup_{n\to\infty} \|e^{-it_n H} G_n^M S_n^M \phi - e^{it_n^N H} G_n^N S_n^N U_\infty \phi\|_\Sigma \lesssim \varepsilon$$

for every $\varepsilon > 0$. This proves the claim (2.43). Similar considerations deal with the second claim (2.44). □

Lemma 2.4.10. *Suppose the frames $\{(t_n^M, x_n, M_n)\}$ and $\{(t_n^N, y_n, N_n)\}$ are equivalent. Put $t_n = t_n^M - t_n^N$. Then for $f, g \in \Sigma$ we have*

$$\langle (G_n^N)^{-1} e^{-it_n H} G_n^M f, g\rangle_{\dot{H}^1} = \langle f, (G_n^M)^{-1} e^{it_n H} G_n^N g\rangle_{\dot{H}^1} + R_n(f,g),$$

where $|R_n(f,g)| \leq C|t_n| \|G_n^M f\|_\Sigma \|G_n^N g\|_\Sigma$.

Remark. It follows from Lemma 2.4.8 that

$$\lim_{n\to\infty} \langle (G_n^N)^{-1} e^{-it_n H} G_n^M f, g\rangle_{\dot{H}^1} = \lim_{n\to\infty} \langle f, (G_n^M)^{-1} e^{it_n H} G_n^N g\rangle_{\dot{H}^1}$$

for fixed $f, g \in \Sigma$. The content of this lemma lies in the quantitative error bound.

Proof. We have

$$\langle (G_n^N)^{-1} e^{-it_n H} G_N^M f, g\rangle_{\dot{H}^1} = \langle f, (G_n^M)^{-1} e^{it_n H} G_n^N g\rangle_{\dot{H}^1} + R_n(f,g)$$

where $R_n(f,g) = \langle [\nabla, e^{-it_nH}]G_n^M f, \nabla G_n^N g\rangle_{L^2} - \langle \nabla G_n^M f, [\nabla, e^{it_nH}]G_n^N g\rangle_{L^2}$. The claim follows from Cauchy-Schwartz and the commutator estimate

$$\|[\nabla, e^{-itH}]\|_{\Sigma \to L^2} = O(t),$$

which is a consequence of the standard identities

$$\begin{aligned} e^{itH} i\nabla e^{-itH} &= i\nabla \cos t - x \sin t \\ e^{itH} x e^{-itH} &= i\nabla \sin t + x \cos t. \end{aligned}$$

□

Next we prove a converse to Lemma 2.4.8.

Lemma 2.4.11 (Weak convergence)**.** *Assume the frames $\mathcal{F}^M = \{(t_n^M, x_n, M_n)\}$ and $\mathcal{F}^N = \{(t_n^N, y_n, N_n)\}$ are orthogonal. Then for any $f \in \Sigma$,*

$$(e^{-it_n^N H} G_n^N)^{-1} e^{-it_n^M H} G_n^M f \to 0 \quad \textit{weakly in } \dot{H}^1.$$

Proof. Put $t_n = t_n^M - t_n^N$, and suppose that $|M_n^2 t_n| \to \infty$. Then

$$\|(G_n^N)^{-1} e^{-it_nH} G_n^M f\|_{L^{\frac{2d}{d-2}}} \to 0$$

for $f \in C_c^\infty$ by a change of variables and the dispersive estimate, thus for general $f \in \Sigma$ by a density argument. Therefore $(G_n^N)^{-1} e^{-it_nH} G_n^M f$ converges weakly in $\dot{H}^1$ to 0. Now consider the case where $M_n^2 t_n \to t_\infty \in \mathbf{R}$. The orthogonality of $\mathcal{F}^M$ and $\mathcal{F}^N$ implies that either $N_n^{-1} M_n$ converges to 0 or ∞, or $M_n|x_n - y_n|$ diverges as $n \to \infty$. In either case, one verifies easily that $(G_n^N)^{-1} G_n^M$ converge to zero weakly as operators on $\dot{H}^1$. By Lemma 2.4.8, $(G_n^N)^{-1} e^{-it_nH} G_n^M f = (G_n^N)^{-1} G_n^M (G_n^M)^{-1} e^{-it_nH} G_n^M f$ converges to zero weakly in $\dot{H}^1$. □

Corollary 2.4.12. *Let $\{(t_n^M, x_n, M_n, M_n')\}$ and $\{(t_n^N, y_n, N_n, N_n')\}$ be orthogonal with corresponding operators G_n^M, S_n^M and G_n^N, S_n^N. Then*

$$(e^{-it_n^N H} G_n^N)^{-1} e^{-it_n^M H} G_n^M S_n^M \phi \rightharpoonup 0 \quad \textit{in} \quad \dot{H}^1$$

whenever $\phi \in H^1$ if $\mathcal{F}^M$ is of type 2a and $\phi \in \dot{H}^1$ if $\mathcal{F}^M$ is of type 2b.

Proof. If $\phi \in C_c^\infty$, then $S_n^M \phi = \phi$ for all large n, and the claim follows from Lemma 2.4.11. The case of general ϕ in H^1 or $\dot{H}^1$ then follows from an approximation argument similar to the one used to prove Corollary 2.4.9. □

2.4.3 End of proof of inverse Strichartz

We return to the proof of Proposition 2.4.3. Thus far, we have identified a frame $\{(t_n, x_n, N_n, N_n')\}$ and an associated profile ϕ_n such that the sequence $N_n^2 t_n$ has a limit in $[-\infty, \infty]$ as $n \to \infty$. The ϕ_n were shown to satisfy properties (2.36), (2.37), and (2.38) if either $(t_n, x_n, N_n) = (0, 0, 1)$ or $N_n \to \infty$ and $N_n^2 t_n \to \pm\infty$. Thus, it remains to prove that if $N_n \to \infty$ and $N_n^2 t_n$ remains bounded, then we may modify the frame so that t_n is identically zero and find a profile ϕ_n corresponding to this new frame which satisfies all the properties asserted in the proposition. The following lemma will therefore complete the proof of the proposition.

Lemma 2.4.13. *Let $f_n \in \Sigma$ satisfy the hypotheses of Proposition 2.4.3. Suppose $\{(t_n, x_n, N_n, N_n')\}$ is an augmented frame with $N_n \to \infty$ and $N_n^2 t_n \to t_\infty \in \mathbf{R}$ as $n \to \infty$. Then there is a profile $\phi_n' = G_n S_n \phi'$ associated to the frame $\{(0, x_n, N_n, N_n')\}$ such that properties* (2.36), (2.37), *and* (2.38) *hold with ϕ_n' in place of ϕ_n.*

Proof. Let $\phi_n = e^{it_n H} G_n S_n \phi$ be the profile defined by (2.42). We have already seen that ϕ_n satisfies properties (2.36) and (2.38), and that

$$\phi = \dot{\mathrm{H}}^1\text{-w-lim}_{n\to\infty} G_n^{-1} e^{-it_n H} f_n.$$

As the sequence $G_n^{-1} f_n$ is bounded in $\dot{H}^1$, it has a weak subsequential limit

$$\phi' = \dot{\mathrm{H}}^1\text{-w-lim}_{n\to\infty} G_n^{-1} f_n.$$

For any $\psi \in C_c^\infty$, apply Lemma 2.4.10 with $f = G_n^{-1} e^{-it_n H} f_n$ to see that

$$\begin{aligned}\langle \phi', \psi \rangle_{\dot{H}^1} &= \lim_{n\to\infty} \langle G_n^{-1} f_n, \psi \rangle_{\dot{H}^1} = \lim_{n\to\infty} \langle G_n^{-1} e^{it_n H} G_n G_n^{-1} e^{-it_n H} f_n, \psi \rangle_{\dot{H}^1} \\ &= \lim_{n\to\infty} \langle G_n^{-1} e^{-it_n H} f_n, G_n^{-1} e^{-it_n H} G_n \psi \rangle_{\dot{H}^1} = \langle \phi, U_\infty \psi \rangle_{\dot{H}^1},\end{aligned}$$

where $U_\infty = \text{s-lim}_{n\to\infty} G_n^{-1} e^{-it_n H} G_n$ is the strong operator limit guaranteed by Lemma 2.4.8. As U_∞ is unitary on $\dot{H}^1$, we have the relation $\phi = U_\infty \phi'$.

Put $\phi_n' = G_n S_n \phi'$. By Corollary 2.4.9,

$$\|\phi_n - \phi_n'\|_\Sigma = \|e^{it_n H} G_n S_n \phi - G_n S_n U_\infty^{-1} \phi\|_\Sigma \to 0 \quad \text{as} \quad n \to \infty.$$

Hence ϕ_n' inherits property (2.36) from ϕ_n. The same proof as for ϕ_n shows that Σ decoupling (2.38) holds as well. It remains to verify the last decoupling property (2.37). As $G_n^{-1} f_n$ converges weakly in $\dot{H}^1$ to ϕ', by Rellich-Kondrashov and a diagonalization argument we may assume after passing to a subsequence that $G_n^{-1} f_n$ converges to ϕ' almost everywhere on $\mathbf{R}^d$. By the Lemma 2.2.6, the fact that $\lim_{n\to\infty} \|G_n S_n \phi' - G_n \phi'\|_{\frac{2d}{d-2}} = 0$, and a change of variables,

$$\begin{aligned}
&\lim_{n\to\infty} \left[\|f_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|f_n - \phi_n'\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|\phi_n'\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} \right] \\
&= \lim_{n\to\infty} \left[\|G_n^{-1} f_n\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|G_n^{-1} f_n - \phi'\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} - \|\phi'\|_{\frac{2d}{d-2}}^{\frac{2d}{d-2}} \right] \\
&= 0.
\end{aligned}$$

□

Remark. As $\lim_{n\to\infty} \|\phi_n - \phi_n'\|_\Sigma = 0$, we see by Sobolev embedding that the decoupling (2.37) also holds for the original profile $\phi_n = e^{it_n H} G_n S_n \phi$ with nonzero time parameter t_n.

2.4.4 Linear profile decomposition

As before, I will denote a fixed interval containing 0 of length at most 1, and all spacetime norms are taken over $I \times \mathbf{R}^d$ unless indicated otherwise.

Proposition 2.4.14. *Let f_n be a bounded sequence in Σ. After passing to a subsequence, there exists $J^* \in \{0, 1, \dots\} \cup \{\infty\}$ such that for each finite $1 \le j \le J^*$, there exist an augmented frame $\mathcal{F}^j = \{(t_n^j, x_n^j, N_n^j, (N_n^j)')\}$ and a function ϕ^j with the following properties.*

- *Either $t_n^j \equiv 0$ or $(N_n^j)^2 (t_n^j) \to \pm\infty$ as $n \to \infty$.*

- ϕ^j *belongs to* Σ, H^1, *or* $\dot H^1$ *depending on whether* $\mathcal{F}^j$ *is of type 1, 2a, or 2b, respectively.*

For each finite $J \le J^*$, *we have a decomposition*

$$f_n = \sum_{j=1}^J e^{it_n^j H} G_n^j S_n^j \phi^j + r_n^J = \sum_{j=1}^J \phi_n^j + r_n^J, \tag{2.45}$$

where G_n^j, S_n^j *are the* $\dot H^1$*-isometry and spatial cutoff operators associated to* $\mathcal{F}^j$. *This decomposition has the following properties:*

$$(G_n^J)^{-1} e^{-it_n^J H} r_n^J \overset{\dot H^1}{\rightharpoonup} 0 \quad \textit{for all } J \le J^*, \tag{2.46}$$

$$\sup_J \lim_{n\to\infty} \Big| \|f_n\|_\Sigma^2 - \sum_{j=1}^J \|\phi_n^j\|_\Sigma^2 - \|r_n^J\|_\Sigma^2 \Big| = 0, \tag{2.47}$$

$$\sup_J \lim_{n\to\infty} \Big| \|f_n\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \sum_{j=1}^J \|\phi_n^j\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \|r_n^J\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} \Big| = 0. \tag{2.48}$$

Whenever $j \ne k$, *the frames* $\{(t_n^j, x_n^j, N_n^j)\}$ *and* $\{(t_n^k, x_n^k, N_n^k)\}$ *are orthogonal:*

$$\lim_{n\to\infty} \tfrac{N_n^j}{N_n^k} + \tfrac{N_n^k}{N_n^j} + N_n^j N_n^k |t_n^j - t_n^k| + \sqrt{N_n^j N_n^k}|x_n^j - x_n^k| = \infty. \tag{2.49}$$

Finally, we have

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{-it_n H} r_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} = 0, \tag{2.50}$$

Remark. One can also show a posteriori using (2.49) and (2.50) the fact, which we will neither prove nor use, that

$$\sup_J \lim_{n\to\infty} \Big| \|e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} - \sum_{j=1}^J \|e^{-itH}\phi_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} - \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{2(d+2)}{d-2}} \Big| = 0.$$

The argument uses similar ideas as in the proofs of [Ker01][Lemma 2.7] or Lemma 2.6.3; we omit the details.

Proof. Proceed inductively using Proposition 2.4.3. Let $r_n^0 = f_n$. Assume that we have a decomposition up to level $J \ge 0$ obeying properties (2.46) through (2.48). After passing to a subsequence, define

$$A_J = \lim_n \|r_n^J\|_\Sigma \quad \text{and} \quad \varepsilon_J = \lim_n \|e^{-it_n H} r_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}.$$

If $\varepsilon_J = 0$, stop and set $J^* = J$. Otherwise apply Proposition 2.4.3 to the sequence r_n^J to obtain a frame $(t_n^{J+1}, x_n^{J+1}, N_n^{J+1}, (N_n^{J+1})')$ and functions

$$\phi^{J+1} \in \dot{H}^1, \quad \phi_n^{J+1} = e^{it_n^{J+1}H} G_n^{J+1} S_n^{J+1} \phi^{J+1} \in \Sigma$$

which satisfy the conclusions of Proposition 2.4.3. In particular ϕ^{J+1} is the $\dot{H}^1$ weak limit of the sequence $(G_n^{J+1})^{-1} e^{-it_n^{J+1}H} r_n^J$. Let $r_n^{J+1} = r_n^J - \phi_n^{J+1}$. By the induction hypothesis, (2.47) and (2.48) are satisfied with J replaced by $J+1$. Also,

$$(G_n^{J+1})^{-1} e^{-it_n^{J+1}H} r_n^{J+1} = [(G_n^{J+1})^{-1} e^{-it_n^{J+1}H} r_n^J - \phi^{J+1}] + (1 - S_n^{J+1})\phi^{J+1}.$$

As $n \to \infty$, the first term goes to zero weakly in $\dot{H}^1$ while the second term goes to zero strongly. Thus (2.46) holds at level $J+1$ as well. After passing to a subsequence, we may define

$$A_{J+1} = \lim_n \|r_n^{J+1}\|_\Sigma \quad \text{and} \quad \varepsilon_{J+1} = \lim_n \|e^{-itH} r_n^{J+1}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}.$$

If $\varepsilon_{J+1} = 0$, stop and set $J^* = J+1$. Otherwise continue the induction. If the algorithm never terminates, set $J^* = \infty$. From (2.47) and (2.48), the parameters A_J and ε_J satisfy the inequality

$$A_{J+1}^2 \le A_J^2 [1 - C(\tfrac{\varepsilon_J}{A_J})^{\frac{d(d+2)}{4}}].$$

If $\limsup_{J \to J^*} \varepsilon_J = \varepsilon_\infty > 0$, then as A_J are decreasing there would exist infinitely many J's so that

$$A_{J+1}^2 \le A_J^2 [1 - C(\tfrac{\varepsilon_\infty}{A_0})^{\frac{d(d+2)}{4}}],$$

which implies that $\lim_{J \to J^*} A_J = 0$. But this contradicts the Strichartz inequality which dictates that $\limsup_{J \to J^*} A_J \gtrsim \limsup_{J \to J^*} \varepsilon_J = \varepsilon_0$. We conclude that

$$\lim_{J \to J^*} \varepsilon_J = 0.$$

Thus (2.50) holds.

It remains to prove the assertion (2.49). Suppose otherwise, and let $j < k$ be the first two indices for which $\mathcal{F}^j$ and $\mathcal{F}^k$ are equivalent. Thus $\mathcal{F}^\ell$ and $\mathcal{F}^k$ are orthogonal for all

$j < \ell < k$. By the construction of the profiles, we have

$$r_n^{j-1} = e^{it_n^j H} G_n^j S_n^j \phi^j + e^{it_n^k H} G_n^k S_n^k \phi^k + \sum_{j<\ell<k} e^{it_n^\ell H} G_n^\ell S_n^\ell \phi^\ell + r_n^k,$$

therefore

$$(e^{it_n^j H} G_n^j)^{-1} r_n^{j-1} = (e^{it_n^j H} G_n^j)^{-1} e^{it_n^j H} G_n^j S_n^j \phi^j + (e^{it_n^j H} G_n^j)^{-1} e^{it_n^k H} G_n^k S_n^k \phi^k$$
$$+ \sum_{j<\ell<k} (e^{it_n^j H} G_n^j)^{-1} e^{it_n^\ell H} G_n^\ell S_n^\ell \phi^\ell + (e^{it_n^j H} G_n^j)^{-1} r_n^k.$$

As $n \to \infty$, the left side converges to ϕ^j weakly in $\dot{H}^1$. On the right side, we apply Corollary 2.4.9 to see that the first and second terms converge in $\dot{H}^1$ to ϕ^j and $U_\infty^{jk}\phi^k$, respectively, for some isomorphism U_∞^{jk} of $\dot{H}^1$. By Corollary 2.4.12, each of the terms in the summation converges to zero weakly in $\dot{H}^1$. Taking for granted the claim that

$$(e^{it_n^j H} G_n^j)^{-1} r_n^k \to 0 \quad \text{weakly in } \dot{H}^1, \tag{2.51}$$

it follows that

$$\phi^j = \phi^j + U_\infty^{jk} \phi^k,$$

so $\phi^k = 0$, which contradicts the nontriviality of ϕ^k. Therefore, the proof of the proposition will be complete upon verifying the weak limit (2.51). As that sequence is bounded in $\dot{H}^1$, it suffices to check that

$$\langle (e^{it_n^j H} G_n^j)^{-1} r_n^k, \psi \rangle_{\dot{H}^1} \to 0 \quad \text{for any } \psi \in C_c^\infty(\mathbf{R}^d).$$

Write $(e^{it_n^j H} G_n^j)^{-1} r_n^k = (e^{it_n^j H} G_n^j)^{-1} (e^{it_n^k H} G_n^k)(e^{it_n^k H} G_n^k)^{-1} r_n^k$, and use Lemma 2.4.10 and the weak limit (2.46) to see that

$$\begin{aligned}
\lim_{n\to\infty} \langle (e^{it_n^j H} G_n^j)^{-1} r_n^k, \psi \rangle_{\dot{H}^1} &= \lim_{n\to\infty} \langle (e^{it_n^k H} G_n^k)^{-1} r_n^k, (e^{it_n^k H} G_n^k)^{-1} (e^{it_n^j H} G_n^j) \psi \rangle_{\dot{H}^1} \\
&= \lim_{n\to\infty} \langle (G_n^k)^{-1} e^{-it_n^k H} r_n^k, (U_\infty^{jk})^{-1} \psi \rangle_{\dot{H}^1} \\
&= 0.
\end{aligned}$$

□

2.5 The case of concentrated initial data

The next step in the proof of Theorem 2.1.2 is to establish wellposedness when the initial data consists of a highly concentrated "bubble". The picture to keep in mind is that of a single profile ϕ_n^j in Proposition 2.4.14 as $n \to \infty$. In the next section we combine this special case with the profile decomposition to treat general initial data. Although we state the following result as a conditional one to permit a unified exposition, by Theorem 2.1.1 the result is unconditionally true in most cases.

Proposition 2.5.1. *Let $I = [-1, 1]$. Assume that Conjecture 2.1.1 holds. Suppose*

$$\mathcal{F} = \{(t_n, x_n, N_n, N_n')\}$$

is an augmented frame with $t_n \in I$ and $N_n \to \infty$, such that either $t_n \equiv 0$ or $N_n^2 t_n \to \pm\infty$; that is, $\mathcal{F}$ is type 2a or 2b in Definition 2.4.2. Let $G_n, \tilde{G}_n$, and S_n be the associated operators defined in (2.33) and (2.34). Suppose ϕ belongs to H^1 or $\dot{H}^1$ depending on whether $\mathcal{F}$ is type 2a or 2b respectively. Then, for n sufficiently large, there is a unique solution $u_n : I \times \mathbf{R}^d \to \mathbf{C}$ to the defocusing equation (2.1), $\mu = 1$, with initial data

$$u_n(0) = e^{it_n H} G_n S_n \phi.$$

This solution satisfies a spacetime bound

$$\limsup_{n\to\infty} S_I(u_n) \le C(E(u_n)).$$

Suppose in addition that $\{(q_k, r_k)\}$ is any finite collection of admissible pairs with $2 < r_k < d$. Then for each $\varepsilon > 0$ there exists $\psi^\varepsilon \in C_c^\infty(\mathbf{R} \times \mathbf{R}^d)$ such that

$$\limsup_{n\to\infty} \sum_k \|u_n - \tilde{G}_n[e^{-\frac{itN_n^{-2}|x_n|^2}{2}} \psi^\varepsilon]\|_{L_t^{q_k} \Sigma_x^{r_k}(I\times\mathbf{R}^d)} < \varepsilon. \tag{2.52}$$

Assuming also that $\|\nabla\phi\|_{L^2} < \|\nabla W\|_{L^2}$ and $E_\Delta(\phi) < E_\Delta(W)$, we have the same conclusion as above for the focusing equation (2.1), $\mu = -1$.

The proof proceeds in several steps. First we construct an approximate solution on I in the sense of Proposition 2.3.3. Roughly speaking, when N_n is large and $t = O(N_n^{-2})$, solutions to (2.1) are well-approximated up to a phase factor by solutions to the energy-critical NLS with no potential, which by Conjecture 2.1.1 exist globally and scatter. In the long-time regime $N_n^{-2} << |t| \leq 1$, the solution to (2.1) has dispersed and resembles a linear evolution $e^{-itH}\phi$. By patching these approximations together, we obtain an approximate solution over the entire time interval I with arbitrarily small error as N_n becomes large. It then follows by Proposition 2.3.3 that for n large, (2.1) admits a solution on I with controlled spacetime bound. The last claim about approximating the solution by functions in $C_c^\infty(\mathbf{R} \times \mathbf{R}^d)$ will follow essentially from our construction of the approximate solutions.

We shall need a commutator estimate. In the sequel, $P_{\leq N}, P_N$ will denote the standard Littlewood-Paley projectors based on $-\Delta$.

Lemma 2.5.2. *Let v be a global solution to*

$$(i\partial_t + \tfrac{1}{2}\Delta)v = F(v),\ \ v(0) \in \dot{H}^1(\mathbf{R}^d)$$

where $F(z) = \pm|z|^{\frac{4}{d-2}}z$. Then on any compact time interval I,

$$\lim_{N\to\infty} \|P_{\leq N}F(v) - F(P_{\leq N}v)\|_{L_t^2 H_x^{1,\frac{2d}{d+2}}(I\times\mathbf{R}^d)} = 0$$

Proof. Recall [TVZ07, Lemma 3.11] that as a consequence of the spacetime bound (2.7), ∇v is finite in all Strichartz norms:

$$\|\nabla v\|_{S(\mathbf{R})} < C(\|v(0)\|_{\dot{H}^1}) < \infty. \tag{2.53}$$

It suffices to show separately that

$$\lim_{n\to\infty} \|P_{\leq N}F(v) - F(P_{\leq N}v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} = 0, \tag{2.54}$$

$$\lim_{n\to\infty} \|\nabla[P_{\leq N}F(v) - F(P_{\leq N}v)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}} = 0. \tag{2.55}$$

Write

$$\begin{aligned}\|\nabla[P_{\leq N}F(v) - F(P_{\leq N}v)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}} &\leq \|\nabla P_{>N}F(v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\quad + \|\nabla[F(v) - F(P_{\leq N}v)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}}.\end{aligned} \tag{2.56}$$

As $P_{>N} = 1 - P_{\leq N}$ and

$$\|\nabla F(v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} \leq C(\|v(0)\|_{\dot{H}^1}),$$

dominated convergence implies that

$$\lim_{N\to\infty} \|\nabla P_{>N} F(v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} = 0.$$

To treat the second term on the right side of (2.56), observe first that with $F(z) = |z|^{\frac{4}{d-2}} z$,

$$|F_z(z) - F_z(w)| + |F_{\overline{z}}(z) - F_{\overline{z}}(w)| \lesssim \begin{cases} |z-w|(|z|^{\frac{6-d}{d-2}} + |w|^{\frac{6-d}{d-2}}), & 3 \leq d \leq 5 \\ |z-w|^{\frac{4}{d-2}}, & d \geq 6. \end{cases}$$

Combining this with the pointwise bound

$$\begin{aligned} |\nabla[F(v) - F(P_{\leq N} v)]| &\leq (|F_z(v) - F_z(P_{\leq N} v)| + |F_{\overline{z}}(v) - F_{\overline{z}}(P_{\leq N} v)|)|\nabla v| \\ &\quad + (|F_z(P_{\leq N} v)| + |F_{\overline{z}}(P_{\leq N} v)|)|\nabla P_{>N} v|, \end{aligned}$$

Hölder, and dominated convergence, when $d \geq 6$ we have

$$\begin{aligned} &\|\nabla[F(v) - F(P_{\leq N} v)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\lesssim \||P_{>N} v|^{\frac{4}{d-2}} |\nabla v|\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + \||P_{\leq N} v|^{\frac{4}{d-2}} |\nabla P_{>N} v|\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\lesssim \|P_{>N} v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} + \|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|P_{>N} \nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} \\ &\to 0 \text{ as } N \to \infty. \end{aligned} \tag{2.57}$$

If $3 \leq d \leq 5$, the first term in the second line of (2.57) is replaced by

$$\begin{aligned} &\||P_{>N} v|(|v|^{\frac{6-d}{d-2}} + |P_{\leq N} v|^{\frac{6-d}{d-2}})|\nabla v|\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\leq \|P_{>N} v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}} \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} \end{aligned}$$

which goes to 0 by dominated convergence. This establishes (2.55). The proof of (2.54)is similar. Write

$$\|P_{\leq N} F(v) - F(P_{\leq N} v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \leq \|P_{>N} F(v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + \|F(v) - F(P_{\leq N} v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}}.$$

By Hölder, Bernstein, and the chain rule,

$$\|P_{>N}F(v)\|_{L_t^2L_x^{\frac{2d}{d+2}}} \lesssim N^{-1}\|v\|^{\frac{4}{d-2}}_{L_{t,x}^{\frac{2(d+2)}{d-2}}}\|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}}L_x^{\frac{2d(d+2)}{d^2+4}}} = O(N^{-1}).$$

Using Bernstein, Hölder, and Sobolev embedding, and the pointwise bound

$$|F(v) - F(P_{\leq N}v)| \lesssim |P_{>N}v|(|v|^{\frac{4}{d-2}} + |P_{\leq N}v|^{\frac{4}{d-2}}),$$

we obtain

$$\begin{aligned}\|F(v) - F(P_{\leq N}v)\|_{L_t^2L_x^{\frac{2d}{d+2}}} &\leq \|(|v|^{\frac{4}{d-2}} + |P_{\leq N}v|^{\frac{4}{d-2}})P_{>N}v\|_{L_t^2L_x^{\frac{2d}{d+2}}} \\ &\lesssim_{|I|} (\|\nabla v\|^{\frac{4}{d-2}}_{L_t^\infty L_x^2} + +\|\nabla v\|^{\frac{4}{d-2}}_{L_t^\infty L_x^2})\|\nabla P_{>N}v\|_{L_t^\infty L_x^2}.\end{aligned}$$

As $v \in C_t^0\dot{H}_x^1(I\times\mathbf{R}^d)$, the orbit $\{v(t)\}_{t\in I}$ is compact in $\dot{H}^1(\mathbf{R}^d)$. The Riesz characterization of L^2 compactness therefore implies that the right side goes to 0 as $N\to\infty$. □

Now suppose that $\phi_n = e^{it_nH}G_nS_n\phi$ as in the statement of Proposition 2.5.1. If $\mu = -1$, assume also that $\|\phi\|_{\dot{H}^1} < \|W\|_{\dot{H}^1},\ E(\phi) < E_\Delta(W)$. We first construct functions $\tilde{v}_n$ which obey all of the conditions of the Proposition 2.3.3 except possibly the hypothesis in (2.31) about matching initial data. A slight modification of the $\tilde{v}_n$ will then yield genuine approximate solutions.

If $t_n\equiv 0$, let v be the solution to the potential-free problem (2.6) provided by Conjecture 2.1.1 with $v(0)=\phi$. If $N_n^2t_n\to\pm\infty$, let v be the solution to (2.6) which scatters in $\dot{H}^1$ to $e^{\frac{it\Delta}{2}}\phi$ as $t\to\mp\infty$. Note the reversal of signs.

Put

$$\tilde{N}_n' = (\tfrac{N_n}{N_n'})^{\frac{1}{2}}, \tag{2.58}$$

let $T>0$ denote a large constant to be chosen later, and define

$$\tilde{v}_n^T(t) = \begin{cases} e^{-\frac{it|x_n|^2}{2}}\tilde{G}_n[S_nP_{\leq\tilde{N}_n'}v](t+t_n) & |t|\leq TN_n^{-2} \\ e^{-i(t-TN_n^{-2})H}\tilde{v}_n^T(TN_n^{-2}), & TN_n^{-2}\leq t\leq 2 \\ e^{-i(t+TN_n^{-2})H}\tilde{v}_n^T(-TN_n^{-2}) & -2\leq t\leq -TN_n^{-2}. \end{cases} \tag{2.59}$$

The awkward time translation by t_n is needed to undo the time translation built into the operator $\tilde{G}_n$; see (2.33). We shall suppress the superscript T unless the role of that parameter needs to be emphasized. Introducing the notation

$$\begin{aligned} v_n(t,x) &= [\tilde{G}_n v](t+t_n,x) = N_n^{\frac{d-2}{2}} v(N_n^2 t, N_n(x-x_n)), \\ \chi_n(x) &= \chi(N_n'(x-x_n)), \end{aligned}$$

where χ is the function used to define the spatial cutoff operator S_n in (2.34), and using the identity $\tilde{G}_n\chi = \chi_n \tilde{G}_n$, we can also write the top expression in (2.59) as

$$\tilde{v}_n(t) = e^{-\frac{it|x_n|^2}{2}} \chi_n P_{\leq \tilde{N}_n' N_n} v_n, \quad |t| \leq TN_n^{-2}.$$

As discussed previously, during the initial time window $\tilde{v}_n$ is essentially a modulated solution to (2.6) with cutoffs applied in both space, to place the solution in $C_t\Sigma_x$, and frequency, to enable taking an extra derivative in the error analysis below.

If $\phi \in \dot{H}^1$, use Lemma 2.4.2 and the fact that $\|v\|_{L_t^\infty \dot{H}_x^1} \leq C(\|\phi\|_{\dot{H}^1})$ (energy conservation) to deduce

$$\limsup_n \|\tilde{v}_n\|_{L_t^\infty \Sigma_x(|t| \leq TN_n^{-2})} \leq C(\|\phi\|_{\dot{H}^1}),$$

therefore

$$\limsup_n \|\tilde{v}_n\|_{L_t^\infty \Sigma_x([-2,2])} \leq C(\|\phi\|_{\dot{H}^1}). \tag{2.60}$$

From (2.7), (2.60), and Strichartz, we obtain

$$\|\tilde{v}_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([-2,2]\times\mathbf{R}^d)} \leq C(\|\phi\|_{\dot{H}^1}) \quad \text{for } n \text{ large}. \tag{2.61}$$

Due to mass conservation, a similar bound holds when $\phi \in H^1$. Now let

$$e_n = (i\partial_t - H)\tilde{v}_n - F(\tilde{v}_n).$$

We show that

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|H^{\frac{1}{2}} e_n\|_{N([-2,2])} = 0, \tag{2.62}$$

so that by taking T large enough the $\tilde{v}_n$ will satisfy the second error condition in (2.31) for all n sufficiently large.

First we deal with the time interval $|t| \le TN_n^{-2}$.

Lemma 2.5.3. $\lim_{T\to\infty} \limsup_{n\to\infty} \|H^{\frac{1}{2}} e_n\|_{N(|t|\le TN_n^{-2})} = 0$.

Proof. When $-TN_n^{-2} \le t \le TN_n^{-2}$, compute

$$\begin{aligned}
e_n &= e^{-\frac{it|x_n|^2}{2}}[\chi_n P_{\le \tilde{N}_n' N_n} F(v_n) - \chi_n^{\frac{d+2}{d-2}} F(P_{\le \tilde{N}_n' N_n} v_n) \\
&+ \frac{|x_n|^2 - |x|^2}{2}(P_{\le \tilde{N}_n' N_n} v_n)\chi_n + \frac{1}{2}(P_{\le \tilde{N}_n' N_n} v_n)\Delta\chi_n + (\nabla P_{\le \tilde{N}_n' N_n} v_n)\cdot\nabla\chi_n] \\
&= e^{-\frac{it|x_n|^2}{2}}[(a) + (b) + (c) + (d)],
\end{aligned}$$

and estimate each term separately in the dual Strichartz space $N(\{|t| \le TN_n^{-2}\})$. Write

$$\begin{aligned}
(a) &= \chi_n P_{\le \tilde{N}_n' N_n} F(v_n) - \chi_n^{\frac{d+2}{d-2}} F(P_{\le \tilde{N}_n' N_n} v_n) \\
&= \chi_n [P_{\le \tilde{N}_n' N_n} F(v_n) - F(P_{\le_{\tilde{N}_n' N_n}} v_n)] + \chi_n(1 - \chi_n^{\frac{4}{d-2}})F(P_{\le \tilde{N}_n' N_n} v_n) \\
&= (a') + (a'').
\end{aligned}$$

By the Leibniz rule and a change of variables,

$$\begin{aligned}
&\|\nabla(a')\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} \\
&\le \|\nabla[P_{\le \tilde{N}_n'} F(v) - F(P_{\le \tilde{N}_n'} v)]\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le T)} \\
&+ \|[P_{\le \tilde{N}_n' N_n} F(v_n) - F(P_{\le \tilde{N}_n' N_n} v_n)]\nabla\chi_n\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})}.
\end{aligned} \tag{2.63}$$

By Lemma 2.5.2, the first term disappears in the limit as $n \to \infty$. That lemma also applies to the second term after a change of variables to give

$$\begin{aligned}
&\|[P_{\le \tilde{N}_n' N_n} F(v_n) - F(P_{\le \tilde{N}_n' N_n} v_n)]\nabla\chi_n\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} \\
&\lesssim N_n' \|P_{\le \tilde{N}_n' N_n} F(v_n) - F(P_{\le \tilde{N}_n' N_n} v_n)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} \\
&\lesssim \tfrac{N_n'}{N_n} \|P_{\le \tilde{N}_n'} F(v) - F(P_{\le \tilde{N}_n'} v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le T)} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \|\nabla(a')\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} = 0.$$

By changing variables, using the bound $|x_n| \lesssim N_n$, and referring to Lemma 2.5.2 once more,

$$\||x|(a')\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim N_n \|P_{\le \tilde{N}_n' N_n} F(v_n) - F(P_{\le \tilde{N}_n' N_n} v_n)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})}$$

$$\lesssim \|P_{\le \tilde{N}_n'} F(v) - F(P_{\le \tilde{N}_n'} v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le T)} \to 0 \text{ as } n \to \infty.$$

It follows from Lemma 2.2.2 that

$$\lim_{n\to\infty} \|H^{\frac{1}{2}}(a')\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} = 0.$$

To estimate (a''), we use the Leibniz rule, a change of variables, Hölder, Sobolev embedding, the bound (2.53), and dominated convergence to obtain

$$\|\nabla(a'')\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \||P_{\le \tilde{N}_n N_n} v_n|^{\frac{4}{d-2}} \nabla P_{\le \tilde{N}_n' N_n} v_n\|_{L_t^2 L_x^{\frac{2d}{d+2}}(|t|\le TN_n^{-2},\ |x-x_n|\sim(N_n')^{-1})}$$

$$+ \tfrac{N_n'}{N_n} \|P_{\le \tilde{N}_n' N_n} v_n\|_{L_t^\infty L_x^{\frac{2d}{d-2}}}^{\frac{d+2}{d-2}}$$

$$\lesssim \|\nabla v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}}} \|P_{\le \tilde{N}_n'} v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(|t|\le T,\ |x|\sim \frac{N_n}{N_n'})}^{\frac{4}{d-2}} + O(\tfrac{N_n'}{N_n})$$

$$\lesssim C(E(v))(\|P_{>\tilde{N}_n'} v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|v\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(|t|\le T, |x|\gtrsim \frac{N_n}{N_n'})})^{\frac{4}{d-2}} + O(\tfrac{N_n'}{N_n})$$

$$= o(1) + O(\tfrac{N_n'}{N_n}).$$

Similarly,

$$\||x|(a'')\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \sim \|F(P_{\le \tilde{N}_n'} v)\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d}{d-2}}(|t|\le T, |x|\sim \frac{N_n}{N_n'})}$$

$$\lesssim (\|P_{>\tilde{N}_n'} v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d}{d-2}}(|t|\le T)} + \|v\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d}{d-2}}(|t|\le T, |x|\sim \frac{N_n}{N_n'})})^{\frac{d+2}{d-2}}$$

$$= o(1).$$

Therefore

$$\lim_{N\to\infty} \|H^{\frac{1}{2}}(a'')\|_{L_t^2 L_{t,x}^{\frac{2d}{d+2}}(|t|\le TN_n^{-2})} = 0$$

as well. This completes the analysis for (a).

For (b), note that on the support of the function we have $||x_n|^2 - |x|^2| = |x_n - x||x_n + x| \sim N_n(N_n')^{-1}$. Thus by Hölder and Sobolev embedding,

$$\begin{aligned}\|\nabla(b)\|_{L_t^1 L_x^2(|t|\le TN_n^{-2})} &\lesssim \tfrac{N_n}{N_n'}\|\nabla P_{\le \tilde N_n' N_n} v_n\|_{L_t^1 L_x^2(|t|\le TN_n^{-2})} \\ &\quad + N_n \|P_{\le \tilde N_n' N_n} v_n\|_{L_t^1 L_x^2(|t|\le TN_n^{-2},\ |x-x_n|\sim (N_n')^{-1})} \\ &\lesssim (N_n' N_n)^{-1}\|\nabla v_n\|_{L_t^\infty L_x^2} \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Using Hölder and Sobolev embedding, we have

$$\begin{aligned}\||x|(b)\|_{L_t^1 L_x^2(|t|\le TN_n^{-2})} &\sim \tfrac{N_n^2}{N_n'}\|P_{\le \tilde N_n' N_n} v_n\|_{L_t^1 L_x^2(|t|\le TN_n^{-2}, |x-x_n|\lesssim (N_n')^{-1})} \\ &\lesssim \begin{cases} (N_n')^{-2}\|\nabla v_n\|_{L_t^\infty L_x^2}, & \lim_{n\to\infty} N_n^{-1}|x_n| = 0 \\ \|v_n\|_{L_t^\infty L_x^2} = O(N_n^{-1}), & \lim_{n\to\infty} N_n^{-1}|x_n| > 0,\end{cases}\end{aligned}$$

which vanishes as $n\to\infty$ in either case. Thus $\|H^{1/2}(b)\|_{L_t^1 L_x^2} \to 0$. The term (c) is dealt with similarly. Finally, to estimate (d), apply Hölder, Bernstein, and the definition (2.58) of the frequency cutoffs $\tilde N_n'$ to obtain

$$\begin{aligned}\|\nabla(d)\|_{L_t^1 L_x^2(|t|\le TN_n^{-2})} &\lesssim N_n' \||\nabla|^2 P_{\le \tilde N_n' N_n} v_n\|_{L_t^1 L_x^2} + \||\nabla P_{\le \tilde N_n' N_n} v_n|(|\nabla|^2 \chi_n)\|_{L_t^1 L_x^2} \\ &\lesssim \left[\left(\tfrac{N_n'}{N_n}\right)^{\frac12} + \left(\tfrac{N_n'}{N_n}\right)^2\right]\|\nabla v_n\|_{L_t^\infty L_x^2} \to 0.\end{aligned}$$

Using Hölder in time, we get

$$\||x|(d)\|_{L_t^1 L_x^2(|t|\le TN_n^{-2})} \lesssim \tfrac{N_n'}{N_n}\|\nabla v_n\|_{L_t^\infty L_x^2} \to 0.$$

This completes the proof of the lemma. □

Next, we estimate the error over the time intervals $[-2, TN_n^{-2}]$ and $[TN_n^{-2}, 2]$.

Lemma 2.5.4. $\lim_{T\to\infty}\limsup_{n\to\infty} \|H^{\frac12} e_n\|_{N([-2,TN_n^{-2}]\cup[TN_n^{-2},2])} = 0.$

Proof. We consider just the forward time interval as the other interval is treated similarly. Since $\tilde v_n^T$ solves the linear equation, the error e_n is just the nonlinear term:

$$e_n = (i\partial_t - H)\tilde v_n^T - F(\tilde v_n^T) = -F(\tilde v_n^T).$$

By the chain rule (Corollary 2.2.4) and Strichartz,

$$\|H^{\frac{1}{2}}e_n\|_{N([TN_n^{-2},2])} \lesssim \|\tilde{v}_n^T\|^{\frac{4}{d-2}}_{L_{t,x}^{\frac{2(d+2)}{d-2}}([TN_n^{-2},2])} \|\tilde{v}_n^T(TN_n^{-2})\|_{\Sigma}.$$

By definition $\tilde{v}_n^T(TN_n^{-2}) = e^{-\frac{iTN_n^{-2}|x_n|^2}{2}}\tilde{G}_nS_nP_{\le \tilde{N}_n'}v(TN_n^{-2}-t_n)$, so Lemma 2.4.2 implies that

$$\limsup_{n\to\infty} \|\tilde{v}_n^T(TN_n^{-2})\|_{\Sigma} \lesssim \begin{cases} \|v|_{L_t^\infty \dot{H}_x^1}, & \lim_{n\to\infty} N_n^{-1}|x_n| = 0, \\ \|v\|_{L_t^\infty H_x^1}, & \lim_{n\to\infty} N_n^{-1}|x_n| > 0 \end{cases}$$

is bounded in either case. Using Strichartz and interpolation, it suffices to show

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n^T\|_{L_T^\infty L_x^{\frac{2d}{d-2}}([TN_n^{-2},2])} = 0.$$

As we are assuming Conjecture 2.1.1, there exists $v_\infty \in \dot{H}^1$ so that

$$\lim_{t\to\infty} \|v(t) - e^{\frac{it\Delta}{2}}v_\infty\|_{\dot{H}_x^1} = 0;$$

if $v(0) \in H^1$ the same limit holds with respect to the H^1 norm. Then one also has

$$\lim_{t\to\infty} \limsup_{n\to\infty} \|P_{\le \tilde{N}_n'}v(t) - e^{\frac{it\Delta}{2}}v_\infty\|_{\dot{H}_x^1} = 0,$$

(with the obvious modification if $v(0) \in H^1$) and Lemma 2.4.2 implies that

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n(TN_n^{-2}) - e^{-\frac{iTN_n^{-2}|x_n|^2}{2}}G_nS_n(e^{\frac{iT\Delta}{2}}v_\infty)\|_{\Sigma} = 0.$$

An application of Strichartz and Corollary 2.4.9 yields

$$\begin{aligned} \tilde{v}_n(t) &= e^{-i(t-TN_n^{-2})H}[\tilde{v}_n(TN_n^{-2})] \\ &= e^{-i(t-TN_n^{-2})H}[e^{-\frac{iTN_n^{-2}|x_n|^2}{2}}G_nS_ne^{\frac{iT\Delta}{2}}v_\infty] + \text{error} \\ &= e^{-itH}[G_nS_nv_\infty] + \text{error} \end{aligned}$$

where $\lim_{T\to\infty}\limsup_{n\to\infty}\|\text{error}\|_{\Sigma} = 0$ uniformly in t. By Sobolev embedding,

$$\begin{aligned} &\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n\|_{L_t^\infty L_x^{\frac{2d}{d-2}}([TN_n^{-2},2])} \\ &= \lim_{T\to\infty} \limsup_{n\to\infty} \|e^{-itH}[G_nS_nv_\infty]\|_{L_t^\infty L_x^{\frac{2d}{d-2}}([TN_n^{-2},2])}. \end{aligned}$$

A standard density argument using the dispersive estimate for e^{-itH} shows that the last limit is zero. $\square$

Lemmas 2.5.3 and 2.5.4 together establish (2.62).

Lemma 2.5.5 (Matching initial data)**.** *Let* $u_n(0) = e^{it_nH}G_nS_n\phi$ *as in Proposition 2.5.1. Then*

$$\lim_{T\to\infty}\limsup_{n\to\ \infty}\|\tilde{v}_n^T(-t_n) - u_n(0)\|_\Sigma = 0.$$

Proof. If $t_n \equiv 0$, then by definition $\tilde{v}_n^T(0) = G_nS_nP_{\le N_n'}\phi$, so Lemma 2.4.2 and the definition (2.58) of the frequency parameter N_n' imply

$$\lim_{n\to\infty}\|\tilde{v}_n^T(0) - u_n(0)\|_\Sigma \lesssim \lim_{n\to\infty}\left\{\begin{array}{ll}\|P_{>N_n'}\phi\|_{H^1}, & \lim_{n\to\infty}N_n^{-1}|x_n| > 0\\ \|P_{>N_n'}\phi\|_{\dot{H}^1}, & \lim_{n\to\infty}N_n^{-1}|x_n| = 0\end{array}\right\} = 0.$$

Next we consider the case $N_n^2t_n \to \infty$; the case $N_n^2t_n \to -\infty$ works similarly. Arguing as in the previous lemma and recalling that in this case, the solution v was chosen to scatter *backward* in time to $e^{\frac{it\Delta}{2}}\phi$, for n large we have

$$\tilde{v}_n^T(-t_n) = e^{it_nH}[G_nS_n\phi] + \text{error}$$

where $\lim_{T\to\infty}\limsup_{n\to\infty}\|\text{error}\|_\Sigma \to 0$. The claim follows. □

For each fixed $T > 0$, set

$$\tilde{u}_n^T(t) = \tilde{v}_n^T(t - t_n), \tag{2.64}$$

which is defined for $t \in [-1, 1]$. Then for a fixed large value of T, this is an approximate solution for all n sufficiently large in the sense of Proposition 2.3.3. Indeed, by (2.60) and (2.61), $\tilde{u}_n^T$ satisfy the hypotheses (2.30) with $E, L = C(\|\phi\|_{\dot{H}^1})$. Lemmas 2.5.3, 2.5.4, 2.5.5, Sobolev embedding, and Strichartz show that for any $\varepsilon > 0$, there exists $T > 0$ so that $\tilde{u}_n^T$ satisfies the hypotheses (2.31) for all large n. Invoking Proposition 2.3.3, we obtain the first claim of Proposition 2.5.1 concerning the existence of solutions.

The remaining assertion of Proposition 2.5.1 regarding approximation by smooth functions will follow from the next lemma. Recall the notation

$$\|f\|_{L_t^q\Sigma_x^r} = \|H^{\frac{1}{2}}f\|_{L_t^qL_x^r}.$$

Lemma 2.5.6. *Fix finitely many admissible* (q_k, r_k) *with* $2 \le r_k < d$. *For every* $\varepsilon > 0$, *there exists a smooth function* $\psi^\varepsilon \in C_c^\infty(\mathbf{R} \times \mathbf{R}^d)$ *such that for all* k

$$\limsup_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n^T - \tilde{G}_n[e^{-\frac{itN_n^{-2}|x_n|^2}{2}} \psi^\varepsilon](t + t_n)\|_{L_T^{q_k} \Sigma_x^{r_k}([-2,2])} < \varepsilon.$$

Proof. We continue using the notation defined at the beginning. Let

$$\tilde{w}_n^T = \begin{cases} e^{-\frac{it|x_n|^2}{2}} \tilde{G}_n[S_n v](t + t_n), & |t| \le TN_n^{-2} \\ e^{-i(t - TN_n^{-2})H}[\tilde{w}_n^T(TN_n^{-2})], & t \ge TN_n^{-2} \\ e^{-i(t+TN_n^{-2})H}[\tilde{w}_n^T(-TN_n^{-2})], & t \le -TN_n^{-2} \end{cases}$$

This is essentially $\tilde{v}_n^T$ in (2.59) without the frequency cutoffs. We see first that $\tilde{v}_n^T$ can be well-approximated by $\tilde{w}_n^T$ in spacetime:

$$\begin{aligned} &\limsup_{n\to\infty} \|\tilde{v}_n^T - \tilde{w}_n^T\|_{L_t^{q_k} \Sigma_x^{r_k}([-2,2])} = 0, \\ &\sup_{T>0} \limsup_{n\to\infty} \|\tilde{w}_n^T\|_{L_t^{q_k} \Sigma_x^{r_k}([-2,2])} < \infty. \end{aligned} \tag{2.65}$$

Indeed by dominated convergence,

$$\|\nabla(v - P_{\le \tilde{N}_n'} v)\|_{L_t^{q_k} L_x^{r_k}(\mathbf{R}\times\mathbf{R}^d)} \to 0 \text{ as } n \to \infty,$$

thus (2.65) follows from Lemma 2.4.2 and the Strichartz inequality for e^{-itH}.

A consequence of the dispersive estimate is that most of the spacetime norm of $\tilde{w}_n^T$ is concentrated in the time interval $|t| \le TN_n^{-2}$:

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{w}_n^T\|_{L_t^{q_k} \Sigma_x^{r_k}([-2, -TN_n^{-2}] \cup [TN_n^{-2}, 2])} = 0. \tag{2.66}$$

To see this, it suffices by symmetry to consider the forward interval. Recall that v scatters forward in $\dot{H}^1$ (and in H^1 if $v(0) \in H^1$) to some $e^{\frac{it\Delta}{2}} v_\infty$. By Lemma 2.4.2,

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|(\tilde{G}_n S_n v(TN_n^{-2} + t_n) - G_n S_n(e^{\frac{iT\Delta}{2}} v_\infty)\|_\Sigma = 0.$$

By Strichartz,

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|e^{\frac{iTN_n^{-2}|x_n|^2}{2}} \tilde{w}_n^T - e^{-i(t-TN_n^{-2})H}[G_n S_n(e^{\frac{iT\Delta}{2}} v_\infty)]\|_{L_t^{q_k} \Sigma_x^{r_k}([TN_n^{-2}, 2])} = 0$$

By Corollary 2.4.9 and Strichartz, for each $T > 0$ we have

$$\lim_{n\to\infty} \|e^{-i(t-TN_n^{-2})H}[G_n S_n(e^{\frac{iT\Delta}{2}} v_\infty)] - e^{\frac{iT(r_\infty)^2}{2}} e^{-itH}[G_n S_n v_\infty]\|_{L_t^{q_k}\Sigma_x^{r_k}} = 0.$$

For each $\varepsilon > 0$, choose $v_\infty^\varepsilon \in C_c^\infty$ such that $\|v_\infty - v_\infty^\varepsilon\|_{\dot{H}^1} < \varepsilon$. By the dispersive estimate,

$$\|e^{-itH}[G_n v_\infty^\varepsilon]\|_{L_t^{q_k} L_x^{r_k}([TN_n^{-2},2])} \lesssim T^{-\frac{1}{q_k}} \|v_\infty^\varepsilon\|_{L_x^{r_k'}}$$

Combining the above with Strichartz and Lemma 2.4.2, we get

$$\limsup_{n\to\infty} \|\tilde{w}_n^T\|_{L_t^{q_k}\Sigma_x^{r_k}([TN_n^{-2},2])} \lesssim o(1) + \varepsilon + O_{\varepsilon,q_k}(T^{-\frac{1}{q_k}}) \text{ as } T \to \infty.$$

Taking $T \to \infty$, we find

$$\limsup_{T\to\infty} \limsup_{n\to\infty} \|\tilde{w}_n^T\|_{L_t^{q_k}\Sigma_x^{r_k}([TN_n^{-2},2])} \lesssim \varepsilon$$

for any $\varepsilon > 0$, thereby establishing (2.66).

Choose $\psi^\varepsilon \in C_c^\infty(\mathbf{R} \times \mathbf{R}^d)$ such that $\sum_{k=1}^N \|v - \psi^\varepsilon\|_{L_t^{q_k}\dot{H}_x^{1,r_k}} < \varepsilon$. By combining Lemma 2.4.2 with (2.65) and (2.66), we get

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{v}_n(t,x) - e^{-\frac{it|x_n|^2}{2}} \tilde{G}_n \psi^\varepsilon(t+t_n)\|_{L_t^{q_k}\Sigma_x^{r_k}([-2,2])} \lesssim \varepsilon.$$

This completes the proof of the lemma, hence Proposition 2.5.1. □

Remark. From the proof it is clear that that the proposition also holds if the interval $I = [-1,1]$ is replaced by any smaller interval.

2.6 Palais-Smale and the proof of Theorem 2.1.2

In this section we prove a Palais-Smale-type compactness property for sequences of blowing up solutions to (2.1). This will quickly lead to Theorem 2.1.2.

For a maximal solution u to (2.1), define

$$S_*(u) = \sup\{S_I(u) : I \text{ is an open interval with } \leq 1\},$$

where we set $S_I(u) = \infty$ if u is not defined on I. All solutions in this section are assumed to be maximal. Set

$$\begin{aligned}
\Lambda_d(E) &= \sup\{S_*(u) : u \text{ solves } (2.1),\ \mu = +1,\ E(u) = E\} \\
\Lambda_f(E) &= \sup\{S_*(u) : u \text{ solves } (2.1),\ \mu = -1,\ E(u) = E, \\
&\qquad \|\nabla u(0)\|_{L^2} < \|\nabla W\|_{L^2}\}.
\end{aligned}$$

Finally, define

$$\mathcal{E}_d = \{E : \Lambda_d(E) < \infty\}, \quad \mathcal{E}_f = \{E : \Lambda_f(E) < \infty\}.$$

By the local theory, Theorem 2.1.2 is equivalent to the assertions

$$\mathcal{E}_d = [0, \infty), \quad \mathcal{E}_f = [0, E_\Delta(W)).$$

Suppose Theorem 2.1.2 failed. By the small data theory, $\mathcal{E}_d$, $\mathcal{E}_f$ are nonempty and open, and the failure of Theorem 2.1.2 implies the existence of a critical energy $E_c > 0$, with $E_c < E_\Delta(W)$ in the focusing case, such that $\Lambda_d(E),\ \Lambda_f(E) = \infty$ for $E > E_c$ and $\Lambda_d(E),\ \Lambda_f(E) < \infty$ for all $E < E_c$.

Define the spaces

$$\dot{X}^1 = \begin{cases} L^{10}_{t,x} \cap L^5_t \Sigma_x^{\frac{30}{11}}([-\frac{1}{2}, \frac{1}{2}] \times \mathbf{R}^3), & d = 3 \\ L_{t,x}^{\frac{2(d+2)}{d-2}} \cap L_t^{\frac{2(d+2)}{d}} \Sigma_x^{\frac{2(d+2)}{d}}([-\frac{1}{2}, \frac{1}{2}] \times \mathbf{R}^d), & d \geq 4. \end{cases}$$

When $d = 3$, also define

$$\dot{Y}^1 = \dot{X}^1 \cap L_t^{\frac{10}{3}} \Sigma_x^{\frac{10}{3}}([-\tfrac{1}{2}, \tfrac{1}{2}] \times \mathbf{R}^3).$$

Remark. The case $d = 3$ is singled out for technical reasons. Our choice of Strichartz norm $L^5_t \Sigma^{30/11}_x$ is guided by the fact that $\frac{30}{11} < 3$, which is needed for Sobolev embedding. In higher dimensions the symmetric Strichartz norm suffices since $\frac{2(d+2)}{d} < d$ for all $d \geq 4$. This distinction necessitates a separate but essentially parallel treatment of various estimates when $d = 3$.

Proposition 2.6.1 (Palais-Smale). *Assume Conjecture 2.1.1 holds. Suppose that* $u_n : (t_n - \frac{1}{2}, t_n + \frac{1}{2}) \times \mathbf{R}^d \to \mathbf{C}$ *is a sequence of solutions with*

$$\lim_{n\to\infty} E(u_n) = E_c, \ \lim_{n\to\infty} S_{(t_n - \frac{1}{2}, t_n]}(u_n) = \lim_{n\to\infty} S_{[t_n, t_n + \frac{1}{2})}(u_n) = \infty.$$

In the focusing case, assume also that $E_c < E_\Delta(W)$ *and* $\|\nabla u_n(t_n)\|_{L^2} < \|\nabla W\|_{L^2}$. *Then there exists a subsequence such that* $u_n(t_n)$ *converges in* Σ.

Let us first see how this would imply the main theorem.

Proof of Theorem 2.1.2. Suppose the theorem failed. In the defocusing case, there exist $E_c \in (0, \infty)$ and a sequence of solutions u_n with $E(u_n) \to E_c$, $S_{(-\frac{1}{4}, 0]}(u_n) \to \infty$, and $S_{[0, \frac{1}{4})}(u_n) \to \infty$. The same is true in the focusing case except E_c is restricted to the interval $(0, E_\Delta(W))$ and $\limsup_n \|u_n(0)\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$. By Proposition 2.6.1, after passing to a subsequence $u_n(0)$ converges in Σ to some ϕ. Then $E(\phi) = \lim_n E(u_n(0)) = E_c$.

Let $u_\infty : (-T_{min}, T_{max}) \to \mathbf{C}$ be the maximal lifespan solution to (2.1) with $u_\infty(0) = \phi$. By comparing u with the u_n and applying Proposition 2.3.3, we see that $S_{([0, \frac{1}{2})}(u_\infty) = S_{(-\frac{1}{2}, 0]}(u_\infty) = \infty$. So $-1/2 \le -T_{min} < T_{max} \le 1/2$. But Proposition 2.6.1 implies that the orbit $\{u_\infty(t) : t \in (-T_{min}, T_{max})\}$ is precompact in Σ, thus there is a sequence of times t_n increasing to T_{max} such that $u_\infty(t_n)$ converges in Σ to some ψ. Taking a local solution with initial data equal to ψ, we can then invoke Proposition 2.3.3 to extend u_∞ to some larger interval $(-T_{min}, T_{max} + \eta)$, contradicting the maximality of u_∞. □

Proof of Proposition 2.6.1. By replacing $u_n(t)$ with $u_n(t + t_n)$, we may assume $t_n \equiv 0$. Note that by energy conservation and Corollary 2.7.2, this time translation does not change the hypotheses of the focusing case.

Observe (referring to the discussion in Section 2.7 for the focusing case) that the sequence $u_n(0)$ is bounded in Σ. Applying Proposition 2.4.14, after passing to a subsequence we have a decomposition

$$u_n(0) = \sum_{j=1}^{J} e^{it_n^j H} G_n S_n \phi^j + w_n^J = \sum_{j=1}^{J} \phi_n^j + w_n^J$$

with the properties stated in that proposition. In particular, the remainder has asymptotically trivial linear evolution:

$$\lim_{J\to J^*}\limsup_{n\to\infty}\|e^{-itH}w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}, \tag{2.67}$$

and the energies asymptotically decouple:

$$\sup_J \lim_{n\to\infty}|E(u_n)-\sum_{j=1}^J E(\phi_n^j)-E(w_n^J)|=0. \tag{2.68}$$

Observe that $\liminf_n E(\phi_n^j)\geq 0$. This is obvious in the defocusing case. In the focusing case, (2.47) and the discussion in Section 2.7 imply that

$$\sup_j \limsup_n \|\phi_n^j\|_\Sigma \leq \|u_n\|_\Sigma < \|\nabla W\|_{L^2},$$

so the claim follows from Lemma 2.7.1. Therefore, there are two possibilities.

Case 1: $\sup_j \limsup_{n\to\infty} E(\phi_n^j)=E_c$.

By combining (2.68) with the fact that the profiles ϕ_n^j are nontrivial in Σ, it follows that $J^*=1$ and

$$u_n(0)=e^{it_nH}G_nS_n\phi+w_n, \ \lim_{n\to\infty}\|w_n\|_\Sigma=0.$$

We argue that $N_n\equiv 1$ (thus $x_n=0$ and $t_n=0$). Suppose $N_n\to\infty$.

Proposition 2.5.1 implies that for all large n, there exists a unique solution u_n on $[-\frac{1}{2},\frac{1}{2}]$ with $u_n(0)=e^{it_nH}G_nS_n\phi$ and $\limsup_{n\to\infty} S_{(-\frac{1}{2},\frac{1}{2})}(u_n)\leq C(E_c)$. By perturbation theory (Proposition 2.3.3),

$$\limsup_{n\to\infty} S_{[-\frac{1}{2},\frac{1}{2}]}(u_n)\leq C(E_c),$$

which is a contradiction. Therefore, $N_n\equiv 1, \ t_n^j\equiv 0, \ x_n^j\equiv 0$, and

$$u_n(0)=\phi+w_n$$

for some $\phi\in\Sigma$. This is the desired conclusion.

Case 2: $\sup_j \limsup_{n\to\infty} E(\phi_n^j)\leq E_c-2\delta$ for some $\delta>0$.

By the definition of E_c, there exist solutions $v_n^j : (-\frac{1}{2}, \frac{1}{2}) \times \mathbf{R}^d \to \mathbf{C}$ with

$$\|v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([-\frac{1}{2},\frac{1}{2}])} \lesssim_{E_c,\delta} E(\phi_n^j)^{\frac{1}{2}}.$$

By standard arguments (c.f. [TVZ07, Lemma 3.11]), this implies the seemingly stronger bound

$$\|v_n^j\|_{\dot{X}^1} \lesssim_{E_c,\delta} E(\phi_n^j)^{\frac{1}{2}}. \tag{2.69}$$

In the case $d = 3$, we also have $\|v_n^j\|_{\dot{Y}^1} \lesssim E(\phi_n^j)^{\frac{1}{2}}$. Put

$$u_n^J = \sum_{j=1}^{J} v_n^j + e^{-itH} w_n^J. \tag{2.70}$$

We claim that for sufficiently large J and n, u_n^J is an approximate solution in the sense of Proposition 2.3.3. To prove this claim, we check that u_n^J has the following three properties:

(i) $\lim_{J \to J^*} \limsup_{n\to\infty} \|u_n^J(0) - u_n(0)\|_{\Sigma} = 0$.

(ii) $\limsup_{n\to\infty} \|u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([-T,T])} \lesssim_{E_c,\delta} 1$ uniformly in J.

(iii) $\lim_{J \to J^*} \limsup_{n\to\infty} \|H^{\frac{1}{2}} e_n^J\|_{N([-\frac{1}{2},\frac{1}{2}])} = 0$, where

$$e_n = (i\partial_t - H)u_n^J - F(u_n^J).$$

There is nothing to check for part (i) as $u_n^J(0) = u_n(0)$ by construction. The verification of (ii) relies on the asymptotic decoupling of the nonlinear profiles v_n^j, which we record in the following two lemmas.

Lemma 2.6.2 (Orthogonality)**.** *Suppose that two frames $\mathcal{F}^j = (t_n^j, x_n^j, N_n^j)$, $\mathcal{F}^k = (t^k, x_n^k, N_n^k)$ are orthogonal, and let $\tilde{G}_n^j$, $\tilde{G}_n^k$ be the associated spacetime scaling and translation operators as defined in* (2.33)*. Then for all ψ^j, ψ^k in $C_c^\infty(\mathbf{R} \times \mathbf{R}^d)$,*

$$\|(\tilde{G}_n^j\psi^j)(\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-2}}} + \|(\tilde{G}_n^j\psi^j)\nabla(\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}} + \||x|(\tilde{G}_n^j\psi^j)(\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}}$$
$$+ \||x|^2(\tilde{G}_n^j\psi^j)(\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d}}} + \|(\nabla\tilde{G}_n^j\psi^j)(\nabla\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d}}} \to 0$$

as $n \to \infty$. When $d = 3$, we also have

$$\||x|^2(\tilde{G}_n^j\psi^j)(\tilde{G}_n^k\psi^k)\|_{L_t^5 L_x^{\frac{15}{11}}} + \|(\nabla\tilde{G}_n^j\psi^j)(\nabla\tilde{G}_n^k\psi^k)\|_{L_t^5 L_x^{\frac{15}{11}}} \to 0.$$

Proof. The arguments for each term are similar, and we only supply the details for the second term. Suppose $N_n^k(N_n^j)^{-1} \to \infty$. By the chain rule, a change of variables, and Hölder,

$$\begin{aligned}\|(\tilde{G}_n^j\psi^j)\nabla(\tilde{G}_n^k\psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}} &= \|\psi^j\nabla(\tilde{G}_n^j)^{-1}\tilde{G}_n^k\psi^k\|_{L_{t,x}^{\frac{d+2}{d-1}}}\\ &\le \|\psi^j\chi_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}\|\nabla\psi^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}},\end{aligned}$$

where χ_n is the characteristic function of the support of $\nabla(\tilde{G}_n^j)^{-1}\tilde{G}_n^k\psi^k$. As the support of χ_n has measure shrinking to zero, we have

$$\lim_{n\to\infty}\|\psi^j\chi_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} = 0.$$

A similar argument deals with the case where $N_n^j(N_n^k)^{-1} \to \infty$. Therefore, we may suppose that

$$\tfrac{N_n^k}{N_n^j} \to N_\infty \in (0,\infty).$$

Make the same change of variables as before, and compute

$$\nabla(\tilde{G}_n^j)^{-1}\tilde{G}_n^k\psi^k(t,x) = (\tfrac{N_n^k}{N_n^j})^{\frac{d}{2}}(\nabla\psi^k)[\tfrac{N_n^k}{N_n^j}t + (N_n^k)^2(t_n^j - t_n^k), \tfrac{N_n^k}{N_n^j}x + N_n^k(x_n^j - x_n^k)].$$

The decoupling statement (2.49) implies that

$$(N_n^k)^2(t_n^j - t_n^k) + N_n^k|x_n^j - x_n^k| \to \infty.$$

Therefore, the supports of ψ^j and $\nabla(\tilde{G}_n^j)^{-1}\tilde{G}_n^k\psi^k$ are disjoint for large n. □

Lemma 2.6.3 (Decoupling of nonlinear profiles)**.** *Let v_n^j be the nonlinear solutions defined above. Then when $d \ge 4$,*

$$\begin{aligned}&\|v_n^jv_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|v_n^j\nabla v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} + \||x|v_n^jv_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}}\\ &\qquad + \|(\nabla v_n^j)(\nabla v_n^k)\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \||x|^2v_n^jv_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \to 0\end{aligned}$$

as $n \to \infty$. When $d = 3$, the same statement holds with the last two expressions replaced by

$$\|(\nabla v_n^j)(\nabla v_n^k)\|_{L_t^5L_x^{\frac{15}{11}}} + \||x|^2v_n^jv_n^k\|_{L_t^5L_x^{\frac{30}{11}}} \to 0.$$

Proof. We spell out the details for the $\|v_n^j|x|v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}}$ term. Consider first the case $d \geq 4$. As $2 < \frac{2(d+2)}{d} < d$, by Proposition 2.5.1 we can approximate v_n^j in $\dot{X}^1$ by test functions

$$c_n^j \tilde{G}_n \psi^j, \quad \psi^j \in C_c^\infty(\mathbf{R} \times \mathbf{R}^d), \quad c_n^j(t) = e^{-\frac{i(t-t_n^j)|x_n^j|^2}{2}}.$$

By Hölder and a change of variables,

$$\begin{aligned}
&\|v_n^j|x|v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} \leq \|(v_n^j - c_n^j \tilde{G}_n^j \psi^j)|x|v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} \\
&+ \||x|\tilde{G}_n^j \psi^j (v_n^k - c_n^k \tilde{G}_n^k \psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}} + \||x|\tilde{G}_n^j \psi^j \tilde{G}_n^k \psi^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} \\
&\leq \|(v_n^j - c_n^j \tilde{G}_n^j \psi^j)\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|v_n^k\|_{\dot{X}^1} \\
&+ \|\psi^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|(v_n^k - c_n^k \tilde{G}_n^k \psi^k)\|_{\dot{X}^1} + \|(\tilde{G}_n^j \psi^j)|x|(\tilde{G}_n^k \psi^k)\|_{L_{t,x}^{\frac{d+2}{d-1}}}
\end{aligned}$$

By first choosing ψ^j, then ψ^k, then invoking the previous lemma, we obtain for any $\varepsilon > 0$ that

$$\limsup_{n\to\infty} \|v_n^j|x|v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}} \leq \varepsilon.$$

When $d = 3$, we also approximate v_n^j in $\dot{X}^1$ (which is possible because the exponent $\frac{30}{11}$ in the definition of $\dot{X}^1$ is less than 3), and estimate

$$\begin{aligned}
&\|v_n^j|x|v_n^k\|_{L_{t,x}^{\frac{5}{2}}} \\
&\leq \|(v_n^j - c_n^j \tilde{G}_n^j \psi^j)|x|v_n^k\|_{L_{t,x}^{\frac{5}{2}}} + \||x|\tilde{G}_n^j \psi^j (v_n^k - c_n^k \tilde{G}_n^k \psi^k)\|_{L_{t,x}^{\frac{5}{2}}} + \||x|\tilde{G}_n^j \psi^j \tilde{G}_n^k \psi^k\|_{L_{t,x}^{\frac{5}{2}}} \\
&\leq \|(v_n^j - c_n^j \tilde{G}_n^j \psi^j)\|_{L_{t,x}^{10}} \|v_n^k\|_{\dot{Y}^1} \\
&+ \|\psi^j\|_{L_t^5 L_x^{30}} \|v_n^k - c_n^k \tilde{G}_n^k \psi^k\|_{\dot{X}^1} + \|(\tilde{G}_n^j \psi^j)|x|(\tilde{G}_n^k \psi^k)\|_{L_{t,x}^{\frac{5}{2}}}
\end{aligned}$$

which, just as above, can be made arbitrarily small as $n \to \infty$. Similar approximation arguments deal with the other terms. □

Let us verify Claim (ii) above. In fact we shall show that

$$\limsup_{n\to\infty} \|u_n^J\|_{\dot{X}^1([-\frac{1}{2},\frac{1}{2}])} \lesssim_{E_c,\delta} 1 \text{ uniformly in } J. \tag{2.71}$$

First, observe that

$$S(u_n^J) = \iint |\sum_{j=1}^{J} v_n^j + e^{-itH} w_n^J|^{\frac{2(d+2)}{d-2}} \, dxdt \lesssim S(\sum_{j=1}^{J} v_n^j) + S(e^{-itH} w_n^J).$$

By the properties of the LPD, $\lim_{J\to J^*} \limsup_{n\to\infty} S(e^{-itH} w_n^J) = 0$. Recalling (2.69), write

$$\begin{aligned} S(\sum_{j=1}^{J} v_n^j) &= \Big\| (\sum_{j=1}^{J} v_n^j)^2 \Big\|_{L_{t,x}^{\frac{d+2}{d-2}}}^{\frac{d+2}{d-2}} \le (\sum_{j=1}^{J} \|v_n^j\|^2_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \sum_{j\neq k} \|v_n^j v_n^k\|_{L_{t,x}^{\frac{d+2}{d-2}}})^{\frac{d+2}{d-2}} \\ &\lesssim (\sum_{j=1}^{J} E(\phi_n^j) + o_J(1))^{\frac{d+2}{d-2}} \end{aligned}$$

where the last line used Lemma 2.6.3. As energy decoupling implies $\limsup_{n\to\infty} \sum_{j=1}^{J} E(\phi_n^j) \le E_c$, we obtain $\lim_{J\to J^*} \limsup_{n\to\infty} S(u_n^J) \lesssim_{E_c,\delta} 1$.

By mimicking this argument one also obtains

$$\limsup_{n\to\infty} (\|\nabla u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \||x| u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}}) \lesssim_{E_c,\delta} 1 \text{ uniformly in } J.$$

Property (ii) is therefore verified in the case $d \ge 4$. The case $d = 3$ is dealt with similarly.

Remark. The above argument shows that for each J and each $\eta > 0$, there exists $J' \le J$ such that

$$\limsup_{n\to\infty} \| \sum_{j=J'}^{J} v_n^j \|_{\dot{X}^1([-\frac{1}{2},\frac{1}{2}])} \le \eta.$$

It remains to check property (iii) above, namely, that

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|H^{1/2} e_n^J\|_{N([-\frac{1}{2},\frac{1}{2}])} = 0. \tag{2.72}$$

Let $F(z) = |z|^{\frac{4}{d-2}} z$ and decompose

$$e_n^J = [\sum_{j=1}^{J} F(v_n^j) - F(\sum_{j=1}^{J} v_n^j)] + [F(u_n^J - e^{-itH} w_n^J) - F(u_n^J)] = (a) + (b). \tag{2.73}$$

Consider (a) first. Suppose $d \ge 6$. Using the chain rule $\nabla F(u) = F_z(u)\nabla u + F_{\overline{z}}(u)\overline{\nabla u}$ and the estimates

$$|F_z(z)| + |F_{\overline{z}}(z)| = O(|z|^{\frac{4}{d-2}}), \; |F_z(z) - F_z(w)| + |F_{\overline{z}}(z) - F_{\overline{z}}(w)| = O(|z-w|^{\frac{4}{d-2}}),$$

we compute

$$|\nabla(a)| \lesssim \sum_{j=1}^{J}\sum_{k\neq j} |v_n^k|^{\frac{4}{d-2}}|\nabla v_n^j|.$$

By Hölder, Lemma 2.6.3, and the induction hypothesis (2.69),

$$\|\nabla(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \lesssim \sum_{j=1}^{J}\sum_{k\neq j} \||v_n^k||\nabla v_n^j|\|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}} \|\nabla v_n^k\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-6}{d-2}} = o_J(1)$$

as $n \to \infty$. When $3 \leq d \leq 5$, we have instead

$$|\nabla(a)| \lesssim \sum_{j=1}^{J}\sum_{k\neq j} |v_n^k||\nabla v_n^j| O\Big(\Big|\sum_{k=1}^{J} v_n^k\Big|^{\frac{6-d}{d-2}} + |v_n^j|^{\frac{6-d}{d-2}}\Big),$$

thus

$$\|\nabla(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \lesssim_J \left(\sum_{j=1}^{J} \|v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}}\right) \sum_{j=1}^{J}\sum_{k\neq j} \||v_n^k||\nabla v_n^j|\|_{L_{t,x}^{\frac{d+2}{d-1}}} = o_J(1).$$

Similarly, writing

$$|(a)| \leq \sum_{j=1}^{J}\Big| |v_n^j|^{\frac{4}{d-2}} - |\sum_{k=1}^{J} v_n^k|^{\frac{4}{d-2}}\Big| |v_n^j| \lesssim \sum_{j=1}^{J}\sum_{k\neq j} |v_n^j||v_n^k|^{\frac{4}{d-2}},$$

we have

$$\|x(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \sum_{j=1}^{J}\sum_{k\neq j} \||x|v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-6}{d-2}} \||x|v_n^j v_n^k\|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}} = o_J(1).$$

When $3 \leq d \leq 5$,

$$|(a)| \lesssim \sum_{j=1}^{J}\sum_{k\neq j} |v_n^j||v_n^k| O\Big(\Big|\sum_{k=1}^{J} v_n^k\Big|^{\frac{6-d}{d-2}} + |v_n^j|^{\frac{6-d}{d-2}}\Big),$$

hence also

$$\||x|(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} = o_J(1).$$

Summing up,

$$\|H^{1/2}(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \lesssim \|\nabla(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} + \|x(a)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} = o_J(1).$$

Next we estimate (b), restricting temporarily to dimensions $d \geq 4$. When $d \geq 6$, write

$$\begin{aligned}(b) &= F(u_n^J - e^{-itH}w_n^J) - F(u_n^J)\\ &= (|u_n^J - e^{-itH}w_n^J|^{\frac{4}{d-2}} - |u_n^J|^{\frac{4}{d-2}})\sum_{j=1}^{J} v_n^j - (e^{-itH}w_n^J)|u_n^J|^{\frac{4}{d-2}}\\ &= O(|e^{-itH}w_n^J|^{\frac{4}{d-2}})\sum_{j=1}^{J} v_n^j - (e^{-itH}w_n^J)|u_n^J|^{\frac{4}{d-2}},\end{aligned}$$

and apply Hölder's inequality:

$$\begin{aligned}\||x|(b)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} &\lesssim \|e^{-itH}w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|\sum_{j=1}^{J}|x|v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}}\\ &\quad + \||x|u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{4}{d-2}} \||x|e^{-itH}w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-6}{d-2}} \|e^{-itH}w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}}\end{aligned} \tag{2.74}$$

When $d = 4, 5$,

$$(b) = (e^{-itH}w_n^J)O(|u_n^J|^{\frac{6-d}{d-2}} + |u_n^J - e^{-itH}w_n^J|^{\frac{6-d}{d-2}})\sum_{j=1}^{J} v_n^j - (e^{-itH}w_n^J)|u_n^J|^{\frac{4}{d-2}},$$

thus

$$\begin{aligned}&\||x|(b)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}}\\ &\lesssim \|e^{-itH}w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \||x|\sum_{j=1}^{J} v_n^j\|_{L_{t,x}^{\frac{2(d+2)}{d}}} (\|u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}} + \|e^{-itH}w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}})\\ &+ \|e^{-itH}w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|xu_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \|u_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{6-d}{d-2}}.\end{aligned}$$

Using (2.71), Strichartz, and the decay property (2.67), we get

$$\lim_{J\to J*}\limsup_{n\to\infty} \||x|(b)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} = 0.$$

It remains to bound $\nabla(b)$. By the chain rule,

$$\begin{aligned}\nabla(b) &\lesssim |e^{-itH}w_n^J|^{\frac{4}{d-2}}\Big|\sum_{j=1}^{J}\nabla v_n^j\Big| + |u_n^J|^{\frac{4}{d-2}}|\nabla e^{-itH}w_n^J|\\ &= (b') + (b'').\end{aligned}$$

The first term (b') can be handled in the manner of (2.74) above. To deal with (b''), fix a small parameter $\eta > 0$, and use the above remark to obtain $J' = J'(\eta) \le J$ such that

$$\| \sum_{j=J'}^{J} v_n^j \|_{\dot{X}^1} \le \eta.$$

By the subadditivity of $z \mapsto |z|^{\frac{4}{d-2}}$ (which is true up to a constant when $d = 4, 5$) and Hölder,

$$\begin{aligned}
\|(b'')\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} &= \| | \sum_{j=1}^{J} v_n^j + e^{-itH} w_n^J |^{\frac{4}{d-2}} |\nabla e^{-itH} w_n^J| \|_{L_{t,x}^{\frac{2(d+2)}{d+4}}} \\
&\lesssim \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|H^{1/2} e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \| \sum_{j=J'}^{J} v_n^j \|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{4}{d-2}} \|H^{1/2} e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} \\
&+ C_{J'} \sum_{j=1}^{J'-1} \|\nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}}^{\frac{d-6}{d-2}} \| |v_n^j| |\nabla e^{-itH} w_n^J| \|_{L_{t,x}^{\frac{d+2}{d-1}}}^{\frac{4}{d-2}}.
\end{aligned}$$

By Strichartz and the decay of $e^{-itH} w_n^J$ in $L_{t,x}^{\frac{2(d+2)}{d-2}}$, the first term goes to 0 as $J \to \infty, \ n \to \infty$. By Strichartz and the definition of J', the second term is bounded by

$$\eta^{\frac{4}{d-2}} \|w_n^J\|_{\Sigma}$$

which can be made arbitrarily small since $\limsup_{n\to\infty} \|w_n^J\|_{\Sigma}$ is bounded uniformly in J. To finish, we check that for each fixed j

$$\lim_{J \to J^*} \limsup_{n \to \infty} \| |v_n^j| \nabla e^{-itH} w_n^J \|_{L_{t,x}^{\frac{d+2}{d-1}}} = 0. \tag{2.75}$$

For any $\varepsilon > 0$, there exist $\psi^j \in C_c^\infty(\mathbf{R} \times \mathbf{R}^d)$ such that if

$$c_n^j = e^{-\frac{i(t-t_n^j)|x_n^j|^2}{2}}$$

then

$$\limsup_{n\to\infty} \|v_n^j - c_n^j \tilde{G}_n^j \psi^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([-\frac{1}{2},\frac{1}{2}])} < \varepsilon,$$

Note that $\tilde{G}_n^j \psi^j$ is supported on the set

$$\{|t - t_n^j| \lesssim (N_n^j)^{-2}, \ |x - x_n^j| \lesssim (N_n^j)^{-1}\}.$$

Thus for all n sufficiently large,

$$\begin{aligned}
&\|v_n^j \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}} \\
&\le \|v_n^j - c_n^j \tilde{G}_n^j \psi^j\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \|\nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d}}} + \|\tilde{G}_n^j \psi^j \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}} \\
&\lesssim_{E_c} \varepsilon + \|(\tilde{G}_n^j \psi^j) \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}}.
\end{aligned}$$

By Hölder, noting that $\frac{d+2}{d-1} \le 2$ whenever $d \ge 4$,

$$\begin{aligned}
\|(\tilde{G}_n^j \psi^j) \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}} &\lesssim_\varepsilon (N_n^j)^{\frac{d-2}{2}} \|\nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}(|t-t_n^j| \lesssim (N_n^j)^{-2}, |x-x_n^j| \lesssim (N_n^j)^{-1})} \\
&\lesssim N_n^j \|\nabla e^{-itH} w_n^J\|_{L_{t,x}^2(|t-t_n^j| \lesssim (N_n^j)^{-2}, |x-x_n^j| \lesssim (N_n^j)^{-1})};
\end{aligned}$$

Corollary 2.2.10 implies

$$\|v_n^j \nabla e^{-itH} w_n^J\|_{L_{t,x}^{\frac{d+2}{d-1}}} \lesssim \varepsilon + C_{\varepsilon, E_c} \|e^{-itH} w_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}}^{\frac{1}{3}}.$$

Sending $n \to \infty$, then $J \to J^*$, then $\varepsilon \to 0$ establishes (2.75), and with it, Property (iii).

When $d = 3$, we estimate (b) in (2.73) instead in the $L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}$ dual Strichartz norm. Write

$$(b) = (e^{-itH} w_n^J) v_n^j O(|u_n^J|^3 + |u_n^J - e^{-itH} w_n^J|^3) \sum_{j=1}^J v_n^j - (e^{-itH} w_n^J)|u_n^J|^4,$$

and apply Hölder's inequality:

$$\begin{aligned}
\||x|(b)\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} &\lesssim \|e^{-itH} w_n^J\|_{L_{t,x}^{10}} \|u_n^J\|_{L_{t,x}^{10}}^3 \|H^{1/2} u_n^J\|_{L_t^5 L_x^{\frac{30}{11}}} \\
&\quad + \|e^{-itH} w_n^J\|_{L_{t,x}^{10}} (\|u_n^J\|_{L_{t,x}^{10}}^3 + \|e^{-itH} w_n^J\|_{L_{t,x}^{10}}^3) \|H^{\frac{1}{2}} \sum_{j=1}^J v_n^j\|_{L_t^5 L_x^{\frac{30}{11}}}.
\end{aligned} \tag{2.76}$$

Using (2.67) and (2.71), we have

$$\lim_{J \to J*} \limsup_{n \to \infty} \||x|(b)\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{11}}} = 0.$$

It remains to bound $\nabla(b)$. By the chain rule,

$$\begin{aligned}
\nabla(b) &= O\left((|u_n^J - e^{-itH} w_n^J|^4 - |u_n^J|^4) \nabla \sum_{j=1}^J v_n^j \right) + |u_n^J|^4 |\nabla e^{-itH} w_n^J| \\
&= (b') + (b'').
\end{aligned}$$

The first term (b') can be treated in the manner of $\||x|(b)\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}}$ above. We now concern ourselves with (b''). Fix a small parameter $\eta > 0$, and use the above remark to obtain $J' = J'(\eta) \leq J$ such that

$$\|\sum_{j=J'}^{J} v_n^j\|_{\dot{X}^1} \leq \eta.$$

Thus by the triangle inequality and Hölder,

$$\begin{aligned}
\|(b'')\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} &= \||\sum_{j=1}^{J} v_n^j + e^{-itH} w_n^J|^4 (e^{-itH} w_n^J)\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} \\
&\lesssim \|e^{-itH} w_n^J\|_{L_{t,x}^{10}}^4 \|H^{\frac{1}{2}} e^{-itH} w_n^J\|_{L_t^5 L_x^{\frac{30}{11}}} \\
&+ \||\sum_{j=J'}^{J} v_n^j|^4 |\nabla e^{-itH} w_n^J|\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} + C_{J'} \sum_{j=1}^{J'} \||v_n^j|^4 \nabla e^{-itH} w_n^J\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} \\
&\lesssim \|e^{-itH} w_n^J\|_{L_{t,x}^{10}}^4 \|H^{\frac{1}{2}} e^{-itH} w_n^J\|_{L_t^5 L_x^{\frac{30}{11}}} \\
&+ \|\sum_{j=J'}^{J} v_n^j\|_{\dot{X}^1}^4 \||\nabla e^{-itH} w_n^J|\|_{L_t^5 L_x^{\frac{30}{11}}} + C_{J'} \sum_{j=1}^{J'} \||v_n^j|^4 \nabla e^{-itH} w_n^J\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}}
\end{aligned}$$

By Strichartz and the decay of $e^{-itH} w_n^J$ in $L_{t,x}^{10}$, the first term goes to 0 as $J \to \infty,\ n \to \infty$. By Strichartz and the definition of J', the second term is bounded by

$$\eta^4 \|w_n^J\|_{\Sigma}$$

which can be made arbitrarily small since $\limsup_{n\to\infty} \|w_n^J\|_{\Sigma}$ is bounded uniformly in J. To finish, we show that for each fixed j

$$\lim_{J \to J^*} \limsup_{n \to \infty} \||v_n^j|^4 \nabla e^{-itH} w_n^J\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{11}}} = 0.$$

By Hölder,

$$\||v_n^j|^4 \nabla e^{-itH} w_n^J\|_{L_t^{\frac{5}{3}} L_x^{\frac{30}{23}}} \leq \|v_n^j\|_{L_{t,x}^{10}}^3 \|v_n^j \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}},$$

so by (2.69) it suffices to show

$$\lim_{J \to J^*} \limsup_{n \to \infty} \|v_n^j \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}} = 0. \tag{2.77}$$

For any $\varepsilon > 0$, there exists $\psi^j \in C_c^\infty(\mathbf{R}\times\mathbf{R}^3)$ and functions $c_n^j(t)$, $|c_n^j| \equiv 1$ such that

$$\limsup_{n\to\infty} \|v_n^j - c_n^j \tilde{G}_n^j \psi^j\|_{L^{10}_{t,x}([-\frac{1}{2},\frac{1}{2}])} < \varepsilon,$$

Note that $\tilde{G}_n^j \psi^j$ is supported on the set

$$\{|t-t_n^j| \lesssim (N_n^j)^{-2},\ |x-x_n^j| \lesssim (N_n^j)^{-1}\}.$$

Thus for all n sufficiently large,

$$\begin{aligned}
&\|v_n^j \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}} \\
&\leq \|v_n^j - c_n^j \tilde{G}_n^j \psi^j\|_{L^{10}_{t,x}} \|\nabla e^{-itH} w_n^J\|_{L_t^5 L_x^{\frac{30}{11}}} + \|\tilde{G}_n^j \psi^j \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}} \\
&\lesssim_{E_c} \varepsilon + \|(\tilde{G}_n^j \psi^j) \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}}.
\end{aligned}$$

From the definition of the operators $\tilde{G}_n^j$, we have

$$\|(\tilde{G}_n^j \psi^j)\nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}} \lesssim_\varepsilon N_n^{\frac{1}{2}} \|\nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}} (|t-t_n^j|\lesssim (N_n^j)^{-2}, |x-x_n^j| \lesssim (N_n^j)^{-1})}.$$

Corollary 2.2.10 implies

$$\|v_n^j \nabla e^{-itH} w_n^J\|_{L_t^{\frac{10}{3}} L_x^{\frac{15}{7}}} \lesssim \varepsilon + C_\varepsilon \|e^{-itH} w_n^J\|_{L^{10}_{t,x}}^{\frac{1}{9}} \|w_n^J\|_{\Sigma}^{\frac{8}{9}}.$$

Sending $n \to \infty$, then $J \to J^*$, then $\varepsilon \to 0$ establishes (2.77), and with it, Property (iii). This completes the treatment of the case $d = 3$.

By perturbation theory, $\limsup_{n\to\infty} S_{(-T,T)} \leq C(E_c) < \infty$, contrary to the Palais-Smale hypothesis. This rules out Case 2 and completes the proof of Proposition 2.6.1. □

2.7 Proof of Theorem 2.1.3

We begin by recalling some facts about the *ground state*

$$W(x) = (1 + \tfrac{|x|^2}{d(d-2)})^{-\frac{d-2}{2}} \in \dot{H}^1(\mathbf{R}^d)$$

This function satisfies the elliptic equation

$$\tfrac{1}{2}\Delta W + W^{\frac{4}{d-2}} W = 0.$$

It is well-known (c.f. Aubin [Aub76] and Talenti [Tal76]) that the functions witnessing the sharp constant in the Sobolev inequality

$$\|f\|_{L^{\frac{2d}{d-2}}(\mathbf{R}^d)} \leq C_d \|\nabla f\|_{L^2(\mathbf{R}^d)},$$

are precisely those of the form $f(x) = \alpha W(\beta(x - x_0)),\ \alpha \in \mathbf{C},\ \beta > 0,\ x_0 \in \mathbf{R}^d$.

For the reader's convenience, we reiterate the definitions of the energy associated to the focusing energy-critical NLS with and without potential:

$$E_\Delta(u) = \int_{\mathbf{R}^d} \tfrac{1}{2}|\nabla u|^2 - (1 - \tfrac{2}{d})|u|^{\frac{2d}{d-2}}\, dx,$$
$$E(u) = E_\Delta(u) + \tfrac{1}{2}\|xu\|_{L^2}^2.$$

Lemma 2.7.1 (Energy trapping [KM06])**.** *Suppose* $E_\Delta(u) \leq (1 - \delta_0) E_\Delta(W)$ *.*

- *Either* $\|\nabla u\|_{L^2} < \|\nabla W\|_{L^2}$ *or* $\|\nabla u\|_{L^2} > \|\nabla W\|_{L^2}$*.*
- *If* $\|\nabla u\|_{L^2} < \|\nabla W\|_{L^2}$*, then there exists* $\delta_1 > 0$ *depending on* δ_0 *such that*

  $$\|\nabla u\|_{L^2} \leq (1 - \delta_1)\|\nabla W\|_{L^2},$$

  and $E_\Delta(u) \geq 0$*.*
- *If* $\|\nabla u\|_{L^2} > \|\nabla W\|_{L^2}$ *then there exists* $\delta_2 > 0$ *depending on* δ_0 *such that*

  $$\|\nabla u\|_{L^2} \geq (1 + \delta_2)\|\nabla W\|_{L^2},$$

  and $\frac{1}{2}\|\nabla u\|_{L^2}^2 - \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} \leq -\delta_0 E_\Delta(W)$*.*

Now suppose $E(u) < E_\Delta(W)$ and $\|\nabla u\|_{L^2} \leq \|\nabla W\|_{L^2}$. The energy inequality can be written as

$$\|u\|_\Sigma^2 + (1 - \tfrac{2}{d})(\|W\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}}) \leq \|\nabla W\|_{L^2}^2.$$

By the variational characterization of W, the difference of norms on the left side is nonnegative; therefore

$$\|u\|_\Sigma \leq \|\nabla W\|_{L^2}.$$

Combining the above with conservation of energy and a continuity argument, we obtain

Corollary 2.7.2. *Suppose $u : I \times \mathbf{R}^d \to \mathbf{C}$ is a solution to the focusing equation* (2.1) *with $E(u) \leq (1-\delta_0) E_\Delta(W)$. Then there exist $\delta_1, \delta_2 > 0$, depending on δ_0, such that*

- *If $\|u(0)\|_{\dot{H}^1} \leq \|W\|_{\dot{H}^1}$, then*

$$\sup_{t \in I} \|u(t)\|_\Sigma \leq (1-\delta_1)\|W\|_{\dot{H}^1} \quad \textit{and} \quad E(u) \geq 0.$$

- *If $\|u(0)\|_{\dot{H}^1} \geq \|W\|_{\dot{H}^1}$, then*

$$\inf_{t \in I} \|u(t)\|_\Sigma \geq (1+\delta_2)\|W\|_{\dot{H}^1} \quad \textit{and} \quad \tfrac{1}{2}\|\nabla u\|_2^2 - \|u\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} \leq -\delta_0 E_\Delta(W).$$

Proof of Theorem 2.1.3. Let u be the maximal solution to (2.1) with

$$u(0) = u_0, \quad E(u_0) < E_\Delta(W), \quad \|\nabla u_0\|_2 \geq \|\nabla W\|_2.$$

Let $f(t) = \int_{\mathbf{R}^d} |x|^2 |u(t,x)|^2\, dx$. It can be shown [Caz03] that f is C^2 on the interval of existence and

$$f''(t) = \int |\nabla u(t,x)|^2 - 2|u(t,x)|^{\frac{2d}{d-2}} - \tfrac{1}{2}|x|^2|u(t,x)|^2\, dx.$$

By the corollary, f'' is bounded above by some fixed $C < 0$. Therefore

$$f(t) \leq A + Bt + \tfrac{C}{2}t^2$$

for some constants A and B. It follows that u has a finite lifespan in both time directions. □

2.8 Bounded linear potentials

In this section we show using a perturbative argument that

$$i\partial_t u = (-\tfrac{1}{2}\Delta + V)u + |u|^{\frac{4}{d-2}}u, \quad u(0) = u_0 \in H^1(\mathbf{R}^d) \tag{2.78}$$

is globally wellposed whenever V is a real-valued function with

$$V_{max} := \|V\|_{L^\infty} + \|\nabla V\|_{L^\infty} < \infty.$$

This equation defines the Hamiltonian flow of the energy functional

$$E(u(t)) = \int_{\mathbf{R}^d} \tfrac{1}{2}|\nabla u(t,x)|^2 + V|u(t,x)|^2 + \tfrac{d-2}{d}|u|^{\frac{2d}{d-2}}\, dx = E(u(0)). \qquad (2.79)$$

Solutions to (2.78) also conserve *mass*:

$$M(u(t)) = \int_{\mathbf{R}^d} |u(t,x)|^2\, dx = M(u(0)).$$

It will be convenient to assume V is positive and bounded away from 0. This hypothesis allows us to bound the H^1 norm of u purely in terms of E instead of both E and M, and causes no loss of generality because for sign-indefinite V we could simply consider the conserved quantity $E + CM$ for some positive constant C.

Theorem 2.8.1. *For any $u_0 \in H^1(\mathbf{R}^d)$, (2.78) has a unique global solution $u \in C^0_{t,loc}H^1_x(\mathbf{R}\times\mathbf{R}^d)$. Further, u obeys the spacetime bounds*

$$S_I(u) \leq C(\|u_0\|_{H^1}, |I|)$$

for any compact interval $I \subset \mathbf{R}$.

The proof follows the strategy pioneered by [TVZ07] and treats the term Vu as a perturbation to (2.6), which is globally wellposed. Thus Duhamel's formula reads

$$u(t) = e^{\frac{it\Delta}{2}}u(t_0) - i\int_0^t e^{\frac{i(t-s)\Delta}{2}}[|u(s)|^{\frac{4}{d-2}}u(s) + Vu(s)]ds. \qquad (2.80)$$

We record mostly without proof some standard results in the local theory of (2.78). Introduce the notation

$$\|u\|_{X(I)} = \|\nabla u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}} (I\times\mathbf{R}^d)}.$$

Lemma 2.8.2 (Local wellposedness)**.** *Fix $u_0 \in H^1(\mathbf{R}^d)$, and suppose $T_0 > 0$ is such that*

$$\|e^{\frac{it\Delta}{2}}u_0\|_{X([-T_0,T_0])} \leq \eta \leq \eta_0$$

where $\eta_0 = \eta_0(d)$ is a fixed parameter. Then there exists a positive

$$T_1 = T_1(\|u_0\|_{H^1}, \eta, V_{max})$$

such that (2.78) *has a unique (strong) solution* $u \in C_t^0 H_x^1([-T_1, T_1] \times \mathbf{R}^d)$. *Further, if* $(-T_{min}, T_{max})$ *is the maximal lifespan of* u, *then* $\|\nabla u\|_{S(I)} < \infty$ *for every compact interval* $I \subset (-T_{min}, T_{max})$, *where* $\|\cdot\|_{S(I)}$ *is the Strichartz norm defined in Section 2.2.1.*

Proof sketch. Run the usual contraction mapping argument using the Strichartz estimates to show that

$$\mathcal{I}(u)(t) = e^{\frac{it\Delta}{2}} u_0 - i\int_0^t e^{\frac{i(t-s)\Delta}{2}} [|u(s)|^{\frac{4}{d-2}} u(s) + Vu(s)] dx$$

has a fixed point in a suitable function space. Estimate the terms involving V in the $L_t^1 L_x^2$ dual Strichartz norm and choose the parameter T_1 to make those terms sufficiently small after using Hölder in time. □

Lemma 2.8.3 (Blowup criterion)**.** *Let* $u : (T_0, T_1) \times \mathbf{R}^d \to \mathbf{C}$ *be a solution to* (2.78) *with*

$$\|u\|_{X((T_0,T_1))} < \infty.$$

If $T_0 > -\infty$ *or* $T_1 < \infty$, *then* u *can be extended to a solution on a larger time interval.*

Our argument uses the stability theory for the energy-critical NLS (2.6).

Lemma 2.8.4 (Stability [TV05])**.** *Let* $\tilde{u} : I \times \mathbf{R}^d \to \mathbf{C}$ *be an approximate solution to equation* (2.6) *in the sense that*

$$i\partial_t \tilde{u} = -\tfrac{1}{2}\Delta u \pm |\tilde{u}|^{\frac{4}{d-2}} \tilde{u} + e$$

for some function e. *Assume that*

$$\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \le L, \quad \|\nabla u\|_{L_t^\infty L_x^2} \le E, \tag{2.81}$$

and that for some $0 < \varepsilon < \varepsilon_0(E, L)$,

$$\|\tilde{u}(t_0) - u_0\|_{\dot{H}^1} + \|\nabla e\|_{N(I)} \le \varepsilon, \tag{2.82}$$

where $\|\cdot\|_{N(I)}$ *was defined in Section 2.2.1. Then there exists a unique solution* $u : I \times \mathbf{R}^d \to \mathbf{C}$ *to* (2.6) *with* $u(t_0) = u_0$ *which further satisfies the estimates*

$$\|\tilde{u} - u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|\nabla(\tilde{u} - u)\|_{S(I)} \le C(E, L)\varepsilon^c \tag{2.83}$$

where $0 < c = c(d) < 1$ *and* $C(E, L)$ *is a function which is nondecreasing in each variable.*

Proof of Theorem 2.8.1. It suffices to show that for T sufficiently small depending only on $E = E(u_0)$, the solution u to (2.78) on $[0,T]$ satisfies an *a priori* estimate

$$\|u\|_{X([0,T])} \leq C(E). \tag{2.84}$$

From Lemma 2.8.3 and energy conservation, it will follow that u is a global solution with the desired spacetime bound.

By Theorem 2.1.1, the equation

$$(i\partial_t + \tfrac{1}{2}\Delta)w = |w|^{\frac{4}{d-2}}w, \quad w(0) = u(0).$$

has a unique global solution $w \in C^0_{t,loc}\dot{H}^1_x(\mathbf{R} \times \mathbf{R}^d)$ with the spacetime bound (2.7). Fix a small parameter $\eta > 0$ to be determined shortly, and partition $[0,\infty)$ into $J(E,\eta)$ intervals $I_j = [t_j, t_{j+1})$ so that

$$\|w\|_{X(I_j)} \leq \eta. \tag{2.85}$$

For some $J' < J$, we then have

$$[0,T] = \bigcup_{j=0}^{J'-1} ([0,T] \cap I_j).$$

We make two preliminary estimates. By Hölder in time,

$$\|Vu\|_{N(I_j)} + \|\nabla(Vu)\|_{N(I_j)} \lesssim C_V T \|u\|_{L^\infty_t H^1_x(I_j)} \leq \varepsilon \tag{2.86}$$

for any ε provided that $T = T(E, V, \varepsilon)$ is sufficiently small. Further, observe that

$$\|e^{\frac{i(t-t_j)\Delta}{2}} w(t_j)\|_{X(I_j)} \leq 2\eta \tag{2.87}$$

for η sufficiently small depending only on d. Indeed, from the Duhamel formula

$$w(t) = e^{\frac{i(t-t_j)\Delta}{2}} w(t_j) - i\int_{t_j}^{t} e^{\frac{i(t-s)\Delta}{2}} (|w|^{\frac{4}{d-2}}w)(s)ds,$$

Strichartz, and the chain rule, it follows that

$$\begin{aligned}
\|e^{\frac{i(t-t_j)\Delta}{2}} w(t_j)\|_{X(I_j)} &\leq \|w\|_{X(I_j)} + c_d \|\nabla(|w|^{\frac{4}{d-2}}w)\|_{L^2_t L^{\frac{2d}{d+2}}_x (I_j)} \\
&\leq \eta + c_d \|w\|_{X(I_j)}^{\frac{d+2}{d-2}} \\
&\leq \eta + c_d \eta^{\frac{d+2}{d-2}}.
\end{aligned}$$

Choosing η sufficiently small relative to c_d yields (2.87).

Take $\varepsilon < \eta$ in (2.86) (taking T small) and apply the Duhamel formula (2.80), Strichartz, Hölder, and (2.87) to obtain

$$\begin{aligned}
\|u\|_{X(I_0)} &\le \|e^{\frac{it\Delta}{2}} u(0)\|_{X(I_0)} + c_d \|u\|_{X(I_0)}^{\frac{d+2}{d-2}} + C\|Vu\|_{L_t^1 H_x^1(I_0)} \\
&\le 2\eta + c_d \|u\|_{X(I_0)}^{\frac{d+2}{d-2}} + C_V T \|u\|_{L_t^\infty H_x^1(I_0)} \\
&\le 3\eta + c_d \|u\|_{X(I_0)}^{\frac{d+2}{d-2}}.
\end{aligned}$$

By a continuity argument,

$$\|u\|_{X(I_0)} \le 4\eta. \tag{2.88}$$

Choose ε sufficiently small in (2.86) so that the smallness condition (2.82) is satisfied, and invoke Lemma 2.8.4 with $\|u(0) - w(0)\|_{\dot{H}^1} = 0$ to find that

$$\|\nabla(u - w)\|_{S(I_0)} \le C(E)\varepsilon^c. \tag{2.89}$$

On the interval I_1, use (2.87), (2.89), and the usual estimates to obtain

$$\begin{aligned}
\|u\|_{X(I_1)} &\le \|e^{\frac{i(t-t_1)\Delta}{2}} u(t_1)\|_{X(I_1)} + c_d \|u\|_{X(I_1)}^{\frac{d+2}{d-2}} + C_V T \|u\|_{L_t^\infty H_x^1(I_1)} \\
&\le C(E)\varepsilon^c + 2\eta + c\|u\|_{X(I_1)}^{\frac{d+2}{d-2}} + \eta,
\end{aligned}$$

where the $C(E)$ in the last line has absorbed the Strichartz constant c; this redefinition of $C(E)$ will cause no trouble because the number of times it will occur depends only on E, d, and V. By taking ε sufficiently small relative to η and using continuity, we see that

$$\|u\|_{X(I_1)} \le 4\eta.$$

As before, taking T sufficiently small yields

$$\begin{aligned}
\|\nabla(Vu)\|_{\dot{N}^0(I_1)} &\le \varepsilon \\
\|e^{\frac{i(t-t_1)\Delta}{2}} [u(t_1) - w(t_1)]\|_{X(I_1)} &\le C(E)\varepsilon^c
\end{aligned}$$

for any $\varepsilon \leq \varepsilon_0(E, L)$. Therefore by Lemma 2.8.4,

$$\|\nabla(u - w)\|_{S(I_1)} \leq C(E)\varepsilon^c.$$

The parameters η, ε, T are chosen so that each depends only on the preceding parameters and on the fixed quantities d, E, V. After iterating at most J' times and summing the bounds over $0 \leq j \leq J' - 1$, we conclude that for T sufficiently small depending on E and V,

$$\|u\|_{X([0,T])} \leq 4J'\eta \leq C(E).$$

This establishes the bound (2.84). □

CHAPTER 3

Extensions to more general potentials

3.1 Introduction

In this chapter, we describe the modifications to extend the results for the harmonic oscillator to a broader class of potentials. Consider the equation

$$\begin{cases} i\partial_t u = (-\tfrac{1}{2}\Delta + V)u + \mu |u|^{\frac{4}{d-2}} u, \quad \mu = \pm 1, \\ u(0) = u_0 \in \Sigma(\mathbf{R}^d), \end{cases} \tag{3.1}$$

whose flow preserves the energy

$$E(u(t)) = \int_{\mathbf{R}^d} \tfrac{1}{2}|\nabla u(t)|^2 + V|u(t)|^2 + \mu(1 - \tfrac{2}{d})|u(t)|^{\frac{2d}{d-2}}\, dx = E(u(0)).$$

The equation is *defocusing* if $\mu = 1$ and *focusing* if $\mu = -1$. We consider the following three assumptions on V:

(V1) $V = V(x)$ is smooth and nonnegative.

(V2) For each $k \geq 2$ there exists a constant $M_k > 0$ such that

$$\sup_{|x| \leq 1} |V(x)| + \sup_{x \in \mathbf{R}^d} |\partial^k V(x)| \leq M_k \quad \text{for all} \quad k \geq 2.$$

(V3) $V(x) \geq \delta |x|^2$ for some $\delta > 0$.

These hypotheses on V ensure that

$$\delta |x|^2 \leq V(x) \leq \delta^{-1}(1 + |x|^2)$$

for some constant $\delta > 0$. Therefore Σ is the energy space, and is also the form domain $Q(H) = D(H^{1/2})$ for the positive operator $H = -\frac{1}{2}\Delta + V$. It will sometimes be more convenient to work with the equivalent norm

$$\|f\|_{Q(H)} := \|H^{1/2} f\|_{L^2} = (\|\nabla f\|_{L^2}^2 + \|V^{1/2} f\|_{L^2}^2)^{1/2},$$

because it is preserved by the propagator e^{-itH}.

As before, the term "energy-critical" refers to the fact that if one ignores the potential V in the equation, the equation

$$\begin{aligned} (i\partial_t + \tfrac{1}{2}\Delta)u &= \mu |u|^{\frac{4}{d-2}} u, \quad u(0) \in \dot{H}^1(\mathbf{R}^d) \\ E_\Delta(u) &= \int_{\mathbf{R}^d} \tfrac{1}{2}|\nabla u|^2 + \mu(1 - \tfrac{2}{d})|u|^{\frac{2d}{d-2}}\, dx \end{aligned} \tag{3.2}$$

is invariant under the scaling $u \mapsto u^\lambda(t,x) = \lambda^{-\frac{d-2}{2}} u(\lambda^{-2}t, \lambda^{-1}x)$. Roughly speaking, if a solution u to (3.1) is concentrated in a width λ neighborhood of some point x_0, where $\lambda \ll 1$, it sees the potential V as approximately equal to $V(x_0)$ and behaves for $|t| \le O(\lambda^2)$ as a solution to the equation (3.2). The asymptotic behavior for the constant coefficient problem (3.2) was summarized in Theorem 2.1.1. This result will be a basic stepping stone to understanding the behavior of concentrated solutions to the variable-coefficient problem.

Corresponding to Conjecture 2.1.2 we have

Conjecture 3.1.1. *When $\mu = 1$, equation (3.1) is globally wellposed. That is, for each $u_0 \in Q(H)$ there is a unique global solution $u : \mathbf{R} \times \mathbf{R}^d \to \mathbf{C}$ with $u(0) = u_0$. This solution obeys the spacetime bound*

$$S_I(u) := \int_I \int_{\mathbf{R}^d} |u(t,x)|^{\frac{2(d+2)}{d-2}}\, dx\, dt \le C(|I|, \|u_0\|_\Sigma) \tag{3.3}$$

for any compact interval $I \subset \mathbf{R}$.

If $\mu = -1$, then the same is true provided also that

$$E(u_0) < E_\Delta(W) \quad \textit{and} \quad \|\nabla u_0\|_{L^2} \le \|\nabla W\|_{L^2}.$$

The restriction on the kinetic energy focusing case is necessary, for as in the case of the harmonic oscillator, we have:

Theorem 3.1.1. *If $\mu = -1$, $E(u_0) < E_\Delta(W)$, and $\|\nabla u_0\|_{L^2} \le \|\nabla W\|_{L^2}$, then the solution to*

To prove this one need only make notational changes to the discussion in Section 2.8, and we refer the reader to there for details.

Just as with Theorem 2.1.2; by Theorem 2.1.1, however, it is actually unconditional except in the focusing case for nonradial data in dimensions $d = 3$ and 4.

Theorem 3.1.2. *Assume Conjecture 2.1.1. Then Conjecture 3.1.1 holds.*

Besides the literature surveyed in Chapter 2, this result also has a predecessor in the work of Carles [Car11], who considered a large class of subquadratic potentials for the energy-subcritical problem

$$i\partial_t u = (-\tfrac{1}{2}\Delta + V)u + \mu|u|^p u, \ p < \tfrac{4}{d-2}.$$

Taking initial data in Σ, he established global wellposedness in the defocusing case when $4/d \le p < 4/(d-2)$ and in the focusing case when $0 < p < 4/d$. Carles did not require that V be bounded from below, and also allowed $V = V(t,x)$ to depend on time. Prior to that work, Oh [Oh89] had shown large data global existence in the focusing case when $p < 4/d$ and the potential is time-independent and subquadratic. We consider a more restricted class of potentials but focus on the subtleties of the critical exponent $p = 4/(d-2)$ described in the introduction of the previous chapter.

To prove Theorem (3.1) we apply the induction on energy paradigm and closely follow the arguments for the harmonic oscillator, save for one key difference. In constructing the linear profile decomposition it was essential to compare the Schrödinger flows for the harmonic oscillator and the free particle on concentrated initial data. We exploited the classical formula for the fundamental solution:

$$e^{-itH}(x,y) = \tfrac{1}{(2\pi i \sin t)^{d/2}} e^{\frac{i}{\sin t}(\frac{x^2+y^2}{2}\cos t - xy)} \quad \text{(Mehler's Formula [Fol89]).} \tag{3.4}$$

No such explicit formula is available for more general potentials. Instead we appeal to more robust techinques from microlocal analysis.

Fujiwara showed [Fuj79, Fuj80] that for a general class of subquadratic potentials, the unitary propagator e^{-itH} can be represented for small t as an oscillatory integral operator

$$e^{-itH}f(x) = \tfrac{1}{(2\pi it)^{d/2}} \int_{\mathbf{R}^d} k(t,x,y)e^{iS(t,x,y)}f(y)\,dy,$$

where for small t the integral kernel is close to that of the free propagator $e^{\frac{it\Delta}{2}}$. In fact, Fujiwara considered time-dependent potentials. Using this representation, we are able to obtain suitable replacements for the arguments in Chapter 2 that relied on the precise form of Mehler's formula. A key step is to prove that in the relevant region of phase space, the bicharacteristics for the symbol $h(x,\xi) = \frac{1}{2}|\xi|^2 + V(x)$ are well-approximated over short time intervals by those of $h(x,\xi) = \frac{1}{2}|\xi|^2$.

Chapter outline

In Section 3.2 we set our notation and review some basic estimates regarding equation (3.1). We also recall Fujiwara's construction of the propagator as a Fourier integral operator and collect some relevant background concerning such operators. Section 3.3 states some standard (but essential) local theory. Section 3.4 discusses the linear profile decomposition mentioned above, focusing on how to modify the arguments that previously invoked Mehler's formula.

The scaling limit analysis of Section 3.5 and the compactness arguments of Section 3.6 parallel the ones given in Chapter 2. As will be the case throughout the chapter, we describe mainly the required adjustments and refer to the previous chapter for the rest of the details.

Acknowledgments

The author is indebted to his advisors Rowan Killip and Monica Visan for their helpful discussions as well as their feedback the manuscript. This work was supported in part by NSF grants DMS-0838680 (RTG), DMS-1265868 (PI R. Killip), DMS-0901166, and DMS-1161396 (both PI M. Visan).

3.2 Preliminaries

3.2.1 Notation and basic estimates

We write $X \lesssim Y$ to mean $X \leq CY$ for some constant C. Similarly $X \sim Y$ means $X \lesssim Y$ and $Y \lesssim X$. Denote by $L^p(\mathbf{R}^d)$ the Banach space of functions $f : \mathbf{R}^d \to \mathbf{C}$ with finite norm

$$\|f\|_{L^p(\mathbf{R}^d)} = \left(\int_{\mathbf{R}^d} |f|^p \, dx \right)^{\frac{1}{p}}.$$

Sometimes we use the more compact notation $\|f\|_p$. If $I \subset \mathbf{R}^d$ is an interval, the mixed Lebesgue norms on $I \times \mathbf{R}^d$ are defined by

$$\|f\|_{L_t^q L_x^r(I \times \mathbf{R}^d)} = \left(\int_I \left(\int_{\mathbf{R}^d} |f(t,x)|^r \, dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}} = \|f(t)\|_{L_t^q(I; L_x^r(\mathbf{R}^d))},$$

where one regards $f(t) = f(t, \cdot)$ as a function from I to $L^r(\mathbf{R}^d)$.

Throughout the chapter we shall use the capital letters D and X to denote the operators $f \mapsto -i\partial f$ and $f \mapsto xf$, respectively.

Introduce the following function spaces

$$\mathcal{B}_k(\mathbf{R}^n) = \{ f \in C^\infty(\mathbf{R}^n) : D^\ell f \in L^\infty \text{ for all } \ell \geq k \},$$
$$\mathcal{B}(\mathbf{R}^n) = \mathcal{B}_0(\mathbf{R}^n)$$

The notation $\mathcal{B}(\mathbf{R}^d \times \mathbf{R}^d)$ goes back to Schwartz [Sch66] and is equivalent to the more modern notation $S^0_{0,0}$, where in general $S^k_{\rho,\delta}(\mathbf{R}^d \times \mathbf{R}^d)$ denotes the smooth symbols on $\mathbf{R}^d_x \times \mathbf{R}^d_\xi$ such that

$$D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m - |\alpha|\rho + |\beta|\delta}$$

for all multiindices α and β. Note, however, that in general $\mathcal{B}_k$ does not coincide with any $S^k_{\rho,\delta}$.

We recall Fujiwara's construction of the propagator for H. The symbol

$$h(\xi, x) = \tfrac{1}{2}|\xi|^2 + V(x)$$

defines the Hamiltonian flow

$$\begin{cases} \dot{x} = \nabla_\xi h, \quad \dot{\xi} = -\nabla_x h \\ \quad (x(0), \xi(0)) = (y, \eta). \end{cases} \tag{3.5}$$

The integral curves $(x(t), \xi(t))$ are the *bicharacteristics* for h. Suppose that V is subquadratic in the sense in the sense of hypothesis (V2). Then the vector field $(-\nabla_x h, \nabla_\xi h)$ is Lipschitz, hence complete, so x and ξ are well-defined functions $x(t, y, \eta),\ \xi(t, y, \eta)$ of t and the initial data.

Proposition 3.2.1 ([Fuj79, Proposition 1.7]). *Let V be a potential such that*

$$\sup_{|x|\le 1} |V(x)| + \sup_{x\in \mathbf{R}^d} |\partial^k V(x)| < \infty \text{ for all } k \ge 2,$$

and put $H(\xi, x) = \frac{1}{2}|\xi|^2 + V(x)$. Then the map $(y, \eta) \mapsto (x, y)$ obeys the derivative estimates

$$\tfrac{\partial x}{\partial y} = I + t^2 a(t, y, \eta), \quad \tfrac{\partial x}{\partial \eta} = t(I + t^2 b(t, y, \eta))$$

for some matrix-valued $a, b \in \mathcal{B}(\mathbf{R}^d_y \times \mathbf{R}^d_\eta)$.

Further, there exists δ_0 such that whenever $0 \ne |t| \le \delta_0$, for pairs $x, y \in \mathbf{R}^d$ there is a unique trajectory $(x(\tau), \xi(\tau))$ such that $x(0) = y$ and $x(t) = x$.

Remark. To get the second statement from the first, one invokes the Hadamard global inverse function theorem to see that $(y, \eta) \mapsto (x, y)$ is a diffeomorphism for $0 \ne t$ sufficiently small.

According to this result, when $|t| \le \delta_0$ and $t \ne 0$ one can define the *action*

$$S(t, x, y) = \int_0^t \frac{1}{2}|\xi(\tau)|^2 - V(x(\tau))\, d\tau, \tag{3.6}$$

where $(x(\tau), \xi(\tau))$ is the unique bicharacteristic with $x(0) = y$ and $x(t) = x$.

Theorem 3.2.2 (Fundamental solution [Fuj79, Fuj80]). *Let V be subquadratic as in the previous proposition. Then there exists $\delta_0 > 0$ such that:*

- *The action $S(t,x,y)$ is well-defined by (3.6) for all $0<|t|<\delta_0$ and satisfies*

$$S(t,x,y)=\tfrac{|x-y|^2}{2t}+t\omega(t,x,y),$$

where the term $\omega(t,\cdot,\cdot)$ belongs to $\mathcal{B}_2$ uniformly for $|t|\le\delta_0$. That is, there exist constants C_k such that

$$|D^k_{x,y}\omega(t,x,y)|\le C_k(1+|x|+|y|)^{\max(2-k,0)}$$

for all k.

- *For all $0<|t|<\delta_0$ and all $f\in C_c^\infty(\mathbf{R}^d)$ we have*

$$e^{-itH}f(x)=\tfrac{1}{(2\pi it)^{d/2}}\int_{\mathbf{R}^d}e^{iS(t,x,y)}a(t,x,y)f(y)\,dy,$$

where

$$\|D^k_{x,y}[a(t,\cdot,\cdot)-1]\|_{L^\infty(\mathbf{R}^d_x\times\mathbf{R}^d_y)}=O_k(t^2)\quad\textit{for all}\quad k\ge 0.$$

The above integral representation immediately yields a dispersive estimate:

Corollary 3.2.3 (Dispersive estimate)**.** *For $|t|\le\delta_0$, we have*

$$\|e^{-itH}f\|_\infty\lesssim|t|^{-\frac{d}{2}}\|f\|_1.$$

We call a pair (q,r) *admissible* if $q\ge 2$ and $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$. Define the norm

$$\|u\|_{S(I)}:=\|u\|_{L^2_tL_x^{\frac{2d}{d-2}}(I\times\mathbf{R}^d)}+\|u\|_{L^\infty_tL^2_x(I\times\mathbf{R}^d)}.$$

By interpolation, this norm controls $\|u\|_{L^q_tL^r_x}$ for all admissible pairs (q,r). Define

$$\|F\|_{N(I)}=\inf\{\|F_1\|_{L_t^{q_1'}L_x^{r_1'}}+\|F_2\|_{L_t^{q_2'}L_x^{r_2'}}:(q_k,r_k)\text{ admissible, }F=F_1+F_2\}$$

where (q_k',r_k') denotes the Hölder dual of (q_k,r_k).

Lemma 3.2.4 (Strichartz [KT98])**.** *Let I be a compact time interval containing t_0, and let $u:I\times\mathbf{R}^d\to\mathbf{C}$ be a solution to the inhomogeneous Schrödinger equation*

$$(i\partial_t-H)u=F.$$

Then there is a constant C, depending only on the length of the interval I, such that

$$\|u\|_{S(I)}\le C(\|u_0\|_{L^2}+\|F\|_{N(I)}).$$

Proof. This follows from [KT98] as a consequence of two ingredients: the dispersive estimate of the previous corollary, and the unitarity of e^{-itH} on $L^2(\mathbf{R}^d)$. $\square$

As V is nonnegative, we have access to the spectral multipler theorem of Hebisch [Heb90]:

Theorem 3.2.5. *If $F : (0,\infty) \to \mathbf{C}$ is a bounded function which obeys the derivative estimates*

$$|\partial^k F(\lambda)| \lesssim_k |\lambda|^{-k} \quad \textit{for all} \quad 0 \le k \le \tfrac{d}{2} + 1,$$

then the operator $F(H)$, defined initially on L^2 by the Borel functional calculus, is bounded from L^p to L^p for all $1 < p < \infty$.

The following equivalence of Sobolev norms was first proven for the quadratic potential by Killip-Visan-Zhang [KVZ09, Lemma 2.7]. We adapt their result to the potentials considered here.

Proposition 3.2.6 (Equivalence of norms)**.** *For any $1 < p < \infty$ and $s \in [0,1]$, we have*

$$\|H^s f\|_p \sim_{p,s} \|(-\Delta)^s f\|_p + \|V^s f\|_p$$

for all Schwartz functions f.

The proof uses the following fact, which is classical when V is quadratic; we verify it at the end for the sake of completeness.

Lemma 3.2.7. *Let $H = -\frac{1}{2}\Delta + V$ where V satisfies the hypotheses (V1) through (V3). Then the smooth vectors for H are precisely the Schwartz functions:*

$$D(H^\infty) := \bigcap_{n\ge 0} D(H^n) = \mathcal{S}(\mathbf{R}^d).$$

Proof of Proposition 3.2.6. We show first that

$$\|(-\Delta)^s f\|_p + \|V^s f\|_p \lesssim_p \|H^s f\|_p \quad \text{for all} \quad f \in \mathcal{S}(\mathbf{R}^d). \tag{3.7}$$

As $f = H^{-s}H^s f$ and $H^s f \in \mathcal{S}(\mathbf{R}^d)$ by Lemma 3.2.7, it suffices to prove

$$\|(-\Delta)^s H^{-s} f\|_p + \|V^s H^{-s} f\|_p \lesssim_p \|f\|_p \quad \text{for all} \quad f \in \mathcal{S}(\mathbf{R}^d). \tag{3.8}$$

By hypothesis, there is some $\delta > 0$ such that $V(x) \geq \delta|x|^2$. Killip-Visan-Zhang [KVZ09] showed that

$$\|V^s H_\delta^{-s} f\|_p \lesssim_p \|f\|_p,$$

where $H_\delta = -\frac{1}{2}\Delta + \delta|x|^2$. On the other hand, the parabolic maximium principle implies

$$0 \leq e^{-tH}(x,y) \leq e^{-tH_\delta}(x,y)$$

Combining this with the identity

$$H^{-s}(x,y) = \tfrac{1}{\Gamma(s)} \int_0^\infty e^{-tH}(x,y) t^{s-1} dt,$$

we obtain the kernel inequality

$$0 \leq H^{-s}(x,y) \leq H_\delta^{-s}(x,y)$$

In particular, $V^s H^{-s}$ and $V^s H_\delta^{-s}$ have nonnegative integral kernels. We may therefore bound

$$\|V^s H^{-s} f\|_p \leq \|V^s H^{-s}|f|\|_p \leq \|V^s H_\delta^{-s}|f|\|_p \lesssim_p \|f\|_p.$$

This yields half of (3.8). Specializing to the case $s = 1$ and writing $-\Delta = 2(H - V)$, we obtain

$$\|(-\Delta)H^{-1} f\|_p \lesssim_p \|f\|_p. \tag{3.9}$$

Using Theorem 3.2.5 and the Stein-Weiss interpolation theorem applied to the analytic family $(-\Delta)^z H^{-z}$, we obtain (3.7) and (3.8) for all $p \in (1,\infty)$ and $s \in [0,1]$.

Dualizing those estimates yields

$$\|H^{-s}(-\Delta)^s f\|_p + \|H^{-s} V^s f\|_p \lesssim_{p,s} \|f\|_p \quad \text{for all} \quad p \in (1,\infty),\ s \in [0,1]$$

Writing $H^s f = H^{s-1} H f = \frac{1}{2} H^{s-1}(-\Delta)^{1-s}(-\Delta)^s f + H^{s-1} V^{1-s} V^s f$, we have

$$\|H^s f\|_p \lesssim_{p,s} \|(-\Delta)^s f\|_p + \|V^s f\|_p \quad \text{for all} \quad p \in (1,\infty),\ s \in [0,1].$$

This completes the proof of the proposition modulo the lemma. □

Proof of Lemma 3.2.7. The inclusion $\mathcal{S}(\mathbf{R}^d) \subset D(H^\infty)$ is clear. To prove the opposite inclusion, we show by an induction argument the equivalent assertion that

$$D(H^\infty) \subset \bigcap_{k\geq 0} \{u : x^\alpha \partial^\beta u \in L^2 \text{ for all } |\alpha| + |\beta| \leq k\}. \tag{3.10}$$

We have the following identities:

$$\begin{aligned} H\partial_j u &= \partial_j H u - (\partial_j V)u \\ Hmu &= mHu - \tfrac{1}{2}(\Delta m)u - \nabla m \cdot \nabla u \end{aligned} \tag{3.11}$$

Define for each $n \geq 1$ the following statements:

$$P_1(n) = \text{“}m : D(H^{n-1}) \to D(H^{n-1}) \text{ for all } m \in \mathcal{B}\text{”}$$

$$P_2(n) = \text{“}\partial_j : D(H^n) \to D(H^{n-1})\text{”}$$

$$P_3(n) = \text{“}\partial_j V : D(H^n) \to D(H^{n-1})\text{”}.$$

As $D(H) \subset D(H^{1/2}) = \{u : \|\nabla u\|_{L^2} + \|xu\|_{L^2} < \infty\}$, these hold for $n = 1$.

Assume that they hold for some n. For $u \in D(H^n)$ and $m \in \mathcal{B}$, use (3.11) and the statements $P_1(n),\ P_2(n)$ to see that $H(mu) \in D(H^{n-1})$, so $mu \in D(H^n)$ and $P_1(n+1)$ holds since m was chosen arbitrarily in $\mathcal{B}$. Similar reasoning shows that $P_2(n)$ and $P_3(n)$ imply $P_2(n+1)$, and that $P_1(n),\ P_2(n),\ P_3(n)$ yield $P_3(n+1)$. Hence, by induction these statements hold for all $n \geq 1$.

Next, apply (3.8) in the special case $s = 1,\ p = 2$ to see that

$$V : D(H) \to D(H^0) = L^2.$$

Suppose $u \in D(H^n)$ and $n \geq 2$. We have

$$H(Vu) = VHu - \tfrac{1}{2}(\Delta V)u - \nabla V \cdot \nabla u.$$

By induction, $VHu \in D(H^{n-2})$, while $P_1(n),\ P_2(n)$, and $P_3(n-1)$ imply that the second and third terms also belong to $D(H^{n-2})$. Thus $Vu \in D(H^{n-1})$

Summing up, we find that

$$V : D(H^n) \to D(H^{n-1}) \text{ for all } n \geq 1.$$

Together with the coercivity hypothesis (V3), these mapping properties yield the claim (3.10).

□

By the equivalence of norms, H^γ inherits many properties of the fractional derivative $(-\Delta)^\gamma$, including Sobolev embedding:

Lemma 3.2.8 ([KVZ09, Lemma 2.8]). *Suppose* $\gamma \in [0,1]$ *and* $1 < p < \frac{d}{2\gamma}$, *and define* p^* *by* $\frac{1}{p^*} = \frac{1}{p} - \frac{2\gamma}{d}$. *Then*

$$\|f\|_{L^{p^*}(\mathbf{R}^d)} \lesssim \|H^\gamma f\|_{L^p(\mathbf{R}^d)}.$$

Similarly, the fractional chain and product rules carry over to the present setting:

Corollary 3.2.9 ([KVZ09, Proposition 2.10]). *Let* $F(z) = |z|^{\frac{4}{d-2}} z$. *For any* $0 \leq \gamma \leq \frac{1}{2}$ *and* $1 < p < \infty$,

$$\|H^\gamma F(u)\|_{L^p(\mathbf{R}^d)} \lesssim \|F'(u)\|_{L^{p_0}(\mathbf{R}^d)} \|H^\gamma f\|_{L^{p_1}(\mathbf{R}^d)}$$

for all $p_0, p_1 \in (1, \infty)$ *with* $p^{-1} = p_0^{-1} + p_1^{-1}$.

Using Proposition 3.2.6 and the Christ-Weinstein fractional product rule for $(-\Delta)^\gamma$ (e.g. [Tay00]), we obtain

Corollary 3.2.10. *For* $\gamma \in (0,1]$, $r, p_i, q_i \in (1, \infty)$ *with* $r^{-1} = p_i^{-1} + q_i^{-1}$, $i = 1, 2$, *we have*

$$\|H^\gamma (fg)\|_r \lesssim \|H^\gamma f\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \|H^\gamma g\|_{q_2}.$$

3.2.2 Microlocal technology

We recall some properties of Fourier integral operators tailored to the Schrödinger equation. The operators we shall use were developed by Fujiwara [Fuj75] and Asada-Fujiwara [AF78].

Definition 3.2.1. Call $\phi \in \mathcal{B}_2(\mathbf{R}^d_x \times \mathbf{R}^d_y)$ a *phase function* satisfies the nondegeneracy condition

$$\inf_{x,y} |\det D^2_{xy} \phi(x,y)| > 0. \tag{3.12}$$

Given a phase $\phi(x,y)$ and an amplitude $a(x,y) \in \mathcal{B}(\mathbf{R}^d_x \times \mathbf{R}^d_y)$, define for each $\lambda \neq 0$ the integral operator

$$A(\lambda)f(x) = \left(\frac{\lambda}{2\pi i}\right)^{\frac{d}{2}} \int_{\mathbf{R}^d} e^{i\lambda\phi(x,y)} a(x,y) f(y) dy. \tag{3.13}$$

Remark. Asada and Fujiwara studied Fourier integral operators in the more general form

$$f \mapsto \left(\frac{\lambda}{2\pi i}\right)^{\frac{m+n}{2}} \int_{\mathbf{R}^m} \int_{\mathbf{R}^n} e^{i\phi(x,\theta,y)} a(x,\theta,y) f(y)\, dy d\theta$$

where $a(x,\theta,y) \in \mathcal{B}(\mathbf{R}^n_x \times \mathbf{R}^m_\theta \times \mathbf{R}^n_y)$ and the phase ϕ satisfies the nondegeneracy condition

$$\left| \det \begin{pmatrix} D^2_{xy}\phi & D^2_{x\theta}\phi \\ D^2_{\theta y}\phi & D^2_{\theta\theta}\phi \end{pmatrix} \right| \geq \delta$$

The integral operator $A(\lambda)$ considered above corresponds to the case $m = 0$.

The operator $A(\lambda)$ is bounded on L^2. More precisely, Fujiwara proved:

Theorem 3.2.11 (Fujiwara [Fuj75])**.** $\|A(\lambda)\|_{L^2 \to L^2} \leq C\|a\|_{C^{2d+1}}$

Let ϕ be a phase function. By the global inverse function theorem, the maps

$$\chi_1(x,y) = (y, -\partial_y\phi) \quad \text{and} \quad \chi_2(x,y) = (x, \partial_x\phi)$$

are diffeomorphisms of $\mathbf{R}^d \times \mathbf{R}^d$. It follows that the relation

$$(y, -\partial_y\phi) \mapsto (x,y) \mapsto (x, \partial_x\phi)$$

defines a diffeomorphism

$$\chi = \chi_2 \circ \chi_1^{-1} : \mathbf{R}^d_y \times \mathbf{R}^d_\eta \to \mathbf{R}^d_x \times \mathbf{R}^d_\xi,$$

which is in fact symplectic (or "canonical") in the sense that $d\xi \wedge dx = d\eta \wedge dy$. The map $\chi(y,\eta) = (x(y,\eta), \xi(y,\eta))$ is the canonical transformation generated by the phase function $\phi(x,y)$.

For a smooth symbol $p \in \mathcal{B}_k(\mathbf{R}^d_x \times \mathbf{R}^d_\theta \times \mathbf{R}^d_y)$ and $\lambda \neq 0$, let $\mathrm{Op}(p,\lambda)$ denote the the (semiclassical) pseudodifferential operator

$$\mathrm{Op}(p,\lambda)f(x) = \left(\frac{\lambda}{2\pi}\right)^d \iint e^{i\lambda(x-y)\theta} p(x,\theta,y) f(y)\, dy d\theta.$$

These operators obey the following Egorov-type relation:

Theorem 3.2.12 ([AF78, Theorem 6.1]). *Let $\phi(x,y)$ be a phase function, $a(x,y) \in \mathcal{B}(\mathbf{R}^d \times \mathbf{R}^d)$, be an amplitude, and $A(\lambda)$ the corresponding Fourier integral operator. Let $\chi : \mathbf{R}^{2d} \to \mathbf{R}^{2d}$ be the canonical transformation generated by ϕ. Let $p(x,\theta,y),\ q(x,\theta,y) \in \mathcal{B}_1(\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d)$ be such that*

$$q(y,\eta,y) = p(x,\xi,x)|_{(x,\xi)=\chi(y,\eta)}$$

Then

$$\mathrm{Op}(\lambda p, \lambda) A(\lambda) - A(\lambda)\, \mathrm{Op}(\lambda q, \lambda) = R(\lambda),$$

for some Fourier integral operator $R(\lambda)$ with phase function ϕ. The operator norm of $R(\lambda)$ satisfies

$$\|R(\lambda)\|_{L^2 \to L^2} \lesssim \lambda^{-1} \|a\|_{0,M} (\|p\|_{1,M} + \|q\|_{1,M})$$

for some positive integer M, where

$$\|f\|_{r,s} = \sup_{r \le k \le s} \|D^k f\|_{L^\infty}.$$

3.3 Local Theory

We record some standard local-wellposedness results for (3.1). These are direct translations of the theory for the scale-invariant equation (3.2). By Lemma 3.2.8 and Corollaries 3.2.9 and 3.2.10, essentially the same proofs as in that case will work here. We refer the reader to [KV13] for those proofs.

Proposition 3.3.1 (Local wellposedness). *Let $u_0 \in \Sigma(\mathbf{R}^d)$ and fix a compact time interval $0 \in I \subset \mathbf{R}$. Then there exists a constant $\eta_0 = \eta_0(d, |I|)$ such that whenever $\eta < \eta_0$ and*

$$\|H^{\frac{1}{2}} e^{-itH} u_0\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}} (I \times \mathbf{R}^d)} \le \eta,$$

there exists a unique solution $u : I \times \mathbf{R}^d \to \mathbf{C}$ to (2.1) which satisfies the bounds

$$\|H^{\frac{1}{2}} u\|_{L_t^{\frac{2(d+2)}{d-2}} L_x^{\frac{2d(d+2)}{d^2+4}} (I \times \mathbf{R}^d)} \le 2\eta \quad \text{and} \quad \|H^{\frac{1}{2}} u\|_{S(I)} \lesssim \|u_0\|_\Sigma + \eta^{\frac{d+2}{d-2}}.$$

Corollary 3.3.2 (Blowup criterion)**.** *Suppose $u : (T_{min}, T_{max}) \times \mathbf{R}^d \to \mathbf{C}$ is a maximal lifespan solution to* (2.1)*, and fix $T_{min} < t_0 < T_{max}$. If $T_{max} < \infty$, then*

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([t_0,T_{max}))} = \infty.$$

If $T_{min} > -\infty$, then

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}((T_{min},t_0])} = \infty.$$

Proposition 3.3.3 (Stability)**.** *Fix $t_0 \in I \subset \mathbf{R}$ an interval of unit length and let $\tilde{u} : I \times \mathbf{R}^d \to \mathbf{C}$ be an approximate solution to* (2.1) *in the sense that*

$$i\partial_t \tilde{u} = Hu \pm |\tilde{u}|^{\frac{4}{d-2}} \tilde{u} + e$$

for some function e. Assume that

$$\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} \leq L, \quad \|H^{\frac{1}{2}} u\|_{L_t^\infty L_x^2} \leq E, \tag{3.14}$$

and that for some $0 < \varepsilon < \varepsilon_0(E, L)$ one has

$$\|H^{1/2}(\tilde{u}(t_0) - u_0)\|_{L^2} + \|H^{\frac{1}{2}} e\|_{N(I)} \leq \varepsilon, \tag{3.15}$$

Then there exists a unique solution $u : I \times \mathbf{R}^d \to \mathbf{C}$ to (2.1) *with $u(t_0) = u_0$ and which further satisfies the estimates*

$$\|\tilde{u} - u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} + \|H^{\frac{1}{2}}(\tilde{u} - u)\|_{S(I)} \lesssim C(E, L)\varepsilon^c \tag{3.16}$$

where $0 < c = c(d) < 1$ and $C(E, L)$ is a function which is nondecreasing in each variable.

3.4 Concentration compactness

Let $0 \in I$ be a compact interval so that $|I| \leq \delta_0$, where δ_0 is the constant in Theorem 3.2.2. As is now standard in the analysis of energy-critical equations, the induction on energy argument relies on a linear profile decomposition for the Strichartz inequality

$$\|e^{-itH} f\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I \times \mathbf{R}^d)} \lesssim \|H^{1/2} f\|_{L^2}.$$

In the previous chapter, we obtained such a decomposition when $H = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2$. Let us highlight the main modification required to adapt that proof to the present setting. One of the key steps in both proofs is to compare the linear evolutions of a spatially localized initial state under the propagators e^{-itH} and $e^{\frac{it\Delta}{2}}$ with and without a potential, respectively (see Proposition 3.4.4 below). For the harmonic oscillator we relied on the Mehler formula to decompose

$$e^{-itH} = m_t(X) e^{i\frac{i\sin(t)\Delta}{2}} m_t(X)$$

where $m_t(x) = \exp(i(\frac{\cos t - 1}{2\sin t})x^2)$. While this factorization clearly manifests the relation between the two propagators, it is not available for the more general potentials considered here. Instead we work directly with the Fourier integral representation from Theorem (3.2.2).

The rest of the proof for the harmonic oscillator can be imported after essentially notational changes. We shall state the main definitions and lemmas to indicate the general flow but refer to the previous chapter for the details.

Definition 3.4.1. A *frame* is a sequence $(t_n, x_n, N_n) \in I \times \mathbf{R}^d \times 2^{\mathbf{N}}$ conforming to one of the following scenarios:

1. $N_n \equiv 1,\ t_n \equiv 0$, and $x_n \equiv 0$.
2. $N_n \to \infty$ and $N_n^{-1} V(x_n)^{1/2} \to r_\infty \in [0, \infty)$.

Remark. The quantity $N_n^{-1} V(x_n)^{1/2}$ is the analog of the ratio $N_n^{-1}|x_n|$ that was considered in Section 2.4.1.

These parameters will specify the temporal center, spatial center, and (inverse) length scale of a function. The hypothesis that V grows essentially quadratically ensures that $|x_n| \lesssim N_n$, which reflects the fact that we only consider functions obeying some uniform bound in $Q(H)$, and such functions cannot be centered arbitrarily far from the origin. We need to augment the frame $\{(t_n, x_n, N_n)\}$ with an auxiliary parameter N_n', which corresponds to a sequence of spatial cutoffs adapted to the frame.

Definition 3.4.2. An *augmented frame* is a sequence $(t_n, x_n, N_n, N_n') \in I \times \mathbf{R}^d \times 2^{\mathbf{N}} \times \mathbf{R}$ belonging to one of the following types:

1. $N_n \equiv 1,\ t_n \equiv 0,\ x_n \equiv 0,\ N_n' \equiv 1.$

2. $N_n \to \infty,\ N_n^{-1} V(x_n)^{1/2} \to r_\infty \in [0, \infty)$, and either

   (a) $N_n' \equiv 1$ if $r_\infty > 0$, or

   (b) $N_n^{1/2} \leq N_n' \leq N_n,\ N_n^{-1} V(x_n)^{1/2} (\frac{N_n}{N_n'}) \to 0$, and $\frac{N_n}{N_n'} \to \infty$ if $r_\infty = 0$.

The frame $\{(t_n, x_n, N_n)\}$ is the *underlying frame.*

Given an augmented frame (t_n, x_n, N_n, N_n'), we define scaling and translation operators on functions of space and of spacetime by

$$\begin{aligned} (G_n \phi)(x) &= N_n^{\frac{d-2}{2}} \phi(N_n(x - x_n)) \\ (\tilde{G}_n f)(t, x) &= N_n^{\frac{d-2}{2}} f(N_n^2(t - t_n), N_n(x - x_n)). \end{aligned} \tag{3.17}$$

We also define spatial cutoff operators S_n by

$$S_n \phi = \begin{cases} \phi, & \text{for frames of type 1} \quad (\text{i.e. } N_n \equiv 1), \\ \chi(\frac{N_n}{N_n'} \cdot) \phi, & \text{for frames of type 2} \quad (\text{i.e. } N_n \to \infty), \end{cases} \tag{3.18}$$

where χ is a smooth compactly supported function equal to 1 on the ball $\{|x| \leq 1\}$. An easy computation yields the following mapping properties of these operators:

$$\begin{aligned} \lim_{n \to \infty} S_n = I \text{ strongly in } \dot{H}^1 &\text{and in } Q(H), \\ \limsup_{n \to \infty} \|G_n\|_{Q(H) \to Q(H)} &< \infty. \end{aligned} \tag{3.19}$$

The following technical lemma is the counterpart of Lemma 2.4.2 and is proved in the same manner (in particular we use the equivalence of norms furnished by Proposition 3.2.6).

Lemma 3.4.1 (Approximation)**.** *Let (q, r) be an admissible pair of exponents with $2 \leq r < d$, and let $\mathcal{F} = \{(t_n, x_n, N_n, N_n')\}$ be an augmented frame of type 2.*

1. *Suppose $\mathcal{F}$ is of type 2a in Definition 3.4.2. Then for $\{f_n\} \subseteq L_t^q H_x^{1,r}(\mathbf{R}\times\mathbf{R}^d)$, we have*

$$\limsup_n \|H^{1/2}\tilde{G}_n S_n f_n\|_{L_t^q L_x^r} \lesssim \limsup_n \|f_n\|_{L_t^q H_x^{1,r}}.$$

2. *Suppose $\mathcal{F}$ is of type 2b and $f_n \in L_t^q \dot{H}_x^{1,r}(\mathbf{R}\times\mathbf{R}^d)$. Then*

$$\limsup_n \|H^{1/2}\tilde{G}_n S_n f_n\|_{L_t^q L_x^r} \lesssim \limsup_n \|f_n\|_{L_t^q \dot{H}_x^{1,r}}.$$

Here $H^{1,r}(\mathbf{R}^d)$ and $\dot{H}^{1,r}(\mathbf{R}^d)$ denote the inhomogeneous and homogeneous L^r Sobolev spaces, respectively, equipped with the norms

$$\|f\|_{H^{1,r}} = \|\langle\nabla\rangle\|_{L^r(\mathbf{R}^d)}, \quad \|f\|_{\dot{H}^{1,r}} = \||\nabla|f\|_{L^r(\mathbf{R}^d)}.$$

Proposition 3.4.2 (Inverse Strichartz)**.** *Let I be a compact interval containing 0 of length at most δ_0, and suppose f_n is a sequence of functions in $Q(H)$ satisfying*

$$0 < \varepsilon \le \|e^{-itH} f_n\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I\times\mathbf{R}^d)} \lesssim \|H^{1/2} f_n\|_{L^2} \le A < \infty.$$

Then, after passing to a subsequence, there exists an augmented frame

$$\mathcal{F} = \{(t_n, x_n, N_n, N_n')\}$$

and a sequence of functions $\phi_n \in Q(H)$ such that one of the following holds:

1. *$\mathcal{F}$ is of type 1 (i.e. $N_n \equiv 1$) and $\phi_n = \phi$ where $\phi \in Q(H)$ is a weak limit of f_n in $Q(H)$.*

2. *$\mathcal{F}$ is of type 2, either $t_n \equiv 0$ or $N_n^2 t_n \to \pm\infty$, and $\phi_n = e^{it_nH} G_n S_n \phi$ where $\phi \in \dot{H}^1(\mathbf{R}^d)$ is a weak limit of $G_n^{-1} e^{-it_nH} f_n$ in $\dot{H}^1$. Moreover, if $\mathcal{F}$ is of type 2a, then ϕ also belongs to $L^2(\mathbf{R}^d)$.*

The functions ϕ_n have the following properties:

$$\liminf_n \|H^{1/2}\phi_n\|_{L^2} \gtrsim A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(d+2)}{8}} \tag{3.20}$$

$$\lim_{n\to\infty} \|f_n\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \|f_n - \phi_n\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \|\phi_n\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} = 0. \tag{3.21}$$

$$\lim_{n\to\infty} \|H^{1/2} f_n\|_{L^2}^2 - \|H^{1/2}(f_n - \phi_n)\|_{L^2}^2 - \|H^{1/2}\phi_n\|_{L^2}^2 = 0 \tag{3.22}$$

Proof. We recall that the proof of the analogous result in Section 2.4.1 used the following ingredients:

- Littlewood-Paley theory adapted to the operator $H = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2$, which relied on Theorem 3.2.5.
- A refined Strichartz inequality, proved using the Littlewood-Paley theory.
- A comparison of the linear propagators for the Hamiltonians of the free particle and the harmonic oscillator when acting on concentrated initial data. This was the only part of the proof that invoked the exact form of Mehler's formula (3.4).

The reader will easily verify that adapting the first two to our situation requires little more than replacing all instances of $|x|^2/2$ in the proofs with V. In the following section, we supply the details for the third. Given suitable replacements for these three ingredients, the rest of the proof carries over without difficulty, and we refer the reader to the previous chapter for the details. $\square$

Proposition 3.4.3 (Linear profile decomposition)**.** *Let $0 \in I$ be an interval with $|I| \leq \delta_0$, and let f_n be a bounded sequence in $Q(H)$. After passing to a subsequence, there exists $J^* \in \{0, 1, \dots\} \cup \{\infty\}$ such that for each finite $1 \leq j \leq J^*$, there exist an augmented frame $\mathcal{F}^j = \{(t_n^j, x_n^j, N_n^j, (N_n^j)')\}$ and a function ϕ^j with the following properties.*

- *Either $t_n^j \equiv 0$ or $(N_n^j)^2(t_n^j) \to \pm\infty$ as $n \to \infty$.*
- *ϕ^j belongs to $Q(H)$, H^1, or $\dot{H}^1$ depending on whether $\mathcal{F}^j$ is of type 1, 2a, or 2b, respectively.*

For each finite $J \leq J^*$, we have a decomposition

$$f_n = \sum_{j=1}^{J} e^{it_n^j H} G_n^j S_n^j \phi^j + r_n^J, \tag{3.23}$$

where G_n^j, S_n^j are the $\dot{H}^1$-isometry and spatial cutoff operators associated to $\mathcal{F}^j$. Writing ϕ_n^j for $e^{it_n^j H} G_n^j S_n^j \phi^j$, this decomposition has the following properties:

$$(G_n^J)^{-1} e^{-it_n^J H} r_n^J \overset{\dot{H}^1}{\rightharpoonup} 0 \quad \textit{for all } J \leq J^*, \tag{3.24}$$

$$\sup_J \lim_{n\to\infty} \Big| \|H^{1/2} f_n\|_{L^2}^2 - \sum_{j=1}^J \|H^{1/2}\phi_n^j\|_{L^2}^2 - \|H^{1/2} r_n^J\|_{L^2}^2 \Big| = 0, \tag{3.25}$$

$$\sup_J \lim_{n\to\infty} \Big| \|f_n\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \sum_{j=1}^J \|\phi_n^j\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} - \|r_n^J\|_{L_x^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} \Big| = 0. \tag{3.26}$$

Whenever $j \neq k$, the frames $\{(t_n^j, x_n^j, N_n^j)\}$ and $\{(t_n^k, x_n^k, N_n^k)\}$ are orthogonal:

$$\lim_{n\to\infty} \tfrac{N_n^j}{N_n^k} + \tfrac{N_n^k}{N_n^j} + N_n^j N_n^k |t_n^j - t_n^k| + \sqrt{N_n^j N_n^k}|x_n^j - x_n^k| = \infty. \tag{3.27}$$

Finally, we have

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{-it_n H} r_n^J\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} = 0, \tag{3.28}$$

Proof. The argument is similar to the one for Proposition 2.4.14. One inductively applies inverse Strichartz to extract the frames $\mathcal{F}^j$ and profiles ϕ^j. To prove the decoupling assertion (3.27), one uses the convergence lemmas discussed in the next section, which completely parallel the ones used in Section 2.4.2. □

3.4.1 Convergence of linear propagators

The main purpose of this section is to prove Proposition 3.4.4, which together with its corollary can be regarded as a comparison of the linear evolutions $e^{-itH}\phi$ and $e^{\frac{it\Delta}{2}}\phi$ for ϕ concentrated at a point.

While the proposition is simply a translation of Lemma 2.4.8, its proof is more involved and requires a closer study of the bicharacteristics.

Definition 3.4.3. Two frames $\mathcal{F}^1 = \{(t_n^1, x_n^1, N_n^1)\}$ and $\mathcal{F}^2 = \{(t_n^2, x_n^2, N_n^2)\}$ (where the superscripts are indices, not exponents) are *equivalent* if

$$\tfrac{N_n^1}{N_n^2} \to R_\infty \in (0,\infty),\ N_n^1(x_n^2 - x_n^1) \to x_\infty \in \mathbf{R}^d,\ (N_n^1)^2(t_n^1 - t_n^2) \to t_\infty \in \mathbf{R}.$$

They are *orthogonal* if any of the above statements fails. Note that replacing the N_n^1 in the second and third expressions above by N_n^2 yields an equivalent definition of orthogonality.

Two augmented frames are said to be equivalent if their underlying frames are equivalent.

Proposition 3.4.4 (Strong convergence)**.** *Suppose* $\mathcal{F}^M = (t_n^M, x_n, M_n)$ *and* $\mathcal{F}^N = (t_n^N, y_n, N_n)$ *are equivalent frames. Define*

$$R_\infty = \lim_{n\to\infty} \tfrac{M_n}{N_n} \qquad\qquad t_\infty = \lim_{n\to\infty} M_n^2(t_n^M - t_n^N),$$
$$x_\infty = \lim_{n\to\infty} \; M_n(y_n - x_n), \quad r_\infty = \lim_n M_n^{-1} V(x_n)^{1/2};$$

(The last limit exists by the definition of a frame.) Let G_n^M, G_n^N *be the scaling and translation operators associated with the frames* $\mathcal{F}^M$ *and* $\mathcal{F}^N$ *respectively. Then the sequence* $(e^{-it_n^N H} G_n^N)^{-1} e^{-it_n^M H} G_n^M$ *converges strongly as bounded operators on* Σ *to the operator* U_∞ *defined by*

$$U_\infty \phi = e^{-it_\infty (r_\infty)^2} R_\infty^{\frac{d-2}{2}} [e^{\frac{it_\infty \Delta}{2}} \phi](R_\infty \cdot + x_\infty).$$

Proof. Write

$$(e^{-it_n^N H} G_n^N)^{-1} e^{-it_n^M H} G_n^M = (G_n^N)^{-1} G_n^M (G_n^M)^{-1} e^{-it_n H} G_n^M$$

where $t_n = t_n^M - t_n^N$. As $(G_n^N)^{-1} G_n^M$ converges strongly to the operator $f \mapsto R_\infty^{\frac{d-2}{2}} f(R_\infty \cdot + x_\infty)$, it suffices to show that

$$(G_n^M)^{-1} e^{-it_n H} G_n^M \to e^{-it_\infty (r_\infty)^2} e^{\frac{it_\infty \Delta}{2}}. \tag{3.29}$$

We proceed in two steps. Recall from Theorem 3.2.2 that the phase in the Fourier integral formula for e^{-itH} is the classical action and has the form

$$S(t,x,y) = \tfrac{|x-y|^2}{2t} + t\omega(t,x,y), \quad \omega(t,\cdot,\cdot) \in \mathcal{B}_2.$$

First we extract the lowest order term from the remainder. This additional information will reveal the limit of the sequence once everything has been expressed in terms of oscillatory integrals. Convergence will then follow from the theory in Section 3.2.2.

The leading terms of the action are obtained by replacing the classical trajectories with straight lines in the integral (3.6). Proceeding in the spirit of Fujiwara [Fuj79], we have the following lemma.

Lemma 3.4.5. *Let* $H(\xi, x) = \frac{1}{2}|\xi|^2 + V(x)$ *with* V *subquadratic, and let* $S(t, x, y)$ *be the action (which is well-defined for all* x *and* y *so long as* $|t| \leq \delta_0$ *where* δ_0 *is the constant in Theorem 3.2.2). Then*

$$S(t, x, y) = \frac{|x - y|^2}{2t} - \int_0^t V(y + (\tfrac{x-y}{t})\tau)\, d\tau + O\Big(t^3(1 + |x|^2 + |y|^2)\Big).$$

Proof. Start by rewriting the ODE system (3.5) in integral form:

$$\begin{aligned} \xi(t) &= \eta - \int_0^t \partial_x V(x(\theta))\, d\theta, \\ x(t) &= y + \int_0^t \xi(\tau)\, d\tau = y + t\eta - \int_0^t (t - \theta)\partial_x V(x(\theta))\, d\theta. \end{aligned} \tag{3.30}$$

As $\partial_x V$ grows at most linearly, Gronwall's inequality implies that for all initial data (y, η).

$$|x(t)| \leq C(1 + |y| + |t\eta|).$$

Fix a time $t > 0$ and positions $x, y \in \mathbf{R}^d$. By Proposition 3.2.1, there is a unique initial momentum $\eta = \eta(t, x, y)$ such that the bicharacteristic $(x(\tau), \xi(\tau))$ satisfies $x(0) = y$ and $x(t) = x$.

Referring to the definition (3.6) of the action, we estimate the error from replacing the true trajectory $x(t)$ by the straight line path from y to x. Rearranging the above expression for $x(t)$, we have

$$\eta = \frac{x - y}{t} + \frac{1}{t}\int_0^t (t - \theta)\partial_x V(x(\theta))\, d\theta. \tag{3.31}$$

For τ between 0 and t,

$$|x(\tau)| \leq |y| + \left|\frac{x - y}{t}\tau\right| + C\int_0^t |t - \theta|(1 + |x(\theta)|)\, d\theta,$$

hence $|x(\tau)| \leq C(1 + |x| + |y|)$. The preceding computations reveal that

$$\begin{aligned} \Big|x(\tau) - y - \tau\big(\frac{x - y}{t}\big)\Big| &\leq \frac{\tau}{t}\int_0^t |t - \theta||\partial_x V(x(\theta))|\, d\theta + \int_0^\tau |\tau - \theta||\partial_x V(x(\theta))|\, d\theta \\ &\leq C(\tau t + \tau^2)(1 + |x| + |y|). \end{aligned}$$

By the fundamental theorem of calculus,

$$\begin{aligned} \int_0^t |V(x(\tau)) - V(y + \tau(\tfrac{x-y}{t}))|\, d\tau &\leq C\int_0^t (\tau t + \tau^2)(1 + |x| + |y|)^2 d\tau \\ &\leq Ct^3(1 + |x| + |y|)^2. \end{aligned} \tag{3.32}$$

Next, by combining the first line of (3.30) with (3.31), we find that

$$\xi(\tau) = \frac{x-y}{t} + \frac{1}{t}\int_0^t (t-\theta)\partial_x V(x(\theta))\,d\theta - \int_0^\tau \partial_x V(x(\theta))\,d\theta.$$

It is easy to see that second and third terms are bounded by $O(t(1+|x|+|y|))$. Therefore,

$$\begin{aligned}\int_0^t \frac{1}{2}|\xi(\tau)|^2\,d\tau &= \frac{|x-y|^2}{2t} + \frac{x-y}{t}\int_0^t (t-\theta)\partial_x V(x(\theta))\,d\theta \\ &\quad - \frac{x-y}{t}\int_0^t\int_0^\tau \partial_x V(x(\theta))\,d\theta d\tau + O(t^3(1+|x|+|y|)^2) \\ &= \frac{|x-y|^2}{2t} + O(t^3(1+|x|+|y|)^2).\end{aligned}$$

Combining this with (3.32) establishes the lemma. □

By Theorem 3.2.2 and a change of variable,

$$(G_n^M)^{-1}e^{-it_nH}G_n^M f(x) = \left(\frac{\lambda_n}{2\pi i}\right)^{\frac{d}{2}} \int_{\mathbf{R}^d} e^{i\lambda_n\phi_n(x,y)}a_n(x,y)f(y)\,dy, \tag{3.33}$$

where

$$\begin{aligned}\lambda_n &= (M_n^2 t_n)^{-1} \\ a_n(x,y) &= a(t_n, x_n + M_n^{-1}x, x_n + M_n^{-1}y) \\ \phi_n(x,y) &= \tfrac{1}{2}|x-y|^2 + \lambda_n^{-1}t_n\omega(t_n, x_n + M_n^{-1}x, x_n + M_n^{-1}y) \\ &= \phi_0(x,y) + \lambda_n^{-1}t_n\omega_n(x,y).\end{aligned}$$

Theorem 3.2.2 and Lemma 3.4.5 imply the following estimates:

$$\begin{aligned}&t_n\omega_n(x,y) = -\int_0^{t_n} V(x_n + M_n^{-1}y + \tfrac{x-y}{M_n t_n}\tau)\,d\tau + O(t_n^3(|x_n|^2 + M_n^{-2}|x|^2 + M_n^{-2}|y|^2)) \\ &= -t_nV(x_n) + O(M_n^{-2}(1+|x|^2+|y|^2)), \\ &|D_{x,y}^k\omega_n(x,y)| \lesssim \begin{cases} M_n^{-1}(1+|x_n+M_n^{-1}x| + |x_n+M_n^{-1}y|), & k=1 \\ M_n^{-k}, & k\geq 2\end{cases} \\ &|D_{x,y}^m[a_n(x,y)-1]| \lesssim_k M_n^{-2-k} \quad \text{for all} \quad k\geq 0.\end{aligned} \tag{3.34}$$

We need the following adaptation of [Fuj79, Proposition 4.15].

Lemma 3.4.6. *The operators $(G_n^M)^{-1}e^{-it_nH}G_n^M$ are uniformly bounded on Σ.*

Proof. We appeal to Theorem 3.2.12. Let $\chi_n : (y, -\partial_y\phi_n) \mapsto (x, \partial_x\phi_n)$ be the canonical relation generated by the phase function ϕ_n. In terms of the variables y and η,

$$\begin{aligned}\chi_n(y,\eta) &= (y+\eta,\eta) + \lambda_n^{-1}t_n(\partial_y\omega_n, \partial_x\omega_n + \partial_y\omega_n)(x(t_n,y,\eta),y) \\ &= (y+\eta,\eta) + (r_{1,n}(y,\eta), r_{2,n}(y,\eta)).\end{aligned}$$

First we show that

$$\|D(G_n^M)^{-1}e^{-it_nH}G_n^M f\|_{L^2} \lesssim \|f\|_\Sigma. \tag{3.35}$$

Put $p(x,\theta,y) = \theta$ and $q_n(x,\theta,y) = \theta + r_{2,n}(y,\theta)$. By construction,

$$p(x,\xi,x)|_{(x,\xi)=\chi_n(y,\eta)} = q_n(y,\eta,y).$$

By the representation (3.33) and Theorem 3.2.12,

$$\begin{aligned}D(G_n^M)^{-1}e^{-it_nH}G_n^M &= \mathrm{Op}(\lambda_n p, \lambda_n)(G_n^M)^{-1}e^{-it_nH}G_n^M \\ &= (G_n^M)^{-1}e^{-it_nH}G_n^M\,\mathrm{Op}(\lambda_n q_n, \lambda_n) + R_n(\lambda_n).\end{aligned} \tag{3.36}$$

In light of the estimates (3.34) and Theorem 3.2.11, it suffices to obtain a uniform bound

$$\|\mathrm{Op}(\lambda_n q_n, \lambda_n)\|_{\Sigma\to L^2} \lesssim 1$$

By definition

$$\mathrm{Op}(\lambda_n q_n, \lambda_n)f(x) = Df + \mathrm{Op}(\lambda_n r_{2,n}, \lambda_n).$$

By the fundamental theorem of calculus, (3.34) and Proposition 3.2.1,

$$\begin{aligned}&\lambda_n r_{2,y}(y,\eta) = t_n(\partial_x\omega_n + \partial_y\omega_n)(t_n, x(t_n,0,0),0) \\ &+ t_n y\int_0^1 (\partial_{xy}^2\omega_n)(t_n, x(t_n,sy,s\eta),sy)\tfrac{\partial x}{\partial y} + (\partial_y^2\omega_n)(t_n,x(t_n,sy,s\eta),sy)\,ds \\ &+ t_n\eta\int_0^1 (\partial_{xy}^2\omega_n)(t_n,x(t_n,sy,s\eta),sy)\tfrac{\partial x}{\partial \eta}\,ds \\ &= c_n + y r_{2,n}^1(y,\eta) + \eta r_{2,n}^2(y,\eta),\end{aligned}$$

where $|c_n| \lesssim M_n^{-2}$ and $\|D^k r^1_{2,n}\|_{L^\infty} \lesssim M_n^{-4},\ \|D^k r^2_{2,n}\|_{L^\infty} \lesssim M_n^{-6}$ for all k. Thus

$$\begin{aligned}\operatorname{Op}(\lambda_n r_{2,n}, \lambda_n) &= c_n I + \operatorname{Op}(y r^1_{2,n}(y,\eta), \lambda_n) + \operatorname{Op}(\eta r^2_{2,n}(y,\eta), \lambda_n) \\ &= c_n I + \operatorname{Op}(r^1_{2,n}, \lambda_n) X + \operatorname{Op}(\lambda_n^{-1} r^2_{2,n}, \lambda_n) D + \operatorname{Op}(\lambda_n^{-1}(D_y r^2_{2,n}), \lambda_n).\end{aligned}$$

The Calderón-Vaillancourt theorem [CV71] now implies

$$\|\operatorname{Op}(\lambda_n r_{2,n}, \lambda_n) f\|_{L^2} \lesssim M_n^{-2}\|f\|_{L^2} + M_n^{-4}\|Xf\|_{L^2} + M_n^{-6}\|Df\|_{L^2} \lesssim M_n^{-2}\|f\|_\Sigma.$$

Altogether we obtain (3.35).

By setting $p(x,\theta,y) = x,\ q(x,\theta,y) = y + \theta + r_{1,n}(y,\eta)$ and making a similar analysis as above, we obtain

$$\|X(G_n^M)^{-1} e^{-it_n H} G_n^M f\|_{L^2} \lesssim \|f\|_\Sigma.$$

This concludes the proof of the lemma. □

Now we verify the limit (3.29). As $e^{\frac{iM_n^2 t_n \Delta}{2}} \to e^{\frac{it_\infty \Delta}{2}}$ strongly, it suffices to show that

$$(G_n^M)^{-1} e^{-it_n H} G_n^M f - e^{-it_\infty (r_\infty)^2} e^{\frac{iM_n^2 t_n \Delta}{2}} f$$

converges to 0 for all $f \in \Sigma$. By Lemma 3.4.6 we may assume $f \in C_c^\infty$. The above difference may be written as

$$\begin{aligned}\left(\tfrac{\lambda_n}{2\pi i}\right)^{\frac{d}{2}} \int e^{i\lambda_n \phi_n}[a_n - 1] f(y)\, dy &+ \left(\tfrac{\lambda_n}{2\pi i}\right)^{\frac{d}{2}} \int [e^{i\lambda_n \phi_n} - e^{-it_\infty (r_\infty)^2} e^{i\lambda_n \phi_0}] f(y)\, dy \\ &= A_n f + B_n f.\end{aligned}$$

Using Theorem 3.2.11, the estimates (3.34), and arguing as in the proof of Lemma 3.4.6, one sees that $\|A_n f\|_\Sigma \lesssim M_n^{-2}\|f\|_\Sigma$.

It remains to bound $B_n f$. By hypothesis f is supported in some ball $B(0,R)$, and the estimates (3.34) show that the integral kernel of B_n converges to 0 in C^∞_{loc}. It follows that $|xB_n f|$ and $|\nabla B_n f|$ converge to 0 locally uniformly. On the other hand, integration by parts reveals that for all n sufficiently large,

$$|xB_n f| + |\nabla B_n f| \lesssim_N |x|^{-N}$$

for any $N > 0$ and for all $|x| \geq 4R$. Hence $\|B_n f\|_\Sigma \to 0$ by dominated convergence. This completes the proof of the proposition. □

In the remainder of this section we collect other lemmata regarding equivalent and orthogonal frames. They can be proved in much the same manner as their counterparts in Section 2.4.2.

Corollary 3.4.7. *Let $\{(t_n^M, x_n, M_n, M_n')\}$ and $\{t_n^N, y_n, N_n, N_n')\}$ be equivalent augmented frames. Let S_n^M, S_n^N be the associated spatial cutoff operators. Then*

$$\lim_{n\to\infty} \|e^{-it_n^M H} G_n^M S_n^M \phi - e^{-it_n^N H} G_n^N S_n^N U_\infty \phi\|_\Sigma = 0 \tag{3.37}$$

and

$$\lim_{n\to\infty} \|e^{-it_n^M H} G_n^M S_n^M \phi - e^{-it_n^N H} G_n^N U_\infty S_n^N \phi\|_\Sigma = 0 \tag{3.38}$$

whenever $\phi \in H^1$ if the frames conform to case 2a and $\phi \in \dot{H}^1$ if they conform to case 2b in Definition 3.4.2.

Proof. Run an approximation argument using Lemma 3.4.1 in the manner of Corollary 2.4.9. □

The following "approximate adjoint" identity is the analogue of Lemma 2.4.10.

Lemma 3.4.8. *Suppose the frames $\{(t_n^M, x_n, M_n)\}$ and $\{(t_n^N, y_n, N_n)\}$ are equivalent. Put $t_n = t_n^M - t_n^N$. Then for $f, g \in \Sigma$ we have*

$$\langle (G_n^N)^{-1} e^{-it_n H} G_n^M f, g\rangle_{\dot{H}^1} = \langle f, (G_n^M)^{-1} e^{it_n H} G_n^N g\rangle_{\dot{H}^1} + R_n(f, g),$$

where $|R_n(f,g)| \le C|t_n| \|G_n^M f\|_\Sigma \|G_n^N g\|_\Sigma$.

Proof. The proof of Lemma 3.4.6 yields the following commutator estimate:

$$\|[D, e^{-itH}]\|_{\Sigma \to L^2} = O(t).$$

We have

$$\langle D(G_n^N)^{-1} e^{-it_n H} G_n^M f, Dg\rangle_{L^2} = \langle Df, D(G_n^M)^{-1} e^{it_n H} G_n^N g\rangle_{L^2} + R_n(f, g)$$

where $R_n(f,g) = \langle [D, e^{-it_n H}] G_n^M f, D G_n^N g\rangle_{L^2} - \langle D G_n^M f, [D, e^{it_n H}] G_n^N g\rangle_{L^2}$. The claim then follows from Cauchy-Schwarz and the above estimate. □

The next lemma is a converse to Proposition 3.4.4.

Lemma 3.4.9 (Weak convergence)**.** *Assume the frames $\mathcal{F}^M = \{(t_n^M, x_n, M_n)\}$ and $\mathcal{F}^N = \{(t_n^N, y_n, N_n)\}$ are orthogonal. Then, for any $f \in \Sigma$,*

$$(e^{-it_n^N H} G_n^N)^{-1} e^{-it_n^M H} G_n^M f \to 0 \quad \textit{weakly in} \quad \dot{H}^1.$$

Proof. Put $t_n = t_n^M - t_n^N$, and suppose that $|M_n^2 t_n| \to \infty$. Then

$$\|(G_n^N)^{-1} e^{-it_n H} G_n^M f\|_{L^{\frac{2d}{d-2}}} \to 0$$

for $f \in C_c^\infty$ by a change of variables and the dispersive estimate, thus for general $f \in \Sigma$ by a density argument. Therefore $(G_n^N)^{-1} e^{-it_n H} G_n^M f$ converges weakly in $\dot{H}^1$ to 0. We consider next the case where $M_n^2 t_n \to t_\infty \in \mathbf{R}$. The orthogonality of $\mathcal{F}^M$ and $\mathcal{F}^N$ implies that either $N_n^{-1} M_n$ converges to 0 or ∞, or $M_n |x_n - y_n|$ diverges as $n \to \infty$. In either case, one verifies easily that $(G_n^N)^{-1} G_n^M$ converges to zero weakly as operators on $\dot{H}^1$. Applying Proposition 3.4.4, we see that $(G_n^N)^{-1} e^{-it_n H} G_n^M f = (G_n^N)^{-1} G_n^M (G_n^M)^{-1} e^{-it_n H} G_n^M f$ converges to zero weakly in $\dot{H}^1$. □

Corollary 3.4.10. *Let $\{(t_n^M, x_n, M_n, M_n')\}$ and $\{(t_n^N, y_n, N_n, N_n')\}$ be augmented frames such that $\{(t_n^M, x_n, M_n)\}$ and $\{(t_n^N, y_n, N_n)\}$ are orthogonal. Let G_n^M, S_n^M and G_n^N, S_n^N be the associated operators. Then*

$$(e^{-it_n^N H} G_n^N)^{-1} e^{-it_n^M H} G_n^M S_n^M \phi \rightharpoonup 0 \quad \textit{in} \quad \dot{H}^1$$

whenever $\phi \in H^1$ if $\mathcal{F}^M$ is of type 2a and $\phi \in \dot{H}^1$ if $\mathcal{F}^M$ is of type 2b.

Proof. If $\phi \in C_c^\infty$, then $S_n^M \phi = \phi$ for all large n, and the claim follows from Lemma 3.4.9. The case of general ϕ in H^1 or $\dot{H}^1$ then follows from an approximation argument similar to the one used in the proof of Corollary 3.4.7. □

3.5 The case of concentrated initial data

With the main complications behind us, we sketch the rest of the global wellposedness argument in the remaining two sections. The next step is to rule out blowup for equation (3.1) when the initial data is highly concentrated in space.

Proposition 3.5.1. *Let $I = [-\delta_0/2, \delta_0/2]$, where δ_0 is the constant in Theorem 3.2.2. Assume that Conjecture 2.1.1 holds. Let*

$$\mathcal{F} = \{(t_n, x_n, N_n, N_n')\}$$

be an augmented frame with $t_n \in I$ and $N_n \to \infty$, such that either $t_n \equiv 0$ or $N_n^2 t_n \to \pm\infty$; that is, $\mathcal{F}$ is type 2a or 2b in Definition 3.4.2. Let $G_n, \tilde{G}_n$, and S_n be the associated operators as defined in (3.17) *and* (3.18). *Suppose ϕ belongs to H^1 or $\dot{H}^1$ depending on whether $\mathcal{F}$ is type 2a or 2b respectively. Then, for n sufficiently large, there is a unique solution $u_n : I \times \mathbf{R}^d \to \mathbf{C}$ to the defocusing equation* (3.1), *$\mu = 1$, with initial data*

$$u_n(0) = e^{it_n H} G_n S_n \phi.$$

This solution satisfies a spacetime bound

$$\limsup_{n\to\infty} S_I(u_n) \le C(E(u_n)).$$

Suppose in addition that $\{(q_k, r_k)\}$ is any finite collection of admissible pairs with $2 < r_k < d$. Then for each $\varepsilon > 0$ there exists $\psi^\varepsilon \in C_c^\infty(\mathbf{R} \times \mathbf{R}^d)$ such that

$$\limsup_{n\to\infty} \sum_k \|H^{1/2}(u_n - \tilde{G}_n[e^{-itN_n^{-2}V(x_n)}\psi^\varepsilon])\|_{L_t^{q_k} L_x^{r_k}(I\times\mathbf{R}^d)} < \varepsilon. \tag{3.39}$$

Assuming also that $\|\nabla\phi\|_{L^2} < \|\nabla W\|_{L^2}$ and $E_\Delta(\phi) < E_\Delta(W)$, we have the same conclusion as above for the focusing equation (3.1), *$\mu = -1$.*

Proof sketch. We only give a rough idea as one can proceed just as in Proposition 2.5 and replace every instance of $|x_n|^2/2$ wih $V(x_n)$. The idea is to show that for n large enough, one can fashion a sufficiently accurate approximate solution $\tilde{u}_n$ in the sense of Proposition 3.3.3, such that $S_I(\tilde{u}_n)$ are bounded. This bound will then be transferred to the exact solution u_n by the stability theory.

While u_n remains highly concentrated (over time scales on the order of N_n^{-2}), it will be approximated by a modified solution to the scale-invariant equation (3.2) (whose solutions admit global spacetime bounds). By the time this approximation breaks down, the solution u_n will be so dispersed that it evolves essentially linearly.

If $t_n \equiv 0$, let v be the global solution to (3.2) furnished by Conjecture 2.1.1 with $v(0) = \phi$. If $N_n^2 t_n \to \pm\infty$, let v be the (unique) solution to (3.2) which scatters in $\dot{H}^1$ to $e^{\frac{it\Delta}{2}}\phi$ as $t \to \mp\infty$. Note the reversal of signs. By standard arguments, if $\phi \in H^1$ the scattering also occurs with respect to the H^1 norm.

The approximate solution is obtained as follows. Let $\tilde{G}_n,\ S_n$ be the frame operators as defined in (3.17) and (3.18), and define for each n a smooth frequency cutoff

$$P_{\le \tilde{N}_n'} = \varphi(-\Delta/(\tilde{N}_n')^2), \quad \tilde{N}_n' = (\tfrac{N_n}{N_n'})^{\frac{1}{2}},$$

where $\varphi : \mathbf{R} \to \mathbf{R}$ denotes a smooth function equal to 1 on the ball $B(0,1)$ and supported in $B(0,1.1)$. Fix a large $T > 0$, and define

$$\tilde{v}_n^T(t) = \begin{cases} e^{-itV(x_n)}\tilde{G}_n[S_n P_{\le \tilde{N}_n'} v](t+t_n) & |t| \le TN_n^{-2} \\ e^{-i(t-TN_n^{-2})H}\tilde{v}_n^T(TN_n^{-2}), & TN_n^{-2} \le t \le \delta_0 \\ e^{-i(t+TN_n^{-2})H}\tilde{v}_n^T(-TN_n^{-2}), & -\delta_0 \le t \le -TN_n^{-2} \end{cases}. \tag{3.40}$$

Inside the "window of concentration", $\tilde{v}_n^T$ is essentially a modulated solution to (3.2) with cutoffs applied in both space, to place the solution in $C_t\Sigma_x$, and frequency, to enable taking an extra derivative in the error analysis for the stability theory. The time translation by t_n is needed to undo the time translation built into the operator $\tilde{G}_n$; see (3.17).

Essentially the same computations as in Section 2.5 yield the estimate

$$\limsup_n \|H^{1/2}\tilde{v}_n^T\|_{L_t^\infty L_x^2([-\delta_0,\delta_0])} + \|\tilde{v}_n^T\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}([-\delta_0,\delta_0]\times\mathbf{R}^d)} \lesssim C(\|\phi\|_{H^1},\ \|\phi\|_{\dot{H}^1}),$$

depending on whether ϕ belongs to H^1 or merely $\dot{H}^1$. One also sees that

$$\lim_{T\to\infty}\limsup_n \|H^{1/2}[(i\partial_t - H)(\tilde{v}_n^T) - F(\tilde{v}_n^T)]\|_{N([-\delta_0,\delta_0])} = 0,$$

where $F(z) = \mu|z|^{\frac{4}{d-2}}z$ is the nonlinearity.

$$\lim_{T\to\infty} \limsup_{n} \|H^{1/2}[\tilde{v}_n^T(-t_n) - u_n(0)]\|_{L_x^2} = 0.$$

Then for some fixed large T and all large n, $\tilde{u}_n(t,x) := \tilde{v}_n^T(t-t_n, x)$ is an approximate solution on the time interval $[-\delta_0/2, \delta_0/2]$ in the sense of Proposition 3.3.3. Thus one obtains the first part of Proposition 3.5.1. The last claim regarding approximation by smooth functions is proven by applying Lemma 3.4.1 to the functions $\tilde{v}_n^T$ in the manner of Lemma 2.5.6. □

3.6 A compactness property for blowup sequences

In this section we state a Palais-Smale compactness property for sequences of blowing up solutions to (3.1). This will quickly lead to the proof of Theorem 3.1.2.

Let δ_0 be the constant in Theorem 3.2.2. For a maximal solution u to (3.1), define

$$S_*(u) = \sup\{S_I(u) : I \text{ is an open interval with } |I| \le \delta_0/2\},$$

where we set $S_I(u) = \infty$ if u is not defined on I. All solutions in this section are assumed to be maximal. Set

$$\begin{aligned}\Lambda_d(E) &= \sup\{S_*(u) : u \text{ solves (2.1)}, \ \mu = +1, \ E(u) = E\} \\ \Lambda_f(E) &= \sup\{S_*(u) : u \text{ solves (2.1)}, \ \mu = -1, \ E(u) = E, \\ &\qquad \|\nabla u(0)\|_{L^2} < \|\nabla W\|_{L^2}\}.\end{aligned}$$

Finally, define

$$\mathcal{E}_d = \{E : \Lambda_d(E) < \infty\}, \quad \mathcal{E}_f = \{E : \Lambda_f(E) < \infty\},$$

By the local theory, Theorem 3.1.2 is equivalent to the assertions

$$\mathcal{E}_d = [0, \infty), \quad \mathcal{E}_f = [0, E_\Delta(W)).$$

Suppose Theorem 3.1.2 failed. By the small data theory, $\mathcal{E}_d$, $\mathcal{E}_f$ are nonempty and open, and the failure of Theorem 3.1.2 implies the existence of a critical energy $E_c > 0$, with $E_c < E_\Delta(W)$ in the focusing case such that $\Lambda_d(E),\ \Lambda_f(E) = \infty$ for $E > E_c$ and $\Lambda_d(E),\ \Lambda_f(E) < \infty$ for all $E < E_c$. We have the following compactness property.

Proposition 3.6.1 (Palais-Smale)**.** *Assume Conjecture 2.1.1 holds. Suppose that*

$$u_n : (t_n - \delta_0, t_n + \delta_0) \times \mathbf{R}^d \to \mathbf{C}$$

is a sequence of solutions with

$$\lim_{n\to\infty} E(u_n) = E_c,\ \lim_{n\to\infty} S_{(t_n-\delta_0,t_n]}(u_n) = \lim_{n\to\infty} S_{[t_n,t_n+\delta_0)}(u_n) = \infty,$$

where δ_0 is the constant in Theorem 3.2.2. In the focusing case, assume also that $E_c < E_\Delta(W)$ and $\|\nabla u_n(t_n)\|_{L^2} < \|\nabla W\|_{L^2}$. Then there exists a subsequence such that $u_n(t_n)$ converges in $Q(H)$.

Proof. We refer to the presentation following Proposition 2.6.1. The proof uses a local smoothing estimate for the propagator e^{-itH}, which can be obtained just as in Corollary 2.2.10. In the focusing case, one also uses energy trapping arguments (see Section 2.7) to see that the hypotheses are in fact equivalent to $\|H^{1/2}u_n(t_n)\|_{L^2} < \|\nabla W\|_{L^2}$. □

Proof of Theorem 3.1.2. Suppose the theorem failed, and let E_c be as above. Then, after applying suitable time translations, there is a sequence of solutions u_n with $E(u_n) \to E_c$ and $S_{(-\delta_0/4,\delta_0/4)}(u_n) \to \infty$. Choose t_n such that $S_{(-\delta_0/4,t_n)}(u_n) = \frac{1}{2}S_{(-\delta_0/4,\delta_0/4)}(u_n)$. By Proposition 3.6.1, after passing to a subsequence we have $\|u(t_n) - \phi\|_\Sigma \to 0$ for some $\phi \in \Sigma$. Then $E(\phi) = \lim_n E(u_n(t_n)) = E_c$.

Let $v : (-T_{min}, T_{max}) \to \mathbf{C}$ be the maximum-lifespan solution to (3.1) with $v(0) = \phi$. By comparing $v(t,x)$ with the solutions $u_n(t + t_n, x)$ and applying Proposition 3.3.3, we see that $S_{(0,\delta_0/2)}(v) = S_{(-\delta_0/2,0)}(v) = \infty$. Thus $-\delta_0/2 \le -T_{min} < T_{max} \le \delta_0/2$. But the orbit $\{v(t)\}_{t\in(-T_{min},T_{max})}$ is a precompact subset of Σ, by Proposition 3.6.1, so there is some sequence of times t_n increasing to T_{max} such that $v(t_n)$ converges in Σ to some ψ. By considering a local solution with initial data ψ and invoking stability theory, we see that v can actually be extended to some larger interval $(-T_{min}, T_{max} + \eta)$, in contradiction to the maximality of v. □

CHAPTER 4

Energy-critical NLS on perturbations of $\mathbf{R}^3$

4.1 Introduction

Let g be a smooth Riemannian metric on $\mathbf{R}^3$. We consider the large-data Cauchy problem for the nonlinear Schrödinger equation

$$(i\partial_t + \Delta_g)u = F(u), \; u(0,x) = u_0(x) \in \dot{H}^1, \tag{4.1}$$

where $F(u) = |u|^4 u$ is a defocusing quintic power-type nonlinearity, and Δ_g is the Laplace-Beltrami operator. More precise assumptions on g shall be prescribed shortly.

This equation admits a conserved energy

$$E(u) = \int_{\mathbf{R}^3} \frac{1}{2} g^{jk} \partial_j u \overline{\partial_j u} + \frac{1}{6}|u|^6 \, dg, \tag{4.2}$$

where $dg = \sqrt{|g|}dx$ is the Riemannian measure. One recovers the scale-invariant energy-critical NLS discussed earlier by taking the standard Euclidean metric $g = \delta$. As discussed in previous chapters, all solutions to that equation scatter.

Although the exact scaling symmetry is lost for general g, it reemerges at small length scales in the sense that solutions concentrated at a point x_0 resemble, for short times, solutions to the scale-invariant equation with the constant metric $g(x_0)$. In addition, for any $\phi \in \dot{H}^1$, putting $\phi_\lambda = \lambda^{-1/2}\phi(\lambda^{-1}\cdot)$, the Sobolev norm $\|\phi_\lambda\|_{\dot{H}^1}$ is essentially independent of λ for small λ. Therefore, boundedness of solutions in norm is necessary (and guaranteed by energy conservation) but not sufficient to deduce global existence.

Unlike when introducing a potential, even mild deviations of g from the flat metric cannot be regarded as perturbations to the Euclidean equation. Indeed, disturbing the highest order terms can destroy fundamental smoothing and decay estimates for the linear equation. This breakdown can be traced to the geometry of the geodesic flow.

For a general metric g, some geodesics may remain in a compact set for all time, and the linear local smoothing estimate $L^2 \to L^2 H^{1/2}_{loc}$ is known to fail in such cases [Doi96]. Also, on a curved background, multiple geodesics emanating from a point may converge at another point. Linear solutions exhibit weaker decay amid such refocusing; in particular, as observed in [HTW06], the Euclidean dispersive estimate

$$\|e^{it\Delta}\|_{L^1(\mathbf{R}^d) \to L^\infty(\mathbf{R}^d)} \lesssim |t|^{-d/2}$$

necessarily fails whenever the metric admits conjugate pairs. In general one can only recover a frequency-localized version which holds at most for times inversely proportional to frequency [BGT04]; the time window stops the flow well before refocusing of geodesics can occur. For comparison, the Schrödinger operators considered earlier in this thesis do admit the analogous dispersive bound at least for small t, with the valid time interval improving the milder the potential; see Theorem 3.2.2.

While trapping does not occur if g is close to flat, arbitrarily small perturbations of the flat metric may cause rays to refocus. Therefore, to draw interesting conclusions about equation (4.1) one must proceed without assuming the dispersive estimate. This constraint has substantial implications for both the linear and nonlinear analysis.

The standard abstract approach to linear Strichartz estimates combines the dispersive estimate with a TT^* argument [KT98]. This method was used to deduce local-in-time estimates in the previous chapter, but is not directly applicable in geometries with unfavorable dispersion. Nonetheless, lossless Strichartz inequalities have been obtained in such settings, starting with the influential work of Staffilani and Tataru [ST02] and generalized substantially since [RT07, HTW06, Tat08, BT08, MMT08]. The basic strategy in these papers is to exploit microlocal versions of the dispersive estimate through suitable parametrices and to control the errors using local smoothing, which holds in greater generality compared to the dispersive estimate.

Linear pointwise decay also plays a key role in the study of nonlinear solutions, in particular, when trying to control highly concentrated profiles that arise as potential obstructions to global existence. Such an analysis occurred in Section 2.5 for the harmonic oscillator, and we briefly recall the main idea here.

Suppose u_n is a sequence of solutions to the defocusing harmonic oscillator on $\mathbf{R}^3$ with initial data $u_n(0) = \lambda_n^{-1/2}\phi(\lambda_n^{-1}\cdot)$ for some $\lambda_n \to 0$ and some compactly supported ϕ. For short times (more precisely, when $|t| \leq T\lambda_n^2$ for any $T > 0$), the harmonic oscillator solution u_n is perceives the potential as essentially constant and is well-approximated by the solution $\tilde{u}_n$ to the Euclidean energy-critical equation with the same initial data. Hence, u_n is well-behaved for $t \leq O(\lambda_n^2)$ since the same is true of the Euclidean solution.

For $t \geq T\lambda_n^2$, the dispersive estimate for the harmonic oscillator the scattering of Euclidean solutions ensure that for large T and small λ_n, the nonlinearity $|u_n|^4 u_n$ is a negligible perturbation of the linear harmonic oscillator. That is, for such t, u_n evolves essentially according to the linear flow applied to $u(T\lambda_n^2)$, which is perfectly well behaved. Thus, linear decay allows one to control concentrated nonlinear solutions for times when the Euclidean approximation no longer holds.

We investigate the situation where g coincides with the flat metric outside the unit ball and all geodesics escape to infinity. This is the simplest nontrivial generalization of the Euclidean metric and is a natural counterpart to the scenario considered recently by Killip, Visan, and Zhang [KVZb], who proved scattering for the analogue of equation (4.1) in the exterior of a hard convex obstacle, where the geodesics are straight lines that reflect off the obstacle. We prove

Theorem 4.1.1. *Let g be a smooth, nontrapping metric $\mathbf{R}^3$ which coincides with the Euclidean metric outside the unit ball. For any $u_0 \in \dot{H}^1$, there is a unique global solution to* (4.1)*. Moreover, there exists $\varepsilon > 0$ such that if $\|g - \delta\|_{C^3} \leq \varepsilon$ then the solutions obey global spacetime bounds*

$$\|u\|_{L^{10}_{t,x}(\mathbf{R}\times\mathbf{R}^3)} \leq C(E(u_0)).$$

The smallness assumption for scattering is probably artificial, but we do not see at this time how to dispense with it.

We use the Kenig-Merle concentration compactness and rigidity method, following in particular the mold of [KVZb]. Assuming that the scattering fails, we show that there must exist a global-in-time blowup solution u_c with minimal energy among all counterexamples to the theorem. In view of this minimality, u_c is also shown to be almost-periodic in the sense that $u(t)$ is trapped in some compact subset of $\dot{H}^1$. However, under the smallness assumption on the metric, a Morawetz inequality will imply that solutions to equation (4.1) can never be almost-periodic. Without the smalless assumption, the question of suitable Morawetz estimates remains open at this time, and the argument merely yields global wellposedness with $L^{10}L^{10}$ bounds on unit time intervals.

The heart of the matter is how to overcome the reduced linear dispersion, which is the main obstacle to analyzing the linear and nonlinear profile decompositions. In Section 4.4, we prove a weak analogue of the usual dispersive estimate which nonetheless suffices for our purposes. This can be regarded as a long-time variant of the Burq-Gerard-Tzvetkov dispersion estimate [BGT04] in which we track the microlocalized Schrödinger flow on timescales that permit refocusing.

Several recent works have exploited analogous weak dispersion estimates to study energy-critical NLS in non-Euclidean geometries, although the decay manifests for different reasons. En route to showing global wellposedness for the quintic NLS on $\mathbf{T}^3$, Ionescu and Pausader introduce an "extinction lemma" [IP12, Lemma 4.2] to control concentrated nonlinear profiles at times beyond the "Euclidean window". Afterwards, Pausader, Tzvetkov, and Wang [PTW14] obtained the analogous result on $\mathbf{S}^3$, also relying crucially on an extinction lemma. The arguments there take advantage of the special structure of the underlying manifold, using for instance Fourier analysis on the torus (which, when combined with number theoretic arguments, yield good bounds on the Schrödinger propagator) or the concentration properties of spherical harmonics.

In a different vein, Killip-Visan-Zhang also obtained an extinction lemma in the exterior of a convex obstacle. To analyze the linear evolution of a profile concentrating near the obstacle, they construct a gaussian wavepacket parametrix and carefully study how the wavepackets reflect off the obstacle. The essential geometric fact in their favor is that any two rays diverge after reflecting off the obstacle.

When the hard obstacle is replaced by a lens, refracted rays can certainly refocus. However, one can recover some dispersion by a different mechanism. Due to the uncertainty principle, a solution which is initially highly concentrated in space must be widely distributed in momentum (frequency). Thus, it will spread out along geodesics as the slower parts lag behind[1]. We make this heuristic precise in Section 4.4 by using a wavepacket parametrix and studying the geodesic flow.

Outline of chapter. In Section 4.2 we collect some technical points concerning Sobolev spaces and some linear theory. From the linear estimates it is a standard matter to obtain the perturbative theory, and we merely state the main results.

Sections 4.3 and 4.4 lie at the core of our argument. In Section 4.3 we study linear solutions in various situations where the variation in the metric is intuitively negligible (for instance, when considering initial data supported far from the origin), and show that they behave essentially like Euclidean solutions. The most interesting case is of course when the solution starts concentrated near the origin, where it experiences nontrivial interaction with the curvature. To completely analyze this situation, we need the extinction lemma which is the subject of Section 4.4.

With those considerations out of the way, we can then construct the linear profile decomposition in Sections 4.5. We also show in Section 4.6 that highly concentrated nonlinear profiles are well-behaved; here the extinction lemma and the existing scattering result for the Euclidean quintic equation both play a critical role.

In Section 4.7, we use a nonlinear profile decomposition and induction on energy to reduce Theorem 4.1.1 to considering almost-periodic minimal-energy counterexamples. This

[1]This is an observation of D. Tataru.

will already imply global wellposedness. Some care is needed to control the interaction between linear and nonlinear profiles; see the discussion preceding Lemma 4.7.6.

Finally, in Section 4.8 we prove scattering under the smallness assumption via a Bourgain-Morawetz inequality.

Acknowledgments. The author wishes to thank Rowan Killip and Monica Visan for many helpful conversations. This project was partially supported by the Rosenfeld-Abrams Dissertation Year Fellowship fund. A portion of this work was completed while in residence at the Mathematical Sciences Research Institute during the Fall 2015 program on deterministic and stochastic PDE.

4.2 Preliminaries

4.2.1 Sobolev spaces

The energy space $\dot{H}^1$ is defined as the completion of test functions $C_0^\infty(\mathbf{R}^3)$ with respect to the quadratic form

$$\|u\|_{\dot{H}^1}^2 = \int_{\mathbf{R}^3} |du|_g^2 \, dg(x) = \int_{\mathbf{R}^3} g^{jk} \partial_j u \overline{\partial_k u} \, dg(x).$$

We want to compare this with the usual Euclidean Sobolev norm. As $|du|_g$ is pointwise comparable to the Euclidean gradient $|du|$ and $\sqrt{|g|}$ is bounded above and below, clearly

$$\|u\|_{\dot{H}^1} \sim \|(-\Delta_\delta)^{1/2} u\|_{L^2(dx)} = \|u\|_{\dot{H}^1(\delta)},$$

where $\dot{H}^1(\delta)$ is the Euclidean homogeneous Sobolev space. To distinguish the two norms we denote the first by $\dot{H}^1(g)$. Thus the spaces $\dot{H}^1(g)$ and $\dot{H}^1(\delta)$ are equal as sets and have equivalent inner products. In particular, the $\dot{H}^1(g) \hookrightarrow L^6$ Sobolev embedding holds. When the distinction is irrelevant (as it usually is), we write $\dot{H}^1(\mathbf{R}^3)$ or just $\dot{H}^1$. The advantage of $\dot{H}^1(g)$ is that Δ_g is self-adjoint with respect to the inner product.

For $1 < p < \infty$, define the homogeneous Sobolev spaces $\dot{H}^{1,p}(\delta)$ and $\dot{H}^{1,p}(g)$ as the completion of C_0^∞ under the norms

$$\|u\|_{\dot{H}^{1,p}(\delta)} := \|(-\Delta_\delta)^{1/2}u\|_{L^p}, \quad \|u\|_{\dot{H}^{1,p}(g)} := \|(-\Delta_g)^{1/2}u\|_{L^p}. \tag{4.3}$$

As noted in the introduction, these two definitions coincide when $p = 2$. Less trivially, these norms are equivalent for all $1 < p < \infty$. This is a consequence of the following boundedness result for the Riesz transform $d(-\Delta_g)^{-1/2}$ on asymptotically Euclidean manifolds.

Proposition 4.2.1 ([CCH06, Remark 5.2])**.** *Let (M, g) be a Riemannian manifold such that for some $R > 0$, $M \setminus B(0, R)$ is Euclidean. Then the Riesz transform $d(-\Delta_g)^{-1/2}$ is bounded from $L^p(M)$ to $L^p(M; T^*M)$ for all $1 < p < \infty$.*

By a well-known duality argument (see for example [CD03, Section 2.1]), this implies the reverse inequality whose proof we give for completeness:

Corollary 4.2.2.

$$\|(-\Delta_g)^{1/2}u\|_{L^p} \lesssim_p \|du\|_{L^p}, \ \forall u \in C_0^\infty, \ 1 < p < \infty.$$

Proof. By duality, it suffices to show

$$|\langle(-\Delta_g)^{1/2}u, v\rangle| \lesssim \|du\|_{L^p}\|\|v\|_{L^{p'}}.$$

Then

$$\begin{aligned} \langle(-\Delta_g)^{1/2}u, v\rangle &= \langle u, (-\Delta_g)^{1/2}v\rangle = \langle u, (-\Delta_g)(-\Delta_g)^{-1/2}v\rangle \\ &= \langle du, d(-\Delta_g)^{-1/2}v\rangle \lesssim \|du\|_{L^p}\|v\|_{L^{p'}}. \end{aligned}$$

Note that while the intermediate manipulations are justified for v spectrally localized away from 0 and ∞, we may then pass to general $v \in L^{p'}$ using (4.6). □

Noting also that

$$\|df\|_{L^p} = \|d(-\Delta_g)^{-1/2}(-\Delta_g)^{1/2}f\|_{L^p} \lesssim \|(-\Delta_g)^{1/2}f\|_{L^p},$$

we summarize the previous two estimates in the following

Corollary 4.2.3 (Equivalence of Sobolev norms)**.** *For all* $1 < p < \infty$ *and* $f \in C_0^\infty$,

$$\|(-\Delta_\delta)^{1/2} u\|_{L^p} \sim_p \|df\|_{L^p} \sim_p \|(-\Delta_g)^{1/2} u\|_{L^p}.$$

The geometric $\dot{H}^1$ norm, defined by the metric Laplacian, is better adapted to the equation as it is conserved by the linear flow. On the other hand, the Euclidean $\dot{H}^1$ is analytically more convenient. The corollary allows us to pass freely between the two and exploit the best properties of each. In particular, the Euclidean $\dot{H}^1$ norm is controlled by the energy for both linear and nonlinear solutions, while the geometric $\dot{H}^1$ norm obeys Leibniz and chain rule estimates:

Corollary 4.2.4.

$$\|(-\Delta_g)^{1/2} F(u)\|_p \lesssim \|F'(u)\|_q \|(-\Delta_g)^{1/2} u\|_r$$

whenever $p^{-1} = q^{-1} + r^{-1}$.

In particular, we have

$$\|(-\Delta_g)^{1/2} (|u|^4 u)\|_{L^2 L^{\frac{6}{5}}} \lesssim \|u\|_{L^{10} L^{10}}^4 \|(-\Delta_g)^{1/2}\|_{L^{10} L^{\frac{30}{13}}}.$$

4.2.2 Strichartz estimates

Local-in-time Strichartz estimates without loss for compact nontrapping metric perturbations were first established by Staffilani and Tataru [ST02]. As later observed, their argument can be combined with the global local smoothing estimate of Rodnianski and Tao to deduce global-in-time Strichartz estimates [RT07]. As mentioned in the introduction, these results have since been extended to long-range metrics.

Proposition 4.2.5. *[[ST02, RT07]] For any function* $u : I \times \mathbf{R}^3 \to \mathbf{C}$,

$$\|u\|_{L^\infty L^2 \cap L^2 L^6} \lesssim \|u(0)\|_{L^2} + \|(i\partial_t + \Delta_g) u\|_{L^1 L^2 + L^2 L^{6/5}}$$

In particular, by Sobolev embedding and Corollary 4.2.3,

$$\|u\|_{L^{10} L^{10}} \lesssim \|(-\Delta_g)^{1/2} u\|_{L^{10} L^{\frac{30}{13}}} \lesssim \|u(0)\|_{\dot{H}^1} + \|\nabla (i\partial_t + \Delta_g) u\|_{L^1 L^2 + L^2 L^{6/5}}.$$

In the sequel we adopt the notation

$$Z(I) = L_t^{10} L_x^{10}(I \times \mathbf{R}^3), \quad N(I) = (L_t^1 L_x^2 + L_t^2 L_x^{6/5})(I \times \mathbf{R}^3).$$

4.2.3 Some harmonic analysis

In this section we set up a Littlewood-Paley theory, which will underlie the linear profile decomposition. We use the heat semigroup and follow essentially standard arguments that combine a suitable spectral multiplier theorem with heat kernel bounds.

Gaussian heat kernel bounds for Δ_g are classical. We quote a result of Aronson, who in fact considered uniformly elliptic operators on Euclidean space; see the book [Gri09] for a comprehensive survey.

Theorem 4.2.6 ([Aro67])**.** *There exist a constant $c > 0$ such that*

$$e^{t\Delta_g}(x, y) \leq c_1 t^{-\frac{3}{2}} e^{-\frac{d_g(x,y)^2}{ct}},$$

where $d_g(x, y)$ is the Riemannian distance between x and y.

In view of this bound, we have access to a very general spectral multiplier theorem. For simplicity we state just the special case that we shall need.

Theorem 4.2.7 ([TOS02, Theorem 3.1])**.** *For any F satisfying the homogeneous symbol estimates*

$$|\lambda^k \partial^k F(\lambda)| \leq C_k \text{ for all } 0 \leq k \leq \lceil \frac{n}{2} \rceil + 1,$$

the operator $F(-\Delta_g)$ maps $L^1 \to L^{1,\infty}$ and $L^p \to L^p$ for all $1 < p < \infty$.

For a dyadic number $N \in 2^{\mathbf{Z}}$, define Littlewood-Paley projections in terms of the heat kernel

$$\tilde{P}_{\leq N} = e^{\Delta_g / N^2}, \quad \tilde{P}_N = e^{\Delta_g / N^2} - e^{4\Delta_g / N^2}.$$

Later on (see Lemma 4.7.6), we also introduce Littlewood-Paley projections $P_{\leq N}$ and P_N using compactly supported spectral multipliers.

We have the Bernstein estimates

Proposition 4.2.8.

$$\|\tilde{P}_{\leq N}\|_{L^p\to L^p} \leq 2,\ 1<p<\infty. \tag{4.4}$$

$$\|\tilde{P}_{\leq N}\|_{L^p\to L^q} \leq cN^{\frac{d}{p}-\frac{d}{q}},\ 1\leq p\leq q\leq\infty. \tag{4.5}$$

$$f=\sum_N \tilde{P}_N f \quad in\ L^p,\ 1<p<\infty.. \tag{4.6}$$

Also, for all $1<p<\infty$, the following square function estimate holds

$$\|(-\Delta_g)^{\frac{s}{2}} f\|_{L^p} \sim_p \Big\|(\sum_N |N^s(\tilde{P}_N)^k f|^2)^{1/2} f\Big\|_{L^p}, \tag{4.7}$$

whenever $2k>s$.

Proof. By the pointwise bound (4.2.6) on the heat kernel,

$$\|e^{t\Delta_g}\|_{L^1\to L^\infty} \leq ct^{-3/2}. \tag{4.8}$$

By duality,

$$\|e^{t\Delta_g}\|_{L^1\to L^2} = \|e^{t\Delta_g}\|_{L^2\to L^\infty} = \|e^{2t\Delta_g}\|_{L^1\to L^\infty}^{1/2} \leq ct^{-\frac{3}{4}}.$$

Since $\int e^{t\Delta_g}(x,y)\,dg(y) = \int e^{t\Delta_g}(x,y)\,dg(x) \equiv 1$, we have

$$\|e^{t\Delta_g}\|_{L^p\to L^p} \leq 1,\ 1\leq p\leq\infty.$$

The claims (4.4) and (4.5) follow from interpolating these estimates.

The convergence in (4.6) follows from the functional calculus when $p=2$. On the other hand, Theorem 4.2.7 ensures boundedness in L^p for all $1<p<\infty$. By interpolation, one gets convergence for all such p.

Finally, the square function estimate (4.7) follows the standard argument using random signs and the multiplier theorem 4.2.7. The lower bound on k ensures that the symbol for $(\tilde{P}_N)^k$ (which is not quite compactly supported) vanishes at the origin to higher order than the symbol for $(-\Delta_g)^{s/2}$; see [KVZa] for details. □

4.2.4 Local wellposedness

We summarize some standard results concerning the local existence, uniqueness, and stability of solutions. These are proved by the usual contraction mapping and bootstrap arguments for the Euclidean NLS (see [KV13] and the references therein). As remarked in the introduction, these arguments apply equally well in dimensions $3 \le d \le 6$. When $d > 6$, however, the stability theorem is proved in the Euclidean setting using exotic Strichartz estimates [TV05, KV13]. These are derived using the Euclidean dispersive estimate, which is unavailable to us.

Proposition 4.2.9. *There exists $\varepsilon_0 > 0$ such that for any $u_0 \in \dot{H}^1$, and for any interval $I \ni 0$ such that*

$$\|(-\Delta_g)^{1/2} e^{it\Delta_g} u_0\|_{L^{10} L^{\frac{30}{13}}(I \times \mathbf{R}^3)} \le \varepsilon \le \varepsilon_0,$$

there is a unique solution to (4.1) *on I with $u(0,x) = u_0$, which also satisfies*

$$\|(-\Delta_g)^{1/2} u\|_{L^{10} L^{\frac{30}{13}}} \le 2\varepsilon. \tag{4.9}$$

In particular, solutions with sufficiently small energy are global and scatter.

Proof. Run contraction mapping on the space X defined by the conditions

$$\|(-\Delta_g)^{1/2} u\|_{L^{10} L^{\frac{30}{13}}} \le 2\varepsilon, \quad \|(-\Delta_g)^{1/2} u\|_{L^\infty L^2} \le \|u_0\|_{\dot{H}^1} + \varepsilon$$

equipped with the metric $\rho(u,v) = \|(-\Delta_g)^{1/2}(u-v)\|_{L^{10} L^{\frac{30}{13}}}$. For each $u \in X$, let $\mathcal{I}(u)$ be the solution to the linear equation

$$(i\partial_t + \Delta_g)\mathcal{I}(u) = |u|^4 u$$

We check that for ε sufficiently small, the map $u \mapsto \mathcal{I}(u)$ is a contraction on X. By the Duhamel formula, Strichartz, the Leibniz rule, and Sobolev embedding,

$$\begin{aligned} \|(-\Delta_g)^{1/2}\mathcal{I}(u)\|_{L^{10} L^{\frac{30}{13}}} &\le \|(-\Delta_g)^{1/2} e^{it\Delta_g} u_0\|_{L^{10} L^{\frac{30}{13}}} + c\|(-\Delta_g)^{1/2}(|u|^4 u)\|_{L^2 L^{\frac{6}{5}}} \\ &\le \varepsilon + c\|(-\Delta_g)^{1/2} u\|^5_{L^{10} L^{\frac{30}{13}}} \le \varepsilon + c(2\varepsilon)^5, \end{aligned}$$

$$\|(-\Delta_g)^{1/2}\mathcal{I}(u)\|_{L^\infty L^2} \le \|(-\Delta_g)^{1/2}u_0\|_{L^2} + c(2\varepsilon)^5$$

Thus $\mathcal{I}$ maps X into itself.

For $u, v \in X$, the difference $\mathcal{I}(u) - \mathcal{I}(v)$ solves the equation with right hand side

$$|u|^4u - |v|^4v = \big(|u|^4 + \overline{u}v(|u|^2 + |v|^2)\big)(u - v) + v^2(|u|^2 + |v|^2)(\overline{u} - \overline{v}).$$

Hence, applying the Leibniz rule and Sobolev embedding repeatedly,

$$\begin{aligned}
&\|(-\Delta_g)^{1/2}[\mathcal{I}(u) - \mathcal{I}(v)]\|_{L^{10}L^{\frac{30}{13}}} \\
&\lesssim \|(-\Delta_g)^{1/2}(u - v)\|_{L^{10}L^{\frac{30}{13}}}(\|(-\Delta_g)^{1/2}u\|^4_{L^{10}L^{\frac{30}{13}}} + \|(-\Delta_g)^{1/2}v\|^4_{L^{10}L^{\frac{30}{13}}}) \\
&\lesssim (2\varepsilon)^4\|(-\Delta_g)^{1/2}(u - v)\|_{L^{10}L^{\frac{30}{13}}}.
\end{aligned}$$

□

Proposition 4.2.10. *Let $\tilde{u}$ solve the perturbed equation*

$$(i\partial_t + \Delta_g)\tilde{u} = \tilde{u}^4\tilde{u} + e, \tag{4.10}$$

and let $0 \in I$ be an interval such that

$$\|\tilde{u}\|_{Z(I)} \le L, \quad \|\nabla\tilde{u}\|_{L^\infty L^2} \le E.$$

Then there exists $\varepsilon_0(E, L)$ such that if $\varepsilon \le \varepsilon_0$ and

$$\|\tilde{u}(0) - u_0\|_{\dot{H}^1} + \|\nabla e\|_{N(I)} \le \varepsilon,$$

there is a unique solution u to (4.1) on I with $u(0) = u_0$, with

$$\begin{aligned}
\|u - \tilde{u}\|_{Z(I)} + \|\nabla(u - \tilde{u})\|_{L^2L^6\cap L^\infty L^2} &\le C(E, L)\varepsilon \\
\|\nabla u\|_{L^2L^6\cap L^\infty L^2(I\times\mathbf{R}^3)} &\le C(E, L).
\end{aligned}$$

4.3 Convergence of propagators

Theorem 4.3.1. *Let (λ_n, x_n) be a sequence of length scales and spatial centers conforming to one of the following scenarios:*

(a) $\lambda_n \to \infty$.

(b) $|x_n| \to \infty$.

(c) $x_n \to x_\infty$, $\lambda_n \to 0$.

Let $\Delta := \delta^{jk}\partial_j\partial_k$ *in the first two cases and* $\Delta := g^{jk}(x_\infty)\partial_j\partial_k$ *in the third. Then for any* $\phi \in \dot{H}^1$, *writing* $\phi_n = \lambda_n^{-\frac{d-2}{2}}\phi(\frac{\cdot - x_n}{\lambda_n})$, *we have*

$$\lim_{n\to\infty} \|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty L^6} = 0.$$

In cases (a), (b), the convergence actually occurs in $L^\infty \dot{H}^1$.

Proof. By approximation in $\dot{H}^1$, we may assume that ϕ is Schwartz.

Suppose first that $\lambda_n \to \infty$. By the Strichartz inequality the equivalence of Sobolev norms, and the Leibniz rule,

$$\begin{aligned}\|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty L^6} &\lesssim (-\Delta_g)^{1/2}(\Delta_g - \Delta)e^{it\Delta}\phi_n\|_{L^2L^{6/5}} \\ &\lesssim \|\chi\nabla e^{it\Delta}\phi_n\|_{L^2L^{6/5}} + \|\chi\nabla^2 e^{it\Delta}\phi_n\|_{L^2L^{6/5}} + \|\chi\nabla^3 e^{it\Delta}\phi_n\|_{L^2L^{6/5}}\end{aligned}$$

where $\chi(x)$ is the characteristic function of the unit ball. By Hölder and the Euclidean dispersive estimate,

$$\|\chi\nabla e^{it\Delta}\phi_n\|_{L^2L^{6/5}} = \lambda_n^2\|\chi(\lambda_n\cdot)e^{it\Delta}\phi\|_{L^2L^{6/5}} \lesssim \lambda_n^{-\frac{1}{2}}\|e^{it\Delta}\phi\|_{L^2L^\infty} \lesssim \lambda_n^{-1/2}\|\phi\|_{L^1}.$$

The terms involving two or more derivatives enjoy even better decay since $\lambda_n \to \infty$.

Assume now that $|x_n| \to \infty$, $\lambda_n \equiv \lambda_0 \in (0,\infty)$. By the Duhamel formula and Sobolev embedding,

$$\|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty L^2} \lesssim \|(\Delta_g - \Delta)e^{it\Delta}\phi_n\|_{L^1L^2} \lesssim \|\chi e^{it\Delta}\nabla\phi_n\|_{L^1L^2} + \|\chi e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2},$$

where χ is a bump function supported on the unit ball. For any fixed $T > 0$, decompose

$$\|\chi e^{it\Delta}\nabla\phi_n\|_{L^1L^2} \le \|\chi_n e^{it\Delta}\phi\|_{L^1L^2(\{|t|\le T\})} + \|\chi_n e^{it\Delta}\phi\|_{L^1L^2(\{|t|>T\})},$$

where $\chi_n = \chi(\cdot + x_n)$. The first term vanishes as $n \to \infty$ because the orbit $\{e^{it\Delta}\phi\}_{|t| \le T}$ is compact in L^2. We use Hölder's inequality and the dispersive estimate to bound the second term by

$$\|e^{it\Delta}\phi\|_{L^1L^\infty(\{|t|>T\})} \lesssim T^{-\frac{1}{2}}\|\phi\|_{L^1}.$$

As T may be chosen arbitrarily large, we conclude that $\lim_{n\to\infty}\|\chi e^{it\Delta}\nabla\phi_n\|_{L^1L^2} = 0$, and similar considerations estimate the term $\|\chi e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2}$. Finally, we have

$$\|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty L^6} \le \|\cdots\|_{L^\infty L^2}^{\frac{1}{3}}\|\cdots\|_{L^\infty L^\infty}^{\frac{2}{3}},$$

and the uniform norms may be estimated via Sobolev embedding:

$$\|e^{it\Delta_g}\phi_n\|_{L^\infty L^\infty} \lesssim \|(1-\Delta_g)e^{it\Delta_g}\phi_n\|_{L^\infty L^2} \lesssim \|(1-\Delta_g)\phi_n\|_{L^2} \lesssim 1.$$

Consider now the scenario where $|x_n| \to \infty$ and $\lambda_n \to 0$. We may assume that ϕ is compactly supported. Let χ be a smooth function such that $\chi(x) = 1$ when $|x| \ge 11/10$ and $\chi(x) = 0$ for $|x| \le 1$. First we show

$$\lim_{n\to\infty}\|(1-\chi)e^{it\Delta}\phi_n\|_{L^\infty L^6} = 0. \tag{4.11}$$

The function $\chi e^{it\Delta}\phi_n$ solves the equation

$$(i\partial_t + \Delta)(\chi e^{it\Delta}\phi_n) = [\chi, \Delta]e^{it\Delta}\phi_n.$$

Thus, by Sobolev embedding and the Duhamel formula,

$$\|(1-\chi)e^{it\Delta}\phi_n\|_{L^\infty L^6} \lesssim \|\nabla[\chi, \Delta]e^{it\Delta}\phi_n\|_{L^1L^2}.$$

The right side has the form

$$\|\beta e^{it\Delta}\nabla\phi_n\|_{L^1L^2} + \|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2}$$

where β is a bump function localizing to the unit ball. We focus on the potentially more dangerous second term. Fix $T > 0$ large, and split

$$\|\chi e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2} \le \|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2(\{|t|\le T\lambda_n\})} + \|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2(\{|t|>T\lambda_n\})}.$$

By Hölder in time and a change of variable, the first term may be written as

$$\|\beta(x_n + \lambda_n \cdot) e^{it\Delta} \nabla^2 \phi\|_{L^\infty L^2(\{|t| \le T\lambda_n^{-1}\})},$$

which goes to zero as $n \to \infty$ by approximate finite speed of propagation or, more precisely, by the Fraunhofer formula

$$\lim_{t\to\infty} \|e^{it\Delta} f - (2it)^{-\frac{3}{2}} \hat{f}(\tfrac{x}{2t}) e^{\frac{|x|^2}{4t}}\|_{L^2} = 0.$$

By the dispersive estimate,

$$\|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2(\{|t|>T\lambda_n\})} \lesssim \|e^{it\Delta}\nabla^2\phi_n\|_{L^1L^\infty(\{|t|>T\lambda_n\})} \lesssim \lambda_n^{\frac{1}{2}}(T\lambda_n)^{-\frac{1}{2}}\|\phi\|_{L^1} \lesssim T^{-\frac{1}{2}}.$$

Hence, choosing T arbitrarily large,

$$\lim_{n\to\infty} \|\beta e^{it\Delta}\nabla^2\phi_n\|_{L^1L^2} = 0,$$

establishing (4.11).

Since $\Delta = \Delta_g$ on the support of the cutoff χ, we also have

$$(i\partial_t + \Delta_g)(\chi e^{it\Delta}\phi_n) = [\chi, \Delta] e^{it\Delta}\phi_n,$$

so by the Duhamel formula, Sobolev embedding, and the equivalence of $\dot{H}^1$ Sobolev norms,

$$\begin{aligned} \|e^{it\Delta_g}\phi_n - \chi e^{it\Delta}\phi_n\|_{L^\infty L^6} &= \Big\|\int_0^t e^{i(t-s)\Delta_g}[\Delta,\chi]e^{is\Delta}\phi_n\,ds\|_{L^\infty L^6} \lesssim \|(-\Delta_g)^{1/2}[\chi,\Delta]e^{it\Delta}\phi_n\|_{L^1L^2} \\ &\lesssim \|\nabla[\chi,\Delta]e^{it\Delta}\phi_n\|_{L^1L^2} \end{aligned}$$

which was just estimated.

Finally, consider the last case where the profile ϕ_n is concentrating at a point. For $T > 0$, split

$$\|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty L^6} \le \|\cdots\|_{L^\infty L^6(\{|t|\le T\lambda_n^2\})} + \|\cdots\|_{L^\infty L^6(\{|t|>T\lambda_n^2\})}. \tag{4.12}$$

For the short time contribution, let χ be a bump function centered at the origin, fix $0 < \theta < 1$, and define

$$\chi_n = \chi\Big(\frac{\cdot - x_n}{\lambda_n^\theta}\Big),\ v_n = e^{it\Delta}\phi_n.$$

Then

$$(i\partial_t + \Delta)(\chi_n v_n) = [\Delta, \chi_n]v_n = 2\langle \nabla_\infty \chi_n, \nabla v_n\rangle_\infty + (\Delta\chi_n)v_n,$$

where the inner product on the right is respect to the metric $g(x_\infty)$, hence

$$\begin{aligned}\|(1-\chi_n)v_n\|_{L^\infty L^6(\{|t|\leq T\lambda_n^2\})} &\lesssim \|(1-\chi_n)v_n\|_{\dot H^1} + \|\nabla[\Delta,\chi_n]v_n\|_{L^1L^2(\{|t|\leq T\lambda_n^2\})}\\ &\lesssim o(1) + T\lambda_n\|\phi\|_{H^2}.\end{aligned}$$

Further, writing $(i\partial_t + \Delta) = (i\partial_t + \Delta_g) + (\Delta - \Delta_g)$, we obtain by the Duhamel formula and Sobolev embedding

$$\begin{aligned}\|e^{it\Delta_g}\phi_n - \chi_n e^{it\Delta}\phi_n\|_{L^\infty L^6(\{|t|\leq T\lambda_n^2\})} &\lesssim \|(1-\chi_n)\phi_n\|_{\dot H^1} + \|\nabla[\Delta,\chi_n]v_n\|_{L^1L^2(\{|t|\leq T\lambda_n^2\})}\\ &\quad + \|\nabla(\Delta_g - \Delta)(\chi_n v_n)\|_{L^1L^2(\{|t|\leq T\lambda_n^2\})}.\end{aligned}$$

The first two terms were estimated before. Writing out $\Delta_g - \Delta$ explicitly and using the Leibniz rule, we see that the worst contributions to the last term are quantities of the form

$$\|(g - g(x_\infty))\chi_n \nabla^3 v_n\|_{L^1L^2(\{|t|\leq T\lambda_n^2\})} \lesssim T\lambda_n^2\lambda_n^{-2}(|x_n - x_\infty| + \lambda_n^\theta)\|e^{it\Delta}\nabla^3\phi\|_{L^\infty L^2},$$

which is acceptable.

The long time contribution to (4.12) is bounded by

$$\|e^{it\Delta_g}\phi_n\|_{L^\infty L^6(\{|t|>T\lambda_n^2\})} + \|e^{it\Delta}\phi_n\|_{L^\infty L^6(\{|t|>T\lambda_n^2\})},$$

which are dealt with respectively by the extinction lemma in the next section and the usual dispersive estimate

$$\|e^{it\Delta}\phi_n\|_{L^\infty L^6(\{|t|>T\lambda_n^2\})} \lesssim T^{-1}\|\phi\|_{L^{6/5}}.$$

□

The proof of the last case yields the following corollary, which asserts that on short time intervals, the convergence in Case (c) of the theorem occurs in the energy norm as well.

Corollary 4.3.2. *Let (λ_n, x_n) be a sequence such that $x_n \to x_\infty$ and $\lambda_n \to 0$. Then for any $T > 0$*

$$\lim_{n\to\infty} \|e^{it\Delta_g}\phi_n - e^{it\Delta}\phi_n\|_{L^\infty \dot H^1([-T\lambda_n^2, T\lambda_n^2]\times\mathbf{R}^3)} = 0.$$

4.4 An extinction lemma

The purpose of this section is to prove a long-time weak dispersion estimate for linear profiles concentrating within a bounded distance of the origin, which arise in the last case of Theorem 4.3.1. For profiles with width h, we want to establish decay for times $t \geq Th^2$ as $h \to 0$ and $T \to \infty$. The analysis naturally splits into two cases, when $t \leq O(h)$ and $t \gg h$. We use semiclassical techniques for short times, while for longer times we invoke the global geometry to see that the solution is essentially Euclidean. Our tools consist of the frequency-localized dispersion estimate of Burq-Gerard-Tzvetkov [BGT04], a wavepacket parametrix, and a non-concentration estimate for the geodesic flow.

Proposition 4.4.1. *Let $d \geq 3$, and suppose $x_h \to x_0 \in \mathbf{R}^d$ as $h \to 0$. For any $\phi \in \dot{H}^1$, denoting $\phi_h = h^{-\frac{d-2}{2}}\phi(h^{-1}(\cdot - x_h))$, we have*

$$\lim_{T\to\infty} \limsup_{h\to 0} \|e^{it\Delta_g}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}(([Th^2,\infty)\times\mathbf{R}^d)} = 0.$$

Proof. We begin with several reductions. After a translation we may assume that $x_0 = 0$. Letting $\rho = |g|^{\frac{1}{4}}$ be the square root of the Riemannian density, we have $e^{it\Delta_g} = \rho^{-1}e^{-itA}\rho$, where the conjugated operator

$$A = \rho(-\Delta_g)\rho^{-1} = -\partial_j g^{jk}\partial_k + V$$

is self-adjoint on $L^2(dx)$ and V is a compactly supported potential. Thus

$$\begin{aligned}\|e^{it\Delta_g}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}} &\lesssim \|e^{-itA}\rho\phi_h\|_{L^{\frac{2d}{d-2}}} \lesssim \rho(x_h)\|e^{-itA}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}} + \|e^{-itA}(\rho-\rho(x_h))\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}} \\ &\lesssim \|e^{-itA}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}} + o(1) \text{ as } h \to 0,\end{aligned}$$

and it suffices to show

$$\lim_{T\to\infty} \limsup_{h\to 0} \|e^{-itA}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}([Th^2,\infty)\times\mathbf{R}^d)} = 0. \tag{4.13}$$

Further, we shall assume that ϕ is Schwartz and that

$$\operatorname{supp}\hat{\phi} \subset \{\varepsilon < |\xi| < \varepsilon^{-1}\} \tag{4.14}$$

for some $\varepsilon > 0$; the rescaled initial data ϕ_h are therefore frequency-localized to $\{h^{-1}\varepsilon < |\xi| < h^{-1}\varepsilon^{-1}\}$.

By the semiclassical dispersion estimate of Burq-Gerard-Tzvetkov [BGT04, Lemma A3] (see also [KT05, Proposition 4.7]), there exists $c > 0$ such that

$$\|e^{-itA}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}([Th^2,ch]\times\mathbf{R}^d)} \lesssim |Th^2|^{-1}\|\phi_h\|_{L^{\frac{2d}{d+2}}} = T^{-1}\|\phi\|_{L^{\frac{2d}{d+2}}}.$$

Hence, it remains to prove the long-time extinction

$$\lim_{h\to 0}\|e^{-itA}\phi_h\|_{L^\infty L^{\frac{2d}{d-2}}([ch,\infty)\times\mathbf{R}^d)} = 0. \tag{4.15}$$

Wavepacket decomposition

We begin by recalling the FBI transform and its basic properties. See for example [ST02] and the references therein. For each $h > 0$ and (x_0, ξ_0), define

$$\psi^h_{(x_0,\xi_0)}(y) = 2^{-\frac{d}{2}}\pi^{-\frac{3d}{4}}h^{\frac{3d}{4}}e^{\frac{i\xi_0(y-x_0)}{h}}e^{-\frac{(y-x_0)^2}{2h}},$$

which is a Gaussian wavepacket localized in phase space to the box

$$\{(x,\xi) : |x-x_0| \le h^{1/2},\ |\xi - h^{-1}\xi_0| \le h^{-1/2}\}.$$

The FBI transform at scale h is an isometry $T_h : L^2(\mathbf{R}^d) \to L^2(\mathbf{R}^d \times \mathbf{R}^d)$ defined by

$$T_hf(x,\xi) = \langle\psi^h_{(x,\xi)}, f\rangle = c_dh^{-\frac{3d}{4}}\int e^{\frac{i\xi(x-y)}{h}}e^{-\frac{(x-y)^2}{2h}}f(y)\,dy = c_dh^{-\frac{5d}{4}}\int e^{\frac{ix\eta}{h}}e^{-\frac{(\xi-\eta)^2}{2h}}\hat{f}(\tfrac{\eta}{h})\,d\eta.$$

From the adjoint formula $T_h^*F(y) = \int \psi^h_{(x,\xi)}(y)F(x,\xi)\,dxd\xi$, one obtains, for each $f \in L^2(\mathbf{R}^d)$ and $h > 0$, a decomposition

$$f = T_h^*T_hf = \int \langle\psi^h_{(x,\xi)}, f\rangle\psi^h_{(x,\xi)}\,dxd\xi.$$

into wavepackets of spatial width $h^{1/2}$. Such a decomposition is useful for studying semiclassical Schrödinger dynamics as the Schrödinger evolution of each wavepacket $\psi^h_{(x_0,\xi_0)}$ will remain coherent and behave essentially as a classical particle on time scales of order h.

Returning to our problem, write

$$\phi_h = \int \psi^h_{(x,\xi)} T_h \phi_h(x,\xi)\, dx d\xi.$$

We may restrict attention to just the wavepackets from the region

$$B = \{(x,\xi) : |x - x_h| \leq h^\theta,\ \tfrac{\varepsilon}{10} \leq \xi \leq \tfrac{10}{\varepsilon}\} \tag{4.16}$$

for any $\theta < \frac{1}{2}$. Indeed, if $|x - x_h| > h^\theta$ then

$$\begin{aligned} |T_h\phi_h(x,\xi)| &\lesssim h^{-\frac{3d}{4}} h^{-\frac{d-2}{2}} \int e^{-\frac{(x-x_h-y)^2}{2h}} |\phi(\tfrac{y}{h})|\, dy \\ &\lesssim h^{1-\frac{5d}{4}} \int_{|y|\leq |x-x_h|/4} + h^{1-\frac{5d}{4}} \int_{|y|>|x-x_h|/4} \\ &\lesssim_N h^{1-\frac{5d}{4}+\theta d} e^{-\frac{(x-x_h)^2}{ch}} + h^{1-\frac{5d}{4}} h^N |x-x_h|^{-N} \\ &\lesssim_{M,N} h^M |x-x_h|^{-N} \end{aligned}$$

for any $M, N \geq 0$. Similarly,

$$|T_h\phi_h(x,\xi)| \lesssim h^{1-\frac{3d}{4}} \int e^{-\frac{(\eta-\xi)^2}{2h}} |\hat{\phi}(\eta)|\, d\eta \lesssim \begin{cases} h^{1-\frac{3d}{4}} e^{-\frac{\varepsilon^2}{ch}}, & |\xi| < \varepsilon/10 \\ h^{1-\frac{3d}{4}} e^{-\frac{\xi^2}{ch}}, & |\xi| > 10/\varepsilon \end{cases}$$

In view of these bounds, we decompose

$$\phi_h = T_h^* 1_B T_h \phi_h + T_h^*(1 - 1_B) T_h \phi_h = f_h^1 + f_h^2, \tag{4.17}$$

where by the triangle inequality we obtain, for any $k \geq 0$,

$$\|\partial^k f_h^2\|_{L^2} \lesssim \int_{B^c} (h^{-\frac{d+k}{2}} + h^{-\frac{d}{2}} |h^{-1}\xi|^k) |T_h\phi_h(x,\xi)|\, dx d\xi = O(h^\infty).$$

By Sobolev embedding, it therefore suffices to show

$$\lim_{h\to 0} \|e^{-itA} f_h^1\|_{L^\infty L^{\frac{2d}{d-2}}((ch,\infty)\times \mathbf{R}^3)} = 0. \tag{4.18}$$

To prove this, we fix a large $T > 0$ and consider separately the time intervals $[ch, Th]$ and $[Th, \infty)$. On semiclassical time scales, the quantum evolution of wavepackets is modeled by the geodesic flow. More precisely, if $\psi^h_{(x,\xi)}$ is a typical wavepacket, then for $|t| \leq Th$ its Schrödinger evolution $e^{-itA}\psi^h_{(x,\xi)}$ will have width $C_T h^{1/2}$ and travel along the geodesic starting at x with initial momentum $h^{-1}\xi$ (that is, with velocity $h^{-1}g^{ab}\xi_b$).

If T is sufficiently large, then by the nontrapping assumption on the metric, all the wavepackets $e^{-iThA}\psi^h_{(x,\xi)}$ with $(x,\xi) \in B$ will have exited the curved region (this is why it is convenient to assume that ϕ is frequency-localized away from 0), and for $t \geq Th$ the solution $e^{-itA}\psi^h_{(x,\xi)}$ will radiate to infinity while dispersing essentially as a Euclidean free particle. The decay for $e^{-itA}f^1_h$ will then be a consequence of the dispersive properties of the Euclidean propagator $e^{it\Delta_{\mathbf{R}^3}}$.

It will be notationally convenient in the sequel to rescale time semiclassically, that is, replace t by th, so that each wavepacket $\psi^h_{(x,\xi)}$ travels at speed $O(1)$ under the propagator e^{-ithA}. The desired estimate then becomes

$$\lim_{h\to 0} \|e^{-ithA}f^1_h\|_{L^\infty L^{\frac{2d}{d-2}}((c,\infty)\times\mathbf{R}^3)} = 0.$$

Frequency-localization

We show next that the operator A may be replaced, up to acceptable errors, by a frequency-localized version. This will let us bring to bear the results of Koch and Tataru [KT05] concerning the evolution of wavepackets at fixed frequency.

Choose frequency cutoffs $\chi_j \in C_0^\infty(\mathbf{R}^d \setminus \{0\})$ such that

$$\{\xi : \varepsilon \leq |\xi| \leq \varepsilon^{-1}\} \prec \chi_1 \prec \chi_2 \prec \chi_3;$$

that is, $\chi_1(\xi) = 1$ on the annulus $\varepsilon \leq |\xi| \leq \varepsilon^{-1}$ and $\chi_j = 1$ near the support of χ_{j-1}. Set $A(h) = h^2 A$, let $a = g^{ij}\xi_i\xi_j$ be the principal symbol of A, and define the operator

$$A'(h) = (\chi_3 a)^w(X, hD) = (2\pi h)^{-d}\int_{\mathbf{R}^d} e^{\frac{i(x-y)\xi}{h}} a\Big(\frac{x+y}{2}, \xi\Big)\chi_3(\xi)\, d\xi.$$

We check that the propagator $e^{-\frac{itA'(h)}{h}}$, which preserves L^2, is also bounded on $\dot{H}^1$ when restricted to frequency h^{-1}.

Lemma 4.4.2.

$$\|e^{-\frac{itA'(h)}{h}}\chi_1(hD)\|_{\dot{H}^1\to\dot{H}^1} \le c_k(1+|t|).$$

Proof. Let $u_h = e^{-\frac{itA'(h)}{h}}\chi_1(hD)$ be the solution to the evolution equation

$$[hD_t + A'(h)]u_h = 0, u_h(0) = \chi_1(hD)\phi.$$

Differentiating this equation, we obtain

$$[hD_t + A'(h)](hD)u_h = [hD, A'(h)]u_h.$$

By pseudodifferential calculus, $\|[hD, A'(h)]\|_{L^2\to L^2} \le ch$, so

$$\begin{aligned}
\|hDu_h(t)\|_{L^2} &\le \|hDu_h(0)\|_{L^2} + h^{-1}\int_0^t \|[hD, A'(h)]u_h(s)\|_{L^2}\ ds \\
&\le \|hDu_h(0)\|_{L^2} + c|t|\|\chi_1(hD)\phi\|_{L^2} \\
&\le c(1+|t|)\|hD\chi_1(hD)\phi\|_{L^2}.
\end{aligned}$$

□

Lemma 4.4.3. *For each $T > 0$ and for all $|t| \le T$,*

$$\|(e^{-\frac{itA(h)}{h}} - e^{-\frac{itA'(h)}{h}})\chi_1(hD)\|_{\dot{H}^1\to\dot{H}^1} \le c_T h|t|$$

Proof. Write

$$e^{-\frac{itA(h)}{h}} - e^{-\frac{itA'(h)}{h}} = (e^{-\frac{itA(h)}{h}} - e^{-\frac{it\tilde{A}(h)}{h}}) + (e^{-\frac{it\tilde{A}(h)}{h}} - e^{-\frac{itA'(h)}{h}}),$$

where

$$\tilde{A}(h) = a^w(X, hD) = -h^2\partial_j g^{jk}\partial_k - \frac{h^2}{4}(\partial_j\partial_k g^{jk}).$$

By the Duhamel formula,

$$\|e^{-\frac{itA(h)}{h}}\phi - e^{-\frac{it\tilde{A}(h)}{h}}\phi\|_{L^2\to L^2} \le h\int_0^t \left|\frac{1}{4}\partial_j\partial_k g^{jk} + V\right|\|e^{-\frac{isA(h)}{h}}\phi\|_{L^2}\, ds \le ch|t|\|\phi\|_{L^2}.$$

Introducing the frequency-localization, we claim that the linear evolutions $e^{-\frac{itA(h)}{h}}$ and $e^{-\frac{it\tilde{A}(h)}{h}}$ essentially preserve frequency-support in the sense that

$$\|(1-\chi_2(hD))e^{-\frac{itA(h)}{h}}\chi_1(hD)\|_{L^2\to H^\sigma} \leq C_{N,\sigma}h^N$$

and similarly with A replaced by $\tilde{A}$. To see this, choose $\tilde{\chi}_1(\lambda)$ such that $\chi_1(\xi) \prec \tilde{\chi}_1(|\xi|_g^2) \prec \chi_2(\xi)$. By semiclassical functional calculus (see [BGT04]),

$$\|[1-\tilde{\chi}_1(A(h))]\chi_1(hD)\|_{L^2\to H^\sigma} \leq C_{N,\sigma}h^N,$$

whence

$$\begin{aligned}\|[1-\chi_2(hD)]e^{-\frac{itA(h)}{h}}\chi_1(hD)\|_{L^2\to H^\sigma} &\leq \|[1-\chi_2(hD)]\tilde{\chi}_1(A(h))e^{-\frac{itA(h)}{h}}\chi_1(hD)\|_{L^2\to H^\sigma} + C_{N,\sigma}h^N \\ &\leq C_{N,\sigma}h^N.\end{aligned}$$

The same proof goes through for the propagator $e^{-\frac{it\tilde{A}(h)}{h}}$. Thus

$$\begin{aligned}\|D(e^{-\frac{itA(h)}{h}} - e^{-\frac{it\tilde{A}(h)}{h}})\chi_1(hD)\phi\|_{L^2} &\leq h^{-1}\|(e^{-\frac{itA(h)}{h}} - e^{-\frac{\tilde{A}(h)}{h}})\chi_1(hD)\phi\|_{L^2} + O(h^\infty) \\ &\lesssim |t|h\|\chi_1(hD)h^{-1}\phi\|_{L^2} \lesssim |t|h\|D\phi\|_{L^2}.\end{aligned}$$

Now we prove

$$\|(e^{-\frac{it\tilde{A}(h)}{h}} - e^{-\frac{itA'(h)}{h}})\chi_1(hD)\|_{\dot{H}^1\to\dot{H}^1} \leq ch|t|.$$

For each $\phi \in \dot{H}^1$, the function $u_h = e^{-\frac{it\tilde{A}(h)}{h}}\chi_1(hD)\phi$ solves the equation $[hD_t + A'(h)]u_h = r_h$, where

$$r_h = [(\chi_3 - 1)a]^w(X, hD)\chi_2(hD)u_h + [(\chi_3-1)a]^w(X, hD)(1-\chi_2(hD))u_h.$$

As the symbols $(\chi_3 - 1)a$ and χ_2 have disjoint supports, the first term on the left is $O(h^\infty)$ in any Sobolev norm. The frequency localization of u_h implies that the second term is similarly negligible. By the Duhamel formula and Lemma 4.4.2, for any $T > 0$ and $|t| \leq T$ we have

$$\|(e^{-\frac{itA'(h)}{h}} - e^{-\frac{it\tilde{A}(h)}{h}})\chi_1(hD)\phi\|_{\dot{H}^1} \leq c_T\int_0^t \|r_h(s)\|_{\dot{H}^1}\,ds \leq c_{T,N}|t|h^N\|\phi\|_{\dot{H}^1}.$$

□

Returning to the decomposition (4.17) and recalling that $\phi_h = \chi_1(hD)\phi_h$, we have

$$\|(1-\chi_1(hD))f_h^1\|_{\dot{H}^1} = O(h^\infty),$$

By the previous lemma and Sobolev embedding, (4.18) will follow from the claims

$$\lim_{h\to 0} \|e^{-\frac{itA'(h)}{h}} f_h^1\|_{L_t^\infty L^{\frac{2d}{d-2}}((c,T)\times\mathbf{R}^d)} = 0 \tag{4.19}$$

$$\lim_{h\to 0} \|e^{-\frac{i(t-T)A(h)}{h}} e^{-\frac{iTA'(h)}{h}} f_h^1\|_{L^\infty L^{\frac{2d}{d-2}}((T,\infty)\times\mathbf{R}^d)} = 0. \tag{4.20}$$

Evolution of a wavepacket

For each $(x,\xi)\in B\subset T^*\mathbf{R}^d$, let $t\mapsto (x^t,\xi^t)$ denote the bicharacteristic starting at (x,ξ).

Proposition 4.4.4 (Short-time). *Let $\psi^h_{(x_0,\xi_0)}$ be a Gaussian wavepacket.*

Then

$$e^{-\frac{itA'(h)}{h}}\psi^h_{(x_0,\xi_0)}(x) = h^{-\frac{3d}{4}} v\Big(x_0,\xi_0,t,\frac{x-x_0^t}{h^{1/2}}\Big) e^{\frac{i}{h}[\xi_0^t(x-x_0^t)+\gamma(t,x_0,\xi_0)]},$$

where $\gamma(t,x_0,\xi_0) = \int_0^t (\xi a_\xi - a)(x_0^s,\xi_0^s)\,ds$, and $v(x_0,\xi_0,t,\cdot)$ is Schwartz uniformly in (x_0,ξ_0) and locally uniformly in t.

Proof. This was proved in [KT05] when $h=1$. We reduce to that case by a change of variable.

For fixed (x_0,ξ_0), let u be the solution to

$$[hD_t + A'(h)]u = 0,\ \ u(0) = \psi^h_{(x_0,\xi_0)},$$

and define the profile v by

$$u(t,x) = h^{-\frac{3d}{4}} v\Big(t,\frac{x-x_0^t}{h^{1/2}}\Big) e^{\frac{i}{h}[\xi_0^t(x-x_0^t)+\gamma(t,x_0,\xi_0)]}.$$

Then v solves the equation $[D_t + (a^h_{(x_0,\xi_0)})^w(t,X,D)v = 0, = v(0) = \psi^1_{(0,0)}$, where

$$a^h_{(x_0,\xi_0)}(t,x,\xi) = h^{-1}[a(t,h^{1/2}x+x_0^t,h^{1/2}\xi+\xi_0^t) - h^{1/2}\xi a_\xi(x_0^t,\xi_0^t) - h^{1/2}x a_x(x_0^t,\xi_0^t) - a(x_0^t,\xi_0^t)].$$

As $a^h_{(x_0,\xi_0)}$ vanishes to second order at $(0,0)$ and satisfies $|\partial_x^\alpha\partial_\xi^\beta a^h_{(x_0,\xi_0)}| \le c_{\alpha\beta}$ for all $|\alpha|+|\beta|\ge 2$, the claim follows from Lemma 4.4.5 below. $\square$

The following lemma was the key step in the proof of [KT05, Proposition 4.3]

Lemma 4.4.5. *Let $a(t,\cdot,\cdot)$ be a time-dependent symbol which vanishes to second order at $(0,0)$ and satisfies $|\partial_x^\alpha \partial_\xi^\beta a| \le c_{\alpha\beta}$ whenever $|\alpha|+|\beta| \ge 2$, and $S(t,s)$ be the solution operator for the evolution equation*

$$[D_t + a^w(t,X,D)]u = 0.$$

Then $S(t,s) : \mathcal{S}(\mathbf{R}^d) \to \mathcal{S}(\mathbf{R}^d)$ locally uniformly in $t-s$.

By the nontrapping hypothesis (G2), there exists $T = O(\varepsilon^{-1})$ such that for each $(x,\xi) \in B$, $|x^t| \ge 10$ for all $t \ge T$.

Proposition 4.4.6 (Long-time)**.** *For each $(x_0,\xi_0) \in B$ and $t \ge T$, we have a decomposition*

$$e^{-\frac{i(t-T)A(h)}{h}} e^{-\frac{iTA'(h)}{h}} \psi^h_{(x_0,\xi_0)} = v_1 + v_2,$$

where

$$|\partial^k v_1(t,x)| \le C_{k,N} h^{-\frac{3d}{4}-k} |t|^{-d/2} \Big(1 + \frac{|x - x_0^t|}{h^{1/2}|t|}\Big)^{-N}$$

and $\|v_2\|_{H^k} = O(h^\infty)$ for all k.

Proof. It suffices to verify the following two assertions:

$$|\partial_x^k e^{i(t-T)h\Delta} e^{-\frac{iTA'(h)}{h}} \psi^h_{(x_0,\xi_0)}(x)| \le C_{k,N} h^{-\frac{3d}{4}-k} |t|^{-d/2} \Big(1 + \frac{|x - x_0^t|}{h^{1/2}|t|}\Big)^{-N} \tag{4.21}$$

$$\|(e^{-i(t-T)hA} - e^{i(t-T)h\Delta}) e^{-\frac{iTA'(h)}{h}} \psi^h_{(x_0,\xi_0)}\|_{H^k} \le C_{k,N} h^N. \tag{4.22}$$

For ϕ Schwartz, by a stationary phase argument we have

$$|e^{it\Delta}\phi(x)| \le C_N \langle t\rangle^{-d/2} \Big\langle \frac{x}{2t} \Big\rangle^{-N}.$$

Indeed, let χ be a smoothed characteristic function of the unit ball, and partition

$$e^{it\Delta}\phi(x) = \int e^{i(x\xi - t|\xi|^2)} \chi\Big(\xi - \frac{x}{2t}\Big) \hat\phi(\xi)\, d\xi + \int e^{i(x\xi - t|\xi|^2)} [1 - \chi\Big(\xi - \frac{x}{2t}\Big)] \hat\phi(\xi)\, d\xi.$$

Integrating by parts in ξ, the second term is bounded, for any $N \geq 0$, by $|t|^{-N}\langle \frac{x}{2t} \rangle^{-N}$. By the stationary phase expansion, the first term equals

$$(2\pi i t)^{-d/2} e^{\frac{i|x|^2}{4t}} \hat{\phi}\Big(\frac{x}{2t}\Big) + t^{-\frac{d}{2}-1} R(t,x),$$
$$|R(t,x)| \leq c \|(1+|D_\xi|^2)^k \chi\Big(\xi - \frac{x}{2t}\Big)\hat{\phi}(\xi)\|_{L^2_\xi} \leq C_N \Big\langle \frac{x}{2t} \Big\rangle^{-N}.$$

By the previous proposition and standard identities for the Euclidean propagator,

$$e^{i(t-T)h\Delta} e^{-\frac{iTA'(h)}{h}} \psi^h_{(x_0,\xi_0)} = h^{-\frac{3d}{4}} e^{\frac{i}{h}[\xi_0^t(x-x_0^t)+\gamma(t,x_0,\xi_0)]} e^{i(t-T)h\Delta} \Psi_{(x_0,\xi_0)}\Big(\frac{x-x_0^t}{h^{1/2}}\Big),$$

where $\Psi_{(x_0,\xi_0)}$ is Schwartz uniformly in (x_0,ξ_0), and we have used the fact that $(x_0^t,\xi_0^t) = (x_0^T + 2(t-T)\xi_0^T, \xi_0^T)$ for all $t \geq T$. This settles (4.21) for $k=0$. When $k>0$, we note that differentiating the above equation brings down at worst a factor of $|h^{-1}\xi_0| \leq c\varepsilon^{-1}h^{-1}$.

To prove (4.22), we note first that e^{-itA} is uniformly bounded on each Sobolev space H^k. Indeed, for a sufficiently large $C>0$ the operators $(1-\Delta)^k(C+A)^{-k}$ and $(C+A)^k(1-\Delta)^{-k}$ are pseudodifferential operators of order 0, which implies that

$$\|u\|_{H^k} = \|(1-\Delta)^{k/2}u\|_{L^2} \sim \|(C+A)^{k/2}u\|_{L^2}.$$

Using the Duhamel formula, triangle inequality, and the above pointwise estimates, we can therefore bound the left side of (4.22) by

$$\int_T^\infty \sum_{m=0}^2 \|\chi D^m e^{i(t-T)h\Delta} e^{-\frac{iTA'(h)}{h}} \psi^h_{(x_0,\xi_0)}\|_{H^k} \leq C_N h^N \int_T^\infty |t|^{-d/2}\, dt \leq C_N h^N.$$

□

The geodesic flow and short-time extinction

By Proposition 4.4.4, on bounded time intervals each wavepacket may be regarded essentially as a particle moving under the geodesic flow. We will obtain the short-time decay (4.19) by showing that not too many wavepackets pile up near any point at any time. Heuristically, by the uncertainty principle the wavepackets have a broad distribution of initial momenta, and slower wavepackets will lag behind faster ones along each geodesic.

To make this rigorous, we need to study the bicharacteristics for the symbol $a = g^{jk}(x)\xi_j\xi_k$. Let $(x,\xi) \mapsto (x^t(x,\xi), \xi^t(x,\xi))$ be the flow on $T^*\mathbf{R}^d$ induced by the ODE

$$\begin{cases} \dot{x}^t = a_\xi = 2g(x^t)\xi^t, \\ \dot{\xi}^t = -a_x = -(\xi^t)^*(\partial_x g)(x^t)\xi^t \end{cases} \qquad (x^0, \xi^0) = (x, \xi);$$

The curve $t \mapsto x^t(x,\xi)$ is the geodesic starting at x with tangent vector $g^{ab}\xi_b$, and for fixed y the mapping $\eta \mapsto x^1(y,\eta)$ is the exponential map with basepoint y (although the exponential map is technically defined on the tangent space). A standard fact from geometry is the identity

$$x^t(x,\xi) = x^1(x, t\xi), \tag{4.23}$$

which follows from the observation that $s \mapsto (x^{ts}(x,\xi), t\xi^{ts}(x,\xi))$ is the bicharacteristic with initial data $(x, t\xi)$.

Lemma 4.4.7. *Let g be a nontrapping metric on $\mathbf{R}^d$. Then, for all $x, z \in \mathbf{R}^d$ with $|x| \le 1$ and all $0 \le r \le 1$,*

$$m(\{\xi \in \mathbf{R}^d : |x^1(x,\xi) - z| \le r\}) \le c_{x,z} r,$$

where m denotes Lebesgue measure on $\mathbf{R}^d_\xi$, and the constant $c_{x,z}$ is locally uniformly bounded in x and z. If also g is Euclidean outside a compact set, then

$$m(\{\xi \in \mathbf{R}^d : |x^1(x,\xi) - z| \le r\}) \le c_x (1 + |z|)^{d-1} r.$$

The basic idea is that the preimage of a small ball under the exponential map will always be thin in the radial direction, though not necessarily in the other directions. This is a consequence of the fact that the exponential map always has nontrivial radial derivative. Note that for z near x, the above bound can be improved to $O(r^d)$ as the map $\xi \mapsto x^1(x,\xi)$ is a diffeomorphism for ξ near 0.

Proof. Fix x and z. For each ξ we have

$$x^1(x, \xi + \zeta) = x^1(x,\xi) + (\partial_\xi x^1)\zeta + r(\zeta), \ |r(\zeta)| = O(|\zeta|^2).$$

Differentiating the scaling relation (4.23) in t, we have

$$\partial_\xi x^1(x,\xi)\xi = \dot{x}^1(x,\xi) = g(x^1)\xi^1(x,\xi)$$

which implies that

$$|\xi^1||\partial_\xi x^1(x,\xi)\xi| \geq \xi^1 \cdot (\partial_\xi x^1)\xi = 2g(x^1)^{jk}(\xi^1)^j(\xi^1)^k \gtrsim |\xi^1|^2.$$

Using also the fact that $g^{jk}(x)\xi_j\xi_k$ is conserved along the flow, it follows that

$$|\partial_\xi x^1(x,\xi)\xi| \geq c|\xi|.$$

Thus, if ζ_0 is such that that $|r(\zeta)| \leq \frac{c}{2}|\zeta|$ for $|\zeta| \leq \zeta_0$, then

$$\frac{c}{2}|\zeta| \leq |x^1(x,\xi+\zeta) - x^1(x,\xi)| \leq 2c|\zeta|$$

for all ζ parallel to ξ with length at most ζ_0.

Let $S_{x,z}$ denote the set on the left side in the lemma; the nontrapping hypothesis implies that $S_{x,z}$ is compact. By the preceding considerations, the intersection of each ray $t \mapsto \frac{t\xi}{|\xi|}$ with $S_{x,z}$ has measure $O(r)$. The first inequality now follows by integrating in polar coordinates.

Under the additional hypothesis that g is flat outside a compact set, observe that for each x there exists $R > 0$ such that

$$\sup_{|\xi|_g=1} |x^t(x,\xi)| - R < 2t < \inf_{|\xi|_g=1} |x^t(x,\xi)| + R, \tag{4.24}$$

where $|\xi|_g^2 = g^{jk}(x)\xi_j\xi_k$. Indeed, for T sufficiently large and $t \geq T$,

$$x^t(x,\xi) = x^T(x,\xi) + 2(t-T)\xi^T(x,\xi),$$

and $|\xi^T| = |\xi^T|_g = |\xi^0|_g = |\xi|_g = 1$. Set $r = \sup_{|\xi|_g=1} |x^T(x,\xi)|$ to get

$$2|t-T| - r \leq |x^t(x,\xi)| \leq 2|t-T| + r.$$

Therefore, $S_{x,z}$ is contained in an annulus $\{|z| - R_x \leq |\xi| \leq |z| + R_x\}$, which is covered by $c(1+|z|)^{d-1}r^{1-d}$ cones of width r. Arguing as before, we obtain the improved bound. $\square$

We are now ready to establish the short-time extinction (4.19). We have

$$|e^{-\frac{itA'(h)}{h}} f_h^1(x)| \le \int_B |e^{-\frac{itA'(h)}{h}} \psi^h_{(x_0,\xi_0)}(x)||T_h\phi_h(x_0,\xi_0)|\, dx_0 d\xi_0.$$

By Proposition 4.4.4, each $e^{-\frac{itA'(h)}{h}} \psi^h_{(x_0,\xi_0)}$ concentrates in a radius $h^{1/2}$ ball at $x^t(x_0,\xi_0)$, so the integral is $O(h^\infty)$ for all $|x| \gg T\varepsilon^{-1}$.

For $|x| \lesssim T\varepsilon^{-1}$, modulo $O(h^\infty)$ we may restrict the integral to the region

$$\{(x_0,\xi_0) \in B : |x^t(x_0,\xi_0) - x| < h^\alpha\}$$

for any $\alpha < \frac{1}{2}$. By Hölder and the rapid decay of ϕ_h,

$$|T_h\phi_h(x_0,\xi_0)| \le h^{-\frac{3d}{4}} \Big(\int_{|y|\le h^{1-}} e^{-\frac{|y|^2}{h}}\, dy \Big)^{\frac{d+2}{2d}} \lesssim h^{1-\frac{d}{4}-\varepsilon}$$

for any $\varepsilon > 0$. Combining this with Proposition 4.4.7 and the definition (4.16) of B,

$$\int_B |e^{-\frac{itA'(h)}{h}} \psi^h_{(x_0,\xi_0)}(x)||T_h\phi_h(x_0,\xi_0)|\, dx_0 d\xi_0 \le h^{1-\frac{d}{4}} h^{-\frac{3d}{4}} h^{d\theta} h^{\alpha} = h^{1+\alpha-d(1-\theta)}.$$

As θ and α may be chosen arbitrarily close to $\frac{1}{2}$, it follows that

$$|e^{-\frac{itA'(h)}{h}} f_h^1(x)| \lesssim_\varepsilon h^{\frac{3}{2}-\frac{d}{2}-\varepsilon}$$

for any $\varepsilon > 0$. Therefore

$$\|e^{-\frac{itA'(h)}{h}} f_h^1\|_{L^{\frac{2d}{d-2}}} \le \|e^{-\frac{itA'(h)}{h}} f_h^1\|_{L^2}^{1-\frac{2}{d}} \|e^{-\frac{itA'(h)}{h}} f_h^1\|_{L^\infty}^{\frac{2}{d}} \lesssim_\varepsilon h^{1-\frac{2}{d}+\frac{3}{d}-1-\varepsilon} \lesssim_\varepsilon h^{\frac{1}{d}-\varepsilon}.$$

Remark. When g is the Euclidean metric, the exponential map is a diffeomorphism, so the bound in Proposition 4.4.7 is $O(r^d)$. Consequently,

$$\int_B |e^{-\frac{itA'(h)}{h}} \psi^h_{(x_0,\xi_0)}(x)||T_h\phi_h(x_0,\xi_0)|\, dx_0 d\xi_0 \lesssim h^{1-\frac{d}{4}} h^{-\frac{3d}{4}} h^{d(\theta+\alpha)} \lesssim_\varepsilon h^{1-\varepsilon}$$

for any $\varepsilon > 0$, and we find that

$$\|e^{-\frac{itA'(h)}{h}} f_h^1\|_{L^{\frac{2d}{d-2}}} \lesssim_\varepsilon h^{1-\varepsilon},$$

recovering modulo arbitrarily small losses the $O(h)$ decay rate predicted by the $L^{\frac{2d}{d+2}} \to L^{\frac{2d}{d-2}}$ dispersive estimate for the Euclidean propagator $e^{it\Delta}$. The epsilon loss can be avoided if, instead of truncating crudely in phase space (4.16), we account for the contribution from each dyadic annulus $\{2^{k-1}h^{1/2} \le |x| < 2^k h^{1/2}\}$, using the rapid decay of each wavepacket on the $h^{1/2}$ scale. We omit the details.

Remark. Rather than exhibiting decay in $L^{\frac{2d}{d-2}}$, one can adapt these arguments to get $L^1 \to L^\infty$ bounds.

Long-time extinction

To prove (4.20), use Proposition (4.4.6) to write

$$e^{-\frac{i(t-T)A(h)}{h}} e^{-\frac{iTA'(h)}{h}} f_h^1 = \int_B v^h_{(x_0,\xi_0)}(t) T_h \phi_h(x_0,\xi_0)\, dx_0 d\xi_0 + \int_B r^h_{(x_0,\xi_0)}(t) T_h \phi_h(x_0,\xi_0)\, dx_0 d\xi_0,$$

where

$$|v_{(x_0,\xi_0)}(t,x)| \le C_N h^{-\frac{3d}{4}} |t|^{-d/2} \Big(1 + \frac{|x - x_0^t|}{h^{1/2}|t|}\Big)^{-N}, \quad \|r_{(x_0,\xi_0)}\|_{H^1} = O(h^\infty).$$

The second integral is clearly negligible in $L^\infty L^6$.

To estimate the first integral, we proceed as in the short-time estimate, interpolating between L^2 and L^∞ to exhibit decay in L^6. For fixed x, modulo $O(h^\infty)$ we may restrict the integral to the region

$$B' = \{(x_0,\xi_0) \in B : |x^t(x_0,\xi_0) - x| \le h^\alpha(1+|t|)\}$$

for any $\alpha < \frac{1}{2}$. As $x^t = x^T + 2(t-T)\xi^T$ when $t \ge T$, for each $(x_0,\xi_0) \in B$ with $|\xi_0|_g = 1$, the ray $r \mapsto (x_0, r\xi_0)$ intersects the above set in an interval of width $O(h^\alpha)$. The region B' therefore has measure at most $O(h^{d\theta}h^\alpha)$, and we obtain

$$\int_{B'} |v^h_{(x_0,\xi_0)}(t,x)||T_h\phi_h(x_0,\xi_0)|\, dx_0 d\xi_0 \le c_\varepsilon h^{1-\frac{d}{4}} h^{-\frac{3d}{4}} |t|^{-d/2} h^{d\theta+\alpha} = h^{1+\alpha-d(1-\theta)}|t|^{-d/2}.$$

Hence, recalling that θ may be chosen arbitrarily close to $\frac{1}{2}$, for any $\varepsilon > 0$ we have

$$\|e^{-\frac{i(t-T)A(h)}{h}} e^{-\frac{iTA'(h)}{h}} f_h^1\|_{L^\infty L^6} \lesssim T^{-1} h^{1-\frac{2}{d}} h^{\frac{2(1+\alpha)}{d} - 2(1-\theta)} \lesssim_\varepsilon T^{-1} h^{\frac{1}{d}-\varepsilon}.$$

This completes the proof of Proposition 4.4.1. □

4.5 Linear profile decomposition

The profile decomposition will follow from repeated application of the following inverse Strichartz theorem.

Proposition 4.5.1. *Let* $\{f_n\} \subset \dot{H}^1$ *be a sequence such that* $\|f_n\|_{\dot{H}^1} \leq A$ *and* $\|e^{it\Delta_g} f\|_{L^\infty L^6} \geq \varepsilon$*. Then there exist a function* $\phi \in \dot{H}^1$ *and parameters* t_n*,* x_n*,* λ_n *such that after passing to a subsequence,*

$$\lim_{n\to\infty} G_n^{-1} e^{it_n\Delta_g} \rightharpoonup \phi \text{ in } \dot{H}^1(g), \tag{4.25}$$

where $G_n\phi = \lambda_n^{-\frac{1}{2}} \phi(\frac{\cdot - x_n}{\lambda_n})$*. Setting* $\phi_n = e^{-it_n\Delta_g} G_n\phi$*, we have*

$$\liminf_n \|\phi_n\|_{\dot{H}^1(g)} \gtrsim \varepsilon^{\frac{9}{4}} A^{-\frac{5}{4}}. \tag{4.26}$$

$$\lim_n \|f_n\|_{\dot{H}^1}^2 - \|f_n - e^{-it_n\Delta_g} G_n\phi\|_{\dot{H}^1}^2 - \|e^{-it_n\Delta_g} G_n\phi\|_{\dot{H}^1}^2 = 0. \tag{4.27}$$

$$\lim_n \|f_n\|_{L^6}^6 - \|f_n - \phi_n\|_{L^6}^6 - \|\phi_n\|_{L^6}^6 = 0 \tag{4.28}$$

Finally, the t_n *may be chosen so that either* $t_n \equiv 0$ *or* $\lambda_n^{-2} t_n \to \infty$*.*

Proof. The proof has the same structure as in the previous chapters, we give a complete exposition as some of the technical details are different.

We have the following inverse Sobolev lemma:

Lemma 4.5.2. *If* $\|f\|_{\dot{H}^1} \leq A$ *and* $\|e^{it\Delta_g} f\|_{L^\infty L^6} \geq \varepsilon$*, then there exist* t*,* x*,* N*, such that*

$$|(\tilde{P}_N)^2 e^{it\Delta_g} f)(x)| \gtrsim N^{\frac{1}{2}} \varepsilon^{\frac{9}{4}} A^{1-\frac{9}{4}}. \tag{4.29}$$

Proof sketch. A Littlewood-Paley theory argument yields the usual Besov refinement of Sobolev embedding (see [GMO97])

$$\|e^{it\Delta_g} f\|_{L^\infty L^6} \lesssim \|f\|_{\dot{H}^1}^{\frac{1}{3}} \|(\tilde{P}_N)^2 e^{it\Delta_g} f\|_{L^\infty L^6}^{\frac{2}{3}}.$$

Then one interpolates $L^\infty L^6$ between $L^\infty L^2$ and $L^\infty L^\infty$, using the elementary inequality

$$\|(\tilde{P}_N)^2 f\|_2 \lesssim N^{-1} \|\tilde{P}_N f\|_{\dot{H}^1},$$

which follows from the corresponding pointwise inequality of symbols. □

Select (t_n, x_n, N_n) according to this lemma, and set $\lambda_n = N_n^{-1}$. After passing to a subsequence, we may assume $\lambda_n \to \lambda_\infty \in [0, \infty]$ and $x_n \to x_\infty \in \mathbf{R}^d \cup \{\infty\}$. We may extract a weak limit

$$G_n^{-1} e^{it_n \Delta_g} f_n \rightharpoonup \phi \text{ in } \dot{H}^1(g).$$

As $\dot{H}^1(g)$ and $\dot{H}^1(\delta)$ have equivalent norms, their duals may be identified; hence the weak limit also holds in $\dot{H}^1(\delta)$.

Define $\phi_n = e^{-it_n \Delta_g} G_n \phi$.

We verify that this profile has positive energy. From Theorem 4.2.6 and the facts that $d_g(x, y) \sim |x - y|$, $dg = \sqrt{|g|}\, dx \sim dx$, there exist constants $c_1, c_2 > 0$ such that

$$N_n^{\frac{1}{2}} \varepsilon^{\frac{9}{4}} A^{-\frac{5}{4}} \leq c_1 N_n^3 \int e^{-c_2 N_n^2 |x_n - y|^2} |e^{it_n \Delta_g} f_n|(y) dy.$$

Thus

$$c \varepsilon^{\frac{9}{4}} A^{-\frac{5}{4}} \leq \int e^{-|y|^2} G_n^{-1} |e^{it_n \Delta_g} f_n|(y)\, dy.$$

As $G_n^{-1} e^{it_n \Delta_g} f_n \rightharpoonup \phi$ in $\dot{H}^1$, $|G_n^{-1} e^{it_n \Delta_g} f_n| \rightharpoonup |\phi|$ in $\dot{H}^1$. Indeed, by the Rellich-Kondrashov theorem, the sequences $G_n^{-1} e^{it_n \Delta_g}$ and $|G_n^{-1} e^{it_n \Delta_g}|$ converge to their $\dot{H}^1$ weak limits in L^2_{loc}.

Taking $n \to \infty$ in the above inequality and bounding $e^{-|y|^2}$ in $\dot{H}^{-1}$ by its $L^{6/5}$ norm, we get

$$\varepsilon^{\frac{9}{4}} A^{-\frac{5}{4}} \leq \int e^{-|y|^2} |\phi|\, dy \lesssim \| |\phi| \|_{\dot{H}^1} \lesssim \|\phi\|_{\dot{H}^1}.$$

The claim (4.26) follows from the equivalence of $\dot{H}^1(\delta)$ and $\dot{H}^1(g)$.

To prove the decoupling (4.27), write

$$\begin{aligned} \|f_n\|_{\dot{H}^1}^2 - \|f_n - e^{-it_n \Delta_g} G_n \phi\|_{\dot{H}^1}^2 - \|e^{-it_n \Delta_g} \phi\|_{\dot{H}^1} &= 2 \operatorname{Re} \langle e^{it_n \Delta_g} f_n - G_n \phi, G_n \phi \rangle_{\dot{H}^1} \\ &= 2 \operatorname{Re} \langle G_n^{-1} e^{it_n \Delta_g} f_n - \phi, \phi \rangle_{\dot{H}^1(g_n)}, \end{aligned}$$

where $g_n(x) = g(x_n + \lambda_n x)$.

To see that the right side goes to 0, we consider two cases. If $\lambda_\infty < \infty$, then by Arzelà-Ascoli, after passing to a subsequence the metrics g_n converge boundedly and locally uniformly to some metric g_∞. If on the other hand $\lambda_\infty = \infty$, then g_n converges weakly to the Euclidean metric as $g_n(x) = \delta$ outside the shrinking balls $|x - x_n| \leq \lambda_n^{-1}$.

To streamline the presentation, in the sequel we let g_∞ denote the locally uniform limit in the first case and $g_\infty = \delta$ in the second case.

When $\lambda_\infty < \infty$, then

$$\langle G_n^{-1} e^{it_n \Delta_g} f_n - \phi, \phi \rangle_{\dot{H}^1(g_n)} = \langle G_n^{-1} e^{it_n \Delta_g} f_n - \phi, \phi \rangle_{\dot{H}^1(g_\infty)} + o(1) \to 0$$

by dominated convergence.

If $\lambda_\infty = \infty$, writing

$$\langle u, v \rangle_{\dot{H}^1(g_n)} = \int \nabla u \cdot \overline{\nabla v}\, dx + \int_{|x - x_n| \leq \lambda_n^{-1}} \langle du, dv \rangle_{g_n} \, dg_n - \int_{|x - x_n| \leq \lambda_n^{-1}} \nabla u \cdot \overline{\nabla v}\, dx,$$

we have

$$\langle G_n^{-1} e^{it_n \Delta_g} f_n - \phi, \phi \rangle_{\dot{H}^1(g_n)} = \langle G_n^{-1} e^{it_n \Delta_g} f_n - \phi, \phi \rangle_{\dot{H}^1(\delta)} + o(1),$$

which vanishes since $G_n^{-1} e^{it_n \Delta_g} f_n - \phi \rightharpoonup 0$ in $\dot{H}^1(\delta)$.

We show next that if $t_n \lambda_n^{-2}$ is bounded, then after modifying the profile ϕ slightly we may arrange for $t_n \equiv 0$.

Suppose that $t_n \lambda_n^{-2} \to t_\infty$. By Theorem 4.3.1 and its corollary, we have

$$\phi_n = e^{-it_n \Delta_g} G_n \phi = G_n e^{-it_\infty \Delta} \phi + r_n, \; \|r_n\|_{\dot{H}^1} = o(1).$$

Define the modified profile $\tilde{\phi} = e^{-it_\infty \Delta}$, $\tilde{\phi}_n = G_n \tilde{\phi}$. Clearly (4.26) holds with $\tilde{\phi}_n$ in place of ϕ_n. We claim that

$$G_n^{-1} f_n \rightharpoonup \tilde{\phi} \text{ in } \dot{H}^1(g). \tag{4.30}$$

Suppose λ_n is bounded above. Passing to a subsequence, the metrics g_n converge locally uniformly to some metric g_∞. Then as

$$\begin{aligned}
\langle G_n^{-1} f_n - e^{-it_\infty \Delta}\phi, \psi\rangle_{\dot{H}^1(g_n)} &= \langle f_n - G_n e^{-it_\infty \Delta}\phi, G_n\psi\rangle)_{\dot{H}^1(g)} + o(1) \\
&= \langle e^{it_n \Delta_g} f_n - G_n \phi, e^{it_n\Delta_g} G_n \psi\rangle_{\dot{H}^1(g)} + o(1) \\
&= \langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, G_n^{-1} e^{it_n \Delta_g} G_n\psi\rangle_{\dot{H}^1(g_n)} + o(1) \\
&= \langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, e^{it_\infty\Delta}\psi\rangle_{\dot{H}^1(g_n)} + o(1) \\
&= \langle G_n^{-1} e^{it_n\Delta_g} f_n - \phi, e^{it_\infty\Delta}\psi\rangle_{\dot{H}^1(g_\infty)} + o(1) \\
&= o(1),
\end{aligned}$$

we have for all $\psi \in \dot{H}^1$

$$\langle G_n^{-1} f_n - e^{-it_\infty\Delta}\phi, \psi\rangle_{\dot{H}^1(g_\infty)} = o(1)$$

which implies weak convergence in $\dot{H}^1(g)$ since the norms defined by g_∞ and g are equivalent.

If instead $\lambda_n \to \infty$, then as before

$$\langle G_n^{-1} f_n - e^{-it_\infty\Delta}\phi, \psi\rangle_{\dot{H}^1(\delta)} = \langle G_n^{-1} f_n - e^{-it_\infty\Delta}\phi, \psi\rangle_{\dot{H}^1(g_n)} + o(1) \to 0.$$

Having verified the weak limit (4.30), the same argument as before establishes the decoupling of kinetic energies (4.27).

To establish the asymptotic additivity of nonlinear energy (4.28), we use the refined Fatou lemma of Brezis and Lieb:

Lemma 4.5.3 ([BL83])**.** *Suppose $f_n \in L^p(\mu)$ converge a.e. to some $f \in L^p(\mu)$ and* $\sup_n \|f_n\|_{L^p} < \infty$*. Then*

$$\int_{\mathbf{R}^d} \Big| |f_n|^p - |f_n - f|^p - |f|^p \Big| \, d\mu \to 0.$$

Proof sketch. Partition $\mathbf{R}^d$ into B and B^c, where B is a large ball that captures essentially all of the L^p norm of f. The integral over B converges to 0 by Egorov's theorem. Over B^c,

f is essentially negligible and the terms $|f_n|^p$ and $|f_n - f|^p$ nearly cancel:

$$\int_{B^c} \Big| |f_n|^p - |f_n - f|^p - |f|^p \Big| \, d\mu \le c \int_{B^c} |f|(|f_n|^{p-1} + |f_n - f|^{p-1}) \, d\mu + \int_{B^c} |f|^p \, d\mu$$
$$\le c\|f\|_{L^p(B^c)}(\|f_n\|_{L^p}^{p-1} + \|f_n - f\|_{L^p(B^c)}^{p-1}) + \|f\|_{L^p(B^c)}^p.$$

□

Assume $t_n \equiv 0$. Then $\phi_n = G_n \phi$ and $G_n^{-1} f_n$ converges weakly in $\dot{H}^1$ to ϕ. By Rellich-Kondrashov and a diagonalization argument, after passing to a subsequence we have $G_n^{-1} f_n \to \phi$ pointwise a.e. By a change of variable, the left side of (4.28) is bounded by

$$\int \Big| |G_n^{-1} f_n|^6 - |G_n^{-1} f_n| - \phi|^6 - |\phi|^6 \Big| \, dg_n \le \int \, dg_\infty + \int \, d|g_n - g_\infty|,$$

where we write $d|g_n - g_\infty|$ for the density $|\sqrt{|g_n|} - \sqrt{|g_\infty|}| \, dx$. The first term vanishes by the Brezis-Lieb lemma, while to deal with the second integral we note that $\int |\phi|^6 \, d|g_n - g_\infty| \to 0$ and argue as in the proof of that lemma.

Suppose $t_n \lambda_n^{-2} \to \infty$ (the case $t_n \lambda_n^{-2} \to -\infty$ is similar). Different arguments are required depending on the behavior of the parameters, but in each case we conclude that

$$\lim_{n \to \infty} \|\phi_n\|_{L^6} = 0,$$

which clearly implies (4.28).

If $\lambda_\infty = \infty$ or $x_\infty = \infty$, then by Theorem 4.3.1 we have

$$\phi_n = e^{-it_n \Delta} G_n \phi + r_n, \ \ \|r_n\|_{L^6} = o(1),$$

and the decay in L^6 follows from the dispersive estimate for the Euclidean propagator.

If $0 < \lambda_\infty < \infty$ and $x_\infty \in \mathbf{R}^3$, then $G_n \phi \to \phi'$, and we appeal to Lemma 4.5.4 below to find $\tilde{\phi} \in \dot{H}^1$ such that

$$\lim_{t \to \infty} \|e^{it\Delta_g} \phi' - e^{it\Delta} \tilde{\phi}\|_{\dot{H}^1} \to 0.$$

We bound by the triangle inequality

$$\|e^{it_n \Delta_g} G_n \phi\|_{L^6} \le \|e^{it\Delta_g}(G_n \phi - \phi')\|_{L^6} + \|e^{it\Delta_g} \phi' - e^{it\Delta} \tilde{\phi}\|_{L^6} + \|e^{it_n \Delta} \tilde{\phi}\|_{L^6}$$

and use the dispersive estimate and Sobolev embedding.

For the remaining case where $\lambda_\infty = 0$ and $x_\infty \in \mathbf{R}^3$, we invoke the extinction lemma.

Lemma 4.5.4 (Linear asymptotic completeness)**.** *The limits* $\lim_{t\to\pm\infty} e^{-it\Delta_\delta} e^{it\Delta_g}$ *exist strongly in* $\dot{H}^1$*.*

Proof. Suppose first that $\phi \in C_0^\infty$. By the Duhamel formula,

$$e^{-it\Delta} e^{it\Delta_g} \phi = \phi + i \int_0^t e^{-is\Delta}(\Delta_g - \Delta) e^{is\Delta_g} \phi \, ds,$$

and we need to show that

$$\lim_{t\to\infty} \int_0^t e^{-is\Delta}(\Delta_g - \Delta) e^{is\Delta_g} \phi \, ds$$

exists in $\dot{H}^1$. We use (the dual of) the endpoint Strichartz estimate $e^{it\Delta_g} : L^2 \to L^2 L^6$. For $t_1 < t_2$, we have

$$\begin{aligned} &\Big\| \int_{t_1}^{t_2} e^{-is\Delta}(\Delta_g - \Delta) e^{is\Delta_g} \, ds \Big\|_{\dot{H}^1} \lesssim \|\nabla(\Delta_g - \Delta) e^{it\Delta_g} \phi\|_{L^2 L^{6/5}([t_1,t_2])} \\ &\lesssim \|\chi \nabla e^{it\Delta_g} \phi\|_{L^2 L^{6/5}} + \|\chi \nabla^2 e^{it\Delta_g} \phi\|_{L^2 L^{6/5}} + \|\chi \nabla^3 e^{it\Delta_g} \phi\|_{L^2 L^{6/5}} \end{aligned}$$

for some bump function χ. Using Hölder, the equivalence of Sobolev spaces, and the Strichartz inequality, each term is bounded by

$$\|\chi\|_{L^{3/2}} \|(1-\Delta)^{3/2} e^{it\Delta_g} \phi\|_{L^2 L^6([t_1,t_2])} \lesssim \|(1-\Delta_g)^{3/2} e^{it\Delta_g} \phi\|_{L^2 L^6([t_1,t_2])} \lesssim \|\phi\|_{H^3}.$$

As $t_1, t_2 \to \infty$, the left side goes to 0. Thus

$$\lim_{t\to\infty} e^{-it\Delta_\delta} e^{it\Delta_g} \phi$$

exists in $\dot{H}^1$ for any $\phi \in C_0^\infty$.

For general $\phi \in \dot{H}^1$, select for each $\varepsilon > 0$ some $\phi_\varepsilon \in C_0^\infty$ with $\|\phi - \phi_\varepsilon\|_{\dot{H}^1} < \varepsilon$. Write $W(t) = e^{-it\Delta_\delta} e^{it\Delta_g}$,

$$W(t)\phi = W(t)\phi_\varepsilon + W(t)(\phi - \phi_\varepsilon).$$

As $W(t)$ are bounded on $\dot{H}^1$ uniformly in t, we have for all $t_1 < t_2$

$$\|W(t_2)\phi - W(t_1)\phi\|_{\dot{H}^1} \le \|W(t_2)\phi_\varepsilon - W(t_1)\phi_\varepsilon\|_{\dot{H}^1} + c\varepsilon;$$

so $W(t)\phi$ also converges in $\dot{H}^1$. □

Definition 4.5.1. Two frames $(\lambda_n^1, t_n^1, x_n^1)$ and $(\lambda_n^2, t_n^2, x_n^2)$ are *orthogonal* if

$$\frac{\lambda_n^1}{\lambda_n^2} + \frac{\lambda_n^2}{\lambda_n^1} + \frac{|t_n^1 - t_n^2|}{\lambda_n^1 \lambda_n^2} + \frac{|x_n^1 - x_n^2|}{\sqrt{\lambda_n^1 \lambda_n^2}} = \infty.$$

They are *equivalent* if

$$\frac{\lambda_n^1}{\lambda_n^2} \to \lambda_\infty \in (0,\infty),\ \frac{t_n^1 - t_n^2}{\lambda_n^1 \lambda_n^2} \in \mathbf{R},\ \frac{x_n^1 - x_n^2}{\sqrt{\lambda_n^1 \lambda_n^2}} \to x_\infty \in \mathbf{R}^3.$$

Lemma 4.5.5. *If frames $(\lambda_n^1, t_n^1, x_n^1)$ and $(\lambda_n^2, \lambda_n^2, \lambda_n^2)$ are orthogonal, then*

$$(e^{-it_n^2\Delta_g} G_n^2)^{-1} e^{-it_n^1 \Delta_g} G_n^1$$

converges in weak $\dot{H}^1$ to zero. If they are equivalent, then $(e^{-it_n^2\Delta_g} G_n^2)^{-1} e^{-it_n^1 \Delta_g} G_n^1$ converges strongly to some injective $U_\infty : \dot{H}^1 \to \dot{H}^1$.

Proof. Assume the frames are orthogonal, and put $t_n = t_n^2 - t_n^1$. Suppose first that $|(\lambda_n^1)^{-2} t_n| \to \infty$. By passing to a subsequence, we may assume $\lambda_n^1 \to \lambda_\infty^1 \in [0,\infty]$ and $x_n^1 \to x_\infty^1 \in \mathbf{R}^3 \cup \{\infty\}$. Then

$$\|(G_n^2)^{-1} e^{i(t_n^2 - t_n^1))\Delta_g} G_n^1 \phi\|_{L^6} \to 0 \text{ for each } \phi \in \dot{H}^1.$$

Indeed, if $\lambda_\infty^1 \in (0,\infty)$ and $x_\infty^1 \in \mathbf{R}^3$ this follows from by Lemma 4.5.4 and the Euclidean dispersive estimate. For all other configurations of λ_∞^1 and x_∞^1, we appeal to Theorem 4.3.1 to see that

$$\|e^{i(t_n^2 - t_n^1)\Delta_g} G_n^1 \phi - e^{i(t_n^2 - t_n^1)} G_n^1 e^{it\Delta} \phi\|_{L^6} \to 0.$$

where Δ is, up to a linear change of variable, the Euclidean Laplacian. The decay in L^6 therefore follows from the Euclidean dispersive estimate.

As $(G_n^2)^{-1}e^{i(t_n^2-t_n^1)\Delta_g}G_n^1\phi$ forms a bounded sequence in $\dot{H}^1$, to determine its weak limit it suffices to test against compactly supported functions. For $\psi \in C_0^\infty$, we have

$$|\langle (G_n^2)^{-1}e^{i(t_n^2-t_n^1)\Delta_g}G_n^1\phi, \psi\rangle_{L^2}| \leq \|(G_n^2)^{-1}e^{i(t_n^2-t_n^1)\Delta_g}G_n^1\phi\|_{L^6}\|\psi\|_{L^{6/5}} \to 0.$$

Assume now that $(\lambda_n^1)^{-2}(t_n^2 - t_n^1) \to t_\infty \in \mathbf{R}$. This implies that

$$\frac{\lambda_n^1}{\lambda_n^2} + \frac{\lambda_n^2}{\lambda_n^1} + \frac{|x_n^1 - x_n^2|}{\sqrt{\lambda_n^1\lambda_n^2}} \to \infty. \tag{4.31}$$

As before, we may assume that $\lambda_n^1 \to \lambda_\infty^1 \in [0,\infty]$ and $x_n^1 \to x_\infty \in \mathbf{R}^3 \cup \{\infty\}$.

If $\lambda_\infty^1 \in (0,\infty)$ and $x_\infty^1 \in \mathbf{R}^3$, then it must be the case that

$$\lim_{n\to\infty} \frac{\lambda_n^1}{\lambda_n^2} \in \{0,\infty\},$$

Since the functions $f_n := e^{i(t_n^2-t_n^1)\Delta_g}G_n^1\phi$ form a precompact subset of $\dot{H}^1$, the sequences ∇f_n and $\xi\hat{f}_n(\xi)$ are tight in L^2. It follows that

$$\langle (G_n^2)^{-1}e^{i(t_n^2-t_n^1)\Delta_g}G_n^1\phi, \psi\rangle_{\dot{H}^1(\delta)} = \langle e^{i(t_n^2-t_n^1)\Delta_g}G_n^1\phi, G_n^2\psi\rangle_{\dot{H}^1(\delta)} \to 0.$$

From the equivalence of $\dot{H}^1(\delta)$ and $\dot{H}^1(g)$ we conclude weak convergence to zero in $\dot{H}^1(g)$.

For all other configurations of the limiting parameters λ_∞^1 and x_∞^1, we appeal to Theorem 4.3.1 and Corollary 4.3.2 to see that

$$\|(G_n^2)^{-1}e^{i(t_n^2-t_n^2)\Delta_g}G_n^1\phi - (G_n^2)^{-1}e^{i(t_n^2-t_n^1)\Delta}G_n^1\phi\|_{\dot{H}^1} \to 0,$$

where Δ is the Euclidean Laplacian modulo a linear change of variable. Thus

$$\langle (G_n^2)^{-1}e^{i(t_n^2-t_n^1)\Delta_g}G_n^1\phi, \psi\rangle_{\dot{H}^1(\delta)} = \langle (G_n^2)^{-1}G_n^1e^{it_\infty\Delta}\phi, \psi\rangle_{\dot{H}^1(\delta)} + o(1),$$

and under the assumption (4.31), the operator $(G_n^2)^{-1}G_n^1$ converges in weak $\dot{H}^1$ to zero.

Now suppose the frames are equivalent. This implies that $(\lambda_n^1)^{-2}(t_n^2 - t_n^1) \to t_\infty \in \mathbf{R}$. If $\lambda_\infty^1 \in (0,\infty)$ and $x_\infty^1 \in \mathbf{R}^3$, then $t_n \to (\lambda_\infty^1)^{-2}t_\infty \in \mathbf{R}$, $\lambda_n^2 \to \lambda_\infty^2 \in (0,\infty)$, $x_n^2 \to x_\infty^2 \in \mathbf{R}^3$, and $(G_n^2)^{-1}e^{i(t_n^2-t_n^1)\Delta_g}G_n^1$ converges strongly to $(G_\infty^2)^{-1}e^{it_\infty\Delta_g}G_\infty^1\phi$ where G_∞^j is the scaling and translation operator corresponding to $(\lambda^j_\infty, x^j_\infty)$. For all other values of λ^1_∞ and x^1_∞, we appeal to Theorem 4.3.1 to see that

$$(G^2_n)^{-1} e^{i(t^2_n - t^1_n)\Delta_g} G^1_n \to G_\infty e^{it_\infty \Delta}$$

where G_∞ is the scaling and translation operator associated to the parameters $(\lambda_\infty, \sqrt{\lambda_\infty} x_\infty)$, and

$$\lambda_\infty = \lim_{n\to\infty} \frac{\lambda^1_n}{\lambda^2_n}, \quad x_\infty = \lim_{n\to\infty} \frac{x^1_n - x^2_n}{\sqrt{\lambda^1_n \lambda^2_n}}.$$

In both cases the limiting operator is clearly invertible. □

This completes the proof of Proposition 4.5.1. □

We are now ready to give the linear profile decomposition.

Proposition 4.5.6. *Let f_n be a bounded sequence in $\dot{H}^1$. After passing to a subsequence, there exist $J^* \in \{1, 2, \dots\} \cup \{\infty\}$, profiles ϕ^j, and parameters $(\lambda^j_n, t^j_n, x^j_n)$ such that for each finite J we have a decomposition*

$$f_n = \sum_{j=1}^{J} e^{-it^j_n \Delta_g} G^j_n \phi^j + r^J_n,$$

where $G^j_n \phi(x) = (\lambda^j_n)^{-\frac{1}{2}} \phi(\frac{\cdot - x_n}{\lambda^j_n})$, satisfying the following properties:

$$(G^J_n)^{-1} r^J_n \rightharpoonup 0 \text{ in } \dot{H}^1. \tag{4.32}$$

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{it\Delta_g} r^J_n\|_{L^\infty L^6} = 0. \tag{4.33}$$

$$E(f_n) = \sum_{j=1}^{J} E(\phi^j_n) + E(r^J_n) + o(1) \text{ as } n \to \infty. \tag{4.34}$$

$$\frac{\lambda^j_n}{\lambda^k_n} + \frac{\lambda^k_n}{\lambda^j_n} + \frac{|x^j_n - x^k_n|}{\sqrt{\lambda^j_n \lambda^k_n}} + \frac{|t^j_n - t^k_n|}{\lambda^j_n \lambda^k_n} \to \infty \text{ for all } j \neq k. \tag{4.35}$$

Moreover, the times t^j_n may be chosen for each j so that either $t^j_n \equiv 0$ or $\lim_{n\to\infty} (\lambda^j_n)^{-2} t^j_n \to \pm\infty$.

Proof. We iteratively apply Proposition 4.5.1 to construct the profiles. Let $r_n^0 = f_n$. Passing to a subsequence, we may assume the existence of the limits

$$A_J = \lim_{n\to\infty} \|r_n^J\|_{\dot{H}^1}, \quad \varepsilon_J = \lim_{n\to\infty} \|e^{it\Delta_g} r_n^J\|_{L^\infty L^6}.$$

If $\varepsilon_J = 0$ then stop and set $J* = J$. Otherwise, apply Proposition 4.5.1 to the sequence r_n^J to obtain a set of parameters $(t_n^{J+1}, x_n^{J+1}, \lambda_n^{J+1}$ and a profile

$$\phi^{J+1} = \text{w-lim}(G_n^{J+1})^{-1} e^{it_n^{J+1}\Delta} r_n^J, \quad \phi_n^{J+1} = G_n^{J+1}\phi^{J+1}. \tag{4.36}$$

Set $r_n^{J+1} = r_n^J - G_n^{J+1}\phi^{J+1}$, and continue the procedure replacing J by $J+1$.

If ε^J never equals zero, then set $J^* = \infty$. In this case, the kinetic energy decoupling (4.27), the lower bound (4.26) imply

$$A_{J+1}^2 \le A_J^2 \Big[1 - c\big(\frac{\varepsilon_J}{A_J}\big)^{\frac{9}{2}}\Big]$$

which in view of the Sobolev embedding $\varepsilon_J \le cA_J$ compels $\varepsilon_J \to 0$ as $J \to \infty$.

It remains to verify the decoupling of parameters.

Suppose (4.35) failed. Choose $j < k$ with k minimal such that the frames $(\lambda_n^j, t_n^j, x_n^j)$ and $(\lambda_n^k, t_n^k, x_n^k)$ are not orthogonal. After passing to a subsequence, we may arrange for the frames $(\lambda_n^j, t_n^j, x_n^j)$, $(\lambda_n^\ell, t_n^\ell, x_n^\ell)$ to be equivalent when $\ell = k$ and orthogonal for $j < \ell < k$. By construction,

$$r_n^{j-1} = e^{-it_n^j\Delta_g} G_n^j \phi^j + e^{-it_n^k\Delta_g} G_n^k \phi^k + \sum_{j<\ell<k} e^{-it_n^\ell\Delta_g} G_n^\ell \phi^\ell,$$

hence

$$(G_n^j)^{-1} e^{it_n^j\Delta_g} r_n^{j-1} = \phi^j + (e^{-it_n^j\Delta_g} G_n^j)^{-1} e^{-it_n^k\Delta_g} G_n^k \phi^k + \sum_{j<\ell<k} (e^{-it_n^j\Delta_g} G_n^j)^{-1} e^{-it_n^\ell\Delta_g} G_n^\ell \phi^\ell.$$

By Lemma 4.5.5, $U_\infty = \lim_{n\to\infty}(e^{-it_n^j\Delta_g} G_n^j)^{-1} e^{-it_n^k\Delta_g} G_n^k$ is an invertible operator on $\dot{H}^1$, and we obtain

$$\phi^j = \phi^j + U_\infty \phi^k.$$

Thus $\phi^k = 0$, contrary to the nontriviality of the profile guaranteed by (4.26). $\square$

4.6 Euclidean nonlinear profiles

Proposition 4.6.1. *Let (λ_n, t_n, x_n) be a frame such that $\lambda_n \to \lambda_\infty \in [0, \infty]$, $x_n \to x_\infty \in \mathbf{R}^3 \cup \{\infty\}$, and either $t_n \equiv 0$ or $\lambda_n^{-2} t_n \to \pm\infty$. Assume that the limiting parameters conform to one of the following scenarios:*

(i) $\lambda_\infty = \infty$.

(ii) $x_\infty = \infty$.

(iii) $x_\infty \in \mathbf{R}^3$, $\lambda_\infty = 0$.

Then, for n sufficiently large, there exists a unique global solution u_n to the equation (4.1) with $u_n(0) = e^{-it_n\Delta_g} G_n \phi$ and which also has finite global Strichartz norm

$$\|\nabla u_n\|_{L^{10} L^{\frac{30}{13}}(\mathbf{R}\times\mathbf{R}^3)} \leq C(E(u_n(0))).$$

Moreover, for any $\varepsilon > 0$ there exists $\psi^\varepsilon \in C_0^\infty(\mathbf{R} \times \mathbf{R}^3)$ such that

$$\limsup_{n\to\infty} \|\nabla\big[u_n - G_n \psi^\varepsilon(\lambda_n^{-2}(t - t_n))\big]\|_{L^{10} L^{\frac{30}{13}}(\mathbf{R}\times\mathbf{R}^3)} < \varepsilon.$$

In particular, by Sobolev embedding the spacetime bound and approximation statement hold in $Z = L^{10} L^{10}$ as well.

Proof. The proof is analogous to that of Proposition 2.5.1 for the harmonic oscillator. In each regime, for n large one expects the solution to the variable-coefficient equation (4.1) to resemble a solution to a constant coefficient NLS

$$(i\partial_t + \Delta)u = F(u)$$

where Δ is the Laplacian for a limiting geometry. In the first two cases, the limiting geometry is the standard one on $\mathbf{R}^3$, while in the last case the geometry is given by the constant metric $g(x_\infty)$. We use solutions to the constant coefficient NLS to build good approximate solutions to (4.1). As the former obey good spacetime bounds, we deduce by stability theory that the same is true of the true solutions to (4.1).

Let $g_\infty = g(x_\infty)$ in the last case and $g_\infty = \delta$ in all other cases, and denote by Δ the associated Laplacian.

If $t_n \equiv 0$, let v be the global scattering solution to the constant coefficient defocusing NLS

$$(i\partial_t + \Delta)v = F(v) \tag{4.37}$$

with $v(0) = \phi$. If $\lambda_n^{-2} t_n \to \pm\infty$, let v instead be the unique solution to the above equation such that

$$\lim_{t \to \mp\infty} \|v(t) - e^{it\Delta}\phi\|_{\dot{H}^1} = 0.$$

In all cases, the Euclidean solution enjoys the global in time spacetime bounds

$$\|\nabla v\|_{L^2 L^6 \cap L^\infty L^2} \le C(E(\phi)) < \infty. \tag{4.38}$$

See [TVZ07, Lemma 3.11].

Fix a small parameter $0 < \theta \ll 1$, and let χ be a smooth bump function equal to 1 on the unit ball. Define spatial and Fourier space cutoffs χ_n and P_n as follows.

If $\lambda_n \to 0$ and $x_n \to x_\infty \in \mathbf{R}^3$, let $d_n = |x_n - x_\infty|$ and define

$$\chi_n = \chi\Big(\frac{(d_n + \lambda_n)^{1/3}(x - x_n)}{\lambda_n}\Big), \; P_n = \chi(\lambda_n(\lambda_n + d_n)^{1/6} D).$$

If $\lambda_n \to 0$ and $|x_n| \to \infty$, let

$$\chi_n = \chi\Big(\frac{x - x_n}{\lambda_n^{2/3}}\Big), \; P_n = \chi(\lambda_n^{4/3} D).$$

If $\lambda_n \to \lambda_\infty \in (0, \infty)$ and $d_n = |x_n| \to \infty$, set

$$\chi_n = \chi\Big(\frac{x - x_n}{d_n^{1/2}}\Big), \; P_n = \chi(d_n^{1/2} D).$$

If $\lambda_n \to \infty$, set

$$\chi_n = \chi\Big(\frac{x - x_n}{\lambda_n^{4/3}}\Big), \; P_n = \chi(\lambda_n^{5/6} D).$$

There is of course some latitude in the choice of exponents. Define the rescaled Euclidean solutions

$$v_n(t) = \lambda_n^{-1/2} v(\lambda_n^{-2} t, \lambda_n^{-1}(\cdot - x_n)) = G_n v(\lambda_n^{-2} t).$$

For $T > 0$ to be chosen later, set

$$\tilde{u}_n = \begin{cases} \chi_n P_n v_n, & |t| \le T\lambda_n^2 \\ e^{i(t - T\lambda_n^2)\Delta_g} \tilde{u}_n(T\lambda_n^2), & t \ge T\lambda_n^2 \\ e^{i(t + T\lambda_n^2)\Delta_g} \tilde{u}_n(-T\lambda_n^2), & t \le -T\lambda_n^2 \end{cases}$$

In the next two lemmas we prepare to invoke Proposition 4.2.10.

Lemma 4.6.2.

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\nabla[(i\partial_t + \Delta_g)\tilde{u}_n - F(\tilde{u}_n)]\|_N \to 0.$$

Lemma 4.6.3.

$$\lim_{T\to\infty} \limsup_{n\to\infty} \|\tilde{u}_n(-t_n) - e^{-it_n\Delta_g} G_n \phi\|_{\dot{H}^1} = 0$$

Proof of Lemma 4.6.2. In view of the definition of $\tilde{u}_n$, we estimate separately the contributions on $\{|t| \le T\lambda_n^2\}$ and $\{|t| > T\lambda_n^2\}$.

The Euclidean window. When $|t| \le T\lambda_n^2$, write

$$\begin{aligned} &(i\partial_t + \Delta_g)\tilde{u}_n - F(\tilde{u}_n) \\ &= \chi_n P_n (i\partial_t + \Delta) v_n + (\Delta_g - \Delta)\chi_n P_n v_n + [i\partial_t + \Delta, \chi_n P_n] v_n - F(\chi_n P_n v_n) \\ &= (\Delta_g - \Delta)\chi_n P_n v_n + [\Delta, \chi_n P_n] v_n + \chi_n P_n F(v_n) - F(\chi_n P_n v_n) \\ &= (a) + (b) + (c). \end{aligned}$$

Consider first the scenario where $\lambda_\infty = 0$ and $x_\infty \in \mathbf{R}^3$. We have

$$\begin{aligned} &\|\nabla(\Delta_g - \Delta)\chi_n P_n v_n\|_{L^1 L^2} \\ &\lesssim \|(g^{jk} - g^{jk}(x_\infty))\nabla\partial_j\partial_k \chi_n P_n v_n\|_{L^1 L^2} + \|(\partial g^{jk})\partial_j\partial_k \chi_n P_n v_n\|_{L^1 L^2} \\ &+ \|\nabla g^{jk} \Gamma_{jk}^m \partial_m \chi_n P_n v_n\|_{L^1 L^2} \end{aligned}$$

By Hölder in time and the definition of the cutoffs, the first term is bounded by

$$\Big(\frac{\lambda_n}{(\lambda_n+d_n)^{1/3}}+d_n\Big)(T\lambda_n^2)\lambda_n^{-2}(\lambda_n+d_n)^{-1/3}\|v_n\|_{L^\infty \dot H^1} \le T(\lambda_n+d_n)^{1/3}\|v\|_{L^\infty \dot H^1} \to 0.$$

Similarly, the second and third terms are at most

$$(T\lambda_n^2)\lambda_n^{-1}(\lambda_n+d_n)^{-1/6}\|v_n\|_{L^\infty \dot H^1} \le T\lambda_n(\lambda_n+d_n)^{-1/6}\|v\|_{L^\infty \dot H^1} \to 0.$$

Hence (a) is acceptable.

Next, we have by Hölder and Sobolev embedding

$$\begin{aligned}\|\nabla[\Delta,\chi_n P_n]v_n\|_{L^1L^2} &\le \|\nabla(\Delta\chi_n)P_n v_n\|_{L^1L^2} + \|\nabla\langle\nabla\chi_n,\nabla P_n v_n\rangle\|_{L^1L^2}\\ &\lesssim T(\lambda_n+d_n)^{\frac{2}{3}}\|\nabla v\|_{L^\infty L^2} + T(d_n+\lambda_n)^{\frac{1}{6}}\|\nabla v\|_{L^\infty L^2}.\end{aligned}$$

Thus (b) is also acceptable.

To bound the nonlinear commutator (c), write

$$\chi_n P_n F(v_n) - F(\chi_n P_n v_n) = (\chi_n P_n - 1)F(v_n) + F(v_n) - F(\chi_n P_n v_n).$$

Estimate

$$\begin{aligned}&\|\nabla(1-\chi_n P_n)F(v_n)\|_{L^2L^{6/5}}\\ &\le \|\nabla(1-\chi_n)F(v_n)\|_{L^2L^{6/5}} + \|(\nabla\chi_n)(1-P_n)F(v_n)\|_{L^2L^{6/5}}\\ &+\|(1-\chi_n)(1-P_n)\nabla F(v_n)\|_{L^2L^{6/5}}.\end{aligned}$$

By a change of variable, the last term is at most

$$\|(1-P_n)\nabla F(v_n)\|_{L^2L^{6/5}([-T\lambda_n^2,T\lambda_n^2])} = \|(1-\tilde P_n)\nabla F(v)\|_{L^2L^{6/5}([-T,T])},$$

where $\tilde P_n = \chi((\lambda_n+d_n)^{\frac{1}{6}}D)$, which goes to zero by the estimate

$$\|\nabla F(v)\|_{L^2L^{6/5}} \lesssim \|v\|_{L^{10}L^{10}}^4 \|\nabla v\|_{L^{10}L^{\frac{30}{13}}} < C(E(\phi))$$

and dominated convergence.

By Hölder and Sobolev embedding, the first two terms are bounded by

$$(\lambda_n + d_n)^{1/3}\lambda_n^{-1}(T\lambda_n^2)^{\frac{1}{2}}\|v_n\|_{L^\infty L^6}^5 \lesssim T^{\frac{1}{2}}(\lambda_n + d_n)^{1/3}\|\nabla v\|_{L^\infty L^2} \to 0$$

Also, as

$$\begin{aligned} F(v_n) - F(\chi_n P_n v_n) = (1-\chi_n P_n)v_n \int_0^1 F_z((1-\theta)\chi_n P_n v_n + \theta v_n)\, d\theta \\ + \overline{(1-\chi_n P_n)v_n} \int_0^1 F_{\bar{z}}((1-\theta)\chi_n P_n v_n + \theta v_n)\, d\theta, \end{aligned}$$

we obtain by the Leibniz rule, Hölder, Sobolev embedding, and the L^p continuity of the Littlewood-Paley projections

$$\begin{aligned} &\|\nabla[F(v_n) - F(\chi_n P_n v_n)]\|_{L^2 L^{\frac{6}{5}}} \\ &\lesssim \|\nabla(1-\chi_n P_n)v_n\|_{L^{10} L^{\frac{30}{13}}}\|v\|_{L^{10}L^{10}}^4 + \|(1-\chi_n P_n)v_n\|_{L^{10}L^{10}}\|v_n\|_{L^{10}}^3\|\nabla \chi_n P_n v_n\|_{L^{10}L^{\frac{30}{13}}} \\ &\lesssim \|\nabla v\|_{L^{10}L^{\frac{30}{13}}}^4 \|\nabla(1-\tilde{\chi}_n \tilde{P}_n v)\|_{L^{10}L^{\frac{30}{13}}}, \end{aligned}$$

where $\tilde{P}_n = \chi((\lambda_n + d_n)^{\frac{1}{6}} D)$ and $\chi_n = \chi((\lambda_n + d_n)x)$. By dominated convergence, this also vanishes as $n \to \infty$.

Now we consider the case where $\lambda_n \to \infty$, and estimate the errors (a), (b), and (c) as before.

Since $\Delta_g - \Delta = (g^{jk} - \delta^{jk})\partial_j\partial_k - g^{jk}\Gamma_{jk}^m \partial_m$, we have

$$\|\nabla(\Delta_g - \Delta)\chi_n P_n v_n\|_{L^2 L^{6/5}} \leq \sum_{j=1}^{3} \|\tilde{\chi}\nabla^j \chi_n P_n v_n\|_{L^2 L^{6/5}}$$

where $\tilde{\chi}$ is a spatial cutoff. As the v_n are being rescaled to low frequencies, the terms with the fewest derivatives applied to v_n are least favorable. Estimate

$$\begin{aligned} \|\tilde{\chi}\nabla\chi_n P_n v_n\|_{L^2 L^{6/5}} &\leq \|\tilde{\chi}(\nabla\chi_n)P_n v_n\|_{L^2 L^{6/5}} + \|\tilde{\chi}\nabla P_n v_n\|_{L^2 L^{6/5}} \\ &\lesssim \lambda_n^{-\frac{4}{3}}(T\lambda_n^2)^{\frac{2}{5}}\|\chi\|_{L^{\frac{15}{11}}}\|P_n v_n\|_{L^{10}L^{10}} + \|\tilde{\chi}\|_{L^{\frac{6}{5}}}\|\nabla P_n v_n\|_{L^2 L^\infty} \\ &\lesssim T^{\frac{2}{5}}\lambda_n^{-\frac{8}{15}}\|v\|_{L^{10}L^{10}} + \lambda_n^{-\frac{5}{12}}\|\nabla v\|_{L^2 L^6}, \end{aligned}$$

which is acceptable by (4.38). Also,

$$\begin{aligned}
&\|\nabla[\Delta, \chi_n P_n]v_n\|_{L^1L^2} \leq \|\nabla(\Delta\chi_n)P_n v_n\|_{L^1L^2} + \|\nabla\langle\nabla\chi_n, \nabla P_n v_n\rangle\|_{L^1L^2} \\
&\lesssim \|(\nabla^3\chi_n)P_n v_n\|_{L^1L^2} + \|(\nabla^2\chi_n)\nabla P_n v_n\|_{L^1L^2} + \|(\nabla\chi_n)\nabla^2 P_n v_n\|_{L^1L^2} \\
&\lesssim (\lambda_n^{-\frac{4}{3}})^2(T\lambda_n^2)\|\nabla P_n v_n\|_{L^\infty L^2} + \chi_n^{-\frac{4}{3}}(T\lambda_n^2)\lambda_n^{-\frac{5}{6}}\|\nabla P_n v_n\|_{L^\infty L^2} \\
&\lesssim T(\lambda_n^{-\frac{2}{3}} + \lambda_n^{-\frac{1}{6}})\|\nabla v\|_{L^\infty L^2}.
\end{aligned}$$

Finally, the same argument as above yields

$$\|\nabla[\chi_n P_n F(v_n) - F(\chi_n P_n v_n)]\|_{L^2 L^{\frac{6}{5}}} \to 0.$$

The remaining cases $\lambda_\infty < \infty$, $|x_n| \to \infty$ are dealt with similarly.

The long-time contribution. When $t \geq T\lambda_n^2$,

$$\|\nabla[(i\partial_t + \Delta_g)\tilde{u}_n - F(\tilde{u}_n)]\|_{L^2L^{6/5}} \lesssim \|\tilde{u}_n\|^4_{L^{10}L^{10}((T\lambda_n^2,\infty))}\|(-\Delta_g)^{1/2}\tilde{u}_n\|_{L^{10}L^{\frac{30}{13}}}.$$

The last norm on the right is bounded by Strichartz and energy conservation. To estimate the L^{10} norm, let $v_+ \in \dot{H}^1$ be the forward scattering state for the Euclidean solution v, defined by

$$\lim_{t\to\infty} \|v(t) - e^{it\Delta}v_+\|_{\dot{H}^1} = 0,$$

and write $v_{+n} = G_n v_+$. Then

$$\begin{aligned}
\tilde{u}_n(t) &= e^{i(t-T\lambda_n^2)\Delta_g}\chi_n P_n v_n(T\lambda_n^2) \\
&= e^{it\Delta_g}(v_{+n}) + e^{i(t-T\lambda_n^2)\Delta_g}[e^{iT\lambda_n^2\Delta}(v_{+n}) - e^{iT\lambda_n^2\Delta_g}(v_{+n})] \\
&+ e^{i(t-T\lambda_n^2)\Delta_g}(\chi_n P_n - 1)v_n(T\lambda_n^2) + e^{i(t-T\lambda_n^2)\Delta_g}[v_n(T\lambda_n^2) - e^{iT\lambda_n^2\Delta}(v_{+n})],
\end{aligned}$$

and we see that if T is sufficiently large, each term becomes acceptably small for n large. Indeed, by interpolating Theorem 4.3.1 or Proposition 4.4.1 with a Strichartz estimate,

$$\lim_{T\to\infty}\lim_{n\to\infty} \|e^{it\Delta_g}v_{+n}\|_{L^{10}L^{10}((T\lambda_n^2,\infty)} = 0.$$

The remaining terms are also acceptable due to Strichartz, Theorem 4.3.1, dominated convergence, and the definition of the scattering state v_{+n}. □

Proof of Lemma 4.6.3. If $t_n \equiv 0$ then there is nothing to prove. So suppose $\lambda_n^{-2} t_n \to \infty$. Recall that by definition,

$$\lim_{t \to -\infty} \|v(t) - e^{it\Delta}\phi\|_{\dot{H}^1} = 0.$$

Referring to the definition of $\tilde{u}_n$, for n large enough

$$\begin{aligned} \tilde{u}_n(-t_n) &= e^{-it_n\Delta_g} e^{iT\lambda_n^2\Delta_g} \chi_n P_n v_n(-T\lambda_n^2) = e^{-it_n\Delta_g} e^{iT\lambda_n^2\Delta_g} G_n v(-T) + r_n \\ &= e^{-it_n\Delta_g} e^{iT\lambda_n^2\Delta_g} e^{-iT\lambda_n^2\Delta_g} G_n \phi + r_n \\ &= e^{-it_n\Delta_g} G_n \phi + r_n, \end{aligned}$$

where, by Theorem 4.3.1 and Corollary 4.3.2, in each line

$$\lim_{T \to \infty} \limsup_{n \to \infty} \|r_n\|_{\dot{H}^1} = 0.$$

□

By the preceding lemmas, for T large enough and n large, the function $\tilde{u}_n(t - t_n, x)$ is a good approximate solution to (4.1) in the sense of Proposition 4.2.10. Thus for any $\varepsilon > 0$ and all n sufficiently large, there is a unique global solution u_n to (4.1) with

$$\|u_n\|_{Z(\mathbf{R})} + \|\nabla u_n\|_{L^{10} L^{\frac{30}{13}}(\mathbf{R} \times \mathbf{R}^3)} \leq C(E(u_n(0))).$$

Finally, for any $\varepsilon > 0$ there exists $\psi^\varepsilon \in C^\infty(\mathbf{R} \times \mathbf{R}^3)$ such that $\|\nabla(v - \psi^\varepsilon)\|_{L^{10} L^{\frac{30}{13}}(\mathbf{R} \times \mathbf{R}^3)} < \varepsilon$. In view of the definition of $\tilde{u}_n$ and the fact that, as proved above,

$$\lim_{T \to \infty} \limsup_{n \to \infty} \|\tilde{u}_n\|_{Z([-T\lambda_n^2, T\lambda_n^2]^c)} = 0,$$

another application of Proposition 4.2.10 yields

$$\lim_{T \to \infty} \limsup_{n \to \infty} \|\nabla u_n\|_{L^{10} L^{\frac{30}{13}}([-T\lambda_n^2, T\lambda_n^2]^c \times \mathbf{R}^3)} = 0.$$

Therefore

$$\lim_{T \to \infty} \limsup_{n \to \infty} \|\nabla[\tilde{u}_n - G_n \psi^\varepsilon(\lambda_n^{-2} t)]\|_{L^{10} L^{\frac{30}{13}}(\mathbf{R} \times \mathbf{R}^3)} \lesssim \varepsilon,$$

as required. □

4.7 Nonlinear profile decomposition

In this section, we show that failure of Theorem (4.1.1) would imply the existence of an "almost-periodic" solution in the sense that it remains in a precompact subset of $\dot{H}^1$. This will already preclude finite time blowup and hence prove the global existence part of the theorem. In the next section, we rule out almost-periodic solutions under a smallness assumption on the metric and obtain global spacetime bounds in that setting.

Although we have worked mainly with the $Z = L^{10}L^{10}$ norm, in the sequel we shall also need the stronger norm

$$Y = L^{10}\dot{H}^{1,\frac{30}{13}}.$$

Let

$$\Lambda(E) : \sup\{\|u\|_{Z(\mathbf{R})} : E(u) \leq E,\ u \text{ solves } (4.1)\}$$
$$E_c = \sup\{E : \Lambda(E) < \infty\}.$$

$$\Lambda'(E) : \sup\{\|u\|_{Z(I)} : |I| \leq 1,\ E(u) \leq E,\ u \text{ solves}(4.1)\}$$
$$E_c' = \sup\{E : \Lambda'(E) < \infty\}.$$

The small data theory implies that $E_c, E_c' > 0$. Global existence (resp. scattering) would follow if we show that $E_c' < \infty$ (resp. $E_c < \infty$).

Proposition 4.7.1. *Suppose $E_c < \infty$. Let u_n be a sequence of solutions to* (4.1) *with $E(u_n) \to E_c$ such that for some sequence of times t_n, $\|u_n\|_{Z((-\infty,t_n))} \to \infty$ and $\|u_n\|_{Z((t_n,\infty))} \to \infty$. Then some subsequence of $u(t_n)$ converges in $\dot{H}^1$.*

The method of proof yields an analogous statement for global existence:

Proposition 4.7.2. *Suppose $E_c' < \infty$, and fix any $\delta > 0$. Let u_n be a sequence of solutions to* (4.1) *with $E(u_n) \to E_c$ such that for some sequence of times t_n, $\|u_n\|_{Z((t_n-\delta,t_n))} \to \infty$ and $\|u_n\|_{Z((t_n,t_n+\delta))} \to \infty$. Then some subsequence of $u(t_n)$ converges in $\dot{H}^1$.*

We prove the global-in-time proposition; as the reader may verify, a nearly identical argument yields the local-in-time version.

Proof of Prop. 4.7.1. By translating in time, we may assume without loss that $t_n \equiv 0$. After passing to a subsequence, we obtain a decomposition

$$u_n(0) = \sum_{j=1}^{J} e^{-it_n^j \Delta_g} G_n^j \phi^j + r_n^J \tag{4.39}$$

into asymptotically independent profiles with the properties described in Proposition 4.5.6. In particular,

$$\lim_{n\to\infty} \Big[E(u_n(0)) - \sum_{j=1}^{J} E(e^{-it_n^j \Delta_g} G_n^j \phi_n^j) - E(r_n^J) \Big] = 0, \tag{4.40}$$

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{it\Delta_g} r_n^J\|_{L^\infty L^6} = 0. \tag{4.41}$$

Lemma 4.7.3. *There exists j such that* $\limsup_{n\to\infty} E(e^{-it_n^j \Delta_g} G_n^j \phi^j) = E_c$.

This will be proved below using a nonlinear profile decomposition. For the moment, we assume the result and observe how it yields the proposition. By the lemma, $u_n(0)$ takes the form

$$u_n(0) = e^{-it_n \Delta_g} G_n \phi + r_n$$

where $\|r_n\|_{\dot{H}^1} \to 0$ and G_n is associated to some frame (λ_n, x_n). After passing to a subsequence, we may assume that $\lambda_n \in \lambda_\infty \in [0, \infty]$, $x_n \to x_\infty \in \mathbf{R}^3 \cup \{\infty\}$, and $\lambda_n^{-2} t_n \to t_\infty \in \mathbf{R} \cup \{\pm\infty\}$.

We claim that $\lambda_\infty \in (0, \infty)$, $x_\infty \in \mathbf{R}^3$, and $t_\infty = 0$, which would clearly imply that $u_n(0)$ converges in $\dot{H}^1$. If either of the first two statements failed, Proposition 4.6.1 would imply that $\limsup_{n\to\infty} \|u_n\|_{Z(\mathbf{R})} < \infty$, contrary to the assumptions u_n. Thus $\lambda_\infty \in (0, \infty)$ and $x_\infty \in \mathbf{R}^3$. If $t_n \to \infty$, then, writing G_∞ for the operator associated to the parameters $(\lambda_\infty, x_\infty)$, we have by the Strichartz estimate

$$\|(-\Delta_g)^{1/2} e^{it\Delta_g} u_n(0)\|_{L^{10} L^{\frac{30}{13}}((-\infty,0)\times\mathbf{R}^3)} \le \|(-\Delta_g)^{1/2} e^{it\Delta} G_\infty \phi\|_{L^{10} L^{\frac{30}{13}}((-\infty,-t_n)\times\mathbf{R}^3)} + o(1) \to 0,$$

which implies by the small data theory that $\lim_{n\to\infty} \|u_n\|_{Z(-\infty,0)} = \infty$, contrary to the hypothesis that u_n blows up forwards and backwards in time. □

Corollary 4.7.4. *If $E_c < \infty$, then there exists a global solution u_c to* (4.1) *with $E(u_c) = E_c$ and $\|u\|_{Z((-\infty,0])} = \|u\|_{Z([0,\infty))} = \infty$. Moreover, u is* almost-periodic *in the sense that $\{u_c(t) : t \in \mathbf{R}\}$ is precompact in $\dot{H}^1$.*

Proof. Let u_n be a sequence of solutions with $E(u_n) \to E_c$ and $\|u_n\|_Z \to \infty$. Choose t_n such that $\|u_n\|_{(-\infty,t_n]} = \|u_n\|_{[t_n,\infty)}$. By the previous proposition, there exists $\phi \in \dot{H}^1$ such that after passing to a subsequence, $u_n(t_n) \to \phi$ in $\dot{H}^1$. Let u_c be the maximal solution with $u_c(0) = 0$. Proposition 4.7.1 and the stability theory imply that u_c is global and blows up forwards and backwards in time. Another application of the previous proposition yields precompactness of the orbit $\{u(t) : t \in \mathbf{R}\}$ in $\dot{H}^1$. □

An immediate consequence of Proposition 4.7.2 and the stability theory is that the equation (4.1) is globally wellposed.

Corollary 4.7.5. *Under the hypotheses of Theorem 4.1.1, solutions of* (4.1) *are global in time.*

Proof. If $E_c' < \infty$, then there exists a sequence of solutions u_n with $E(u_n) \to E_c'$ and $\|u_n\|_{Z((-\frac{1}{2},0))}, \|u_n\|_{Z((0,\frac{1}{2}))} \to \infty$. By Proposition 4.7.2, some subsequence of $u_n(0)$ converges to some $\phi \in \dot{H}^1$. Let $u_c : (T_-, T_+) \times \mathbf{R}^3 \to \mathbf{C}$ be the maximal-lifespan solution with $u_c(0) = \phi$. By the Proposition 4.2.10, u_c has infinite Z-norm on $(-\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, so the interval of definition for u_c is contained in $(-\frac{1}{2}, \frac{1}{2})$. As in the previous corollary, the solution curve $u_c(t)$ is precompact in $\dot{H}^1$, so along some sequence of times $t_n \to T_+$ the functions $u_c(t_n)$ converge to some ϕ_+ in $\dot{H}^1$. But then we may use a local solution u_+ with $u_+(0) = \phi_+$ and the stability theory to continue u_c to a larger time interval $(-T, T_+ + \delta)$, contradicting its maximality. □

We prove Proposition 4.7.1 in the remainder of this section. While the overall argument is quite standard, involving a nonlinear profile decomposition, some remarks are warranted concerning how to control the interaction between nonlinear profiles and the linear evolution of the remainder in the decomposition. This is normally accomplished using local smoothing, which prevent high-frequency linear solutions from lingering in a confined region. In Euclidean space, the local smoothing estimate takes the form

$$\|\nabla e^{it\Delta_{\mathbf{R}^3}}\phi\|_{L^2(\mathbf{R}\times\{|x|\le R\})} \lesssim R^{1/2}\|\phi\|_{\dot{H}^{1/2}}.$$

However, most existing local smoothing estimates on manifolds work at a fixed spatial scale, and since the metric is not scale-invariant, it is not obvious how the constants depend on the size of the physical localization.

The following lemma is analogous to Lemma 7.1 of Ionescu-Pausader concerning NLS on the torus [IP12], although the proof there is quite different due to trapping.

Let $\chi(\lambda)$ be a smooth function on the real line equal to 1 when $\lambda \le 1$ and vanishing when $\lambda \ge 1.2$, and define the spectral multipliers $P_{\le N} = \chi(\sqrt{-\Delta_g}/N)$. By Theorem 4.2.7, these satisfy the Littlewood-Paley estimates of Proposition 4.2.8 except when $p = 1$ or $p = \infty$ (which will not be needed).

Lemma 4.7.6. *For any $R, N, T > 0$, $B \ge 1$, and $(t_0, x_0) \in \mathbf{R}\times\mathbf{R}^3$,*

$$\|\nabla e^{it\Delta_g} P_{>BN}\phi\|_{L^2(|t-t_0|\le TN^{-2},\ |x-x_0|\le RN^{-1})} \le CB^{-1/2}N^{-1}\|\phi\|_{\dot{H}^1}.$$

Proof. By invariance under time translation, we may take $t_0 = 0$. We adapt the standard proof of local smoothing on Euclidean space via a Morawetz multiplier but need to deal with error terms arising from the background curvature. These will be controlled by a separate local smoothing estimate adapted to the metric.

Let $a(x) = \langle x\rangle$. We compute (all derivatives are partial derivatives)

$$\begin{aligned}
\partial a &= \frac{x}{\langle x\rangle}, & \partial^2 a &= \langle x\rangle^{-3}P_r + \langle x\rangle^{-1}P_\theta,\\
\Delta a &\ge \frac{3}{\langle x\rangle^3}, & \Delta^2 a &= -\frac{15}{\langle x\rangle^7}\\
|\partial^k a| &\le \frac{c_k}{\langle x\rangle^{k-1}},
\end{aligned}$$

where P_r and $P_\theta = I - P_r$ are the radial and tangential projections, respectively.

Now write $D = d + \Gamma$ for the Levi-Civita covariant derivative, where Γ are the Christoffel symbols for the metric g; by our assumptions on g, Γ is supported in the unit ball.

If u is a solution to the equation

$$(i\partial_t + \Delta_g)u = \mu|u|^4 u, \ \ \mu \in \mathbf{R},$$

define the Morawetz action

$$M(t) = \int_{\mathbf{R}^3} a(x)|u(t,x)|^2 \, dg.$$

Then as in the Euclidean setting, we have

$$\partial_t M(t) = 2 \operatorname{Im} \int \overline{u} D^\alpha a D_\alpha u \, dg,$$

and the Morawetz identity

$$\partial_t^2 M = 4 \operatorname{Re} \int (D^2_{\alpha\beta} a) D^\alpha \overline{u} D^\beta u \, dg - \int (\Delta_g^2 a)|u|^2 \, dg + \frac{4\mu}{3} \int (\Delta_g a)|u|^6 \, dg. \tag{4.42}$$

We apply this identity with $\mu = 0$ and $u = e^{it\Delta_g} P_{>BN}\phi$; later on we will use this when $\mu = 1$. For $N > 0$ and $x_0 \in \mathbf{R}^3$, let $a_{N,x_0}(x) = a\big(N(x - x_0)\big)$. We compute

$$\begin{aligned} Da_{N,x_0} &= N(\partial a)\Big(N(x - x_0)\Big) \\ D^2_{\alpha\beta} a_{N,x_0} &= N^2 (\partial_\alpha \partial_\beta a)\Big(N(x - x_0)\Big) - N\Gamma^\mu_{\alpha\beta}\partial_\mu a\Big(N(x - x_0)\Big). \end{aligned}$$

Then

$$\begin{aligned} \Delta_g^2 a_{N,x_0} &= g^{\alpha\beta}(\partial_\alpha\partial_\beta - \Gamma^\mu_{\alpha\beta}\partial_\mu) g^{\alpha'\beta'}(\partial_{\alpha'}\partial_{\beta'} - \Gamma^{\mu'}_{\alpha'\beta'}\partial_{\mu'}) a_{N,x_0} \\ &= N^4 g^{\alpha\beta} g^{\alpha'\beta'} (\partial_\alpha\partial_\beta\partial_{\alpha'}\partial_{\beta'} a)\Big(N(x - x_0)\Big) + (N^3 P_3 a + N^2 P_2 a + N^1 P_1 a)\Big(N(x - x_0)\Big), \end{aligned}$$

where P_k is a differential operator of order k with coefficients supported in the unit ball; hence

$$\Big|P_k a\Big(N(x - x_0)\Big)\Big| \le c_k \, 1_{\{|x| \le 1\}}(x) \Big\langle N(x - x_0)\Big\rangle^{1-k}.$$

Inserting these bounds into the Morawetz identity and integrating in time over the interval $|t| \le TN^{-2}$, we obtain

$$\iint_{|t|\le TN^{-2}} \langle N(x-x_0)\rangle^{-3} |\nabla u|^2 \, dxdt \lesssim N^{-1} \iint 1_{\{|t|\le TN^{-2}, |x|\le 1\}}(t,x)(|\nabla u|^2 + |u|^2)\, dxdt$$
$$+ N^2 \iint_{|t|\le TN^{-2}} |u|^2 \, dxdt + N^{-1} \|u\|_{L^\infty L^2} \|u\|_{L^\infty \dot{H}^1}.$$

By the unitarity of the propagator and the spectral localization of u, we have

$$\|u\|_{L^\infty L^2} \|u\|_{L^\infty \dot{H}^1} \lesssim (BN)^{-1} \|\phi\|_{\dot{H}^1}^2.$$

Also, by Hölder in time, the second term on the right may be bounded by

$$T\|u\|_{L^\infty L^2}^2 \lesssim T(BN)^{-2} \|\phi\|_{\dot{H}^1}^2.$$

Finally, the first term on the right is controlled by the following scale-1 local smoothing estimate of Rodnianski and Tao [RT07]

$$\|\langle x\rangle^{-\frac{1}{2}-\sigma} \nabla e^{it\Delta_g} \phi\|_{L^2(\mathbf{R}\times\mathbf{R}^3)} + \|\langle x\rangle^{-\frac{3}{2}-\sigma} u\|_{L^2(\mathbf{R}\times\mathbf{R}^3)} \lesssim_\sigma \|\phi\|_{\dot{H}^{1/2}}, \ \sigma > 0,$$

who strengthened an earlier local-in-time version by Doi [Doi96]. Summing up, we obtain

$$\|\nabla u\|_{L^2(\{|t|\le TN^{-2}, |x-x_0|\le RN^{-1}\})}^2 \lesssim (B^{-1}N^{-2} + T(BN)^{-2}) \|\phi\|_{\dot{H}^1}^2.$$

□

Proof of Lemma 4.7.3. Assuming that the claim fails, the asymptotic additivity of energy implies the existence of some $\delta > 0$ such that $\limsup_{n\to\infty} E(e^{-it_n^j \Delta_g} G_n^j \phi^j) \le E_c - \delta$ for all j. We shall deduce that

$$\limsup_{n\to\infty} \|u_n\|_{Z(\mathbf{R})} \le C(E_c, \delta) < \infty, \tag{4.43}$$

which contradicts the hypotheses on u_n.

For each $j \le J$, let u_n^j be the maximal-lifespan nonlinear solution with $u_n^j(0) = e^{-it_n^j\Delta_g} G_n^j \phi^j$; by the definition of E_c, for all n sufficiently large we have $\|u_n^j\|_{Z(\mathbf{R})} \le C$, hence $\|u_n^j\|_{Y(\mathbf{R})} \le C'$. Define

$$\tilde{u}_n^J = \sum_{j=1}^J u_n^j + e^{it\Delta_g} r_n^J.$$

The bound (4.43) will be a consequence of Proposition 4.2.10 and the following three assertions:

1. $\limsup_{J\to J^*} \limsup_{n\to\infty} \|\tilde{u}_n^J\|_{Y(\mathbf{R})} \leq C(E_c, \delta) < \infty$.

2. $\lim_{J\to J^*} \limsup_{n\to\infty} \|u_n(0) - \tilde{u}_n^J(0)\|_{\dot{H}^1} = 0$.

3. $\lim_{J\to J^*} \limsup_{n\to\infty} \|\nabla[(i\partial_t + \Delta_g)\tilde{u}_n^J - F(\tilde{u}_n^J)]\|_{N(\mathbf{R})} = 0$, where $F(z) = |z|^4 z$.

Proof of claim (1). As the Strichartz estimate and the hypothesis of bounded energy imply that the remainder $e^{it\Delta_g} r_n^J$ is bounded in Y, it suffices to show that

$$\limsup_{J\to J^*} \limsup_{n\to\infty} \Big\| \sum_{j=1}^J u_n^j \Big\|_{Y(\mathbf{R})} < \infty.$$

For each J, we have

$$\Big\| \sum_{j=1}^J u_n^j \Big\|_Y^2 = \Big\| \big(\sum_{j=1}^J \nabla u_n^j\big)^2 \Big\|_{L^5 L^{\frac{15}{13}}} \leq \sum_{j=1}^J \|\nabla u_n^j\|_{L^{10} L^{\frac{30}{13}}}^2 + c_J \sum_{j\neq k} \|(\nabla u_n^j)(\nabla u_n^k)\|_{L^5 L^{\frac{15}{13}}}. \quad (4.44)$$

By Lemma (4.7.7), the cross-terms vanish as $n \to \infty$. By the asymptotic additivity of energy, there is some J_0 such that $\limsup_{n\to\infty} \|\nabla u_n^j(0)\|_{L^2}$ is smaller than the small-data threshold in Proposition 4.2.9 for all $j \geq J_0$. In view of the small-data estimate (4.9), for any $J > J_0$ we have

$$\limsup_{n\to\infty} \Big\| \sum_{j=1}^J u_n^j \Big\|_Y^2 \leq C_{J_0}(E_c) + \limsup_{n\to\infty} \sum_{j=J_0}^J E(u_n^j) \leq C_{J_0}(E_c) + E_c.$$

For future reference, we observe this also proves that for any $\varepsilon > 0$, there exists $J'(\varepsilon, E_c)$ with

$$\limsup_{n\to\infty} \Big\| \sum_{J' \leq j \leq J} u_n^j \Big\|_Y < \eta. \quad (4.45)$$

for all J.

Lemma 4.7.7. *For all $j \neq k$,*

$$\lim_{n\to\infty} \|u_n^j u_n^k\|_{L^5 L^5} + \|u_n^j \nabla u_n^k\|_{L^5 L^{\frac{15}{8}}} + \|\nabla u_n^j \nabla u_n^k\|_{L^5 L^{\frac{15}{13}}} = 0.$$

Proof of Lemma 4.7.7. The argument is well-known, and we will just illustrate it by estimating the middle term. By Proposition 4.6.1, for each $\varepsilon > 0$ there exist $\psi^j, \psi^k \in C_0^\infty(\mathbf{R} \times \mathbf{R}^3)$ such that

$$\|\nabla[u_n^j(t) - G_n^j \psi^j((\lambda_n^j)^{-2}(t - t_n^j))]\|_{L^{10}L^{\frac{30}{13}}} + \|\nabla[u_n^k(t) - G_n^k \psi^k((\lambda_n^k)^{-2}(t - t_n^k))]\|_{L^{10}L^{\frac{30}{13}}} < \varepsilon$$

for all n sufficiently large. Letting v_n^j, v_n^k denote the compactly supported approximations, we have by Hölder

$$\begin{aligned}\|u_n^j(\nabla u_n^k)\|_{L^5 L^{\frac{15}{8}}} \le \|u_n^j - v_n^j\|_{L^{10}L^{10}} \|\nabla u_n^k\|_{L^{10}L^{\frac{30}{13}}} + \|v_n^j\|_{L^{10}L^{10}} \|\nabla(u_n^k - v_n^k)\|_{L^{10}L^{\frac{30}{13}}} \\ + \|v_n^j \nabla v_n^k\|_{L^5 L^{\frac{15}{8}}}.\end{aligned}$$

The last term vanishes due to the pairwise orthogonality of the frames $(\lambda_n^j, t_n^j, x_n^j)$ and $(\lambda_n^k, t_n^k, x_n^k)$. Thus

$$\limsup_{n\to\infty} \|u_n^j \nabla u_n^k\|_{L^5 L^{\frac{15}{8}}} \le C(E_c, \delta)\varepsilon$$

for any $\varepsilon > 0$. □

Claim (2) is immediate.

Proof of Claim (3). Write

$$\begin{aligned}(i\partial_t + \Delta_g)\tilde{u}_n^J - F(\tilde{u}_n^J) = \sum_{j=1}^J F(u_n^j) - F\Big(\sum_{j=1}^J u_n^j\Big) \\ + F\Big(\sum_{j=1}^J u_n^j\Big) - F\Big(\sum_{j=1}^J u_n^j + e^{it\Delta_g} r_n^J\Big),\end{aligned}$$

and expand

$$\begin{aligned}&F\Big(\sum_{j=1}^J u_n^j\Big) - \sum_{j=1}^J F(u_n^j) = \Big|\sum_{j=1}^J u_n^j\Big|^4 \Big(\sum_{j=1}^J u_n^j\Big) - \sum_{j=1}^J |u_n^j|^4 u_n^j \\ &= \sum_{j=1}^J \Big(\Big|\sum_{j=1}^J u_n^j\Big|^4 - |u_n^j|^4\Big) u_n^j \\ &= \sum_{j=1}^J \sum_{k\neq j} \Big(u_n^j u_n^k \int_0^1 G_z\Big(\sum_{\ell=1}^J u_n^\ell - \theta u_n^j\Big)\, d\theta + u_n^j \overline{u_n^k} \int_0^1 G_{\overline{z}}\Big(\sum_{\ell=1}^J u_n^\ell - \theta u_n^j\Big)\, d\theta\Big),\end{aligned}$$

where $G(z)=|z|^4$. By the Leibniz rule, Hölder, and Lemma 4.7.7,

$$\begin{aligned}\|F\Big(\sum_{j=1}^J u_n^j\Big)-\sum_{j=1}^J F(u_n^j)\|_{L^2L^{\frac{6}{5}}} &\le \sum_{j=1}^J\sum_{j\neq k}\|\nabla(u_n^ju_n^k)\|_{L^5L^{\frac{15}{8}}}\Big\|\int_0^1 G'\Big(\sum_{\ell=1}^J u_n^\ell-\theta u_n^j\Big)\,d\theta\Big\|_{L^{\frac{10}{3}}L^{\frac{10}{3}}}\\ &\le c_J\sum_{j=1}^J\sum_{j\neq k}\|u_n^j\nabla u_n^k\|_{L^5L^{\frac{15}{8}}}\Big(\Big\|\sum_{\ell=1}^J u_n^\ell\Big\|_{L^{10}L^{10}}^3+\|u_n^j\|_{L^{10}L^{10}}^3\Big)\\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Similarly, write

$$\begin{aligned}F\Big(\sum_{j=1}^J u_n^j+e^{it\Delta_g}r_n^J\Big)-F\Big(\sum_{j=1}^J u_n^j\Big) &= \Big(\Big|\sum_{j=1}^J u_n^j+e^{it\Delta_g}r_n^J\Big|^4-\Big|\sum_{j=1}^J u_n^j\Big|^4\Big)\sum_{j=1}^J u_n^j\\ &\quad+\Big|\sum_{j=1}^J u_n^j+e^{it\Delta_g}r_n^J\Big|^4 e^{it\Delta_g}r_n^J\\ &=(I)+(II).\end{aligned}$$

First consider (I). As before, writing $G(z)=|z|^4$, we have by the Leibniz rule, Hölder, and Sobolev embedding

$$\begin{aligned}\Big\|\nabla(I)\|_{L^2L^{\frac{6}{5}}} &\le \Big\|\nabla(e^{it\Delta_g}r_n^J)\int_0^1 G'\Big(\sum_{j=1}^J u_n^j+\theta e^{it\Delta_g}r_n^J\Big)\sum_j u_n^j\Big\|_{L^2L^{\frac{6}{5}}}\\ &\quad+\Big\|(e^{it\Delta_g}r_n^J)\nabla\int_0^1 G'\Big(\sum_{j=1}^J u_n^j+\theta e^{it\Delta_g}r_n^J\Big)\sum_{j=1}^J u_n^j\Big\|_{L^2L^{\frac{6}{5}}}\\ &\lesssim \|\nabla(e^{it\Delta_g}r_n^J)\Big|\sum_{j=1}^J u_n^j\Big|^4\Big\|_{L^2L^{\frac{6}{5}}}+\|\nabla(e^{it\Delta_g}r_n^J)\|_{L^{10}L^{\frac{30}{13}}}\|e^{it\Delta_g}r_n^J\|_{L^{10}L^{10}}^3\Big\|\sum_{j=1}^J u_n^j\Big\|_{L^{10}L^{10}}\\ &\quad+\|e^{it\Delta_g}r_n^J\|_{L^{10}L^{10}}(\|\nabla u_n^J\|_{L^{10}L^{\frac{30}{13}}}^4+\|\nabla e^{it\Delta_g}r_n^J\|_{L^{10}L^{\frac{30}{13}}}^4).\end{aligned}$$

By (4.41) and interpolation, $\lim_{J\to J^*}\limsup_{n\to\infty}\|e^{it\Delta_g}r_n^J\|_{L^{10}L^{10}}=0$; therefore, all but the first term are acceptable.

To deal with the first term, we recall that for any ε, there exists by (4.45) a threshold $J'(\varepsilon)$ such that for all n large,

$$\Big\|\sum_{j=J'}^J u_n^j\Big\|_{L^{10}L^{10}}<\varepsilon.$$

With ε fixed but arbitrary, this implies that

$$\begin{aligned}&\|\nabla(e^{it\Delta_g} r_n^J)\Big|\sum_{j=1}^{J} u_n^j\Big|^4\|_{L^2L^{\frac{6}{5}}}\\&\leq c_{J'}\sum_{j=1}^{J'}\|(u_n^j)^4\nabla e^{it\Delta_g} r_n^J\|_{L^2L^{\frac{6}{5}}}+\varepsilon^4 E_c^{1/2}.\end{aligned}$$

It therefore remains to show that

$$\limsup_{J\to J^*}\limsup_{n\to\infty}\|(u_n^j)^4\nabla e^{it\Delta_g} r_n^J\|_{L^2L^{\frac{6}{5}}}\lesssim\varepsilon \text{ for each } j\leq J'. \tag{4.46}$$

Select $\psi^j\in C_0^\infty(\mathbf{R}\times\mathbf{R}^3)$ so that $\|u_n^j-v_n^j\|_Y<\varepsilon$, where $v_n^j=G_n^j\psi^j((\lambda_n^j)^{-2}(t-t_n^j))$. Then we may replace u_n^j by v_n^j in the above sum since for all n sufficiently large, since

$$\begin{aligned}\|(u_n^j)^4\nabla e^{it\Delta_g} r_n^J\|_{L^2L^{\frac{6}{5}}}&\leq c\|u_n^j-v_n^j\|_{L^{10}L^{10}}(\|u_n^j\|^3_{L^{10}L^{10}}+\|v_n^j\|^3_{L^{10}L^{10}})\|\nabla e^{it\Delta_g} r_n^J\|_{L^{10}L^{\frac{30}{13}}}\\&\leq C(E_c)\varepsilon.\end{aligned}$$

Let χ_n^j denote the characteristic function of $\operatorname{supp}(v_n^j)$. Putting $N_n^j=(\lambda_n^j)^{-1}$, we estimate using Hölder, Littlewood-Paley theory, and Lemma 4.7.6

$$\begin{aligned}\|(v_n^j)^4\nabla e^{it\Delta_g} r_n^J\|_{L^2L^{\frac{6}{5}}}&\lesssim (N_n^j)^2\|\chi_n^j\nabla e^{it\Delta_g}P_{\leq BN_n^j} r_n^J\|_{L^2L^{\frac{6}{5}}}+(N_n^j)^2\|\chi_n e^{it\Delta_g}P_{>BN_n^j} r_n^J\|_{L^2L^{\frac{6}{5}}}\\&\lesssim (N_n^j)^{-1}\|\nabla e^{it\Delta_g}P_{\leq BN_n^j} r_n^J\|_{L^\infty L^6}+N_n^j\|\chi_n e^{it\Delta_g}P_{>BN_n^j} r_n^J\|_{L^2L^2}\\&\lesssim B\|e^{it\Delta_g} r_n^J\|_{L^\infty L^6}+B^{-1/2}.\end{aligned}$$

As the remainder vanishes in $L^\infty L^6$ and B is arbitrary, it follows that

$$\lim_{J\to J^*}\limsup_{n\to\infty}\|(v_n^j)^4\nabla e^{it\Delta_g} r_n^J\|_{L^2L^{\frac{6}{5}}}=0.$$

Altogether, we obtain (4.46), hence (I) is acceptable.

The contribution of (II) is estimated similarly. By the Leibniz rule,

$$\|\nabla(II)\|_{L^2L^{\frac{6}{5}}}\lesssim\|e^{it\Delta_g} r_n^J(u_n^J)^3\nabla u_n^J\|_{L^2L^{\frac{6}{5}}}+\|(u_n^J)^4\nabla e^{it\Delta_g} r_n^J\|_{L^2L^{\frac{6}{5}}}.$$

The first term is acceptable due to the undifferentiated $e^{it\Delta_g} r_n^J$, while the second term is handled is above. This completes the proof of Claim 3, and therefore finishes the proof of Lemma 4.7.3 asserting the existence of a critical profile. Consequently, Proposition 4.7.1 is proved. □

4.8 Scattering for small metric perturbations

In this final section we prove scattering for metrics g with $\|g-\delta\|_{C^3} \leq \varepsilon$ for some ε depending on the diameter of $\mathrm{supp}(g-\delta)$. If the curvature is sufficiently mild, we can adapt the one-particle Bourgain-Morawetz inequality [Bou99] for the Euclidean nonlinear Schrödinger equation to preclude the existence of almost-periodic solutions, which, when combined with Corollary 4.7.4, yields scattering.

Proposition 4.8.1. *There exists $\varepsilon > 0$ such that if $\|g-\delta\|_{C^3} \leq \varepsilon$, then for any solution u to the nonlinear equation* (4.1) *and any time interval I,*

$$\int_I \int_{|x| \leq |I|^{1/2}} \frac{|u|^6}{\langle x \rangle} \, dxdt \leq c|I|^{1/2} E(u).$$

Proof. Let $a = \langle x \rangle$ as in the proof of Lemma 4.7.6, and write $a_R = a(x)\chi(\frac{\cdot}{R})$ where χ is a smooth cutoff equal to 1 on the ball $|x| \leq 1$ and supported in $|x| \leq 2$. Then

$$\partial a_R = O(1), \qquad \Delta a_R = \Big(\frac{2}{\langle x \rangle} + \frac{1}{\langle x \rangle^3} \Big) 1_{\{|x| \leq R\}} + O(R^{-1} 1_{\{|x| \sim R\}})$$

$$\partial^2 a_R = \partial^2 a 1_{\{|x| \leq R\}} + O(R^{-1} 1_{\{|x| \sim R\}}), \quad \Delta^2 a_R = -\frac{15}{\langle x \rangle^7} 1_{\{|x| \leq R\}} + O(R^{-3} 1_{\{|x| \sim R\}}).$$

Let $D = d + \Gamma$ denote the covariant derivative, where Γ is supported in the unit ball and $\|\Gamma\|_{C^2} = O(\varepsilon)$. It follows that if ε is sufficiently small, the above formulas continue to hold with the partial derivatives replaced by the covariant derivative D and Δ by the metric Laplacian Δ_g. Applying the Morawetz identity (4.42) with action $M(t) = \int a_R(x) |u(t,x)|^2 \, dg$, we obtain $|\partial_t M| \leq cR \|\nabla u\|_{L^2}^2$ and

$$\begin{aligned} \int_{|x| \leq R} \frac{|u|^6}{\langle x \rangle} \, dx &\leq \partial_t^2 M + cR^{-3} \int_{|x| \sim R} |u|^2 \, dx + cR^{-1} \int_{|x| \sim R} |\nabla u|^2 + |u|^6 \, dx \\ &\leq \partial_t^2 M + cR^{-1} E(u). \end{aligned}$$

Setting $R = |I|^{1/2}$ and integrating in time, we obtain

$$\int_I \int_{|x| \leq |I|^{1/2}} \frac{|u|^6}{\langle x \rangle} \, dxdt \leq \sup_t 2|\partial_t M| + c|I| R^{-1} E(u) \lesssim |I|^{1/2} E(u).$$

□

By Corollary 4.7.4, if there is a finite energy solution to (4.1) that failed to scatter, then there exists a nonzero almost-periodic solution u_c, i.e. which remains in a precompact subset of $\dot{H}^1$.

Corollary 4.8.2. *If* $\|g-\delta\|_{C^3} \leq \varepsilon$*, the equation* (4.1) *does not admit nonzero almost-periodic solutions. Hence, all finite-energy solutions to* (4.1) *scatter.*

Proof. Suppose $0 \neq u_c$ is almost-periodic. Then there exists $\eta > 0$ and a radius R such that

$$\|u_c(t)\|_{L^6(\{|x| \leq R\})} \geq \eta \text{ for all } t.$$

For if not, there would exist radii $R_n \to \infty$ and times t_n such that $\|u_c(t_n)\|_{L^6(\{|x| \leq R_n\})} \to 0$. By compactness, it follows that some subsequence of $u_c(t_n)$ converges in $\dot{H}^1$ to 0. But this yields the contradiction that $E(u_c) = 0$.

We now apply Proposition 4.8.1 on time intervals I with $|I|^{1/2} > R$, and deduce

$$\eta |I| R^{-1} \leq \int_I \int_{|x| \leq |I|^{1/2}} \frac{|u_c|^6}{\langle x \rangle} \, dx dt \lesssim |I|^{1/2} E(u_c).$$

But this yields a contradiction for I sufficiently large. □

CHAPTER 5

Mass-critical inverse Strichartz theorems

5.1 Introduction

5.1.1 Background

In this final chapter, we prove inverse Strichartz theorems for several Schrödinger equations at L^2 regularity. The classical Strichartz estimate states that if $u(t) = e^{\frac{it\Delta}{2}}u(0)$ solves the linear Schrödinger equation

$$i\partial_t u = -\frac{1}{2}\Delta u,\ u(0,\cdot) \in L^2(\mathbf{R}^d),$$

then

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(\mathbf{R}\times\mathbf{R}^d)} \leq C\|u(0)\|_{L^2(\mathbf{R}^d)}. \tag{5.1}$$

Inverse theorems for this inequality characterize the initial data that give rise to solutions with nontrivial spacetime norm, and underpin the large-data theory for critical nonlinear Schrödinger equations. They reveal how blowup solutions concentrate by identifying characteristic length scales and other properties associated to symmetries or approximate symmetries of the equation.

Let us first recall these theorems in the context of the constant-coefficient energy-critical NLS in three space dimensions

$$i\partial_t u = -\frac{1}{2}\Delta u + |u|^4 u,\ u(0) \in \dot{H}^1(\mathbf{R}^3).$$

From Sobolev embedding and the L^2 inequality (5.1) when $d = 3$, linear solutions satisfy the

$\dot{H}^1$ Strichartz estimate

$$\|e^{\frac{it\Delta}{2}}u(0)\|_{L^{10}L^{10}(\mathbf{R}\times\mathbf{R}^3)} \lesssim \|\nabla u(0)\|_{L^2(\mathbf{R}^3)}. \tag{5.2}$$

According to the perturbative theory [CW90], if the linear evolution of $u(0)$ has spacetime norm less than some threshold $\varepsilon > 0$, then the nonlinear solution initialized at $u(0)$ exists for all time and scatters. To prove scattering for arbitrary initial data, one must therefore consider solutions u where the left hand side of (5.2) exceeds the threshold ε. Using a reverse Strichartz theorem, one then deduces that $u(0)$ must concentrate a significant fraction of its energy in some spacetime "bubble", with a definite width λ_0 and center (t_0, x_0), determining when and where the concentration occurs. The time parameter t_0 corresponds to the fact that the inequality is invariant under pullback $u(0) \mapsto e^{-\frac{it_0\Delta}{2}}u(0)$ by the linear flow. As the equation is invariant under scaling and translation, this already constitutes a significant finding.

By organizing this structural information into profile decompositions and imposing a minimal-energy hypothesis, one is led to consider "minimal" blowup solutions $u(t)$ which for each t concentrate essentially all of their energy in a "bubble" with width $\lambda(t)$ and position $x(t)$. This forms the basis of the Bourgain-Kenig-Merle concentration compactness and rigidity paradigm.

Similar considerations apply for the mass-critical NLS

$$i\partial_t u = -\frac{1}{2}\Delta u + |u|^{\frac{4}{d}}u, \;\; u(0) \in L^2(\mathbf{R}^d), \tag{5.3}$$

but with a twist due to Galilei invariance; if u satisfies the constant-coefficient mass-critical equation, then for each ξ_0

$$u_{\xi_0}(t,x) = e^{i(x\xi_0 - \frac{1}{2}t|\xi_0|^2)}u(t, x - t\xi_0)$$

is also a solution with the same mass (L^2-norm) and with Fourier transform shifted by frequency ξ_0. Note that while the Galilei boost also preserves the class of solutions to the energy-critical NLS, the energy grows with $|\xi_0|$. Thus, assuming finite energy eliminates this degree of freedom by effectively forcing solutions to be centered at frequency 0.

Due to Galilei invariance, a reverse theorem for the L^2 Strichartz inequality (5.1) is much more subtle compared to the energy-critical situation since it must locate a significant frequency ξ_0 in addition to a length λ_0, time t_0, and position x_0. When inverting the $\dot{H}^1$ inequality (5.2), one can use Littlewood-Paley theory to separate the contributions from each dyadic length scale. This argument does not work when the frequency center is also one of the parameters to be determined, as concentration could occur anywhere in frequency space, not just in annuli about the origin. One needs to exploit orthogonality not merely in space but in spacetime.

The existing proofs of L^2 inverse Strichartz theorems [CK07, MVV99, MV98, BV07] do this with the aid of Fourier restriction estimates, viewing solutions to the constant-coefficient Schrödinger equation as the spacetime Fourier transform of measures on the characteristic paraboloid $\tau + \frac{1}{2}|\xi|^2 = 0$. See also the exposition in [KV13]. These inverse theorems play a foundational role in the proofs of large-data scattering for equation (5.3) [TVZ08, Doda, Dodb, Dod12].

Although variable-coefficient equations generally lack scaling or translation-invariance, they may still not have a preferred length scale or location in spacetime. For example, we saw for the energy-critical harmonic oscillator that solutions with large norm may concentrate in arbitrarily small regions of space. In these cases, the broken symmetries stand in the way of classical tools for extracting the essential properties of blowup solutions, in particular the Fourier transform.

The loss of symmetries becomes particularly problematic when one considers mass-critical equations, such as the harmonic oscillator

$$i\partial_t u = (-\frac{1}{2}\Delta + \sum_{j=1}^{d} \omega_j^2 x_j^2)u + |u|^{\frac{4}{d}}u, \ u(0) \in L^2(\mathbf{R}^d). \tag{5.4}$$

Although the equation lacks a global scaling symmetry, highly concentrated solutions nonetheless accumulate spacetime norm on the same timescales as for the constant-coefficient mass-critical equation. As noted in Lemma 5.2.5 below, the problem also admits a more complicated analogue of translation and Galilei symmetry. This is connected to the well-known fact that u solves equation (5.3) if and only if its *Lens transform*

$$\mathcal{L}u(t,x) = \frac{1}{(\cos t)^{d/2}} u\Big(\tan t, \frac{x}{\cos t}\Big) e^{-\frac{i|x|^2 \tan t}{2}}.$$

solves equation (5.4) with $\omega_j \equiv \frac{1}{2}$.

While the Lens transform may be inverted to deduce that equation (5.4) is globally well-posed when $\omega_j \equiv \frac{1}{2}$, this connection with equation (5.3) disappears if the ω_j are not all equal. Studying the equation in greater generality therefore requires a more robust line of attack, such as the concentration-compactness and rigidity paradigm. To implement that strategy one needs appropriate inverse L^2 Strichartz estimates. This is no small matter since the Fourier-analytic techniques underpinning the proofs of the constant-coefficient theorems—most notably, Fourier restriction theory—are incompatible with variable-coefficient equations.

We present an alternate approach to these inverse estimates in one space dimension. By eschewing Fourier analysis for physical space arguments, we can uniformly treat a family of Schrödinger operators that includes the free particle and the harmonic oscillator.

5.1.2 The setup

Consider a (possibly time-dependent) Schrödinger operator on the real line

$$H(t) = -\frac{1}{2}\partial_x^2 + V(t,x),$$

where the potential conforms to the following hypotheses:

- For each $k \geq 2$, there exists there exists $M_k < \infty$ so that

$$\|V(t,x)\|_{L_t^\infty L_x^\infty(|x|\leq 1)} + \|\partial_x^k V(t,x)\|_{L_{t,x}^\infty} + \|\partial_x^k \partial_t V(t,x)\|_{L_{t,x}^\infty} \leq M_k. \tag{5.5}$$

- There exists some $\varepsilon > 0$ so that

$$|\langle x\rangle^{1+\varepsilon}\partial_x^3 V| + |\langle x\rangle^{1+\varepsilon}\partial_x^3\partial_t V| \in L_{t,x}^\infty. \tag{5.6}$$

  This implies by the fundamental theorem of calculus that the second derivative $\partial_x^2 V(t,x)$ converges as $x \to \pm\infty$. Here and in the sequel we write $\langle x\rangle := (1+|x|^2)^{1/2}$.

In particular, the potentials $V = 0$ and $V = \frac{1}{2}x^2$ both fall into this class.

The first set of conditions on the space derivatives of V are quite natural in view of classical Fourier integral operator constructions, from which one can deduce dispersive and Strichartz estimates; see Theorem 5.2.3. We also assume the conditions on $\partial_t V$ to have control of the time regularity of solutions. In contrast, the decay hypothesis on the $\partial_x^3 V$ is technical; see the discussion surrounding Lemma 5.4.7 below.

The unitary propagator $U(t, s)$ for such Hamiltonians is known to obey Strichartz estimates at least locally in time:

$$\|U(t,s)f\|_{L^6_{t,x}(I\times\mathbf{R})} \lesssim_I \|f\|_{L^2(\mathbf{R})} \tag{5.7}$$

for any compact interval I and any fixed $s \in \mathbf{R}$; see Corollary 5.2.4. Note that $U(t, s) = e^{-i(t-s)H}$ is a one-parameter group if one assumes that $V = V(x)$ is time-independent, but our methods do not require this assumption.

Our main result is an inverse form of this inequality which asserts that if the left side is nontrivial relative to the right side, then the initial data must concentrate somewhere. Such concentration will be detected by probing the solution with suitably scaled, translated, and modulated test functions.

For $\lambda > 0$ and $(x_0, \xi_0) \in T^*\mathbf{R} \cong \mathbf{R}_x \times \mathbf{R}_\xi$, define the scaling and phase space translation operators

$$S_\lambda f(x) = \lambda^{-1/2} f(\lambda^{-1}x), \quad \pi(x_0,\xi_0)f(x) = e^{i(x-x_0)\xi_0} f(x - x_0).$$

Throughout this chapter, let ψ denote a real, even Schwartz function $\psi \in \mathcal{S}(\mathbf{R})$ with $\|\psi\|_{L^2} = (2\pi)^{-1/2}$. Its phase space translate $\pi(x_0, \xi_0)\psi$ is localized in space near x_0 and in frequency near ξ_0.

Theorem 5.1.1. *There exists $\beta > 0$ such that if $0 < \varepsilon \le \|U(t,0)f\|_{L^6([-\frac{1}{2},\frac{1}{2}]\times\mathbf{R})}$ and $\|f\|_{L^2} \le A$, then*

$$\sup_{z\in T^*\mathbf{R},\ 0<\lambda\le 1,\ |t|\le 1/2} |\langle \pi(z)S_\lambda\psi, U(t,0)f\rangle_{L^2(\mathbf{R})}| \ge C\varepsilon(\tfrac{\varepsilon}{A})^\beta$$

for some constant C depending on the seminorms in (5.5) *and* (5.6).

In Section 5.5 we use this to construct a linear profile decomposition via standard arguments. It will follow essentially from repeatedly applying the following corollary. For simplicity we state it assuming the potential is time-independent (so that $U(t,0) = e^{-itH}$).

Corollary 5.1.2. *Let* $\{f_n\} \subset L^2(\mathbf{R})$ *be a sequence such that* $0 < \varepsilon \le \|e^{-itH} f_n\|_{L^6_{t,x}([-\frac{1}{2},\frac{1}{2}]\times\mathbf{R})}$ *and* $\|f\|_{L^2} \le A$ *for some constants* $A, \varepsilon > 0$. *Then, after passing to a subsequence, there exist a sequence of parameters*

$$\{(\lambda_n, t_n, z_n)\}_n \subset (0,1] \times [-1/2, 1/2] \times T^*\mathbf{R}$$

and a function $0 \ne \phi \in L^2$ *such that,*

$$\begin{aligned} S_{\lambda_n}^{-1} \pi(z_n)^{-1} e^{-it_n H} f_n &\rightharpoonup \phi \text{ in } L^2 \\ \|\phi\|_{L^2} &\gtrsim \varepsilon(\tfrac{\varepsilon}{A})^{\beta} \end{aligned} \tag{5.8}$$

Further,

$$\|f_n\|_2^2 - \|f_n - e^{it_n H}\pi(z_n) S_{\lambda_n}\phi\|_2^2 - \|e^{it_n H}\pi(z_n) S_{\lambda_n}\phi\|_2^2 \to 0. \tag{5.9}$$

Proof. By Theorem 5.1.1, there exist (λ_n, t_n, z_n) such that $|\langle \pi(z_n) S_{\lambda_n}\psi, e^{-it_n H} f_n\rangle| \gtrsim \varepsilon(\frac{\varepsilon}{A})^{\beta}$. As the sequence $S_{\lambda_n}^{-1}\pi(z_n)^{-1} e^{-it_n H} f_n$ is bounded in L^2, it has a weak subsequential limit $\phi \in L^2$. Passing to this subsequence, we have

$$\|\phi\|_2 \ge |\langle \psi, \phi\rangle| = \lim_{n\to\infty} |\langle \psi, S_{\lambda_n}^{-1}\pi(z_n)^{-1} e^{-it_n H} f_n\rangle| \gtrsim \varepsilon(\tfrac{\varepsilon}{A})^{\beta}.$$

To obtain (5.9), write the left side as

$$2\,\mathrm{Re}(\langle f_n - e^{it_n H}\pi(z_n) S_{\lambda_n}\phi, e^{it_n H}\pi(z_n) S_{\lambda_n}\phi\rangle = 2\,\mathrm{Re}(\langle S_{\lambda_n}^{-1}\pi(z_n)^{-1} e^{-it_n H} f_n - \phi, \phi\rangle) \to 0,$$

by the definition of ϕ. □

The restriction to a compact time interval in the above statements is dictated by the generality of our hypotheses. For a generic subquadratic potential, the $L^6_{t,x}$ norm of a solution need not be finite on $\mathbf{R}_t \times \mathbf{R}_x$. For example, solutions to the harmonic oscillator $V = x^2$ are periodic in time. However, the conclusions may be strengthened in some cases. In particular, our methods specialize to the case $V = 0$ to yield

Theorem 5.1.3. *If* $0 < \varepsilon \le \|e^{\frac{it\Delta}{2}} f\|_{L^6_{t,x}(\mathbf{R}\times\mathbf{R})} \lesssim \|f\|_{L^2} = A$, *then*

$$\sup_{z\in T^*\mathbf{R},\ \lambda>0,\ t\in\mathbf{R}} |\langle \pi(z) S_\lambda \psi, e^{\frac{it\Delta}{2}} f\rangle| \gtrsim \varepsilon(\tfrac{\varepsilon}{A})^\beta.$$

This yields the analogue of Corollary 5.1.2, which can be upgraded to a linear profile decomposition for the 1d free particle as in the proof of Proposition 5.5.2. Such a profile decomposition was obtained originally by Carles-Keraani [CK07] using different methods.

5.1.3 Ideas of proof

Let us first make a few reductions. We shall assume in the sequel that the initial data f is Schwartz. This assumption will justify certain applications of Fubini's theorem and may be removed a posteriori by an approximation argument. Further, we prove the theorem with the time interval $[-\frac{1}{2}, \frac{1}{2}]$ replaced by $[-\delta_0, \delta_0]$, where δ_0 is to be chosen later (in Theorem 5.2.3) according to the seminorms M_k of the potential. Indeed, the interval $[-\frac{1}{2}, \frac{1}{2}]$ can then be tiled by subintervals of length δ_0.

With these preliminary remarks out of the way, let us describe the main ideas of the proof of Theorem 5.1.1. We want to locate the parameters describing a bubble of concentration in the initial data. The relevant parameters in our setting are length scale λ_0, spatial center x_0, frequency center ξ_0, and a time parameter t_0 describing when the concentration occurs. Each of those parameters is associated with a noncompact symmetry or approximate symmetry of the Strichartz inequality. For instance, when $V = 0$ or $V = \frac{1}{2}x^2$, both sides of (5.7) are preserved by translations $f \mapsto f(\cdot - x_0)$ and modulations $f \mapsto e^{i(\cdot)\xi_0} f$ of the initial data (see Lemma 5.2.5 below).

The existing approaches to inverse Strichartz inequalities for the free particle can be very roughly summarized as follows. First, one uses Fourier analysis to isolate a scale λ_0 and frequency center ξ_0. For example, Carles-Keraani prove in their Proposition 2.1 that for some $1 < p < 2$,

$$\|e^{it\partial_x^2} f\|_{L^6_{t,x}(\mathbf{R}\times\mathbf{R})} \lesssim_p \left(\sup_J |J|^{\frac{1}{2}-\frac{1}{p}} \|\hat{f}\|_{L^p(J)}\right)^{1/3} \|f\|_{L^2(\mathbf{R})},$$

where J ranges over all intervals and $\hat{f}$ is the Fourier transform of f. Then one uses a separate argument to determine x_0 and t_0. This strategy ultimately relies on the fact that the propagator for the free particle is diagonalized by the Fourier transform.

General Schrödinger operators do not enjoy that luxury as the momenta of particles may vary with time and in a position-dependent manner. Thus it is natural to consider the position and frequency parameters together. To this end, we use a wavepacket decomposition as a partial substitute for the Fourier transform. Unlike the Fourier transform, however, the wavepacket transform requires that one first choose a length scale. This is not so easy because the Strichartz inequality (5.7) which we are trying to invert has no intrinsic length scale; the rescaling

$$f \mapsto \lambda^{-d/2} f(\lambda^{-1}\cdot),\ 0 < \lambda \ll 1$$

preserves both sides of the inequality exactly $V = 0$ and at least approximately for sub-quadratic V such as the harmonic oscillator $V = |x|^2$.

The key ingredient that gets us started is a refinement of the Strichartz inequality in the time variable due to Killip-Visan. Using a direct physical space argument, which we describe in Section 5.3, they show that if $u(t,x)$ is a solution with nontrivial $L^6_{t,x}$ norm, then there exists a time interval J such that u is large in $L^q_{t,x}(J \times \mathbf{R})$ for some $q < 6$. Unlike the $L^6_{t,x}$ norm, the $L^q_{t,x}$ norm of the solution has a preferred length scale directly related to the width of J. Having obtained a significant time t_0 and width λ_0, we then use an interpolation and rescaling argument to reduce matters to a refined $L^2_x \to L^4_{t,x}$ estimate. This is then proved using a wavepacket decomposition, integration by parts, and study of the Hamilton flow, revealing the parameters x_0 and ξ_0 simultaneously.

This chapter is structured as follows. Section 5.2 collects some preliminary definitions and lemmas. The heart of the argument is presented in Sections 5.3 and 5.4. Finally, Section 5.5

discusses the linear profile decomposition.

As the identification of a time interval works in any number of spatial dimensions, Sections 5.2 and 5.3 are written for a general subquadratic Schrödinger operator on $\mathbf{R}^d$. However, our subsequent reduction to L^4 relies on $d = 1$. A naive attempt to extend our argument to higher dimensions would have us to prove a refined L^p estimate for some $2 < p < 4$, but our techniques currently exploit the fact that 4 is an even integer.

Acknowledgements

It is a pleasure to thank Rowan Killip and Monica Visan for many helpful discussions, in particular for sharing the arguments presented in Section 5.3. This work was partially supported by NSF grants DMS-1265868 (PI R. Killip), DMS-1161396, and DMS-1500707 (both PI M. Visan).

5.2 Preliminaries

5.2.1 Wavepackets

We briefly recall the (continuous) wavepacket decomposition; see for instance [Fol89]. Fix a real, even Schwartz function $\psi \in \mathcal{S}(\mathbf{R}^d)$ with $\|\psi\|_{L^2} = (2\pi)^{-d/2}$. For $f \in L^2(\mathbf{R}^d)$ and $z = (x, \xi) \in T^*\mathbf{R}^d = \mathbf{R}^d_x \times \mathbf{R}^d_\xi$, define

$$Tf(z) = \int_{\mathbf{R}^d} e^{i(x-y)\xi} \psi(x-y) f(y)\, dy = \langle f, \psi_z \rangle_{L^2(\mathbf{R}^d)}.$$

By taking the Fourier transform in the x variable, we get

$$\mathcal{F}_x Tf(\eta, \xi) = \int_{\mathbf{R}^d} e^{-iy\eta} \hat{\psi}(\eta - \xi) f(y)\, dy = \hat{\psi}(\eta - \xi) \hat{f}(\eta).$$

Thus T maps $\mathcal{S}(\mathbf{R}^d) \to \mathcal{S}(\mathbf{R}^d \times \mathbf{R}^d)$ and is an isometry $L^2(\mathbf{R}^d) \to L^2(T^*\mathbf{R}^d)$. The hypothesis that ψ is even implies the adjoint formula

$$T^*F(y) = \int_{T^*\mathbf{R}^d} F(z) \psi_z(y)\, dz$$

and the inversion formula

$$f = T^*Tf = \int_{T^*\mathbf{R}^d} \langle f, \psi_z \rangle_{L^2(\mathbf{R}^d)} \psi_z \, dz.$$

5.2.2 Bicharacteristics

We collect here some relevant properties of the classical phase space flow for a subquadratic potential.

Let $V(t,x)$ satisfy $\partial_x^k V(t,\cdot) \in L^\infty(\mathbf{R}^d)$ for all $k \geq 2$, uniformly in t, and let $\Phi(t,s)$ denote the (time-dependent) Hamiltonian flow on $T^*\mathbf{R}^d$ generated by the symbol $h(t,x,\xi) = \frac{1}{2}|\xi|^2 + V(t,x)$. Note that this is well-defined for all s and t since the Hamilton vector field $\xi\partial_x - (\partial_x V)\partial_\xi$ is globally Lipschitz. For $z = (x,\xi)$, write $z^t = (x^t(z), \xi^t(z)) = \Phi(t,0)(z)$ for the bicharacteristic starting at z.

Fix $z_0, z_1 \in T^*\mathbf{R}^d$. We obtain by integrating the vector field

$$x_0^t - x_1^t = x_0^s - x_1^s + (t-s)(\xi_0^s - \xi_1^s) - \int_s^t (t-\tau)(\partial_x V(\tau, x_0^\tau) - \partial_x V(\tau, x_1^\tau)) \, d\tau$$
$$\xi_0^t - \xi_1^t = \xi_0^s - \xi_1^s - \int_s^t (\partial_x V(\tau, x_0^\tau) - \partial_x V(\tau, x_1^\tau)) \, d\tau.$$

As $|\partial_x V(\tau, x_0^\tau) - \partial_x V(\tau, x_1^\tau)| \leq \|\partial_x^2 V\|_{L^\infty} |x_0^\tau - x_1^\tau|$, we have for $|t-s| \leq 1$

$$|x_0^t - x_1^t| \leq (|x_0^s - x_1^s| + |t-s||\xi_0^s - \xi_1^s|) e^{\|\partial_x^2 V\|_{L^\infty}},$$
$$|\xi_0^t - \xi_1^t - (\xi_0^s - \xi_1^s)| \leq (|t-s||x_0^s - x_1^s| + |t-s|^2 |\xi_0^s - \xi_1^s|) \|\partial_x^2 V\|_{L^\infty} e^{\|\partial^2 V\|_{L^\infty}},$$
$$|x_0^t - x_1^t - (x_0^s - x_1^s) - (t-s)(\xi_0^s - \xi_1^s)| \leq (|t-s|^2 |x_0^s - x_1^s| + |t-s|^3 |\xi_0^s - \xi_1^s|) e^{\|\partial_x^2 V\|_{L^\infty}},$$

In the sequel, we always assume that $|t-s| \leq 1$, and all implicit constants will depend on $\partial_x^2 V$ or finitely many higher derivatives. We also remark that this time restriction may be dropped if $\partial_x^2 V \equiv 0$. The preceding computations immediately yield the following dynamical consequences:

Lemma 5.2.1. *Assume the preceding setup.*

- *There exists $\delta > 0$, depending on $\|\partial_x^2 V\|_{L^\infty}$, such that $|t-s| \leq \delta$ implies*

$$|x_0^t - x_1^t - (x_0^s - x_1^s) - (t-s)(\xi_0^s - \xi_1^s)| \leq \frac{1}{100}(|x_0^s - x_1^s| + |t-s||\xi_0^s - \xi_0^s|).$$

Hence if $|x_0^s - x_1^s| \le r$ *and* $C \ge 2$*, then* $|x_0^t - x_1^t| \ge Cr$ *for* $\frac{2Cr}{|\xi_0^s - \xi_1^s|} \le |t-s| \le \delta$*. Informally, two particles colliding with sufficiently large relative velocity will interact only once during a length* δ *time interval.*

- *If* $|x_0^s - x_1^s| \le r$*, then*

$$|\xi_0^t - \xi_1^t - (\xi_0^s - \xi_1^s)| \le \min\Big(\delta, \frac{2Cr}{|\xi_0^s - \xi_1^s|}\Big) Cr \|\partial_x^2 V\|_{L^\infty} e^{\|\partial_x^2 V\|_{L^\infty}}$$

for all t *such that* $|x_0^t - x_1^t| \le Cr$*. That is, the relative velocity of two particles remains essentially constant during an interaction.*

The following technical lemma will be used in Section 5.4.2.

Lemma 5.2.2. *There exists a constant* $C > 0$ *so that if* $Q_\eta = (0, \eta) + [-1,1]^{2d}$ *and* $r \ge 1$*, then*

$$\bigcup_{|t-t_0| \le \min(|\eta|^{-1},1)} \Phi(t,0)^{-1}(z_0^t + rQ_\eta) \subset \Phi(t_0, 0)^{-1}(z_0^{t_0} + CrQ_\eta).$$

In other words, if the bicharacteristic z^t starting at $z \in T^*\mathbf{R}^d$ passes through the cube $z_0^t + rQ_\eta$ in phase space during some time window $|t - t_0| \le |\eta|^{-1}$, then it must also pass through the twice-larger cube $z_0^{t_0} + 2rQ_\eta$ at time t_0.

Proof. If $z^s \in z_0^s + rQ_\eta$, then (5.2.2) and $|t-s| \le \min(|\eta|^{-1}, 1)$ imply that

$$|x^t - x_0^t| \lesssim |x^s - x_0^s| + \min(|\eta|^{-1}, 1)(|\eta| + r) \lesssim r,$$
$$|\xi^t - \xi_0^t - (\xi^s - \xi_0^s)| \lesssim r \min(|\eta|^{-1}, 1).$$

□

5.2.3 The Schrödinger propagator

In this section we collect some basic facts regarding the quantum propagator for subquadratic potentials.

Theorem 5.2.3 (Fujiwara [Fuj79, Fuj80]). *Let $V(t,x)$ satisfy*

$$M_k := \|\partial_x^k V(t,x)\|_{L^\infty} + \|V(t,x)\|_{L_t^\infty L_x^\infty(|x|\le 1)} < \infty$$

for all $k \ge 2$. There exists a constant $\delta_0 > 0$ such that for all $0 < |t-s| \le \delta_0$ the propagator $U(t,s)$ for $H = -\frac{1}{2}\Delta + V(t,x)$ has Schwartz kernel

$$U(t,s)(x,y) = \Big(\frac{1}{2\pi i(t-s)}\Big)^{d/2} a(t,s,x,y)e^{iS(t,s,x,y)},$$

where for each $m > 0$ there is a constant $\gamma_m > 0$ such that

$$\|a(t,s,x,y) - 1\|_{C^m(\mathbf{R}_x^d \times \mathbf{R}_y^d)} \le \gamma_m |t-s|^2.$$

Moreover

$$S(t,s,x,y) = \frac{|x-y|^2}{2(t-s)} + (t-s)r(t,s,x,y),$$

with

$$|\partial_x r| + |\partial_y r| \le C(M_2)(1 + |x| + |y|),$$

and for each multindex α with $|\alpha| \ge 2$, the quantity

$$C_\alpha = \|\partial_{x,y}^\alpha r(t,s,\cdot,\cdot)\|_{L^\infty}$$

is finite. The map $U(t,s) : \mathcal{S}(\mathbf{R}^d) \to \mathcal{S}(\mathbf{R}^d)$ is a topological isomorphism, and all implicit constants depend on finitely many seminorms M_k.

Definition 5.2.1. A pair of exponents (q,r) is (Schrödinger)-*admissible* if $(q,r,d) \ne (2,\infty,2)$, $2 \le q \le \infty$, and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$.

Corollary 5.2.4 (Dispersive and Strichartz estimates). *If V satisfies the hypotheses of the previous theorem, then $U(t,s)$ admits the fixed-time bounds*

$$\|U(t,s)\|_{L_x^1(\mathbf{R}^d) \to L_x^\infty(\mathbf{R}^d)} \lesssim |t-s|^{-d/2}$$

whenever $|t-s| \le \delta_0$. For any compact time interval I and any admissible exponents (q,r),

$$\|U(t,s)f\|_{L_t^q L_x^r(I\times\mathbf{R}^d)} \lesssim_I \|f\|_{L^2(\mathbf{R}^d)}.$$

Proof. It follows from the general machinery of Keel-Tao [KT98], the above pointwise bound for $U(t,s)$, and the unitarity of $U(t,s)$ on L^2 that for any fixed s,

$$\|U(t,s)f\|_{L_t^q L_x^r(\{|t-s|\le\delta_0\}\times\mathbf{R}^d)} \lesssim \|f\|_{L^2}.$$

If $I=[T_0,T_1]$ is a general time interval, partition it into subintervals $[t_{j-1},t_j]$ of length at most δ_0. For each such subinterval we can write $U(t,s)=U(t,t_{j-1})U(t_{j-1},s)$, so

$$\|U(t,s)f\|_{L_t^q L_x^r([t_{j-1},t_j]\times\mathbf{R}^d)} \lesssim \|U(t_{j-1},s)f\|_{L^2} = \|f\|_{L^2}.$$

The corollary follows from summing over the subintervals. □

Recall that solutions to the free particle equation $i\partial_t u = -\frac{1}{2}\Delta u,\ u(0)=\phi$ transform according to the following rule with respect to phase space translations of the initial data:

$$e^{\frac{it\Delta}{2}}\pi(x_0,\xi_0)\phi(x) = e^{i[(x-x_0)\xi_0-\frac{1}{2}t|\xi_0|^2]}(e^{\frac{it\Delta}{2}}\phi)(x-x_0-t\xi_0). \tag{5.10}$$

Physically, $\pi(x_0,\xi_0)\phi$ represents the state of a quantum particle with position x_0 and momentum ξ_0. The above relation states that the time evolution of $\pi(x_0,\xi_0)\phi$ in the absence of a potential oscillates in space and time at frequency ξ_0 and $-\frac{1}{2}|\xi_0|^2$, respectively, and tracks the classical trajectory $t\mapsto x_0+t\xi_0$.

In the presence of a potential, the time evolution of such modified initial data admits a more complicated but structurally similar description:

Lemma 5.2.5. *If $U(t,s)$ is the propagator for $H=-\frac{1}{2}\Delta+V(t,x)$, then*

$$\begin{aligned} U(t,s)\pi(z_0^s)\phi(x) &= e^{i[(x-x_0^t)\xi_0^t+\int_s^t \frac{1}{2}|\xi_0^\tau|^2-V(\tau,x_0^\tau)\,d\tau]}U^{z_0}(t,s)\phi(x-x_0^t)\\ &= e^{i\alpha(t,s,z_0)}\pi(z_0^t)U^{z_0}(t,s)\phi(x), \end{aligned}$$

where

$$\alpha(t,s,z) = \int_s^t \frac{1}{2}|\xi_0^\tau|^2 - V(\tau,x_0^\tau)\,d\tau,$$

is the classical action, $U^{z_0}(t,s)$ is the propagator for $H^{z_0}=-\frac{1}{2}\Delta+V^{z_0}(t,x)$,

$$V^{z_0}(t,x) = V(t,x_0^t+x) - V(t,x_0^t) - x\partial_x V(t,x_0^t) = \langle x, Qx\rangle$$

where

$$Q(t,x) = \int_0^1 (1-\theta)\partial_x^2 V(t, x_0^t + \theta x)\, d\theta,$$

and $z_0^t = (x_0^t, \xi_0^t)$ *is the trajectory of* z_0 *under the Hamiltonian flow of the symbol* $h = \frac{1}{2}|\xi|^2 + V(t,x)$*. The propagator* $U^{z_0}(t,s)$ *is continuous on* $\mathcal{S}(\mathbf{R}^d)$ *uniformly in* z_0 *and* $|t-s| \le \delta_0$*.*

Proof. The formula for $U(t,s)\pi(z_0^s)\phi$ is verified by direct computation. To obtain the last statement, we notice that $\|\partial_x^k V^{z_0}\|_{L^\infty} = \|\partial_x^k V\|_{L^\infty}$ for $k \ge 2$, and appeal to the last part of Theorem 5.2.3. □

Remarks.

- This reduces to (5.10) when $V = 0$ and also yields analogous relations when V is a polynomial of degree at most 2. When $V = Ex$ is the potential for a constant electric field, we recover the well-known Avron-Herbst formula by setting $z_0 = 0$ (hence $V^{z_0} = 0$). For $V = \sum_j \omega_j x_j^2$ we get the "phase space translation" symmetry mentioned in the introduction.

- Direct computation shows that the above identity extends to semilinear equations of the form

  $$i\partial_t u = (-\frac{1}{2}\Delta + V)u + |u|^p u.$$

  That is, if u is the solution with $u(0) = \pi(z_0)\psi$, then

  $$u(t) = e^{i\int_0^t L(\tau, z_0^\tau)\, d\tau}\pi(z_0^t)u_{z_0}(t)$$

  where u_{z_0} solves

  $$i\partial_t u_{z_0} = (-\frac{1}{2}\Delta + V^{z_0})u_{z_0} + |u_{z_0}|^p u_{z_0},\ u_{z_0}(0) = \psi,$$

  with the potential V^{z_0} defined as above.

- One can combine this lemma with a wavepacket decomposition to represent a solution $U(t,0)f$ as a sum of wavepackets

  $$U(t,0)f = \int_{z_0 \in T^*\mathbf{R}^d} \langle f, \psi_{z_0}\rangle U(t,0)(\psi_{z_0})\, dz_0,$$

where the oscillation of each wavepacket $U(t,0)(\psi_{z_0})$ is largely captured in the phase

$$(x - x_0^t)\xi_0^t + \int_0^t \frac{1}{2}|\xi_0^\tau|^2 - V(\tau, x_0^\tau)\, d\tau.$$

Our arguments will make essential use of this information. Analogous wavepacket representations have been constructed by Koch and Tataru [KT05, Theorem 4.3] for a broad class of pseudodifferential operators.

5.3 Locating a length scale

In this section we present an unpublished argument of Killip-Visan that identifies both a characteristic length and a temporal center for our sought-after bubble of concentration. Recall that the usual TT^* proof of the nonendpoint Strichartz inequality combines the dispersive estimate with the Hardy-Littlewood-Sobolev inequality in time. By using instead an inverse HLS inequality, one can locate a time interval on which the solution is large in a non-admissible spacetime norm.

Proposition 5.3.1. *Let (q, r) be admissible with $2 < q < \infty$, and suppose $u = U(t,0)f$ solves*

$$i\partial_t u = \left(-\tfrac{1}{2}\Delta + V\right)u,\ \ u(0) = f \in L^2(\mathbf{R}^d)$$

with $\|f\|_{L^2(\mathbf{R}^d)} = 1$ and $\|u\|_{L_t^q L_x^r([-\delta_0,\delta_0]\times\mathbf{R}^d)} \ge \varepsilon$, where δ_0 is the constant from Theorem 5.2.3. Then there is a time interval $J \subset [-\delta_0, \delta_0]$ such that

$$\|u\|_{L_t^{q-1} L_x^r (J\times\mathbf{R}^d)} \gtrsim |J|^{\frac{1}{q(q-1)}} \varepsilon^{1+\frac{4(q+2)}{q(q-2)}}$$

Remark. That this estimate singles out a special length is easiest to see when $V = 0$. For $\lambda > 0$, let $f_\lambda = \lambda^{-d/2} f(\lambda^{-1})$ be a rescaling of some fixed $f \in L^2$, and let $u_\lambda(t,x) = \lambda^{-d/2}u(\lambda^{-2}t, \lambda^{-1}x) = e^{\frac{it\Delta}{2}}(f_\lambda)$ be their linear evolutions (here $u := u_1$). Both sides of the Strichartz inequality

$$\|u_\lambda\|_{L_t^q L_x^r} \lesssim \|f_\lambda\|_{L^2}$$

remain constant as λ varies.

We claim (supposing for example that $J = [0,1]$ in the lemma)

$$\|u_\lambda\|_{L_t^{q-1} L_x^r([0,1]\times\mathbf{R}^d)} \to 0 \text{ as } \lambda \to 0$$

Indeed, as $\|u\|_{L_t^q L_x^r(\mathbf{R}\times\mathbf{R}^d)} \lesssim \|f\|_{L^2} < \infty$, for each $\eta > 0$ there exists $T > 0$ so that (suppressing the region of integration in x) $\|u\|_{L_t^q, L_x^r(\{|t|>T\})} < \eta$. Then

$$\begin{aligned}
\|u_\lambda\|_{L_t^{q-1} L_x^r([0,1])} &\leq \|u_\lambda\|_{L_t^{q-1} L_x^r([0,\lambda^2 T])} + \|u_\lambda\|_{L_t^{q-1} L_x^r([\lambda^2 T,1])} \\
&\leq (\lambda^2 T)^{\frac{1}{q(q-1)}} \|u_\lambda\|_{L_t^q L_x^r([0,\lambda^2 T])} + \|u_\lambda\|_{L_t^q L_x^r([\lambda^2 T,1])} \\
&\leq (\lambda^2 T)^{\frac{1}{q(q-1)}} \|u\|_{L_t^q L_x^r} + \eta,
\end{aligned}$$

which yields the claim. Thus, a lower bound on $\|u_\lambda\|_{L_t^{q-1} L_x^r([0,1])}$ is incompatible with concentration of the solution at arbitrarily small scales. Similar considerations preclude $\lambda \to \infty$.

To prove the proposition we shall use the following inverse Hardy-Littlewood-Sobolev inequality. For $0 < s < d$, denote by $I_s f(x) = (|D|^{-s} f)(x) = c_{s,d} \int_{\mathbf{R}^d} \frac{f(x-y)}{|y|^{d-s}}\, dy$ the fractional integration operator.

Lemma 5.3.2 (Inverse HLS)**.** *For $1 < p < \infty$ and $0 < s < d/p$,*

$$\|I_s f\|_{L_x^{\frac{pd}{d-ps}}(\mathbf{R}^d)} \lesssim \|f\|_{L^p}^{1-\frac{ps}{d}+\frac{s(p-1)}{d-s}} \left(\sup_B |B|^{\frac{1}{p}-1} \int_B f(y)\, dy \right)^{\frac{s(d-ps)}{d(d-s)}},$$

where the sup is taken over all balls.

Proof. We use a variant of the usual proof of the HLS inequality due to Hedberg [Hed72]; see also [Ste93, §VIII.4.2]. Let

$$\delta = \sup_B |B|^{\frac{1}{p}-1} \int_B f(y)\, dy \leq \|f\|_{L^p}.$$

For $r_1 < r_2$ to be fixed shortly, decompose the integral as

$$\begin{aligned}
I_s f(x) = c_{s,d} \int_{\mathbf{R}^d} \frac{f(x-y)}{|y|^{d-s}} &\leq \int_{|y|\leq r_1} + \int_{r_1 \leq |y| \leq r_2} + \int_{|y|\geq r_2} \\
&\lesssim r_1^s Mf(x) + \delta r_1^{s-d} r_2^{d-\frac{d}{p}} + r_2^{s-\frac{d}{p}} \|f\|_{L^p},
\end{aligned}$$

where Mf is the Hardy-Littlewood maximal function and Hölder was used to estimate the second and third integrals. Choosing r_1 and r_2 to equate the terms, we find that

$$(\tfrac{r_2}{r_1})^{d-s} = \tfrac{\|f\|_p}{\delta}, \quad r_2 = (\tfrac{\delta}{Mf})^{\frac{p}{d}}(\tfrac{\|f\|_p}{\delta})^{\frac{p}{d-s}}$$

which yields the pointwise bound

$$I_s(f) \lesssim \delta^{\frac{ps}{d}-\frac{s(p-1)}{d-s}} Mf^{1-\frac{ps}{d}} \|f\|_{L^p}^{\frac{s(p-1)}{d-s}}.$$

The conclusion follows. $\square$

Proof of Proposition 5.3.1. Define the map $T : L^2_x \to L^q_t L^r_x$ by $Tf(t) = U(t,0)f$, which by Corollary 5.2.4 is continuous. By duality, $\varepsilon \le \|u\|_{L^q_t L^r_x}$ implies $\varepsilon \le \|T^*\phi\|_{L^2_x}$, where

$$\phi = \frac{|u|^{r-1}}{\|u(t)\|_{L^r_x}^{r-1}} \frac{\|u(t)\|_{L^r_x}^{q-1}}{\|u\|_{L^q_t L^r_x}^{q-1}}$$

satisfies $\|\phi\|_{L^{q'}_t L^{r'}_x} = 1$, and

$$T^*\phi = \int U(0,s)\phi(s)\, ds.$$

By the dispersive estimate of Corollary 5.2.4,

$$\varepsilon^2 \le \langle T^*\phi, T^*\phi\rangle = \langle \phi, TT^*\phi\rangle_{L^2_x} = \int \overline{\phi(t)} U(t,s)\phi(s)\, dxdsdt \lesssim \int\int \frac{G(t)G(s)}{|t-s|^{2/q}}\, dsdt,$$

where $G(t) = \|\phi(t)\|_{L^{r'}_x}$. Writing the last term as $\|I_s G\|_{L^2_t}^2$, where $s = \frac{1}{2} - \frac{1}{q} = \frac{1}{q'} - \frac{1}{2}$, and appealing to the previous lemma with $p = q'$, we can bound the above by

$$(\sup_J |J|^{-1/q} \|G\|_{L^1(J)})^{\frac{q'(\frac{1}{q'}-\frac{1}{2})}{1-(\frac{1}{q'}-\frac{1}{2})}} = (\sup_J |J|^{-1/q} \|u\|_{L^q_t L^r_x}^{1-q} \|u\|_{L^{q-1}_t L^r_x(J\times\mathbf{R})}^{q-1})^{\frac{q(q-2)}{2(q-1)(q+2)}}.$$

Upon rearranging, we get

$$\sup_J |J|^{-\frac{1}{q(q-1)}} \|u\|_{L^{q-1}_t L^r_x(J\times\mathbf{R})} \gtrsim \varepsilon^{(1+\frac{4(q+2)}{q(q-2)})}.$$

$\square$

5.4 A refined L^4 estimate

5.4.1 Reduction to L^4

Now we specialize to the one-dimensional setting $d = 1$, and apply Proposition 5.3.1 to the Strichartz pair

$$(q, r) = \Big(\frac{7+\sqrt{33}}{2}, \frac{5+\sqrt{33}}{2}\Big)$$

determined by the conditions $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ and $q - 1 = r$.

Corollary 5.4.1. *With (q, r) as above, choose*

$$\frac{\frac{1}{6} - \frac{1}{q}}{\frac{1}{4} - \frac{1}{q}} < \theta < 1.$$

Suppose

$$\varepsilon = \|U(t, 0)f\|_{L^6_{t,x}([-\delta_0,\delta_0]\times\mathbf{R})} \lesssim \|f\|_{L^2} = A.$$

Then there exists a time interval J such that

$$\|U(t, 0)f\|_{L^{q-1}_t L^r_x(J\times\mathbf{R})} \gtrsim A|J|^{\frac{1}{q(q-1)}} \Big(\frac{\varepsilon}{A}\Big)^{\frac{1}{\theta}(1+\frac{4(q+2)}{q(q-2)})}.$$

Proof. Let (q_0, r_0) be any Strichartz pair with $4 < q < 6$. Then with

$$\theta = \frac{\frac{1}{6} - \frac{1}{q}}{\frac{1}{q_0} - \frac{1}{q}},$$

we have

$$\varepsilon \leq \|U(t, 0)f\|_{L^6_{t,x}} \leq \|U(t, 0)f\|^{1-\theta}_{L^{q_0}_t L^{r_0}_x} \|U(t, 0)f\|^{\theta}_{L^q_t L^r_x} \lesssim A^{1-\theta}\|U(t, 0)f\|^{\theta}_{L^q_t L^r_x}.$$

The claim now follows from the previous lemma. □

Let $J = [t_0 - \lambda^2, t_0 + \lambda^2]$ be the interval from the above corollary, and set

$$u(t, x) = \lambda^{-1/2} u_\lambda(\lambda^{-2}(t - t_0), \lambda^{-1}x),$$

where u_λ solves

$$i\partial_t u_\lambda = (-\tfrac{1}{2}\partial_x^2 + V_\lambda)u_\lambda = 0,\ u_\lambda(0, x) = \lambda^{1/2}u(t_0, \lambda x).$$

and $V_\lambda(t,x) = \lambda^2 V(t_0+\lambda^2 t, \lambda x)$ also satisfies the hypotheses (5.5) and (5.6) for all $0 < \lambda \leq 1$. By the corollary and a change of variables,

$$\|u_\lambda\|_{L^{q-1}_{t,x}([-1,1]\times\mathbf{R})} \gtrsim A(\tfrac{\varepsilon}{A})^{\frac{1}{\theta}(1+\frac{4(q+2)}{q(q-2)})}.$$

As $4 < q-1 < 6$, Theorem 5.1.1 will follow by interpolating between the $L^2_x \to L^6_{t,x}$ Strichartz estimate and the following refined $L^2_x \to L^4_{t,x}$ estimate. Recall that ψ is the test function fixed in the introduction.

Proposition 5.4.2. *Let V be a potential satisfying the hypotheses* (5.5) *and* (5.6)*. Then there exists $\delta_0 > 0$ (depending on the seminorms in* (5.5)*) so that if $\eta(t)$ is a bump function vanishing when $|t| \geq \delta_0$ then*

$$\|U_V(t,0)f\|_{L^4_{t,x}(\eta(t)dxdt)} \lesssim \|f\|_2^{1-\beta} \sup_z |\langle \psi_z, f\rangle|^\beta$$

for some absolute constant $0 < \beta < 1$.

5.4.2 Proof of Proposition 5.4.2

We fix the potential V and drop the subscript V from the propagator. Assuming the setup of the proposition, we decompose f into wavepackets $f = \int_{T^*\mathbf{R}} \langle f, \psi_z\rangle \psi_z \, dz$, and expand the L^4 norm:

$$\|U(t,0)f\|^4_{L^4_{t,x}} \leq \int_{(T^*\mathbf{R})^4} K(z_1,z_2,z_3,z_4) \prod_{j=1}^4 |\langle f, \psi_{z_j}\rangle| \, dz_1 dz_2 dz_3 dz_4,$$

where

$$K = |\langle U(t,0)(\psi_{z_1})U(t,0)(\psi_{z_2}), U(t,0)(\psi_{z_3})U(t,0)(\psi_{z_4})\rangle_{L^2_{t,x}(\eta(t)dxdt)}|. \tag{5.11}$$

There is no difficulty with interchanging the order of integration as f was assumed to be Schwartz.

Proposition 5.4.3. *For some $0 < \theta < 1$ the kernel*

$$K(z_1,z_2,z_3,z_4)\max(\langle z_1 - z_2\rangle^\theta, \langle z_3 - z_4\rangle^\theta)$$

is bounded as a map on $L^2(T^*\mathbf{R} \times T^*\mathbf{R})$.

We defer the proof for the moment and observe how this proposition implies the previous one. Writing $a_z = |\langle f, \psi_z \rangle|$, we have

$$\|U(t,0)f\|_{L^4}^4 \lesssim \Big(\int_{(T^*\mathbf{R})^2} a_{z_1}^2 a_{z_2}^2 \langle z_1 - z_2 \rangle^{-2\theta} \, dz_1 dz_2 \Big)^{1/2} \Big(\int_{(T^*\mathbf{R})^2} a_{z_3}^2 a_{z_4}^2 \, dz_3 dz_4 \Big)^{1/2}$$
$$\lesssim \|f\|_{L^2}^2 \Big(\int_{(T^*\mathbf{R})^2} a_{z_1}^2 a_{z_2}^2 \langle z_1 - z_2 \rangle^{-2\theta} \, dz_1 dz_2 \Big)^{1/2}$$

By Young's inequality, the convolution kernel $k(z_1, z_2) = \langle z_1 - z_2 \rangle^{-2\theta}$ is bounded from L_z^p to $L_z^{p'}$ for some $p \in (1,2)$, and the integral on the right is bounded by

$$\Big(\int_{T^*\mathbf{R}} a_z^{2p} \, dz \Big)^{2/p} \le \|f\|_{L^2}^{4/p} \sup_z a_z^{4/p'}.$$

This yields the desired estimate

$$\|U(t,0)f\|_{L^4} \lesssim \|f\|_{L^2}^{\frac{1}{2} + \frac{1}{2p}} \sup_z a_z^{\frac{1}{2p'}}.$$

Thus it remains to prove Proposition 5.4.3. By Lemma 5.2.5,

$$U(t,0)(\psi_{z_j})(x) = e^{i\alpha_j} U_j(t,0)\psi(x - x_j^t),$$

where

$$\alpha_j(t,x) = (x - x_j^t)\xi_j^t + \int_0^t \frac{1}{2}|\xi_j^\tau|^2 - V(\tau, x_j^\tau) \, d\tau$$

and U_j is the propagator for $H_j = -\frac{1}{2}\partial_x^2 + V_j(t,x)$, where

$$V_j(t,x) = x^2 \int_0^1 (1-s)\partial_x^2 V(t, x_j^t + sx) \, ds, \tag{5.12}$$

and the envelopes $U_j(t,0)\psi(x - x_j^t)$ concentrate along the classical trajectories $t \mapsto x_j^t$:

$$|\partial_x^k U_j(t,0)\psi(x - x_j^t)| \lesssim_{k,N} \langle x - x_j^t \rangle^{-N}. \tag{5.13}$$

The kernel K thus admits the crude bound

$$K(\vec{z}) \lesssim_N \int \prod_{j=1}^4 \langle x - x_j^t \rangle^{-N} \, \eta(t) dx dt \lesssim \max(\langle z_1 - z_2 \rangle, \langle z_3 - z_4 \rangle)^{-1},$$

and Proposition 5.4.3 will follow from

Proposition 5.4.4. *For $\delta > 0$ sufficiently small the kernel $K^{1-\delta}$ is bounded on $L^2(T^*\mathbf{R} \times T^*\mathbf{R})$.*

Proof. We partition the 4-particle phase space $(T^*\mathbf{R})^4$ according to the degree of interaction between the particles. Define

$$E_0 = \{\vec{z} \in (T^*\mathbf{R})^4 : \min_{|t|\le\delta_0} \max_{k,k'} |x_k^t - x_{k'}^t| \le 1\}$$

$$E_m = \{\vec{z} \in (T^*\mathbf{R})^4 : 2^{m-1} < \min_{|t|\le\delta_0} \max_{k,k'} |x_k^t - x_{k'}^t| \le 2^m\},\ m \ge 1,$$

and decompose

$$K = K1_{E_0} + \sum_{m\ge1} K1_{E_m} = K_0 + \sum_{m\ge1} K_m.$$

Then

$$K^{1-\delta} = K_0^{1-\delta} + \sum_{m\ge1} K_m^{1-\delta}.$$

Heuristically, the K_0 term corresponds to the 4-tuples of wavepackets that all collide at some time $t \in [-\delta_0, \delta_0]$. Due to the rapid decay in (5.13), this will be the dominant term. We shall show that for any $N > 0$,

$$\|K_m^{1-\delta}\|_{L^2\to L^2} \lesssim_N 2^{-mN}, \tag{5.14}$$

which immediately implies the proposition upon summing in m. In turn, this will be a consequence of the following pointwise bound:

Lemma 5.4.5. *For each m and $\vec{z} \in E_m$, let $t(\vec{z})$ be a time witnessing the minimum in the definition of E_m. Then for any $N_1, N_2 > 0$,*

$$\begin{aligned}|K_m(\vec{z})| &\lesssim_{N_1,N_2} 2^{-mN_1} \\ &\times \min\Big(\frac{\langle \xi_1^{t(\vec{z})} + \xi_2^{t(\vec{z})} - \xi_3^{t(\vec{z})} - \xi_4^{t(\vec{z})}\rangle^{-N_2}}{1 + |\xi_1^{t(\vec{z})} - \xi_2^{t(\vec{z})}| + |\xi_3^{t(\vec{z})} - \xi_4^{t(\vec{z})}|}, \frac{1 + |\xi_1^{t(\vec{z})} - \xi_2^{t(\vec{z})}| + |\xi_3^{t(\vec{z})} - \xi_4^{t(\vec{z})}|}{\big|(\xi_1^{t(\vec{z})} - \xi_2^{t(\vec{z})})^2 - (\xi_3^{t(\vec{z})} - \xi_4^{t(\vec{z})})^2\big|^2}\Big),\end{aligned}$$

This will be proved below. For the moment, let us use it to deduce (5.14). By Schur's test, it will suffice to show that

$$\int K_m(z_1, z_2, z_3, z_4)^{1-\delta}\, dz_1 dz_2 + \int K_m(z_1, z_2, z_3, z_4)^{1-\delta}\, dz_3 dz_4 \lesssim_N 2^{-mN}. \tag{5.15}$$

We estimate just the first integral as the second is handled similarly.

Fix (z_3, z_4) in the image of the projection $E_m \subset (T^*\mathbf{R})^4 \to T^*\mathbf{R}_{z_3} \times T^*\mathbf{R}_{z_4}$, and let

$$E_m(z_3, z_4) = \{(z_1, z_2) \in (T^*\mathbf{R})^2 : (z_1, z_2, z_3, z_4) \in E_m\}.$$

Choose t_1 minimizing $|x_3^{t_1} - x_4^{t_1}|$; the definition of E_m implies that $|x_3^{t_1} - x_4^{t_1}| \leq 2^m$.

Suppose $(z_1, z_2) \in E_m(z_3, z_4)$. By Lemma 5.2.1, any "collision time" $t(z_1, z_2, z_3, z_4)$ must belong to the interval

$$I = \Big\{t \in [-\delta_0, \delta_0] : |t - t_1| \lesssim \min\Big(1, \frac{2^m}{|\xi_3^{t_1} - \xi_4^{t_1}|}\Big)\Big\},$$

and for such t,

$$|\xi_3^t - \xi_4^t - (\xi_3^{t_1} - \xi_4^{t_1})| \lesssim \min\Big(2^m, \frac{2^{2m}}{|\xi_3^{t_1} - \xi_4^{t_1}|}\Big).$$

The contribution of each $(z_1, z_2) \in E_m(z_3, z_4)$ to the integral (5.15) will depend on their relative momenta at the collision time. We now organize $E_m(z_1, z_2)$ accordingly.

Write $Q_\xi = (0, \xi) + [-1, 1]^2 \subset T^*\mathbf{R}$, and denote by $\Phi(t, s)$ the classical propagator for the Hamiltonian

$$h = \frac{1}{2}|\xi|^2 + V(t, x).$$

Using the shorthand $z^t = \Phi(t, 0)(z)$, define for μ_1, μ_2

$$Z_{\mu_1,\mu_2} = \bigcup_{t \in I} (\Phi(t, 0) \otimes \Phi(t, 0))^{-1}\Big(\frac{z_3^t + z_4^t}{2} + 2^m Q_{\mu_1}\Big) \times \Big(\frac{z_3^t + z_4^t}{2} + 2^m Q_{\mu_2}\Big),$$

where $\Phi(t, 0) \otimes \Phi(t, 0)(z_1, z_2) = (z_1^t, z_2^t)$ is the product flow on $T^*\mathbf{R} \times T^*\mathbf{R}$. These correspond to the wavepackets (z_1, z_2) with momenta (μ_1, μ_2) relative to the wavepackets (z_3, z_4) at the time of interaction. We have

$$E_m(z_3, z_4) \subset \bigcup_{\mu_1, \mu_2 \in \mathbf{Z}} Z_{\mu_1,\mu_2}.$$

Lemma 5.4.6. $|Z_{\mu_1,\mu_2}| \lesssim 2^{4m} \max(|\mu_1|, |\mu_2|)|I|$, *where* $|\cdot|$ *on the left denotes Lebesgue measure on* $(T^*\mathbf{R})^2$.

Proof. Without loss assume $|\mu_1| \geq |\mu_2|$. Partition the interval I into subintervals of width $|\mu_1|^{-1}$. For each t' in the partition, Lemma 5.2.2 implies that

$$\bigcup_{|t-t'|\leq|\mu_1|^{-1}} \Phi(t,0)^{-1}\Big(\frac{z_3^t+z_4^t}{2}+2^m Q_{\mu_1}\Big) \subset \Phi(t',0)^{-1}\Big(\frac{z_3^{t'}+z_4^{t'}}{2}+C2^m Q_{\mu_1}\Big)$$
$$\bigcup_{|t-t'|\leq|\mu_1|^{-1}} \Phi(t,0)^{-1}\Big(\frac{z_3^t+z_4^t}{2}+2^m Q_{\mu_2}\Big) \subset \Phi(t',0)^{-1}\Big(\frac{z_3^{t'}+z_4^{t'}}{2}+C2^m Q_{\mu_2}\Big),$$

hence

$$\bigcup_{|t-t'|\leq|\mu_1|^{-1}} (\Phi(t,0)\otimes\Phi(t,0))^{-1}\Big(\frac{z_3^t+z_4^t}{2}+2^m Q_{\mu_1}\Big)\times\Big(\frac{z_3^t+z_4^t}{2}+2^m Q_{\mu_2}\Big)$$
$$\subset (\Phi(t',0)\otimes\Phi(t',0))^{-1}\Big(\frac{z_3^{t'}+z_4^{t'}}{2}+C2^m Q_{\mu_1}\Big)\times\Big(\frac{z_3^{t'}+z_4^{t'}}{2}+C2^m Q_{\mu_2}\Big).$$

By Liouville's theorem, the right side has measure $O(2^{4m})$ in $(T^*\mathbf{R})^2$. The claim follows by summing over the partition. □

For each $(z_1,z_2)\in E_m(z_3,z_4)\cap Z_{\mu_1,\mu_2}$, suppose t in I is such that $z_j^t \in \frac{z_3^t+z_4^t}{2}+2^m Q_{\mu_j}$. As

$$\xi_j^t = \frac{\xi_3^t+\xi_4^t}{2}+\mu_j+O(2^m),\ j=1,2,$$

the second assertion of Lemma 5.2.1 implies that

$$\xi_1^{t(\vec{z})}+\xi_2^{t(\vec{z})}-\xi_3^{t(\vec{z})}-\xi_4^{t(\vec{z})} = \mu_1+\mu_2+O(2^m)$$
$$\xi_1^{t(\vec{z})}-\xi_2^{t(\vec{z})} = \mu_1-\mu_2+O(2^m),$$

hence by Lemma 5.4.5

$$\begin{aligned}|K_m| &\lesssim_N 2^{-3mN}\\ &\quad\times\min\Big(\frac{\langle\mu_1+\mu_2+O(2^m)\rangle^{-N}}{1+\big||\mu_1-\mu_2|+|\xi_3^{t_1}-\xi_4^{t_1}|+O(2^m)\big|},\frac{1+|\mu_1-\mu_2|+|\xi_3^{t_1}-\xi_4^{t_1}|+O(2^m)}{\big|(\mu_1-\mu_2)^2-(\xi_3^{t_1}-\xi_4^{t_1})^2+O(2^{2m})\big|^2}\Big)\\ &\lesssim_N 2^{(5-N)m}\min\Big(\frac{\langle\mu_1+\mu_2\rangle^{-N}}{1+|\mu_1-\mu_2|+|\xi_3^{t_1}-\xi_4^{t_1}|},\frac{1+|\mu_1-\mu_2|+|\xi_3^{t_1}-\xi_4^{t_1}|}{\big|(\mu_1-\mu_2)^2-(\xi_3-\xi_4)^2\big|^2}\Big).\end{aligned}$$

Applying Lemma 5.4.6, writing $\max(|\mu_1|,|\mu_2|)\leq|\mu_1+\mu_2|+|\mu_1-\mu_2|$, and absorbing

$|\mu_1+\mu_2|$ into the factor $\langle \mu_1+\mu_2\rangle^{-N}$,

$$\int K_m(z_1,z_2,z_3,z_4)^{1-\delta}\, dz_1dz_2 \leq \sum_{\mu_1,\mu_2\in\mathbf{Z}} \int K_m(z_1,z_2,z_3,z_4)^{1-\delta} 1_{Z_{\mu_1,\mu_2}}(z_1,z_2)\, dz_1dz_2$$

$$\lesssim \sum_{\mu_1,\mu_2\in\mathbf{Z}} 2^{-mN} \min\Big(\frac{\langle \mu_1+\mu_2\rangle^{-N}}{1+|\mu_1-\mu_2|+|\xi_3^{t_1}-\xi_4^{t_1}|}, \frac{1+|\mu_1-\mu_2|+|\xi_3^{t_1}-\xi_4^{t_1}|}{\big|(\mu_1-\mu_2)^2-(\xi_3-\xi_4)^2\big|^2}\Big)^{1-\delta} \frac{1+|\mu_1-\mu_2|}{1+|\xi_3^{t_1}-\xi_4^{t_1}|}.$$

When $|\mu_1-\mu_2|\leq 1$, we choose the term in the minimum to see that the sum is of size 2^{-mN}. Hence we may restrict attention to the terms where $|\mu_1-\mu_2|\geq 1$.

When $|\mu_1-\mu_2|\geq 2|\xi_3^{t_1}-\xi_4^{t_1}|$, the above expression is bounded by

$$\sum_{\mu_1,\mu_2\in\mathbf{Z}} 2^{-mN} \min\Big(\langle \mu_1+\mu_2\rangle^{-N}, \frac{1}{\big||\mu_1-\mu_2|+1\big|^2}\Big)^{1-\delta} \lesssim_N 2^{-mN}.$$

Otherwise, one has the bound

$$\sum_{\mu_1,\mu_2\in\mathbf{Z}} 2^{-mN} \min\Big(\frac{\langle \mu_1+\mu_2\rangle^{-N}}{1+|\mu_1-\mu_2|}, \frac{1}{\big||\mu_1-\mu_2|-|\xi_3^{t_1}-\xi_4^{t_1}|\big|^2}\Big)^{1-\delta} \lesssim_N 2^{-mN}.$$

Therefore

$$\int K_m(z_1,z_2,z_3,z_4)^{1-\delta} dz_1dz_2 \lesssim_N 2^{-mN},$$

and the same considerations apply with the roles of (z_1,z_2) and (z_3,z_4) reversed. The estimate (5.14) now follows from Schur's test. Modulo Lemma 5.4.5, this completes the proof of Proposition 5.4.4. □

5.4.3 Proof of Lemma 5.4.5

Note that from (5.13) and the definition of E_m one immediately gets the cheap bound

$$|K_m(\vec{z})| \lesssim_N 2^{-mN}.$$

However one can often do better by exploiting how the wavepackets oscillate in space and time. As the argument is essentially the same for all m, we shall for simplicity take $m=0$ in the sequel.

Suppose that $t(\vec{z})=0$. By Lemma 5.2.5,

$$K_0(\vec{z}) = \Big|\int e^{i\Phi} \prod_{j=1}^4 U_j(t,0)\psi(x-x_j^t)\,\eta(t)dxdt\Big|,$$

where $\sigma = (+,+,-,-)$, $\prod_{j=1}^4 c_j = c_1 c_2 \overline{c}_3 \overline{c}_4$, and

$$\Phi = \sum_j \sigma_j \Big[(x - x_j^t)\xi_j^t + \int_0^t \frac{1}{2}|\xi_j^\tau|^2 - V(t, x_j^\tau)\, d\tau \Big].$$

To save space we abbreviate $U_j(t,0)$ as U_j.

Let $1 = \theta_0 + \sum_{\ell \geq 1} \theta_\ell$ be a partition of unity such that θ_0 is supported in the unit ball and θ_ℓ is supported in the annulus $\{2^{\ell-1} < |x| < 2^{\ell+1}\}$. Also choose $\chi \in C_0^\infty$ equal to 1 on $|x| \leq 8$. Further decompose $K_0 \leq \sum_{\vec{\ell}} K_0^{\vec{\ell}}$, where

$$K_0^{\vec{\ell}} = \Big| \int e^{i\Phi} \prod_j U_j \psi(x - x_j^t) \theta_{\ell_j}(x - x_j^t)\, \eta(t) dx dt \Big|$$

Fix $\vec{\ell}$, and write $\ell^* = \max \ell_j$. By Lemma 5.2.1, the integrand is nonzero only in the spacetime region

$$\{(t,x) : |t| \lesssim \min(1, \tfrac{2^{\ell^*}}{\max|\xi_j - \xi_k|}), |x - x_j^t| \lesssim 2^{\ell_j}\}, \tag{5.16}$$

and for all t subject to the above restriction we have

$$|x_j^t - x_k^t| \lesssim 2^{\ell^*},\ \ |\xi_j^t - \xi_j| \lesssim \min(2^{\ell^*}, \frac{2^{2\ell^*}}{\max|\xi_j - \xi_k|}). \tag{5.17}$$

We estimate $K_0^{\vec{\ell}}$ using integration by parts. The relevant derivatives of the phase function are

$$\partial_x \Phi = \sum_j \sigma_j \xi_j^t, \quad \partial_x^2 \Phi = 0,$$
$$-\partial_t \Phi = \sum_j \sigma_j h(t, z_j^t) + \sum_j \sigma_j (x - x_j^t) \partial_x V(t, x_j^t).$$

Integrating by parts repeatedly in x yields, for any $N \geq 0$,

$$\begin{aligned} |K_0^{\vec{\ell}}| &\lesssim_N \int |\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t|^{-N} \big| \partial_x^N \prod U_j \psi(x - x_j^t) \theta_{\ell_j}(x - x_j^t) \big|\, \eta(t) dx dt \\ &\lesssim_N \frac{2^{-\ell^* N} \langle \xi_1 + \xi_2 - \xi_3 - \xi_4 \rangle^{-N}}{1 + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|} \end{aligned} \tag{5.18}$$

where we have used (5.17) to replace $\xi_1^t + \xi_2^t - \xi_3^t - \xi_4^t$ with $\xi_1 + \xi_2 - \xi_3 - \xi_4 + O(2^{\ell^*})$.

One can also use other vector fields besides ∂_x. A naive choice might be ∂_t, but better decay can be obtained by accounting for the bulk motion of the wavepackets in addition to the phase. If one pretends that the envelope $U_j\psi(x-x_j^t)$ "$\approx$" $\phi(x-x_j^t)$ is simply transported along the classical trajectory, then

$$(\partial_t + \xi_j^t \partial_x) U_j \psi(x - x_j^j) \text{“} \approx \text{”} = (-\xi_j^t + \xi_j^t)\phi'(x - x_j^t) = 0.$$

In view of this heuristic, we introduce a vector field adapted to the average bicharacteristic for the four wavepackets. This will be most effective when the wavepackets all follow nearby bicharacteristics; when they are far apart in phase space, we can exploit the strong spatial localization and the fact that two wavepackets widely separated in momentum will interact only for a short time.

Define

$$\overline{x^t} = \frac{1}{4}\sum_j x_j^t, \quad \overline{\xi^t} = \frac{1}{4}\sum_j \xi_j^t,$$
$$x_j^t = \overline{x^t} + \overline{x^t}_j, \quad \xi_j^t = \overline{\xi^t} + \overline{\xi^t}_j.$$

The variables $(\overline{x^t}_j, \overline{\xi^t}_j)$ describe the bicharacteristic for the jth wavepacket relative to the average $(\overline{x^t}, \overline{\xi^t})$. We have

$$\begin{aligned}
\frac{d}{dt}\overline{x^t}_j &= \overline{\xi^t}_j = O(\max_{j,k} |\xi_j^t - \xi_k^t|) = O\Big(|\xi_j - \xi_k| + \min(2^{\ell^*}, \frac{2^{2\ell^*}}{\max|\xi_j - \xi_k|})\Big) \\
\frac{d}{dt}\overline{\xi^t}_j &= \frac{1}{4}\sum_k \partial_x V(t, x_k^t) - \partial_x V(t, x_j^t) \\
&= \frac{1}{4}\sum_k [(x_k^t - x_j^t)\int_0^1 \partial_x^2 V(t, (1-\theta)x_j^t + \theta x_k^t)\, d\theta] \\
&= O(2^{\ell^*}).
\end{aligned} \tag{5.19}$$

Note that

$$\max_j |\overline{x^t}_j| \sim \max_{j,k} |x_j^t - x_k^t|, \quad \max_j |\overline{\xi^t}_j| \sim \max_{j,k} |\xi_j^t - \xi_k^t|. \tag{5.20}$$

Consider the operator

$$D = \partial_t + \overline{\xi^t}\partial_x.$$

Then

$$\begin{aligned}
-D\Phi &= \sum \sigma_j h(t, z_j^t) + \sum \sigma_j [(x - x_j^t)\partial_x V(t, x_j^t) - \overline{\xi^t}\xi_j^t] \\
&= \frac{1}{2}\sum \sigma_j |\overline{\xi^t}_j|^2 + \sum \sigma_j (V(t, x_j^t) + (x - x_j^t)\partial_x V(t, x_j^t)].
\end{aligned}$$

This is more transparent when expressed in the relative variables $\overline{x}_j$ and $\overline{\xi}_j$. Each term in the second sum can be written as

$$
\begin{aligned}
&V(t, \overline{x^t} + \overline{x^t}_j) + (x - x_j^t)\partial_x V(t, \overline{x^t} + \overline{x^t}_j) \\
&= V(t, \overline{x^t} + \overline{x^t}_j) - V(t, \overline{x^t}) - \overline{x^t}_j \partial_x V(t, \overline{x^t}) \\
&+ V(t, \overline{x^t}) + \overline{x^t}_j \partial_x V(t, \overline{x^t}) + (x - x_j^t)\partial_x V(t, \overline{x^t}) \\
&+ (x - x_j^t)(\partial_x V(t, \overline{x^t} + \overline{x^t}_j) - \partial_x V(t, \overline{x^t})) \\
&= V^{\overline{z}}(t, \overline{x^t}_j) + V(t, \overline{x^t}) + (x - x_j^t)\partial_x V^{\overline{z}}(\overline{x^t}_j) + (x - \overline{x^t})\partial_x V(t, \overline{x^t}),
\end{aligned}
$$

where

$$
V^{\overline{z}}(t, x) = V(t, \overline{x^t} + x) - V(t, \overline{x^t}) - x\partial_x V(t, \overline{x^t}) = x^2 \int_0^1 (1 - s)\partial_x^2 V(t, \overline{x^t} + sx)\, ds. \quad (5.21)
$$

The terms without the subscript j cancel upon summing, and we obtain

$$
-D\Phi = \sum \sigma_j \frac{1}{2} |\overline{\xi^t}_j|^2 + \sum \sigma_j [V^{\vec{z}}(t, \overline{x^t}_j) + (x - x_j^t)\partial_x V^{\overline{z}}(t, \overline{x^t}_j)]. \quad (5.22)
$$

Thus, the contribution to $D\Phi$ from V depends essentially only on the relative displacements $x_j^t - x_k^t$; by (5.16), (5.17), and (5.20), the second sum is at most $O(2^{2\ell^*})$.

Note also that

$$
(\overline{\xi^t}_j)^2 = (\overline{\xi}_j)^2 + O(2^{2\ell^*}),
$$

as can be seen via (5.19), the fundamental theorem of calculus, and the time restriction (5.16). It follows that if

$$
\Big| \sum_j \sigma_j (\overline{\xi}_j)^2 \Big| \geq C \cdot 2^{2\ell^*} \quad (5.23)
$$

for some large constant $C > 0$, then on the support of the integrand

$$
\begin{aligned}
|D\Phi| &\gtrsim \Big| \sum_j \sigma_j (\overline{\xi_j})^2 \Big| = \frac{1}{2} \Big| (\overline{\xi}_1 + \overline{\xi}_2)^2 - (\overline{\xi}_3 + \overline{\xi}_4)^2 + (\overline{\xi}_1 - \overline{\xi}_2)^2 - (\overline{\xi}_3 - \overline{\xi}_4)^2 \Big| \\
&\gtrsim \big| |\xi_1 - \xi_2|^2 - |\xi_3 - \xi_4|^2 \big|^2,
\end{aligned} \quad (5.24)
$$

where the last inequality follows from the fact that $\overline{\xi}_1 + \overline{\xi}_2 + \overline{\xi}_3 + \overline{\xi}_4 = 0$.

The second derivative of the phase is

$$
\begin{aligned}
-D^2\Phi &= \sum \sigma_j \overline{\xi^t}_j \Big(\frac{1}{4}\sum_k \partial_x V(t,x_k^t) - \partial_x V(t,x_j^t)\Big) + \sum_j \sigma_j (x - x_j^t)\xi_j^t \partial_x^2 V(t,x_j^t) \\
&\quad + \overline{\xi^t} \sum \sigma_j \partial_x V(t,x_j^t) + \sum \sigma_j [\partial_t V(t,x_j^t) + (x-x_j^t)\partial_t\partial_x V(t,x_j^t)] \\
&= \sum \sigma_j \overline{\xi^t}_j \Big(\frac{1}{4}\sum_k \partial_x V(t,x_k^t) - \partial_x V(t,x_j^t)\Big) + \sum \sigma_j (x - x_j^t)\overline{\xi^t}_j \partial_x^2 V(t,x_j^t) \\
&\quad + \sum \sigma_j [\partial_t V(t,x_j^t) + (x-x_j^t)\partial_t\partial_x V(t,x_j^t)] + \overline{\xi^t} \sum \sigma_j [\partial_x V(t,x_j^t) + (x-x_j^t)\partial_x^2 V(t,x_j^t)].
\end{aligned}
$$

We rewrite the last two sums as before to obtain

$$
\begin{aligned}
-D^2\Phi &= \sum \sigma_j \overline{\xi^t}_j \Big(\frac{1}{4}\sum_k \partial_x V(t,x_k^t) - \partial_x V(t,x_j^t)\Big) + \sum \sigma_j (x - x_j^t)\overline{\xi^t}_j \partial_x^2 V(t,x_j^t) \\
&\quad + \sum \sigma_j [(\partial_t V)^{\overline{z}}(t,\overline{x^t}_j) + (x-x_j^t)\partial_x(\partial_t V)^{\overline{z}}(t,\overline{x^t}_j)] \\
&\quad + \overline{\xi^t} \sum \sigma_j [(\partial_x V)^{\overline{z}}(t,\overline{x^t}_j) + (x-x_j^t)\partial_x(\partial_x V)^{\overline{z}}(t,\overline{x^t}_j)],
\end{aligned} \tag{5.25}
$$

where

$$
\begin{aligned}
(\partial_t V)^{\overline{z}}(t,x) &= x^2 \int_0^1 (1-s)\partial_x^2 \partial_t V(t,\overline{x^t}+sx)\,ds \\
(\partial_x V)^{\overline{z}}(t,x) &= x^2 \int_0^1 (1-s)\partial_x^3 V(t,\overline{x^t}+sx)\,ds.
\end{aligned}
$$

Assume that (5.23) holds. Write $e^{i\Phi} = \frac{D\Phi}{i|D\Phi|^2}\cdot De^{i\Phi}$ and integrate by parts to get

$$
\begin{aligned}
K_0^{\vec{\ell}} &\le \Big|\int e^{i\Phi}\frac{D^2\Phi}{(D\Phi)^2}\prod U_j\psi(x-x_j^t)\theta_{\ell_j}(x-x_j^t)\,\eta(t)dxdt\Big| \\
&\quad + \Big|\int e^{i\Phi}\frac{1}{(D\Phi)}D\prod U_j\psi(x-x_j^t)\theta_{\ell_j}(x-x_j^t)\,\eta(t)dxdt\Big| \\
&\le \Big|\int e^{i\Phi}\frac{D^2\Phi}{(D\Phi)^2}\prod U_j\psi(x-x_j^t)\theta_{\ell_j}(x-x_j^t)\,\eta(t)dxdt\Big| \\
&\quad + \Big|\int e^{i\Phi}\frac{2D^2\Phi}{(D\Phi)^3}D\prod U_j\psi(x-x_j^t)\theta_{\ell_j}(x-x_j^t)\,\eta(t)dxdt\Big| \\
&\quad + \Big|\int e^{i\Phi}\frac{1}{(D\Phi)^2}D^2\prod U_j\psi(x-x_j^t)\theta_{\ell_j}(x-x_j^t)\,\eta(t)dxdt\Big| \\
&= I + II + III.
\end{aligned}
$$

Note that after the first integration by parts, we only repeat the procedure for the second term. The point of this is to avoid higher derivatives of Φ, which may be unacceptably large due to factors of $\overline{\xi^t}$.

Consider first the contribution from I. Write $I \le I_a + I_b + I_c$, where I_a, I_b, I_c correspond respectively to the first, second, and third lines in the expression (5.25) for $D^2\Phi$.

In view of (5.13), (5.17), (5.19), and (5.24), we have

$$\begin{aligned}
I_a &\lesssim_N \int \frac{2^{\ell^*}\sum_j |\overline{\xi^t}_j|}{|D\Phi|^2} \prod_j 2^{-\ell_j N}\chi\Big(\frac{x - x_j^t}{2^{\ell_j}}\Big)\,\eta(t)dxdt \\
&\lesssim \frac{2^{2\ell^*}(1 + \sum |\bar{\xi}_j|)}{\big|(\xi_1-\xi_2)^2 - (\xi_3-\xi_4)^2\big|^2} \cdot \int \prod_j 2^{-\ell_j N}\chi\Big(\frac{x - x_j^t}{2^{\ell_j}}\Big)\,\eta(t)dxdt \\
&\lesssim_N 2^{-\ell^* N} \cdot \frac{\langle \xi_1+\xi_2-\xi_3-\xi_4\rangle + |\xi_1-\xi_2| + |\xi_3-\xi_4|}{\big|(\xi_1-\xi_2)^2 - (\xi_3-\xi_4)^2\big|^2} \cdot \frac{1}{1 + |\xi_1-\xi_2| + |\xi_3-\xi_4|},
\end{aligned}$$

where we have observed that

$$\begin{aligned}
\sum_j |\bar{\xi}_j| &\sim \big(\sum_j |\bar{\xi}_j|^2\big)^{1/2} \sim \big(|\bar{\xi}_1+\bar{\xi}_2|^2 + |\bar{\xi}_1-\bar{\xi}_2|^2 + |\bar{\xi}_3+\bar{\xi}_4|^2 + |\bar{\xi}_3-\bar{\xi}_4|^2\big)^{1/2} \\
&\lesssim |\xi_1+\xi_2-\xi_3-\xi_4| + |\xi_1-\xi_2| + |\xi_3-\xi_4|.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_b &\lesssim \int \frac{2^{2\ell^*}}{|D\Phi|^2} \prod_j 2^{-\ell_j N}\chi\Big(\frac{x - x_j^t}{2^{\ell_j}}\Big)\,\eta(t)dxdt \\
&\lesssim_N \frac{2^{-\ell^* N}}{\big|(\xi_1-\xi_2)^2 - (\xi_3-\xi_4)^2\big|^2} \cdot \frac{1}{1 + |\xi_1-\xi_2| + |\xi_3-\xi_4|}.
\end{aligned}$$

To estimate I_c, use the decay hypothesis $|\partial_x^3 V| \lesssim \langle x\rangle^{-1-\varepsilon}$ to obtain

$$\begin{aligned}
I_c &\lesssim \int \frac{2^{2\ell^*}|\overline{\xi^t}|}{|\partial_t\Phi|^2}\Big(\int_0^1 \sum_j \langle \overline{x^t} + s\overline{x^t}_j\rangle^{-1-\varepsilon}\,ds\Big) \prod_j 2^{-\ell_j N}\chi\Big(\frac{x - x_j^t}{2^{\ell_j}}\Big)\,\eta(t)dxdt \\
&\lesssim \frac{2^{-\ell^* N}}{|(\xi_1-\xi_2)^2 - (\xi_3-\xi_4)^2|^2} \int_0^1 \sum_j \int_{|t|\le\delta_0} |\overline{\xi^t}|\langle \overline{x^t} + s\overline{x^t}_j\rangle^{-1-\varepsilon}\,dtds.
\end{aligned}$$

The integral on the right is estimated in the following technical lemma.

Lemma 5.4.7.

$$\int_0^1 \sum_j \int_{|t|\le\delta_0} |\overline{\xi^t}|\langle \overline{x^t} + s\overline{x^t}_j\rangle^{-1-\varepsilon}\,dtds = O(2^{(2+\varepsilon)\ell^*}).$$

Proof. It will be convenient to replace the average bicharacteristic $(\overline{x^t}, \overline{\xi^t})$ with the ray $(\overline{x}^t, \overline{\xi}^t)$ starting from the average initial data. We claim that

$$|\overline{x^t} - \overline{x}^t| + |\overline{\xi^t} - \overline{\xi}^t| = O(2^{\ell^*})$$

during the relevant t, for Hamilton's equations imply that

$$\begin{aligned}
\overline{x^t} - \overline{x}^t &= -\int_0^t (t-\tau)\Big(\frac{1}{4}\sum_k \partial_x V(\tau, x_k^\tau) - \partial_x V(\tau, \overline{x}^\tau)\Big)\, d\tau \\
&= -\int_0^t (t-\tau)\Big(\frac{1}{4}\sum_k (\overline{x^\tau}_k + \overline{x^\tau} - \overline{x}^\tau)\int_0^1 \partial_x^2 V(\tau, \overline{x}^\tau + s(x_k^\tau - \overline{x}^\tau))\, ds\Big)\, d\tau \\
&= -\int_0^t (t-\tau)(\overline{x^\tau} - \overline{x}^\tau)\Big(\int_0^1 \frac{1}{4}\sum_k \partial_x^2 V(\tau, \overline{x}^\tau + s(x_k^\tau - \overline{x}^\tau))\, ds\Big) + O(2^{\ell^*} t^2),
\end{aligned}$$

and we can invoke Gronwall. Similar considerations yield the bound for $|\overline{\xi^t} - \overline{\xi}^t|$. As also $\overline{x^t}_j = O(2^{\ell^*})$, we are reduced to showing

$$\int_{|t|\le \delta_0} |\overline{\xi}^t| \langle \overline{x}^t \rangle^{-1-\varepsilon}\, dt = O(1). \tag{5.26}$$

Integrating the ODE

$$\tfrac{d}{dt}\overline{x}^t = \overline{\xi}^t, \quad \tfrac{d}{dt}\overline{\xi}^t = -\partial_x V(t, \overline{x}^t),$$

yields the estimates

$$\begin{aligned}
|\overline{x}^t - \overline{x}^s - (t-s)\overline{\xi}^s| &\le C|t-s|^2(1 + |\overline{x}^s| + |(t-s)\overline{\xi}^s|) \\
|\overline{\xi}^t - \xi^s| &\le C|t-s|(1 + |\overline{x}^s| + |(t-s)\overline{\xi}^s|)
\end{aligned}$$

for some constant C depending on $\sup_t |\partial_x V(t,0)|$. By subdividing the time interval $[-\delta_0, \delta_0]$ if necessary, we may assume in (5.26) that $(1+C)|t| \le 1/10$.

Consider separately the cases $|\overline{x}| \le |\overline{\xi}|$ and $|\overline{x}| \ge |\overline{\xi}|$. When $|\overline{x}| \le |\overline{\xi}|$,

$$2|\overline{\xi}| \ge |\overline{\xi}^t| \ge |\overline{\xi}| - \tfrac{1}{10}(1 + 2|\overline{\xi}|) \ge \tfrac{1}{2}|\overline{\xi}|$$

(assuming as we may that $|\overline{\xi}| \ge 1$), the bound (5.26) follows from the change of variables $y = x^t$. If $|\overline{x}| \ge |\overline{\xi}|$, then $|\overline{x}^t| \ge \frac{1}{2}|\overline{x}|$, $|\overline{\xi}^t| \le 2|\overline{x}|$, which also yields the desired bound. □

Returning to I_c, we conclude that

$$I_c \lesssim_N \frac{2^{-\ell^* N}}{|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2|^2}.$$

Overall

$$I \le I_a + I_b + I_c \lesssim_N 2^{-\ell^* N} \frac{\langle \xi_1 + \xi_2 - \xi_3 - \xi_4 \rangle}{|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2|^2}.$$

For II, we have

$$D[U_j \psi(x - x_j^t)] = -iH_j U_j \psi(x - x_j^t) - \overline{\xi^t}_j \partial_x U_j \psi(x - x_j^t) \tag{5.27}$$

and estimating as in I,

$$\begin{aligned} II &\lesssim_N \frac{1 + \sum_j |\bar{\xi}_j|}{|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2|} \int \frac{|D^2 \Phi|}{|D\Phi|^2} \prod 2^{-\ell_j N} \chi\Big(\frac{x - x_j^t}{2^{\ell_j}}\Big) \eta dx dt \\ &\lesssim_N 2^{-\ell^* N} \Big(\frac{\langle \xi_1 + \xi_2 - \xi_3 - \xi_4 \rangle + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|}{|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2|} \Big) \frac{\langle \xi_1 + \xi_2 - \xi_3 - \xi_4 \rangle}{|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2|^2}, \end{aligned}$$

It remains to consider III. The derivatives can distribute in various ways:

$$\begin{aligned} III \lesssim \frac{1}{|D\Phi|^2} \Big(& \int |D^2[U_1 \psi(x - x_1^t)] \prod_{j=2}^4 U_j \psi(x - x_j^t) \prod_{k=1}^4 \theta_{\ell_k}(x - x_k^t) \eta| \, dx dt \\ & + \int \Big| D[U_1 \psi(x - x_1^t)] D[U_2 \psi(x - x_2^t)] \prod_{j=3}^4 U_j \psi(x - x_j^t) \prod_{k=1}^4 \theta_{\ell_k}(x - x_k^t) \eta \Big| \, dx dt \\ & + \int \Big| D \prod_j U_j \psi(x - x_j^t) D \prod_k \theta_{\ell_k}(x - x_k^t) \eta \Big| \, dx dt \\ & + \int \Big| \prod_j U_j \psi(x - x_j^t) D^2 \prod_k \theta_{\ell_k}(x - x_k^t) \eta \Big| \, dx dt \Big), \end{aligned} \tag{5.28}$$

where the first two terms represent sums over the appropriate permutations of indices.

We focus on the terms involving double derivatives of U_j as the other terms can be dealt with as in the estimate for II. From (5.27),

$$\begin{aligned} D^2[U_j \psi(x - x_j^t)] = & -i\partial_t V_j(t, x - x_j^t) U_j \psi(x - x_j^t) + (H_j)^2 U_j \psi(x - x_j^t) \\ & + 2i\overline{\xi^t}_j \partial_x H_j U_j \psi(x - x_j^t) + \Big(\frac{1}{4} \sum_k \partial_x V(t, x_k^t) - \partial_x V(t, x_j^t)\Big) \partial_x U_j \psi(x - x_j^t) \\ & + (\overline{\xi^t}_j)^2 \partial_x^2 U_j \psi(x - x_j^t). \end{aligned} \tag{5.29}$$

Recalling from (5.12) that

$$\partial_t V_j(t,x) = x^2\Big[\xi_j^t \int_0^1 (1-s)\partial_x^3 V(x_j^t+sx)ds + \int_0^1 (1-s)\partial_t\partial_x^2 V(t,x_j^t+sx)\,ds\Big],$$

it follows that

$$\begin{aligned}
&\int\Big|\partial_t V_1(t,x-x_1^t)U_1\psi(x-x_1^t)\prod_{j=2}^4 U_j\psi(x-x_j^t)\prod_{k=1}^4\theta_{\ell_k}(x-x_k^t)\Big|\eta(t)dxdt\\
&\lesssim 2^{2\ell_1}\int\Big[\int_0^1 |\xi_1^t\partial^3 V_x(t,x_1^t+s(x-x_1^t))|\,ds\\
&+\int_0^1 |\partial_t\partial_x^2 V(t,x_j^t+s(x-x_j^t))\,ds\Big]\prod_j 2^{-\ell_j N}\chi\Big(\frac{x-x_j^t}{2^{\ell_j}}\Big)\,\eta dxdt\\
&\lesssim_n 2^{-\ell^* N},
\end{aligned}$$

where the terms involving $\partial_x^3 V$ are handled as in I_c above. Also, from (5.13) and (5.17),

$$\begin{aligned}
&\int\Big|(\overline{\xi}_1^t)^2\partial_x^2 U_1\psi(x-x_1^t)\prod_{j=2}^4 U_j\psi(x-x_j^t)\prod_{k=1}^4\theta_{\ell_k}(x-x_k^t)\Big|\eta(t)dxdt\\
&\lesssim_N \frac{2^{-\ell^* N}(1+|\overline{\xi}_1|^2)}{1+|\xi_1-\xi_2|+|\xi_3-\xi_4|}.
\end{aligned}$$

The intermediate terms in (5.29) and the other terms in the the expansion (5.28) yield similar upper bounds. We conclude overall that

$$\begin{aligned}
&III\\
&\lesssim_N 2^{-\ell^* N}\Big(\frac{1}{|(\xi_1-\xi_2)^2-(\xi_3-\xi_4)^2|^2}+\frac{(1+\sum_j|\overline{\xi}_j|)^2}{|(\xi_1-\xi_2)^2-(\xi_3-\xi_4)^2|^2\cdot(1+|\xi_1-\xi_2|+|\xi_3-\xi_4|)}\Big)\\
&\lesssim 2^{-\ell^* N}\Big(\frac{1}{|(\xi_1-\xi_2)^2-(\xi_3-\xi_4)^2|^2}+\frac{\langle\xi_1+\xi_2-\xi_3-\xi_4\rangle^2+(|\xi_1-\xi_2|+|\xi_3-\xi_4|)^2}{\big||\xi_1-\xi_2|^2-|\xi_3-\xi_4|^2\big|^2\cdot(1+|\xi_1-\xi_2|+|\xi_3-\xi_4|)}\Big)\\
&\lesssim 2^{-\ell^* N}\frac{\langle\xi_1+\xi_2-\xi_3-\xi_4\rangle^2+|\xi_1-\xi_2|+|\xi_3-\xi_4|}{|(\xi_1-\xi_2)^2-(\xi_3-\xi_4)^2|^2}.
\end{aligned}$$

Note also that in each of the integrals I, II, and III we may integrate by parts in x to obtain arbitrarily many factors of $|\xi_1+\xi_2-\xi_3-\xi_4|^{-1}$. All instances of $\langle\xi_1+\xi_2-\xi_3-\xi_4\rangle$ in the above estimates may therefore be replaced by 1.

Combining I, II, and III, we obtain under the hypothesis (5.23)

$$|K_0^{\vec{\ell}}| \lesssim_N 2^{-\ell^* N}\frac{1+|\xi_1-\xi_2|+|\xi_3-\xi_4|}{\big|(\xi_1-\xi_2)^2-(\xi_3-\xi_4)^2\big|^2}.$$

In general,

$$|K_0^{\vec{\ell}}| \lesssim_N 2^{-\ell^* N} \min\Big(1, \frac{1 + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|}{\big|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2\big|^2}\Big), \tag{5.30}$$

Combining this with (5.18),

$$|K_0^{\vec{\ell}}| \lesssim_{N_1, N_2} 2^{-\ell^* N_1} \min\Big(\langle \xi_1 + \xi_2 - \xi_3 - \xi_4 \rangle^{-N_2}, \frac{1 + |\xi_1 - \xi_2| + |\xi_3 - \xi_4|}{\big|(\xi_1 - \xi_2)^2 - (\xi_3 - \xi_4)^2\big|^2}\Big) \tag{5.31}$$

for any $N_1, N_2 > 0$. Lemma 5.4.5 now follows from summing in $\vec{\ell}$, at least when $t(\vec{z}) = 0$.

For general $t(\vec{z})$, use Lemma 5.2.5 to write

$$\begin{aligned} U(t, 0)\psi_{z_j} &= U(t, t(\vec{z}))U(t(\vec{z}), 0)(\psi_{z_j}) = U(t, t(\vec{z}))e^{i\alpha(t(\vec{z}), 0, z_j)}\pi(z_j^{t(\vec{z})})U^{z_j}(t(\vec{z}), 0)\psi \\ &= e^{i\alpha(t, 0, z_j)}\pi(z_j^t)U^{z_j^{t(\vec{z})}}(t, t(\vec{z}))\psi_{t(\vec{z}), j}, \end{aligned}$$

where

$$\psi_{t(\vec{z}), j} = U^{z_j}(t(\vec{z}), 0)\psi$$

is bounded in $\mathcal{S}(\mathbf{R})$ uniformly in z_j and $t(\vec{z})$, and where we have used the additivity of the action

$$\alpha(t, s, z^s) + \alpha(s, 0, z) = \alpha(t, 0, z).$$

The argument then proceeds analogously as before.

5.5 An L^2 Linear Profile Decomposition

In this discussion we assume for simplicity that $V = V(x)$ is time-independent and satisfies hypotheses (5.5) and (5.6). The propagator is then a one-parameter unitary group e^{-itH}. Let δ_0 be the constant from Theorem 5.2.3, so that the dispersive estimate

$$\|e^{-itH}\|_{L^1(\mathbf{R}) \to L^\infty(\mathbf{R})} \lesssim |t|^{-1/2}$$

holds for all $|t| \le \delta_0$. All spacetime norms in this section will be taken over the time interval $[-\delta_0, \delta_0]$.

Given a bounded sequence $\{f_n\}_n \subset L^2$, we can apply Corollary 5.1.2 inductively to obtain a full profile decomposition. But first we introduce some systematic (and standard) notation and terminology.

- A *frame* is a sequence $\{(\lambda_n, t_n, z_n)\} \subset (0, 1] \times [-\delta_0, \delta_0] \times T^*\mathbf{R}$.
- Two frames $(\lambda_n^j, t_n^j, z_n^j)$ and $(\lambda_n^k, t_n^k, z_n^k)$ are *orthogonal* if

  $$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + [(\lambda_n^j)^{-2} + (\lambda_n^k)^{-2}]|t_n^j - t_n^k| + |s_{\lambda_n^j}((z_n^j)^{t_n^k - t_n^j} - z_n^k)| + |s_{\lambda_n^k}((z_n^k)^{t_n^j - t_n^k} - z_n^j)| \to \infty$$

  Here $s_\lambda(x, \xi) = (\lambda^{-1}x, \lambda\xi)$ is a (volume-preserving) rescaling of phase space, and $t \mapsto z^t$ is the bicharacteristic starting at z.
- Two frames $(\lambda_n^j, t_n^j, z_n^j)$ and $(\lambda_n^k, t_n^k, z_n^k)$ are *equivalent* if the following limits exist as $n \to \infty$:

  $$\frac{\lambda_n^j}{\lambda_n^k} \to \lambda_\infty \in (0, \infty), \qquad (\lambda_n^j)^{-2}(t_n^j - t_n^k) \to t_\infty$$
  $$s_{\lambda_n^j}((z_n^j)^{t_n^k - t_n^j} - z_n^k) \to z_\infty, \quad s_{\lambda_n^k}((z_n^k)^{t_n^j - t_n^k} - z_n^j) \to z'_\infty.$$

  If two frames are not orthogonal, then they are equivalent after passing to a subsequence.

Lemma 5.5.1. *Let* $\Gamma^j = (\lambda_n^j, t_n^j, z_n^j)$ *and* $\Gamma^k = (\lambda_n^k, t_n^k, z_n^k)$ *be two frames, and denote by*

$$g_n^j = \pi(z_n^j) S_{\lambda_n^j}, \quad g_n^k = \pi(z_n^k) S_{\lambda_n^k}.$$

the associated symmetry operators on L^2*.*

(a) If Γ^j *and* Γ^k *are equivalent, then after passing to a subsequence*

$$(e^{it_n^k H} g_n^k)^{-1} e^{it_n^j H} g_n^j$$

converges strongly in L^2*.*

(b) *If Γ^j and Γ^k are orthogonal, then*

$$\langle (e^{it_n^k H} g_n^k)^{-1} e^{it_n^j H} g_n^j \phi, \psi \rangle \to 0$$

for all ϕ, ϕ in L^2.

Proof. Write $t_n = t_n^k - t_n^j$. Then by Lemma 5.2.5 and the identity $\pi(z)S_\lambda = S_\lambda \pi(s_\lambda(z))$,

$$\begin{aligned}(g_n^k)^{-1} e^{-it_n H} g_n^j &= S_{\lambda_n^k}^{-1} \pi(-z_n^k) e^{-it_n H} \pi(z_n^j) S_{\lambda_n^j} \\ &= e^{i\alpha(t_n, 0, z_n^j)} S_{(\lambda_n^k)^{-1}\lambda_n^j} \pi(s_{\lambda_n^j}((z_n^j)^{t_n} - z_n^k)) U_{\lambda_n^j}^{z_n^j}((\lambda_n^j)^{-2} t_n, 0) \\ &= e^{i\alpha_n} S_n \pi_n U_n,\end{aligned}$$

where $U_{\lambda_n^j}^{z_n}(t,s)$ is the propagator for the Hamiltonian with potential $(\lambda_n^j)^2 V^{z_n^j}((\lambda_n^j)^2 t, \lambda_n^j x)$.

(a) Equivalence of the frames implies that S_n, π_n, and U_n all converge strongly, while the phase $e^{i\alpha_n}$ is bounded.

(b) By continuity, it suffices to prove the claim with ϕ and ψ Schwartz. Suppose first that both $(\lambda_n^j)^{-2} t_n$ and $(\lambda_n^k)^{-2} t_n$ diverge to infinity. Assuming without loss that $\lambda_n^j \geq \lambda_n^k$, the dispersive estimate yields

$$|\langle e^{-it_n H} g_n^j \phi, g_n^k \psi \rangle| \lesssim |t_n|^{-1/2} (\lambda_n^k)^{1/2} (\lambda_n^j)^{1/2} \|\phi\|_1 \|\psi\|_1 \leq \lambda_n^j |t_n|^{-1/2} \|\phi\|_1 \|\psi\|_1 \to 0.$$

Suppose now that $(\lambda_n^j)^{-2} t_n$ stays bounded; if $\sup_n |(\lambda_n^k)^{-2} t_n| < \infty$, the same following considerations apply after taking adjoints. By Lemma 5.2.5, the operators U_n are uniformly continuous on $\mathcal{S}(\mathbf{R})$, so for fixed Schwartz ϕ the sequence $\{U_n \phi\}_n$ is precompact in L^2. If $(\lambda_n^k)^{-1} \lambda_n^j \to \{0, \infty\}$, then $\pi_n^{-1} S_n^{-1} \phi$ converges weakly to zero as it becomes increasingly concentrated or dispersed. If on the other hand $(\lambda_n^k)^{-1} \lambda_n^j \to \lambda_\infty \in (0, \infty)$, then also $(\lambda_n^k)^{-2} t_n \to t'_\infty$, and inequivalence implies that, after interchanging j and k if necessary,

$$|s_{\lambda_n^j}((z_n^j)^{t_n^k - t_n^j} - z_n^k)| \to \infty,$$

Hence $\pi_n \to 0$ weakly on L^2, and

$$\langle (g_n^k)^{-1} e^{it_n H} g_n^j \phi, \psi \rangle = \langle e^{i\alpha_n} \pi_n U_n \phi, S_n^{-1} \psi \rangle \to 0$$

since S_n^{-1} converges strongly and $\{U_n\phi\}_n$ is precompact.

□

Proposition 5.5.2. *Suppose $\{f_n\} \subset L^2(\mathbf{R})$ is bounded. Then, after passing to a subsequence, there exist $J^* \in \{0, 1, \ldots, \} \cup \{\infty\}$, functions $\phi^j \in L^2(\mathbf{R})$, mutually orthogonal frames $\Gamma^j = \{(\lambda_n^j, t_n^j, z_n^j)\}$, and for every finite $J \leq J^*$ a sequence r_n^J, which obey the following properties:*

For each finite $J \leq J^*$,

$$f_n = \sum_{j=1}^{J} e^{it_n^j H} \pi(z_n^j) S_{\lambda_n^j} \phi^j + r_n^J = \sum_{j=1}^{J} e^{it_n^j H} g_n^j \phi^j + r_n^J.$$

$$\lim_{n\to\infty} \|f_n\|_2^2 - \sum_{j=1}^{J} \|g_n^j \phi^j\|_2^2 - \|r_n^J\|_2^2 = 0 \tag{5.32}$$

$$(g_n^J)^{-1} e^{-it_n^J H} r_n^J \rightharpoonup 0 \text{ as } n \to \infty. \tag{5.33}$$

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{-itH} r_n^J\|_{L^6_{t,x}} = 0. \tag{5.34}$$

Proof. Let $r_n^0 = f_n$, and define inductively

$$\varepsilon_J = \limsup_{n\to\infty} \|e^{-itH} r_n^J\|_{L^6_{t,x}}, \quad A_J = \limsup_{n\to\infty} \|r_n^J\|_{L^2},$$

where the lim sup for the A_J is evaluated along a subsequence that realizes the lim sup for ε_J. After passing to a subsequence in n, the limsups may be replaced by genuine limits. If $\varepsilon_J > 0$, apply Corollary 5.1.2 to obtain a frame $\Gamma^{J+1} = \{(\lambda_n^{J+1}, t_n^{J+1}, z_n^{J+1})\}_n$ and a profile

$$\phi^{J+1} = \lim_n g_n^{J+1} r_n^J.$$

where the limit is taken in the L^2 sense. Set

$$r_n^{J+1} = r_n^J - e^{it_n^{J+1} H} g_n^{J+1} \phi^{J+1}.$$

Continue either until $\limsup_{n\to\infty} \|e^{-itH} r_n^J\|_{L^6_{t,x}} = 0$ (in which case set $J^* = J$) or forever ($J^* = \infty$). The decoupling (5.32) of L^2 norms follows from applying the corresponding assertion (5.9) in Corollary 5.1.2 at each step of the construction.

To see (5.34) in the case that $J^* = \infty$, note that by L^2 decoupling and the lower bound (5.8) for the L^2 norm of each profile,

$$A_{J+1}^2 \leq A_J^2 - C\varepsilon_J^\alpha A_J^{-2\beta} = A_J^2(1 - C\varepsilon_J^{2\alpha} A_J^{-2\beta-2}),$$

which, together with the Strichartz estimate $\varepsilon_J \lesssim A_J$ and the boundedness of f_n in L^2, implies that $\lim_{J\to\infty} \varepsilon_J = 0$.

To prove the mutual inequivalence of frames, suppose on the contrary that two frames are equivalent (after possibly passing to a subsequence). Choose k minimal so that Γ^j and Γ^k are equivalent for some $j < k$. By definition,

$$r_n^{j-1} = e^{it_n^j H} g_n^j \phi^j + e^{it_n^k H} g_n^k \phi^k + \sum_{j<\ell<k} e^{it_n^\ell H} g_n^\ell \phi^\ell + r_n^k,$$

so

$$(g_n^k)^{-1} e^{i(t_n^j - t_n^k)H} g_n^j [(g_n^j)^{-1} e^{-it_n^j H} r_n^{j-1} - \phi^j] = \phi^k + \sum_{j<\ell<k} (g_n^k)^{-1} e^{i(t_n^\ell - t_n^k)H} g_n^\ell \phi^\ell + (g_n^k)^{-1} e^{-it_n^k H} r_n^k$$

Taking $n \to \infty$, recalling the definition of ϕ^j, and invoking the previous Lemma, we deduce that $\phi^k = 0$. But each ϕ^k is nonzero by construction. □

References

[AF78] Kenji Asada and Daisuke Fujiwara. "On some oscillatory integral transformations in $L^2(R^n)$." *Japan. J. Math. (N.S.)*, **4**(2):299–361, 1978.

[Aro67] D. G. Aronson. "Bounds for the fundamental solution of a parabolic equation." *Bull. Amer. Math. Soc.*, **73**:890–896, 1967.

[Aub76] Thierry Aubin. "Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire." *J. Math. Pures Appl. (9)*, **55**(3):269–296, 1976.

[BGT04] N. Burq, P. Gérard, and N. Tzvetkov. "Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds." *Amer. J. Math.*, **126**(3):569–605, 2004.

[BL83] Haïm Brézis and Elliott Lieb. "A relation between pointwise convergence of functions and convergence of functionals." *Proc. Amer. Math. Soc.*, **88**(3):486–490, 1983.

[Bou99] J. Bourgain. "Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case." *J. Amer. Math. Soc.*, **12**(1):145–171, 1999.

[BT08] Jean-Marc Bouclet and Nikolay Tzvetkov. "On global Strichartz estimates for non-trapping metrics." *J. Funct. Anal.*, **254**(6):1661–1682, 2008.

[BV07] Pascal Bégout and Ana Vargas. "Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equation." *Trans. Amer. Math. Soc.*, **359**(11):5257–5282, 2007.

[Car02] R. Carles. "Remarks on nonlinear Schrödinger equations with harmonic potential." *Ann. Henri Poincaré*, **3**(4):757–772, 2002.

[Car03] Rémi Carles. "Nonlinear Schrödinger equations with repulsive harmonic potential and applications." *SIAM J. Math. Anal.*, **35**(4):823–843 (electronic), 2003.

[Car05] Rémi Carles. "Global existence results for nonlinear Schrödinger equations with quadratic potentials." *Discrete Contin. Dyn. Syst.*, **13**(2):385–398, 2005.

[Car11] Rémi Carles. "Nonlinear Schrödinger equation with time-dependent potential." *Commun. Math Sci.*, **9**(4):937–964, 2011.

[Caz03] Thierry Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics.* New York University Courant Institute of Mathematical Sciences, New York, 2003.

[CCH06] Gilles Carron, Thierry Coulhon, and Andrew Hassell. "Riesz transform and L^p-cohomology for manifolds with Euclidean ends." *Duke Math. J.*, **133**(1):59–93, 2006.

[CD03] Thierry Coulhon and Xuan Thinh Duong. "Riesz transform and related inequalities on noncompact Riemannian manifolds." *Comm. Pure Appl. Math.*, **56**(12):1728–1751, 2003.

[CK07] Rémi Carles and Sahbi Keraani. "On the role of quadratic oscillations in nonlinear Schrödinger equations. II. The L^2-critical case." *Trans. Amer. Math. Soc.*, **359**(1):33–62 (electronic), 2007.

[CKS08] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. "Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$." *Ann. of Math. (2)*, **167**(3):767–865, 2008.

[CV71] Alberto-P. Calderón and Rémi Vaillancourt. "On the boundedness of pseudo-differential operators." *J. Math. Soc. Japan*, **23**:374–378, 1971.

[CW90] Thierry Cazenave and Fred B. Weissler. "The Cauchy problem for the critical nonlinear Schrödinger equation in H^s." *Nonlinear Anal.*, **14**(10):807–836, 1990.

[Doda] Benjamin Dodson. "Global well-posedness and scattering for the defocusing, L^2-critical nonlinear Schrödinger equation when $d = 1$." Preprint.

[Dodb] Benjamin Dodson. "Global well-posedness and scattering for the defocusing, L^2-critical nonlinear Schrödinger equation when $d = 2$." Preprint.

[Dod12] Benjamin Dodson. "Global well-posedness and scattering for the defocusing, L^2-critical nonlinear Schrödinger equation when $d \geq 3$." *J. Amer. Math. Soc.*, **25**(2):429–463, 2012.

[Doi96] Shin-ichi Doi. "Smoothing effects of Schrödinger evolution groups on Riemannian manifolds." *Duke Math. J.*, **82**(3):679–706, 1996.

[Fol89] Gerald B. Folland. *Harmonic analysis in phase space*, volume 122 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[Fuj75] Daisuke Fujiwara. "On the boundedness of integral transformations with highly oscillatory kernels." *Proc. Japan Acad.*, **51**:96–99, 1975.

[Fuj79] Daisuke Fujiwara. "A construction of the fundamental solution for the Schrödinger equation." *J. Analyse Math.*, **35**:41–96, 1979.

[Fuj80] Daisuke Fujiwara. "Remarks on convergence of the Feynman path integrals." *Duke Math. J.*, **47**(3):559–600, 1980.

[GMO97] Patrick Gerard, Yves Meyer, and Frédérique Oru. "Inégalités de Sobolev précisées." In *Séminaire sur les Équations aux Dérivées Partielles, 1996–1997*, pp. Exp. No. IV, 11. École Polytech., Palaiseau, 1997.

[Gri09] Alexander Grigor'yan. *Heat kernel and analysis on manifolds*, volume 47 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.

[Heb90] Waldemar Hebisch. "A multiplier theorem for Schrödinger operators." *Colloq. Math.*, **60/61**(2):659–664, 1990.

[Hed72] Lars Inge Hedberg. "On certain convolution inequalities." *Proc. Amer. Math. Soc.*, **36**:505–510, 1972.

[HTW06] Andrew Hassell, Terence Tao, and Jared Wunsch. "Sharp Strichartz estimates on nontrapping asymptotically conic manifolds." *Amer. J. Math.*, **128**(4):963–1024, 2006.

[IP12] Alexandru D. Ionescu and Benoit Pausader. "The energy-critical defocusing NLS on $\mathbb{T}^3$." *Duke Math. J.*, **161**(8):1581–1612, 2012.

[IPS12] Alexandru D. Ionescu, Benoit Pausader, and Gigliola Staffilani. "On the global well-posedness of energy-critical Schrödinger equations in curved spaces." *Anal. PDE*, **5**(4):705–746, 2012.

[Jao16] Casey Jao. "The energy-critical quantum harmonic oscillator." *Comm. Partial Differential Equations*, (1):79–133, 2016.

[Ker01] Sahbi Keraani. "On the defect of compactness for the Strichartz estimates of the Schrödinger equations." *J. Differential Equations*, **175**(2):353–392, 2001.

[Ker06] Sahbi Keraani. "On the blow up phenomenon of the critical nonlinear Schrödinger equation." *J. Funct. Anal.*, **235**(1):171–192, 2006.

[KKS12] Rowan Killip, Soonsik Kwon, Shuanglin Shao, and Monica Visan. "On the mass-critical generalized KdV equation." *Discrete Contin. Dyn. Syst.*, **32**(1):191–221, 2012.

[KM06] Carlos E. Kenig and Frank Merle. "Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case." *Invent. Math.*, **166**(3):645–675, 2006.

[KSV12] Rowan Killip, Betsy Stovall, and Monica Visan. "Scattering for the cubic Klein-Gordon equation in two space dimensions." *Trans. Amer. Math. Soc.*, **364**(3):1571–1631, 2012.

[KT98] Markus Keel and Terence Tao. "Endpoint Strichartz estimates." *Amer. J. Math.*, **120**(5):955–980, 1998.

[KT05] Herbert Koch and Daniel Tataru. "Dispersive estimates for principally normal pseudodifferential operators." *Comm. Pure Appl. Math.*, **58**(2):217–284, 2005.

[KV10] Rowan Killip and Monica Visan. "The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher." *Amer. J. Math.*, **132**(2):361–424, 2010.

[KV13] Rowan Killip and Monica Vişan. "Nonlinear Schrödinger equations at critical regularity." In *Evolution equations*, volume 17 of *Clay Math. Proc.*, pp. 325–437. Amer. Math. Soc., Providence, RI, 2013.

[KVZa] Rowan Killip, Monica Visan, and Xiaoyi Zhang. "Harmonic analysis outside a convex obstacle." *Preprint*.

[KVZb] Rowan Killip, Monica Visan, and Xiaoyi Zhang. "Quintic NLS in the exterior of a strictly convex obstacle." *To appear in Amer. J. Math.*

[KVZ09] Rowan Killip, Monica Visan, and Xiaoyi Zhang. "Energy-critical NLS with quadratic potentials." *Comm. Partial Differential Equations*, **34**(10-12):1531–1565, 2009.

[MMT08] Jeremy Marzuola, Jason Metcalfe, and Daniel Tataru. "Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations." *J. Funct. Anal.*, **255**(6):1497–1553, 2008.

[MV98] F. Merle and L. Vega. "Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D." *Internat. Math. Res. Notices*, (8):399–425, 1998.

[MVV99] A. Moyua, A. Vargas, and L. Vega. "Restriction theorems and maximal operators related to oscillatory integrals in $\mathbf{R}^3$." *Duke Math. J.*, **96**(3):547–574, 1999.

[Oh89] Yong-Geun Oh. "Cauchy problem and Ehrenfest's law of nonlinear Schrödinger equations with potentials." *J. Differential Equations*, **81**(2):255–274, 1989.

[PTW14] Benoit Pausader, Nikolay Tzvetkov, and Xuecheng Wang. "Global regularity for the energy-critical NLS on $\mathbb{S}^3$." *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **31**(2):315–338, 2014.

[RT07] I. Rodnianski and T. Tao. "Longtime decay estimates for the Schrödinger equation on manifolds." In *Mathematical aspects of nonlinear dispersive equations*, volume 163 of *Ann. of Math. Stud.*, pp. 223–253. Princeton Univ. Press, Princeton, NJ, 2007.

[RV07] E. Ryckman and M. Visan. "Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$." *Amer. J. Math.*, **129**(1):1–60, 2007.

[Sch66] Laurent Schwartz. *Théorie des distributions*. Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X. Nouvelle édition, entiérement corrigée, refondue et augmentée. Hermann, Paris, 1966.

[ST02] Gigliola Staffilani and Daniel Tataru. "Strichartz estimates for a Schrödinger operator with nonsmooth coefficients." *Comm. Partial Differential Equations*, **27**(7-8):1337–1372, 2002.

[Ste93] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[Tal76] Giorgio Talenti. "Best constant in Sobolev inequality." *Ann. Mat. Pura Appl. (4)*, **110**:353–372, 1976.

[Tat08] Daniel Tataru. "Parametrices and dispersive estimates for Schrödinger operators with variable coefficients." *Amer. J. Math.*, **130**(3):571–634, 2008.

[Tay00] Michael E. Taylor. *Tools for PDE*, volume 81 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000. Pseudodifferential operators, paradifferential operators, and layer potentials.

[TOS02] Xuan Thinh Duong, El Maati Ouhabaz, and Adam Sikora. "Plancherel-type estimates and sharp spectral multipliers." *J. Funct. Anal.*, **196**(2):443–485, 2002.

[TV05] Terence Tao and Monica Visan. "Stability of energy-critical nonlinear Schrödinger equations in high dimensions." *Electron. J. Differential Equations*, pp. No. 118, 28, 2005.

[TVZ07] Terence Tao, Monica Visan, and Xiaoyi Zhang. "The nonlinear Schrödinger equation with combined power-type nonlinearities." *Comm. Partial Differential Equations*, **32**(7-9):1281–1343, 2007.

[TVZ08] Terence Tao, Monica Visan, and Xiaoyi Zhang. "Minimal-mass blowup solutions of the mass-critical NLS." *Forum Math.*, **20**(5):881–919, 2008.

[Vis07] Monica Visan. "The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions." *Duke Math. J.*, **138**(2):281–374, 2007.

[Vis14] Monica Visan. "Dispersive Equations." In *Dispersive Equations and Nonlinear Waves*, volume 45 of *Oberwolfach Seminars*. Springer Basel, 2014.

[Zha00] Jian Zhang. "Stability of attractive Bose-Einstein condensates." *J. Statist. Phys.*, **101**(3-4):731–746, 2000.