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ABSTRACT

An investigation is made of the possible time-dependence of decay laws for unstable particles. The probability, $P(t)$, that an unstable particle has not decayed at time t is expressed in terms of S-matrix quantities. It is concluded that, contrary to popular belief, the exponential decay law $P = e^{-\Gamma t}$ is only one of a discrete set of possible decay laws.

INTRODUCTION

It is generally accepted that the intuitive notions of stable particles (or bound states) and unstable particles (or resonances in scattering reactions) make their appearance in S-matrix theory as singularities of S-matrix elements when the latter are regarded as functions of a complex energy variable.[1] Thus poles of the S-matrix on the real energy axis correspond to stable particles while those occurring near the real axis on so-called unphysical sheets are resonances, or if you prefer, unstable particles. These concepts, which are carried over into the relativistic regime, are based largely on experience gained in the laboratory of non-relativistic quantum theory and direct examination of solutions of the time dependent Schrödinger equation. There are in addition some rather convincing discussions based on approximations in quantum electrodynamics[2] and simple field theoretical models.[3] Finally, there are a number of papers which attempt to relate unstable particle decays to properties of propagators in quantum field theory.[4] A very complete discussion of the general decay problem may be found in Chap. 8 of our forthcoming book.[5]

One might conclude from the above remarks that there isn't much motivation for the present work. In spite of the fact that one understands quite well the connection between S-matrix element singularities and resonant states on the one hand and the general features of the time decay of unstable states on the basis of the Schrödinger equation on the other hand, the relationship between these two aspects of the same physical situation is less transparent than might be desired. One of the purposes of this paper is to clarify this, and in so doing we find a remarkably simple and physically satisfying connection between the two approaches. Another purpose is to continue our study of the role of familiar space-time concepts of quantum theory (and common sense) in what is generally called S-matrix theory where such concepts are rather obscure. (It is perhaps worth remarking that in approaching these questions we have neither the zeal of a true S-matrix fanatic nor the rigidness of the axiomatic field theoretician; we are completely dedicated to integration and will not hesitate to use any convenient technique at our disposal.) Finally, we address ourselves to the question of the exponential decay law. We are not concerned with the frequently discussed but essentially trivial and uninteresting fact that in reality, for very long times, one has to do with a power dependence on time. Rather, we are interested in exploring the kinds of decay laws that could be expected on the basis of either provable or possible singularities of S-matrix elements. As we shall see, the conventional association of simple poles of the S-matrix on unphysical sheets is not required by any known physical principle and the possibility of the consequent deviations from simple exponential decay laws is worth studying.

Ordinarily one produces resonances or unstable particles in reactions and observes the subsequent decay products as a function of time measured more or less from the time of production. It is, of course, meaningful and useful to speak of an unstable particle only if it lives for a time long compared to the production reaction time. For only then can one reasonably regard the production and decay as a two-step process, an obvious idealization in which the S-matrix element factors into a product, to a very good approximation.

Our interest in the question of the exponential decay law arose directly from discussions with Professor V. L. Fitch. He pointed out that the supporting evidence for such a law was far from convincing in unstable particle decays. Since we had already been led to considering S-matrix element singularities which naturally give a more complex time behavior, we were stimulated to explore this question in more detail. We would like to suggest that the time-honored study of decay curves (rather than the simple determination of mean lifetimes) might be worthwhile.

We describe in Sec. II a simple and straightforward treatment of the decay of unstable states within the framework of conventional non-relativistic quantum theory. The result of these considerations is such that a more general formulation is attempted in Sec. III which would seem to have validity in the relativistic regime. In Sec. IV a specific calculation is presented and the possibility of non-exponential decays is discussed in detail. A highly idealized experiment for the detection of unstable particle decays is described in Section V and a short summary given in Section VI.

II. DECAY OF UNSTABLE STATES ACCORDING TO NON-RELATIVISTIC QUANTUM MECHANICS.

The understanding of the decay of a radioactive nucleus is an old problem and its description is properly regarded as one of the important successes of quantum theory. One imagines that at time zero the unstable system is spatially confined and one asks for the probability that after a certain time the system will be found in the initial state. The simplicity of this physical situation is unfortunately frequently obscured by the detailed considerations of barrier penetration, introduction of complex eigenvalues, etc. We shall attempt to formulate the problem in such a simple way that the extension of the description to the relativistic regime of unstable particle production and decay is almost immediate.

We imagine that we are dealing with a system which decays into two particles and work in the barycentric coordinate system of the decaying state. The wave function at $t = 0$ is called $\Psi(0)$ and is taken to have a definite angular momentum, ℓ. It is important for our purposes to think of $\Psi(0)$ as being localized in space within a distance characterized by a parameter $1/\beta$. [For example, β might represent an exponential fall-off rate for $\Psi(0)$.] We shall later discuss in more detail the significance of the choice for β; for the present it will be convenient to assume the restriction that

$$\frac{1}{\beta} << v \, \Delta t \, , \qquad (2.1)$$

where v is the velocity of the decay products and Δt is a measure of the "lifetime" of the state $\Psi(0)$.

The meaning of the condition (2.1) is, of course, the requirement that the initial packet be small in spatial extent compared with the distance which the decay products can travel during the characteristic time Δt. Were this not the case a detailed study of the decay as a function of time would not appear possible.

The wave function describing the relative motion of the decay products is $\psi_K^+(r)$ where r is the relative coordinate and the superscript + carries the usual connotation of outgoing spherical waves, and K is the wave number related to the energy E and reduced mass m according to $E = K^2/2m$ (with $\hbar = 1$). Quite specifically

$$\psi_K^+ \rightarrow (2/\pi)^{1/2}\, i^{\ell} \exp(i\delta_{\ell})\, \frac{\sin(Kr - \frac{\ell\pi}{2} + \delta_{\ell})}{Kr}\,, \qquad (2.2)$$

for large r; δ_{ℓ} is the phase shift corresponding to the scattering of the decay products. The factors are chosen to correspond to the continuism normalization

$$\int_0^\infty r^2\, dr\, [\psi_{K'}^+(r)]^* \,\psi_K^+(r) = \frac{1}{K^2}\,\delta(K' - K)\,. \qquad (2.3)$$

We shall assume that the ψ_K^+ form a complete set so that the prepared decaying state may be expressed as

$$\Psi(0) = \int_0^\infty K^2 dK\, c(K)\, \psi_K^+(r)\,, \qquad (2.4a)$$

or

$$c(\kappa) = (\psi_\kappa^+,\ \Psi(0)) = \int_0^\infty dr\ r^2[\psi_\kappa^+(r)]^*\ \Psi(0)\ . \qquad (2.4b)$$

At any time $t > 0$, the state $\Psi(t)$ is given by

$$\Psi(t) = e^{-iHt}\ \Psi(0) = \int_0^\infty \kappa^2 d\kappa\ c(\kappa)\ e^{-iEt}\ \psi_\kappa^+(r)\ , \qquad (2.5)$$

where H is the complete Hamiltonian for the system. The quantity of interest is the probability amplitude, $A(t)$, for finding the system, at time t, in the state $\Psi(0)$ given by

$$A(t) = (\Psi(0),\ \Psi(t)) = \int_0^\infty \kappa^2 d\kappa\ |c(\kappa)|^2\ e^{-iEt}\ . \qquad (2.6)$$

It is clear that the $c(\kappa)$ must have some special properties which reflect the fact that $\Psi(0)$ corresponds to a more or less localized state (that is, that $\Psi(0)$ is square integrable) and further that we are dealing with a long-lived system which has a reasonably well-defined energy. The latter feature implies that $c(\kappa)$ will be particularly large in the neighborhood of some energy $E \approx E_0$. We must evidently exhibit explicitly this energy dependence of $c(\kappa)$ if we are to have any hope of describing $A(t)$ in a general way. Of course, from the standpoint of the preparation of $\Psi(0)$ in a collision between the decay products one cannot entirely disentangle the confined character of $\Psi(0)$ from the relatively sharp energy E_0 and the assumed long lifetime of the state. We shall see below the connection between these aspects of the problem. Just to set the stage we remark that for a very narrow Breit-Wigner resonance one has[5]

$$A(t) = \frac{\Gamma}{2\pi} \int_0^\infty dE \frac{e^{-iEt}}{(E - E_0)^2 + \Gamma^2/4} \cong \exp(-iE_0 t) \exp(-\Gamma t/2) , \tag{2.7}$$

where Γ is the so-called width of the resonance. In this example

$$|c(\kappa)|^2 = \frac{\Gamma}{2m\kappa} \frac{1}{(E - E_0)^2 + \Gamma^2/4} . \tag{2.8}$$

Our problem is to isolate this typical resonance structure in a general way.

The method we first describe leans heavily on well-known properties of solutions of the Schrödinger equation in non-relativistic quantum mechanics.[5,6] The form of the result suggests, however, that it has much greater generality and in Sec. III we present arguments in support of this contention. There is a very close and scarcely surprising connection between the theory of final state interaction described in Chap. 9 of reference 5, and the decay problem.

We begin by remarking that ψ_κ^+ may be written as

$$\psi_\kappa^+(r) = (2/\pi)^{1/2} \frac{(i\kappa)^\ell}{r} \frac{\varphi(\kappa,r)}{f(-\kappa)} , \tag{2.9}$$

where $\varphi(\kappa,r)$ is a real solution (for real κ) of the Schrödinger equation corresponding to angular momentum ℓ and the boundary condition [see, for example Eq. (6-259) of reference 5]

$$\lim_{r \to 0} (2\ell + 1)!!\, r^{-\ell-1}\, \varphi(\kappa,r) = 1 , \tag{2.10}$$

and f(-κ) is the so-called Jost function. It is in turn defined in terms of φ and a solution of the same Schrödinger equation satisfying the boundary condition

$$\lim_{r \to \infty} e^{i\kappa r} f(\kappa,r) = i^{\ell}, \tag{2.11}$$

according to

$$f(\kappa) = \kappa^{\ell}\left\{f(\kappa,r)\,\frac{\partial\varphi(\kappa,r)}{\partial r} - \varphi(\kappa,r)\,\frac{\partial f(\kappa,r)}{\partial r}\right\} . \tag{2.12}$$

The function $\varphi(\kappa,r)$ is an entire function of κ^2 and, of course, $f(\kappa,r)$ is defined by the boundary condition (2.11) only in the half plane $\mathrm{Im}\,\kappa < 0$. For real κ, we can define another solution $f(-\kappa,r)$ according to

$$f(-\kappa,r) = (-1)^{\ell} f^*(\kappa,r) . \tag{2.13}$$

The function $\varphi(\kappa,r)$ may be expressed in terms of $f(\kappa,r)$ and $f(-\kappa,r)$ by

$$\varphi(\kappa,r) = \frac{1}{2i\kappa^{\ell+1}}\left[-f(-\kappa)\, f(\kappa,r) + (-1)^{\ell} f(\kappa)\, f(-\kappa,r)\right]$$

$$\xrightarrow[r\to\infty]{} \frac{1}{2i\kappa^{\ell+1}}\left[-f(-\kappa)\exp\left(-i\left(\kappa r-\ell\frac{\pi}{2}\right)\right) + f(\kappa)\exp\left(i\left(\kappa r-\ell\frac{\pi}{2}\right)\right)\right], \tag{2.14}$$

from which it follows by comparison with Eq. (2.2) that the S-matrix element $S_\ell \equiv \exp(2i\delta_\ell)$ is given by

$$S_\ell = \frac{f(\kappa)}{f(-\kappa)} \quad . \tag{2.15}$$

The important feature of this expression for S_ℓ for our purpose is that the singularities of S_ℓ are associated with the vanishing of the denominator, $f(-\kappa)$. Bound states make their appearance at points $\kappa = +i\kappa_n$, $\kappa_n > 0$ such that $f(-i\kappa_n) = 0$; provided $f(i\kappa_n) \neq 0$ this leads to a simple pole in the S-matrix element. On the other hand poles of S_ℓ in the lower half κ-plane, say at $\kappa = -\kappa_r - i\gamma$, $\gamma > 0$, are evidently associated with zeros of $f(\kappa)$ in the upper half κ-plane, and it is not possible, in general, to say anything about the multiplicity of these.[7] It can be shown that $f(\kappa)$ is an analytic function in the lower half κ-plane. Under certain circumstances this domain of analyticity may be extended to the upper half plane (for example for potentials which fall off like $\exp(-\mu r)$, one has a strip of analyticity, $\mathrm{Im}\,\kappa < \mu/2$). In such a case we have $f^*(-\kappa^*) = f(\kappa)$, so that if $f(-\kappa_r + i\gamma) = 0$, so is $f(+\kappa_r + i\gamma)$. Similarly if there is a pole of S_ℓ at $-\kappa_r - i\gamma$ there is also one at $+\kappa_r - i\gamma$. The singularity structure of S_ℓ in the neighborhood of a pole then is

$$S_\ell \approx \frac{(\kappa - \kappa_r - i\gamma)(\kappa + \kappa_r - i\gamma)}{(\kappa - \kappa_r + i\gamma)(\kappa + \kappa_r + i\gamma)} \quad . \tag{2.16}$$

It is conventional to consider the function $f(-\kappa)$ which is analytic in the upper half κ-plane as a function of the energy, E, called D(E), defined in the whole E plane cut along the positive real axis, the physical values being obtained as the limit on $\eta \to (0+)$ of $D(E + i\eta)$. The following things are important:[8]

$$\arg D(E + i\eta) = - \delta_\ell(E)$$

$$\lim_{E \to \infty} D(E + i\eta) = 1 \tag{2.17}$$

$$\exp(2i\delta_\ell(E)) = \frac{D(E - i\eta)}{D(E + i\eta)}$$

$$D(E) = \prod_B (1 - \frac{E_B}{E}) \exp\left\{\frac{1}{\pi}\int_0^\infty dE' \frac{\delta_\ell(E')}{E' - E - i\eta}\right\},$$

where the E_B are bound state energies. We shall assume hereafter that there are no bound states and as already instituted in the last of (2.17) interpret D(E) to be the limit as $\eta \to 0$ of $D(E + i\eta)$.

We may now express the expansion coefficients $c(\kappa)$ in terms of $\varphi(\kappa,r)$ and D(E). We write

$$c(\kappa) = (\psi_\kappa^+, \Psi(0)) = (2/\pi)^{1/2} \frac{(-i\kappa)^\ell \{[\varphi(\kappa,r)]/r, \Psi(0)\}}{D^*(E)} . \tag{2.18}$$

This is, of course, just what we are looking for. The zeros of $D^*(= f(\kappa))$ near the real axis are just the resonances anticipated in Eq. (2.8) and this structure of D^* will give the important long-time dependence of $A(t)$ defined by Eq. (2.6). This is not an exact statement since as $t \to \infty$, as is well-known, $A(t)$ shows a power dependence on t whereas we are interested in the essentially exponential regime (see Chap. 8 of reference 5 for a complete discussion). The numerator of $c(\kappa)$ will in general have singularities in the E- or κ-plane far from the real axis. The reason is that $\varphi(\kappa,r)$ is an entire function of κ^2 so that the only singularities of the numerator can arise from a failure of the integral over r, implied in the scalar product, to converge for complex κ. Such singularities are related to the detailed fall-off of the localized state $\Psi(0)$. If the latter be expressed by $\exp(-\beta r)$ we expect in general a branch line extending from $E = -\infty$ to $E = -\beta^2/2m$ (or in the κ plane from $\kappa = i\beta$ to $\kappa = i\infty$); hence the larger β (and thus the greater the localization) the farther are these singularities from the physical region $E > 0$. It is furthermore clear that $(-i\kappa)^{\ell}\ (\frac{\varphi}{r}, \Psi(0))$ must approach zero for large κ sufficiently fast (since $D(E) \to 1$) that the normalization condition

$$A(0) = \int_0^\infty \kappa^2 d\kappa \; |c(\kappa)|^2 = 1 \qquad (2.19)$$

can hold.

On the basis of the above discussion we write

$$c(\kappa) = (-i\kappa)^{\ell} \frac{g(E)}{D^*(E)} \qquad (2.20)$$

where

$$g(E) = \left(\frac{\phi(\kappa,r)}{r},\ \Psi(0)\right)(2/\pi)^{1/2} \qquad (2.21)$$

is regarded as a function of E, since ϕ depends only on κ^2. We anticipate that g(E) is a slowly varying function of E in the neighborhood of the real E axis. It is, of course, g(E) which contains the detailed information about $\Psi(0)$ which would be required for an exact evaluation of A(t). However the factor $[D^*(E)]^{-1}$ is the thing which expresses the fact that $\Psi(0)$ is supposed to be nearly an eigenstate of H; that is, we are dealing with a long-lived resonance, one for which $\Psi(0)$ contains components with energies all in the neighborhood of some E_0. As long as we are in neither the very short nor very long time period for A(t), we can expect that the most important effects are contained in D*(E) and that our predictions will be largely independent of $\Psi(0)$ and hence of the production mechanism.

It is perhaps worthwhile to show the manner in which the recognition of the singular behavior of $c(\kappa)$ indeed allows for a description of the localized $\Psi(0)$ and further, how if this feature is not recognized no such localization would be possible. Using our explicit expressions for $c(\kappa)$ and for the wave function ψ_κ^+ in terms of Jost functions we have (writing $D^*(E) = f(\kappa)$)

$$\Psi(0) = (2/\pi)^{1/2} \int_0^\infty \kappa^2 d\kappa \frac{(-i\kappa)^\ell}{f(\kappa)} g(E) \frac{(i\kappa)^\ell}{2ir\kappa^{\ell+1}} \Big[-f(\kappa,r) + (-1)^\ell \frac{f(\kappa)}{f(-\kappa)} f(-\kappa,r)\Big]$$

(continued)

$$= \frac{(2/\pi)^{1/2}}{2ir} \int_0^\infty \kappa d\kappa\, g(E)\, \kappa^\ell \left[- \frac{f(\kappa,r)}{f(\kappa)} + (-1)^\ell \frac{f(-\kappa,r)}{f(\kappa)} \right] \quad (2.22)$$

$$= \frac{i(2/\pi)^{1/2}}{4r} \int_{-\infty}^{\infty} \kappa d\kappa\, g(E)\, \kappa^\ell \frac{f(\kappa,r)}{f(\kappa)} .$$

In the last line we have used the fact that g(E) is an even function of κ^2. Now we look at $\Psi(0)$ for large r, in the region where $f(\kappa,r) \rightarrow i^\ell \exp\{-i\kappa r\}$. Since $f(\kappa)$ by hypothesis has no zeros in the lower half plane (these of necessity being bound states) and g(E) has no singularities until we reach $\kappa = -i\beta$ where $1/\beta$ is associated with the "size" of $\Psi(0)$, we may lower the contour to this point and it is clear that $\Psi(0)$ will indeed go, as it should, like $\exp(-\beta r)$.

Now suppose we had been so naive to expect the expansion coefficients $c(\kappa)$ to be just any old smoothly varying function of κ. Then

$$\Psi(0) = (2/\pi)^{1/2} \int_0^\infty \kappa^2 d\kappa\, c(\kappa) \frac{i^{\ell-1}}{2\kappa r} \left[-f(\kappa,r) + (-1)^\ell \frac{f(\kappa)}{f(-\kappa)} f(-\kappa,r) \right]$$

$$\xrightarrow[r\rightarrow\infty]{} (2/\pi)^{1/2} \frac{i^{\ell-1}}{2r} \int_0^\infty \kappa d\kappa\, c(\kappa) \quad (2.23)$$

$$\times \left[-\exp(-i\kappa(r - \tfrac{\ell\pi}{2})) + \exp(2i\delta_\ell(\kappa)) \exp(i\kappa(r - \tfrac{\ell\pi}{2}) \right].$$

If we are concerned with a sharp resonance, so that $S_\ell = \exp\{2i\delta_\ell\}$ has the structure (2.16), a simple stationary phase argument shows that $\Psi(0) \sim \exp\{-\frac{\gamma}{2} r\}$ which is ordinarily much too "fat" a wave packet to correspond to physically sensible, initial conditions. Since $\gamma \approx (v\Delta t)^{-1}$, this choice for $c(\kappa)$ would violate our fundamental condition (2.1).

We are now prepared to complete our discussion of the amplitude A(t) for finding the initial state present at time t. We have

$$A(t) = \int_0^\infty \kappa^2 d\kappa \, \kappa^{2\ell} g^2(E) \frac{e^{-iEt}}{|D(E)|^2} = \int_0^\infty dE \frac{B(E)}{|D(E)|^2} e^{-iEt} , \qquad (2.24)$$

where

$$g^2(E) \, \kappa^{2\ell+2} \, d\kappa \equiv B(E) \, dE . \qquad (2.25)$$

The probability that at time t, the unstable system has not decayed is

$$P(t) = |A(t)|^2 \qquad (2.26)$$

and the probability that the decay takes place during the interval dt is clearly

$$p(t) \, dt = - \frac{dP(t)}{dt} dt . \qquad (2.27)$$

Our expression for A(t) involving $|D(E)|^{-2}$, Eq. (2.24) would seem to express the decay amplitude so far as possible in terms of S-matrix quantities. It should be noted that whereas a knowledge of $D(E)$ implies a knowledge of S_ℓ, the converse is not true, since S_ℓ involves only arg $D(E) = - \delta_\ell(E)$, and

$$S_\ell(E) = \frac{D^*(E)}{D(E)} \, . \tag{2.28}$$

In many ways, $D(E)$ can be regarded as a "more fundamental" quantity than $S_\ell(E)$. It enters quite naturally into a variety of problems such as the electromagnetic structure of particles and in the theory of multichannel scattering processes, just to name two. One might even conjecture that the formulation of a D-matrix theory rather than an S-matrix theory might be very worthwhile. This is not the purpose of the present paper so we shall not pursue the question further. (Another reason for not doing so is that we don't know precisely how to do it! Needless to say, $D(E)$ is the same quantity that occurs in the so-called N/D method of solving partial wave dispersion relations.) We shall return in Sec. IV to the explicit evaluation of $P(t)$ after we address ourselves to the general validity of our expression for the decay probability given by Eqs. (2.24) and (2.26).

III. A MORE GENERAL FORMULATION OF THE DECAY PROBLEM

Our treatment of the decay problem would appear superficially to depend rather heavily on detailed properties of solutions of the Schrödinger equation. In fact we feel that this is not at all the case and that the same conclusions can be drawn without explicitly mentioning things which might be unpalatable for a pure S-matrix theorist. The point is simply that our principal problem was the isolation of the factor $[f(\kappa)]^{-1} = [D^*(E)]^{-1}$ in the expression for the amplitude of the decaying state. The latter in turn necessarily is determined, since we are dealing with continuum states largely with the behavior of asymptotic wave functions which are quite legitimate targets of discussion for S-matrix theorists. That is, we argue that asymptotic wave functions must exist in any acceptable physical theory.

We recall the well-known fact (see, for example, Sec. 5.2 of reference 5) that if one prepares a pre-collision packet of asymptotic states for a scattering process with certain wave-packet amplitudes $c(\kappa)$ then the interacting state vector at the time of interaction is a superposition with precisely the same amplitudes $c(\kappa)$ of the exact eigenfunctions. This implies that a study of the asymptotic wave functions suffices to determine the nature of the expansion coefficients. In our problem the desire to represent a spatially confined decaying system requires the presence in the asymptotic wave packet amplitude of a factor which will permit such a description. We cannot specify by this argument that we require exactly $[f(\kappa)]^{-1}$ but this is a sufficient condition to insure the possibility of describing a localized state. We certainly cannot designate any other reasonable factor reflecting the presence of a resonance without disastrous effects on the asymptotic states.

Another way to see the above advertised behavior is to consider the following simple example: Consider the scattering of two particles which can form a long-lived resonant state and then decay into the initial pair. We prepare a pre-collision packet which is so arranged that the colliding particles reach the origin of coordinates at a time we agree to call zero. The wave function at any positive time t after the collision is over is represented by

$$\Psi(t) = \int d^3\kappa' \int d^3\kappa \; e^{i\underset{\sim}{\kappa}' \cdot \underset{\sim}{r} - iE(\kappa')t} \Big\{ \delta(\underset{\sim}{\kappa}' - \underset{\sim}{\kappa}) - 2\pi i \, \delta[E(\kappa') - E(\kappa')] \, T_{\underset{\sim}{\kappa}'\underset{\sim}{\kappa}} \Big\} \, a(\underset{\sim}{\kappa} - \underset{\sim}{\kappa}_0) \qquad (3.1)$$

where $a(\underset{\sim}{\kappa} - \underset{\sim}{\kappa}_0)$ describes the initial pre-collision packet, and $T_{\underset{\sim}{\kappa}'\underset{\sim}{\kappa}}$ is the T-matrix element describing the scattering. If we imagine a resonance in a particular angular momentum state, the important part of $T_{\underset{\sim}{\kappa}'\underset{\sim}{\kappa}}$ will contain a term $N/D \; P_\ell(\hat{\underset{\sim}{\kappa}}' \cdot \hat{\underset{\sim}{\kappa}})$ and the resonant character of the reaction appears in the factor D. Thus the scattered wave function amplitude has the factor $a(\underset{\sim}{\kappa} - \underset{\sim}{\kappa}_0) \, [D(E)]^{-1}$; $a(\underset{\sim}{\kappa} - \underset{\sim}{\kappa}_0)$ knows nothing about the resonance, but D(E) of course does. The numerator function $N(\kappa)$ is also expected to be smooth in the resonance region. We see the natural occurrence of D(E) in the scattered wave function.

The asymptotic wave packet states may be shown to be an essentially complete orthonormal set in a well-defined sense (see reference 5, Chap. 3 and 4). Thus the asymptotic form of a resonant state may surely be represented in the form originally suggested, Eq. (2.4a). The condition (2.1) instructs us to require that $\Psi(0)$ vanish in the asymptotic region for distances greater than β^{-1}. That this suggests very strongly the form (2.20) for $c(\kappa)$ may be seen on repeating the argument given in connection with Eqs. (2.22) and (2.23), but using only the asymptotic form, for large r of these equations.

IV. IMPLICATIONS OF A LONG-LIVED UNSTABLE STATE

It is apparent from our general expression (2.6) that any decay time can be achieved for any unstable physical system. The reason for this is that Eq. (2.6) involves only the wave packet expansion coefficients--and does not contain any reference to the dynamical characteristics of the decaying system. In the previous two sections we have attempted to explain why many classes of unstable physical systems show similar characteristics. That is, for considerable variation of initial boundary conditions such systems exhibit remarkably uniform properties--so much so, in fact, that one tends to think of unstable "particles" as having unique properties.[9]

The physical conditions required for such uniform properties seem to require that the decaying system (1) have a fairly sharply defined energy near, say, E_0; (2) that it have a long lifetime Δt; and (3) that it be confined in space as required by the condition (2.1).

In Chap. 8 of reference (5) we investigated the consequence of a long-lived state in a scattering experiment. For the case that both incident and final channels contain two particles, and when the lifetime Δt is large compared to the free flight time of the interacting particles across their region of mutual interaction, the eigenvalues of the S-matrix were shown to have the unique form

$$S(E) = \left[\frac{E - E_0 - i\,\frac{\Gamma}{2}}{E - E_0 + i\,\frac{\Gamma}{2}}\right]^r e^{2i\nu(E)} \quad . \tag{4.1}$$

Here $r = 1, 2, \ldots$ is a positive integer, and $\nu(E)$ represents the "background", or "potential", scattering (as it is sometimes called). The constant Γ in Eq. (4.1) is the level width, or more precisely, $\frac{\hbar}{\Gamma}$ is the Wigner lifetime[5] of the interacting system. When $\Delta t (\cong \frac{\hbar}{\Gamma})$ is very large (in the sense just described) we may treat $\nu(E)$ as a constant and ignore it.

The case r = 1 in Eq. (4.1) corresponds, of course, to a conventional Breit-Wigner resonance. It was shown in reference (5), Chap. 8, that $r \geq 2$ corresponds to a more general class of resonances.

We return now to our discussion of the decay problem and ask what are the general characteristics of an unstable system having a long lifetime Δt and initially confined in space as required by (2.1). We have just said that the condition of a long lifetime permits us to write

$$S_r(E) \cong \left[\frac{E - E_0 - i\frac{\Gamma}{2}}{E - E_0 + i\frac{\Gamma}{2}}\right]^r , \qquad (4.2)$$

where Γ is "small". This and the condition (2.1) permits us to treat B(E) as constant in Eq. (2.24).

We shall restrict ourselves to the case that there are no bound states having energies near E_0, within a range large compared to Γ. Then we may write[10]

$$D_r(E) = e^{-\Delta} ,$$

$$\Delta = \frac{1}{2\pi i} \int_{E_M}^{\infty} \frac{dE' \ln S_r(E')}{E' - E - i\eta} , \qquad (4.3)$$

corresponding to a given integer r in Eq. (4.2). Evidently, we have

$$D_r(E) = [D_1(E)]^r , \qquad (4.4)$$

where $D_1(E)$ corresponds to a conventional Breit-Wigner resonance.

Near the energy E_0 we may write[11]

$$D_1(E) \cong N(E - E_0 + i \frac{\Gamma}{2}) , \qquad (4.5)$$

where N is a constant. From Eq. (4.4) we obtain the general result

$$D_r(E) = N^r(E - E_0 + i \frac{\Gamma}{2})^r . \qquad (4.6)$$

The decay characteristics of the system described may now be obtained from Eq. (2.24):

$$A_r(t) = \int_0^\infty dE \frac{\rho_r(E)\ e^{-iEt}}{[(E - E_0)^2 + \frac{\Gamma^2}{4}]^r} \qquad (4.7)$$

where

$$\rho_r(E) \equiv \frac{B(E)}{N^r} .$$

Since B(E) is considered to be nearly constant over an interval comparable to Γ at $E = E_0$, we may re-write this in the approximate form [here $E_1 \equiv E_0 - i \frac{\Gamma}{2}$]

$$A_r(t) \approx \rho_r(E_0) \int_0^\infty \frac{dE\ e^{-Et}}{[(E - E_1)(E - E_1^*)]^r}$$

$$\approx \rho_r(E_0) \int_{-\infty}^\infty \frac{dE\ e^{-iEt}}{[(E - E_1)(E - E_1^*)]^r}$$

$$= - \frac{2\pi i\ \rho_r(E_0)}{(r - 1)!} \left[\frac{\partial^{r-1}}{\partial E^{r-1}} \frac{e^{-iEt}}{(E - E_1^*)^r}\right]_{E = E_1} \tag{4.8}$$

$$= \frac{2\pi\ \rho_r(E_0)}{(r - 1)!} \exp(-iE_0t) \exp(- \frac{\Gamma t}{2} \sum_{\ell=0}^{r-1} \frac{t^{r-1-\ell}}{\Gamma^{r+\ell}} \frac{(r + \ell - 1)!}{(r - \ell)!\ \ell!} .$$

On choosing $\rho_r(E_0)$ to satisfy the condition (2.19) that $A_r(0) = 1$, we find

$$A_r(t) = \exp(-iE_0t - \frac{\Gamma t}{2}) \sum_{\ell=0}^{r-1} (\Gamma t)^{r-1-\ell} \frac{(r + \ell - 1)!\ (r - 1)!}{(r - \ell - 1)!\ \ell!(2r - 2)!} \tag{4.9}$$

Except for small corrections associated with the mode of formation of the state [that is, with the detailed properties of B(E) in Eq. (2.24)], the decay laws

$$P_r(t) = |A_r(t)|^2 ,$$

$r = 1, 2, \ldots$, are believed to represent the most general allowed for long-lived systems which are initially localized in accordance with Eq. (2.1).[12]

We list the decay amplitudes for $r = 1, 2, \ldots 5$:

$$(A_1) = e^{-\Gamma t/2}$$

$$(A_2) = e^{-\Gamma t/2} \{1 + \frac{\Gamma t}{2}\}$$

$$(A_3) = e^{-\Gamma t/2} \{1 + \frac{\Gamma t}{2} + \frac{\Gamma^2 t^2}{12})\} \qquad (4.10$$

$$(A_4) = e^{-\Gamma t/2} \{1 + \frac{\Gamma t}{2} + \frac{\Gamma^2 t^2}{10} + \frac{\Gamma^3 t^3}{120}\}$$

$$(A_5) = e^{-\Gamma t/2} \{1 + \frac{\Gamma t}{2} + \frac{3}{28} (\Gamma t)^2 + \frac{(\Gamma t)^3}{84} + \frac{(\Gamma t)^4}{1680}\}.$$

In Fig. 1 we show $\ln P_r(t) = \ln |A_r(t)|^2$ as a function of Γt. The obvious feature of these curves is that $P_r(t)$ tends to stay closer to unity as time increases the larger the value of r. We know of no examples showing other than the pure exponential behavior characteristic of $r = 1$ but a careful study of decay curves may be worthwhile. [As noted in the Introduction, there are relatively few measurements of $P(t)$ for the unstable particles.]

There is a natural tendency to interpret a pole of the S-matrix of order higher than the first as an accidental degeneracy; this implies, however, that the "primeval" poles are simple and we can find no deep theoretical basis for such an allegation.

V. A COMMENT ON THE OBSERVATION OF DECAY LAWS

For the simple exponential decay corresponding to $r = 1$, [see Eqs. (4.10)]

$$P_1(t) = e^{-\Gamma t} , \qquad (5.1)$$

the choice of $t = 0$ has no effect on the shape of the decay law. This is evidently not the case for $r \geq 2$, although the exponential factor tends to dominate the time dependence of these for $\Gamma t >> 1$ [see Fig. (1)]. To compare these laws with experimental observations one must therefore discuss the initial conditions with some care. We shall now illustrate this with a somewhat idealized example.

Referring to Fig. (2), we imagine that the unstable particle is created within a sphere S in a bubble chamber. The size of this sphere is limited by the range of secondary electrons along the path of charged particles. We have seen that the actual size of S is not relevant, as long as it is compatible with the condition (2.1), that is that the region be small compared to $v \Delta t$. Since we are studying the decay as a function of time, we assume that the time of creation (say $t = 0$) of the particle is known to within an interval small compared with $\Delta t = \frac{\hbar}{\Gamma}$.

We next suppose that the decaying particle passes through (and is registered by) counter C_1 at time t_1 and then is stopped in the block B. Here it decays and the decay product is counted in C_2 at time t_2. Errors in registering the times t_1 and t_2 are again considered small compared with $\Delta t = \frac{\hbar}{\Gamma}$.

The wave function in the interval $0 < t < t_1$ then has the form (2.5) with $c(\kappa)$ given by Eq. (2.20). To take account of the information provided by C_1 that an unstable particle passed through it at time t_1, we introduce a projection operator[13]

$$E(r) = 1 \text{ for } r \text{ within counter}$$
$$= 0 \text{ for } r \text{ outside counter.}$$

Then, immediately following the time t_1 the wave function is[14]

$$\Psi'(0) \equiv N_E\, E(r)\, \Psi(t_1) , \qquad (5.2)$$

where N_E is the normalization constant. We may treat $\Psi'(0)$ as a new initial wave function and follow the argument leading to Eq. (2.5) to obtain, for $t > t_1$,

$$\Psi'(t) = \int_0^\infty \kappa^2 d\kappa\, c'(\kappa)\, \exp(-iE(t-t_1))\, \psi_\kappa^+(r) , \qquad (5.3)$$

where

$$c'(\kappa) = (\psi_\kappa^+(r), \Psi'(0)) . \qquad (5.4)$$

The probability amplitude for decay is then obtained for $t > t_1$ using Eq. (2.6) with the $c(\kappa)$ replaced by $c'(\kappa)$. The arguments of Sec. II and III would lead us to expect that the result (4.9) would again be obtained with t replaced by $(t - t_1)$, unless the size of counter C_1 is such that the condition (2.1) is poorly satisfied.

We repeat that the example just given is quite idealized and was presented only to emphasize that attention to initial and subsequent information may be important in studying particle decays.

VI. CONCLUSIONS

We have given a formulation of decay of unstable states which involves in its essentials only what might be termed S-matrix quantities. In fact a knowledge of the S-matrix does not suffice in general since what enters is really the so-called denominator function D(E) which contains more information; it is, however, something which can be legitimately sought in a pure S-matrix theory. It is obvious that a detailed description of a decay process requires a precise specification of the production mechanism. There does not appear in principle to be any difficulty in formulating the problem although one can expect simplicity only under the circumstance that the overall S-matrix factors into a production part and a decay part.

We have explored the possibility of finding decay laws more complicated than a simple exponential, resulting from resonance poles which are not of first order. There seem to be no very convincing arguments to say that there are in nature only first order poles. (If one were to find experimentally only pure exponential decays, we would be led to a postulate in S-matrix theory which could be called the principle of minimal policity.) It is clear that non-exponential decays might result if the production mechanism had some wild energy dependence. In general what one might expect however is something like what happens when the decay products of a radioactive decay are themselves unstable. This gives a mixture of pure exponentials but nothing oscillatory or very spectacular. (See reference 5, Chap. 8 for a complete treatment.) If there did happen to be two nearby resonances in the decay channel one would find an oscillatory time dependence superimposed on the decaying exponentials.

APPENDIX

We describe here the evaluation of Eq. (4.3) for $r = 1$ to obtain the expression (4.5). First, we consider a model for which $D_1(E)$ may be evaluated explicitly and then give a more general argument.

For the model chosen we write the scattering phase shift as (here $0 < E < \infty$)

$$\delta = \frac{1}{2i} \ln S = \tan^{-1} \frac{a(E)^{1/2}}{E_0 - E} , \qquad \text{(A-1)}$$

where $a << (E_0)^{1/2}$. This corresponds to a $\nu(E)$ in Eq. (4.1) which has the value $\nu \approx - \pi/2$ for $E << E_0$. Then

$$D_1(E) = \exp \left\{ - \frac{1}{\pi} \int_0^\infty \frac{dE' \tan^{-1}(a(E')^{1/2} / E_0 - E')}{E' - E - i\eta} \right\} . \qquad \text{(A-2)}$$

The evaluation of $D_1(E)$ is most easily carried out noting that it must be analytic in the entire energy plane except along the real positive axis, must be real for $E < 0$, must approach unity as $E \rightarrow \infty$, and must have the prescribed phase. The function which has all of these virtues is

$$D_1(E) = \frac{E - E_0 + i\, a(E)^{1/2}}{E} . \qquad \text{(A-3)}$$

The approximate form (4.5) follows on setting $\Gamma = 2a(E_0)^{1/2}$, and restricting E to values close to E_0.

To argue more generally, we substitute the expression (4.2) into (4.3), setting $r = 1$. It is convenient to define the zero of energy so that $E_M = 0$ and to introduce a cutoff M for the upper limit. Then we have

$$\Delta = \frac{1}{2\pi}\,[I_+ - I_-]\,, \tag{A-4}$$

where

$$I_{\pm} = \int_0^M dE'\, \frac{\ln(E' - E_0 \pm i\,\frac{\Gamma}{2})}{E' - E - i\eta}\,. \tag{A-5}$$

The substitution

$$z \equiv E' - E_0 \pm i\,\frac{\Gamma}{2}$$

permits us to write

$$I_{\pm} = \int_{C_{\pm}} \frac{dz\,\ln z}{z - a_{\pm}}\,, \tag{A-6}$$

where

$$a_{\pm} = E - E_0 \pm i\,\frac{\Gamma}{2}\,. \tag{A-7}$$

The contours $C_{\pm}$ are illustrated in Fig. (3).

As $|E - E_0|$ and Γ become very small the points $a_{\pm}$ approach the branch point. Only I_+ becomes singular in this case. Its singularity may be exhibited by moving the contour up into the positive imaginary z-plane and keeping the residue of the pole at a_+. The leading (singular) term in I_+ is

$$I_+ \cong 2\pi i \ln a_+ ,$$

or

$$\Delta \cong - \ln a_+ , \qquad \text{(A-8)}$$

from which Eq. (4.5) follows.

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   We shall follow the notation of this article reasonably closely and record below some of the principal results which we need.

7. We are indebted to Professors Bargmann and Wigner for a discussion of this point.
8. See, for example, Eq. (6-281) of reference (5).
9. In a strict sense, for example, no two neutrons are quite the same, since the set of wave packet amplitudes $c(K)$ describing the set of "neutron-like" systems is not countable and since (presumably) the precise conditions of creation of a given "neutron" cannot be duplicated.

10. See, for example, Eq. (6-283) of reference (5). The integral (4.3) can be made finite on introducing suitable subtractions.
11. This is shown in the Appendix.
12. An exception can occur, however, if two or more "resonances" happen to be separated by distances comparable to their respective widths.
13. Strictly speaking, we require that the coordinates of both decay products lie within the counter. For simplicity of presentation we are ignoring the center-of-mass coordinate of the unstable system.
14. See, for example, M. L. Goldberger and K. M. Watson, Phys. Rev. (in press) where such sequential observations are discussed.

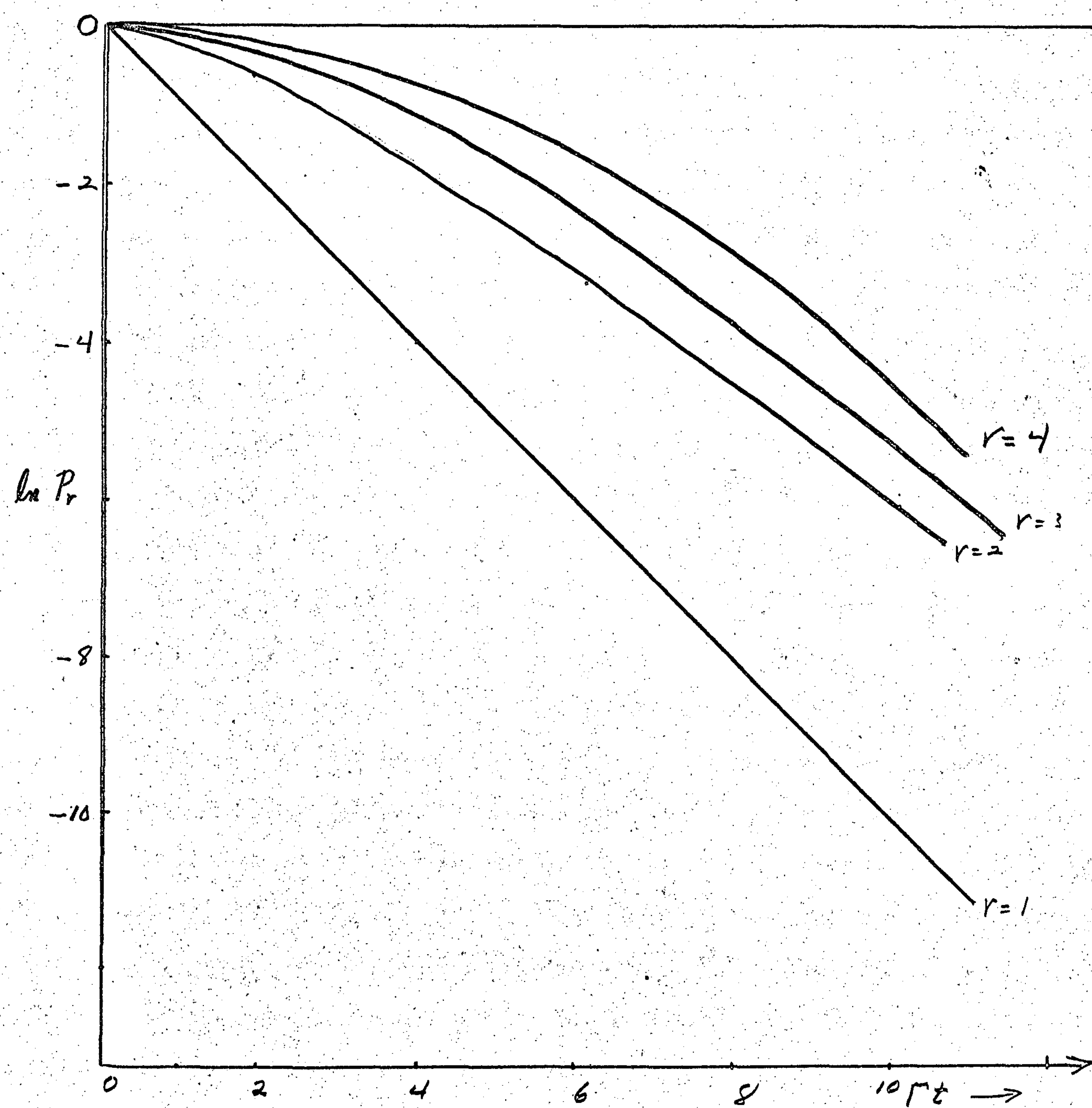

Fig. 1

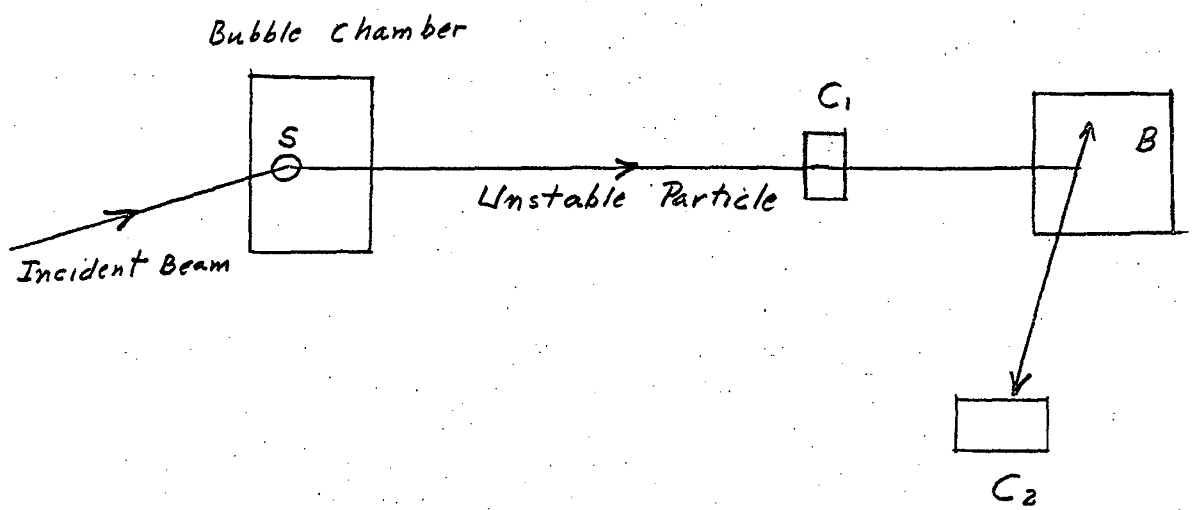

Fig. 2

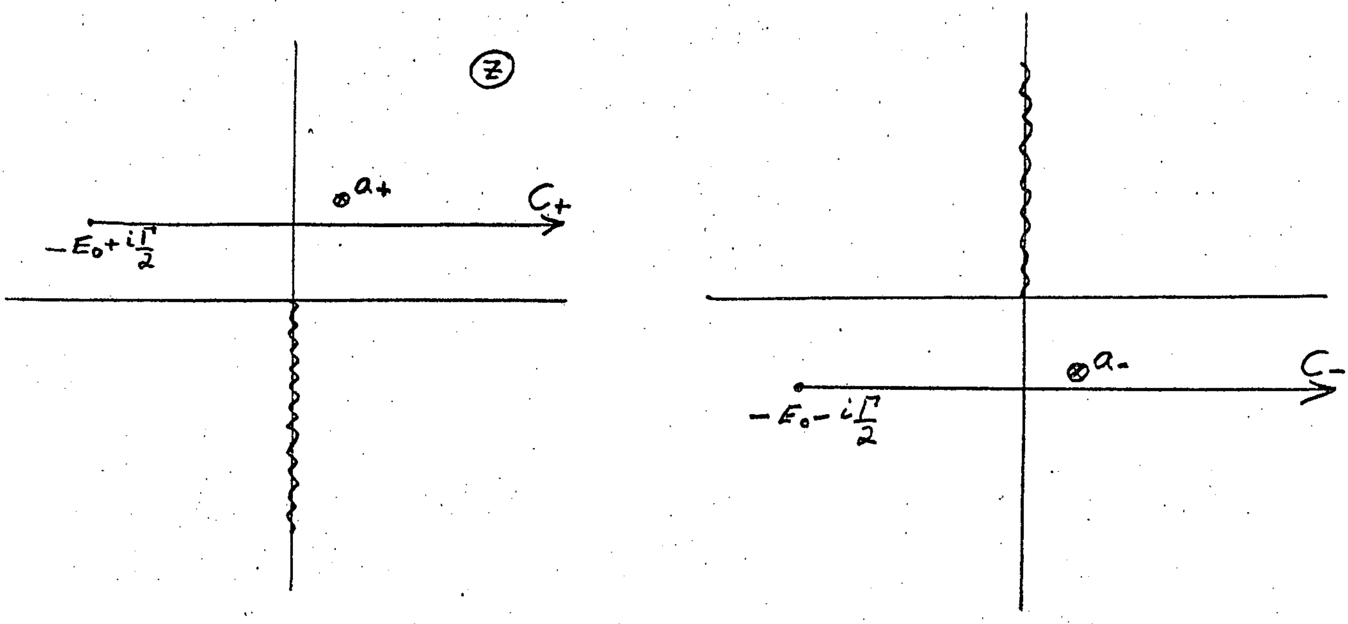

Fig. 3