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Refined Catalan and Narayana cyclic sieving

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Abstract. We prove several new instances of the cyclic sieving phenomenon (CSP) on Catalan objects of type A and type B. Moreover, we refine many of the known instances of the CSP on Catalan objects. For example, we consider triangulations refined by the number of "ears", non-crossing matchings with a fixed number of short edges, and non-crossing configurations with a fixed number of loops and edges.

Keywords. Dyck paths, cyclic sieving, Narayana numbers, major index, q-analog
Mathematics Subject Classifications. 05E18, 05A19, 05A30

1. Introduction

The original inspiration for this paper is a natural interpolation between type A and type B Catalan numbers. For $n \geqslant 0$ consider the expression

$$\binom{2n}{n} - \binom{2n}{n-s-1}. \tag{1.1}$$

For $s = 0$, we recover the n^{th} Catalan number and for $s = 1$, we recover the $(n+1)^{\text{th}}$ Catalan number. When $s = n$, we obtain the central binomial coefficient $\binom{2n}{n}$, which is known as the n^{th} type B Catalan number, see [Arm09]. There are several combinatorial families of objects which are counted by the expression in (1.1), certain standard Young tableaux and lattice paths to name a few. The expression in (1.1) has the q-analog given by the difference of q-binomials

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^{s+1} \begin{bmatrix} 2n \\ n-s-1 \end{bmatrix}_q. \tag{1.2}$$

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For $s \in \{0, 1, n\}$, the polynomials in (1.2) appear in instances of the *cyclic sieving phenomenon*. Furthermore, it follows from [APRU21, Theorem 46] that there exist group actions such that the polynomials in (1.2) exhibit cyclic sieving for all $s \in \{0, 1, \dots, n\}$.

Definition 1.1 (Cyclic sieving, [RSW04])**.** Let X be a set and C_n be the cyclic group of order n acting on X. Let $f(q) \in \mathbb{N}[q]$. We say that the *triple* $(X, C_n, f(q))$ *exhibits the cyclic sieving phenomenon (CSP)* if for all $d \in \mathbb{Z}$,

$$|\{x \in X : g^d \cdot x = x\}| = f(\xi^d) \tag{1.3}$$

where ξ is a primitive n^{th} root of unity.

Note that it follows immediately from the definition that $|X| = f(1)$. In the study of cyclic sieving, it is mainly the case that the C_n-action and the polynomial $f(q)$ are natural in some sense. The group action could be some form of rotation or cyclic shift of the elements of X. The polynomial usually has a closed form and is also typically the generating polynomial for some combinatorial statistic defined on X. See B. Sagan's article [Sag11] for a survey of various types of CSP instances.

Many known instances of the cyclic sieving phenomenon involve a set X whose size is a Catalan number. Once such a CSP triple is obtained, one can ask if X can be partitioned $X = \sqcup_j X_j$ in such a way that the group action on X induces a group action on X_j for all j, and, in that case, also ask if there is a *refinement* of the CSP triple in question.

Definition 1.2 (Refinement of cyclic sieving)**.** The family $\{(X_j, C_n, f_j(q))\}_j$ of CSP triples is said to *refine* the CSP triple $(X, C_n, f(q))$ if

- $\bigsqcup X_j = X$,
- $\sum_j f_j(q) = f(q)$ and
- the C_n-action on X_j coincides with the C_n-action on X restricted to X_j, for all j.

Typically, the sets X_i are of the form $X_j = \{x \in X : \text{st}(x) = j\}$ for some statistic $\text{st} : X \to \mathbb{N}$ that is preserved by the group action. Examples of such statistics are the number of cyclic descents of a word, the number of blocks of a partition, and the number of ears of a triangulation of an n-gon—all with the group action being (clockwise) cyclic rotation. Throughout the paper, *we shall consistently use the order of the group (or group generator) as subscript*. For example, rotation by $2\pi/n$ is denoted rot_n.

For $s \in \{0, n\}$, the q-analog in (1.2) admits a natural refinement, so that the type A and type B q-Narayana polynomials are recovered. The q-Narayana polynomials can be used to refine the aforementioned instances of the CSP. It is therefore natural to ask if there is a q-analog of (1.1) for arbitrary $s \in \{0, 1, \dots, n\}$ which also exhibits similar combinatorial properties as the type A and type B q-Narayana polynomials. We discuss partial results and motivations behind this problem in Section 3.

In the process of analyzing this intriguing question, we discovered several new instances of the cyclic sieving phenomenon. Some concern new q-analogs of Catalan numbers, while others refine known instances. In the tables in Section 2.5, we present a comprehensive (but most likely incomplete) overview of the current state-of-the-art regarding the cyclic sieving phenomenon involving Catalan and Narayana objects of type A and B.

1.1. Overview of our results

We only highlight some of the results in our paper; in addition we also prove several other results which fill gaps in the literature. In Section 4, the main result is the following theorem, which is a new refined CSP instance on Catalan objects. It can be stated either in terms of promotion (denoted ∂_{2n}) on two-row standard Young tableaux with k cyclic descents, $\mathrm{SYT}_{\mathrm{cdes}}(n^2, k)$, or non-crossing perfect matchings with k short edges, $\mathrm{NCM}_{\mathrm{sh}}(n, k)$.

Theorem 1.3 (Theorem 4.8)**.** *Let $k, n \geqslant 2$ be natural numbers and let*

$$\mathbf{Syt}(n,k;q) \coloneqq \frac{q^{k(k-2)}(1+q^n)}{[n+1]_q} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q.$$

Then

$$\sum_k \mathbf{Syt}(n,k;q) = \mathbf{Cat}(n;q),$$

and the triples

$$(\mathrm{SYT}_{\mathrm{cdes}}(n^2,k), \langle \partial_{2n} \rangle, \mathbf{Syt}(n,k;q))$$

and

$$(\mathrm{NCM}_{\mathrm{sh}}(n,k), \langle \mathrm{rot}_{2n} \rangle, \mathbf{Syt}(n,k;q))$$

exhibit the cyclic sieving phenomenon.

In Section 5, we study the set of so-called non-crossing (1,2)-configurations on n vertices, which we denote by $\mathrm{NCC}(n+1)$. The cardinality of this set is the Catalan number $\mathrm{Cat}(n+1) = \binom{2n}{n} - \binom{2n}{n-2}$. We define a simple “rotate-and-flip” action on $\mathrm{NCC}(n+1)$ which has order $2n$ and is reminiscent of promotion.

Theorem 1.4 (Theorem 5.4)**.** *The triple*

$$\left(\mathrm{NCC}(n+1), \langle \mathrm{twist}_{2n} \rangle, \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^2 \begin{bmatrix} 2n \\ n-2 \end{bmatrix}_q \right)$$

exhibits the cyclic sieving phenomenon.

Note that we use a quite non-standard q-analog of the Catalan numbers here, which has not appeared in the context of cyclic sieving before. Cyclic sieving on non-crossing (1,2)-configurations was studied earlier by M. Thiel [Thi17], with rotation as the group action. In Theorem 5.9 and Corollary 5.10, we refine Thiel's result. In particular, we obtain a new CSP instance involving the q-Narayana polynomial $\mathbf{Nar}(n+1,k;q)$.

In Section 6, we study various instances of cyclic sieving involving the type B Catalan numbers, $\binom{2n}{n}$. Some results have more or less appeared in earlier works, but we make some of the results more explicit. One novel result is a type B version of Theorem 1.4, where we consider the twist action on *type B non-crossing (1,2)-configurations*. Briefly, such objects are obtained from elements in $\mathrm{NCC}(n)$ by choosing to mark one edge.

Theorem 1.5 (Theorem 6.6)**.** *The triple*

$$\left(\mathrm{NCC}^B(n+1), \langle \mathrm{twist}^2_{2n} \rangle, \begin{bmatrix} 2n \\ n \end{bmatrix}_q \right)$$

exhibits the cyclic sieving phenomenon.

As in type A, we also obtain a refined cyclic sieving result in Theorem 6.10 where we consider rotation instead.

In Section 7, we briefly consider two-column semistandard Young tableaux, and note in Theorem 7.3 that $(\mathrm{SSYT}(2^k, n), \langle \hat{\partial}_n \rangle, \mathbf{Nar}(n+1, k+1; q))$ is a CSP triple, where $\hat{\partial}_n$ denotes the so-called k-promotion and $\mathrm{SSYT}(2^k, n)$ is the set of semistandard Young tableaux of the rectangular shape 2^k whose maximal entry is at most n.

In Section 8, we refine the classical CSP triple on triangulations of an n-gon by taking *ears* into consideration. An ear in a triangulation is a triangle formed by three cyclically consecutive vertices. We let $\mathrm{TRI}_{\mathrm{ear}}(n, k)$ denote the set of triangulations of an n-gon with k ears.

Theorem 1.6 (Theorem 8.1 and Theorem 8.2)**.** *Let $2 \leqslant k \leqslant \frac{n}{2}$ and let*

$$\mathbf{Tri}(n, k; q) := q^{k(k-2)} \frac{[n]_q}{[k]_q} \begin{bmatrix} n-4 \\ 2k-4 \end{bmatrix}_q \mathbf{Cat}(k-2; q) \left(\sum_{j=0}^{n-2k} q^{j(n-2)} \begin{bmatrix} n-2k \\ j \end{bmatrix}_q \right).$$

Then

$$\sum_k \mathbf{Tri}(n, k; q) = \mathbf{Cat}(n-2; q),$$

and

$$(\mathrm{TRI}_{\mathrm{ear}}(n, k), \langle \mathrm{rot}_n \rangle, \mathbf{Tri}(n, k; q))$$

exhibits the cyclic sieving phenomenon.

In the last section, we consider another natural interpolation between type A and type B Catalan objects and prove a cyclic sieving result using standard methods.

Finally, a word about the proofs in this paper. There are traditionally two different approaches to proving instances of the cyclic sieving phenomenon—combinatorial[1] or representation-theoretical (using vector spaces and diagonalization). In this paper we exclusively use the combinatorial approach, meaning that we need to explicitly evaluate the CSP polynomials at roots of unity and also count the number fixed points of the sets under the group actions. It may also involve the use of equivariant bijections to derive new CSP triples from the previously known ones.

2. Preliminaries

We shall use standard notation in the area of combinatorics, see the common references [Sta01, Mac95]. In particular, $[n] := \{1, 2, \dots, n\}$ and it should not be confused with the q-analog $[n]_q$ defined further down.

[1] Or "brute-force".

2.1. Words and paths

Given a word $w = w_1 \cdots w_n \in [k]^n$, a *descent* is an index $i \in [n-1]$ such that $w_i > w_{i+1}$. We let the *major index*, denoted $\operatorname{maj}(w)$, be the sum of the descents of w. An *inversion* in w is a pair of indices $i, j \in [n]$ such that $i < j$ and $w_i > w_j$. We let $\operatorname{inv}(w)$ be the number of inversions of w. Let $\mathrm{BW}(n,k)$ denote the set of binary words of length n with exactly k ones.

Let $\mathrm{PATH}(n)$ be the set of paths from $(0,0)$ to (n,n) using north, $(1,0)$, and east, $(0,1)$, steps. A *peak* is a north step followed by an east step, and a *valley* is an east step followed by a north step. We have an obvious bijection $\mathrm{PATH}(n) \leftrightarrow \mathrm{BW}(2n,n)$ where we identify north steps with zeros. Given $P \in \mathrm{PATH}(n)$, we let $\operatorname{maj}(P)$ be defined as the sum of the positions of the valleys of the path P. Observe that this coincides with the major index of the corresponding binary word, as valleys correspond to descents. We shall also let $\operatorname{pmaj}(P)$ denote the sum of the positions of the *peaks*. For a path $P \in \mathrm{PATH}(n)$, we let the *depth*, $\operatorname{depth}(P)$ be the largest value of $r \geqslant 0$ such that the path touches the line $y = x - r$. Let us define $\mathrm{PATH}_s(n) \subseteq \mathrm{PATH}(n)$ as the set of paths with $\operatorname{depth}(P) \leqslant s$. We set $\mathrm{DYCK}(n) := \mathrm{PATH}_0(n)$.

2.2. *q*-analogs

Roughly, a q-analog of a certain expression is a rational function in the variable q from which we can obtain the original expression in the limit $q \to 1$.

Definition 2.1. Let $n \in \mathbb{N}$. Define the *q-analog of n* as $[n]_q := 1 + q + \cdots + q^{n-1}$. Furthermore, define the *q-factorial of n* as $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$. Lastly, the *$q$-binomial coefficient* is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}$$

if $n \geqslant k \geqslant 0$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q := 0$ otherwise. The *q-multinomial coefficients* are defined in a similar manner. Note that the q-binomial coefficients are polynomials in the variable q, see [Sta11] for more background. Moreover, they satisfy the following.

Theorem 2.2 (See e.g. [Sta11, Prop. 1.7.1])**.** *For $n, k \in \mathbb{Z}$,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{b \in \mathrm{BW}(n,k)} q^{\operatorname{inv}(b)} = \sum_{b \in \mathrm{BW}(n,k)} q^{\operatorname{maj}(b)}.$$

Theorem 2.3 (**q**-Vandermonde identity)**.** *The* q-Vandermonde identity *states for non-negative integers a, b, c, that*

$$\begin{bmatrix} a+b \\ c \end{bmatrix}_q = \sum_j q^{j(a-c+j)} \begin{bmatrix} a \\ c-j \end{bmatrix}_q \begin{bmatrix} b \\ j \end{bmatrix}_q. \tag{2.1}$$

Theorem 2.4 (**q**-Lucas theorem, see e.g. [Sag92])**.** *Let $n, k \in \mathbb{N}$. Let n_1, n_0, k_1, k_0 be the unique natural numbers satisfying $0 \leqslant n_0, k_0 \leqslant d-1$ and $n = n_1 d + n_0$, $k = k_1 d + k_0$. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \equiv \binom{n_1}{k_1} \begin{bmatrix} n_0 \\ k_0 \end{bmatrix}_q \pmod{\Phi_d(q)}$$

where $\Phi_d(q)$ is the d^{th} cyclotomic polynomial. In particular, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_\xi = \binom{n_1}{k_1} \begin{bmatrix} n_0 \\ k_0 \end{bmatrix}_\xi \tag{2.2}$$

if ξ is a primitive d^{th} root of unity.

When ξ is a root of unity, let $\mathrm{o}(\xi)$ denote the smallest positive integer with the property that $\xi^{\mathrm{o}(\xi)} = 1$. The following is a standard lemma that should not need a proof.

Lemma 2.5. *Let $n, k, d \in \mathbb{N}$ and let ξ be a primitive n^{th} root of unity. Then*

$$\lim_{q \to \xi^d} \frac{[n]_q}{[k]_q} = \begin{cases} n/k & \text{if } \mathrm{o}(\xi^d) \mid k, \\ 0 & \text{otherwise}. \end{cases}$$

We will use Theorem 2.4 and Lemma 2.5 in later sections.

Lemma 2.6. *Let ξ be a primitive n^{th} root of unity, and suppose that $f \in \mathbb{N}[q]$ is such that $f(\xi^j) \in \mathbb{Z}$ for all $j \in \mathbb{Z}$. Then for all $j \in \mathbb{Z}$, $f(\xi^j) = f(\xi^{\gcd(j,n)})$.*

Proof. In [AA19, Lem. 2.2], it is proved that f (up to mod $q^n - 1$) is a linear combination of

$$h_d(q) := \sum_{i=0}^{n/d-1} q^{di} = \frac{[n]_q}{[d]_q} \qquad \text{where } d \mid n.$$

It then suffices to verify that

$$h_d(\xi^j) = h_d(\xi^{\gcd(j,n)}) = \begin{cases} \frac{n}{d} & \text{if } \frac{n}{\gcd(j,n)} \mid d, \\ 0 & \text{otherwise} \end{cases}$$

for all $d \mid n$, $j \in \mathbb{Z}$, which is straightforward by using Lemma 2.5. □

Hence, if we know that $f(\xi^d) \in \mathbb{Z}$ for all $d \in \mathbb{Z}$, it suffices to verify (1.3) for all $d \mid n$. There is a related result about computing the number of fixed points.

Lemma 2.7. *Suppose that $C_n = \langle g \rangle$ acts on the set X. If $d \in \mathbb{Z}$, then*

$$|\{x \in X : g^d \cdot x = x\}| = |\{x \in X : g^{\gcd(n,d)} \cdot x = x\}|$$

Proof. Note that all elements of C_n with order o generate the same subgroup $S \subseteq C_n$. If h, h' are both of order o, then $\langle h \rangle = \langle h' \rangle = S$, and $h \cdot x = x$ implies that $h' \cdot x = h^e \cdot x = x$, for some $e \in \mathbb{Z}$. □

Lemma 2.6 and Lemma 2.7 are useful facts and are used implicitly in many papers. We shall use them without further mention throughout the paper.

2.3. Catalan and Narayana numbers

The *Catalan numbers* $\mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n}$ are indexed by natural numbers. These numbers occur frequently in combinatorics, see A000108 in the OEIS, and give the cardinalities of many families of combinatorial objects. For the purpose of this paper, we note that the following sets all have cardinality $\mathrm{Cat}(n)$.

- $\mathrm{DYCK}(n)$: the set of Dyck paths of size n, that is, the subset of paths in $\mathrm{PATH}(n)$ which never touch the line $y = x - 1$,
- $\mathrm{SYT}(n^2)$: the set of standard Young tableaux with two rows of length n,
- $\mathrm{NCP}(n)$: the set of non-crossing partitions on n vertices,
- $\mathrm{NCM}(n)$: the set of non-crossing matchings on $2n$ vertices,
- $\mathrm{TRI}(n)$: the set of triangulations of an $(n+2)$-gon,
- $\mathrm{NCC}(n)$: the set of non-crossing $(1,2)$-configurations on $n-1$ vertices.
- $\mathrm{SSYT}(2^*, n-1)$: the set of two-column semistandard Young tableaux whose maximal entry is at most $n-1$.

Examples of such objects are listed in Appendix A.

Throughout this paper, we use *MacMahon's q-analog* of the Catalan numbers. For any natural number n, the n^{th} q-Catalan number is defined by

$$\mathbf{Cat}(n;q) := \frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q = \begin{bmatrix}2n\\ n\end{bmatrix}_q - q\begin{bmatrix}2n\\ n-1\end{bmatrix}_q \tag{2.3}$$

$$= \sum_{P\in\mathrm{DYCK}(n)} q^{\mathrm{maj}(P)} = \sum_{T\in\mathrm{SYT}(n^2)} q^{\mathrm{maj}(T)-n}. \tag{2.4}$$

A definition of maj on standard Young tableaux can be found in the next section. The *Narayana numbers* $\mathrm{Nar}(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, indexed by two natural numbers n and k such that $1 \leqslant k \leqslant n$, are also well-known and have many applications, see the OEIS entry A001263. The Narayana numbers *refine* the Catalan numbers in the sense that $\sum_k \mathrm{Nar}(n,k) = \mathrm{Cat}(n)$. For our purposes, it suffices to know that the following sets all have cardinality $\mathrm{Nar}(n,k)$.

- $\mathrm{DYCK}(n,k)$: the set of paths in $\mathrm{DYCK}(n)$ with exactly k peaks,
- $\mathrm{SYT}(n^2,k)$: the set of tableaux in $\mathrm{SYT}(n^2)$ with exactly k descents,
- $\mathrm{NCP}(n,k)$: the set of partitions in $\mathrm{NCP}(n)$ with exactly k blocks,
- $\mathrm{NCC}(n,k)$: the set of non-crossing $(1,2)$-configurations in $\mathrm{NCC}(n)$ such that the numbers of proper edges plus the number of loops is equal to $k-1$,
- $\mathrm{NCM}(n,k-1)$: the set of non-crossing matchings in $\mathrm{NCM}(n)$ with $k-1$ even edges.

- $\mathrm{SSYT}(2^{k-1}, n-1)$: the set of two-column tableaux in $\mathrm{SSYT}(2^*, n-1)$ with exactly $k-1$ rows.

The *q-Narayana numbers* are defined as the q-analog

$$\begin{aligned}\mathbf{Nar}(n,k;q) \coloneqq \frac{q^{k(k-1)}}{[n]_q}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\k-1\end{bmatrix}_q &= \sum_{P\in\mathrm{DYCK}(n,k)} q^{\mathrm{maj}(P)}\\ &= \sum_{T\in\mathrm{SYT}(n^2,k)} q^{\mathrm{maj}(T)-n}.\end{aligned}$$

The q-Narayana numbers refine the q-Catalan numbers, that is, $\sum_k \mathbf{Nar}(n,k;q)$ is equal to $\mathbf{Cat}(n;q)$. We also mention that there is a bijection $\mathtt{NCPtoDYCK}$ from $\mathrm{NCP}(n,k)$ to $\mathrm{DYCK}(n,k)$ described in Bijection 8. Thus,

$$\mathbf{Nar}(n,k;q) = \sum_{\pi\in\mathrm{NCP}(n,k)} q^{\mathrm{maj}(\mathtt{NCPtoDYCK}(\pi))}. \tag{2.5}$$

For more background, see [Sim94] and [ZZ11].

2.4. Type B Catalan numbers

We shall now describe the type B analogs of the combinatorial objects we saw in the previous section. The *type B Catalan numbers* $\mathrm{Cat}^B(n)$ are defined as

$$\mathrm{Cat}^B(n) \coloneqq \binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{k}. \tag{2.6}$$

The *type B Narayana numbers* $\mathrm{Nar}^B(n,k)$ are defined as

$$\mathrm{Nar}^B(n,k) \coloneqq \binom{n}{k}^2. \tag{2.7}$$

The type B Narayana numbers clearly refine the B Catalan numbers, as can be seen from (2.6). Among other things, they count the number of elements in $\mathrm{PATH}(n)$ with k valleys. For a more comprehensive list, see A008459 in the OEIS, and also the reference [Arm09] for more background. The *q-analogs of the type B Catalan numbers* and the *type B q-Narayana numbers* are defined as

$$\mathbf{Cat}^B(n;q) \coloneqq \begin{bmatrix}2n\\n\end{bmatrix}_q, \qquad \mathbf{Nar}^B(n,k;q) \coloneqq q^{k^2}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q. \tag{2.8}$$

It is straightforward to verify that $\mathbf{Cat}^B(n;q) = \sum_{k=0}^{n}\mathbf{Nar}^B(n,k;q)$. Moreover, one can show that

$$\sum_{k=0}^{n} t^k\mathbf{Nar}^B(n,k;q) = \sum_{T\in\mathrm{SYT}((2n,n)/(n))} t^{|\mathrm{Des}(T)|}q^{\mathrm{maj}(T)} \tag{2.9}$$

$$= \sum_{P\in\mathrm{PATH}(n)} t^{\mathrm{valleys}(P)}q^{\mathrm{maj}(P)}, \tag{2.10}$$

see [Sul98, Sul02].

The following combinatorial interpretation of the type B q-Narayana numbers is mentioned in I. Macdonald's book [Mac95, p. 400]. Let V be a $2n$-dimensional vector space over F_q, and let U be an n-dimensional subspace of V. Then $\mathbf{Nar}^B(n,k;q)$ is the number of n-dimensional subspaces U' of V such that $\dim(U \cap U') = n - k$.

2.5. Overview of the CSP on Catalan and Narayana objects

Table 2.1 and Table 2.2 list the state-of-the-art of the CSP on Catalan-type objects of type A and B respectively, including the results proven in the present paper. Examples of such objects can be found in Appendix A. We use several bijections (described in Appendix B) between Catalan and Narayana objects, see Figure 2.1.

$$\begin{tikzcd}
\mathrm{SSYT}(2^{k-1}, n-1) \arrow[r, "\text{Bijection 3}"] & \mathrm{NCP}(n,k) \arrow[d, "\text{Bijection 7}"] \arrow[dr, bend left, "\text{Bijection 8}"] & \\
\mathrm{SYT}(n^2,k) \arrow[rr, bend right, "\text{Bijection 4} \circ \text{Bijection 6}"'] & \mathrm{NCM}(n,k-1) & \mathrm{DYCK}(n,k) \arrow[d, "\text{Bijection 2}"] \\
 & & \mathrm{NCC}(n,k)
\end{tikzcd}$$

Figure 2.1: Schematic overview of the Narayana zoo. Note that we have bijections from $\mathrm{SYT}(n^2) \to \mathrm{NCM}(n)$ (Bijection 6) and $\mathrm{NCM}(n) \to \mathrm{DYCK}(n)$ (Bijection 4), but they do not respect the particular Narayana refinements.

The bijections in Figure 2.1 respect a Narayana refinement and so, for example, $\mathrm{SYT}(n^2,k)$, $\mathrm{NCM}(n,k-1)$ and $\mathrm{NCP}(n,k)$ are all equinumerous. Furthermore, composing the natural bijections Bijection 6 and the inverse of Bijection 7, we get that promotion on SYT corresponds to rotation on non-crossing matchings.

However, promotion on standard Young tableaux does not preserve the number of descents, but rotation preserves the number of even edges of matchings. It follows that the cyclic sieving phenomenon on non-crossing matchings with a specified number of even edges does not correspond to one on $\mathrm{SYT}(n^2)$ with a fixed number of descents with promotion as the action. In general, *a specific Narayana refinement might be incompatible with a cyclic group action*. By Bijection 6, here the compatible statistic one should use for $\mathrm{SYT}(n^2)$ is the number of even entries in the first row.

A note on the general philosophy of the paper. Having many different sets of objects and well-behaved bijections between these sets turns out to be a very fruitful approach to proving instances

Type A set & reference	**Group & polynomial**
Triangulations of an n-gon	Rotation rot_n
[RSW04, Thm. 7.1]	$\mathbf{Cat}(n-2;q)$
Triangulations of an n-gon with k ears	Rotation rot_n
Theorem 8.2	Complicated
Two-row standard Young tableaux	Promotion ∂_{2n}
[Rho10, Thm. 1.3]	$\mathbf{Cat}(n;q)$
Non-crossing matchings	Rotation rot_{2n}
See [Hei07, Thm. 5] and [Rho10, Thm. 8.3].	$\mathbf{Cat}(n;q)$
Non-crossing partitions	Kreweras compl., krew_{2n}
[Hei07, Thm. 1]	$\mathbf{Cat}(n;q)$
Non-crossing matchings with k short edges	Rotation rot_{2n}
Theorem 4.8	$\frac{q^{k(k-2)}(1+q^n)}{[n+1]_q}\begin{bmatrix}n+1\\k\end{bmatrix}_q\begin{bmatrix}n-2\\k-2\end{bmatrix}_q$
Two-row SYT with k cyclic descents	Promotion ∂_{2n}
Theorem 4.8	$\frac{q^{k(k-2)}(1+q^n)}{[n+1]_q}\begin{bmatrix}n+1\\k\end{bmatrix}_q\begin{bmatrix}n-2\\k-2\end{bmatrix}_q$
Non-crossing partitions with k parts	Rotation rot_n
[RSW04, Thm. 7.2]	$\mathbf{Nar}(n,k;q)$
Non-crossing matchings with k even edges	Rotation rot_n
Proposition 4.1	$\mathbf{Nar}(n,k;q)$
Non-cross. (1,2)-config.	Rotation rot_n
[Thi17]	$\mathbf{Cat}(n+1;q)$
Non-cross. (1,2)-config. with l loops and e edges	Rotation rot_n
Theorem 5.9	$\frac{q^{e(e+1)+(n+1)l}}{[e+1]_q}\begin{bmatrix}n\\e,e,l,n-2e-l\end{bmatrix}_q$
Non-cross. (1,2)-config. with k edges or loops	Rotation rot_n
Corollary 5.10	$\mathbf{Nar}(n+1,k;q)$
Two-column SSYT, $\mathrm{SSYT}(2^k,n)$	k-promotion $\hat{\partial}_n$
Theorem 7.3	$\mathbf{Nar}(n+1,k+1;q)$.
Non-cross. (1,2)-config.	Twisted rotation twist_{2n}
Theorem 5.4	$\begin{bmatrix}2n\\n\end{bmatrix}_q - q^2\begin{bmatrix}2n\\n-2\end{bmatrix}_q$

Table 2.1: The current state-of-the-art regarding cyclic sieving on type A Catalan objects, including the new results presented in this article. The four major blocks concern a group action of size n or $2n$, and q-analogs of $\mathrm{Cat}(n-2)$, $\mathrm{Cat}(n)$, $\mathrm{Cat}(n+1)$ and $\mathrm{Cat}(n)$, respectively. The expression $\begin{bmatrix}2n\\n\end{bmatrix}_q - q^2\begin{bmatrix}2n\\n-2\end{bmatrix}_q$ is a new q-analog of Catalan numbers in the context of the CSP. The additional partitioning within blocks represents a more subjective logical grouping.

Type B set & reference	**Group & polynomial**
Binary words $\mathrm{BW}(2n, n)$ [RSW04, Prop. 4.4]	Cyclic shift shift_{2n} $\mathbf{Cat}^B(n;q)$
Skew two-row SYT, $\mathrm{SYT}((2n,n)/(n))$ [SW12, Section 3.1]	Promotion ∂_{2n} $\mathbf{Cat}^B(n;q)$
Type B root poset order ideals $\mathrm{OI}_B(n)$ [AST13, Thm. 1.5] and [SW12]	Rowmotion, ρ_{2n} $\mathbf{Cat}^B(n;q)$
Binary words $\mathrm{BW}(2n, n)$ with k cyclic descents Proposition 6.4, [AS18, Thm. 1.5]	Cyclic shift shift_{2n} $q^{k(k-1)}(1+q^n)\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$
Type B non-crossing partitions $\mathrm{NCP}^B(n)$ [BR11, Thm. 1.1], see also Proposition 6.1	Rotation rotB_n $\mathbf{Cat}^B(n;q)$
Marked $(1,2)$-configs. Theorem 6.6	Twisted rotation twist^2_{2n} $\mathbf{Cat}^B(n;q)$
$\mathrm{NCP}^B(n)$ with $2k$ or $2k+1$ blocks (6.2) in Proposition 6.2	Rotation rotB_n $\mathbf{Nar}^B(n,k;q)$
Skew two-column SSYT, $\mathrm{SSYT}(2^k1^k/1^k, n)$ Theorem 7.5	k-promotion $\hat{\partial}_n$ $q^{k(k+1)}\begin{bmatrix} n \\ k \end{bmatrix}_q^2$
Marked $(1,2)$-configs. with e edges and l loops Theorem 6.10	Rotation rot_n Complicated
Marked $(1,2)$-configs. with k edges and loops Corollary 6.11	Rotation rot_n Complicated
Type B triangulations on $2n+2$ vertices Proposition 8.5	Rotation rot_{n+1} $\mathbf{Cat}^B(n;q)$

Table 2.2: The current state-of-the-art regarding cyclic sieving on type B Catalan objects, including the new results presented in this article. The three major groupings concern group actions of order $2n$, n and $n+1$, respectively. The additional partitioning within blocks is to distinguish different refinements and q-analogues.

of the CSP. In this context, well-behaved often times means that the bijection is equivariant. If a group action looks complicated on a certain set, it can perhaps be made easier if one first applies an equivariant bijection and then studies the image. For example, promotion on $\mathrm{SYT}(n^2)$ is complicated while rotation on $\mathrm{NCM}(n)$ is easier. There is a type of converse of the above. If one has two different CSP triples with identical CSP polynomials and whose cyclic groups have the same order, then there exists an equivariant bijection between these two sets (by sending orbits to orbits of the same size).

Remark 2.8. Note that there is a natural s-interpolation also between type A and type B Fuß–Catalan numbers. Recall that the type A and B Fuß–Catalan numbers are

$$\frac{1}{kn+1}\binom{kn+1}{n} \quad \text{and} \quad \binom{kn}{n},$$

respectively, where we recover the classical Catalan numbers when $k = 2$. P. Drube [Dru18] proves that the type A Fuß–Catalan numbers are realized as the number of *set-valued standard Young tableaux* of shape (n, n) where the boxes in the top row are occupied by singleton sets of numbers, while each box in the bottom row is filled with a set of size $k-1$. Although Drube does not consider skew shapes in his article, it is easy to see that the number of ways to fill the skew shape $(2n, n)/(n)$ with the same constraints is exactly $\binom{kn}{n}$. Hence, by considering set-valued SYTs of shape $(n+s, n)/(s)$, with the above constraints on the number of elements in each box, we get a natural s-interpolation between Fuß–Catalan numbers of type A ($s = 0$) and type B ($s = n$). This is an interesting generalisation that we do not investigate in this paper. Note that there exist CSP also for Fuß-Catalan numbers, see [KM13].

3. Type A/B-Narayana numbers and a quest for a q-analog

We shall now discuss a natural interpolation between type A and type B Catalan numbers. The following observation illustrates this interpolation. For any $s \geqslant 0$, the sets below are equinumerous:

1. the set of skew standard Young tableaux $\mathrm{SYT}((n+s, n)/(s))$,
2. the set of lattice paths, $\mathrm{PATH}_s(n)$,
3. the set of order ideals in the type B root poset with at most s elements on the top diagonal.

Note that for $s = 0$, we recover sets of cardinality $\mathrm{Cat}(n)$, and for $s = n$, we recover sets of cardinality $\mathrm{Cat}^B(n)$. Bijective arguments are given below in Proposition 3.1 and Proposition 3.4.

Let $T \in \mathrm{SYT}(\lambda/\mu)$ where the diagram of λ/μ has n boxes. A *descent* of T is an integer $j \in \{1, \dots, n-1\}$ such that $j+1$ appears in a row below j. The *major index* of T is the sum of the descents. The major-index generating function for skew standard Young tableaux is defined as

$$f^{\lambda/\mu}(q) := \sum_{T \in \mathrm{SYT}(\lambda/\mu)} q^{\mathrm{maj}(T)} \tag{3.1}$$

when λ/μ is a skew shape. Our motivation for studying this polynomial is [APRU21, Thm. 46] which states that for any skew shape λ/μ where each row contains a multiple of m boxes, there must exist some cyclic group action C_m of order m such that

$$\big(\mathrm{SYT}(\lambda/\mu), C_m, f^{\lambda/\mu}(q)\big) \tag{3.2}$$

is a CSP triple. We do not know how such a group action looks like except in the case $m = 2$. In that case one can use *evacuation*, defined by Schützenberger [Sch63].

3.1. Skew standard Young tableaux with two rows

We now describe a bijection between skew SYT with two rows and certain lattice paths.

Proposition 3.1. *Given $s \in \{0, \dots, n\}$, there is a bijection*

$$\mathrm{SYT}((n+s, n)/(s)) \longrightarrow \mathrm{PATH}_s(n)$$

which sends descents in the tableau to peaks in the path.

Proof. A natural generalization of the standard bijection works: an i in the upper or lower row corresponds to the i^{th} step in the path being north or east, respectively. Evidently, a descent in the tableau is sent to a peak in the path. □

Recall that for a SYT T the statistic $\mathrm{maj}(T)$ is the sum of the position of the descents, which is then sent to $\mathrm{pmaj}(P)$ which is the sum of the positions of the peaks in the corresponding path P. Let

$$X_{n,s}(q) := \sum_{P \in \mathrm{PATH}_s(n)} q^{\mathrm{pmaj}(P)} \qquad \text{and} \qquad Y_{n,s}(q) := \sum_{P \in \mathrm{PATH}_s(n)} q^{\mathrm{maj}(P)}.$$

By Proposition 3.1 we also have $X_{n,s}(q) = f^{(n+s,n)/(s)}(q)$.

Proposition 3.2 ([Kra89, Thm. 7])**.** *For $n \geqslant 1$ and $n \geqslant s \geqslant 0$,*

$$X_{n,s}(q) = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - \begin{bmatrix} 2n \\ n-s-1 \end{bmatrix}_q \text{ and } Y_{n,s}(q) = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^{s+1} \begin{bmatrix} 2n \\ n-s-1 \end{bmatrix}_q. \tag{3.3}$$

In particular,

$$q^{-n} X_{n,0}(q) = Y_{n,0}(q) = \mathbf{Cat}(n;q) \text{ and } X_{n,n}(q) = Y_{n,n}(q) = \mathbf{Cat}^B(n;q).$$

In light of (3.2), it would be of great interest to explicitly describe a group action C_n so that $X_{n,s}(q)$ or $Y_{n,s}(q)$ is the corresponding polynomial in a CSP triple. We know that such an action exists, as mentioned earlier in (3.2). In Section 5 we give an explicit action in the case $s = 1$, which gives a new CSP triple involving the Catalan numbers. The values of $Y_{n,s}(q)$ at roots of unity are given in the next lemma.

Lemma 3.3. *Let ξ be a primitive $(2n)^{\text{th}}$ root of unity. For all integers $n > s \geqslant 0$ and $d \mid 2n$, we have*

$$Y_{n,s}\left(\xi^d\right) = \chi(d,2)\binom{d}{\frac{d}{2}} - (-1)^d \chi(n-s-1, 2n/d)\binom{d}{\frac{d(n-s-1)}{2n}}.$$

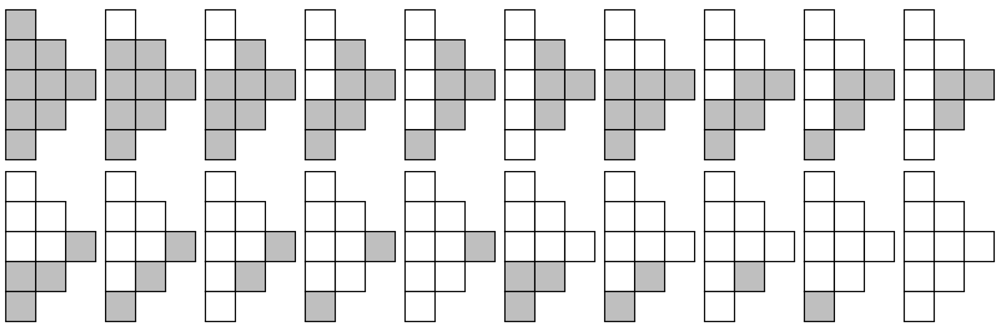
where $\chi(a,b)$ is equal to 1 if b divides a and 0 otherwise.

Proof. The evaluation follows from the q-Lucas theorem, Theorem 2.4. □

It follows from Lemma 3.3 that the values of $\mathbf{Cat}(n;q)$ when evaluated at a $(2n)^{\text{th}}$ root of unity are non-negative integers, and thus also for n^{th} roots of unity. Similarly, one can say use the q-Lucas theorem to prove the same thing for an $(n-1)^{\text{th}}$ and an $(n+2)^{\text{th}}$ root of unity. In Table 2.1, we see that there are CSPs for these four orders, with the q-Catalan numbers as the corresponding CSP polynomials. In contrast, it can be shown that $\mathbf{Nar}(n,k;q)$ evaluates to non-negative integer values at $(n-1)^{\text{th}}$ and n^{th} roots of unity but not at $(n+2)^{\text{th}}$ nor at $(2n)^{\text{th}}$ roots of unity. As an explicit counter-example, consider $\mathbf{Nar}(2,2;q) = q^2$.

3.2. Root lattices in type A/B

The following illustrate the root ideals of B_n where $n = 3$. There are in total $\binom{2\cdot 3}{3} = 20$ such ideals. A root ideal is simply a lower set in the root poset—marked as shaded boxes in the diagrams below. Root ideals are also called *non-nesting partitions of type W*, where W is the Weyl group of some root system.



The *top diagonal* in such a diagram is the set of boxes that neither have a box to the right nor above them. An explicit bijection from the set of skew standard Young tableaux $\mathrm{SYT}((n+s,n)/(s))$ to the root ideals of B_n with at most s elements on the top diagonal is described below. First, let $\mathrm{OI}(n,s)$ be the set of root ideals with at most s elements on the top diagonal.

Bijection 1. Let $a_1, a_2, \dots, a_n$ be on the top row of the skew tableau. We identify this top row using the bijection in Proposition 3.1 with a path $\alpha \in \mathrm{PATH}_s(n)$ and get that $\mathrm{depth}(\alpha) = \max_i\{a_i - 2i + 1\}$. Let j be the smallest value for which the maximum is obtained, so $\mathrm{depth}(\alpha) = a_j - 2j + 1$. We then define the map ϕ by changing the step $a_j - 1$, just before reaching maximal depth for the first time from an east step to a north step. That is, $\phi(\alpha) = a_1, \dots, a_{j-1}, a_j - 1, a_j, \dots, a_n$. This new path ends at $(n-1, n+1)$ and has depth one less than α. We repeat $\mathrm{depth}(\alpha)$ times and get $\phi^{\mathrm{depth}(\alpha)}(\alpha)$ which ends in $(n - \mathrm{depth}(\alpha), n + \mathrm{depth}(\alpha))$ and has depth zero. This path always starts with a north step, and the boxes below will make up a root ideal o in B_n with $\mathrm{depth}(\alpha)$ elements in the top diagonal. Since $\mathrm{depth}(\alpha) \leqslant s$ this gives $o \in \mathrm{OI}(n, s)$ and the desired map. See Figure 3.1 for an example. The inverse ϕ^{-1} is easily obtained by given a root ideal o, letting $\beta(o)$ be the north-east path along its boundary, starting with an extra north step. Now, we change the north step of $\beta(o)$ after the last time the path has reached maximum depth to an east step. The map ϕ has been used many times before, see e.g. [FH85, ALP19]. The inverse of the bijection is obtained by iterating ϕ^{-1} until the path ends in (n, n).

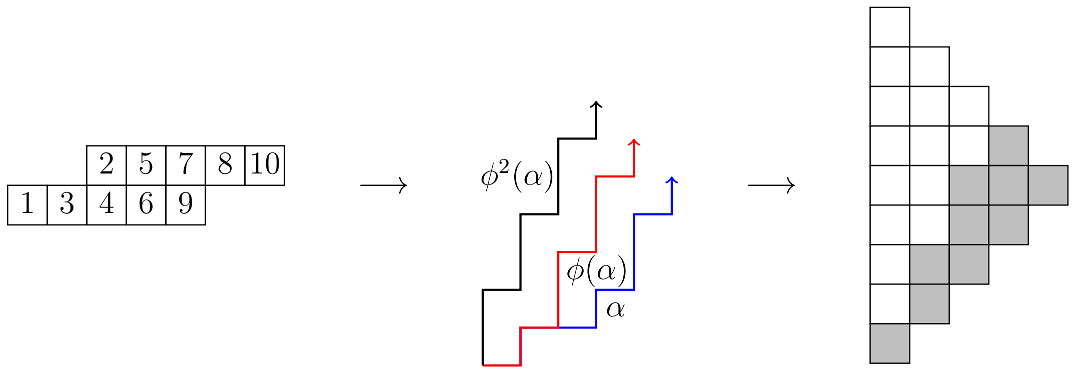


Figure 3.1: An example with: $\phi^2(2, 5, 7, 8, 10) = \phi(2, 4, 5, 7, 8, 10) = 1, 2, 4, 5, 7, 8, 10$.

We naturally define $\mathrm{maj}(o) := \mathrm{maj}(\beta(o))$ and $\mathrm{pmaj}(o) := \mathrm{pmaj}(\beta(o))$ for $o \in \mathrm{OI}(n, s)$.

Proposition 3.4. *The map in Bijection 1 is a bijection so*

$$|\mathrm{OI}(n, s)| = \binom{2n}{n} - \binom{2n}{n-1-s}.$$

Furthermore, we get the following q-polynomials

$$\sum_{o\in \mathrm{OI}(n,s)} q^{\mathrm{pmaj}(o)} = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - \begin{bmatrix} 2n \\ n-s-1 \end{bmatrix}_q,$$

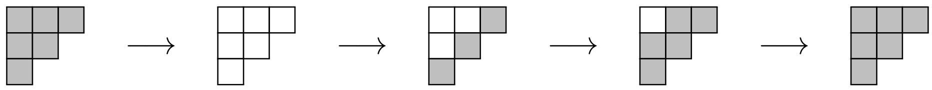
$$\sum_{o\in \mathrm{OI}(n,s)} q^{\mathrm{maj}(o)} = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q \begin{bmatrix} 2n \\ n-s-1 \end{bmatrix}_q + \sum_{d=1}^{s} (1-q) \begin{bmatrix} 2n \\ n-d \end{bmatrix}_q.$$

Proof. The map is clearly a bijection and the first formula follows. For the second statement note that the map ϕ does not change the peaks and thus not pmaj, but it changes the position of one valley and decreases maj by 1. Thus the q-polynomial for pmaj is identical to $X_{n,s}(q)$ in Proposition 3.2. For maj we know from Proposition 3.2 that paths with depth at most s are recorded by $Y_{n,s}(q)$. Thus the paths having depth exactly d is $q^d \begin{bmatrix} 2n \\ n-d \end{bmatrix}_q - q^{d+1} \begin{bmatrix} 2n \\ n-d-1 \end{bmatrix}_q$. A path with depth d is mapped by ϕ^d to a root ideal with exactly d elements in the top diagonal. This gives the sum

$$\sum_{o\in \mathrm{OI}(n,s)} q^{\mathrm{maj}(o)} = \sum_{d=0}^{s} \left(q^{d} \begin{bmatrix} 2n \\ n-d \end{bmatrix}_q - q^{d+1} \begin{bmatrix} 2n \\ n-d-1 \end{bmatrix}_q \right) q^{-d},$$

which simplifies to the formula given. □

Remark 3.5. There is a notion of rowmotion as an action on order ideals. Unfortunately, this action does not have the order we are looking for when restricted to $\mathrm{OI}(n,s)$, see [SW12]. For example, for $n = 3$ and $s = 1$ we have the following orbit of length 4, implying that the action does not have the order we would like (which is $n = 3$).



3.3. Narayana connection

We discuss an open problem regarding the interpolation between type A and type B q-Narayana numbers. This problem is part of a broader set of questions regarding the interplay of cyclic sieving and characters in the symmetric group, see [APRU21]. We argue that the small special case discussed below is interesting in its own right.

Recall that

$$\sum_{D\in \mathrm{DYCK}(n)} q^{\mathrm{maj}(D)} t^{\mathrm{peaks}(D)} = \sum_{k=1}^{n} t^{k} \mathbf{Nar}(n,k;q),$$

where the sum ranges over Dyck paths of size n. Hence, $\sum_{k=1}^{n} \mathbf{Nar}(n,k;q) = Y_{n,0}(q)$. Note that the set of Dyck paths with k peaks is in bijection with the set of non-crossing set partitions of $[n]$ with k blocks.

Problem 3.6 (Main Narayana problem)**.** Refine the expression

$$Y_{n,s}(q) = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^{s+1} \begin{bmatrix} 2n \\ n-s-1 \end{bmatrix}_q$$

for all $s \geqslant 0$ in the same way as the *q-Narayana numbers* $\mathbf{Nar}(n,k;q)$ refine the case $s = 0$.

This problem is not really interesting unless we impose some additional requirements. In Problem 3.6, we are hoping to find a family of polynomials, $N(s,n,k;q) \in \mathbb{N}[q]$ with some of the following properties:

Specializes to $\mathbf{Nar}(n,k;q)$: For $s = 0$, we have

$$N(0,n,k;q) = \mathbf{Nar}(n,k;q).$$

Refines the $Y_{n,s}(q)$ in (3.3): We want that for all $s \geqslant 0$, we have the identity

$$\sum_{k=1}^{n} N(s,n,k;q) = \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^{s+1} \begin{bmatrix} 2n \\ n-1-s \end{bmatrix}_q .$$

Is given by some generalization of the the peak statistic: We hope for some statistic $\mathrm{peaks}_s(P)$ such that $\mathrm{peaks}_0(P)$ is the usual number of peaks of a Dyck path and

$$\sum_{P\in \mathrm{PATH}_s(n)} q^{\mathrm{maj}(P)} t^{\mathrm{peaks}_s(P)} = \sum_{k=1}^{n} t^k N(s,n,k;q). \tag{3.4}$$

We can alternatively consider some other family of combinatorial objects mentioned in (3), such as type B root ideals with at most s elements on the top diagonal, or standard Young tableaux in $\mathrm{SYT}((n+s,n)/(s))$ with some type of generalized descents.

Refines $\mathbf{Cat}^B(n)$ at $s = n$: For $s = n$, we have a natural candidate

$$N(n,n,k;q) = q^{k(k-1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = [n+1]_q \mathbf{Nar}(n,k;q). \tag{3.5}$$

Note that $N(n,n,k;q)$ is not equal to $\mathbf{Nar}^B(n,k;q)$ that appear in Section 2.4. The combinatorial interpretation in this case is as follows:

$$\sum_{P\in \mathrm{PATH}(n)} q^{\mathrm{maj}(P)} t^{\mathrm{modpeaks}(P)} = \sum_{k=1}^{n} t^k q^{k(k-1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k \end{bmatrix}_q$$

where a *modified peak* is any occurrence of 01 (north-east) in the path, plus 1 if the path ends with a north step.

Palindromicity: The Narayana numbers have quite nice properties. First of all,

$$\sum_{k=1}^{n} t^k \mathrm{Nar}(n,k)$$

is a *palindromic* polynomial (in t). For example, for $n = 5$, this sum is $t+10t^2+20t^3+10t^4+t^5$. One would therefore hope that for fixed s the sum $\sum_{k=1}^{n} t^k N(s,n,k;q)$ is palindromic. The $s = n$ candidate given by $[n+1]_q \mathbf{Nar}(n,k;q)$ is also palindromic.

Palindromicity II: Each $N(s,n,k;q)$ is a palindromic polynomial (in q). This is true for $\mathbf{Nar}(n,k;q)$ and the expression in (3.5).

Gamma-positivity: The sum $\sum_{k=1}^{n} t^k N(s,n,k;0)$ is $\gamma(t)$-positive (see the survey [Ath18] for the definition). The corresponding statement seems to hold for the expression in (3.5). One might hope that the general expression $\sum_{k=1}^{n} t^k N(s,n,k;0)$ also has $\gamma(t)$-positivity.

Values at roots of unity and cyclic sieving: We require that $N(s,n,k;\xi)$ is a non-negative integer whenever ξ is an n^{th} (or better, $(2n)^{\text{th}}$) root of unity. This resonates well with the palindromicity properties, and cyclic sieving for (3.2). Taking (3.4) into account, we would like that for every $k \geqslant 0$,

$$(\{P \in \mathrm{PATH}_s(n) : \mathrm{peaks}_s(P) = k\}, \langle \beta_n \rangle, N(s,n,k;q))$$

is a CSP triple for some action β_n of order n. Note that such a refinement is known in the case $s = 0$, as shown in the table below.

Set	Group action	q-statistic	peak-statistic
Dyck paths	—	maj	peaks
Non-crossing partitions†	Rotation	maj	blocks
$\mathrm{SYT}(n^2)$	∂_{2n}^2	maj	—
Non-crossing matchings†	Rotation	maj	—

Table 3.1: We only have the full Narayana refinement picture for the non-crossing partition family. That is, there is a "peak"-statistic and a group action of order n preserving the peak-statistic. Note that promotion on Dyck paths does not preserve the number of peaks. †For these sets maj is computed via a bijection to paths.

Example 3.7. For $n = 2, s = 1$, we have that $Y_{2,1}(q) = q^4 + q^3 + q^2 + q + 1$. We want to refine this into two polynomials corresponding to $k = 1, 2$. The criteria to have non-negative evaluations at roots at unity, here -1, tells us that q^3 and q must be together with at least one other term each. By palindromicity II there are five possibilities for $N(1, 2, 2; q)$: q^4+q^3+q+1, $q^4 + q^3 + q^2 + q$, $q^4 + q^3 + q^2$, $q^4 + q^3$ and q^4.

4. Case $s = 0$ and non-crossing matchings

The goal of this section is to prove two Narayana-refinements of cyclic sieving on non-crossing perfect matchings by considering the number of even edges and short edges. The second result corresponds to a refinement of the CSP on $\mathrm{SYT}(n^2)$ under promotion, where we refine the set by the number of cyclic descents.

4.1. Even edge refinement

Given a non-crossing perfect matching, let $\mathrm{even}(M)$ denote the number of edges $\{i, j\}$ where $i < j$ and i is even. We refer to them as *even edges*, and all non-even edges are called *odd*. Let $\mathrm{NCM}(n, k)$ be the set of $M \in \mathrm{NCM}(n)$ such that $\mathrm{even}(M) = k$.

Note that for parity reasons an edge $\{i, j\}$ must have $i + j$ odd. Thus the set of non-crossing perfect matchings on $2n$ vertices with k even edges is invariant under rotation by rot_n since

- any odd edge $(i, 2n)$ is mapped to the even edge $(2, i + 2)$;
- any even edge $(j, 2n - 1)$ is mapped to the odd edge $(1, j + 2)$.

The first result is essentially just a restatement of [RSW04, Thm. 7.2].

Proposition 4.1. *For $0 \leqslant k \leqslant n$, the triple*

$$(\mathrm{NCM}(n, k), \mathrm{rot}_n, \mathbf{Nar}(n, k + 1; q)) \tag{4.1}$$

exhibits the cyclic sieving phenomenon.

Proof. Mapping non-crossing matchings to non-crossing partitions via the inverse of `NCPtoNCM` takes matchings with k even edges to partitions with $k + 1$ blocks, see Bijection 7. This CSP result was noted already in [RSW04, Thm. 7.2]. □

4.2. Short edge refinement

Definition 4.2. We define *promotion* $\partial_{2n} : \mathrm{SYT}(n^2) \to \mathrm{SYT}(n^2)$ as the following composition of bijections:

$$\partial_{2n} := \mathtt{SYTtoNCM}^{-1} \circ \mathrm{rot}_{2n} \circ \mathtt{SYTtoNCM},$$

where SYTtoNCM is the standard Bijection 6 in Appendix B.1. If $T \in \mathrm{SYT}(n^2)$, we use the shorthand $\partial_{2n}T$ to mean $\partial_{2n}(T)$.

Promotion is originally defined for Young tableaux of all shapes using the so-called jeu-de-taquin, see Appendix C. The notion has been generalized to arbitrary posets by R. Stanley, see [Sta09].

Definition 4.3. Let $T \in \mathrm{SYT}(n^2)$. Define the *cyclic descent set* $\mathrm{cDes}(T)$ as follows. We have $\mathrm{Des}(T) = \mathrm{cDes}(T) \cap [1, 2n-1]$ and let $2n \in \mathrm{cDes}(T)$ if and only if $1 \in \mathrm{cDes}(\partial_{2n}T)$. Denote the number of *cyclic descents* of T by $\mathrm{cdes}(T) := |\mathrm{cDes}(T)|$ and denote $\mathrm{SYT}_{\mathrm{cdes}}(n^2, k)$ the set of $T \in \mathrm{SYT}(n^2)$ such that $\mathrm{cdes}(T) = k$.

The above definition of cyclic descent set can be generalized in a straightforward manner to all rectangular standard Young tableaux—that is, tableaux of shape $\lambda = (a^b)$. In [Hua20], an explicit construction is given, where it is shown that all shapes which are not connected ribbons admit a type of cyclic descent statistic. It follows that one can define the set $\mathrm{SYT}_{\mathrm{cdes}}(\lambda, k)$ for all such shapes λ as well.

The set $\mathrm{SYT}_{\mathrm{cdes}}(n^2, k)$ is in bijection with a certain subset of $\mathrm{DYCK}(n)$ which we shall now describe. We first recall the standard bijection SYTtoDYCK between $\mathrm{SYT}(n^2)$ and $\mathrm{DYCK}(n)$: given a $T \in \mathrm{SYT}(n^2)$, let $\mathtt{SYTtoDYCK}(T) = \mathsf{w}_1\mathsf{w}_2\cdots\mathsf{w}_{2n}$ be the Dyck path where $\mathsf{w}_i = 0$ if i is in the top row and $\mathsf{w}_i = 1$ otherwise.

Call a Dyck path $\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3\cdots\mathsf{w}_{2n}$ *elevated* if $\mathsf{w}_2\mathsf{w}_3\cdots\mathsf{w}_{2n-1}$ is also a Dyck path. A Dyck path which is not elevated is called *non-elevated*. Elevated Dyck paths of size n are in natural bijection with Dyck paths of size $n-1$. The next lemma now easily follows: the "if" direction from the fact that vertex 1 in $\mathtt{SYTtoNCM}(T)$ cannot then be matched to any other vertex than $2n$, and the "only if" direction since vertex 2 must be a start vertex and $2n-1$ an end vertex if $\{1, 2n\}$ is an edge.

Lemma 4.4. *Let $T \in \mathrm{SYT}_{\mathrm{cdes}}(n^2, k)$. Then $2n \in \mathrm{cDes}(T)$ if and only if $\mathtt{SYTtoDYCK}(T)$ is elevated.*

It follows from Lemma 4.4 that the restriction of SYTtoDYCK to $\mathrm{SYT}_{\mathrm{cdes}}(n^2, k)$ is a bijection to the set of $D \in \mathrm{DYCK}(n)$ such that D either is non-elevated and has k peaks or is elevated and has $k-1$ peaks. Hence, we get (using an argument by T. Došlić [Doš10, Prop. 2.1]),

$$\begin{aligned} |\mathrm{SYT}_{\mathrm{cdes}}(n^2, k)| &= \mathrm{Nar}(n, k) - \mathrm{Nar}(n-1, k) + \mathrm{Nar}(n-1, k-1) \\ &= \frac{2}{n+1}\binom{n+1}{k}\binom{n-2}{k-2}. \end{aligned} \tag{4.2}$$

These numbers are a shifted variant of the OEIS entry A108838. Define the following q-analog of these numbers. For any two natural numbers n and k, let

$$\mathbf{Syt}(n,k;q) := \sum_{T\in \mathrm{SYT}_{\mathrm{cdes}}(n^2,k)} q^{\mathrm{maj}(T)-n} = \sum_{D} q^{\mathrm{maj}(D)} \tag{4.3}$$

where the second sum is taken over all $D \in \mathrm{DYCK}(n)$ that are either non-elevated with k peaks or elevated with $k-1$ peaks. For integers $k, n \geqslant 1$ we claim that

$$\mathbf{Syt}(n,k;q) = \mathbf{Nar}(n,k;q) - q^{k-1}\mathbf{Nar}(n-1,k;q) + q^{k-2}\mathbf{Nar}(n-1,k-1;q). \tag{4.4}$$

To see this, consider the restriction of `SYTtoDYCK` to $\mathrm{SYT}_{\mathrm{cdes}}(n^2,k)$. If D is an elevated Dyck path of size n with k peaks and D' is the corresponding Dyck path of size $n-1$, then $\mathrm{maj}(D)-\mathrm{maj}(D') = k-1$, as each of the $k-1$ valleys contribute one less to the major index in D' compared to in D.

The polynomials $\mathbf{Syt}(n,k;q)$ refine the q-Catalan numbers, which is easily seen by comparing their definition with (2.3).

Proposition 4.5. *For all integers n,*

$$\sum_k \mathbf{Syt}(n,k;q) = \mathbf{Cat}(n;q).$$

It is easy to see that $\mathbf{Syt}(0,0;q) = \mathbf{Syt}(1,1;q) = 1$ and $\mathbf{Syt}(n,k;q) = 0$ for all other pairs of natural numbers n, k such that either $n \leqslant 1$ or $k \leqslant 1$. For larger n and k, we have the following closed form for $\mathbf{Syt}(n,k;q)$.

Lemma 4.6. *For all integers $k, n \geqslant 2$,*

$$\mathbf{Syt}(n,k;q) = \frac{q^{k(k-2)}(1+q^n)}{[n+1]_q} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q. \tag{4.5}$$

Proof. We may restrict ourselves to the case when $n \geqslant k$ as both sides of (4.5) are identically zero otherwise. We write $\mathbf{Syt}(n,k;q)$ using the expression in (4.4) and expand the q-Narayana numbers to obtain

$$\begin{aligned} &\frac{q^{k(k-1)}}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q - q^{k-1}\frac{q^{k(k-1)}}{[n-1]_q}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \\ &\qquad\qquad + q^{k-2}\frac{q^{(k-1)(k-2)}}{[n-1]_q}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \end{aligned}$$

$$= \frac{q^{k(k-2)}}{[n+1]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q \left(q^k \frac{[n-1]_q}{[k-1]_q} - q^{2k-1}\frac{[n-k+1]_q[n-k]_q}{[n]_q[k-1]_q} + \frac{[k]_q}{[n]_q} \right).$$

The expression in the parentheses is then rewritten as

$$\frac{q^k[n]_q[n-1]_q - q^{2k-1}[n-k+1]_q[n-k]_q + [k]_q[k-1]_q}{[n]_q[k-1]_q}.$$

We must now show that this is equal to $1 + q^n$ or, equivalently, that the following identity holds.

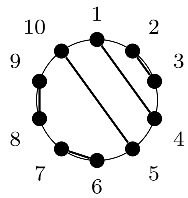
$$q^k[n]_q[n-1]_q - q^{2k-1}[n-k+1]_q[n-k]_q + [k]_q[k-1]_q = (1+q^n)[n]_q[k-1]_q \tag{4.6}$$

This can be proved by elementary algebra. ☐

The edge xy in a non-crossing perfect matching is said to be ***short*** if either $x = i$ and $y = i+1$ for some i or if $x = 1$ and $y = 2n$. If $M \in \mathrm{NCM}(n)$, then we denote $\mathrm{short}(M)$ its number of short edges and $\mathrm{NCM}_{\mathrm{sh}}(n,k)$ the set of $M \in \mathrm{NCM}(n)$ such that $\mathrm{short}(M) = k$. The set $\mathrm{SYT}_{\mathrm{cdes}}(n^2, k)$ is in a natural bijection with $\mathrm{NCM}_{\mathrm{sh}}(n,k)$. To see this, we use the standard bijection SYTtoNCM between $\mathrm{SYT}(n^2)$ and $\mathrm{NCM}(n)$ (see Bijection 6 in Appendix B.1).

1	2	5	6	8
3	4	7	9	10

$\xrightarrow{\texttt{SYTtoNCM}}$



It follows from our definition of promotion that SYTtoNCM is an equivariant bijection in the sense that

$$\texttt{SYTtoNCM}(\partial_{2n}T) = \mathrm{rot}_{2n}(\texttt{SYTtoNCM}(T)). \tag{4.7}$$

From the definition of SYTtoNCM and (4.7), one can prove the following lemma.

Lemma 4.7. *Let $T \in \mathrm{SYT}(n^2)$. Then $x \in \mathrm{cDes}(T)$ if and only if xy, where $x < y$, is a short edge in* $\texttt{SYTtoNCM}(T)$.

Proof. If xy is short in $\texttt{SYTtoNCM}(T)$ and $y \neq 2n$, x must be on row 1 and $y = x+1$ on row 2 in T, so $x \in \mathrm{Des}(T)$ and hence $x \in \mathrm{cDes}(T)$. If $y = 2n$, then $\{1,2\}$ is an edge in $\mathrm{rot}_{2n}(\texttt{SYTtoNCM}(T)) = \texttt{SYTtoNCM}(\partial_{2n}T)$, so 1 is a descent in $\partial_{2n}T$. Thus $x \in \mathrm{cDes}(T)$.

If $x \in \mathrm{cDes}(T)$, either $x \in \mathrm{Des}(T)$ or $x = 2n$. In the former case, x is on row 1 and $x+1$ on row 2 in T. Hence x is a start vertex and $x+1$ an end vertex in $\texttt{SYTtoNCM}(T)$, and the matching is forced to have the short edge $\{x, x+1\}$. In the latter, $1 \in \mathrm{Des}(\partial_{2n}T)$ so $\{1,2\}$ is a short edge in $\texttt{SYTtoNCM}(\partial_{2n}T) = \mathrm{rot}_{2n}(\texttt{SYTtoNCM}(T))$. Hence $\{1, 2n\}$ is a short edge in $\texttt{SYTtoNCM}(T)$. ☐

Theorem 4.8. *Let n, k be natural numbers. The triple*

$$(\mathrm{NCM}_{\mathrm{sh}}(n,k), \langle \mathrm{rot}_{2n} \rangle, \mathbf{Syt}(n,k;q))$$

exhibits the cyclic sieving phenomenon.

Proof. Let ξ be a primitive $(2n)^{\text{th}}$ root of unity. Write $k = k_1\, \mathrm{o}(\xi^d) + k_0$ for the unique natural numbers k_1 and k_0 such that $0 \leqslant k_0 < \mathrm{o}(\xi^d)$. Note that $\mathrm{o}(\xi^d) = 2n/\gcd(2n,d)$. Then, by dividing into cases and applying the q-Lucas theorem (Theorem 2.4) twice, we get

$$\mathbf{Syt}(n,k;\xi^d)=\begin{cases}\frac{2}{n+1}\binom{n+1}{k}\binom{n-2}{k-2} & \text{if } d=2n\\ 2\binom{n/\operatorname{o}(\xi^d)}{k_1}\binom{n/\operatorname{o}(\xi^d)-1}{k_1-1} & \text{if } d\neq 2n, \operatorname{o}(\xi^d)\mid n \text{ and } k_0=0\\ \frac{2n}{n+1}\binom{(n+1)/2}{k_1}\binom{(n-3)/2}{k_1-1} & \text{if } \operatorname{o}(\xi^d)=2,\ n \text{ odd and } 2\mid k\\ 0 & \text{otherwise.}\end{cases}$$

We prove that these evaluations agree with the number of fixed points in $\mathrm{NCM}_{\mathrm{sh}}(n,k)$ under rot_n^d on a case-by-case basis.

Case $\boldsymbol{d=2n}$: Trivial.

Case $\boldsymbol{d\neq 2n}$, $\boldsymbol{\operatorname{o}(\xi^d)\mid n}$ and $\boldsymbol{k_0=0}$: By using Bijection 5, we see that such rotationally symmetric perfect matchings are in bijection with the set $\mathrm{BW}^{k_1}(2n/\operatorname{o}(\xi^d))$, where k_1 denotes the number of cyclic descents. To see that this set has the desired cardinality, we equate the two expressions in (6.4) and (6.5) and then take $q=1$.

Case $\boldsymbol{\operatorname{o}(\xi^d)=2}$, $\boldsymbol{n}$ odd and $\boldsymbol{2\mid k}$: It is easy to check that the assertion holds in the case $n=3$ and $k=2$. It thus remains to show the assertion for $n>3$. Such a non-crossing perfect matching must have a diagonal (an edge that connects two vertices i and $i+n \pmod{2n}$) that divides the matching into two halves. The diagonal can be chosen in n ways. The matching is now determined uniquely by one of its two halves. In one half, we choose a non-crossing matching on $(n-1)$ vertices with $k/2$ short edges. Such a matching is either (i) an element of $\mathrm{NCM}_{\mathrm{sh}}((n-1)/2,k/2)$ which does not have an edge between the two vertices that lie closest to the diagonal or (ii) an element of $\mathrm{NCM}_{\mathrm{sh}}((n-1)/2,k/2+1)$ which does have a short edge between the two vertices that lie closest to the diagonal.

Let us note that, in general, the fraction of elements in $\mathrm{NCM}_{\mathrm{sh}}(n,k)$ that have a short edge adjacent to a given side is equal to $k/2n$. This is easily seen by considering rotations of such a non-crossing perfect matching. So in our case the number of matchings fixed by rot_{2n}^d is equal to

$$n\left(\frac{n-1-k/2}{n-1}|\mathrm{NCM}_{\mathrm{sh}}\left(\frac{n-1}{2},\frac{k}{2}\right)|+\frac{k/2+1}{n-1}|\mathrm{NCM}_{\mathrm{sh}}\left(\frac{n-1}{2},\frac{k}{2}+1\right)|\right).$$

Substituting $r=k/2, m=(n-1)/2$ and using the values from (4.2) (recall $|\mathrm{NCM}_{\mathrm{sh}}(a,b)|=|\mathrm{SYT}_{\mathrm{cdes}}(a^2,b)|$), this is easily shown to equal $\frac{2(2m+1)}{2m+2}\binom{m+1}{r}\binom{m-1}{r-1}$ as is given by $\mathbf{Syt}(n,k;-1)$.

The remaining cases : We need to show that, in all the remaining cases, there are no rotationally symmetric non-crossing perfect matchings. Suppose first that $\operatorname{o}(\xi^d)=2$, n is odd and $2\nmid k$. It is clear that such a non-crossing perfect matching must have a diagonal dividing the matching into two halves. The two halves are identical up to a rotation of π radians and so, in particular, they must have the same number of short edges. In other words, the number of short edges must be even, contradicting $2\nmid k$. Suppose next that $\operatorname{o}(\xi^d)\mid n$ and $k_0\neq 0$. Such a matching is completely determined by how vertices $1, 2, \ldots, d$ are paired up. It follows that the number of short edges must be a multiple of d, contradicting $k_0 \neq 0$.

Suppose lastly that $o(\xi^d) \nmid n$. Since $o(\xi^d) = 2n/\gcd(2n, d)$, $o(\xi^d)\gcd(2n, d) = 2n$ and 2 cannot divide $\gcd(2n, d)$. Hence d is odd. Using this, we see that there cannot be any non-crossing perfect matchings that are fixed under rotation of an odd number of steps, except in the case when $o(\xi^d) = 2$ and n is odd, i.e. when the matching has a diagonal. This exhausts all possibilities and thus the proof is done. □

Theorem 4.8 can be stated in an alternative way as follows. Since `SYTtoNCM` maps cyclic descents to short edges, we see that $\mathrm{SYT}_{\mathrm{cdes}}(\lambda, k)$ is closed under promotion for all rectangular λ. Recall that $(\mathrm{SYT}(\lambda), \langle\partial\rangle, \mathbf{Cat}(n; q))$ exhibits the cyclic sieving phenomenon, for rectangular λ. In the case when $\lambda = (n, n)$, we have the following refinement with regards to the number of cyclic descents.

Corollary 4.9. *Let n, k be natural numbers. The triple*

$$(\mathrm{SYT}_{\mathrm{cdes}}(n^2, k), \langle\partial_{2n}\rangle, \mathbf{Syt}(n, k; q))$$

exhibits the cyclic sieving phenomenon.

In parallel to us, C. Ahlbach, B. Rhoades and J. Swanson [ARS20] have written an unpublished manuscript in which they prove Theorem 4.8, Corollary 4.9 and state Conjecture 4.10 in a similar way to what we do here.

It follows from [Rho10, Lemma 3.3] that the number of cyclic descents remains fixed under promotion of rectangular standard Young tableaux. Experiments suggests that Corollary 4.9 generalizes to all rectangular standard Young tableaux. This would be a refinement of the famous CSP result on rectangular tableaux, see [Rho10, Theorem 1.3]. More precisely, we denote $f_k^\lambda(q) := \sum_T q^{\mathrm{maj}(T)}$ where the sum is taken over all standard Young tableaux of shape λ with exactly k cyclic descents.

Conjecture 4.10. Let n, m, k be natural numbers and put $\lambda = (n^m)$. The triple

$$\big(\mathrm{SYT}_{\mathrm{cdes}}(\lambda, k), \langle\partial_{nm}\rangle, q^{-\kappa(\lambda)} f_k^\lambda(q)\big)$$

exhibits the cyclic sieving phenomenon. Here, $\kappa(\lambda) := \sum_i (i-1)\lambda_i$.

5. Case $s = 1$ and non-crossing (1,2)-configurations

For $s = 1$, there is a nice Catalan family, given by non-crossing $(1, 2)$-configurations described in [Sta15, Family 60]. In the first subsection, we introduce a twisted rotation action on such configurations, and prove a new instance of Catalan CSP. In the second subsection, we refine a CSP result of Thiel, where the group action is given by rotation.

5.1. A new Catalan CSP under twisted rotation

A *non-crossing $(1,2)$-configuration* of size n is constructed by placing vertices $1,\dots,n-1$ around a circle, and then drawing some non-intersecting edges between the vertices. Here, we allow vertices to have a loop, which is counted as an edge. There are $\mathrm{Cat}(n)$ elements in this family. Let $\mathrm{NCC}(n)$ be the set of such objects of size n, and let $\mathrm{NCC}(n,k)$ be the subset of those with $k-1$ edges, loops included. See Appendix A for a figure when $n=3$.

Bijection 2 (Laser construction). See Figure 5.1 for an example. Let $P \in \mathrm{DP}(n)$. Define the non-crossing $(1,2)$-configuration $\texttt{DYCKtoNCC}(P)$ as follows. First, number the east-steps with $1,2,\dots,n-1$. Secondly, if there is a valley at (i,j), draw a line (a laser) from (i,j) to $(i+\Delta, j+\Delta)$, where Δ is the smallest positive integer such that $(i+\Delta, j+\Delta)$ lies on P. Now, consider an east-step ending in (i_1,j_1) on P. If there is a laser drawn from (i_1,j_1), then let (i_2,j_2) be the vertex of P where this laser ends. Then there is an edge between j_1 and j_2-1 in $\texttt{DYCKtoNCC}(P)$ (this can be a loop). The remaining vertices in $\texttt{DYCKtoNCC}(P)$ will be unmarked, that is, unpaired and without a loop.

Proposition 5.1. *The map* $\texttt{DYCKtoNCC}$ *is a bijection* $\mathrm{DP}(n) \to \mathrm{NCC}(n)$.

A proof of Proposition 5.1 can essentially be found in [Bod19, Prop. 6.5]. M. Bodnar studies so called *$n+1, n$-Dyck paths* and shows that these are in bijection with $\mathrm{NCC}(n)$. It is not hard, however, to see that the set of $n+1, n$-Dyck paths is in bijection with $\mathrm{DYCK}(n)$ by removing the first north-step.

Note that there is a natural correspondence between Dyck paths of size n and paths of size $n-1$ that stay weakly above the diagonal $y = x-1$. If $P = \mathsf{w}_1\mathsf{w}_2\cdots\mathsf{w}_{2n-1}\mathsf{w}_{2n}$ is a Dyck path of size n, then let $P' = \mathsf{w}_2\cdots\mathsf{w}_{2n-1} \in \mathrm{PATH}_1(n-1)$. Furthermore, P and P' have the same number of valleys.

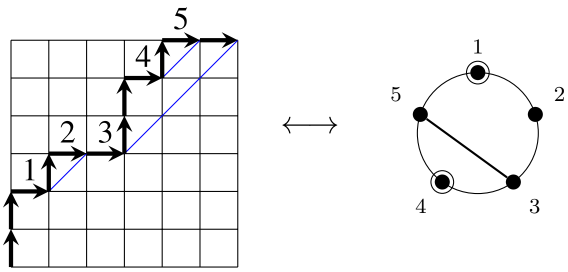


Figure 5.1: An example of the bijection $\texttt{DYCKtoNCC}$ in Bijection 2.

Lemma 5.2. *We have that* $|\mathrm{NCC}(n,k)| = \mathrm{Nar}(n,k)$*, the Narayana numbers.*

Proof. The bijection $\texttt{DYCKtoNCC}$ maps Dyck paths with $k-1$ valleys to non-crossing $(1,2)$-configurations with $k-1$ edges. It remains to note that a Dyck path with $k-1$ valleys has k peaks. $\square$

Remark 5.3. Recall that *Motzkin numbers* M_i count the number of ways to draw non-intersecting chords on i vertices arranged around a circle, see A001006 in the OEIS. The set

$$\{C \in \mathrm{NCC}(n+1) : \mathrm{loops}(C) = l\}$$

has cardinality $\binom{n}{l} M_{n-l}$ sice we can first choose the l vertices which have loops, and then proceed by choosing one of the M_{n-l} possible arrangements of non-intersecting chords on the remaining $n-l$ vertices.

Let rot_n denote rotation by one step, acting on $\mathrm{NCC}(n+1)$. Furthermore, let γ denote the the action of removing the mark on vertex 1 if it is marked, and marking it if it is unmarked. It does not do anything if 1 is connected to an edge. We refer to this as a *flip*.

Let the *twist action* be defined as $\mathrm{twist}_{2n} := \mathrm{rot}_n \circ \gamma$. It is straightforward to see that twist_{2n} generates a cyclic group of order $2n$ acting on $\mathrm{NCC}(n+1)$. Alternatively, we can act by $(\mathrm{rot}_n \circ \gamma)^{n-1}$, which closely resembles *promotion*. Recall that promotion on SYT may be defined as a sequence of swaps, for $i = 1, 2, \dots, n-1$, where swap i interchanges the labels i and $i+1$ if possible.

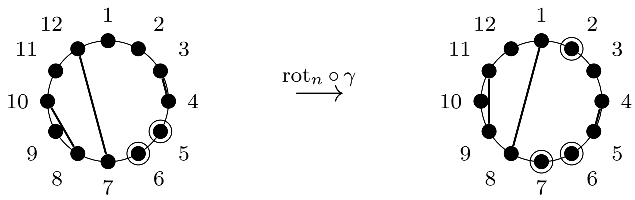


Figure 5.2: The result of $\mathrm{rot}_n \circ \gamma$ on an element in $\mathrm{NCC}(13)$.

Theorem 5.4 (A new cyclic sieving on Catalan objects)**.** *The triple*

$$\left(\mathrm{NCC}(n+1), \langle \mathrm{twist}_{2n} \rangle, \begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^2 \begin{bmatrix} 2n \\ n-2 \end{bmatrix}_q \right)$$

exhibits the cyclic sieving phenomenon. Note that

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^2 \begin{bmatrix} 2n \\ n-2 \end{bmatrix}_q = \frac{[2]_q}{[n+2]_q} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q.$$

Proof. We compute the number of fixed points under twist_{2n}^m, where we may without loss of generality assume $m \mid 2n$. There are two cases to consider, m odd and m even. In the first, we must, according to Lemma 3.3, show that the number of fixed points under twist_{2n}^m is

$$\begin{cases} \binom{m}{(m-1)/2} & \text{if } m = n/2 \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

For the first expression we reason as follows. Since m is odd, any fixed point for such m must consist of a diagonal (an edge from i to $i+m$) and two rotationally symmetrical halves, both consisting of $m-1$ vertices. In such a non-crossing configuration, no vertex can be isolated. To see this, note that if vertex j is isolated, then so is $j+m \pmod n$. But it is clear that j and $j+m \pmod n$ cannot both be marked or unmarked. Thus, all vertices in the non-crossing configuration are incident to an edge i.e. it is really a non-crossing matching. The diagonal can be chosen in m ways and a non-crossing matching on one of the two halves can be chosen in $\mathrm{Cat}((m-1)/2)$ ways, so there are $m\mathrm{Cat}((m-1)/2) = \binom{m}{(m-1)/2}$ fixed points.

In the second case above, if $n = m$ then at least one vertex has to be isolated since m is odd, which implies there can be no fixed points. For $n/m > 2$, we use a "parity" argument. Since any isolated vertices among $S = \{n-m+2, n-m+3, \dots, n, 1\}$ change from unmarked to marked and vice versa under twist_{2n}^m, the number of isolated vertices has to be even. Since m is odd, this implies there must be an odd number of edges from S to $[n] \setminus S$ in a fixed point. However, note that S and the edges out of S completely determine the configuration. Hence the edges must have their other endpoints in the two neighboring intervals of length m. But this violates being rotationally symmetric under rotations of m steps since the number of edges is odd.

In the case $2 \mid m$, according to Lemma 3.3 we must show that the number of fixed points under twist_{2n}^m is equal to

$$\begin{cases} \binom{2n}{n} - \binom{2n}{n+2} & \text{if } m = 2n, \\ \mathrm{Cat}(n/2) & \text{if } m = n \text{ is even}, \\ \binom{m}{m/2} & \text{if } 2 \mid m. \end{cases}$$

Counting fixpoints for the first case is trivial. For the second expression, note that twist_{2n}^n is simply the action of flipping the markings on all vertices. A fixed point can therefore not have any isolated vertices. What remains are non-crossing matchings, which there are $\mathrm{Cat}(n/2)$ many of.

It remains to prove the third expression. Let $m = 2d$.

Case n and n/d is even. In this case, the only possible invariant configurations are non-crossing matchings that are rotationally symmetrical, when rotating $2d$ steps. Recall from Bijection 5 that such matchings are in bijection with $\mathrm{BW}(2d, d)$ and this set clearly has cardinality $\binom{2d}{d}$.

Case n/d is odd. Here we can have fixed points under twist_{2n}^m with unpaired vertices, for examples see Figure 5.3. Here the orbit of a vertex j under the operation twist_{2n}^m is $\{j + dk \pmod n\}_{k\in\mathbb{Z}}$ and if $1 \leqslant i \leqslant d$ is a unpaired, unmarked (that is without loop) vertex the vertices $\{j + 2dk \pmod n\}_{0\leqslant k<n/2d}$ must be unmarked whereas $\{j + 2dk \pmod n\}_{n/2d<k<n/d}$ will be marked. Note that in the latter case $j+2dk \pmod n \equiv j+d+2dr$ for $r = k-(n/d+1)/2$. Thus it suffices to understand the vertices from 1 to d. We now claim that for every $0 \leqslant i \leqslant \lfloor d/2 \rfloor$ we get a valid fixed point by choosing i left vertices and i right vertices and match them in a non-crossing manner as in the previous case. Then we can choose to put a loop at any subset of the remaining $d-2i$ unpaired vertices. Every fixed point will appear exactly once from each such choice. This gives a total of $\sum_{i=0}^{\lfloor d/2 \rfloor} \binom{d}{i}\binom{d-i}{i} 2^{d-2i}$ fixed points. Finally we need to prove that this sum is equal to $\binom{2d}{d}$. We will use a bijection to all possible subsets A of size d from two

rows with numbers 1 to d, the numbers in the top row being blue and the bottom row red. For a given i we choose i numbers and let both the red and the blue belong to A and then i numbers such that neither blue nor red belong to A. Finally the term 2^{d-2i} corresponds to choosing any subset of the remaining $d-2i$ numbers such that the red numbers in that subset belong to A and the blue in the complement should be in A. $\square$

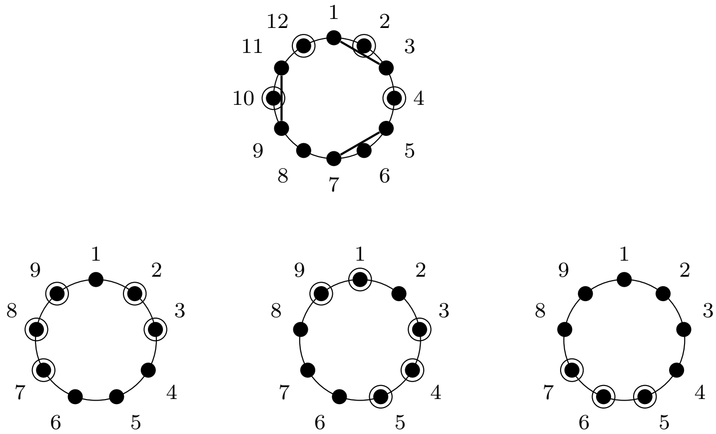


Figure 5.3: A fixed point under twist_{2n}^{m}, $n = 12$, $m = 8$ and $2n/m = 3$ and below an orbit under twist_{2n}^{2} of size 3.

Problem 5.5. It would be nice to refine the CSP triple in Theorem 5.4 to the hypothetical Narayana case discussed in Section 3. That is, we want the following equality to hold:

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q - q^2 \begin{bmatrix} 2n \\ n-2 \end{bmatrix}_q = \sum_{k=1}^{n} N(1, n, k; q)$$

where $N(1, n, k; 1)$ is the number of NCC on n vertices with $(k-1)$ edges. This is a natural consideration, as indicated by Lemma 5.2.

5.2. A refinement of Thiel's result

Recall that rot_n acts on non-crossing $(1,2)$-configurations of size $(n+1)$ via a $2\pi/n$-rotation. M. Thiel proved the following.

Proposition 5.6 (See [Thi17].)**.** *Let $n \in \mathbb{N}$. The triple*

$$(\mathrm{NCC}(n+1), \langle \mathrm{rot}_n \rangle, \mathbf{Cat}(n+1; q))$$

exhibits the cyclic sieving phenomenon.

Denote $\mathrm{NCC}(n+1, e, l)$ the number of non-crossing $(1,2)$-configurations with n vertices, e proper edges and l loops. We determine an element in $\mathrm{NCC}(n+1, e, l)$ by first choosing the

$2e$ vertices that are incident to some proper edge in $\binom{n}{2e}$ ways, then choosing a non-crossing matching among these $2e$ vertices in $\mathrm{Cat}(e)$ ways and finally choosing l of the remaining $n-2e$ vertices to be loops. Hence,

$$|\mathrm{NCC}(n+1,e,l)| = \binom{n}{2e}\mathrm{Cat}(e)\binom{n-2e}{l}. \tag{5.1}$$

For any $e, l \in \mathbb{N}$, define the following q-analog of the above the expression:

$$\mathbf{Ncc}(n,e,l;q) := q^{e(e+1)+(n+1)l}\begin{bmatrix} n \\ 2e \end{bmatrix}_q \mathbf{Cat}(e;q)\begin{bmatrix} n-2e \\ l \end{bmatrix}_q \tag{5.2}$$

$$= q^{e(e+1)+(n+1)l}\frac{1}{[e+1]_q}\begin{bmatrix} n \\ e,e,l,n-2e-l \end{bmatrix}_q. \tag{5.3}$$

Example 5.7. Consider the case with $n=4$ and $k=2$ proper edges and loops. We have

$$\begin{aligned}
\mathbf{Ncc}(4,2,0;q) &= q^6(1+q^2)\\
\mathbf{Ncc}(4,1,1;q) &= q^7(1+2q+3q^2+3q^3+2q^4+q^5)\\
\mathbf{Ncc}(4,0,2;q) &= q^{10}(1+q+2q^2+q^3+q^4)
\end{aligned}$$

It is easily verified that these polynomials refine $\mathbf{Nar}(5,3;q)$. That is,

$$\begin{aligned}
\mathbf{Nar}(5,3;q) &= q^6(1+q+3q^2+3q^3+4q^4+3q^5+3q^6+q^7+q^8)\\
&= \mathbf{Ncc}(4,2,0;q)+\mathbf{Ncc}(4,1,1;q)+\mathbf{Ncc}(4,0,2;q).
\end{aligned}$$

Lemma 5.8. *For every $k \geqslant 0$ we have the identity*

$$\mathbf{Nar}(n+1,k+1;q) = \sum_{e+l=k}\mathbf{Ncc}(n,e,l;q).$$

Proof. Unraveling the definitions, it suffices to show that

$$\frac{q^{k(k+1)}}{[n+1]_q}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \sum_{e+l=k} q^{e(e+1)+(n+1)l}\begin{bmatrix} n \\ 2e \end{bmatrix}_q\begin{bmatrix} n-2e \\ l \end{bmatrix}_q\frac{1}{[e+1]_q}\begin{bmatrix} 2e \\ e \end{bmatrix}_q.$$

Expanding the q-binomials gives

$$\begin{aligned}
&\frac{q^{k(k+1)}}{[n+1]}\frac{[n+1]!}{[k+1]![n-k]!}\frac{[n+1]!}{[k]![n-k+1]!}\\
&= \sum_{0\leqslant e\leqslant k}\frac{q^{e(e+1)+(n+1)(k-e)}[n]!}{[2e]![n-2e]!}\frac{[n-2e]!}{[k-e]![n-k-e]!}\frac{[2e]!}{[e+1]![e]!},
\end{aligned}$$

where we have omitted the q-subscripts for brevity. We start doing cancellations,

$$q^{k(k+1)}\frac{[n]!}{[k+1]![n-k]!}\frac{[n+1]!}{[k]![n-k+1]!} = [n]!\sum_{0\leqslant e\leqslant k}\frac{1}{[e+1]}\frac{q^{e(e+1)+(n+1)(k-e)}}{[k-e]![n-k-e]![e]![e]!},$$

and additional cancellations and some rewriting gives

$$q^{k(k+1)}\frac{[n+1]!}{[k]![k+1]![n-k]![n-k+1]!}=\sum_{0\leqslant e\leqslant k}\frac{q^{e(e+1)+(n+1)(k-e)}}{[k-e]![n-k-e]![e]![e+1]!}.$$

Further rewriting now gives

$$q^{k(k+1)}\frac{[n+1]!}{[n-k]![k+1]!}=\sum_{0\leqslant e\leqslant k}q^{e(e+1)+(n+1)(k-e)}\frac{[k]!}{[k-e]![e]!}\frac{[n-k+1]!}{[n-k-e]![e+1]!}.$$

Thus, the identity we wish to prove is equivalent to showing that

$$\begin{bmatrix}n+1\\k+1\end{bmatrix}_q=\sum_{0\leqslant e\leqslant k}q^{(k-e)(n-e-k)}\begin{bmatrix}k\\k-e\end{bmatrix}_q\begin{bmatrix}n-k+1\\e+1\end{bmatrix}_q.$$

However, this follows from the q-Vandermonde identity (Theorem 2.3) by substituting $a=n+1-k$, $b=k$, $c=k+1$ and $j=k-e$. □

It is clear that one can restrict the action of rot_n to $\mathrm{NCC}(n+1,e,l)$. The following result is a refinement of Proposition 5.6.

Theorem 5.9. *Let $n,e,l\in\mathbb{N}$. The triple*

$$(\mathrm{NCC}(n+1,e,l),\langle\mathrm{rot}_n\rangle,\mathbf{Ncc}(n,e,l;q))$$

exhibits the cyclic sieving phenomenon.

Proof. Let ξ be a primitive n^{th} root of unity and let $d\mid n$. Write $e=e_1(n/d)+e_0$ and $l=l_1(n/d)+l_0$ for the unique natural numbers e_1,e_0,l_1,l_0 such that $0\leqslant e_0<n/d$ and $0\leqslant l_0<n/d$. Using the q-Lucas theorem (Theorem 2.4) repeatedly, we get

$$\mathbf{Ncc}(n,e,l;\xi^d)=\begin{cases}\binom{n}{2e}\mathrm{Cat}(e)\binom{n-2e}{l} & \text{if } d=n\\ \binom{d}{2e_1}\binom{2e_1}{e_1}\binom{d-2e_1}{l_1} & \text{if } d\neq n, e_0=0 \text{ and } l_0=0\\ \binom{d}{e}\binom{e}{e_1}\binom{d-e}{l_1} & \text{if } d=n/2,\ e_0=1 \text{ and } l_0=0\\ 0 & \text{otherwise.}\end{cases}$$

We prove that these evaluations agree with the number of fixed points in $\mathrm{NCC}(n+1,e,l)$ under rot_n^d on a case-by-case basis.

Case $d=n$: Trivial.

Case $d \neq n$, $e_0 = 0$ and $l_0 = 0$: A $(2,1)$-configuration that is fixed by rot_n^d is completely determined by its first d vertices. Among these d vertices, there must be $2e(n/d) = 2e_1$ vertices that are incident to an edge and $l(n/d) = l_1$ loops. Choose $2e_1$ from the first d vertices in $\binom{d}{2e_1}$ ways. The number of ways to arrange these edges in an admissible way is equal to the number of perfect matchings that are invariant when rotating $2e_1$ steps. By Bijection 5, we know that there are $\binom{2e_1}{e_1}$ such matchings. Lastly, choose l_1 loops among the remaining $d - 2e_1$ vertices in $\binom{d-2e_1}{l_1}$ ways. These choices are all independent and the desired result follows.

Case $d = n/2$, $e_0 = 1$ and $l_0 = 0$: Such a $(2,1)$-configuration must have a diagonal (an edge from i to $i + d$) that splits the $(2,1)$-configuration into two halves. The diagonal can be chosen in d ways. The $(2,1)$-configuration is now determined uniquely by one of its halves. Such a half must have $d-1$ vertices with $(e-1)/2 = e_1$ edges and l_1 loops. Choose the $2e_1$ vertices that are incident to an edge from the $d-1$ vertices in $\binom{d-1}{2e_1}$. The number of the ways to arrange these edges in an admissible way is equal to the number of non-crossing perfect matchings on $2e_1$ vertices, namely $\mathrm{Cat}(e_1)$. Finally, choose l_1 loops from the remaining $d-e$ vertices in $\binom{d-e}{l_1}$ ways. Since these choices are independent, the number of fixed points is given by

$$d\binom{d-1}{2e_1}\mathrm{Cat}(e_1)\binom{d-e}{l_1} = \binom{d}{e}\binom{e}{e_1}\binom{d-e}{l_1}$$

where the equality follows from some simple manipulations of binomial coefficients.

The remaining cases : Suppose that $P \in \mathrm{NCC}(n+1, e, l)$ is invariant under rot_n^d, where $d \neq n$. If $l_0 \neq 0$, then there would be $l/(n/d)$ loops among the first d vertices but n/l is not an integer, so there cannot be such a P. So assume that $l_0 = 0$. If $d \neq n/2$ and $e_0 \neq 0$, then for each edge ij in P, there must be edges $(i+d)(j+d), (i+2d)(j+2d), \dots, (i+n-d)(j+n-d)$ in P (where addition is taken modulo n). So the number of edges must be a multiple of n/d which cannot be the case if $e_0 \neq 0$.

This exhausts all possibilities and thus the proof is done. □

Recall that $\mathrm{NCC}(n+1, k)$ is the set of non-crossing $(1,2)$-configurations P on n vertices such that the number of loops plus proper edges of P is equal to $k-1$. In other words,

$$\mathrm{NCC}(n+1, k) = \bigcup_{i=0}^{k-1} \mathrm{NCC}(n+1, i, k-1-i).$$

By applying Lemma 5.8, we obtain the following result.

Corollary 5.10. *For every $n, k \in \mathbb{N}$ such that $0 \leqslant k \leqslant n+1$,*

$$(\mathrm{NCC}(n+1,k), \langle \mathrm{rot}_n \rangle, \mathbf{Nar}(n+1,k;q))$$

exhibits the cyclic sieving phenomenon.

There is already a known instance of the cyclic sieving phenomenon with the q-Narayana numbers as the polynomial, namely that of non-crossing partitions with a fixed number of blocks and where the group action is rotation [RSW04, Thm 7.2]. Note, however, that in Corollary 5.10 the cyclic group has a different order than the one with non-crossing partitions.

Remark 5.11. We cannot hope to find a Kreweras-like refinement of the above CSP result as in [RS18]. For example, consider $n = 4$ and $k = 2$. There are two partitions of n into k parts, namely $(3, 1)$ and $(2, 2)$. There are 4 non-crossing partitions with parts given by $(3, 1)$ and 2 non-crossing partitions with parts given by $(2, 2)$. But $\mathrm{NCC}(4, 2)$ has two orbits under rotation, both of size 3.

6. Case $s = n$ and type B Catalan numbers

In this section, we prove several instances of the CSP, related to type B Catalan numbers. We first consider a q-Narayana refinement on non-crossing matchings. In the subsequent subsection, we consider a cyclic descent refinement on binary words. Finally, in the last subsection we prove a type B analog of Theorem 5.9.

6.1. Type B Narayana CSP

A *type B non-crossing partition* of size n is a non-crossing partition of $\{1, \dots, n, n+1, \dots, 2n\}$ which is preserved under a half-turn rotation. These were first defined by V. Reiner in [Rei97]. We let this set be denoted $\mathrm{NCP}^B(n)$ and let rotB_n denote the action on $\mathrm{NCP}^B(n)$ by rotation of π/n. Note that we only need to make a half-turn before arriving at the initial position and we prefer to write rotB_n instead of rot_{2n} to emphasize that the order of the action is n.

The earliest reference we can find for the following proposition is the paper by D. Bessis and V. Reiner [BR11, Thm. 1.1]. They prove a much more general theorem about complex reflection groups. We include a short proof as an introduction to the cyclic sieving phenomenon on type B objects.

Proposition 6.1. *The triple*

$$\big(\mathrm{NCP}^B(n), \langle \mathrm{rotB}_n \rangle, \mathbf{Cat}^B(n; q)\big)$$

is a CSP triple.

Proof. There are many ways to prove this. For example, $\mathrm{NCP}^B(n)$ can first be put in bijection with type B non-crossing matchings, which are non-crossing matchings on $4n$ vertices that are symmetric under a half-turn, by using Bijection 7.

We then consider the first $2n$ new vertices, and for each vertex u, we record a 1 if the edge $u \to v$ is oriented clockwise, and 0 otherwise. This is a binary word of length $2n$, with n ones. Furthermore, rotB_n of the non-crossing partition corresponds to shift_{2n}^2 on the binary word. Shifting by one step together with the maj-polynomial $\begin{bmatrix} 2n \\ n \end{bmatrix}_q$ is a well-known CSP instance (see [RSW04, Prop. 4.4]), and then it is direct from the definition of cyclic sieving that shifting two steps on a word of even length also gives cyclic sieving. □

We shall now consider Narayana refinements of type B non-crossing partitions and non-crossing matchings. First, we introduce the following polynomial:

$$\Pi_n(q; t) := \sum_{j=0}^{n} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}_q \left(t^{2j} q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q + t^{2j+1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q \right). \tag{6.1}$$

Note that by using the q-Pascal identity, $\Pi_n(q;1) = \sum_j q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 = \begin{bmatrix} 2n \\ n \end{bmatrix}_q$, so the sum of the polynomials

$$[t^0]\Pi_n(q;t), \quad [t^1]\Pi_n(q;t), \quad \dots, \quad [t^{2n}]\Pi_n(q;t)$$

refines the type B q-Catalan numbers. With the polynomial formulated, cyclic sieving is easy to prove by following the proof of [RSW04, Thm 7.2]. As a side note, the coefficients of the polynomial at $q = 1$ are given by the OEIS entry A088855.

Proposition 6.2. *Let $n, k \geqslant 0$ be integers. Then*

$$\left(\{P \in \mathrm{NCP}^B(n) : \mathrm{blocks}(P) = k\}, \langle \mathrm{rotB}_n \rangle, [t^k]\Pi_n(q;t)\right),$$

and

$$\left(\{P \in \mathrm{NCP}^B(n) : 2k \leqslant \mathrm{blocks}(P) \leqslant 2k+1\}, \langle \mathrm{rotB}_n \rangle, q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q^2\right) \tag{6.2}$$

exhibit the cyclic sieving phenomenon.

Moreover, for every $n \geqslant 1$ and k, $0 \leqslant k \leqslant n$,

$$\left(\{M \in \mathrm{NCM}^B(n) : \mathrm{even}(M) = k-1\}, \langle \mathrm{rotB}_n \rangle, [t^k]\Pi_n(q;t)\right)$$

and

$$\left(\{M \in \mathrm{NCM}^B(n) : 2k-1 \leqslant \mathrm{even}(M) \leqslant 2k\}, \langle \mathrm{rotB}_n \rangle, q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q^2\right) \tag{6.3}$$

exhibit the cyclic sieving phenomenon.

Proof. Everything is trivial unless $1 \leqslant k \leqslant n$, so we assume this holds. Using Bijection 7 the first two statements are equivalent to the last two, so we only need to prove the former. The number of half-turn symmetric non-crossing partitions with $2k$ or $2k+1$ blocks is $\binom{n}{k}^2$. This can be proven in different ways, but it suffices to refer to the proof of [RSW04, Thm 7.2].

Divide the numbers into $2d$ intervals $t\frac{n}{d}+1, \dots, (t+1)\frac{n}{d}$, $t \in \{0, \dots, 2d-1\}$. If a partition P satisfies $\mathrm{rotB}_n^{n/d}(P) = P$, a block only contains numbers from two adjacent intervals or it is a central block with numbers from every interval. Let r be the number of blocks that contain numbers from only interval $t = 0$ and blocks of numbers from both that and the next interval ($t = 1$), but no other. Then the total number of blocks is $2dr$ or $2dr+1$, the latter if there is also a central block. In the aforementioned proof, they use a result from [AR04] to also prove that the number of partitions $P \in \mathrm{NCP}^B(n)$ invariant under $\mathrm{rotB}_n^{n/d}$ with $2dr$ and $2dr+1$ blocks is

$$\frac{dr}{n}\binom{n/d}{r}^2 \text{ and } \frac{n-dr}{n}\binom{n/d}{r}^2, \text{ respectively.}$$

We now evaluate $[t^k]\Pi_n(q;t)$ at a primitive d^{th} root of unity. If $k \neq 0, 1 \pmod{d}$, $d \geqslant 2$ then it is clearly zero. For $k = 2dr$, we get

$$[t^{2dr}]\Pi_n(q;t) = q^{(dr)^2+n-dr} \begin{bmatrix} n \\ dr \end{bmatrix}_q \begin{bmatrix} n-1 \\ dr-1 \end{bmatrix}_q$$

which by Theorem 2.4 (the q-Lucas theorem) becomes $\binom{n/d}{r}\binom{n/d-1}{r-1} = \binom{n/d}{r}^2 \frac{r}{n/d}$, which is what we want. A similar calculation gives the case for $k = 2dr + 1$. The expression $q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q^2$ in (6.2) is just the sum of the two cases. The evaluation for $d = 1$ is also straightforward. □

A cyclic sieving result involving type B Kreweras numbers and thus type B Catalan numbers was proven in [RS18, Thm. 1.7]. The downside is that the Kreweras-numbers in type B are not indexed by usual partitions, but partitions of $2n + 1$, where each even part has even multiplicity.

6.2. A second refinement of the type B Catalan numbers

Recall $\mathrm{BW}(2n, n)$ is the set of binary words of length $2n$ with exactly n ones. Define a *cyclic descent* of a binary word $\mathsf{b} = \mathsf{b}_1\mathsf{b}_2 \cdots \mathsf{b}_{2n}$ as an index i such that $\mathsf{b}_i > \mathsf{b}_{i+1}$, where the indices are taken modulo $2n$. The number of cyclic descents of b is denoted $\mathrm{cdes}(\mathsf{b})$. As an example, if $\mathsf{b} = 0110010111$, then $\mathrm{cdes}(\mathsf{b}) = 3$. For any two natural numbers n and k, let $\mathrm{BW}^k(n) \subset \mathrm{BW}(2n, n)$ consist of all $\mathsf{b} \in \mathrm{BW}(2n, n)$ such that $\mathrm{cdes}(\mathsf{b}) = k$. Define

$$\mathbf{Bw}(n, k; q) := \sum_{\mathsf{b} \in \mathrm{BW}^k(n)} q^{\mathrm{maj}(\mathsf{b})}, \tag{6.4}$$

where we use the ordinary major index maj on words as described in Section 2.1. At $q = 1$, $\mathbf{Bw}(n, k; q)$ is A335340 in the OEIS and two times A103371. Note that we have $\mathbf{Bw}(0, 0; q) = 1$ and $\mathbf{Bw}(n, k; q) = 0$ if $k > n$.

Lemma 6.3. *For all integers* $1 \leqslant k \leqslant n$,

$$\mathbf{Bw}(n, k; q) = q^{k(k-1)}(1 + q^n) \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{6.5}$$

Proof. The set $\mathrm{BW}^k(n)$ is in bijection with a certain subset of $\mathrm{PATH}(n)$ which we shall now describe. Call binary words of the form $\mathsf{b} = 0\mathsf{b}_2\mathsf{b}_3 \cdots \mathsf{b}_{2n-1}1$ *elevated* and call binary words that are not elevated *non-elevated* (so a binary word is elevated if $\mathrm{cdes}(\mathsf{b}) = \mathrm{des}(\mathsf{b}) + 1$). Elevated binary words in $\mathrm{PATH}(n)$ are in natural bijection with paths in $\mathrm{PATH}(n-1)$ by letting the elevated binary word $0\mathsf{b}_2\mathsf{b}_3 \cdots \mathsf{b}_{2n-1}1$ correspond to the binary word $\mathsf{b}_2\mathsf{b}_3 \cdots \mathsf{b}_{2n-1}$.

It follows that a word in $\mathrm{BW}^k(n)$ corresponds either to a non-elevated path in $\mathrm{PATH}(n)$ with k valleys or to an elevated path in $\mathrm{PATH}(n)$ with $k-1$ valleys. Using this correspondence and (2.9), one gets that

$$\mathbf{Bw}(n, k; q) = q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 - q^k \cdot q^{k^2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^2 + q^{k-1} \cdot q^{(k-1)^2} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q^2. \tag{6.6}$$

Here, the factors q^k and q^{k-1} appear since by translating a binary word $\mathsf{c} \in \mathrm{BW}(2(n-1), n-1)$ with k descents into its corresponding elevated binary word c' in $\mathrm{BW}(2n, n)$, we have $\mathrm{maj}(P') - \mathrm{maj}(P) = k$ as each descent of c' contributes one more to maj than in c.

It remains to show that the expression in (6.6) coincides with the one in (6.5). To do this, we rewrite

$$q^{k^2}\begin{bmatrix} n \\ k \end{bmatrix}_q^2 - q^k \cdot q^{k^2}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q^2 + q^{k-1}\cdot q^{(k-1)^2}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q^2$$
$$= q^{k(k-1)}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \left(\frac{q^k[n]_q^2 - q^{2k}[n-k]_q^2 + [k]_q^2}{[n]_q[k]_q}\right).$$

It is therefore sufficient to show that the following equality holds:

$$q^k[n]_q^2 - q^{2k}[n-k]_q^2 + [k]_q^2 = (1+q^n)[n]_q[k]_q.$$

This equality can be derived from (4.6) by adding $q^{k+n-1}[n]_q - q^{2k-1}[n-k]_q + q^{k-1}[k]_q = q^{k-1}(1+q^n)[n]_q$ to each side of the equation. This concludes the proof. □

The number of cyclic descents of a binary word is clearly invariant under cyclic shifts of the word so one has a group action of rot_{2n} on $\mathrm{BW}^k(n)$. The following proposition follows from [AS18, Cor. 1.6], although they do not compute the closed-form expression of Equation (6.4).

Proposition 6.4. *For all $n, k \in \mathbb{N}$ such that $1 \leqslant k \leqslant n$, the triple*

$$\left(\mathrm{BW}^k(n), \mathrm{shift}_{2n}, \mathbf{Bw}(n,k;q)\right)$$

exhibits the cyclic sieving phenomenon.

6.3. Type B non-crossing configurations with a twist

Recall that $\mathrm{NCC}(n+1)$ denotes the set of non-crossing $(1,2)$-configurations on n vertices. We shall now modify this family slightly.

Definition 6.5. Let $\mathrm{NCC}^B(n)$ be the set of non-crossing $(1,2)$-configurations on $n-1$ vertices, with the extra option that one of the proper edges may be marked. We let $\mathrm{NCC}^B(n,e,l) \subset \mathrm{NCC}^B(n)$ be the subset with exactly e proper edges, and l loops. Finally, let $\mathrm{NCC}^B(n,k)$ be the subset of $\mathrm{NCC}^B(n)$ with k edges and loops, i.e.

$$\mathrm{NCC}^B(n,k) \coloneqq \bigcup_{e+l=k} \mathrm{NCC}^B(n,e,l).$$

It follows directly from the definition that $|\mathrm{NCC}^B(n,e,l)| = (e+1)|\mathrm{NCC}(n,e,l)|$ and it is not difficult to sum over all possible e, l to prove that $|\mathrm{NCC}^B(n+1,k)| = \mathrm{Nar}^B(n,k) = \binom{n}{k}^2$.

Theorem 6.6. *We let twist_{2n} act on $\mathrm{NCC}^B(n+1)$ as before (the marked edge is also rotated), which gives an action of order $2n$. Then*

$$\left(\mathrm{NCC}^B(n+1), \langle \mathrm{twist}_{2n}^2 \rangle, \mathbf{Cat}^B(n;q)\right) \tag{6.7}$$

is a CSP triple.

Proof. We compute the number of fixed points under $(\mathrm{twist}_{2n}^2)^d$ where we can without loss of generality assume $d \mid n$. Write $n = md$. By Theorem 2.4,

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \binom{2d}{d}$$

at a primitive m^{th} root of unity. The claim follows from Theorem 5.4 except in the cases where a marked edge can appear in a fixed point. Note that in the case $4 \mid n$ and $2d = n/2$ or $3n/2$ a marked edge would have to split the configuration into two non-crossing matchings on an odd number of vertices. Hence there cannot be a marked edge in a fixed point in this case.

The only case left is $n \mid 2d$. First, $2d = 2n$ is trivial. Second, if $2d = n$, no fixed point can have marked vertices, as is noted in the proof of Theorem 5.4. Hence we only have non-crossing matchings on $2d$ vertices with one edge possibly marked, the number of which is $(d+1)\mathrm{Cat}(d) = \binom{2d}{d}$. □

It should be possible to prove Theorem 6.6 bijectively.

Problem 6.7. Find an equivariant bijection between $\mathrm{NCC}^B(n+1)$ and $\mathrm{BW}(2n, n)$ sending twist_{2n}^2 to shift_{2n}^2.

Note that the triple in Theorem 6.6 exhibits the so-called Lyndon-like cyclic sieving [ALP19], which is not intuitively clear (as it is for $\mathrm{BW}(2n, n)$).

Remark 6.8. Theorem 6.6 does not hold when only considering twist_{2n}. For $n = 2$, $\left[\begin{smallmatrix} 2n \\ n \end{smallmatrix}\right]_q$ evaluated at a primitive 4^{th} root of unity gives 0. However, there are 6 elements in $\mathrm{NCC}^B(3)$, two of which are fixed under twist_4; consider an edge between vertices 1 and 2, which may or may not be unmarked. Since there are no loops or isolated vertices, these two elements are fixed. Can one modify the q-analog of $\binom{2n}{n}$ so that it is compatible with twist_{2n}?

Problem 6.9. Is it possible to define a refinement $P(n, e, l; q)$ of $\left[\begin{smallmatrix} 2n \\ n \end{smallmatrix}\right]_q$ so that

$$\left(\bigcup_{l=0}^{n} \mathrm{NCC}^B(n+1, e, l), \langle \mathrm{twist}_{2n}^2 \rangle, \sum_{l \geqslant 0} P(n, e, l; q) \right)$$

is a CSP triple?

Unfortunately the polynomials

$$\mathbf{Ncc}^B(n, e, l; q) := q^{e^2+nl}[e+1]_q \begin{bmatrix} n \\ 2e \end{bmatrix}_q \mathbf{Cat}(e; q) \begin{bmatrix} n-2e \\ l \end{bmatrix}_q$$

do not serve this purpose even though they do satisfy the identities (proof omitted)

$$\mathbf{Ncc}^B(n, e, l; 1) = |\mathrm{NCC}^B(n+1, e, l)|,$$
$$\mathbf{Nar}^B(n, k; q) = \sum_{e+l=k} \mathbf{Ncc}^B(n, e, l; q).$$

6.4. Thiel's CSP for type B

Theorem 6.10. *We let rotation* rot_n *act on* $\mathrm{NCC}^B(n+1,e,l)$*, and let*

$$\mathbf{Ncc}(n,e,l;q) := q^{e(e+1)+(n+1)l} \begin{bmatrix} n \\ 2e \end{bmatrix}_q \mathbf{Cat}(e;q) \begin{bmatrix} n-2e \\ l \end{bmatrix}_q .$$

Then

$$\big(\mathrm{NCC}^B(n+1,e,l), \langle \mathrm{rot}_n \rangle, (1+[e]_q)\mathbf{Ncc}(n,e,l;q)\big) \tag{6.8}$$

is a CSP triple.

Proof. We can split $\mathrm{NCC}^B(n+1,e,l)$ into two sets, the first set A being the case without a marked edge, and the second set B the case with a marked edge. Then it suffices to prove that

$$(A, \langle \mathrm{rot}_n \rangle, \mathbf{Ncc}(n,e,l;q)) \qquad \text{and} \qquad (B, \langle \mathrm{rot}_n \rangle, [e]_q \mathbf{Ncc}(n,e,l;q))$$

are CSP triples. The first one is already proved in Theorem 5.9.

For the second, consider rot_n^d, and without loss of generality write $n = kd$. A single marked edge can only appear in a fixed point if $d = n$ or $d = n/2$. The former is trivial. Now, rewrite the polynomial as

$$q^{e(e+1)+(n+1)l} \begin{bmatrix} n \\ 2e \end{bmatrix}_q \begin{bmatrix} 2e \\ e-1 \end{bmatrix}_q \begin{bmatrix} n-2e \\ l \end{bmatrix}_q ,$$

and apply Theorem 2.4. At a primitive k^{th} root of unity, this evaluates to 0 unless $k \mid 2e$, $k \mid e-1$ and $k \mid l$. The second implies $\gcd(k,e) = 1$, so by the first $k \mid 2$. If $k = 2$, that is $d = n/2$, we get that the number of fixed points should be

$$\binom{\frac{n}{2}}{e}\binom{e}{\frac{e-1}{2}}\binom{\frac{n}{2}-e}{\frac{l}{2}}.$$

This is indeed the case. The marked edge has to split the configuration into two symmetric parts, and connects i to $i+n/2$ for some $1 \leqslant i \leqslant n/2$. The symmetric configurations are on $n/2-1$ vertices, and have $(e-1)/2$ edges and $l/2$ marked vertices each. The number of fixed points is hence

$$\frac{n}{2}\binom{\frac{n}{2}-1}{e-1}\mathrm{Cat}((e-1)/2)\binom{\frac{n}{2}-1-(e-1)}{\frac{l}{2}} = \binom{\frac{n}{2}}{e}\binom{e}{\frac{e-1}{2}}\binom{\frac{n}{2}-e}{\frac{l}{2}}. \qquad \square$$

By summing over the cases when $e + l = k$, we get the following corollary:

Corollary 6.11. *We have a q-analog of the type B Narayana numbers, which admits the CSP triple*

$$\big(\mathrm{NCC}^B(n+1,k), \langle \mathrm{rot}_n \rangle, U_{n,k}(q)\big) ,$$

where $U_{n,k}(q) = \sum_{e=0}^{k} q^{e(e+1)+(n+1)(k-e)}(1+[e]_q)\left[\begin{smallmatrix} n \\ 2e \end{smallmatrix}\right]_q \mathbf{Cat}(e;q)\left[\begin{smallmatrix} n-2e \\ k-e \end{smallmatrix}\right]_q$.

Proof. As in the discussion before Theorem 6.6 it is not difficult to prove that when $q = 1$, we do indeed obtain $\binom{n}{k}^2$, so this is a q-Narayana refinement. $\square$

Now, summing over all k gives cyclic sieving on $\mathrm{NCC}^B(n+1)$ under rotation. We leave it as an open problem to find a nice expression for $\sum_k U_{n,k}(q)$.

7. Two-column semistandard Young tableaux

The *Schur polynomial* $\mathrm{s}_\lambda(x_1,\dots,x_n)$ is defined as the sum

$$\mathrm{s}_\lambda(x_1,\dots,x_n) := \sum_{T\in \mathrm{SSYT}(\lambda,n)} \prod_{j\in\lambda} x_{T(j)}$$

where $\mathrm{SSYT}(\lambda,n)$ is the set of semi-standard Young tableaux of shape λ with maximal entry at most n. The product is taken over all labels in T.

P. Brändén gave the following interpretation of q-Narayana numbers.

Theorem 7.1 (See [Brä04, Thm. 6].). *For* $0\leqslant k\leqslant n-1$,

$$\mathbf{Nar}(n,k+1;q) = \mathrm{s}_{2^k}(q,q^2,\dots,q^{n-1}). \tag{7.1}$$

There is a type B analog of Theorem 7.1.

Theorem 7.2. *For* $0\leqslant k\leqslant n$,

$$q^{k(k+1)}\begin{bmatrix} n\\ k\end{bmatrix}_q \begin{bmatrix} n\\ k\end{bmatrix}_q = \mathrm{s}_{2^k1^k/1^k}(q,q^2,\dots,q^n). \tag{7.2}$$

Proof. We first note that $\mathrm{s}_{2^k1^k/1^k} = (\mathrm{s}_{1^k})^2$. To compute s_{1^k}, we simply sum over all k-subsets of $[n]$. This gives immediately that

$$\mathrm{s}_{1^k}(q,q^2,\dots,q^n) = q^{k(k+1)/2}\begin{bmatrix} n\\ k\end{bmatrix}_q,$$

and the theorem above follows. □

It is then reasonable to interpret

$$\mathrm{s}_{2^k1^s/1^s}(q,q^2,\dots,q^n) \tag{7.3}$$

for $0\leqslant s\leqslant k$ as an interpolation between type A ($s=0$) and type B ($s=k$) q-Narayana polynomials. Note that this approach is different from what is sought after in Section 3. The expression in (7.3) can easily be computed by the dual Jacobi–Trudi identity, see [Mac95]. We find that (7.3) is equal to

$$q^{k(k+1)}\left(\begin{bmatrix} n\\ k\end{bmatrix}_q^2 - q^{(s+1)^2}\begin{bmatrix} n\\ k-s-1\end{bmatrix}_q\begin{bmatrix} n\\ k+s+1\end{bmatrix}_q\right).$$

The first part of the theorem below follows from combining [Rho10, Thm. 1.4] and Theorem 7.1. For a definition of k-promotion, see Appendix C.

Theorem 7.3. *Assume* $1 \leqslant k < n$ *and let* $\hat{\partial}_{n-1}$ *act on* $\mathrm{SSYT}(2^k, n-1)$ *via so-called* k*-promotion, so that* $\hat{\partial}_{n-1}$ *has order* $n-1$*. Then*

$$\left(\mathrm{SSYT}(2^k, n-1), \langle \hat{\partial}_{n-1} \rangle, \mathbf{Nar}(n, k+1; q)\right)$$

is a CSP triple. Moreover, there is a cyclic group $\langle \varphi \rangle$ *of order* n *acting on* $\mathrm{SSYT}(2^k, n-1)$ *such that*

$$\big(\mathrm{SSYT}(2^k, n-1), \langle \varphi \rangle, \mathbf{Nar}(n, k+1; q)\big)$$

is a CSP triple.

Proof. We can define the action φ as follows. Given $T \in \mathrm{SSYT}(2^k, n-1)$, define

$$Q = \{j : 1 \leqslant j \leqslant n-1, |\{i : T_{i,1} \geqslant j\}| = |\{i : T_{i,2} \geqslant j\}|\}$$

and let P be the subset of Q containing numbers not occurring as entries in T. Further, define $\ell = \max P$ and $b = \max Q$. If $P = \varnothing$, let $\ell = 0$. Now $\varphi(T)$ is defined as follows. If $\ell = n-1$, then we just add one to every entry in T and are done. If $\ell \neq n-1$, then we add one to every entry in T and

- in the first column: remove $b+1$, add 1 in increasing order;
- in the second column: remove n, add $\ell + 1$ in increasing order.

This can be seen to be an action with the desired properties by referring to Bijection 3 from $\mathrm{SSYT}(2^k, n-1)$ to $\mathrm{NCP}(n, k+1)$. Then rotation one step of the latter set corresponds to φ where P are the other elements in the same block as n, b is the smallest element in the block of $n-1$ (if $n-1 \notin P$), and ℓ is the largest element in the block of n other than n itself. Clearly, $b+1$ must be removed from the first column since it will be the smallest element in the block of n in $\varphi(T)$, and $\ell + 1$ must be added to the second since it will be the largest element in the block containing 1 in $\varphi(T)$. □

Bijection 3. Let $T \in \mathrm{SSYT}(2^k, n-1)$. Starting from $i = 1$, consider $x_i = T_{i,2}$. Find the largest $y \in T_{*,1}, y \leqslant x_i$, which is not in

$$P_{i-1} \coloneqq \bigcup_{j \in [i-1]} p_j,$$

and set $y_i = y$. Let $p_i = \{z \in \mathbb{N} : y_i \leqslant z \leqslant x_i, z \notin P_{i-1}\}$. Repeat for $i < k+1$. Finally, let $p_{k+1} = [n] \setminus P_k$. Then the blocks $p_1, \ldots, p_{k+1}$ form a non-crossing partition in $\mathrm{NCP}(n, k+1)$ by construction. Note that exactly one element from each of $T_{*,1}$ and $T_{*,2}$ is contained in p_i. Note also that this is in fact the unique non-crossing partition in $\mathrm{NCP}(n, k+1)$ having parts whose smallest and largest elements are y_i and x_i respectively.

The inverse of Bijection 3 can be described as follows. Let $p_1|p_2|\cdots|p_{k+1} \in \mathrm{NCP}(n, k+1)$ and assume $n \in p_{k+1}$. Let the first column of T consist of the smallest elements from p_i, $1 \leqslant i \leqslant k$, in increasing order from top to bottom, and the second column of T of the largest elements from the same blocks, also in increasing order. To see why $T \in \mathrm{SSYT}(2^k, n-1)$, it suffices to suppose that we on some row i have $T_{i,1} > T_{i,2}$, so the smallest element in the block containing $T_{i,2}$ must be $T_{j,1}$ for some $j < i$. Then the smallest element in the block containing $T_{j,2}$ has to be $T_{k,1}$ for some $j < k < i$, the smallest element in the block containing $T_{j+1,2}$ some $T_{k',1}$ for $j < k' < i$, and so on. We match all elements $T_{j,2}, \ldots, T_{i-1,2}$ to elements in the first column between $T_{j,1}$ and $T_{i,1}$, but this yields a contradiction. Note that T is the unique element of $\mathrm{SSYT}(2^k, n-1)$ whose first column consists of the smallest elements of $p_1, \ldots, p_k$ and whose second column consists of the largest elements of these parts. Hence it is clear that this indeed is the inverse.

Below is an example of Bijection 3 and φ, when $n = 8$, $k = 4$.

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline 7 & 7 \\ \hline \end{array} \quad \longleftrightarrow \quad \{\{1,4\},\{2\},\{3\},\{7\},\{5,6,8\}\}, \quad \varphi(T) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 3 & 5 \\ \hline 4 & 7 \\ \hline \end{array}$$

Remark 7.4. The bijection in the proof of [Brä04, Thm. 5] together with Bijection 8 provides a different bijection between $\mathrm{SSYT}(2^k, n-1)$ and $\mathrm{NCP}(n, k+1)$ where, if $T \in \mathrm{SSYT}(2^k, n-1)$, then $T_{1,1}, T_{2,1} - T_{1,1}, \ldots, T_{k,1} - T_{k-1,1}, n - T_{k,1}$ are the sizes of the blocks with $T_{1,2}, T_{2,2}, \ldots, T_{k,2}$ as the largest elements.

There is a type B version of Theorem 7.3.

Theorem 7.5. *Let $\hat{\partial}_n$ act on $\mathrm{SSYT}(2^k 1^k/1^k, n)$ via k-promotion, then*

$$\left(\mathrm{SSYT}(2^k 1^k/1^k, n), \langle \hat{\partial}_n \rangle, q^{k(k+1)} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 \right)$$

is a CSP triple.

Proof. We describe a bijection from $\mathrm{SSYT}(2^k 1^k/1^k, n)$ to $\mathrm{BW}(n,k) \times \mathrm{BW}(n,k)$. Let $T \in \mathrm{SSYT}(2^k 1^k/1^k, n)$. The corresponding pair of binary words $(\mathsf{b}_1, \mathsf{b}_2)$ is constructed as follows. Write $\mathsf{b}_1 = \mathsf{b}_{11} \ldots \mathsf{b}_{1n}$ and let $\mathsf{b}_{1i} = 1$ if T has an i in the left column, and otherwise, let $\mathsf{b}_{1i} = 0$. In an analogous way, let b_2 be determined by the entries in the right column of T. It is easy to see that this bijection is invariant, if the corresponding group action is rot_n on the pair of binary words. Here, rot_n acts by cyclically shifting both of the words one step. It follows from [RSW04, Thm. 1.1] that

$$\left(\mathrm{BW}(n,k) \times \mathrm{BW}(n,k), \mathrm{rot}_n, \begin{bmatrix} n \\ k \end{bmatrix}_q^2 \right)$$

exhibits the cyclic sieving phenomenon. Since the two different CSP polynomials agree at n^{th} roots of unity, this completes the proof. □

It is natural to ask if the first part of Theorem 7.3 and Theorem 7.5 generalize to skew shapes. We would then hope that k-promotion acting on $\mathrm{SSYT}(2^k 1^s/1^s, n)$, for $1 < s < k-1$, has order n. However, this is not the case, as for $n = 4$, $k = 2$, $s = 1$, the tableaux

$$\begin{array}{cc} & \boxed{3} \\ \boxed{1} & \boxed{4} \\ \boxed{2} & \end{array}, \quad \begin{array}{cc} & \boxed{1} \\ \boxed{2} & \boxed{4} \\ \boxed{3} & \end{array}, \quad \begin{array}{cc} & \boxed{2} \\ \boxed{1} & \boxed{3} \\ \boxed{4} & \end{array}$$

form an orbit under k-promotion, but we want a group action of order 4.

Perhaps some other group action gives a CSP triple with (7.3) as the CSP polynomial. In a recent preprint, Y.-T. Oh and E. Park [OP21] the authors show some closely related results, regarding cyclic sieving on SSYT.

8. Triangulations of n-gons with k-ears

We shall now consider type A and type B triangulations of an n-gon. The main result in this section is a refinement of the CSP instance on triangulations of n-gons which are counted by $\mathrm{Cat}(n-2)$, see [RSW04, Thm. 7.1].

8.1. Refined CSP on triangulations by considering ears

Let $\mathrm{TRI}(n)$ denote the set of triangulations of a regular n-gon. If the vertices $i, i+1, i+2 \pmod{n}$ are pairwise adjacent for $T \in \mathrm{TRI}(n)$, we say they form an ***ear*** of T. Let $\mathrm{TRI}_{\mathrm{ear}}(n,k)$ denote the set of triangulations of a regular n-gon with exactly k ears, and let $\mathrm{Tri}(n,k)$ be the cardinality of this set. Note that in particular, $\mathrm{Tri}(3,3) = 1$. For all other pairs of n, k such that $0 \leqslant n/2 < k$, $\mathrm{Tri}(n,k) = 0$ and for all $n \geqslant 3$, $\mathrm{Tri}(n,0) = \mathrm{Tri}(n,1) = 0$. It was shown by F. Hurtado and M. Noy [HN96, Thm. 1] that

$$\mathrm{Tri}(n,k) = \frac{n}{k}\binom{n-4}{2k-4}\mathrm{Cat}(k-2)\cdot 2^{n-2k} \quad \text{whenever} \quad 2 \leqslant k \leqslant \frac{n}{2}. \tag{8.1}$$

We now introduce the following q-analog of the expression in (8.1). For integers n and k satisfying $2 \leqslant k \leqslant \frac{n}{2}$, let

$$\mathbf{Tri}(n,k;q) := q^{k(k-2)}\frac{[n]_q}{[k]_q}\begin{bmatrix} n-4 \\ 2k-4 \end{bmatrix}_q \mathbf{Cat}(k-2;q)\left(\sum_{j=0}^{n-2k} q^{j(n-2)}\begin{bmatrix} n-2k \\ j \end{bmatrix}_q\right).$$

At a first glance, one might hope that there is an easier expression for $\mathbf{Tri}(n,k;q)$. However, note that $\mathbf{Tri}(n,k;q)$ is not palindromic in general. As an example, consider $\mathbf{Tri}(6,2;q) = 1 + q^4 + q^5 + q^8$. This means that one cannot hope to find a formula for $\mathbf{Tri}(n,k;q)$ which only involves products of palindromic polynomials. In particular, it cannot be a product of q-binomial coefficients.

The choice of $\mathbf{Tri}(n,k;q)$ is motivated by the following theorem.

Theorem 8.1. *For all integers $n \geqslant 4$, we have that*

$$\sum_k \mathbf{Tri}(n,k;q) = \mathbf{Cat}(n-2;q). \tag{8.2}$$

In other words, the polynomials $\mathbf{Tri}(n,k;q)$ refine the q-Catalan numbers.

Proof. We first recall some notation from q-hypergeometric series, where we use [GR04, Appendix I-II] as the main reference. We set $(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$ so that

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_r(q;q)_{m-r}} \quad \text{and} \quad [m]_q = \frac{(q;q)_m}{(1-q)(q;q)_{m-1}}.$$

We have, [GR04, I.7–I.26], that for all a,

$$\begin{aligned}
(a;q)_{n+k} &= (a;q)_k(aq^k;q)_n, \\
(a;q)_{n-k} &= \frac{(a;q)_n}{(q^{1-n}/a;q)_k}\left(-\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}, \\
(aq^{-n};q)_n &= q^{-\binom{n}{2}}\left(-\frac{a}{q}\right)^n (q/a;q)_n, \\
(aq^{-n};q)_k &= q^{-\binom{n}{2}}\left(-\frac{a}{q}\right)^n \frac{(q/a;q)_n(a;q)_k}{(q^{1-k}/a;q)_n}, \\
(q^{-n};q)_k &= (-1)^k q^{\binom{k}{2}-nk}\frac{(q;q)_n}{(q;q)_{n-k}}, \\
(aq^n;q)_n &= \frac{(a;q)_{2n}}{(a;q)_n}.
\end{aligned}$$

Moreover, we let

$${}_2\phi_1\left[\begin{matrix} a & b \\ & c \end{matrix};q;z\right] := \sum_{n\geqslant 0}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}z^n.$$

The q-Chu–Vandermonde identity [GR04, II.6, II.7], can be stated in the following two ways:

$${}_2\phi_1\left[\begin{matrix} a & q^{-n} \\ & c \end{matrix};q;q\right] = \frac{(c/a;q)_n}{(c;q)_n}a^n \quad \text{and} \quad {}_2\phi_1\left[\begin{matrix} a & q^{-n} \\ & c \end{matrix};q;cq^n/a\right] = \frac{(c/a;q)_n}{(c;q)_n}. \tag{8.3}$$

We are now ready to prove Theorem 8.1, which is equivalent to proving that for all $n \geqslant 4$,

$$\sum_{k\geqslant 2} q^{k(k-2)}\frac{[n][n-1]}{[k][k-1]}\begin{bmatrix} n-4 \\ 2k-4 \end{bmatrix}_q \begin{bmatrix} 2k-4 \\ k-2 \end{bmatrix}_q \sum_{j=0}^{n-2k} q^{j(n-2)}\begin{bmatrix} n-2k \\ j \end{bmatrix}_q = \begin{bmatrix} 2n-4 \\ n-2 \end{bmatrix}_q. \tag{8.4}$$

After shifting the k-indices by 2, and the n-indices by 4, and multiplying both sides with $(1+q)$, we must show that

$$\sum_{k,j} R(k,j) = \frac{(1+q)\begin{bmatrix} 2n+4 \\ n+2 \end{bmatrix}_q}{[n+3][n+4]}, \tag{8.5}$$

where

$$\begin{aligned}
R(k,j) &= \frac{(1+q)q^{k(k+2)+j(n+2)}}{[k+1][k+2]}\begin{bmatrix} n \\ 2k \end{bmatrix}_q \begin{bmatrix} 2k \\ k \end{bmatrix}_q \begin{bmatrix} n-2k \\ j \end{bmatrix}_q \\
&= \frac{q^{k(k+2)}}{(q;q)_k(q^3;q)_k}\frac{q^{j(n+2)}(q;q)_n}{(q;q)_{n-2k-j}(q;q)_j} \\
&= (-1)^j \frac{q^{n(2k+j)-\binom{2k+j}{2}}q^{k(k+2)}}{(q;q)_k(q^3;q)_k}\frac{q^{j(n+2)}(q^{-n};q)_{2k+j}}{(q;q)_j}.
\end{aligned}$$

The right hand side of (8.5) simplifies to $\frac{(q^5;q)_{2n}}{(q^5;q)_n(q^3;q)_n}$. There is no issue with extending the summation index in (8.5) so that k, j ranges over all integers since $R(k,j)$ vanishes unless $0 \leqslant j \leqslant n-2k$. By shifting the indexing, so that $r := k$, $s := k+j$, it suffices to prove that

$$\sum_{r,s} S(r,s) = \frac{(q^5;q)_{2n}}{(q^5;q)_n(q^3;q)_n} \tag{8.6}$$

where

$$S(r,s) := R(r,s-r) = (-1)^{s+r} \frac{q^{2ns+2s-\binom{s+r}{2}} q^{r^2}}{(q;q)_r(q^3;q)_r} \frac{(q^{-n};q)_{s+r}}{(q;q)_{s-r}}.$$

By using the identities

$$(q^{-n};q)_{s+r} = (q^{s-n};q)_r(q^{-n};q)_s \text{ and } (q;q)_{s-r} = \frac{(q;q)_s}{(q^{-s};q)_r}(-1)^r q^{\binom{r}{2}-rs},$$

we have

$$\begin{aligned} S(r,s) &= (-1)^{s+r} \frac{q^{2ns+2s-\binom{s+r}{2}} q^{r^2}}{(q;q)_r(q^3;q)_r} (q^{s-n};q)_r(q^{-n};q)_s \frac{(q^{-s};q)_r(-1)^r q^{-\binom{r}{2}+rs}}{(q;q)_s} \\ &= \frac{(-q^{2n+2})^s q^{-\binom{s}{2}} (q^{-n};q)_s(q^{s-n};q)_r(q^{-s};q)_r}{(q;q)_r(q^3;q)_r(q;q)_s} \cdot q^r. \end{aligned}$$

Thus, (8.4) is equivalent to

$$\sum_{r,s} \frac{(-q^{2n+2})^s q^{-\binom{s}{2}} (q^{-n};q)_s(q^{s-n};q)_r(q^{-s};q)_r}{(q;q)_r(q^3;q)_r(q;q)_s} \cdot q^r = \frac{(q^5;q)_{2n}}{(q^5;q)_n(q^3;q)_n}.$$

But this follows from substituting $a = q^2q^n$ and $c = q^3$ in the following claim and then expanding the q-hypergeometric series, together with using the fact that $(q^5;q)_{2n} = (q^5;q)_n(q^nq^5;q)_n$.

Claim: For non-negative integers n, we have the identity

$$\sum_{k\geqslant 0} \frac{(-aq^n)^k q^{-\binom{k}{2}} (q^{-n};q)_k}{(q;q)_k} {}_2\phi_1\left[\begin{matrix} \frac{cq^k}{aq} & q^{-k} \\ & c \end{matrix} ; q; q\right] = \frac{(ac;q)_n}{(c;q)_n}. \tag{8.7}$$

To prove the claim start by applying the first q-Chu–Vandermonde identity. The left-hand side becomes

$$\sum_{k\geqslant 0} \frac{(-aq^n)^k q^{-\binom{k}{2}} (q^{-n};q)_k}{(q;q)_k} \frac{(aq/q^k;q)_k}{(c;q)_k} \left(\frac{cq^k}{aq}\right)^k.$$

Now, using the identity $(aq/q^k;q)_k = q^{-\binom{k}{2}}(-a)^k(1/a;q)_k$ we see that the left-hand side of (8.7) is equal to

$$\sum_{k\geqslant 0} \frac{(-aq^n)^k q^{-\binom{k}{2}} (q^{-n};q)_k}{(q;q)_k} \frac{q^{-\binom{k}{2}}(-a)^k(1/a;q)_k}{(c;q)_k} \left(\frac{cq^k}{aq}\right)^k.$$

Simplification gives

$$\sum_{k\geqslant 0}(acq^n)^k\frac{(q^{-n};q)_k(1/a;q)_k}{(c;q)_k(q;q)_k}={}_2\phi_1\left[\begin{matrix}1/a & q^{-n}\\ & c\end{matrix};q;acq^n\right].$$

This is now a special case of the second q-Chu–Vandermonde identity and we are done. □

A curious observation is that Theorem 8.1 refines the q-Catalan numbers in the same spirit as the following q-analog of Touchard's identity [And10, Thm. 1], which states that

$$\mathbf{Cat}(n+1;q)=\sum_{r\geqslant 0}q^{2r^2+2r}\begin{bmatrix}n\\2r\end{bmatrix}_q\mathbf{Cat}(r;q)\frac{(-q^{r+2};q)_{n-r}}{(-q;q)_r}.$$

We let rot_n act on $\mathrm{TRI}(n)$ by rotating a triangulation one step clockwise. As rot_n also preserves the set $\mathrm{TRI}_{\mathrm{ear}}(n,k)$, we have a group action of $\langle\mathrm{rot}_n\rangle$ on $\mathrm{TRI}_{\mathrm{ear}}(n,k)$.

Theorem 8.2. *For all integers $2\leqslant k\leqslant \frac{n}{2}$, the triple*

$$(\mathrm{TRI}_{\mathrm{ear}}(n,k),\langle\mathrm{rot}_n\rangle,\mathbf{Tri}(n,k;q))$$

exhibits the cyclic sieving phenomenon.

Proof. Let ξ be a primitive n^{th} root of unity and let d be an integer. Repeatedly using Theorem 2.4 and Lemma 2.5 yields

$$\mathbf{Tri}(n,k;\xi^d)=\begin{cases}\frac{n}{k}2^{n-2k}\binom{n-4}{2k-4}\mathrm{Cat}(k-2) & \text{if } d=n\\ \frac{n}{k}2^{n/2-k}\binom{n/2-2}{k-2}\binom{k-2}{k/2-1} & \text{if } \mathrm{o}(\xi^d)=2 \text{ and } 2\mid k\\ \frac{n}{k}2^{n/3-2k/3}\binom{n/3-2}{2k/3-2}\binom{2k/3-2}{k/3-1} & \text{if } \mathrm{o}(\xi^d)=3 \text{ and } 3\mid k\\ 0 & \text{otherwise.}\end{cases}$$

We prove that these evaluations agree with the number of fixed points in $\mathrm{TRI}_{\mathrm{ear}}(n,k)$ under rot_n^d on a case-by-case basis.

Case $d=n$: Trivial.

Case $\mathrm{o}(\xi^d)=2$ and $2\mid k$: Such a triangulation must have a diagonal (an edge from some i to $i+d$) that divides the triangulation into two halves. The diagonal can be chosen in $n/2$ ways. The triangulation is now determined uniquely by one of its halves. To choose one half, we choose a triangulation of an $(n/2+1)$-gon with $k/2$ ears whose sides do not coincide with the diagonal. Such a triangulation is either (i) an element of $\mathrm{TRI}_{\mathrm{ear}}(n/2+1,k/2)$ which does not have an ear with an edge coinciding with the diagonal, (ii) an element of $\mathrm{TRI}_{\mathrm{ear}}(n/2+1,k/2+1)$ which has an ear coinciding with the diagonal, or (iii) the unique element in $\mathrm{TRI}_{\mathrm{ear}}(3,3)$ (and this only happens when $n=4$). Based on the rotational symmetry, we note that $2k/n$ of the elements in $\mathrm{TRI}_{\mathrm{ear}}(n,k)$ have an ear that has an edge adjacent to a given side. Thus, the number of triangulations fixed by $\mathrm{rot}_n^{n/2}$ is equal to

$$\frac{n}{2}\left(\frac{n/2+1-k}{n/2+1}\mathrm{Tri}(n/2+1,k/2)+\frac{2(k/2+1)}{n/2+1}\mathrm{Tri}(n/2+1,k/2+1)\right),$$

if $n \geqslant 6$, or 2 if $n=4$. The case $n=4$ is easy to check, and after the substitution $m := n/2+1$, $r := k/2+1$, it is straightforward to verify that the expression above is equal to $\mathbf{Tri}(n,k;\xi^d)$.

Case $\mathrm{o}(\xi^d) = 3$ and $3 \mid k$: Similar to the above case. Such a triangulation must have a central triangle (a triangle with vertices i, $i+n/3$ and $i+2n/3 \pmod{n}$) that divides the triangulation into three parts. The central triangle can be chosen in $n/3$ ways. The triangulation is now determined uniquely by one of its three parts. Choosing one part is done with a similar argument as above, so the number of triangulations fixed by $\mathrm{rot}_n^{n/3}$ is equal to

$$\frac{n}{3}\left(\frac{n/3+1-2k/3}{n/3+1}\mathrm{Tri}(n/3+1,k/3)+\frac{2(k/3+1)}{n/3+1}\mathrm{Tri}(n/3+1,k/3+1)\right),$$

if $n \geqslant 9$, and 2 if $n=6$. By checking the case $n=6$ separately, and then using the substitution $m := n/3+1$, $r := k/3+1$, it is straightforward to show that this expression matches the one for $\mathbf{Tri}(n,k;\xi^d)$.

The remaining cases : If $\mathrm{o}(\xi^d)=2$ (or 3), it is clear that any triangulation fixed by rot_n^d must have a diagonal (or, respectively, a central triangle). Thus each of the two halves (or, respectively, three parts) must have the same number of ears and hence $2 \mid k$ (or $3 \mid k$). If $\mathrm{o}(\xi^d) > 3$, then it is clear that there are no fixed triangulations under the action of rot_n^d. This exhausts all possibilities and thus the proof is done. □

Problem 8.3. One might ask if there are further refinements. One attempt that fails is that $\frac{n}{k}\binom{n-2k}{j}\binom{n-4}{2k-4}\mathrm{Cat}(k-2)$, $0 \leqslant j \leqslant n-2k$, do not refine $\frac{n}{k}2^{n-2k}\binom{n-4}{2k-4}\mathrm{Cat}(k-2)$ since the former is not always an integer, for example, at $n=5, k=2, j=1$.

Remark 8.4. Unfortunately, there is no Narayana refinement of rotation acting on triangulations. To see this, observe that $\mathbf{Nar}(2,2;q)=q^2$ but at a 4^{th} root of unity ξ, we have $\mathbf{Nar}(2,2;\xi)=-1$.

8.2. Triangulations of type B

Let us define *type B triangulations* $\mathrm{TRI}^B(n)$, as the set of elements in $\mathrm{TRI}(2n)$ which are invariant under rotation by a half-turn. In such a triangulation, there is always an edge through the center. There are n choices of such an edge, and then we need to choose a triangulation on one half of the $2n$-gon. This gives $n \cdot \frac{1}{n}\binom{2(n-1)}{n-1} = \binom{2(n-1)}{n-1}$ such type B triangulations. The following result is straightforward to prove but also follows from [EF08, Thm. 4.1].

Proposition 8.5 (See [EF08, Thm. 4.1].). *The triple*

$$\left(\mathrm{TRI}^B(n), \langle \mathrm{rot}_n \rangle, \mathbf{Cat}^B(n-1;q)\right)$$

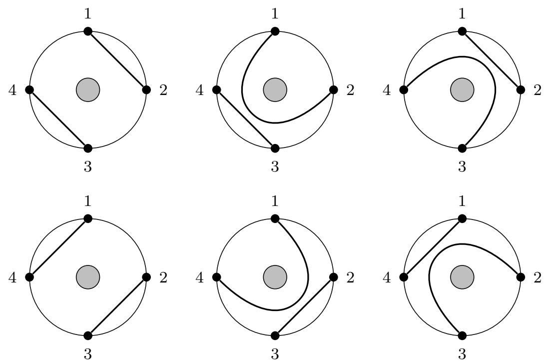
exhibits the cyclic sieving phenomenon.

The polynomial $[n]_q\mathbf{Tri}(n+1,k;q)$ does not seem to give a refinement of Proposition 8.5.

9. Marked non-crossing matchings

A *marked non-crossing matching* is a non-crossing matching where some of the regions have been marked. Let $\mathrm{NCM}^{(r)}(n)$ denote the set of marked non-crossing matchings with exactly r marked regions. Since every non-crossing matching in $\mathrm{NCM}(n)$ has $n+1$ regions, it follows that $|\mathrm{NCM}^{(r)}(n)| = \binom{n+1}{r}|\mathrm{NCM}(n)|$.

In particular, for $r = 1$ we can think of our objects as non-crossing matchings of vertices on the outer boundary of an annulus rather than on a disk.



This model is reminiscent of the *non-crossing permutations* considered in [Kim13], where points on the boundary of an annulus are matched in a non-crossing fashion, but with some other technicalities imposed.

The following generalizes Proposition 4.1.

Theorem 9.1. *Let* $1 \leqslant k \leqslant n$ *and* $0 \leqslant r \leqslant n+1$. *Then*

$$\left(\{M \in \mathrm{NCM}^{(r)}(n) : \mathrm{even}(M) = k\}, \langle \mathrm{rot}_n \rangle, \mathbf{Nar}(n, k+1; q) \begin{bmatrix} n+1 \\ r \end{bmatrix}_q \right)$$

is a CSP triple.

Proof. Consider elements of $\mathrm{NCM}(n)$ with k even edges, fixed by rot_n^d, where we may without loss of generality assume $d \mid n$. As noted in Section 4.1, $\mathrm{even}(M)$ is invariant under rot_n. Write $n = md$. By the q-Lucas theorem (Theorem 2.4),

$$\begin{bmatrix} n+1 \\ r \end{bmatrix}_q = \begin{cases} \binom{d}{r/m} & \text{if } m \mid r, \\ \binom{d}{(r-1)/m} & \text{if } m \mid r-1, \\ 0 & \text{otherwise} \end{cases}$$

at a primitive m^{th} root of unity.

Divide the $2n$ vertices of a non-crossing matching with n/d-fold rotational symmetry (remember that rot_n rotates the vertices two steps) into m segments of length $2d$, say $[2d]$, $[4d] \setminus [2d]$, and so on. The d edges going from $[2d]$ to higher vertices (in $[4d]$ including $[2d]$) each have a unique region to their left. This means that the number of marked regions r must be jm, or $jm+1$ if there is a central region which is marked, see Figure 9.1. We can thus get a fixed

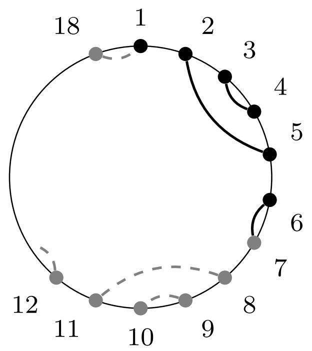


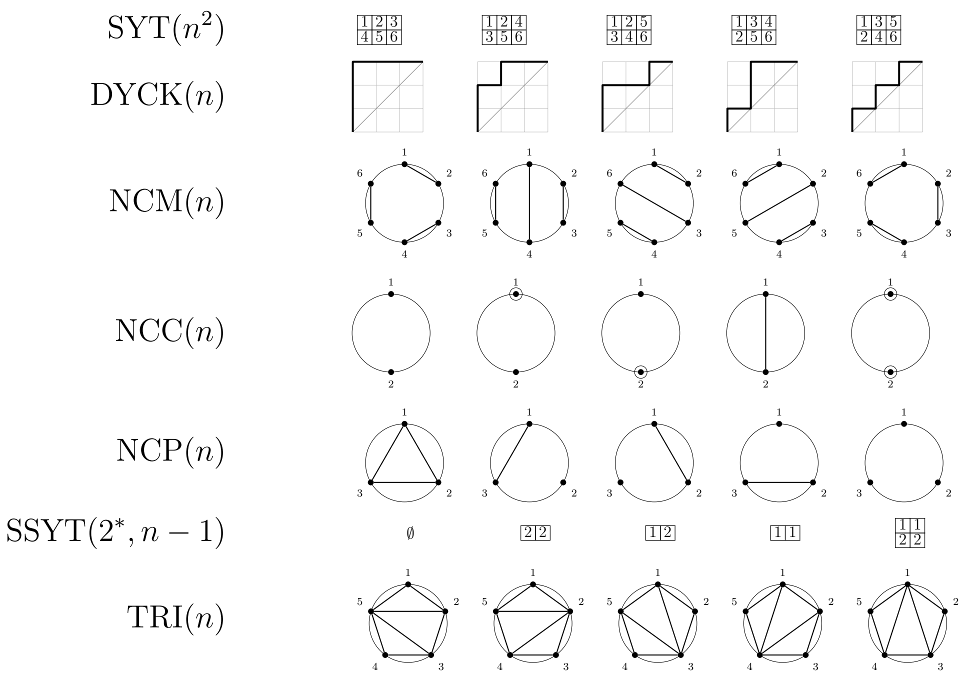
Figure 9.1: Partitioning a non-crossing perfect matching of size $2n = 18$ with n/d-fold rotational symmetry, $d = 3$, into segments of length $2d$. Each of the three edges from $[2d]$ to vertices with bigger labels has a unique region to its left.

point of rot_n^d by choosing to mark $j = r/m$ (or, in the latter case, $(r-1)/m$) of the d regions associated to the edges from $[2d]$ to vertices with larger labels, which gives $\binom{d}{r/m}$ and $\binom{d}{(r-1)/m}$ respectively. Now, we combine this with Proposition 4.1 and the theorem follows. □

A. Catalan objects

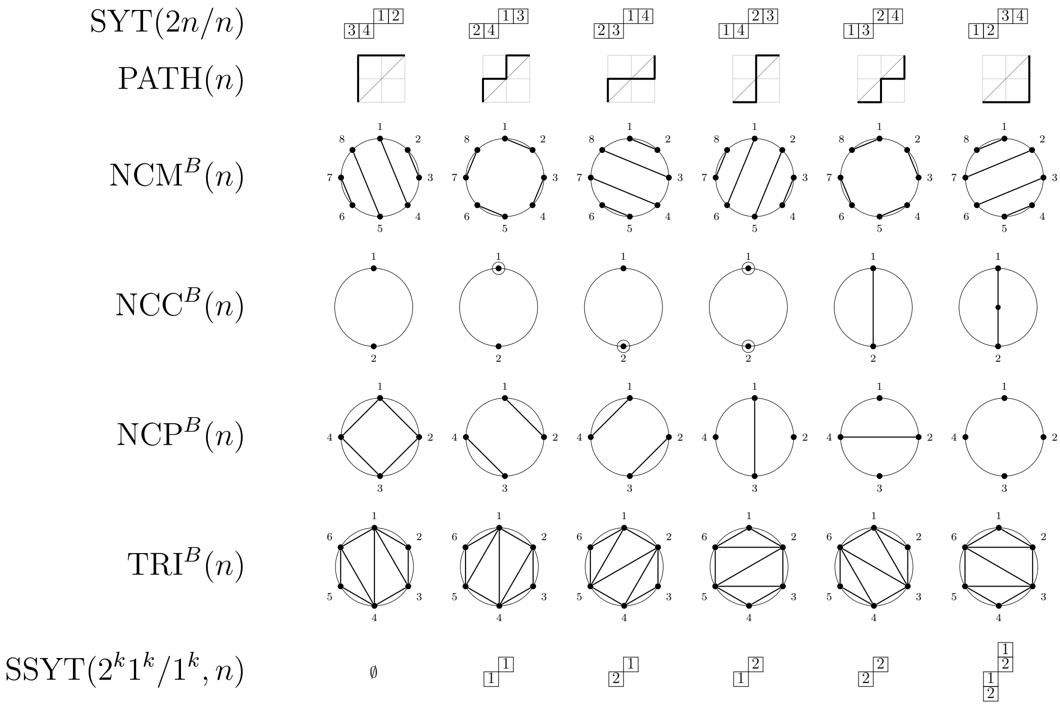
A.1. Type A objects

Below is an overview of the Catalan objects we consider for $n = 3$. Recall that $\mathrm{Cat}(3) = 5$. The orders of the first six sets of objects follow the bijections given in Figure 2.1.



A.2. Type B objects

Below is an overview of the considered type B Catalan objects for $n = 2$. Recall that $\mathrm{Cat}^B(2) = 6$.



B. Bijections

Here we recall several bijections on Catalan objects which have appeared earlier in the literature. We have tried to find the earliest reference for each. The order of the objects in Appendix A serves as examples of Bijections 5, 7 and 8.

B.1. NCM and binary words

Suppose xy is an edge in a non-crossing perfect matching, with $x < y$. We say that x is the *starting vertex* and y is the *end vertex*. Further, denote $\mathrm{NCM}_{\mathrm{sh}}(n, k)$ the subset of all $N \in \mathrm{NCM}(n)$ such that $\mathrm{short}(N) = k$.

We now describe a well-known bijection `NCMtoDYCK` from $\mathrm{NCM}(n)$ to $\mathrm{DYCK}(n)$.

Bijection 4. Take $M \in \mathrm{NCM}(n)$ and construct the Dyck path $\texttt{NCMtoDYCK}(M) = \mathsf{b}_1\mathsf{b}_2 \cdots \mathsf{b}_{2n}$ as follows. For vertices $i = 1, 2, \dots, 2n$ in M, let $\mathsf{b}_i = 0$ if i is a starting vertex and let $\mathsf{b}_i = 1$ if i is an end vertex. It is not hard to see that this procedure ensures that the resulting binary word is a Dyck path.

Let d be a natural number such that $d \mid n$. If a matching $M \in \mathrm{NCM}(n)$ has $2n/d$-fold rotational symmetry, it is sufficient to understand how the vertices $1, 2, \dots, d$ are matched up.

Here, we restrict ourselves to the case when M does not have a diagonal—for the case when M has a diagonal, see the third case in the proof of Theorem 4.8. In this case, there is a bijection $\mathtt{BWtoNCM}$ between $\mathrm{BW}(d, d/2)$ and rotationally symmetric non-crossing perfect matchings.

Bijection 5. Let $\mathsf{c} = \mathsf{c}_1\mathsf{c}_2 \cdots \mathsf{c}_d \in \mathrm{BW}(d, d/2)$. We show how to construct the corresponding $\mathtt{BWtoNCM}(\mathsf{c}) \in \mathrm{NCM}(n)$. Think of the vertices $1, 2, \dots, d$ being arranged on a line. For all vertices $i = 1, 2, \dots d$, we let vertex i be a "left" vertex if $\mathsf{c}_i = 0$ and a "right" vertex if $\mathsf{c}_i = 1$. Then we pair up every left vertex with a right vertex directly to its right if such a vertex exist. Otherwise we recursively pair it with the closest available right vertex to its right without creating a crossing of edges. It is well-known there is a unique way of doing this. There might be some left vertices which are not paired up because there are not enough right vertices to their right. There must also be equally many unpaired right vertices because there are not enough left vertices to their left. Since this will be the same in every interval of vertices $[dk + 1, d(k + 1)]$ there is a unique way to pair the remaining left vertices with the remaining right vertices of the interval to the right and vice versa. This also shows that d must be even.

We can prove a stronger statement about rotationally symmetric NCMs. We study $\mathtt{BWtoNCM}$ restricted to $\mathrm{NCM}_{\mathrm{sh}}(n, k)$. Once again, such a matching is completely determined by how vertices $1, 2, \dots, d$ are paired up. Among these d vertices, there are two possible cases to consider.

Case 1: Exactly $dk/(2n)$ short edges where either vertex 1 is a left vertex or vertex $2d$ is a right vertex.

Case 2: Exactly $dk/(2n) - 1$ short edges where vertex 1 is a right vertex and vertex $2d$ is a left vertex.

If one applies $\mathtt{BWtoNCM}^{-1}$ (that is, left vertex corresponds to $\mathsf{0}$ and right vertex corresponds to $\mathsf{1}$) to the non-crossing perfect matchings in the two above cases, one gets the image $\mathrm{BW}^{dk/(2n)}(d)$.

B.2. NCM and SYT

We recall the definition of this bijection.

Bijection 6. Let $T \in \mathrm{SYT}(n^2)$ and construct $\mathtt{SYTtoNCM}(T)$ as follows. For $i \in \{1, 2, \dots, 2n\}$, let vertex i be a starting vertex if i is in the first row and an end vertex otherwise. It is not hard to show that determining the starting and ending vertices uniquely determines a non-crossing matching.

B.3. NCP and NCM

There is a bijection between non-crossing partitions and non-crossing matchings, $\mathtt{NCPtoNCM} : \mathrm{NCP}(n) \to \mathrm{NCM}(n)$, that directly restricts to a bijection between NCP^B and NCM^B. This bijection has another nice property; it is equivariant with regards to the Kreweras complement on the non-crossing perfect matchings and rotation on the non-crossing perfect matchings [Hei07, Thm. 1].

Bijection 7. Consider $\pi \in \mathrm{NCP}(n)$, and for every vertex $j \in \{1, \dots, n\}$, insert a new vertex $(2j - 1)'$ immediately after j, and $(2j - 2)'$ immediately before j, where we insert $(2n)'$ immediately before 1. There we match $(2j)'$ to the closest point $(2k - 1)'$, $j < k$, such that the edge between those vertices does not cross any of the blocks in π. If no such k exists, we match to

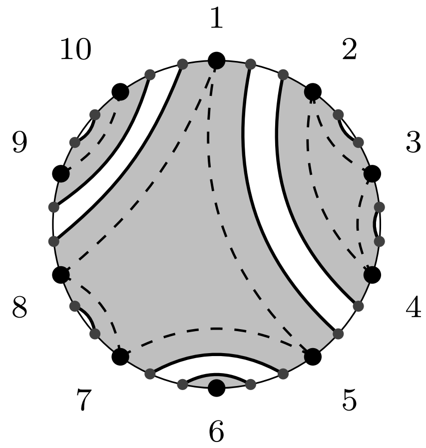
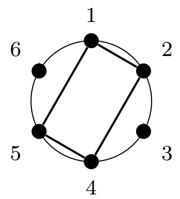


Figure B.1: A non-crossing partition given by the dashed edges and the corresponding non-crossing matching given by the solid edges.

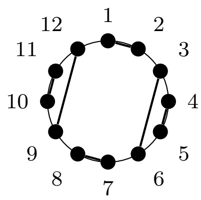
the smallest number $k, 0 < k \leqslant j$. Since $j = k$ is always possible the map is well-defined. This gives a perfect matching σ on $1', 2', \ldots, (2n)'$ which is non-crossing, and in addition, does not cross any of the blocks of π. We can get the inverse of the map `NCPtoNCM` by putting back the vertices $1, 2, \ldots, n$ and letting all vertices that can have an edge between them without crossing any edge of the perfect matching σ belong to the same block.

The following is an example of this bijection.

`NCPtoNCM` :



$\rightarrow$



(B.1)

One can also illustrate the bijection as follows. For each singleton block, add an edge between two copies of the vertex. For each block of size two, split the edge into two non-crossing edges and hence each vertex into two copies. Finally, for each block with $m \geqslant 3$ elements, push apart the edges at the vertices so that we have m non-crossing edges on $2m$ vertices. See Figure B.1.

Note that, by definition, every block except the one whose minimum element is 1 corresponds to an even edge under the bijection. If $j > 1$ is the smallest element of a block and k is the largest, then $(2j - 2)'$ is matched to $(2k - 1)'$. The remaining even vertices are matched to smaller odd vertices. Hence Bijection 7 in fact gives a bijection between the sets $\mathrm{NCP}(n, \ell)$ and $\mathrm{NCM}(n, \ell - 1)$.

B.4. NCP and Dyck paths

We briefly describe a bijection between non-crossing partitions and Dyck paths, `NCPtoDYCK`, with the property that the number of parts is sent to the number of peaks. This bijection is often attributed to P. Biane [Bia97].

Bijection 8. Let $\pi = B_1|B_2|\cdots|B_k$ be a non-crossing partition of size n, where the blocks are ordered increasingly according to maximal member. By convention, we let B_0 be a block of size 0, where the maximal member is also 0. We construct a Dyck path $D \in \mathrm{DYCK}(n)$ from π as follows. For each $j = 1, \ldots, k$, we have a sequence of $\max(B_j) - \max(B_{j-1})$ north steps, immediately followed by $|B_j|$ east steps.

C. Jeu-de-taquin and k-promotion

Here we recall the definition of jeu-de-taquin and how one defines promotion and k-promotion in terms of it.

Jeu-de-taquin was introduced by M. P. Schützenberger [Sch72] and transforms a skew semistandard Young tableau into a non-skew one. There are two types of operations called *slides*. We assume $a < b \leqslant c$ and $a \leqslant b < c$, respectively. The boxes containing b and c may or may not be present.

$$\begin{array}{cc} \bullet & a \\ b & c \end{array} \longrightarrow \begin{array}{cc} a & \bullet \\ b & c \end{array} \qquad\qquad \begin{array}{cc} \bullet & b \\ a & c \end{array} \longrightarrow \begin{array}{cc} a & b \\ \bullet & c \end{array}$$

A $\bullet$ represents an empty box. The sliding procedure applied to a skew tableau T stops when a non-skew shape has been reached. It turns out that the result is independent of the order of the slides.

The operation *k-promotion* is a generalization of promotion to semistandard Young tableaux with entries at most k, see [Ful97]. The k-promotion operator, $\hat{\partial}_k$, may be defined via jeu-de-taquin as follows. Given $T \in \mathrm{SSYT}(\lambda, k)$, replace all entries equal to k by $\bullet$'s. Perform inverse jeu-de-taquin slides on the resulting tableau, always moving the leftmost $\bullet$ not in the north-west corner. Add 1 to all entries in the tableau and finally replace the $\bullet$'s by 1s. The result is a semistandard tableau of shape λ. k-promotion works similarly for skew tableaux (but note that there are several north-west corners). *Promotion* can be seen as k-promotion restricted to standard Young tableaux.

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