Title
Martin-Löf Randomness and Brownian Motion

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Publication Date
2014

Peer reviewed|Thesis/dissertation

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by

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A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Logic and the Methodology of Science

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2014

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Abstract

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We investigate the Martin-Löf random sample paths of Brownian motion, applying techniques from algorithmic randomness to Brownian motion, an active area of research in probability theory.

In Chapter 2, we investigate many classical results about one-dimensional Brownian motion in the context of Martin-Löf randomness. We show that many results which are known to hold almost surely for the Brownian motion process - including results concerning the modulus of continuity, points of increase, time inversion, and law of large numbers - hold for every Martin-Löf random sample path. We also show that scaling invariance and the strong Markov property hold for every Martin-Löf random path, with suitable effectivization.

In Chapter 3, we investigate the zero set of one-dimensional Brownian motion. We prove that the set of zeroes is characterized by having high effective dimension. We also demonstrate that, although the zeroes are highly noncomputable in the sense of effective dimension, many of them are layerwise computable from a Brownian path.

In Chapter 4, we give a new proof that the solution to the Dirichlet problem in the plane is computable. It is a well-known result of Kakutani that the solution to the Dirichlet problem can be found using expected hitting times of Brownian paths to the boundary. We show that the hitting times of Martin-Löf random Brownian paths on a computable boundary are layerwise computable in the path, and thus the expected value of the hitting times is computable, and so the solution to the Dirichlet problem is computable.

In Chapter 5, we further investigate planar Martin-Löf random Brownian motion. We demonstrate that a Martin-Löf random planar Brownian path only hits points such that the path is not random relative to those points (except the origin), which implies that a Martin-Löf random planar path has area zero and that every point except the origin is hit by only measure 0 many paths. We also show that every Martin-Löf random planar Brownian path has points of uncountable multiplicity.

Contents

Acknowledgments

First, I would like to thank my advisor, Ted Slaman, for his support and guidance. He provided excellent mathematical advice and was an unfailingly kind and patient mentor throughout my years in graduate school. I would also like to thank Laurent Bienvenu, with whom I collaborated on much of this work. He is an excellent mathematician and a pleasure to work with. I am also grateful to Jason Rute and Doug Rizzolo for helpful conversations about this material.

I would also like to thank the math department staff, especially Marsha Snow and Barb Waller, for being so kind and so helpful so very many times.

I gratefully acknowledge the support I have received in my last year from a University of California Dissertation Year Fellowship.

I would like to thank the many friends who made graduate school so enjoyable. I am grateful to Adam Boocher, Morgan Brown, Taryn Flock, Mike Hartglass, Ian Herbert, Katrina Honigs, Ralph Morrison, James Tener, Ivan Ventura, Cynthia Vinzant, and many others.

Lastly, I wish to thank my parents, Ann and Ken Allen, my brother, Patrick Allen, and my fiancé, Charley Crissman, for all their love and encouragement. Thank you for always believing in me.

Chapter 1

Introduction and Background

1.1 Introduction

Background material in computability theory will mostly be based on the books "Computability and Randomness" by Nies [27] and "Algorithmic Randomness and Complexity" by Downey and Hirschfeldt [8]. Background in Brownian motion will mostly be based on the book "Brownian Motion" by Mörters and Peres [26] and lecture notes of Peres [28]. Background in probability theory can be found in Durrett [9].

Algorithmically random Brownian motion uses tools from algorithmic randomness to study Brownian motion. Heuristically, Brownian motion is the random continuous function resulting from the limit of discrete random walks as the time interval approaches zero. The paths of Brownian motion are considered typical with respect to *Wiener measure* on a function space, generally $C[0,1]$, $C[\mathbb{R}^{\geq 0}]$, or $C[I,\mathbb{R}^n]$ for $I=[0,1]$ or $[0,\infty)$

Computability theory provides a collection of tools to formulate and study questions about randomness. Intuitively, algorithmically random elements of a measure space are those which appear random to any algorithm. The most widely studied notion of algorithmic randomness is known as Martin-Löf randomness, but there exist stronger and weaker forms of randomness. The Martin-Löf random (or Schnorr random, etc.) elements of a function space with respect to Wiener measure are known as Martin-Löf random (or Schnorr random, etc.) Brownian motion. Fouché showed that the class of Martin-Löf random Brownian motion is the same as the class of complex oscillations, a class of functions defined by Asarin and Pokrovskii and later investigated to a greater degree by Fouché, Kjos-Hanssen, Nerode, and Szabados. [1], [11], [12], [10], [20], [21]

The study of Martin-Löf random Brownian motion provides insight into both classical Brownian motion and the power of algorithmic randomness. In chapters two and five, we will see how Martin-Löf random Brownian motion exhibits much of the "almost everywhere" behavior observed in classical Brownian motion, providing an example of a particular measure one set for which many of the important properties of Brownian motion hold. In chapter three, we will see that the effective properties of Martin-Löf random Brownian paths are also quite fascinating. For example, the zeros of Martin-Löf Brownian motion are characterized by having high "effective dimension" – meaning they are difficult to locate algorithmically. In chapter four, we will show that techniques from algorithmic randomness combined with a beautiful classic result applying Brownian motion to the Dirichlet problem provide a new proof that solution to the Dirichlet problem is computable.

1.2 Notation and Definitions in Computability Theory

We use $\mathbb{N}$ to denote the natural numbers, identified with the least countably infinite ordinal ω. $2^{<\omega}$ will denote the set of finite binary strings and 2^{ω} will denote Cantor space, the set of infinite binary strings, identified both with binary expansions of real numbers in $[0, 1]$ and with subsets of $\mathbb{N}$. For $\sigma \in 2^{<\omega}$, $|\sigma|$ denotes the length of σ.

Definition 1.2.1.

1. A *partial computable function* is a partial function $f : \mathbb{N} \to \mathbb{N}$ that can be computed by some deterministic algorithm (i.e. by a Turing machine), whose domain may not be total.

2. A partial computable f which converges on every input is said to be *computable*.

Definition 1.2.2.

Definition 1.2.3.
A *machine* $\mathcal{M}$ is a partial recursive function $\mathcal{M} : 2^{<\omega} \to 2^{<\omega}$. M is *prefix-free* if for any $\sigma,\ \tau \in 2^{<\omega}$ such that σ is an initial segment of τ, if $\mathcal{M}(\sigma)$ converges then $\mathcal{M}(\tau)$ converges.

A prefix-free machine induces a notion of complexity on $2^{<\omega}$ given by $K_{\mathcal{M}}(\sigma) = \min\{|\tau| : \mathcal{M}(\tau) = \sigma\}$, that is, the complexity of σ relative to $\mathcal{M}$ is the length of the shortest τ that can be used as a code for σ. The following is folklore:

Theorem 1.2.1. *There is a prefix-free machine $\mathcal{U}$ such that for any other prefix-free machine $\mathcal{M}$, there is a constant $c_{\mathcal{M}}$ such that for all $\sigma \in 2^{<\omega}$,*

$$K_{\mathcal{U}}(\sigma) \leq K_{\mathcal{M}}(\sigma) + c_{\mathcal{M}}$$

$\mathcal{U}$ is known as a universal prefix-free machine.

Fixing some universal machine $\mathcal{U}$, we have the following definition:

Definition 1.2.4.
For a finite binary string σ, the *prefix-free Kolmogorov complexity* of σ is

$$K(\sigma) = \min\{|\tau| : \mathcal{U}(\tau) = \sigma\}.$$

This will sometimes be referred to simply as the *Kolmogorov complexity* of a string.

1.3 Randomness in Computability Theory

In trying to define what makes a set "random," one intuitive notion is that such a set should not have any rare properties, and that it should be difficult to describe. In computability theory, there are many rigorous notions of what it means to be "describable," and these notions can be used to form different definitions of randomness. One such notion is the idea of *tests* that a random number should pass. [24] These definitions were developed in the context of Cantor space and the study of randomness is still primarily focused on the study of random real numbers, but most definitions and results given are applicable in any computable probability space.

Definition 1.3.1

- A *Martin-Löf test* on a computable probability space (X, μ) is a uniformly Σ^0_1 sequence $(A_n)_{n\in\mathbb{N}}$ of open sets in X such that $\forall n \in \mathbb{N},\ \ \mu(A_n) \le 2^{-n}$
- A set $Z \in X$ *fails* the test if $Z \in \bigcap_n A_n$, otherwise Z *passes* the test.
- $Z \in X$ is *Martin-Löf random* if Z passes each Martin-Löf test.

We say that a Martin-Löf test $(U_n)_{n\in\mathbb{N}}$ is *universal* if $\bigcap_n A_n \subset \bigcap_n U_n$ for any Martin-Löf test $(A_n)_{n\in\mathbb{N}}$.

Theorem 1.3.1 (Martin-Löf [24]). *Let $U_n = \bigcup_{e\in\mathbb{N}} A^e_{n+e+1}$. Then $(U_n)_{n\in\mathbb{N}}$ is a universal Martin-Löf test.*

Martin-Löf randomness was one of the first rigorous notions of algorithmic randomness developed, and has proved to be one of the most robust. Several equivalent definitions have been found for Martin-Löf randomness.

One useful equivalence is the notion of *Solovay Randomness.*

Definition 1.3.1.

We say that an element x of a computable probability space is *Solovay random* iff for all computable collections of c.e. open sets $\{U_n\}$ such that $\sum_n \mu(U_n) < \infty$, x is in only finitely many U_i.

Theorem 1.3.2 (Solovay). *A real x is Solovay random iff it is Martin-Löf random.*

Martin-Löf randomness in Cantor space can also be characterized using the initial segment complexity of a string.

Theorem 1.3.3 (Levin [22], Schnorr [35]). *The following are equivalent for a set Z.*

- *Z is Martin-Löf random.*

- $\exists b \forall n [K(Z \upharpoonright n) > n - b]$

There also exist stronger and weaker notions of randomness. In computable analysis and in the study of algorithmically random Brownian motion, we frequently encounter *Schnorr tests*.

Definition 1.3.2.
A *Schnorr test* $\{U_n\}, n \in \mathbb{N}$ is a computable collection of c.e. open sets such that $\mu(U_n) \leq 2^{-n}$ and the function $f(n) = \mu(U_n)$ is a computable function of n.

Definition 1.3.3.
A real x is *Schnorr random* iff it passes all Schnorr tests.

Schnorr randomness is a weaker notion than Martin-Löf randomness, and although it seems a natural notion in much of computable analysis, it has attracted less attention in computability theory than Martin-Löf randomness. This is in part because Martin-Löf randomness is enough for many results, as well as the fact that historically, Schnorr randomness has proved harder to work with; for example it has no universal test. See [7] and [6] for more discussion of Schnorr randomness and the many other types of algorithmic randomness.

1.4 Computable analysis

A real x is *computable* if there is a computable function f such that f takes an input $\varepsilon \in \mathbb{Q}$, and outputs a rational r such that $|x - r| < \varepsilon$.

A sequence of reals x_i, $i \in \mathbb{N}$ is *uniformly computable* if there exists a function f which takes input $< i, \varepsilon >$ for $\varepsilon \in \mathbb{Q}$, and outputs a rational r such that $\|x_i - r\| < \varepsilon$.

Definition 1.4.1.
A *computable metric space* is a triple (X, μ, S) where:

- (X, μ) is a separable complete metric space
- $S = \{s_i : i \in \mathbb{N}\}$ is a countable dense subset of X,
- The real numbers $\mu(s_i, s_j)$ are computable uniformly in $< i, j >$.

Definition 1.4.2.
For (X, μ) a probability space, a measurable map $T : X \to X$ is *measure preserving* if for all measurable $A \subset X$, $\mu(T^{-1}A) = \mu(A)$.

Definition 1.4.3.
We say that a measurable set $A \subset X$ is *invariant* under a map $T : X \to X$ if $T^{-1}A = A$ up to a set of measure zero.

Definition 1.4.4.
A measure-preserving map $T : X \to X$ is *ergodic* if every T-invariant measurable subset of X has measure 0 or measure 1.

The following theorem provides a useful relationship between Martin-Löf random reals and ergodic maps. See [3, 13] for proofs and further discussion of the relationship between ergodicity and algorithmic randomness.

Theorem 1.4.1 (Bienvenu et al.). *Let (X, μ) be a computable probability space and $T : X \to X$ a computable, ergodic, measure-preserving map. Let A be a Σ^0_1 set with $\mu(A) < 1$. Then any Martin-Löf random $x \in X$ has $T^n(x) \notin A$ for infinitely many n.*

1.5 Notation and Definitions in Probability Theory

Definition 1.5.1.
A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is a set (a set of outcomes), $\mathcal{F}$ is a σ-algebra on Ω (a set of events), and $\mathbb{P}$ is a probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$.

Definition 1.5.2.
A (real-valued) *random variable* is a function from Ω to $\mathbb{R}$.

Definition 1.5.3.
A random variable X has a *normal distribution* with mean μ and variance σ if

$$\mathbb{P}(X > a) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^\infty e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy, \quad \text{for all } a \in \mathbb{R}$$

Definition 1.5.4.
A *stochastic process* is a collection of random variables indexed by a totally ordered set T (time), that is,

$$\{X_t : t \in T\}$$

where each X_t is a random variable on some probability space Ω.

1.6 Classical Brownian Motion

Brownian motion is of great theoretical interest and practical value. It arises as a limit of random walks as the time interval goes to zero, as a model of the stock market [2] and as a model of the random motion of particles in a fluid, originally described by Robert Brown, from whom Brownian motion takes its name.

We begin by defining one-dimensional Brownian Motion on $C[0, 1]$.

Brownian motion as a random function

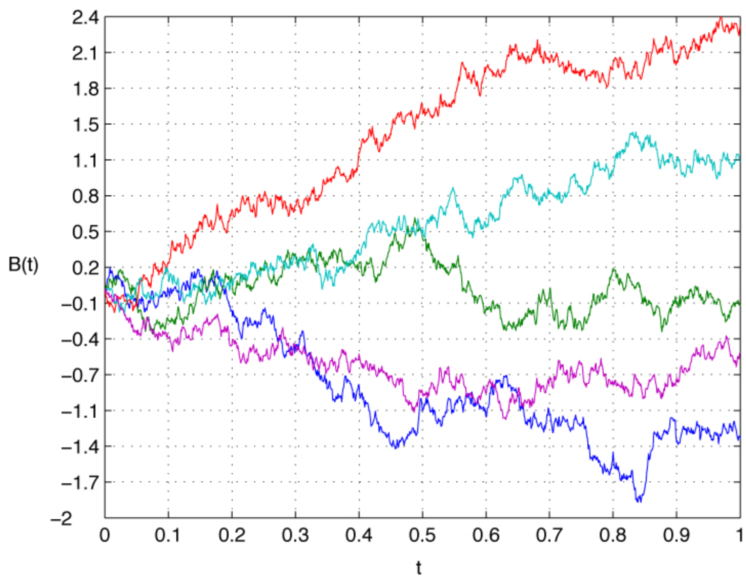

Fig. 1.1. Graphs of five sampled Brownian motions

Image from "Brownian Motion" by Mörters and Peres

Definition 1.6.1.

A real-valued stochastic process $\{B(t) : t \in I\}$ is called *standard Brownian motion* if the following holds:

- B(0) = 0,
- The process has *independent increments*: for all times $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, the increments $B(t_n) - B(t_{n-1}), B(t_{n-1}) - B(t_{n-2}), \ldots B(t_2) - B(t_1)$ are independent random variables,
- for all $t \geq 0$ and $h > 0$, the increments $B(t+h) - B(t)$ are normally distributed with expectation 0 and variance h,
- almost surely, the function $t \mapsto B(t)$ is continuous.

This defines Brownian motion as a stochastic process, that is, a family of uncountably many random variables $\omega \mapsto B(t, \omega)$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, but we can also interpret this as a collection of random functions with the sample paths defined by $t \mapsto B(t, \omega)$. Much of the study of classical Brownian motion focuses on the sample path properties of these random functions, and this thesis is devoted to the study of the properties of Martin-Löf random sample paths.

These requirements induce a measure on a function space called Wiener measure, and which we will denote by $\mathbb{P}$. It is possible to define Brownian motion starting at any point x at time 0, rather than starting at the origin, in which case we will denote the corresponding measure by $\mathbb{P}_x$ (in other words, $\mathbb{P}_x(\mathscr{B} \in \mathcal{A}) = \mathbb{P}(x + \mathscr{B} \in \mathcal{A})$). When we wish to emphasize that we are talking about standard Brownian motion, we will use $\mathbb{P}_0$. *Martin-Löf random Brownian motion* is the collection of functions which are Martin-Löf random with respect to Wiener measure.

We denote by $\mathfrak{p}(t, x, u)$ the transition density of Brownian motion, i.e., the unique function such that

$$\mathbb{P}_x(B(t) \in A) = \int_A \mathfrak{p}(t, x, u)du$$

1.7 Construction of Brownian Motion

The construction presented here is the Franklin-Wiener series representation of Brownian Motion as found in Kahane [18]. We construct an infinite series of the form

$$\xi_0 \Delta_0(t) + \xi_1 \Delta_1(t) + \sum_i \sum_{j<2^i} \xi_{i,j} \Delta_{i,j}(t)$$

where $\xi_{i,j}$ follows a Gaussian distribution of parameters with mean 0 and varaiance 1 and $\Delta_{i,j}(t)$ are sawtooth functions with support on a dyadic interval.

Let $\Delta_0(t)$ be the linear interpolation between the points (0, 0) and (1, 1). $\Delta_1(t)$ is the linear interpolation between points (0, 0), (1/2, 1/2), and (1, 0). $\Delta_{i,j}(t)$ $(0 < j < 2^i)$ is the function that linearly interpolates between $(j/2^i, 0)$, $(j/2^i + 2^{-(i+1)}, 2^{-j/2-1})$, and $((j+1)/2^i, 0)$ and is equal to 0 everywhere else.

To define the weights, we'll use the following function $g : [0, 1] \to \mathbb{R}$

$$x = \int_{-\infty}^{g(x)} \frac{e^{-t^2/2}}{\sqrt{2\pi}} \; dt$$

For a given real $\alpha \in [0, 1]$ we will identify α with its binary representation. Using any computable listing of the bits of α in an $\mathbb{N} \times \mathbb{N}$ grid, we obtain from one real in [0, 1] a countably infinite list of reals in [0, 1] which we will number as $\beta_0, \beta_1, \{\beta_{i,j}\}_{i \in \mathbb{N}, j < 2^i}$.

Then our weights will be $\xi_0 = g(\beta_0),\ \xi_1 = g(\beta_1),\ \xi_{i,j} = g(\beta_{i,j})$, and the series

$$B_\alpha(t) = \xi_0 \Delta_0(t) + \xi_1 \Delta_1(t) + \sum_i \sum_{j<2^i} \xi_{i,j} \Delta_{i,j}(t) \tag{1.1}$$

converges to a continuous function for almost every $\alpha \in 2^\omega$.

It is established in Kahane's book [18] that the resulting class satisfies the definition of Brownian motion on $C[0,1]$.

To extend Brownian motion to $C[\mathbb{R}^{\geq 0}]$, let $\{B_n(t)\}_{n\in\mathbb{N}}$ be independent Brownian motions on $C[0,1]$. Then

$$B(t) = B_{\lfloor t \rfloor}(t - \lfloor t \rfloor) + \sum_{0 \leq i < \lfloor t \rfloor} B_i(1)$$

satisfies the definition of Brownian motion for the space of continuous functions on $[0,\infty)$.

1.8 Layerwise Computability

The driving engine of many of the results that will follow is the notion of *layerwise computability* and the fact that layerwise computable maps preserve Martin-Löf randomness. The intuition behind layerwise computability is that a computable probability space can be thought of as having a canonical layering induced by a universal Martin-Löf test. Intuitively, a layerwise computable map can be thought of as one which is computable on every Martin-Löf random real, given the additional information of when the real stops looking nonrandom with respect to a fixed universal test. These ideas were developed by Hoyrup and Rojas; see their papers [17] and [16] for a more detailed introduction.

Definition 1.8.1.
For U_n a universal Martin-Löf test on a space (X,μ), let $K_n := X \setminus U_n$. A function $T : (X,\mu) \to Y$ is *layerwise computable* if it is computable on every K_n, uniformly on n.

Definition 1.8.2.
The *randomness deficiency* of a Martin-Löf random real α will refer to the smallest n such that $\alpha \in K_n$.

Theorem 1.8.1 (Hoyrup, Rojas)**.** *If $T : (X,\mu) \to Y$ is a layerwise computable map from a computable probability space to a computable metric space, then:*

- *The push-forward measure $\nu := \mu \circ T^{-1} \in \mathcal{M}(Y)$ is computable.*
- *T preserves Martin-Löf randomness; i.e. $T(ML_\mu) \subset ML_\nu$. Moreover, there is a constant c (computable from a description of T) such that $T(K_n) \subset K'_{n+c}$ for all n, where K'_{n+c} is the canonical layering of (Y,ν).*

Layerwise computability was developed in the context of computable analysis, and proves especially helpful in the study of Brownian motion. Many of the maps that arise in the study of classical Brownian motion, such as some of the standard constructions of Brownian motion, are layerwise computable but not computable. It is a common theme that a Brownian path will be well behaved "eventually," or a sequence of random walks will converge "eventually," where, for Martin-Löf random Brownian paths, "eventually" corresponds to when a path passes a particular Martin-Löf test. Knowing that such maps preserve Martin-Löf randomness will prove useful again and again.

Another important result we will need in several occasions is that one can compute the integral of layerwise computable functions.

Theorem 1.8.2 (Hoyrup, Rojas [16]). *Let f be a layerwise computable function defined on some computable probability space $(\mathbb{X}, \mu)$. Then the integral*

$$\int_{x \in \mathbb{X}} f(x) d\mu(x)$$

is computable uniformly in an index for f.

1.9 Two constructions of Martin-Löf random Brownian motion

There are many known constructions for Brownian motion, including Kahane's, above, and the more widely-used limit of random walk construction, among others. When these constructions are layerwise computable, they correspond to different ways to define Martin-Löf random Brownian motion.

Theorem 1.9.1 (Fouché [12]). *If α is Martin-Löf random, then (1.1) converges to a continuous function, and, moreoever, for each α there is an $M_\alpha > 0$ such that for all $m > M_\alpha$, one can compute a piecewise linear function p_m using only the first m bits of α such that $||B_\alpha - p_m|| < \frac{\log m}{\sqrt{m}}$*

Moreover, the constant M_α is layerwise computable from B_α - see [5] for a discussion of this fact. Thus we have that

Corollary 1.9.1. *The Martin-Löf random elements of $C[0, 1]$ with respect to Wiener measure are exactly the functions B_α arising from (1.1) with α a Martin-Löf random real.*

The next construction of Martin-Löf random Brownian motion is the original construction of Asarin and Prokovskiy [1]. They construct a class of functions they call "complex oscillations," which are the limits of random walks with high Kolmogorov complexity. This construction is similar to Levy's construction of Brownian motion as a limit of random walks.

Definition 1.9.1.

For $n \geq 1$, let C_n be the class of continuous functions with slope $\pm\sqrt{n}$ on the intervals $[(i-1)/n, i/n]$, $i = 1, ..., n$. For $x \in C_n$, we can associate a binary string $c(x) \in 2^{<\omega}$ of length n where the ith bit of $c(x)$ is 1 if the slope of x is positive on $[(i-1)/n, i/n]$ and 0 otherwise.

Definition 1.9.2.
A function $x \in C[0,1]$ is a *complex oscillation* if there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

- For all n, $x_n \in C_n$
- There exists a d such that for all n, $K(c(x_n)) \geq n - d$,
- $||x_n - x|| \to 0$ effectively as $n \to \infty$.

The following theorem, due to Asarin and Pokrovskiy [1] and translated from Russian in a paper by Kjos-Hanssen and Szabados [21], establishes the connection between complex oscillations and Martin-Löf random Brownian motion. More discussion about the equivalence of the two definitions of Martin-Löf random Brownian motion discussed here can be found in papers by Fouché . [12]

Theorem 1.9.2 (Asarin)**.** *$B \in C[0,1]$ is a Martin-Löf random Brownian path if and only if there is a constant d such that for all but finitely many $n \in \mathbb{N}$ there is $x_n \in C_n$ such that*

$$||B - x_n|| \leq n^{-1/10} \qquad \textit{and} \qquad K(c(x_n)) \geq n - d$$

Kjos-Hanssen and Szabados improved the bound on convergence to $\frac{c \log n}{\sqrt{n}}$. [21]

1.10 Justification of convergence for Martin-Löf randomness

We end by giving a general theorem for convergence of sums of random variables on Martin-Löf random reals. This result provides another way of proving the result of Fouché [12] and Asarin and Prokovskiy [1] that their demonstrated constructions of Brownian motion converge on every Martin-Löf random real, and also provides a more general statement about layerwise computability in probability.

Proposition 1.10.1. *Let (X_n) be a computable sequence of random variables taking their values in a computable Banach space, and such that the sum $\sum ||X_n||$ is effectively convergent. Then $\sum_n X_n$ is defined almost everywhere and is layerwise computable.*

Proposition 1.10.2. *Let (X_n) be a computable sequence of real-valued random variables such that $\mathbb{E}(X_n) = 0$ for all n and $\sum |X_n|^2$ is effectively convergent. Then the sum $\sum_n X_n$ is defined almost everywhere and is layerwise computable.*

Proof. Fix δ. For all N, define

$$A_N = \left\{ x : (\exists M > N) \left| \sum_{n=N}^{M} X_n(x) \right| > \delta \right\}$$

Note that A_N is uniformly Σ^0_1. Moreover we have, by Chebychev's inequality

$$\begin{aligned}\mathbb{P}(A_N) &\leq 1/\delta^2 \cdot \mathbb{E}\left(\sum_{n\geq N} X_n\right)^2 \\ &\leq 2/\delta^2 \cdot \mathbb{E}\left(\sum_{n\geq N} X_n^2\right)\end{aligned}$$

(the second inequality coming from the fact that $\mathbb{E}(X_n) = 0$ for all n). This means that given δ, ε one can effectively find N such that $\mathbb{P}(A_N) \leq \varepsilon$. Thus knowing where a particular x passes the Martin-Löf test given by $\{A_n\}$, we can compute $\sum_n X_n(x)$, and so $\sum_n X_n$ is layerwise computable.

□

Chapter 2

Continuity, Local Properties, and Invariance

Brownian motion has many fascinating local properties and limiting properties, and satisfies many useful invariant relations. The theme of this chapter will be to summarize some of these results, which typically hold almost surely for the Brownian motion process, and then demonstrate that they hold for every Martin-Löf random Brownian path. It is a common theme that Martin-Löf random Brownian motion reflects many of the most interesting and useful classical results.

2.1 Invariance Properties and the Law of Large Numbers

Scaling invariance is one of the most useful invariance properties of Brownian motion and we will make use of it many times in the following chapters.

Theorem 2.1.1 (Classical Scaling Invariance)**.** *Suppose $B(t) : t \geq 0$ is standard Brownian motion. Let $a > 0$. Then the stochastic process $X(t) : t \geq 0$ defined by $X(t) = \frac{1}{a}B(a^2t)$ is also standard Brownian motion.*

Proof. Proof is from Mörters and Peres [26]. Continuity of paths, independence and stationarity of the increments remain unchanged under the scaling. It remains to observe that $X(t) - X(s) = \frac{1}{a}(B(a^2t) - B(a^2s))$ is normally distributed with expectation 0 and variance $(\frac{1}{a^2}(a^2t - a^2s)) = t - s$. □

Corollary 2.1.1. *Let B be a Martin Löf random Brownian path on $C[\mathbb{R}^{\geq 0}]$. Then $\frac{1}{a}B(a^2t)$ is also a Martin Löf random Brownian path whenever B is random relative to a.*

Proof. The map $f(t) \mapsto \frac{1}{a}f(a^2t)$ is a Wiener-measure preserving map from $C[0, \infty) \to C[0, \infty)$ by Theorem 2.1.1. This map is a-computable and so preserves Martin Löf randomness relative to a for elements of $C[0, \infty)$ by Theorem 1.8.1.

□

Note that Brownian scaling does not hold in general for Martin-Löf random Brownian motion. For example, for a given Martin-Löf random Brownian path $B(t)$, one can choose an a so that $\frac{1}{a}B(a^2t)$ has a computable value for a computable time t, a contradiction for Martin-Löf random Brownian motion [10]. See chapter 3 for a discussion on what values zeros can and cannot have.

Using a similar argument, we can also see that Martin-Löf random Brownian motion is preserved by time inversion. Time inversion is a useful property of Brownian motion that allows us to draw parallels between the behavior of sample paths within an epsilon neighborhood of zero and the behavior as time goes to infinity.

Theorem 2.1.2 (Classical Time Inversion). *Suppose $B(t) : t \geq 0$ is standard Brownian motion. Then the stochastic process defined by*

$$X(t) = \begin{cases} 0 & \text{for } t = 0 \\ tB(1/t) & \text{for } t > 0 \end{cases}$$

is also standard Brownian motion.

And again, using Theorem 1.8.1, we have see that the result holds for every Martin-Löf random path.

Corollary 2.1.2. *Let $B(t)$ be a Martin Löf random Brownian path on $C[0,\infty)$. Then the function defined by*

$$X(t) = \begin{cases} 0 & \text{for } t = 0 \\ tB(1/t) & \text{for } t > 0 \end{cases}$$

is also a Martin-Löf random Brownian path on $C[0,\infty)$.

This allows us to prove the Law of Large Numbers for Martin-Löf random Brownian motion.

Corollary 2.1.3 (Law of Large Numbers). *For a Martin-Löf random Brownian path,*

$$\lim_{t\to\infty} \frac{B(t)}{t} = 0$$

Proof. For $B(t)$ a Martin-Löf random Brownian path,

$$\lim_{t\to\infty} \frac{B(t)}{t} = \lim_{t\to\infty} B(1/t) = B(0) = 0$$

□

2.2 Continuity

The definition of classical Brownian motion requires that the sample paths are continuous almost surely, and we know by Theorem 1.9.1 that every Martin-Löf random Brownian path is continuous. One can go further and show that, almost surely, Brownian motion obeys a *modulus of continuity* for a deterministic function - that is, there is a deterministic function $\phi(h)$ such that

$$\limsup_{h\to 0} \sup_{0\leq t\leq 1-h} \frac{|B(t+h)-B(t)|}{\phi(h)} \leq 1$$

In his papers establishing many of the local properties of Martin-Löf random Brownian motion[11], Fouché shows every Martin-Löf random Brownian path obeys a modulus of continuity $\phi(h)$ such that

$$\phi(h) = O\left(\sqrt{hlog(1/h)}\right) \tag{2.1}$$

It is possible to extend this result with big-O notation to the particular constant ($\sqrt{2}$) from the classical result, and moreover, the sufficiently small h for which the modulus always holds can be seen to be layerwise computable in a Martin-Löf random path.

Proposition 2.2.1. *Let B be a Martin-Löf random Brownian path. Then for all $c < \sqrt{2}$, for all h_0, there exists $h < h_0$ such that*

$$|B(t+h) - B(t)| > c\sqrt{h\log(1/h)}$$

Proof. For a large n (to be specified later), split the interval $[0, 1]$ into chunks of size e^{-n} (omitting the last bit). For each $0 \leq k < e^n$, consider the event

$$A_k : |B((k+1)e^{-n}) - B(ke^{-n})| \geq c\sqrt{e^{-n}n}$$

(i.e., what we want, with $h = e^{-n}$) Note that the A_k are independent by definition of Brownian motion and by time-translation invariance, all have the same probability. Let us estimate the probability of A_0, which is the event: $|B(e^{-n}) - B(0)| \geq c\sqrt{e^{-n}n}$. By scaling, it is also equal to the probability of the event: $|B(1) - B(0)| \geq c\sqrt{n}$. By the estimate given in [26, Lemma 12.9], we have

$$\mathbb{P}(A_0) \geq \frac{c\sqrt{n}}{c^2n+1}e^{-c^2n/2}$$

so, by assumption on c, there exists an $\alpha < 1$ such that for almost all n

$$\mathbb{P}(A_0) \geq e^{-\alpha n}$$

Since the A_k are independent,

$$\mathbb{P}(\text{no } A_k \text{ happens}) \leq (1-e^{-\alpha n})^{e^n} \sim e^{-e^{(1-\alpha)n}}$$

For n taken large enough, this can be made arbitrarily small. Moreover, notice that c can be supposed to be computable, which makes the A_k Π^0_1 classes, hence the event "no A_k happens" corresponds to a Σ^0_1 class. Thus, we have a Solovay test that any Martin-Löf random Brownian path should pass, and for such a Martin-Löf random Brownian path, there are infinitely many n for which some A_k happens. □

Proposition 2.2.2. *Let B be a Martin-Löf random Brownian path. Then for all $c > \sqrt{2}$ there is h_0 such that for all $h < h_0$ and all t*

$$|B(t+h) - B(t)| < c\sqrt{h \log(1/h)}$$

Moreover, h_0 is layerwise computable in B.

The proof given is the same as that of Mörters and Peres Theorem 1.14 [26], with the addition of keeping track of the layerwise computability of h_0. We first look at increments over a class of intervals which is chosen to be sparse but still large enough to approximate arbitrary intervals. More precisely, given $n, m \in \mathbb{N}$, we let $\Lambda_n(m)$ be the collection of all intervals of the form

$$[(k-1+b)2^{-n+a},\ \ (k+b)2^{-n+a}],$$

for $k \in \{1, ...2^n\},\ a,\, b \in \{0, \frac{1}{m}, ..., \frac{m-1}{m}\}$. We further define $\Lambda(m) \coloneqq \bigcup_n \Lambda_n(m)$.

Lemma 2.2.1. *For any fixed m and $c > \sqrt{2}$, for $B(t)$ a Martin-Löf random Brownian path, there exists $n_0 \in \mathbb{N}$, layerwise computable in $B(t)$, such that for any $n \geq n_0$,*

$$|B(t) - B(s)| \leq c\sqrt{(t-s)\log\frac{1}{(t-s)}} \quad \text{for all } [s,t] \in \Lambda_m(n).$$

Proof. From the tail estimate for a standard normal variable X, see, for example Lemma 12.9 in [26], we obtain

$$\mathbb{P}\left\{ \begin{array}{c} \sup_{k\in\{1,\dots,2^n\}} \sup_{a,b\in\{0,\frac{1}{m},\dots,\frac{m-1}{m}\}} \\ |B((k-1+b)2^{-n+a}) - B((k+b)2^{-n+a})| > c\sqrt{2^{-n+a}\log(2^{n+a})} \end{array} \right\}$$

$$\begin{aligned} &\leq 2^n m^2 \mathbb{P}\{X > c\sqrt{\log(2^n)}\} \\ &\leq \frac{m^2}{c\sqrt{\log(2^n)}}\frac{1}{\sqrt{2\pi}}2^{n(1-\frac{c^2}{2})}. \end{aligned} \tag{2.2}$$

Note that c can be taken to be computable, so for fixed $m, n \in \mathbb{N}$ the event

$$\begin{array}{c} \sup_{k\in\{1,\dots,2^n\}} \sup_{a,b\in\{0,\frac{1}{m},\dots,\frac{m-1}{m}\}} \\ |B((k-1+b)2^{-n+a}) - B((k+b)2^{-n+a})| > c\sqrt{2^{-n+a}\log(2^{n+a})} \end{array}$$

is computable in $B(t)$ and the right hand side of 2.2 is summable, giving a Solovay test which every Martin-Löf random Brownian path $B(t)$ will pass.

The standard proof of the equivalence of Solovay randomness and Martin-Löf randomness gives a way of converting a Solovay test $\{S_i\}$ to a Martin-Löf test $\{U_j\}$, and knowing a k such that a Martin-Löf random path $B(t) \notin U_k$ gives us an n_0 where the path no longer appears in any S_n for $n > n_0$. Thus the n_0 given in the proof above is layerwise computable in B. See, for example, [8] for a discussion on the equivalence of Martin-Löf randomness and Solovay randomness.

□

Lemma 2.2.2. *Given $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for every interval $[s,t] \subset [0,1]$ there exists an interval $[s',t'] \in \Lambda(m)$ with $|t-t'| < \varepsilon(t-s)$ and $|s-s'| < \varepsilon(t-s)$.*

Proof. Choose m large enough to ensure that $\frac{1}{m} < \frac{\varepsilon}{4}$ and $2^{1/m} < 1 + \frac{\varepsilon}{2}$. Given an interval $[s,t] \subset [0,1]$, we first pick n such that $2^{-n} \leq t-s < 2^{-n+1}$, then $a \in \{0, \frac{1}{m}, ..., \frac{m-1}{m}\}$ such that $2^{-n+a} \leq t-s < 2^{n+a+1/m}$. Next, pick $k \in \{1, ..., 2^n\}$ such that $(k-1)2^{-n+a} < s \leq k2^{-n+a}$, and $b \in \{0, \frac{1}{m}, ..., \frac{m-1}{m}\}$ such that $(k-1+b)2^{-n+a} < s \leq (k-1+b+\frac{1}{m})2^{-n+a}$. Let $s' = (k-1+b)2^{-n+a}$, then

$$|s-s'| \leq \frac{1}{m}2^{-n+a} \leq \frac{\varepsilon}{4}2^{-n+1} \leq \frac{\varepsilon}{2}(t-s).$$

Choosing $t' = (k+b)2^{-n+a}$ ensures that $[s',t'] \in \Lambda_n(m)$ and, moreover,

$$\begin{aligned}
|t-t'| &\leq |s-s'| + |(t-s)-(t'-s')| \\
&\leq \frac{\varepsilon}{2}(t-s) + (2^{-n+a+1/m} - 2^{-n+a}) \\
&\leq \frac{\varepsilon}{2}(t-s) + \frac{\varepsilon}{2}2^{-n+a} \\
&\leq \varepsilon(t-s),
\end{aligned}$$

as required.

□

Proof of Proposition 2.2.2. Given $c > \sqrt{2}$, pick $0 < \varepsilon < 1$ small enough to ensure that $c^* := c - \varepsilon > \sqrt{2}$ and $m \in \mathbb{N}$ as in Lemma 2.2.2. Using Lemma 2.2.1 we choose $n_0 \in \mathbb{N}$ large enough that, for all $n \geq n_0$ and all intervals $[s',t'] \in \Lambda_n(m)$, almost surely,

$$|B(t') - B(s')| \leq c^* \sqrt{(t'-s') \log \frac{1}{(t'-s')}}.$$

Now let $[s,t] \subset [0,1]$ be arbitrary, with $t - s < \min(2^{-n_0}, \varepsilon)$, and pick $[s', t'] \in \Lambda(m)$ with $|t - t'| < \varepsilon(t - s)$ and $|s - s'| < \varepsilon(t - s)$. Then, recalling 2.1, there is a C such that

$$\begin{aligned} |B(t) - B(s)| &\leq |B(t) - B(t')| + |B(t') - B(s')| + |B(s') - B(s)| \\ &\leq C\sqrt{|t-t'|\log\frac{1}{|t-t'|}} + c^*\sqrt{(t'-s')\log\frac{1}{(t'-s')}} + C\sqrt{|s-s'|\log\frac{1}{|s-s'|}} \\ &\leq (4C\sqrt{\varepsilon} + c^*\sqrt{(1+2\varepsilon)(1-\log(1-2\varepsilon))})\sqrt{(t-s)\log\frac{1}{t-s}}. \end{aligned}$$

By making $\varepsilon > 0$ small, the first factor on the right can be chosen arbitrarily close to c. This completes the proof of the theorem.

□

In addition to asking about behavior of the Brownian motion on small time intervals, one might be interested in the asymptotic behavior of the sample paths. One aspect of the asymptotic behavior is described in a classical result known as the *Law of the Iterated Logarithm*. This result establishes that there is a function $\phi : (1, \infty) \to \mathbb{R}$ such that

$$\limsup_{t\to\infty} \frac{B(t)}{\phi(t)} = 1.$$

Classically, the result holds almost surely, and Kjos-Hanssen and Nerode proved that this classical almost surely result holds for every Martin-Löf random Brownian path.

Theorem 2.2.1 (Kjos-Hanssen, Nerode). *Let $B(t) : t \geq 0$ be a Martin-Löf random Brownian path in $C[0, \infty)$. Then*

$$\limsup_{t\to\infty} \frac{|B(t+h) - B(t)|}{\sqrt{2|h|\log\log(1/|h|)}} = 1.$$

2.3 Strong Markov Property

In discussions of the strong Markov property we will need the concept of a *filtration*, which is an increasing sequence of σ-algebras on a probability space. The following definitions and exposition follow the book by Mörters and Peres [26].

Definition 2.3.1

1. A *filtration* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $(\mathcal{F}(t) : t \geq 0)$ of σ-algebras such that $\mathcal{F}(s) \subset \mathcal{F}(t) \subset \mathcal{F}$ for all $s < t$.

2. A probability space together with a filtration is a *filtered probability space.*

Suppose we have a Brownian motion $\{B(t) : t \geq 0\}$ defined on some probability space, then we can define a filtration $(\mathcal{F}^0(t) : t \geq 0)$ by letting

$$\mathcal{F}^0(t) = \sigma(B(s) : 0 \leq s \leq t)$$

be the σ-algebra generated by the random variable $B(s)$, for $0 \leq s \leq t$. Intuitively, this σ-algebra contains all the information available from observing a process up to time t.

We can also define a slightly larger σ-algebra $\mathcal{F}^+(s)$ defined by

$$\mathcal{F}^+(s) = \bigcap_{t>s} \mathcal{F}^0(t).$$

The family $(\mathcal{F}^+(t) : t \geq 0)$ is again a filtration and $\mathcal{F}^+(s) \supset \mathcal{F}^0(s)$, and intuitively $\mathcal{F}^+(s)$ is a bit larger than $\mathcal{F}^0(s)$, allowing an additional infinitesimal glance into the future.

The following classical result tells us that if an event is dependent only on the "germ" of Brownian motion, that is, on an infinitesimal small interval to the right of the origin, then that event has probability 0 or 1. We will use the Blumenthal 0-1 law several times in the following chapters.

Theorem 2.3.1 (Blumenthal 0-1 law)**.** *Let $x \in \mathbb{R}^d$ and $A \in \mathcal{F}^+(0)$. Then $\mathcal{P}_x(A) \in \{0, 1\}$.*

See Theorem 2.7 in Mörters and Peres for a proof and further discussion of the significance of this result.

Now we turn our attention to the strong Markov property and a suitable effectivation for the context of Martin-Löf random Brownian motion.

The classical strong Markov property states that Brownian motion is started anew at each *almost surely finite stopping time*. A stopping time can be thought of as the first moment where a random event related to the path happens - for example, the first exit time of a planar Brownian motion from a circle with a given radius is an almost surely finite stopping time. In looking at algorithmically random Brownian motion, we will talk about almost surely finite stopping times which are layerwise computable from the path.

Definition 2.3.1.

A random variable T with values in $[0, \infty)$ and defined on a probability space with filtration $(\mathcal{F}(t) : t \geq 0)$ is called a *stopping time* with respect to $(\mathcal{F}(t) : t \geq 0)$ if $\{T \leq t\} \in \mathcal{F}(t)$ for every $t \geq 0$.

We will make use of the following useful facts about stopping times. (See [26])

- Every deterministic time $t \geq 0$ is a stopping time with respect to every filtration $(\mathcal{F}(t) : t \geq 0)$.

- If $(T_n : n \in \mathbb{N})$ is an increasing sequence of stopping times with respect to a filtration $(\mathcal{F}(t) : t \geq 0)$ and $T_n \to T$, then T is also a stopping time with respect to $(\mathcal{F}(t) : t \geq 0)$, because
$$\{T \leq t\} = \bigcap_{n=1}^{\infty} \{T_n \leq t\} \in \mathcal{F}(t)$$

Theorem 2.3.2 (Strong Markov Property)**.** *For every almost surely finite stopping time T, the process*
$$\{B(T+t) - B(T) : t \geq 0\}$$
is standard Brownian motion.

Theorem 2.3.3 (Constructive Strong Markov Property)**.** *Let g be a layerwise computable function from the space of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^{\geq 0}$ describing an almost surely finite stopping time (e.g. $g(f(t))$ describes first zero of $f(t)$ after some rational q; see Chapter 3 for a discussion on layerwise computability of zeros). Then for B a Martin-Löf random Brownian path,*
$$W(t) = B(g(B(t)) + t) - B(g(B(t)))$$
is also a Martin-Löf random Brownian path.

Proof. The strong Markov property states that for T an almost surely finite stopping time, the map $B(t) \mapsto B(t+T) - B(T)$ is a Wiener measure preserving map from $C[I, \mathbb{R}^n]$ to $C[I, \mathbb{R}^n]$. For $T = g(B(t))$ a layerwise computable function of B, this map is layerwise computable and so preserves Martin-Löf random elements of $C[I, \mathbb{R}^n]$ [16]. □

2.4 Points of Increase

Definition 2.4.1.

A function $f : \mathbb{R} \to \mathbb{R}$ has a *global point of increase in the interval* (a, b) if there is a point $t_0 \in (a, b)$ such that $f(t) \leq f(t_0)$ for all $t \in (a, t_0)$ and $f(t_0) \leq f(t)$ for all $t \in (t_0, b)$. A point t_0 is a *local point of increase* for f if there is an interval (a, b) for which t_0 is a global point of increase. We say that a sequence of real numbers $s_0, s_1, \ldots, s_n$ has a *point of increase* at k if $s_0, \ldots, s_{k-1} \leq s_k$ and $s_k \leq s_{k+1}, \ldots, s_n$.

Theorem 2.4.1 (See [26] Theorem 5.14)**.** *Classical Brownian motion almost surely has no local points of increase.*

Theorem 2.4.2 (Theorem 13.1 in [28])**.** *Let $S_0, S_1, \ldots, S_n$ be a random walk where the independent identically distributed increments $X_i = S_i - S_{i-1}$ have mean 0 and finite variance. Then*
$$\mathbb{P}(S_0, S_1, \ldots, S_n \text{ has a point of increase}) \leq \frac{C}{\log n},$$
for $n > 1$, where C does not depend on n.

Corollary 2.4.1. *Martin-Löf random paths of Brownian motion do not have points of increase.*

Proof. Consider any interval (a, b). Without loss of generality, assume a and b are computable.

By construction, sampling the class of Martin-Löf random Brownian motion on any fixed set of reals of the form $\frac{i}{2^k}, \frac{i+1}{2^k}, \ldots, \frac{j}{2^k}$ gives a class of random walks with mean 0 and finite variance.

For a given n, choose k large enough that we can find $n+1$ such dyadic points in (a, b); call them $S_0, \ldots, S_n$. We know that

$$\mathbb{P}(S_0, S_1, \ldots, S_n \text{ has a point of increase}) \leq \frac{C}{\log n},$$

Thus, sampling the dyadic points in (a, b), a computable interval, with increasingly large denominators (i.e. increasingly small intervals) forms a Martin-Löf test on the set of continuous functions, so any Martin-Löf random Brownian path does not have points of increase.

□

Chapter 3

Zero sets of Brownian paths

Definition 3.0.2.

Let $B(t)$ be a one-dimensional standard Brownian path. We will denote the zero set of B by

$$Z_B = \{t : B(t) = 0\}.$$

In classical probability theory, the zero set of Brownian motion is an interesting and much-studied random object. Almost surely, it is a closed set with no isolated points - so it is a perfect set, and therefore uncountable. It also, almost surely, has Lebesgue measure zero, so in a sense it is a small set. An obvious interesting question then is to ask the fractal dimension of the set. Let *dim* denote (classical) Hausdorff dimension. The following result is well known in classical probability theory; see, for example, the book by Mörters and Peres [26].

Theorem 3.0.3. *Let $\{B(t) : 0 \leq t \leq 1\}$ be a linear Brownian motion. Then, almost surely,*

$$dim(Zeros \cap [0,1]) = \frac{1}{2}$$

In this chapter, we will show that the classical results hold in the context of Martin-Löf random Brownian motion, and extend the classical results by (almost) classifying the effective Hausdorff dimension of the zero set of a Martin-Löf random Brownian path.

Effective Hausdorff dimension is a modification of Hausdorff dimension for the computability setting. Intuitively, effective Hausdorff dimension describes how "computably locatable" a point or set is in addition to its size. For example, an algorithmically random point in $\mathbb{R}^n$ has effective Hausdorff dimension n because it can't be computably located any more precisely than a small computable ball, which has (classical) Hausdorff dimension n.

There are many equivalent definitions of effective Hausdorff dimension, but we will use the following definition of Mayordomo [25]. See the book by Downey and Hirschfeldt [8], or papers by Lutz [23] and Reimann [29, 32] for more details.

Definition 3.0.3. The *effective Hausdorff dimension* of $X \in 2^\omega$ is

$$\mathrm{cdim}(x) := \liminf_n \frac{K(X \upharpoonright n)}{n}$$

This definition can be extended to real numbers by identifying them with their binary representation.

The majority of this chapter will be devoted to characterizing the effective dimension of the zeroes of Martin-Löf random paths. This can be broken down in two questions.

1. Given a Martin-Löf random B, what is the set $\{\mathrm{cdim}(x) \mid x > 0 \text{ and } x \in Z_B\}$?

2. Given a real x, can we give a necessary or sufficient condition in terms of the effective dimension of x for the existence of some Martin-Löf random path which has a zero at x?

As to the first question, Kjos-Hanssen and Nerode [20] have showed that with probability 1 over B, $\{\mathrm{cdim}(x) \mid x > 0 \text{ and } x \in Z_B\}$ is dense in $[1/2, 1]$[1]. We make this more precise by showing that for every Martin-Löf random path B (not just almost all paths) $\{\mathrm{cdim}(x) \mid x > 0 \text{ and } x \in Z_B\}$ is contained in $[1/2, 1]$ and contains all the computable reals $> 1/2$ of this interval.

We will answer the second question by proving that having effective dimension at least $1/2$ is necessary, while having effective strictly greater than $1/2$ is sufficient (although having dimension $1/2$ is not sufficient).

3.1 Effective version of Kahane's Theorem

First we will prove an effective version of the following theorem of Kahane's, which we will need in the next section.

Theorem 3.1.1 (Kahane)**.** *Let E_1 and E_2 be two (disjoint) closed subsets of* $[0, 1]$ *such that* $\dim(E_1 \times E_2) > 1/2$. *Then:*

$$\mathcal{P}(B[E_1] \cap B[E_2] \neq \emptyset) > 0$$

(where $B[E]$ is the set $\{B(t) : t \in E\}$ and dim denotes classical Hausdorff dimension). We shall prove the following.

Theorem 3.1.2. *Let E_1 and E_2 be two (disjoint)* Π^0_1 *classes such that* $\dim(E_1 \times E_2) > 1/2$ *then:*

(i) There exists a Martin-Löf random path B such that $B[E_1] \cap B[E_2] \neq \emptyset$

[1] this is actually a stronger form of the theorem proven in [20], but the proof of the latter can easily be adapted

(ii) Given a fixed Martin-Löf random path B, there exists an integer c such that $B[E_1/c] \cap B[E_2/c] \neq \emptyset$

Proof. First of all, observe that item (i) of the theorem follows from item (ii). Indeed, if we have a Martin-Löf random path B and an integer c such that $B[E_1/c] \cap B[E_2/c] \neq \emptyset$, by the scaling property, $\frac{1}{\sqrt{c}}B(ct)$ is also Martin-Löf random and satisfies (i). Thus we only need to prove (ii). For this we will use the classical version of Kahane's theorem, together with Blumenthal's 0-1 law and some recent results from algorithmic randomness.

Consider the scaling map $S : B \mapsto \frac{1}{2}B(4t)$. As we saw in Corollary 2.1.1, S is computable and preserves Wiener measure $\mathbb{P}$ on $C[0,1]$. Moreover, this map is ergodic. Indeed, let $\mathcal{A}$ be an $\mathbb{P}$-measurable event which is invariant under S, i.e we have $B \in \mathcal{A} \Leftrightarrow S(B) \in \mathcal{A}$. By induction, $B \in \mathcal{A} \Leftrightarrow \forall n\, S^n(B) \in \mathcal{A}$. The function $S^n(B)$ on $[0,1]$ only depends on the values of B on $[0, 4^{-n}]$. Therefore the event $\mathcal{A}$, which is equal to $[\forall n\, S^n(B) \in \mathcal{A}]$, only depends on the germ of B. By Blumenthal's 0-1 law, this ensures that $\mathcal{A}$ has probability 0 or 1. Thus S is ergodic.

Now, consider the set

$$\mathcal{U} = \{B \mid B[E_1] \cap B[E_2] = \emptyset\}$$

We claim that $\mathcal{U}$ is a Σ^0_1 subset of $C([0,1])$. This is because of a classical result in computable analysis: the image of a Π^0_1 class by a computable function is a Π^0_1 class. This fact is uniform: from an index of a Π^0_1 class P and a computable function f on can effectively compute the index of the Π^0_1 class $f[P]$. By uniform relativization, there is a computable function γ s.t. given a pair (f, P) where f is a continuous function given as oracle, and P is a Π^0_1 class of index e, $\gamma(e)$ is an index for $f[P]$ as a $\Pi^{0,f}_1$-class. Here we have two Π^0_1 classes E_1 and E_2, say of respective indices e_1 and e_2. By the above discussion $B[E_1]$ and $B[E_2]$ have respective indices $\gamma(e_1)$ and $\gamma(e_2)$ as $\Pi^{0,B}_1$-classes and since the intersection of two Π^0_1 classes is index-computable, there is a computable function θ such that $B[E_1] \cap B[E_2]$ has index $\theta(e_1, e_2)$ as a $\Pi^{0,B}_1$-class. Since one can computably enumerate, uniformly in the oracle B, the indices of $\Pi^{0,B}_1$-classes, it follows that the set $\mathcal{U}$ is Σ^0_1, as wanted.

We can now apply the effective ergodic theorem 1.4.1: since $\mathcal{U}$ has measure less than 1 (by Kahane's theorem) and is a Σ^0_1 set, there are infinitely many n such that $S^n(B) \notin \mathcal{U}$ (in fact, the set of such n's is a subset of $\mathbb{N}$ of positive density), i.e., such that $B[E_1/2^n] \cap B[E_2/2^n] \neq \emptyset$. □

3.2 An initial result toward classifying the dimension spectrum of zeroes

The next theorem is a direct consequence of the effective version of Kahane's theorem.

Theorem 3.2.1. *Given a Martin-Löf random path B and computable real $\alpha > 1/2$, there exists a real x in Z_B of constructive dimension α.*

Proof. Let B be such a path and α such a real. Consider the Bernoulli measure μ_p (i.e., the measure where each bit has probability p of being a zero, independently of all other bits) such that $p < 1/2$ and $-p \log p - (1-p)\log(1-p) = \alpha$. Since α is computable, so is p (and hence μ_p), because the function $x \mapsto -x \log x - (1-x)\log(1-x)$ is computable and increasing on $[0, 1/2]$. Let $E_1 = \{0\}$ and E_2 be the complement of the first level of the universal Martin-Löf test for μ_p (it is a Π^0_1 class since μ_p is computable). It is well-known that every set of positive μ_p-measure has Hausdorff dimension $\geq \alpha$, and moreover that every μ_p random real has constructive Hausdorff dimension α (see for example Reimann [29]). Applying Theorem 3.1.2, there exists some c such that $B[E_1/2^c] \cap B[E_2/2^c] \neq \emptyset$. That is, there is some $x \in E_2$ such that $B(2^c x) = 0$. Multiplying by 2^c just adds c zeros in the binary expansion of x, thus $2^c x$ has the same constructive dimension as x, which is α. □

3.3 Effective Dimension of zeroes

We now address the second of the two above questions: what properties (in terms of effective dimension or Kolmogorov complexity) characterize the reals that belong to Z_B for some Martin-Löf random B?

To find the effective dimension of zeros, we will first need to know the probability that $B(t)$ has a zero in a given interval $[a, a+\varepsilon]$.

Proposition 3.3.1. *[28] For any $\varepsilon \in (0,1)$ and $a > 0$*

$$\mathbb{P}_0 \left(B(s) = 0 \text{ for some } s \in [a, a+\varepsilon]\right) = \frac{2}{\pi} \arctan\left(\sqrt{\frac{\varepsilon}{a}}\right)$$

which is $\sim \frac{2}{\pi\tau}\sqrt{\varepsilon}$ as ε tends to 0.

We will also need the following theorem which estimates the probability for B to hit *two* intervals of the same length.

Proposition 3.3.2. *Let $0 < a < b < 1$ and $\varepsilon > 0$. Suppose that the intervals $[a, a+\varepsilon]$ and $[b, b+\varepsilon]$ are disjoint. Let δ be the distance between them (i.e., $\delta = b - a - \varepsilon$). Let $\mathcal{A}_1$ be the event "$\mathscr{B}(s) = 0$ for some $s_1 \in [a, a+\varepsilon]$" and $\mathcal{A}_2$ be "$\mathscr{B}(s) = 0$ for some $s_2 \in [b, b+\varepsilon]$". Then*

$$\mathbb{P}_0 \left(\mathcal{A}_1 \wedge \mathcal{A}_2\right) \leq \frac{\varepsilon \cdot O(1)}{\sqrt{a\delta}}$$

where the term $O(1)$ is a constant independent of a, b, ε.

Proof. In this proof, we make use of the following notation: given an event $\mathcal{A}$, $\mathcal{A}^{\uparrow\tau}$ the unique (by assumption on $\mathcal{A}$) event such that $t \mapsto B(t+s) \in \mathcal{A}^{\uparrow\tau}$ if and only if $t \mapsto B(t) \in \mathcal{A}$.

Now, let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the above events, and let us write

$$\mathbb{P}_0 \left(\mathcal{A}_1 \wedge \mathcal{A}_2\right) = \mathbb{P}_0(\mathcal{A}_1)\mathbb{P}_0(\mathcal{A}_2 \mid \mathcal{A}_1)$$

The term $\mathbb{P}_0(\mathcal{A}_1)$ is, by Proposition 3.3.1, equal to $O(\sqrt{\frac{\varepsilon}{a}})$. It remains to evaluate the term $\mathbb{P}(\mathcal{A}_2 \mid \mathcal{A}_1)$. The event $\mathcal{A}_2$ only depends on the values of $\mathscr{B}$ on the interval $[b, b+\varepsilon]$, thus

$$\mathbb{P}_0(\mathcal{A}_2 \mid \mathcal{A}_1) = \int_{z\in\mathbb{R}} \mathbb{P}_z(\mathcal{A}_2^{\uparrow(a+\varepsilon)})\, f(z)\, dz$$

where f is the density function of $\mathscr{B}(a+\varepsilon)$ conditioned by $\mathcal{A}_1$. By shift invariance of the Wiener measure, we observe that in this expression, the term $\mathbb{P}_z(\mathcal{A}_2^{\uparrow(a+\varepsilon)})$ is equal to $\mathbb{P}_z(\mathscr{B}$ has a zero in $[\delta, \delta+\varepsilon])$. This is, in turn, always bounded by $\mathbb{P}_0(\mathscr{B}$ has a zero in $[\delta, \delta+\varepsilon])$, by Proposition 3.3.1. Thus

$$\begin{aligned}
\mathbb{P}_0(\mathcal{A}_2 \mid \mathcal{A}_1) &= \int_{z\in\mathbb{R}} \mathbb{P}_z(\mathcal{A}_2^{\uparrow(a+\varepsilon)})\, f(z)\, dz \\
&\le \int_{z\in\mathbb{R}} \mathbb{P}_0(\mathcal{A}_2^{\uparrow(a+\varepsilon)})\, f(z)\, dz \\
&\le \mathbb{P}_0(\mathcal{A}_2^{\uparrow(a+\varepsilon)}) \\
&\le \mathbb{P}_0(\mathscr{B} \text{ has a zero in } [\delta, \delta+\varepsilon]) \\
&\le \frac{2}{\pi} \arctan\left(\sqrt{\frac{\varepsilon}{\delta}}\right) \\
&\le \frac{2}{\pi}\sqrt{\frac{\varepsilon}{\delta}}
\end{aligned}$$

We have thus established the desired result.

□

Theorem 3.3.1. *If B is a Martin-Löf random path, then all members of the set $Z_B \setminus \{0\}$ have effective dimension at least $1/2$.*

Proof. Suppose that for a given B^*, we have $B^*(a) = 0$ for some a such that $\mathrm{cdim}(a) < 1/2$. We will show that B^* is not Martin-Löf random.

Let $\mathrm{cdim}(a) < \rho < 1/2$. Take also some rational b such that $0 < b < a$. By definition of constructive dimension, for all n, there exists a prefix σ of a such that $K(\sigma) \le \rho|\sigma| - n$. For all strings σ such that $0.\sigma > b$, let $I_\sigma = [0.\sigma, 0.\sigma + 2^{-|\sigma|}]$ and the event

$$\mathcal{E}_\sigma : [\mathscr{B} \text{ has a positive and a negative value in } I_\sigma]$$

The event $\mathcal{E}_\sigma$ is a Σ^0_1 subset of $C[0,1]$, uniformly in σ. The probability of $\mathcal{E}_\sigma$ is $O(2^{-|\sigma|/2})$ by Proposition 3.3.1, with the multiplicative constant depending on b. (We will use big-O notation rather than named constants, because our estimates will change the constants in ways that we do not need to keep track of.) Define

$$\mathcal{U}_n = \bigcup \left\{ \mathcal{E}_\sigma \mid K(\sigma) \le \rho|\sigma| - n \right\}$$

By assumption, B^* belongs to almost all $\mathcal{U}_n$. However, we have

$$\begin{aligned}\mathbb{P}(B \in \mathcal{U}_n) &\leq O(1) \cdot \sum \{2^{-|\sigma|/2} \mid K(\sigma) \leq \rho|\sigma| - n\} \\ &\leq O(1) \cdot \sum_{\sigma} 2^{-K(\sigma)-n} \\ &\leq O(2^{-n})\end{aligned}$$

Thus the $\mathcal{U}_n$ form a Martin-Löf test, which shows that B^* is not Martin-Löf random.

□

We now prove an (almost) counterpart of Theorem 3.3.1:

Theorem 3.3.2. *Let $x \in [0,1]$ be of effective dimension strictly greater than $1/2$. Then there exists a Martin-Löf random path B such that $B(x) = 0$.*

The proof is much more difficult and involves the notion of α-energy. Given a measure μ on $\mathbb{R}$ and $\alpha \geq 0$, the α-energy of μ is the quantity

$$\int\int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}$$

This quantity might be finite or infinite, depending on the value of α.

Lemma 3.3.1. *Let $\beta > \alpha \geq 0$. If μ is a measure satisfying the conditions of Frostman's lemma with exponent β (i.e., $\mu(A) \leq c \cdot |A|^\beta$ for every interval A), then μ has finite α-energy.*

Proof. Morters-Peres, proof of Theorem 4.32. □

Lemma 3.3.2. *Let $\beta > 1/2$ and let μ be a finite Borel measure on $[0,1]$ such that for every dyadic interval I, $\mu(I) \leq c \cdot |I|^\beta$ for some fixed constant c (and thus by the previous lemma μ has finite $1/2$-energy). Then there exists a constant $c' > 0$ such that the following holds: for any set $A \subseteq [1/2, 1]$ which is a countable union of closed dyadic intervals*

$$\mathbb{P}_0\big(Z_{\mathscr{B}} \cap A \neq \emptyset\big) \geq c' \cdot \mu(A)^2$$

Proof. It suffices to prove this theorem for a finite number of intervals, and up to splitting them if necessary we can assume that they all have the same length 2^{-n} for some n. Let $I_1, ..., I_k$ be those intervals. Define for all k the random variable X_k by

$$X_k = \mu(I_k) \cdot 2^{(n/2)} \cdot \mathbf{1}_{\{Z_{\mathscr{B}} \cap I_k \neq \emptyset\}}$$

and $Y = \sum_{j=1}^k X_j$. We want to show that $\mathbb{P}(Y > 0) \geq \frac{\mu(A)^2}{C}$ for constant C which does not depend on A, which immediately gives the result (since $Y > 0$ is equivalent to $Z_B \cap A \neq \emptyset$). To do so, we will use the Chebychev-Cantelli inequality

$$\mathbb{P}(Y > 0) \geq \frac{\mathbb{E}(Y)^2}{\mathbb{E}(Y^2)}$$

Let us evaluate separately $\mathbb{E}(Y)$ and $\mathbb{E}(Y^2)$. We have

$$\begin{aligned}
\mathbb{E}(Y) &= \sum_{j=1}^{k} \mathbb{E}(X_j) \\
&\geq \sum_{j=1}^{k} 2^{(n/2)} \cdot \mu(I_j) \cdot C_1 \cdot (\sqrt{2^{-n}}) \\
&\geq C_1 \sum_{j=1}^{k} \cdot \mu(I_j) \\
&\geq C_1 \cdot \mu(A)
\end{aligned}$$

for some constant $C_1 \neq 0$, the second inequality coming from Proposition 3.3.1.

Let us now turn to $\mathbb{E}(Y^2)$, which we need to bound by a constant. We have

$$\mathbb{E}(Y^2) = \sum_{\substack{1 \leq i \leq k \\ 1 \leq j \leq k}} \mathbb{E}(X_i X_j)$$

To evaluate this sum, we decompose it into three parts:

$$\mathbb{E}(Y^2) = \sum_{i=1}^{k} \mathbb{E}(X_i^2) \; + \; 2 \sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ adjacent}}} \mathbb{E}(X_i X_j) \; + \; 2 \sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \mathbb{E}(X_i X_j)$$

The first part is an easy computation. For all i,

$$\begin{aligned}
\mathbb{E}(X_i^2) &= \mu(I_i)^2 \cdot 2^n \cdot \mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset\} \\
&= O\big(\mu(I_i)^2 \cdot 2^n \cdot 2^{-(n/2)}\big) \\
&= O\big(\mu(I_i) \cdot 2^{-\beta n} \cdot 2^n \cdot 2^{-(n/2)}\big) \\
&= \mu(I_i) \cdot O\big(2^{(1/2-\beta)n}\big) \\
&= \mu(I_i) \cdot O(1)
\end{aligned}$$

(for the third equality, we use the fact that $\mu(I_i) \leq |I_i|^\beta$, and for the fifth one the fact that $\beta > 1/2$). Thus

$$\sum_{i=1}^{k} \mathbb{E}(X_i^2) = \sum_{i=1}^{k} \mu(I_i) \cdot O(1) = O(1)$$

For the second part, we use a rough estimate: first notice that

$$\mathbb{E}(X_i X_j) = \mu(I_i) \cdot \mu(I_j) \cdot 2^n \cdot \mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset \, \wedge \, Z_{\mathscr{B}} \cap I_j \neq \emptyset\}$$

and for the second part only, we will use the trivial upper bound:

$$\mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset \, \wedge \, Z_{\mathscr{B}} \cap I_j \neq \emptyset\} \leq \mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset\} = O(2^{-n/2})$$

Combining this with $\mu(I_j) \leq 2^{-\beta n}$, we get:

$$\mathbb{E}(X_i X_j) = \mu(I_i) \cdot O(2^{(1/2-\beta)n}) = \mu(I_i) \cdot O(1)$$

Moreover, each interval I_i has at most two adjacent intervals I_j. Thus,

$$\sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ adjacent}}} \mathbb{E}(X_i X_j) \leq 2 \sum_{i=1}^{k} \mu(I_i) \cdot O(1) = O(1)$$

Finally, for the third part, we will use the fact that the 1/2-energy of μ is finite. Let us, for a pair of nonadjacent intervals I_i, I_j with $\max(I_i) < \min(I_j)$, denote by $g(i,j)$ the length of the gap between the two, i.e., $g(i,j) = \min(I_j) - \max(I_i)$. We have

$$\sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \mathbb{E}(X_i X_j) = \sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \mu(I_i) \cdot \mu(I_j) \cdot 2^n \cdot \mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset \ \wedge \ Z_{\mathscr{B}} \cap I_j \neq \emptyset\} \quad (3.1)$$

By Proposition 3.3.2,

$$\mathbb{P}\{Z_{\mathscr{B}} \cap I_i \neq \emptyset \ \wedge \ Z_{\mathscr{B}} \cap I_j \neq \emptyset\} = \frac{2^{-n} \cdot O(1)}{\sqrt{g(i,j)}} \quad (3.2)$$

(note that we use the fact that I_i and I_j are contained in $[1/2, 1]$, hence $\min(I_i)$ is bounded away from 0).

Thus,

$$\sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \mathbb{E}(X_i X_j) = \sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \frac{\mu(I_i) \cdot \mu(I_j)}{\sqrt{g(i,j)}} \cdot O(1) \quad (3.3)$$

Note that, since I_i and I_j are non-adjacent dyadic intervals of length 2^{-n}, we have $g(i,j) \geq 2^{-n}$. Therefore, for two reals x, y, if $x \in I_i$ and $y \in I_j$, then $|y - x| \leq 3g(i,j)$. By this observation, we have

$$\sum_{\substack{1 \leq i < j \leq k \\ I_i, I_j \text{ nonadjacent}}} \frac{\mu(I_i) \cdot \mu(I_j)}{\sqrt{g(i,j)}} \leq O(1) \cdot \int \int \frac{d\mu(x) d\mu(y)}{|x-y|^{1/2}} \leq O(1)$$

(the last inequality comes from the fact that the 1/2-energy of μ is finite by Lemma 3.3.1).

We have thus established that $\mathbb{E}(Y^2) = O(1)$, which completes the proof. □

Let KM denote the 'a priori' Kolmogorov complexity function (see [8, Section 6.3.2]). Recall that $KM(\sigma) = K(\sigma) + O(\log |\sigma|)$, thus in particular K can be replaced by KM in the definition of effective dimension. The reason we need KM instead of K is the following result of Reimann [30, Theorem 14], which we will apply in the proof of Theorem 3.3.2: Let z be a real such that $KM(z \upharpoonright n) \geq \beta n - O(1)$. Then, there exists a measure μ such that $\mu(A) = O(|A|^\beta)$ for all intervals A, and such that z is Martin-Löf random for the measure μ.

Proof of Theorem 3.3.2. Let z be of dimension $\alpha > 1/2$. Let β be a rational such that $1/2 < \beta < \alpha$. Then for almost all n, $KM(z \restriction n) \geq \beta n$. By Reimann's theorem, let μ be a measure such that $\mu(A) = O(|A|^\beta)$ for all intervals A, and such that z is Martin-Löf random for the measure μ.

For all n, let $\mathcal{K}_n$ be the complement of the n-th level of the universal Martin-Löf test over $(C[0,1], \mathbb{P})$ and consider the set

$$\mathcal{U}_n = \{x \mid \forall B \in \mathcal{K}_n \ B(x) \neq 0\}$$

We claim that $\mathcal{U}_n$ is Σ^0_1 uniformly in n, and $\mu(\mathcal{U}_n) = O(2^{-n/2})$. To see that it is Σ^0_1 suppose that $x \in \mathcal{U}_n$, i.e., $B(x) \neq 0$ for all $B \in \mathcal{K}_n$. The set $\mathcal{K}_n$ being compact, the value of $|B(x)|$ for $B \in \mathcal{K}_n$ reaches a positive minimum. Thus there is a rational a such that $B(x) > a$ for all $B \in \mathcal{K}_n$. By uniform continuity of the members of $\mathcal{K}_n$ (ensured by Proposition 2.2.1), there is a rational closed interval I containing x such that $|B(t)| > a/2$ for all $t \in I$ and $B \in \mathcal{K}_n$. Thus $\mathcal{U}_n$ is the union of intervals (s_1, s_2) such that $\min\{B(t) : t \in [s_1, s_2]\} > b$ for some rational b and all $B \in \mathcal{K}_n$. Moreover, the condition "$\min\{B(t) : t \in [s_1, s_2]\} > b$ for all $B \in \mathcal{K}_n$" is Σ^0_1, because the function $B \mapsto \min\{B(t) : t \in [s_1, s_2]\}$ is layerwise computable (thus uniformly computable on $\mathcal{K}_n$), and the minimum of a computable function on an effectively compact set is lower semi-computable uniformly in a code for that set. This shows that $\mathcal{U}_n$ is Σ^0_1.

To evaluate $\mu(\mathcal{U}_n)$, let us first observe that by definition of $\mathcal{U}_n$,

$$\mathbb{P}_0(Z_{\mathscr{B}} \cap \mathcal{U}_n) \leq \mathbb{P}_0(\mathscr{B} \in \mathcal{K}_n \text{ and } Z_{\mathscr{B}} \cap \mathcal{U}_n) + 2^{-n} \leq 2^{-n}$$

Applying Lemma 3.3.2, it follows that $\mu(\mathcal{U}_n) = O(2^{-n/2})$, as wanted. Since z is Martin-Löf random with respect to μ, it cannot be in all sets $\mathcal{U}_n$, and thus it must be the zero of some Martin-Löf random path.

□

3.4 The case of points of effective dimension $1/2$

In the previous section we showed that no point of constructive dimension less than 1/2 can be the zero of a Martin-Löf random Brownian path, and that every point of dimension greater than 1/2 is necessarily a zero of some Martin-Löf random path. This leaves open the question of what happens at constructive dimension exactly 1/2. Laurent Bienvenu has provided a partial answer to this question, which I will include here for completeness. He has shown that among points of constructive dimension 1/2, some are zeroes of some Martin-Löf random Brownian path, and some are not.

The next theorem, which strengthens Theorem 3.3.1, gives a necessary condition for a point to be a zero of some Martin-Löf random path.

Theorem 3.4.1. *If $x > 0$ is a zero of some Martin-Löf random path, then*

$$\sum_n 2^{-K(x\restriction n)+n/2} < \infty$$

It is interesting to notice the parallel with the so-called 'ample excess lemma' (see [8, Theorem 6.6.1]): a real x is Martin-Löf random if and only if $\sum_n 2^{-K(x\restriction n)+n} < \infty$.

Proof. The proof is an adaptation of that of Theorem 3.3.1. First take a rational a such that $0 < a$. We shall prove the lemma for all $x > a$, which will be enough since a is arbitrary. For each string σ consider, like in Theorem 3.3.1, the interval $I_\sigma = [0.\sigma, 0.\sigma + 2^{-|\sigma|}]$ and the event

$$\mathcal{E}_\sigma : [\mathscr{B} \text{ has a positive and a negative value in } I_\sigma]$$

Now, consider the function $\mathbf{t}$ defined on $C[0,1]$ by

$$\mathbf{t}(B) = \sum_{\sigma \ s.t. \ a<0.\sigma} 2^{-K(\sigma)+|\sigma|/2} \cdot \mathbf{1}_{\mathcal{E}_\sigma}(B)$$

The event $\mathcal{E}_\sigma$ is a Σ^0_1 subset of $C[0,1]$, uniformly in σ. Thus the function $\mathbf{t}$ is lower semi-computable. Moreover, the probability of $\mathcal{E}_\sigma$ is $O(2^{-|\sigma|/2})$ by Proposition 3.3.1 (the multiplicative constant depending on a). Thus the integral of $\mathbf{t}$ is bounded, and therefore $\mathbf{t}$ is an integrable test (see [14]). Let now B be a Martin-Löf random path and suppose $B(x) = 0$ for some $x > a$. Then for almost all n, $a < 0.(x \restriction n)$. Moreover, for every n, B having a zero in $I_{x\restriction n}$, it must in fact have a positive and a negative value on that interval (by Proposition 3.5.2). Thus, by definition of $\mathbf{t}$

$$\mathbf{t}(B) + O(1) \geq \sum_n 2^{-K(x\restriction n)+n/2}$$

(the $O(1)$ accounts for the finitely many terms such that $a \geq 0.(x \restriction n)$). But since B is Martin-Löf random and $\mathbf{t}$ is a integrable test, we have $\mathbf{t}(B) < \infty$, which proves our result. □

This theorem shows in particular that if x is the zero of some Martin-Löf random path, then $K(x \restriction n) - n/2 \to +\infty$.

We now give a sufficient condition which actually is very close to our necessary condition.

Proposition 3.4.1. *Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $\sum_n 2^{-f(n)} < \infty$. Let μ be a Borel measure on $[0,1]$ such that for every interval A of length $\leq 2^{-n}$, $\mu(A) \leq 2^{-\alpha n - f(n)}$. Then μ has finite α-energy.*

Proof. For now, let us fix some x. Define for all n the interval I_n to be $[x-2^{-n+1}, x-2^{-n}] \cap [0,1]$ and $J_n = [x+2^{-n}, x+2^{-n+1}] \cap [0,1]$. Then

$$\begin{aligned}
\int \frac{d\mu(y)}{|x-y|^\alpha} &\leq \sum_n \int_{y \in I_n} \frac{d\mu(y)}{|x-y|^\alpha} + \sum_n \int_{y \in J_n} \frac{d\mu(y)}{|x-y|^\alpha} \\
&\leq \sum_n 2^{\alpha n} \mu(I_n) + \sum_n 2^{\alpha n} \mu(J_n) \\
&\leq \sum_n 2^{\alpha n} 2^{-\alpha n - f(n)} + \sum_n 2^{\alpha n} 2^{-\alpha n - f(n)} \\
&\leq 2 \cdot \sum_n 2^{-f(n)} \\
&< \infty
\end{aligned}$$

Therefore, the μ-integral over x of $\int \frac{d\mu(y)}{|x-y|^\alpha}$ is itself finite, which is what we wanted. □

Theorem 3.4.2. *Let $f : \mathbb{N} \to \mathbb{N}$ be a nondecreasing computable function such that $f(n+1) \leq f(n)+1$ for all n, and such that $\sum_n 2^{-f(n)} < \infty$. Let x be a real such that $KM(x \upharpoonright n) \geq n/2 + f(n) + O(1)$. Then x is the zero of some Martin-Löf random path.*

Proof. Let f be such a function and x such a real. By a result of Reimann [30, Theorem 14], there exists a measure μ such that $\mu(A) \leq 2^{-n/2-f(n)+O(1)}$ for all intervals of length $\leq 2^{-n}$ such that x is Martin-Löf random with respect to μ. By Proposition 3.4.1, μ has finite 1/2-energy. The rest of the proof is identical to the proof of Theorem 3.3.2. □

Theorem 3.4.3. *Let $0 < \alpha < 1$ and let $f : \mathbb{N} \to \mathbb{N}$ be a Lipschitz function such that $f(n) = o(n)$. Then there exists $x \in [0,1]$ such that $K(x \upharpoonright n) = \alpha n + f(n) + O(1)$.*

Remark 3.4.1. This theorem was proven by J. Miller (unpublished) for $f = 0$.

Proof. Fix a large integer m, which we will implicitly define during the construction. We will build the sequence x by blocks of length m. For m large enough, the empty string has complexity less than $3 \log m$. Suppose we have already constructed a prefix σ of x such that $|K(\sigma \upharpoonright n) - \alpha n + f(n)| \leq 3 \log m$ for all $n \leq |\sigma|$ multiple of m. Pick a string τ of length n such that

$$K(\tau \mid \sigma) \geq m$$

We then have

$$K(\sigma\tau) \geq K(\sigma) + m - 2\log m - O(1)$$

On the other hand

$$K(\sigma 0^m) \leq K(\sigma) + 2\log m + O(1)$$

For each $i \leq m$, consider the "mixture" between 0^m and τ: $\rho_i = (\tau \upharpoonright i)0^{m-i}$. Since ρ_i and ρ_{i+1} differ by only one bit in position $\leq m$ from the right, we have $|K(\sigma\rho_i) -$

$K(\sigma\rho_{i+1})| \leq 2\log m + O(1)$. By this 'continuity' property, there must be some i such that $|K(\sigma\rho_i) - \alpha n - f(n)| \leq 2\log m + O(1)$ (here the $O(1)$ constant depends on f, but not on m). Thus, for m large enough, we get $|K(\sigma\rho_i) - \alpha n - f(n)| \leq 3\log m$. Thus, if m is large enough, we can iterate this argument to build a sequence x such that $|K(x \upharpoonright n) - \alpha n - f(n)| \leq 3\log m$ for all n multiple of m. Since $\alpha n + f(n)$ is a Lipschitz function, this is sufficient to ensure $|K(x \upharpoonright n) - \alpha n - f(n)| = O(m)$. □

We can finally prove the promised theorem.

Theorem 3.4.4. *Among reals of effective dimension* $1/2$*, some are zeros of some Martin-Löf random path, and some are not.*

Proof. By Theorem 3.4.3, first consider a real x such that $K(x \upharpoonright n) = n/2 + O(1)$. This real has effective dimension $1/2$ and cannot be a zero of a Martin-Löf random path (Theorem 3.4.1).

Applying Theorem 3.4.3 again, let y be a real such that $K(y \upharpoonright n) = n + 4\log n + O(1)$. Since for every σ, $KM(\sigma) \geq K(\sigma) - K(|\sigma|) - O(1) \geq K(\sigma) - 2\log|\sigma| - O(1)$, it follows that $KM(y \upharpoonright n) \geq n + 2\log n - O(1)$, and thus y is a zero of some Martin-Löf random path (Theorem 3.4.2). Of course, y has effective dimension $1/2$ as well.

□

This section leaves open the existence of a precise characterization of the reals x of dimension $1/2$ for which there exists a Martin-Löf random path B such that $B(x) = 0$. Short of an exact characterization, it would be interesting to know whether this depends on Kolmogorov complexity alone. By this, we mean the following question: if $K(x \upharpoonright n) \leq K(y \upharpoonright n) + O(1)$ and x is a zero of some Martin-Löf random path, is y a zero of some Martin-Löf random path? We ask the same question with KM instead of K.

3.5 Further results about zeroes of Brownian paths

Lemma 3.5.1. *For B a Martin-Löf random Brownian path, B has zeros in all intervals* $(0, \varepsilon)$.

Proof. Fix a real $0 < 2^{-k} < \varepsilon$ for some k. By 3.3.1, we know that the probability of a Martin-Löf random Brownian path not having a zero in any interval of the form $(2^{-k-n}, 2^{-k})$ is

$$1 - \frac{2}{\pi}\arctan\left(\frac{2^{-k} - 2^{-k-n}}{2^{-k-n}}\right)$$

which limits to zero, computably, as $n \to \infty$. This gives a Schnorr test for having a zero before a real 2^{-k}, which every Martin-Löf random Brownian path must pass, so every Martin-Löf random Brownian path has a zero before every real of the form 2^{-k}, so has a zero in every interval $(0, \varepsilon)$. □

Proposition 3.5.1. *For B a Martin-Löf random Brownian path,*

$$Z_B = \{t \geq 0 : B(t) = 0\}$$

is a closed set with no isolated points.

Proof. Z_B is closed because $B(t)$ is continuous.

To see there are no isolated points, consider $\tau_q = \inf\{t \geq q : B(t) = 0\}$, the first zero after some $q \in \mathbb{Q}$. By closure of Z_B, the infimum is a minimum. τ_q is layerwise computable in B and is an almost surely finite stopping time. Thus by the constructive strong Markov property, τ_q is not an isolated zero from the right.

Now, consider zeroes that are not of the form τ_q. Call some such zero t_0. To see it is not isolated from the left, consider a sequence of rationals $q_n \uparrow t_0$. By assumption on t_0, for all n there is some $\tau_{q_n} \in (q_n, t_0)$, so t_0 is not an isolated zero from the left. □

Thus far we have proved that the zero set is a perfect set, and we will see in further sections that all zeroes have high effective dimension, meaning they are difficult to describe computably. However, it turns out that many of the zeroes are easy to describe - in fact, layerwise computable - from the path $B(t)$. This result and its corollaries will prove very useful, and they inspired the proof given in chapter four that the solution to the Dirichlet problem is computable.

First, we will show that the maximum (and minimum) values of a Martin-Löf random path on a given computable interval cannot be computable. This observation was first made by Fouché in [11], though the proof given here is different than that suggested by Fouché .

Lemma 3.5.2. *For $B(t)$ a Martin-Löf random Brownian path and $[r_1, r_2]$ a computable interval, the maximum and minimum of $B(t)$ on $[r_1, r_2]$ are layerwise computable in $B(t)$.*

Proof. To compute the maximum of $B(t)$ on $[r_1, r_2]$ to within ε, we run the following simple algorithm: Pick h_0 small enough so that $B(t)$ obeys a modulus of continuity with constant $c = 2$ (see proposition 2.2.2) and so that $2\sqrt{h_0 \log(1/h_0)} < \varepsilon$. Then we know that the maximum of the values $B(r_1), B(r_1 + h_0), B(r_1 + 2h_0), ..., B(r_2)$ must be within $2\sqrt{h_0 \log(1/h_0)}$, and therefore within ε, of the maximum value of $B(t)$ on $[r_1, r_2]$. The minima are also layerwise computable by the same argument.

Note that this argument does not establish the layerwise computability of the time(s) at which the maximum occurs; the best we can say using this argument is that the time(s) are Π^0_1 in B, and the argument uses the randomness deficiency of B and so is not uniform.

□

Proposition 3.5.2. *Local maxima and local minima of a Martin-Löf random Brownian path are Martin-Löf random reals (in particular, they cannot be computable reals).*

Proof. Fix two rational numbers $x < y$. It is known classically that $\max(\mathscr{B}, 0, y)$ is distributed according to the density function

$$f(a) = 2 \cdot \frac{e^{-a^2/(2y)}}{\sqrt{2\pi y}}$$

for $a \geq 0$, and $f(a) = 0$ for $a < 0$ (see [26, Theorem 2.21]). By the Markov property, $\max(\mathscr{B}, x, y)$ has the same distribution as $\mathscr{B}(x) + \max(\mathscr{B}, 0, y - x)$, and thus is distributed according to the density function

$$g(a) = \frac{e^{-a^2/(2x)}}{\sqrt{2\pi x}} + 2\frac{e^{-a^2/(2(y-x))}}{\sqrt{2\pi(y-x)}}$$

for $a \geq 0$, and $f(a) = 0$ for $a < 0$. It is known that if a computable measure μ on $\mathbb{R}$ admits a continuous positive density function, then its random elements are exactly the Martin-Löf random reals [17]. Since the function

$$B \mapsto \max(B, x, y)$$

is layerwise computable, its image measure is computable, and by the above has a continuous positive density function. Moreover, by the randomness preservation theorem since the function

$$B \mapsto \max(B, x, y)$$

is layerwise computable, the image of a Martin-Löf random B is random for the image measure, hence is Martin-Löf random for the uniform measure.

□

Now we can begin the proof of the main result of this subsection:

Proposition 3.5.3. *For B a Martin-Löf random Brownian path, the first zero of B after any given computable real q is layerwise computable from B.*

We will need the following lemmas:

Lemma 3.5.3. *It is layerwise computable in a Martin-Löf random path $B(t)$ to see that there is not a zero in a given interval $[r_1, r_2]$ with computable endpoints.*

Proof. This follows from 2.2.2 above. Because h_0 is layerwise computable in $B(t)$, we can, layerwise computably in B, find increments of size h in $[r_1, r_2]$ such that $B(r_1), B(r_1 + h), ..., B(r_2) > 2h\sqrt{\log 1/h}$ if B is bounded above zero, or $B(r_1), B(r_1 + h), ..., B(r_2) < -2h\sqrt{\log 1/h}$ if B is bounded below zero. Then 2.2.2 tells us that $B(t)$ must be bounded away from zero on $[r_1, r_2]$. □

Lemma 3.5.4. *It is layerwise computable in $B(t)$ to see that there is a zero in a given interval $[r_1, r_2]$ with computable endpoints.*

Proof. By Proposition 3.5.2, we know that the maxima and minima of $B(t)$ on $[r_1, r_2]$ must have non-computable values, so any zero must have times s_1 and s_2 arbitrarily close on either side where $B(s_1) < 0$ and $B(s_2) > 0$. In fact, there must be computable such points, as continuity of the function guarantees open intervals arbitrarily close to the zero where the function is always positive and always negative.

The construction for $B(t)$ given above has layerwise computable convergence, so we can layerwise computably search through any dense set of computable times in $[r_1, r_2]$ and eventually find times s_1, s_2 where $B(s_1) < 0$ and $B(s_2) > 0$

□

Proof of 3.5.3. Then we can (layerwise in B) compute the first zero after a given computable real q in the following way: We start by dividing the interval $[q, 1]$ into subintervals of size $1/2^{n_0}$ for some suitable n_0, then finding the closest such interval $[l_0, r_0]$ that contains a zero, as the endpoints l_0, r_0 will be computable when q is computable. We divide this interval into intervals of size $1/2^{n_1}$ for $n_1 > n_0$, and finding the closest such interval $[l_1, r_1]$ to l_0 that contains a zero. Continuing in this way, we will have convergent sequences $l_0, l_1, ...$ and $r_0, r_1, ...$ that converge from the left and from the right toward the first zero after q.

□

Note the fact that we are crossing zero did not play a large role in the proof - in lemmas 3.5.3 and 3.5.4 we could just as easily have checked if $B(t)$ was greater or less than a for any computable a. This gives us the following corollary.

Corollary 3.5.1. *For $B(t)$ a standard one-dimensional Martin-Löf random Brownian path, it is layerwise computable to see if $B(t)$ crosses a computable value a in computable time interval $[t_1, t_2]$, and thus the first hitting time of $B(t)$ to any computable value a after any computable time t is layerwise computable in B.*

Chapter 4

Dirichlet Problem

4.1 Brownian motion in higher dimensions

So far we have talked about Brownian motion on $C[0,1]$ or $C[\mathbb{R}^{\geq 0}]$, but it is easy to extend these definitions to Brownian motion in higher dimensions.

Definition 4.1.1.
If $B_1, ..., B_d$ are independent linear Brownian motions started in $x_1, ..., x_d$, then the process $\{B(t) : t \geq 0\}$ given by $B(t) = (B_1(t), ..., B_d(t))$ is *d-dimensional Brownian motion* started in $(x_1, ..., x_d)$. The d-dimensional Brownian motion started at the origin is also called *standard Brownian motion.* One-dimensional Brownian motion is also called *linear*, and two-dimensional Brownian motion is also called *planar Brownian motion.*

Theorem 4.1.1. *A function $B(t) = (B_1(t), ..., B_d(t))$ in the space of continuous functions from $[0,\infty)$ to $\mathbb{R}^d$ with Wiener measure is a Martin-Löf random Brownian path if and only if $B_1(t), ..., B_d(t)$ are mutually Martin-Löf random linear Brownian motion.*

Proof. This follows immediately from Van Lambalgen's theorem which states that (A, B) is a Martin-Löf random element of a product space $(X,\mu)\times(X,\mu)$ if and only if A and B are mutually Martin-Löf random elements of (X,μ). $\square$

4.2 Dirichlet Problem

The Dirichlet problem asks the following question: given a region U in $\mathbb{R}^n$ and a function ϕ defined everywhere on the boundary ∂U of U, is there a unique, continuous function u such that u is harmonic on the interior of U and $u = f$ on ∂U? The Dirichlet problem arises whenever one considers notions of potential - for example, the problem may be thought of as finding the temperature of the interior of a heat-conducting region for which the temperature on the boundary is known, or alternatively, finding the electric potential on the interior of a region for which the charge on the boundary is known.

These physical interpretations of the problem make it clear that there should be a unique solution, and indeed, many ways of finding this unique solution are known. One method of solving the Dirichlet problem which arises from an intuition of heat diffusion in a heat-conducting substance uses the mathematical model of Brownian motion [19].

Definition 4.2.1.
Let $U \subset \mathbb{R}^d$ be a domain. We say that U satisfies the Poincaré cone condition at $x \in \partial U$ if there exists a cone V based at x with opening angle $\alpha > 0$ and $h > 0$ such that $V \cap \mathscr{B}(x,h) \subset U^c$, where $\mathscr{B}(x,h)$ denotes an open ball around x of radius h.

Theorem 4.2.1 (Kakutani)**.** *Suppose $U \subset \mathbb{R}^d$ is a bounded domain such that every boundary point satisfies the Poincaré cone condition, and suppose ϕ is a continuous function on ∂U. Let $\tau(\partial U) = \inf\{t > 0 : B(t) \in \partial U\}$, which is an almost surely finite stopping time. Then the function $u : \overline{U} \to \mathbb{R}$ given by*

$$u(x) = \mathbb{E}_x\left[\phi(B(\tau(\partial U)))\right], \qquad for\ x \in \overline{U},$$

is the unique continuous function harmonic on U with $u(x) = \phi(x)$ for all $x \in \partial U$.

By relativizing Proposition 3.5.3, we can use the layerwise computability of the hitting time of Martin-Löf random Brownian motion to a computable line to show that the solution to the Dirichlet problem is computable in the planar case when the boundary is a computable curve and the condition on the boundary is both computable. Of course, we first need to specify what we mean by that. In particular, there are several notions of computable curve, see [33]. We will take the most general one: We assume that ∂U is computable in the sense that there exists a computable sequence $(C_n)_{n\in\mathbb{N}}$ such that for all n, C_n is a finite set of squares in the 2-dimensional grid $2^{-n}\mathbb{Z} \times 2^{-n}\mathbb{Z}$ whose union is connected, contains ∂U, and every point inside this union of squares is at distance at most 2^{-n+2} of the curve. To formalise the fact that the condition ϕ is computable, we assume that there is a uniformly computable family $(\phi_n)_{n\in\mathbb{N}}$, where each ϕ_n is a function which assigns a real value to each square $\mathbf{c}$, in such a way that this value is within $\varepsilon(n)$ of the values of ϕ on $\partial U \cap \mathbf{c}$, and the values of two adjacent squares are within $\varepsilon(n)$ of each other, ε being a computable function which tends to 0 computably in n.

Theorem 4.2.2 (Computable Dirichlet Problem)**.** *Let U be a bounded domain whose boundary ∂U satisfies the Poincaré cone condition and ϕ a condition on the boundary. Assume ∂U and ϕ are computable in the sense described above. Then the solution to the Dirichlet problem - the unique, continuous function $u : \overline{U} \to \mathbb{R}$ harmonic on U such that $u(x) = \phi(x)$ for all $x \in \partial U$ - is computable.*

The rest of the section will be devoted to proving this result. The plan is to prove the theorem in two steps:

(i) First, we prove it in the particular case where ∂U is a squared curve, i.e., a closed curve which is made of a finite number of vertical and horizontal (i.e, parallel to the x-axis or y-axis) segments with rational endpoints, the list being given explicitly. As we will see, in this case, we can apply the results of the previous sections to compute the first time a Martin-Löf random path hits the curve.

(ii) Then we extend it to all computable functions γ by approximation. That is, we approximate ∂U by a squared curve with arbitrary precision and apply Step 1.

Let us first see how to apply the results of the previous section to planar Brownian motion.

Lemma 4.2.1. *For $B(t)$ a Martin-Löf random planar Brownian motion started at a computable point, seeing when $B(t)$ hits the line parallel to either the $x-axis$ or $y-axis$, if the line is computable, is layerwise decidable in $B(t)$.*

Proof. Without loss of generality, say we are looking for the first time $X(t) = \alpha$, for $B(t) = (X(t), Y(t))$, α computable, $B(t)$ started at $q = (q_x, q_y) \in \mathbb{Q}$. This is equivalent to looking for the first time $X'(t) = X(t) - q_x$, a standard 1-dimensional Brownian motion, crosses $q_x - \alpha$, which follows from Corollary 3.5.1 above. □

Lemma 4.2.2. *For $B(t)$ a Martin-Löf random planar Brownian motion started at a computable point, the first time $B(t)$ passes through a vertical or horizontal line segment with computable endpoints is layerwise computable in $B(t)$.*

Proof. To layerwise computably find the first crossing time through the line segment, we run the following algorithm. Let $r_0 = 0$ be the first time considered. The first crossing of $B(t)$ through the line $y = \alpha$ after r_0 is layerwise computable in B, call this time t_1. If t_1 falls within the line segment, we are done.

Assuming t_1 crosses the line away from the line segment, we will call the distance from the line segment $\varepsilon_1 > 0$. In order for $B(t) = (X(t), Y(t))$ to hit the line segment after t_1, $X(t)$ must change by more than ε_1. By Proposition 2.2.2, we can find an h_1, layerwise in $X(t)$, such that this does not occur in $(t_1, t_1 + h_1)$. We choose $r_1 \in (t_1 + h_1/2, t_1 + h_1)$ to be any rational time, and then continue the algorithm by finding the next crossing time $t_2 > r_1$ through the line $y = \alpha$.

Because the line segment has computable vertices, $B(t)$ will not cross through the vertex of the line by Corollary 5.1.3. This tells us that before hitting the line segment, there is a closest value $\varepsilon_L > 0$ away from the vertex of line segment such that $B(t)$ crosses no closer than ε_L to the vertex. As above, this ε_L is associated with a time h_L within which $X(t)$ will not cross the line segment. As each $\varepsilon_n \geq \varepsilon_L$, each $h_n \geq h_L > 0$, so we are incrementing our time steps by at least $h_L/2$ at each stage. Therefore we are taking time steps small enough so that we do not miss the first crossing time, but time steps which are always bounded away from 0, so we must eventually find the first crossing time of $B(t)$ through the line segment. □

We can now prove our theorem in the restricted case of an explicitly given squared curve.

Proposition 4.2.1. *If U is a planar region such that ∂U is an explicitly given squared curve and ϕ is a computable function on ∂U, then the solution to the Dirichlet problem is computable for U.*

Proof. By Lemma 4.2.2 the first hitting times on each line segment are computable uniformly in starting point x and layerwise in B, and δU is composed of finitely many line segments with computable endpoints, so the first hitting time $\tau_B(\partial U)$ to the boundary is layerwise computable in B, uniformly in the starting point. Since ϕ is computable, $\phi(\tau_B(\partial U))$ is computable uniformly in starting point x and layerwise in B.

By Theorem 1.8.2, the expression

$$u(x) = \mathbb{E}_x\left[\phi(B(\tau_B(\partial U)))\right], \qquad for\ x \in \overline{U}$$

is computable, uniformly in x, and by Kakutani's classical result 4.2.1, this is the solution to the Dirichlet problem. □

Now, all we need to do is extend this last proposition to the general case.

Proof of Theorem 4.2.2. Let u be the solution of Dirichlet's problem (we don't know yet it is computable, but we know it exists) for condition ϕ on ∂U. Given a point $x \in U$, we first compute, for all n, an approximation C_n of ∂U which are squares of $2^{-n}\mathbb{Z} \times 2^{-n}\mathbb{Z}$. Compute the largest set Q of squares of $2^{-n}\mathbb{Z} \times 2^{-n}\mathbb{Z}$ which (a) contains the square $\mathbf{c}$ which contains x, (b) does not contain any square in C_n and (c) is 4-connected, i.e., every square of Q_n should share an edge with another member of Q_n (unless there is only one square). Call V_n the interior of the union of the squares in Q_n. Observe that, by Jordan's curve theorem, V_n must be contained in U, since it contains a point in U, is connected, and is disjoint from ∂U. Observe also that each segment of ∂V_n must be the edge of a square $\mathbf{c} \in C_n$, so we can compute a condition ψ on ∂V_n which is equal to $\phi_n(\mathbf{c})$ on the edge of $\phi_n(\mathbf{c})$ (up to smoothing it out around corners to ensure continuity).

Claim. For every point $z \in \partial V_n$, $|\psi(z) - u(z)| < O(\varepsilon(n) + 2^{-n})$. Indeed, let $\mathbf{c}$ be the member of C_n which has z on its edge. Every point of $\mathbf{c}$ is at distance at most 2^{-n+2} of the curve, so there is a square $\mathbf{c}'$ at distance $O(2^{-n})$ of $\mathbf{c}'$ which contains some point $z' \in \partial U$, and the value of $\phi_n(\mathbf{c}')$ is within $\varepsilon(n)$ of the value of $u(z')$. Thus,

$$\begin{aligned} |\psi(z) - u(z)| &\leq |\psi(z) - \phi_n(\mathbf{c}')| + |\phi_n(\mathbf{c}') - u(z')| + |u(z') - u(z)| \\ &\leq O(\varepsilon(n)) + \varepsilon(n) + O(2^{-n}) \end{aligned}$$

(for the last term, we use the fact that $|z' - z| = O(2^{-n})$ and the fact that u is harmonic, hence Lipschitz), the constants in the O-notations not depending on n, z', z. To be precise, we need to add the possible error induced by the 'smoothing around corners', but it itself is bounded by $O(\varepsilon(n) + 2^{-n})$ since the ϕ_n-values of two adjacent segments of ∂V_n are $O(\varepsilon(n) + 2^{-n})$-close to each other. . Thus, applying the restricted case of our theorem (Proposition 4.2.1) to ψ and V_n, we can compute the value $v_n(x)$ of the solution to Dirichlet's problem on ∂V_n with condition ψ. But since $|\psi - u| = |v_n - u|$ is bounded by $O(\varepsilon(n) + 2^{-n})$ by $O(\varepsilon(n) + 2^{-n})$ on ∂V_n, this implies that $|v_n - u|$ is also bounded $O(\varepsilon(n) + 2^{-n})$ on all of V_n (by the maximum principle, since $v_n - u$ is harmonic). Thus, we have effectively obtained an approximation of $u(x)$ with precision $O(2^{-n} + \varepsilon(n))$ uniformly in n and x, which means that u is computable. □

Then letting $B_x(t)$ be a Martin-Löf Brownian motion started at a point x in U, and picking $g(t)$ so that x is also inside the boundary $g(t)$, let τ_g denote the first hitting time of $B_x(t)$ on $g(t)$. By 4.2.1, $\psi(B_x(\tau_g))$ is layerwise computable in B, uniformly in x, and is within $k \cdot \varepsilon$ of $u(B_x(\tau_g))$, so $u(B_x(\tau_g))$ is also layerwise computable in B, uniformly in x.

So by the result of Hoyrup and Rojas that the integral of a bounded, layerwise computable function is computable [16],

$$u(x) = \mathbb{E}_x \left[u(B(\tau_g)) \right], \qquad for\ x \in \overline{U}$$

is computable, uniformly in x, and by Kakutani's result 4.2.1, this is the solution to the Dirichlet problem.

Chapter 5

Planar Martin-Löf random Brownian Motion

The real plane captures an exceptionally interesting set of behaviors of Brownian paths. It is the smallest Euclidean space where Brownian motion almost surely does not hit points - that is, in one dimension, a Brownian path which runs forever will hit every point uncountably many times, almost surely, but in the plane, any fixed point except the origin is almost surely *not* hit by a Brownian path. However, the real plane is the largest Euclidean space where Brownian motion is neighborhood recurrent, meaning a path will hit any ε-neighborhood in a time set limiting to infinity, almost surely, although in all higher dimensions, a Brownian path is almost surely divergent. This combination of properties leads to a lot of interesting behavior in planar Brownian motion, such as the existence of multiple points of high multiplicity. And, as in the one-dimensional case, many of the interesting "almost surely" properties of planar Brownian motion are reflected in every Martin-Löf random Brownian path.

5.1 Points on planar Martin-Löf random Brownian Motion

Several interesting results about the behavior of planar Brownian motion can be realized as consequences of the fact that a Martin-Löf random planar path will only hit points that derandomize it, with the exception of starting at the origin.

Theorem 5.1.1. *At any time $t > 0$, for $B(t)$ a planar Martin-Löf random Brownian path started at (0, 0), B is not random relative to any point $(B_x(t), B_y(t))$ on the path.*

Proof. For $B(t)$ a standard planar Brownian motion, the probability that $B(t)$ hits an ε-ball around a point $(x, y) \neq (0, 0)$, for $\varepsilon < |x^2 + y^2|$ is equivalent to the probability that a planar Brownian motion started at radius $|x^2 + y^2| = R$ hits an ε-ball around zero, by radial symmetry of the planar Brownian motion. The radial part of d-dimensional Brownian motion is the Bessel process of order ν where $d = 2\nu + 2$, and is well understood. In the planar case we are concerned with the Bessel process of order zero.

Let $\tau_{R,\varepsilon}$ be the first hitting time of the Bessel process of order zero started at R, hitting to ε. Using a result of Haman and Matsumoto [15], we know that

$$\mathbb{P}(\tau_{R,\varepsilon} \leq 1) = \int_0^1 \frac{R-\varepsilon}{\sqrt{2\pi s^3}} e^{-\frac{(R-\varepsilon)^2}{2s}} ds - \int_0^1 \frac{R-\varepsilon}{\sqrt{2\pi s^3}} e^{-\frac{(R-\varepsilon)^2}{2s}} \left[\int_0^\infty \frac{L_{0,R/\varepsilon}(x)}{x} e^{-\frac{x(R-\varepsilon)}{\varepsilon\sqrt{s}}} dx \right] ds$$

where

$$L_{0,R/\varepsilon}(x) = \frac{I_0(Rx/\varepsilon)K_0(x) - I_0(x)K_0(Rx/\varepsilon)}{(K_0(x))^2 + \pi^2(I_0(x))^2}$$

and

$$I_0(x) = \frac{1}{\pi} \int_0^\pi e^{x \cos t} dt,$$

$$K_0(x) = \int_0^\infty \frac{\cos{(xt)}}{\sqrt{t^2+1}} dt$$

These functions are computable, because all the component pieces - cosine, square root, exponentiation, multiplication, and division - are computable, and the integral of a computable function is computable. See the book by Weihrauch [37] for more details. Moreover, this integral goes to zero as ε goes to zero, which is more easily seen using a classical result of Spitzer [36]:

$$\lim_{\varepsilon \to 0} \log \left(\frac{1}{\varepsilon} \right) Pr(\tau_{R,\varepsilon} \leq 1) = \int_{R^2/2}^\infty \frac{e^{-x}}{2x} dx.$$

As the right hand side is a constant, and $log(\frac{1}{\varepsilon}) \to \infty$, we know that $Pr(\tau_{R,\varepsilon} \leq 1) \to 0$.

Thus we have a Schnorr test relative to the point (x, y), so a Martin-Löf random Brownian path $B(t)$ will only pass through points (x, y) such that the path B (or a code for the path) is not random relative to (x, y), before time 1. The argument is the same for any finite time, not just time 1, so the statement of the theorem holds. □

Corollary 5.1.1. *$B(t)$ a standard planar Martin-Löf random Brownian path has zero area.*

Proof. Only Lebesgue measure zero many points derandomize any particular real, so any Martin-Löf random path hits only Lebesgue measure zero many points. □

Corollary 5.1.2. *For any point $(x, y) \neq (0, 0)$, only measure zero many Brownian paths hit (x, y) (Almost surely, Brownian motion does not hit points)*

Proof. A real derandomizes only Lebesgue measure zero many reals. □

Corollary 5.1.3. *At any time $t > 0$, for $B(t)$ a standard planar Martin-Löf random Brownian path, B does not pass through any computable points.*

Proof. A Martin-Löf random path is always random relative to a computable point. □

5.2 Multiple Points of Martin-Löf random Brownian motion

Nearly any question one can formulate about Brownian motion leads to interesting results, and planar Brownian motion has especially fascinating behavior. Although a planar Brownian path hits only Lebesgue-measure zero many points, we know that it is neighborhood recurrent - meaning that, almost surely, for every point x and real $\varepsilon > 0$, there is a sequence $t_n \upharpoonright \infty$ such that a planar Brownian motion path $B(t)$ has $B(t_n) \in \mathscr{B}(x, \varepsilon)$ for all $n \in \mathbb{N}$. Given this fact, it is unsurprising that a Brownian motion path has *multiple points* - that is, points $x \in \mathbb{R}^n$ such that there exist $t_1 \neq t_2$ where $B(t_1) = B(t_2) = x$. However, what is surprising is that not only does planar Brownian motion have double points almost surely, but it has multiple points of every finite multiplicity almost surely, and it is even the case that planar Brownian motion has points of uncountable multiplicity almost surely. We will reproduce this surprising result for Martin-Löf random Brownian paths, following the classical proof in Mörters and Peres [26].

In Chapter four, we showed that it is layerwise computable in a Martin-Löf random Brownian path $B(t)$ to see whether $B(t)$ crossed a computable line segment. For the proof of the existence of uncountable multiple points, we need to show that it is layerwise computable in $B(t)$ to see whether it crosses through the boundary of a circle C with computable center (a, b) and computable radius r.

For a, b computable reals and $B(t) = (X(t), Y(t))$ a planar Martin-Löf random Brownian path, let $D(B(t)) = \sqrt{(X(t) - a)^2 + (Y(t) - b)^2}$ denote the distance from $B(t)$ to (a, b) at a time t. For (a, b) the center of a computable circle C with radius r, we can determine whether $B(t)$ crossed through the boundary of C by seeing if $D(B(t))$ has values both above and below r.

To see whether $D(B(t))$ has values both above and below r, we will first need to show that maxima and minima of $D(B(t))$ are not computable. This tells us that if a Martin-Löf random Brownian path intersects a computable circle, it must cross the boundary of the circle, and that will be an event that we can prove does or does not happen, layerwise computably in B.

First we will show that $D(B(t))$ obeys a modulus of continuity. From here on we will refer to $D(B(t))$ as $D^B(t)$ to indicate that we are thinking of D as a function of time for a fixed $B(t)$. Note that as $B(t)$ is continuous, $D^B(t)$ is also a continuous function of t.

Lemma 5.2.1. *Let $D^B(t)$ be as defined above for $B(t)$ Martin-Löf random and (a, b) computable. Then for all $c > \sqrt{2}$, there is an $h_0 \in \mathbb{R}$, such that for all $h < h_0$ and all t*

$$|D^B(t + h) - D^B(t)| < 2c\sqrt{h \log(1/h)}$$

Moreover, h_0 is layerwise computable in B.

Proof. The result follows immediately from lemma 2.2.2 and the triangle inequality. h_0 is layerwise computable because we can simply take the smaller of the two h that are layerwise computable from the Brownian paths $X(t)$ and $Y(t)$. □

Now that we have the immensely useful modulus of continuity result, we can show that the values of the maxima and minima of $D^B(t)$ on a computable interval $[t_1, t_2]$ are layerwise computable in B, but not computable.

Lemma 5.2.2. *For $B(t) = (X(t), Y(t))$ a planar Martin-Löf random Brownian path and $[t_1, t_2]$ a computable interval, the maximum and minimum of $D^B(t) = \sqrt{(X(t) - a)^2 + (Y(t) - b)^2}$ on $[t_1, t_2]$ are layerwise computable in $B(t)$.*

Proof. To compute the maximum of $D^B(t)$ on $[t_1, t_2]$ to within ε, we run the following simple algorithm: Pick h_0 small enough so that $D^B(t)$ obeys a modulus of continuity with constant $c = 2$ (see lemma 5.2.1) and so that $4\sqrt{h_0 \log(1/h_0)} < \varepsilon$. Then we know that the maximum of the values $D^B(t_1), D^B(t_1 + h_0), D^B(t_1 + 2h_0), ..., D^B(t_2)$ must be within $4\sqrt{h_0 \log(1/h_0)}$, and therefore within ε, of the maximum value of $D^B(t)$ on $[t_1, t_2]$. The minima are also layerwise computable by the same argument. □

Lemma 5.2.3. *The maxima (and minima) of $D^B(t) = \sqrt{(X(t) - a)^2 + (Y(t) - b)^2}$ on a computable interval $[t_1, t_2]$ are not computable for $B(t) = (X(t), Y(t))$ a planar Martin-Löf random Brownian path.*

Proof. Let $m^{[t_1,t_2]}$ denote the maximum of $D^B(t)$ on a computable interval $[t_1, t_2]$. By Lemma 5.2.2, we know that $m^{[t_1,t_2]}$ is layerwise computable in B. Using 1.8.1, this tells us that the map $g^{[t_1,t_2]} : (C(\mathbb{R}^[\geq 0], \mathbb{R}^2), \mathbb{P}) \to \mathbb{R}$ which gives the max of $D^B(t)$ on $[t_1, t_2]$ is a measurable map, which preserves Martin-Löf randomness. That is, we are pushing Wiener measure to its image under $D(B(t))$ and then to the maximum of $D^B(t)$ on $[t_1, t_2]$. Calling that pushforward measure ν, the maximum of $D^B(t)$ on a computable interval is a Martin-Löf random real with respect to the measure ν.

Note that ν having an atom corresponds to a positive (Wiener) measure of paths $B(t)$ having maximum radius r from a point (a, b) in time $[t_1, t_2]$. Classically, we know that the probability of entering a circle with radius $r + \varepsilon$ but not entering the circle with radius $r - \varepsilon$ goes to zero as ε goes to zero - that is,

$$\lim_{\varepsilon \to 0} [\mathbb{P}(B(t) \text{ enters } \mathscr{B}((a, b), r + \varepsilon)) - \mathbb{P}(B(t) \text{ enters } \mathscr{B}((a, b), r + \varepsilon))] \to 0$$

See, for example, the proof of Theorem 5.1.1 and the references given there for a discussion of the probability that a planar Martin-Löf random path hits a ball around a given point away from zero in a given time.

Thus ν is a non-atomic measure, so by a result of Reimann and Slaman [31], $m^{[t_1,t_2]}$, which is Martin-Löf random with respect to ν, cannot be computable. Minima also cannot be computable by the same argument.

□

Now we can prove that it is layerwise computable to see whether a planar Martin-Löf random path $B(t)$ crosses through a circle with computable center and radius. This is similar to the proof that it is layerwise computable in $B(t)$ to see if B crosses through a computable real value, for $B(t)$ a one-dimensional standard Martin-Löf random Brownian path.

Proposition 5.2.1. *For $B(t) = (X(t), Y(t))$ a planar Martin-Löf random Brownian path, it is layerwise computable in $B(t)$ to see whether $B(t)$ crosses the boundary of a circle with computable center (a, b) and computable radius r in a given computable time interval $[t_1, t_2]$*

This is equivalent to seeing whether $D^B(t) = \sqrt{(X(t) - a)^2 + (Y(t) - b)^2}$ crosses through a computable real r in time $[t_1, t_2]$.

Lemma 5.2.4. *It is layerwise computable in $B(t)$ to see that $D^B(t)$ does not cross through a computable value r in time $[t_1, t_2]$*

Proof. This follows from 5.2.1 above. Because h_0 is layerwise computable in $B(t)$, we can, layerwise computably in B, find increments of size h in $[t_1, t_2]$ such that $D^B(t_1), D^B(t_1 + h), ..., D^B(t_2) > r+4h\sqrt{\log 1/h}$ if B is bounded outside the circle, or $D(t_1), D(t_1+h), ..., D(t_2) < r - 4h\sqrt{\log 1/h}$ if B is bounded inside the circle. Then 5.2.1 tells us that $B(t)$ must be bounded away from the boundary of the circle on $[t_1, t_2]$. □

Lemma 5.2.5. *It is layerwise computable in $B(t)$ to see that $D^B(t)$ does cross through a computable value r in time $[t_2, t_2]$.*

Proof. By Lemma 5.2.3, we know that the maxima and minima of $D^B(t)$ on $[t_1, t_2]$ must have non-computable values, so any crossing time must have times s_1 and s_2 arbitrarily close on either side where $D^B(s_1) < r$ and $D^B(s_2) > r$. In fact, there must be computable such points, as continuity of $D^B(t)$ guarantees open intervals arbitrarily close to the crossing time where the function is always greater than r and always less than r.

The construction for $B(t)$ given in Chapter 1 has layerwise computable convergence, so $D^B(t)$ must also have layerwise computable convergence, so we can layerwise computably search through any dense set of computable times in $[t_1, t_2]$ and eventually find times s_1, s_2 where $D^B(s_1) < r$ and $D^B(s_2) > r$

□

These lemmas together prove proposition 5.2.1.

Now we can begin the proof that planar Martin-Löf random Brownian has points of uncountable multiplicity.

Theorem 5.2.1. *[26, Theorem 9.24] Let $\{B(t) : t \geq 0\}$ be a planar Brownian motion. Then, almost surely, there exists a point $x \in \mathbb{R}^2$ such that the set $\{t \geq 0 : B(t) = x\}$ is uncountable.*

And, as always, we can replace "almost surely" with "for every Martin-Löf random path"

Theorem 5.2.2. *Let $\{B(t) : t \geq 0\}$ be a Martin-Löf random planar Brownian motion. Then there exists a point $x \in \mathbb{R}^2$ such that the set $\{t \geq 0 : B(t) = x\}$ is uncountable.*

The proof of Theorem 5.2.2 that follows is the same as the proof of Theorem 5.2.1 given in the book by Mörters and Peres [26], with some modifications in order to demonstrate that failing to have an uncountable multiple point corresponds to failing a Martin-Löf test.

First we describe the rough strategy of the proof. We start by finding two disjoint intervals I_1 and I_2 with $B(I_1) \cap B(I_2) \neq \emptyset$. Inside these we find disjoint subintervals $I_{11}, I_{12} \subset I_1$ and $I_{21}, I_{22} \subset I_2$ such that the four Brownian images $B(I_{ij})$ intersect. Continuing this way, we construct a binary tree T of time intervals where rays in T represent sequences of nested intervals and the intersection of each such sequence will be mapped to the same point by the Brownian motion.

We will use the following notation. For any open or closed sets $A_1, A_2, ...$ and a Brownian motion $B : [0, \infty) \to \mathbb{R}^2$ define stopping times

$$\tau(A_1) \coloneqq \inf\{t \geq 0 : B(t) \in A_1\}$$

$$\tau(A_1, ..., A_n) \coloneqq \inf\{t \geq \tau(A_1, ...A_{n-1}) : B(t) \in A_n\}, \text{ for } n \geq 2,$$

where the infimum of the empty set is taken to be infinity. We say the Brownian path *upcrosses the shell $\mathscr{B}(x, 2r) \setminus \mathscr{B}(x, r)$ twice* before a stopping time T if

$$\tau(\mathscr{B}(x, r), \mathscr{B}(x, 2r)^C, \mathscr{B}(x, r), \mathscr{B}(x, 2r)^C) < T.$$

We call the paths between $\tau(\mathscr{B}(x, r))$ and $\tau(\mathscr{B}(x, r), \mathscr{B}(x, 2r)^C)$, and between $\tau(\mathscr{B}(x, r), \mathscr{B}(x, 2r)^C, \mathscr{B}(x, r))$ and $\tau(\mathscr{B}(x, r), \mathscr{B}(x, 2r)^C, \mathscr{B}(x, r), \mathscr{B}(x, 2r)^C))$ the *upcrossing excursions*. See figure 5.1.

From now on let T be the first exit time of Brownian motion from the unit ball.

Lemma 5.2.6. *There exist (computable) constants $0 < c_0 < C_0$ such that, if $2 < m < n$ are two integers and $\mathscr{B}$ a ball of radius 2^{-n} with center at distance at least 2^{-m} and at most $3 * 2^{-m}$ from the origin, we have*

$$c_0 \frac{m}{n} \leq \mathbb{P}_0\{\tau(\mathscr{B}) < T\} \leq C_0 \frac{m}{n}.$$

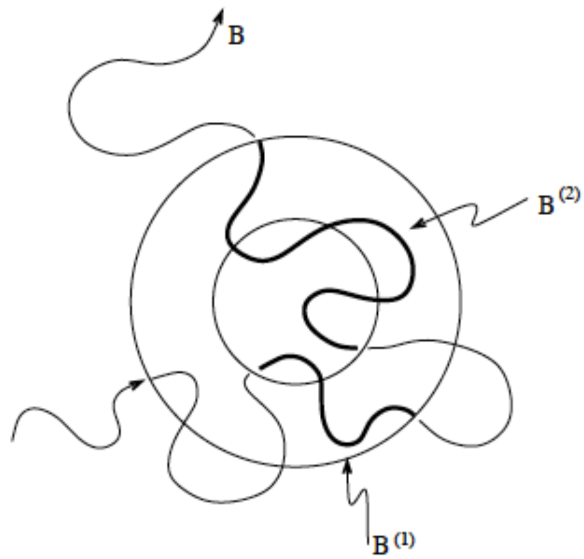

Figure 5.1: The path $B(t)$ upcrosses the shell twice; the upcrossing excursions are bold and marked $B^{(1)}$ and $B^{(2)}$. Image from "Brownian Motion" by Mörters and Peres.

See Mörters and Peres [26] Lemma 9.25 for proof.

The density of $B(T)$ under $\mathbb{P}_z$ is given by the Poisson kernel,

$$\mathcal{P}(z,w) = \frac{1-|z|^2}{|z-w|^2} \quad \text{for any } z \in \mathscr{B}(0,1) \text{ and } w \in \partial\mathscr{B}(0,1).$$

Lemma 5.2.7. *Consider Brownian motion started at $z \in \mathscr{B}(0,r)$ where $r < 1$, and stopped at time T when it exits the unit ball. Let $\tau \leq T$ be a stopping time, and let $A \in \mathcal{F}(\tau)$. Then we have*

- $\mathbb{P}_z(A|B(T)) = \mathbb{P}_z(A)\dfrac{\mathbb{E}_z[\mathcal{P}(B(\tau),B(T))|A]}{\mathcal{P}(z,B(T))}$.
- *If $\mathbb{P}_z(\{B(\tau) \in \mathscr{B}(0,r)\}|A) = 1$, then*

$$\left(\frac{1-r}{1+r}\right)^2 \mathbb{P}_z(A) \leq \mathbb{P}_z(A|B(t)) \leq \left(\frac{1+r}{1-r}\right)^2 \mathbb{P}_z(A) \text{ almost surely.}$$

See Mörters and Peres [26] Lemma 9.26 for proof.

The following lemma, concerning the upcrossings of L-many Brownian excursions, is the driving engine of the proof of the theorem, so we will give the proof in full, though it is identical to that found in [26].

Lemma 5.2.8. *Let $n > 5$ and let $\{x_1, ..., x_{4^{n-5}}\}$ be points such that the balls $\mathscr{B}(x_i, 2^{1-n})$ are disjoint and contained in the shell $\{z : \frac{1}{4} \leq |z| \leq \frac{3}{4}\}$. Consider L independent Brownian upcrossing excursions $W_1, ..., W_L$, started at prescribed points on $\partial\mathscr{B}(0,1)$ and stopped when they reach $\partial\mathscr{B}(0,2)$. Let S denote the number of centers x_i, $1 \leq i \leq 4^{n-5}$ such that the shell $\mathscr{B}(x_i, 2^{-n+1}) \setminus \mathscr{B}(x_i, 2^{-n})$ is upcrossed twice by each of $W_1, ..., W_L$. Then there exist constants $c, c_* > 0$ such that*

$$\mathbb{P}\{S > 4^n (c/n)^L\} \geq \frac{c_*^L}{L!}. \tag{5.1}$$

Moreover, the same estimate (with a suitable constant c_*) is valid if we condition on the endpoints of the excursions $W_1, ..., W_L$, and the constants c, c_* can be taken to be computable.

Proof. By lemma 5.2.6, for any $z \in \partial\mathscr{B}(0,1)$, the probability of Brownian motion starting at z hitting the ball $\mathscr{B}(x_i, 2^{-n})$ before reaching $\partial\mathscr{B}(0,2)$ is at least $\frac{c_0}{n}$, and the probability of the second upcrossing excursion of $\mathscr{B}(x_i, 2^{-n+1}) \setminus \mathscr{B}(x_i, 2^{-n})$, when starting at $\partial\mathscr{B}(x_i, 2^{1-n})$ is at least $1/2$. Thus

$$\mathbb{E}S \geq 4^{n-5} \left(\frac{c_0}{2n}\right)^L. \tag{5.2}$$

We now estimate the second moment of S. Consider a pair of centers x_i, x_j such that $2^{-m} \leq |x_i - x_j| \leq 2^{1-m}$ for some $m < n-1$. For each $k \leq L$, let $V_k = V_k(x_i, x_j)$ denote the event that the balls $\mathscr{B}(x_i, 2^{-n})$ and $\mathscr{B}(x_j, 2^{-n})$ are both visited by W_k. Given that $\mathscr{B}(x_i, 2^{-n})$ is reached first, the conditional probability that W_k will also visit $\mathscr{B}(x_j, 2^{-n})$ is at most $C_0 \frac{m}{n}$, by lemma 5.2.6. We conclude that $\mathbb{P}(V_k) \leq 2C_0^2 \frac{m}{n^2}$, giving us that

$$\mathbb{P}\left(\bigcap_{k=1}^{L} V_k\right) \leq \left(2C_0^2 \frac{m}{n^2}\right)^L.$$

For each $m < n-1$ and $i \leq 4^{n-5}$, the number of centers x_j such that $2^{-m} \leq |x_i - x_j| \leq 2^{1-m}$ is at most a constant multiple of 4^{n-m}. Using that the diagonal terms are of lower order, we deduce that there exists $C_1 > 0$ such that

$$\mathbb{E}S^2 \leq \frac{C_1^L 4^{2n}}{n^{2L}} \sum_{m=1}^{n} \leq \frac{(2C_1)^L 4^{2n} L!}{n^{2L}}, \tag{5.3}$$

where the last inequality follows, e.g., from taking $x = 1/4$ in the binomial identity

$$\sum_{m=0}^{\infty} \binom{m+L}{L} x^m = (1-x)^{-L-1}.$$

Note that C_1 can be taken to be computable.

Now (5.2), (5.3), and the Paley-Zygmund inequality yield (5.1).The final statement of the lemma follows from lemma 5.2.7.

The Paley-Zygmund inequality states the following: For any nonnegative random variable X with $\mathbb{E}[X^2] < \infty$ and $\lambda \in [0, 1)$,

$$\mathbb{P}\{X > \lambda \mathbb{E}[X]\} \geq (1-\lambda)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

□

Proof of Theorem 5.2.2. Let $\{n_i : i \geq 1\}$ denote a computable increasing sequence to be chosen later, and let $N_l = \sum_{i=0}^{l} n_i$ with $N_0 = 0$. Denote $q_i = 4^{n_i - 5}$ and $Q_i = 4^{N_i - 5i}$. We begin by constructing a nested sequence of centers with which we associate a forest, i.e. a collection of trees, in the following manner. The first level of the forest consists of Q_i-many centers, $\{x_1^{(1)}, ..., x_{Q_1}^{(1)}\}$, chosen such that the centers are computable points in $\mathbb{R}^2$ and the balls $\{\mathscr{B}(x_k^{(1)}, 2^{-N_1+1}) : 1 \leq k \leq Q_1\}$ are disjoint and contained in the annulus $\{z : \frac{1}{4} \leq |z| \leq \frac{3}{4}\}$.

Continue this construction recursively. For $l > 1$ suppose that the level $l-1$ of the forest has been constructed. Level l consists of Q_l-many vertices $\{x_1^{(l)}, ..., x_{Q_l}^{(l)}\}$. Each vertex $x_i^{l-1}, 1 \leq i \leq Q_{(l-1)}$, at level $l-1$ has q_l-many children $\{x_j^{(l)} : (i-1)q_l < j \leq iq_l\}$ at level l; the balls of radius 2^{-N_l+1} around these children are disjoint and contained in the annulus

$$\{z : \frac{1}{4} 2^{-N_{l-1}} \leq |z - x_i^{(l-1)}| \leq \frac{3}{4} 2^{-N_{l-1}}\}$$

and the vertices are chosen to be computable, uniformly in l. Recall that $T = \inf\{t > 0 : |B(t)| = 1\}$. We say that a level one vertex $x_k^{(1)}$ *survived* if the Brownian motion upcrosses the annulus $\mathscr{B}(x_k^{(1)}, 2^{-N_1+1}) \setminus \mathscr{B}(x_k^{(1)}, 2^{-N_1})$ twice before T.

A vertex at the second level $x_k^{(2)}$ is said to have *survived* if its parent vertex survived, and in each upcrossing excursion of its parent, the Brownian motion upcrosses the annulus $\mathscr{B}(x_k^{(2)}, 2^{-N_2+1}) \setminus \mathscr{B}(x_k^{(2)}, 2^{-N_2})$ twice.

Recursively, we say a vertex $x_k^{(l)}$, at level l of the forest, *survived* if its parent vertex survived, and in each of the 2^{l-1} upcrossing excursions of its parent, the Brownian motion upcrosses the shell

$$\mathscr{B}(x_k^{(l)}, 2^{-N_l+1}) \setminus \mathscr{B}(x_k^{(l)}, 2^{-N_l})$$

twice. Note at this point that if there is an infinite sequence of surviving vertices

$$x_{k(1)}^{(1)}, x_{k(2)}^{(2)}, x_{k(3)}^{(3)}, ...$$

such that $x_{k(l+1)}^{(l+1)}$ is a child of $x_{k(l)}^{(1)}$, for $l = 1, 2, 3, ...$, then the sequence of compact balls centered in $x_{k(l)}^{(l)}$ with radius 2^{-N_l} is nested. Therefore there exists exactly one point in the intersection of these balls. For any level l, there are 2^l-many disjoint upcrossing excursions of the shell $\mathscr{B}(x_k^{(l)}, 2^{-N_l+1}) \setminus \mathscr{B}(x_k^{(l)}, 2^{-N_l})$. Each of these contains two disjoint excursions at level $l+1$. Thus the time intervals corresponding to these excursions form a binary tree, where the children of an interval at level l are the two intervals at level $l+1$ it contains. An infinite ray in this tree is a nested sequence of compact intervals and their intersection is a time t with $B(t) = x$. Since there are uncountably many rays, x has uncountable multiplicity.

Now, for any $l \geq 1$, let S_l denote the number of vertices at level l of the forest that survived. Using the notation of lemma 5.2.8, let

$$\Gamma_l = 4^{n_l}\left(\frac{c}{n_l}\right)^L \quad \text{and} \quad p_l = \frac{(c_*)^L}{L!},$$

where $L = L(l) = 2^{l-1}$. Lemma 5.2.8 with $n = n_1$ states that

$$\mathbb{P}\{S_1 > \Gamma_1\} \geq p_1 = c_*. \tag{5.4}$$

For $l > 1$, the same lemma, and independence of excursions in disjoint annuli given their endpoints, yield

$$\mathbb{P}(\{S_{l+1} \leq \Gamma_{l+1}\}|\{S_l > \Gamma_l\}) \leq (1 - p_{l+1})^{\Gamma_l} \leq e^{-p_{l+1}\Gamma_l}. \tag{5.5}$$

By choosing each n_l large enough, we can control how small each $e^{-p_{l+1}\Gamma_l}$ becomes, making the right hand side summable in l. Consequently,

$$\mathbb{P}\left(\text{There is not a multiple point in the annulus } \frac{1}{4} \leq |z| \leq \frac{3}{4} \text{ before } B(t) \text{ exits } \mathscr{B}(0,1)\right)$$

$$\begin{aligned}
&\leq \mathbb{P}\left(\exists l[\{S_{l+1} \leq \Gamma_{l+1}\}|\{S_l > \Gamma_l\}]\right) = 1 - \mathbb{P}\left(\bigcap_{l=1}^{\infty}\{S_l > \Gamma_l\}\right) \\
&= 1 - \left(\mathbb{P}(S_1 > \Gamma_1)\prod_{l=1}^{\infty}\mathbb{P}\left(\{S_{l+1} > \Gamma_{l+1}\}|\{S_l > \Gamma_l\}\right)\right) \\
&\leq 1 - c^*\prod_{l=1}^{\infty}(1 - e^{-p_{l+1}\Gamma_l}) = \beta \in (0,1)
\end{aligned}$$

Note that because all constants are computable and we control the sequence $\{n_l\}_{l\in\mathbb{N}}$, β is computable. And because it is layerwise computable in a fixed path $B(t)$ to see whether $B(t)$ has crossed into or out of a circle with computable center and computable radius, the statement $[\{S_{l+1} \leq \Gamma_{l+1}\}|\{S_l > \Gamma_l\}]$ is layerwise computable, or for a fixed randomness deficiency, the statement is computable. Then, for a fixed randomness deficiency, the statement $\exists l[\{S_{l+1} \leq \Gamma_{l+1}\}|\{S_l > \Gamma_l\}]$ is Σ_1^0.

Let H_1 denote the event $\exists l[\{S_{l+1} \leq \Gamma_{l+1}\}|\{S_l > \Gamma_l\}]$ in the annulus $\frac{1}{4} \leq |z| \leq \frac{3}{4}$ before $B(t)$ exits $\mathscr{B}(0,1)$. As argued above, H_1 is a Σ_1^0 event with $\mathbb{P}(H_1) \leq \beta$ for $0 < \beta < 1$.

Now, run the same argument in the annulus $\frac{1}{16} \leq |z| \leq \frac{3}{16}$ before the path $B(t)$ exits $\mathscr{B}(0,\frac{1}{4})$. Let H_2 be the event $\exists l[\{S_{l+1} \leq \Gamma_{l+1}\}|\{S_l > \Gamma_l\}]$ in the annulus $\frac{1}{16} \leq |z| \leq \frac{3}{16}$

before $B(t)$ exits $\mathscr{B}(0, \frac{1}{4})$. This is independent of H_1 because the annuli are disjoint, so the segments of the path before exiting $\mathscr{B}(0, \frac{1}{4})$ and after entering $\frac{1}{4} \leq |z| \leq \frac{3}{4}$ are disjoint. By Brownian scaling, $\mathbb{P}(H_2) = \mathbb{P}(H_1) = \beta$, so the event $[H_1 \text{ and } H_2]$ is a Σ_1^0 event such that

$$\mathbb{P}(H_1 \text{ and } H_2) \leq \beta^2$$

Similarly, for event H_n defined to be $\exists l[\{S_{l+1} \leq \Gamma_{l+1}\}|\{S_l > \Gamma_l\}]$ in the annulus $\frac{1}{2^{2n}} \leq |z| \leq \frac{3}{2^{2n}}$ before $B(t)$ exits $\mathscr{B}(0, \frac{1}{2^{2n-2}})$, H_n is independent of H_i for all $i < n$, so $H_1 \wedge H_2 \wedge ... \wedge H_n$ is a Σ_1^0 event such that

$$\mathbb{P}(H_1 \wedge H_2 \wedge ... \wedge H_n) \leq \beta^n,$$

giving a Martin-Löf test. Thus there must be an infinite number of annuli $\frac{1}{2^{2n}} \leq |z| \leq \frac{3}{2^{2n}}$ such that a Martin-Löf random Brownian path does not have a level l such that $\{S_{l+1} \leq \Gamma_{l+1}\}|\{S_l > \Gamma_l\}$, guaranteeing that $B(t)$ must have an uncountable multiple point in that annulus.

Note that this Martin-Löf test is dependent on a fixed randomness deficiency, so while all Martin-Löf random planar Brownian paths $B(t)$ will "pass" the test, it only guarantees a multiple point of uncountable multiplicity for paths whose randomness deficiency match that of the test. However, for each path, there is such a Martin-Löf test with appropriate randomness deficiency, so for each Martin-Löf random Brownian path $B(t)$, B has multiple points of uncountable multiplicity.

□

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