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Quantum Evolution: Black Holes, Gravitational Dressing, and General Backgrounds

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Physics

by

Julie Perkins

Committee in charge:

Professor Steven B. Giddings, Chair
Professor Donald Marolf
Professor David Stuart

September 2024

The Dissertation of Julie Perkins is approved.

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September 2024

Quantum Evolution: Black Holes, Gravitational Dressing, and General Backgrounds

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by

Julie Perkins

This thesis is dedicated to my family, my mother, father, step father, and especially my husband, whose daily support and encouragement has helped me in countless ways.

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S. B. Giddings and J. Perkins, "Quantum evolution of the hawking state for black holes," *Physical Review D* **106** no. 6, (2022) 065011

S. B. Giddings and J. Perkins, "Perturbative quantum evolution of the gravitational state and dressing in general backgrounds," *Physical Review D* **110** no. 2, (2024) 026012.

Abstract

Quantum Evolution: Black Holes, Gravitational Dressing, and General Backgrounds

by

Julie Perkins

The full theory of Quantum Gravity (QG) that unites gravity and quantum mechanics has not yet been discovered. One of the pressing issues is to correctly account for unitarity in quantum mechanical processes, even on a static curved spacetime. The presence of black holes in particular leads to profound issues with unitary evolution and locality. Though Hawking radiation and gravitational interactions are well studied in the Heisenberg picture, this work uses the Schrödinger picture to examine their evolution. Through the ADM decomposition and the associated Hamiltonian we can study the Schrödinger picture, and define the unitary time evolution operator, as in regular quantum mechanics. By carefully examining the Schrödinger picture, we aim to provide a clearer understanding of QG, and defer the study of changes needed to unitarize the theory to later work. This thesis focuses in particular on massless scalar fields propagating on curved spacetimes of dimension $D \geq 4$. The simplicity of the scalar field is a useful test case, and we expect to be able to generalize to other, more complicated fields. First, Hawking radiation is studied for Schwarzschild black holes in the Schrödinger picture. Using the ADM decomposition, "nice slices" are introduced, which are smooth foliations of the spacetime that are regular across the horizon, rather than the more typical singular ones involving tortoise coordinates. The role of ultra high energy Hawking modes is discussed, and these are found to be a result of the choice of singular coordinates used near the horizon, rather than an indication of transplanckian physics occurring on the horizon scale. In addition, the constraint equations, which are the ADM decomposition of the Einstein equations, are expanded to second order in $\kappa = \sqrt{32\pi G}$ in an arbitrary background spacetime. Observable operators are "dressed" in their gravitational fields, in analogy with the description from quantum field theories, and the creation of such an operator makes an associated field which extends to infinity. A general form for the gravitational dressing is found to leading order using the expansion of the constraint equations.

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Chapter 1

Introduction

The full theory that unites gravity and quantum mechanics has been the subject of research and debate within the theoretical physics community for more than 50 years. Though thoroughly experimentally tested in separate regimes, strong gravitational forces operating on very short scales require novel physics to describe, the exact nature of which is still unsettled. The existence of Black Holes (BHs) in particular provides an area of study that leads to mutually exclusive predictions and paradoxes involving the two theories. Resolving the tensions surrounding BHs and their dynamical evolution presents not only a theoretical, but also an experimental problem, as these are astrophysical objects present in our universe.

Significant progress was made when BHs began to be viewed as thermodynamic objects, with temperature and entropy defined by the physical quantities associated with the BH: mass (or area), charge, and angular momentum. The notion of BH entropy was essential to their modern description; otherwise the usual laws of thermodynamics would be violated by objects falling beyond the event horizon [1]. This realization led to the development of the laws of BH thermodynamics, in analogy with the laws of thermodynamics of macroscopic systems introduced in the preceding century.

Hawking's famous contribution to the development of BH thermodynamics [2] was to associate a temperature with the surface gravity of a BH, which implied that BHs must radiate through the mechanism of pair production, in which a particle is produced on each side of the event horizon. One of these particles is constrained to fall into the BH, while the other may escape to infinity as thermal radiation. Because the role of energy and momentum are switched inside the horizon, the infalling Hawking partner particle with negative momentum *decreases* the overall mass of the BH.

In the language of Quantum Field Theory (QFT), this simple example of a quantum field propagating on a classical BH background leads to the information paradox. The above process of pair production can continue unabated until the BH completely disappears.[1] An observer stationed at infinity with a detector collecting the radiation which escapes will find that it is featureless, thermal graybody radiation,[2] and will contain no information about the BH interior. This leads to a major problem for quantum mechanics. When the BH completely evaporates, all the quantum information stored in the interior also vanishes, with no seeming way for an outside observer to reconstruct it. In technical terms, this leads to a pure state (representing the outside system and the BH) evolving into a mixed state, where the radiation in the exterior is entangled with an object that no longer exists. Any successful theory of quantum gravity will be able to sensibly relate the entropy of the black hole to a precise counting of the quantum mechanical microstates of the system, and it is our view that the correct theory will restore unitarity.[3]

The problems associated with unitarity and the construction of the theory of Quantum Gravity (QG) are deep and involve more than a single paradox. For example, it

[1]For a summary of other ideas on BH formation and end states, see section IV of [3].

[2]The BH does not radiate as a perfect black body, as the surrounding gravitational potential causes some Hawking particles outside the horizon to scatter back into the interior.

[3]For arguments against this view, see [3].

has been shown in the last century how to quantize fields on flat backgrounds, uniting special relativity and quantum mechanics. This lead to the discovery of quantum field theories and, in particular, Quantum Electrodynamics and Quantum Chromodynamics, and eventually the construction of the Standard Model of particle physics. However, in general relativity the background is not fixed and flat, but curved and dynamic, making it difficult to introduce quantization and incorporate time evolution of the system as a whole. Much work in the field, such as the preceding example of Hawking radiation shows, is done using the semiclassical description, where quantum fields are allowed to propagate on fixed classical backgrounds. This is a useful approximation to study in many cases, but in a quantum theory we expect spacetime itself to be quantized, not a continuous manifold. All of these challenges must be incorporated and explained in a full theory of QG.

Hawking radiation has been extensively studied in the Heisenberg picture since its inception [2]. The simplest case is of a massless scalar field propagating on a spacetime with a Schwarzschild BH [4], but can be extended to other types of fields, including those of charged particles [5], and also for spacetimes with rotating and/or charged BHs [6, 7], and with different spacetime dimensions and alternative boundary conditions [8, 9]. Although it has not been detected, Hawking radiation and BH evaporation has been predicted to exist in a wide variety of physical situations, including some relevant to our universe.

Unfortunately, it is expected to be extremely hard to experimentally verify the existence of Hawking radiation, since for large BHs the effect is predicted to be very small compared to the temperature of the cosmic microwave background radiation temperature of ~ 3 K. Indeed, for a BH of one solar mass, which is smaller than any astrophysical BH that has yet been detected, the radiation temperature is $T_H \sim 6 \cdot 10^{-8}$ K. Astrophysical BHs are relatively young, and not near the point of evaporation where the Hawking temperature would be high enough to observe experimentally. On the other hand, there have been numerous searches for primordial BHs which formed in the early universe and are near the point of evaporation. These BHs have not been directly detected so far, but it has been argued that there is some observational evidence for such objects [10].

Recently there has been much new progress in experimental gravitational physics. The Laser Interferomator Gravitational Wave Observatory (LIGO) has achieved the historic detection of gravitational waves [11]. This experiment studies the mergers of BHs and tracks the emission of gravitational waves as they inspiral, merge, and ringdown into a single BH. LIGO has so far confirmed the classical picture of gravitational waves, and has not discovered any quantum effects. Other gravitational wave detectors, which are space based interferomators, such as LISA [12] and DECIGO [13] will also begin operation in the next decade, and will study the mHz and dHz frequency ranges, respectively, while LIGO is most sensitive in the $\sim 10^2$ Hz range. Additionally, BHs have also been studied for the first time using imaging techniques when the Event Horizon Telescope (EHT) captured the first image of a BH [14], and has also taken an image of the super massive BH at the center of our galaxy [15]. The EHT has not identified any quantum corrections to the description of BHs from classical gravity, but these collaborations and future planned detectors will continue to operate and search for new physics in the coming years.

With the growing experimental community studying BHs and gravitational waves, it is crucial to develop a full understanding of the current predictions from QG, and explore its unitarization from the theoretical perspective. Perhaps the most notable development in the field of QG over recent years has been the Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondence conjecture [16]. This proposal states that there is a direct relationship between a CFT on the timelike boundary of an AdS space and the physics in the bulk of the spacetime. This correspondence is defined by the "dictionary" between correlators and other quantities calculated on the boundary and states in the bulk. In such dual theories, the unitary nature of the CFT indicates that the gravitational theory in the bulk is also unitary, including processes like BH evaporation. Similarly, there have been attempts to find holographic theories for other boundary conditions, although the lack of a timelike boundary in other spacetimes makes it difficult to describe the evolution of the CFT. Nevertheless, there have been proposals for both flat space [17] and dS space holography [18].

Another idea which will be suited to the work in this thesis is adding nonlocal modifications to the semiclassical description of BH evaporation [19]. These nonlocal interactions between the BH interior and the exterior transfer information outside of the horizon so information is conserved during the process of BH evaporation. These effects are expected to occur on a length scale outside of the BH of roughly the size of the horizon radius, rather than the Planck scale very near the horizon, where the underlying physics is poorly understood. Crucially, this does not modify the physics far from the BH, which has been thoroughly experimentally verified to match with the classical description of gravity. The mechanism to implement these departures from the semiclassical description is to add an interaction term to the Hamiltonian theory. Such modifications could potentially be seen by experiments such as LIGO and particularly EHT, as they occur on the scale of the size of the horizon radius and may distort the light rays around the BH.

As in the previous example, this thesis will examine the Hamiltonian evolution, and work in the Schrödinger picture to study quantum fields propagating on the spacetime. This is not as natural in general relativity as the covariant description provided by the Lagrangian approach, but allows one to use the familiar quantum evolution of systems in terms of the generator of time translations, at the cost of treating the space and time coordinates differently. We will make use of the ADM decomposition [20],[4] where each time slice is a spacelike hypersurface with an associated induced metric. The choice of

[4]For a useful pedagogical introduction to the ADM decomposition, see [21].

the time coordinate and the resulting embedding is highly nonunique; different choices amount to a change of coordinates.

The above procedure is well defined in stationary spacetimes, but there are subtleties when it comes to the Schrödinger picture for time-dependent cases, or indeed, even flat space when reparameterized in certain coordinates in greater than two spacetime dimensions [22, 23, 24]. The Schrödinger picture unitary operator, which is dependent on the Hamiltonian, is not always well defined, particularly for time-dependent spacetimes. There are many calculations that do not produce the result from the Heisenberg picture, and predict contradictory outcomes, such as infinite particle production. This hinders any discussion of BH formation and evaporation in the Schrödinger picture, but also more generally, any cosmological or non-stationary spacetime.

This work will focus mostly on fixed backgrounds and neglect the effects of back reaction for BH evaporation or dynamical spacetimes. In the Schrödinger picture, Hawking radiation has been studied most extensively in two dimensions. The lower dimensional spacetime simplifies the equations significantly, but can also lead to insights in higher dimensional settings, since the spacetimes under study are spherically symmetric. One troubling aspect of the original derivation of Hawking radiation is the role of ultra-high energy modes, which are produced near the horizon and depend on the singular coordinates traditionally used there. Coordinates that are regular both for incoming and outgoing modes can be constructed which replace this singular description [25]. In the regular basis, the dominant contribution to Hawking radiation occurs at wave number $k \sim \mathcal{O}(1)$, while the transplanckian modes are shown to be an an artifact of the coordinates rather than essential to the description of the Hawking process. Additionally, Hawking radiation is shown to be produced in an atmosphere extending around the BH approximately the size of the horizon radius, which further supports the introduction of nonlocal interactions on this scale as in [19].

The study of the Hawking process most closely focuses on the quantum evolution of fields propagating on the spacetime, but on the other hand, a physical spacetime should eventually be described as part of the quantum system. One approach to begin to account for this is to perturbatively expand the spacetime around a classical background. There have been several attempts using the ADM decomposition in different cases, including asymptotically flat [26], AdS [27], and for spacetimes with black holes [28]. These works seek to expand the constraint equations, that is, the ADM decomposition of the Einstein equation, which relates the curvature of the spacetime to the matter fields. Enforcing the constraints is equivalent to requiring that the Einstein equation be satisfied.

The gravitational interactions for matter fields propagating on a curved spacetime have been studied in particular through the technique of gravitational dressing. Developed in the context of QFT, the operators associated with free particles in the local QFT are "dressed" in their gravitational (or electromagnetic) fields, which are the mechanisms for introducing interactions [29]. For gravitational particles, the fields are diffeomorphism (gauge) invariant, so these dressed operators dependent on the fields must commute with the constraints. Creating a dressed particle necessarily constructs a gravitational (electromagnetic) field which extends to infinity, and is inherently nonlocal.

In the electromagnetic case, the infinite dressing field does not contradict the local structure of the quantum theory, as factorized Hilbert spaces can still be defined [30]. Charged particles in a region can be screened by charges of the opposite sign. Thus, dressed operators will commute at spacelike separation and an algebra of observables which respects locality can be constructed. The situation is more complicated for gravitationally dressed fields. While the gauge symmetries of, e.g., QED, are independent of the spacetime coordinates, the gauge symmetries of gravity are not, which precludes this type of screening and the associated quantum mechanical structure. For gravitationally dressed operators, the algebraic structure differs from the approximate structure expected from the local QFT limit, which matches with experimental observation, due to the nonlocal effects of the dressing fields.

This result can be summarized in the "Dressing Theorem", proved both for asymptotically flat and asymptotically AdS spaces [30, 31]. In the expansion of gauge invariant operators around small $\kappa = \sqrt{32\pi G}$, the nonlocal effects from the gravitational dressing necessarily arise at first order. The nonlocal effects deviate significantly from the results local operators, which has profound implications for the way information is discussed and the BH information problem. If information is not localized and quantum subsystems cannot be defined, information inside a BH is not screened by the event horizon for gravitationally interacting particles. In other words, a particle inside a BH has a gravitational dressing which extends to infinity at first order in κ, so an observer at infinity may be able to collect non-trivial information about the particle even while it is behind the BH horizon.

The above would be a major violation of the usual structure of local operators in QFT. However, locality can be restored by introducing "gravitational splitting", which is an approximate notion of a subsystem [32]. Such dressings are insensitive to the precise details of the state, and have been constructed to leading order in κ [33]. For an appropriate dressing, an observer at spacelike separation would be able to discern the total gravitational charge in a region, but not information about the configuration or state of the particles, which restores the notion of locality which agrees with experiment.

The main developments of this thesis are split into two parts. First studied is the Schrödinger picture of quantum evolution of Hawking radiation, and secondly, gravitationally dressed fields and gauge invariant observables. Both are done in $D \geq 4$ spacetime dimensions for the simple case massless scalar fields, but are expected to generalize. The Schrödinger picture for Hawking radiation will closely follow the work of [25], which is the $D = 2$ case. The ADM decomposition is used, and smooth embeddings, called "nice slices," which are regular across the BH horizon are the foliations introduced to study the system. The case of energy eigenmodes for the scalar field, and also arbitrary regular coordinates, are both considered. In each case, the Hamiltonian governing the evolution of the quantum system is calculated, and for regular modes, it is shown that transplanckian excitations do not play a significant role in the emission of Hawking radiation. Finally, the results are generalized to BHs in asymptotically AdS spacetimes.

The constraint equations are also expanded to second order in κ for a massless scalar field propagating on an arbitrary spacetime. Enforcing the constraints results in a Hamiltonian which depends only on the boundary term. A novel quantization is introduced, called "gauge-invariant canonical quantization," which creates non-trivial physical states built from the vacuum which is annihilated by half the constraints. Gravitational observables are taken to be the observables defined in local QFT which are gravitationally dressed. Such observables are gauge-invariant operators, and thus these gravitationally dressed operators must commute with the constraints. The dressing functions are found order by order in κ in terms of highly nonunique Green's functions by evaluating the commutators with the expansion of the constraint equations.

Chapter two of this thesis will examine the Schrödinger picture for a massless scalar field propagating on a Schwarzschild BH background and Hawking radiation. Chapter three deals with the constraints and gravitational dressing for a massless scalar field in an arbitrary background spacetime. Finally, several appendices follow outlining the use of confluent Heun functions in regular coordinates, Kruskal coordinates, the QED case of gauge-invariant canonical quantization, and gauge transformations and diffeomorphisms.

1.1 Permissions and Attributions

1. The content of chapter 2, Appendix A, and Appendix B is the result of a collaboration with S. Giddings, and has previously appeared in Physical Review D [34]. Additionally, the content of chapter 3, Appendix C and Appendix D is the result of a collaboration with S. Giddings, and has previously appeared in Physical Review D [35]. These works are licensed under the Creative Commons Attribution 4.0 International license (CC BY 4.0), `http://creativecommons.org/licenses/by/4.0/`.

Chapter 2

Quantum evolution of the Hawking state for black holes

2.1 Introduction

Hawking radiation [2] is apparently one of the most mysterious phenomena in the world of physics. While it appears to follow straightforwardly from the basic principles of local quantum field theory (LQFT) extended to curved spacetime backgrounds, its ultimate implications include an apparent internal contradiction among the basic principles of physics. This "black hole information problem," or perhaps more aptly, "unitarity crisis," seems to point to the necessity of revising fundamental principles, in connection with understanding the foundations of quantum gravity.

While black holes (BHs) thus may play a key role in understanding these principles, the phenomenon of particle production in a nontrivial gravitational background is also important because of its role in the very early Universe, and in particular during a possible phase of inflationary expansion, which is believed to have created the fluctuations that lead to the large-scale structure in the visible matter distribution in the cosmos. The seed fluctuations can be derived by methods closely parallel to those used to describe production of Hawking radiation.

In investigating the unitarity crisis, a lot of recent thought has focussed on the view that a more basic understanding of quantum information and its evolution in quantum gravity is important. In particular, an important question is whether it is possible to think of a black hole and its environment as quantum subsystems, at least to a good approximation, of a larger quantum system, which evolve together in time.[1] This does appear to be the correct leading order picture, with possible small modifications that ultimately lead to a description consistent with unitarity.

The original derivation [2] of Hawking radiation was based on an asymptotic description, analogous to that of the S-matrix: it analyzed the asymptotic state of the radiation, but didn't directly describe the time evolution of the quantum state of the BH and its surroundings. Thus, its connection to a description of an evolving quantum system is not direct. Rederivations of the Hawking effect have largely followed in this vein, though various approaches have provided some additional information about the evolving quantum state.

A goal of this paper is to give a direct and more complete treatment of the evolving quantum state of a BH and its surroundings. We will focus on this in the approximation where it is described within LQFT, and thus will not attempt to describe the more complete dynamics that is ultimately believed to be unitary. However, this approximate evolution plays an important background role in describing this unitary dynamics, if the latter is a correction to this evolution that is in certain regimes a small correction. Another role for the present description is in treating Hawking radiation for interacting theories; treatment of interactions in Hawking's original derivation is problematic due to its reliance on evolution via free mode propagation.

[1] For further discussion of this question, see [36].

Specifically, we will describe the time-dependent evolution in a picture analogous to standard Schrödinger picture, by making a choice of time slices that is regular across the horizon, and deriving the resulting evolution of the quantum wavefunction. This was previously done for two-dimensional black holes in [25, 37]; earlier work on dynamical evolution on such slices includes [38, 39, 40, 41]. Description of this evolution also relies on choosing coordinates on the time slices, and a basis of modes.

In outline, we begin in the next section by paremeterizing such slicings, and describing the corresponding Arnowitt-Deser-Misner (ADM) parameterization of the metric. The following section then derives the hamiltonian for scalar matter in such a slicing, and the canonical quantization of the theory. A choice of modes leads to a Fock construction of the Hilbert space, and construction of the evolution operator acting on it. In fact, there are many such descriptions of the evolving quantum state, which depend on the specific choice of slices, coordinates, and mode basis; these are analogous to different "pictures" of the evolution (and are expected to be equivalent).

In a free theory, this evolution can be simplified by in particular using energy eigenstates for the mode basis. Section four describes such modes (and Appendix A finds an explicit form of them in $D = 4$ spacetime dimensions in terms of Heun functions) and outlines their important properties. While the evolution is simplified in such a basis, the basis is *singular*. This connects to Hawking's and related derivations, where modes are traced back to ultraplanckian wavelengths near the horizon. The resulting transplanckian behavior has served as a source of concern and confusion in the literature, but from this viewpoint just arises from choice of a singular basis to describe the state.

These issues may be avoided, as in the next section, by instead working with a regular basis. This does lead to a more complicated description, but one that in principle exhibits evolution without any explicit reference to transplanckian excitations. This section in particular finds the form of the resulting hamiltonian.

Section six then puts the previous treatments together to give a description of the evolution of the quantum state, which we call the Hawking state, resulting from a collapsing BH. Again, this can be done in terms of a regular mode basis, but at the price of a more complicated evolution law. We can, however, learn about its structure by comparing it to the singular and simpler description in terms of energy eigenmodes. In particular, one can see the familiar behavior of asymptotic Hawking excitations, as well as of the internal partner excitations and pairing between inside and outside excitations. We also discuss the internal evolution on "nice slices," and exhibit a "frozen" description of the internal state, in a picture that results from particular choices of internal coordinates.

The final section outlines the extension of these results to interacting theories, and to situations with different asymptotic metrics besides that of Minkowski. It in particular discusses the case of anti de Sitter space. Here, the same quantization procedure is argued to yield a hamiltonian and evolution that furnishes, in AdS/CFT language, a leading order large-N description of quantum evolution of a BH, as well as $1/N$ corrections arising from interactions. Here, again, this is not expected to yield evolution that is ultimately unitary, connecting to the question of the form of additional corrections needed to unitarize the dynamics.

2.2 Geometry and time slicings

We begin by describing the geometry of a BH, and time slices of that geometry.

2.2.1 Schwarzschild parameterizations

The standard form of the Schwarzschild metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{2.1}$$

where for $D > 3$ dimensions

$$f(r) = 1 - \left(\frac{R}{r}\right)^{D-3} \ ; \tag{2.2}$$

here R is the horizon radius. Both for describing field propagation and for giving a smoother description of the geometry, it is useful to introduce conformal coordinates for the t, r plane, by defining $r_*(r) = \int dr/f(r)$ so that

$$ds^2 = f(r)(-dt^2 + dr_*^2) + r^2 d\Omega^2. \tag{2.3}$$

For example in $D = 4$,

$$r_* = \int \frac{dr}{1 - R/r} = r - R + R\ln\left(\frac{r}{R} - 1\right) \ , \tag{2.4}$$

up to an overall additive constant. Then, for the exterior of the horizon, we can define left/right moving Eddington–Finkelstein coordinates

$$x^\pm = t \pm r_* \ . \tag{2.5}$$

While the Schwarzschild coordinates (2.1) and the definition of r_* are clearly singular at the horizon, the latter definition can be extended to $r < R$; for example in $D = 4$

$$r_* = r - R + R\ln\left(1 - \frac{r}{R}\right) \pm \pi i R \ . \tag{2.6}$$

The sign reversal of f in (2.3) indicates that r_* plays the role of a time coordinate for $r < R$, and left/right moving coordinates for the interior are

$$\hat{x}^\pm = r_* \pm t \ . \tag{2.7}$$

The metric (2.1) is of course smooth across the horizon, and this can for example be exhibited by working in the incoming Eddington–Finkelstein coordinates, (x^+, r). This gives

$$ds^2 = -f(r)dx^{+2} + 2dx^+dr + r^2d\Omega^2 \tag{2.8}$$

which smoothly covers the region $r > 0$. The time translation invariance is also inherited by this form of the metric, and becomes invariance under

$$x^+ \to x^+ + \text{constant} \ . \tag{2.9}$$

This invariance plays an important role in the dynamics.

2.2.2 Slicings and ADM description

In order to describe dynamical evolution, we can provide a foliation of the geometry (2.1), (2.8) by time slices. Such a foliation can in general be parameterized as

$$x^\mu = \mathcal{X}^\mu(t, x^i) \tag{2.10}$$

where now t labels slices of the foliation, and x^i is a spatial coordinate. In a Schwarzschild background, the description is simplest for a foliation respecting the spherical symmetry, so that

$$x^+ = \mathcal{X}^+(t, x) \quad , \quad r = \mathcal{X}^r(t, x) \ , \tag{2.11}$$

independent of angles, with general radial coordinate x, and using the standard angular coordinates. We also can anticipate some simplifications for slicings that respect the translation symmetry (2.9), and these take the general form

$$x^+ = t + s(x) \quad , \quad r = r(x) \ , \tag{2.12}$$

which we refer to as a "stationary slicing" [19, 25]. Their specification particularly simplifies if we use r as the radial coordinate,

$$x^+ = t + S(r) \ . \tag{2.13}$$

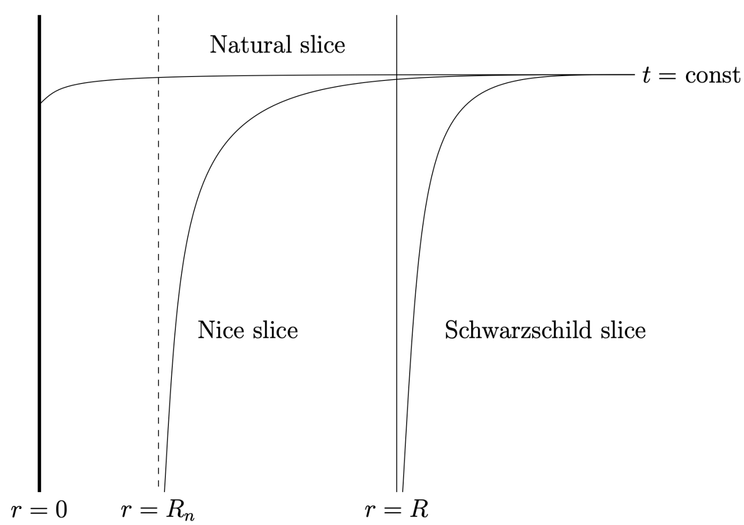

Figure 2.1: Shown in an Eddington-Finkelstein diagram are different kinds of slices. In addition to the familiar Schwarzschild slices, there are nice slices, asymptoting to a constant $r = R_n$, and natural slices which reach $r = 0$. These slices all asymptote to constant Schwarzschild time slices at $r = \infty$. The full family of slices of the geometry is found by translating one of these slices vertically in the figure, corresponding to a time translation in Schwarzschild time.

This family of slicings unifies various descriptions of the Schwarzschild spacetime, and the form of the "slice function" $S(r)$ plays a key role. For example, from the coordinate definition (2.5), we see that $S(r) = r_*(r)$ gives the Schwarzschild time slicing, relevant for observers who stay outside the horizon. Slicings that cross the horizon – for example describing observations of a family of observers, some of whom enter the BH – arise from slice functions that are smooth at the horizon. Freely falling such observers will reach $r = 0$, and that is naturally described by a family of "natural slices" for which $S(r)$ is finite there. A very simple example is $S(r) = r$, corresponding to "straight slices," which lead to some simplifications. However, such natural slices do not give good Cauchy slices, since they cease to describe excitations that have reached $r = 0$, in the absence of a supplementary description there. Cauchy slices can, however, be specified by using an $S(r)$ that asymptotes to minus infinity at some finite $r = R_n < R$. This gives an example of the construction of "nice slices," such as were described by [42, 43]. These different kinds of slices are shown in Fig. 2.1.

Evolution over a general slicing is conveniently described by using ADM variables [20] for the metric,

$$ds^2 = -N^2 dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \ . \tag{2.14}$$

The lapse N, shift N^i, and spatial metric q_{ij} are dependent on the spatial coordinates, and we also define $N_i = q_{ij}N^j$. For the Schwarzschild metric in a stationary slicing described by (2.13), these functions become [19]

$$N^2 = \frac{1}{S'(2 - fS')}, \qquad N_r = 1 - fS', \qquad q_{rr} = S'(2 - fS') \ , \tag{2.15}$$

with $S' = dS/dr$, and with the remaining components q_{ij} of standard angular form. This is also readily generalized for a more general stationary radial coordinate, (2.12). It is also useful to have the unit normal to the time slices, which takes the general form

$$n^\mu = (1, -N^i)/N. \tag{2.16}$$

2.3 Schrödinger Description

2.3.1 Canonical quantization

Our goal is to describe the evolution of the quantum state of a BH. Of course, emitted Hawking radiation changes the mass of the BH, making the geometry non-stationary. However, the average time to emit a Hawking quantum of energy $\sim 1/R$ is R. This means that the fractional change in the mass over the characteristic emission time is $RdM/Mdt \sim 1/RM \sim 1/S_{BH}$, where S_{BH} is the Bekenstein-Hawking entropy. This small parameter justifies using the stationary approximation over times $\ll RS_{BH}$ for large BHs. We will focus on this approximation and describe evolution of quantum fields on the stationary BH background, leaving treatment of quantum backreaction for future work. For simplicity, we consider evolution of a free massless scalar field ϕ in a $D \geq 4$ dimensional Schwarzschild background. Working in ADM variables, with a general slicing, the action takes the form

$$S = -\frac{1}{2}\int d^D x\sqrt{|g|}(\nabla\phi)^2 = \frac{1}{2}\int dt d^{D-1}x\sqrt{q}N\left[(\partial_n\phi)^2 - q^{ij}\partial_i\phi\partial_j\phi\right] \ , \tag{2.17}$$

where we have defined the normal derivative $\partial_n\phi = n^\mu\partial_\mu\phi$. The canonical momentum is then defined by

$$\pi = \frac{1}{\sqrt{q}}\frac{\delta S}{\delta\dot{\phi}} = \frac{1}{N}\left(\partial_t\phi - N^i\partial_i\phi\right) = \partial_n\phi \ . \tag{2.18}$$

Using this, we can write the canonical form of the action

$$S = \int dt d^{D-1}x\sqrt{q}\left(\pi\dot{\phi} - \mathcal{H}\right) \ , \tag{2.19}$$

where the hamiltonian is

$$H = \int d^{D-1}x\sqrt{q}\mathcal{H} = \int d^{D-1}x\sqrt{q}\left[\frac{N}{2}\left(\pi^2 + q^{ij}\partial_i\phi\partial_j\phi\right) + \pi N^i\partial_i\phi\right] . \tag{2.20}$$

Quantization proceeds via the equal time canonical commutation relations

$$[\pi(x^i,t),\phi(x^{i\prime},t)] = -i\frac{\delta^{D-1}(x-x')}{\sqrt{q}} . \tag{2.21}$$

Note that the hamiltonian (2.20) depends on the choices of both foliation and spatial coordinate in the general expression (2.10), through the dependence of N and N^i on these choices. This results in what can effectively be described as different "pictures" for the evolution, generalizing the choice of Heisenberg or Schrödinger picture, as also discussed in [37]. In a given such Schrödinger picture, we take the field and momentum operators to be time independent, and all evolution to be in the state.[2]

Description of the evolution also depends the choice of a mode basis, given by specifying a complete set of pairs of functions $\gamma_I(x^i) = (\phi_I(x^i), \pi_I(x^i))$. Such a pair gives Cauchy data for a solution $\phi_I(x^i,t)$. For quantization, one also specifies a choice of complex structure [47, 48, 24] that separates these into "positive frequency" modes $\gamma_A(x^i) = (\phi_A(x^i), \pi_A(x^i))$ and conjugate "negative frequency" modes $\gamma_A^*(x^i)$. The inner product of two such sets of Cauchy data is

$$(\gamma_1,\gamma_2) = i\int d^{D-1}x\sqrt{q}(\phi_1^*\pi_2 - \pi_1^*\phi_2) , \tag{2.22}$$

[2]In a time-dependent background, there are further subtleties with Schrödinger picture discussed in [23, 44, 45, 24, 46]. The present work avoids these with the time-independent background and slicing.

and extends to an inner product between solutions,

$$(\phi_1, \phi_2) = i \int d^{D-1}x \sqrt{q} n^\mu \phi_1^* \overleftrightarrow{\partial_\mu} \phi_2 \ , \tag{2.23}$$

which is conserved by the equations of motion. Different such choices of modes also lead to different pictures.

The Schrödinger picture field operators, in a given such picture, can be expanded as

$$\phi(x^i) = \sum_A \left[a_A \phi_A(x^i) + a_A^\dagger \phi_A^*(x^i) \right] \quad , \quad \pi(x^i) = \sum_A \left[a_A \pi_A(x^i) + a_A^\dagger \pi_A^*(x^i) \right] \ . \tag{2.24}$$

If the mode basis is orthonormal,

$$(\gamma_A, \gamma_B) = \delta_{AB} \quad , \quad (\gamma_A, \gamma_B^*) = 0 \ , \tag{2.25}$$

the canonical commutators imply the commutation relations

$$[a_A, a_B^\dagger] = \delta_{AB} \quad , \quad [a_A, a_B] = [a_A^\dagger, a_B^\dagger] = 0 \ . \tag{2.26}$$

A Fock space basis for the Hilbert space then arises by acting with the creation operators $a_A^\dagger$ on the vacuum $|0\rangle$ annihilated by the a_A.

Schrödinger picture evolution is then described by the action of the time evolution operator,

$$U(t_2, t_1) = \exp\left[-i \int_{t_1}^{t_2} H dt \right] \ , \tag{2.27}$$

determined by the hamiltonian (2.20), on the state. And, for example, if initially the state is the vacuum state $|0\rangle$ (in a particular basis), it will not necessarily remain in that state, since the hamiltonian in general creates additional excitations.

2.3.2 Hamiltonian and pictures

The hamiltonian (2.20) can be written in other forms which are useful in describing evolution. We begin by using the explicit definition of the momentum to obtain from (2.20)

$$H = \int d^{D-1}x\sqrt{q}\left[\frac{1}{2N}(\partial_t\phi)^2 + \frac{N}{2}g^{ij}\partial_i\phi\partial_j\phi\right] . \tag{2.28}$$

We can rewrite this in terms of a vector $\xi = \partial_t$, which connects points on neighboring slices with equal spatial coordinates. In component form, this becomes from (2.10)

$$\xi^\mu = \frac{\partial\mathcal{X}^\mu}{\partial t}\bigg|_{x^i} , \tag{2.29}$$

at fixed spatial coordinate x^i. Then, using the stress tensor for the minimally coupled scalar field, the Hamiltonian becomes

$$H_\xi = \int d^{D-1}x\sqrt{q}n^\mu\xi^\nu T_{\mu\nu} , \tag{2.30}$$

where n^μ is the unit normal (2.16).

This form of the hamiltonian also exhibits the dependence both on the slicing and on the choice of spatial coordinate x^i along the slices; for example a redefinition $x^{i\prime}(x^j,t)$ changes ξ^μ and thus H_ξ. The different choices of ξ define different Hamiltonians and Schrödinger pictures, which lead to distinct descriptions of the evolution. In particular, the Hamiltonian H_ξ is conserved when ξ is a Killing vector, since the stress tensor is also conserved. Such a Killing vector is present after the matter has collapsed to form a BH. The expression (2.30) can alternately be derived in the covariant canonical formalism; see *e.g.* appendix B of [30] for a review.

Another useful expression can be found by putting the hamiltonian (2.28) in a form which resembles the conserved inner product (2.23). We will connect to the inner product by introducing the canonical momentum $\pi = \partial_n \phi$ into (2.28). Replacing one of the time derivatives using (2.18), the Hamiltonian becomes

$$H = \int d^{D-1}x\sqrt{q}\left(\frac{1}{2}\partial_t\phi\partial_n\phi + \frac{1}{2}\partial_t\phi\frac{N^i}{N}\partial_i\phi + \frac{N}{2}g^{ij}\partial_i\phi\partial_j\phi\right). \tag{2.31}$$

Integrating the last two terms by parts with respect to x^i, and neglecting the boundary term, the expression becomes

$$H = \int d^{D-1}x\sqrt{q}\left[\frac{1}{2}\partial_t\phi\partial_n\phi - \frac{\phi}{2\sqrt{q}}\partial_i\left(\sqrt{q}Ng^{i\mu}\partial_\mu\phi\right)\right]. \tag{2.32}$$

Finally, using the equation of motion to rewrite the second term,[3] the Hamiltonian takes the simplified form

$$H = \frac{1}{2}\int d^{D-1}x\sqrt{q}\left(\partial_t\phi\partial_n\phi - \phi\partial_t\partial_n\phi - \partial_n\phi\phi\frac{\partial_t q}{2q}\right). \tag{2.33}$$

Comparing the above equation to (2.23), we see that for time-independent metric coefficients it reduces to the inner product $i(\phi^*, \xi^\mu\partial_\mu\phi)/2$. This will be useful in the following analysis, particularly when considering special choices of modes.

2.4 Energy eigenmodes and their evolution

The time-translation symmetry (2.9) suggests expanding in a basis of modes that correspond to eigenstates of the time translation generator, which is ∂_{x^+}, or ∂_t in the

[3] At the quantum level this is allowed since the field operators are expected to satisfy the equation of motion.

slice coordinates. We begin by separating off the angular coordinates using[4]

$$\phi(x) \sim u_l(x^+, r)\frac{Y_{lm}(\Omega)}{r^{D/2-1}} \ . \tag{2.34}$$

Then $u_l(x^+, r)$ obeys the equation

$$\partial_r \left(2\partial_+ u + f\partial_r u\right) - V_l(r)u = 0 \tag{2.35}$$

with potential

$$V_l(r) = \left(\frac{D}{2} - 1\right)^2 \frac{R^{D-3}}{r^{D-1}} + \frac{l(l+D-3) + (D-2)(D-4)/4}{r^2} \ ; \tag{2.36}$$

e.g. in the case $D = 4$,

$$V_l(r) = \frac{R}{r^3} + \frac{l(l+1)}{r^2} \ . \tag{2.37}$$

Eigenfunctions of the time translation symmetry then take the form

$$e^{-i\omega x^+} u_{\omega l}(r)\frac{Y_{lm}}{r^{D/2-1}} \ , \tag{2.38}$$

or in the t, r coordinates arising from a stationary slicing (2.13)

$$e^{-i\omega t} U_{\omega l}(r)\frac{Y_{lm}}{r^{D/2-1}} \quad , \quad U_{\omega l}(r) = e^{-i\omega S(r)} u_{\omega l}(r) \ . \tag{2.39}$$

Again, the special case of a Schwarzschild slicing corresponds to $S(r) = r_*(r)$, and the equation (2.35) can be simplified by defining

$$g_{\omega l}(r) = e^{-i\omega r_*} u_{\omega l}(r) \tag{2.40}$$

[4]For spacetime dimension $D > 4$, the spherical harmonics involve multiple angular quantum numbers m_i; for notational simplicity, we use the four-dimensional notation Y_{lm} in the following discussion.

and becomes

$$\frac{d^2 g_{\omega l}}{dr_*^2} + \left[\omega^2 - f(r)V_l(r)\right] g_{\omega l} = 0 \ . \tag{2.41}$$

At large r, or near the horizon, the effective potential fV_l vanishes, and we have

$$g_{\omega l} \sim \exp\{\pm i\omega r_*\} \ . \tag{2.42}$$

Solving for $g_{\omega l}$ becomes a well-known barrier penetration problem. Inside the horizon, we also find the behavior (2.42) near the horizon, $r_* \to -\infty$. The general internal solutions are difficult to find, but one indicator of their behavior is their WKB approximation, which has the form

$$g_{\omega l} \sim e^{\pm i \int dr_* \sqrt{\omega^2 - fV_l}} \ . \tag{2.43}$$

This approximation in particular fails as r approaches zero ($r_* \to -R$), but does illustrate the rapidly varying nature of the solutions.

The behavior of energy eigenmodes can be further understood by examining the differential equation (2.35). The equation has regular singular points at 0 and $Re^{2\pi in/(D-3)}$ for integers $n = 0, 1, ..., D-4$, and an irregular singular point at infinity (see Appendix A). The solutions are not known in general, however, in $D = 4$ (2.35) can be transformed into the confluent Heun equation. In this case of $D = 4$ Schwarzschild BHs the solutions to the Heun equation are well known and have been widely studied in the literature, and incoming and outgoing modes are classified by their behavior near the singular points. We will not need the detailed behavior of the solutions to (2.35) in order to see that they define a basis, but their asymptotic behaviors will be important.

One can describe quantization using a basis of such solutions. One way to characterize solutions is in terms of their behavior at $t \to -\infty$. There are incoming solutions from $r = \infty$, but also can be (singular) solutions that asymptote to $r = R$ (or $r_* = -\infty$) in the far past. Specifically, we can introduce the following basis:

- $\tilde{u}_{\omega l}$: these "in" modes are modes such that the coefficient of $e^{i\omega r_*}$ vanishes near the horizon, *i.e.* $\tilde{g}_{\omega l} \sim e^{-i\omega r_*}$. Eq. (2.40) then shows that these modes are non-singular at the horizon; they have purely ingoing behavior there, $\phi \sim e^{-i\omega x^+}$. At $r_* = r = \infty$ they have both an ingoing piece, which may be normalized to unity, and a reflected outgoing piece. The internal part of the solution also is purely ingoing at the horizon, but takes a more general form for finite r_*.

- $u_{\omega l}$: these "up" modes are modes that in $g_{\omega l}$ have nonvanishing coefficient of $e^{i\omega r_*}$, taken to be unity, as $r \to R_+$, but vanishing coefficient as $r \to R_-$, and with purely outgoing wave $e^{i\omega r_*}$ at $r_* \to \infty$. These both give behavior $\phi \sim e^{-i\omega x^-}$. These also have a reflected $e^{-i\omega r_*}$ piece at the horizon, which continues to the interior similarly to the previous case.

- $\hat{u}^*_{\omega l}$: these "inside" modes are modes that in $\hat{g}_{\omega l}$ have nonvanishing coefficient of $e^{i\omega r_*}$, taken to be unity, as $r \to R_-$, and which vanish outside the horizon. Thus near the horizon $\phi \sim e^{i\omega \hat{x}^-}$; for finite internal r_*, they have general behavior.

In the exterior region $r > R$, the "in" and "up" modes correspond to those of [49], also discussed in [50], but we have also described the interior continuations of these solutions, which have an ingoing part. We have also introduced the "inside" modes. Of course, the "up" and "inside" modes are singular at the horizon, due to the singularity at $r = R$ in the definition of r_*. The corresponding full solutions are denoted

$$\phi_{\omega lm}(x^i, t) = e^{-i\omega x^+} u_{\omega l}(r) \frac{Y_{lm}}{r^{D/2-1}} \ , \tag{2.44}$$

and likewise for $\tilde{\phi}_{\omega lm}$, $\hat{\phi}^*_{\omega lm}$.

These solutions, written in terms of $g_{\omega l}$ via (2.40), are of course orthogonal under the conserved inner product (2.23) unless l, m match, in which case the product becomes

$$(\phi_1, \phi_2) = \int \frac{dr}{f} \left[(\omega_1 + \omega_2) g^*_{\omega_1 l} g_{\omega_2 l} - if(1 - fS') g^*_{\omega_1 l} \overleftrightarrow{\partial_r} g_{\omega_2 l} \right] e^{i(\omega_1 - \omega_2)[t + S(r) - r_*]} \ . \quad (2.45)$$

One can see that the sets of modes $\phi_{\omega lm}$, $\hat{\phi}_{\omega lm}$, $\tilde{\phi}_{\omega lm}$ are mutually orthogonal by considering a general localized wavepacket of such modes. In the far past, $t \to -\infty$, this wavepacket will localize near the horizon for $\phi_{\omega lm}$, $\hat{\phi}_{\omega lm}$, and near infinity for $\tilde{\phi}_{\omega lm}$, and so the inner product vanishes. The modes in a given set are orthogonal for $\omega_1 \neq \omega_2$, since otherwise (2.45) would contradict the time-independence of (ϕ_1, ϕ_2). One also finds by examining their near-horizon behavior that the modes $\hat{\phi}^*_{\omega lm}$ are *negative* norm, and so their conjugates $\hat{\phi}_{\omega lm}$ are positive norm solutions.[5] Our normalization convention is $(\phi_{\omega lm}, \phi_{\omega' l' m'}) = 4\pi\omega\delta(\omega - \omega')\delta_{ll'}\delta_{mm'}$, and similarly for $\hat{\phi}$, $\tilde{\phi}$.

The expansion of the field in terms of the modes inside and outside of the horizon takes the form

$$\phi(x^i, t) = \sum_{lm} \int_0^\infty \frac{d\omega}{4\pi\omega} \left(b_{\omega lm}\phi_{\omega lm} + \tilde{b}_{\omega lm}\tilde{\phi}_{\omega lm} + \hat{b}_{\omega lm}\hat{\phi}_{\omega lm} + h.c. \right) \quad (2.46)$$

and may be, for example, evaluated at $t = 0$, in a given slicing, to give the Schrödinger picture operator (2.24). This expansion also may be expressed compactly as

$$\phi = \sum_A b_A \phi_A + h.c. \quad (2.47)$$

where the integral over frequencies has been included in the general sum over modes labeled by A. Note that for the purely internal $\hat{\phi}_{\omega lm}(r)$ modes, the frequencies have the opposite sign, in accord with the above definitions.

[5] We also take $m \to -m$ in our definition, so that $\hat{\phi}_{\omega lm} \propto Y_{lm}$.

As anticipated, the hamiltonian greatly simplifies in this basis. From (2.33) we found for a stationary slicing

$$H = \frac{1}{2}\int d^{D-1}x\sqrt{q}n^{\mu}[\partial_t\phi\partial_\mu\phi - \phi\partial_t\partial_\mu\phi] = \frac{1}{2}(\phi^*, i\partial_t\phi) \ . \tag{2.48}$$

Then using

$$\partial_t\phi(x) = -i\sum_A \omega_A b_A \phi_A + h.c. \tag{2.49}$$

and the orthogonality between modes, we find

$$H = \sum_A \omega_A b_A^\dagger b_A(\phi_A, \phi_A) + H_0 \tag{2.50}$$

where H_0 is the normal ordering constant

$$H_0 = \sum_A \frac{\omega_A}{2}[b_A, b_A^\dagger](\phi_A, \phi_A) \ . \tag{2.51}$$

Returning to a more explicit labeling of the mode sum, the Hamiltonian becomes

$$H = \sum_{lm}\int \frac{d\omega}{4\pi\omega}\omega(b^\dagger_{\omega lm}b_{\omega lm} - \hat{b}^\dagger_{\omega lm}\hat{b}_{\omega lm} + \tilde{b}^\dagger_{\omega lm}\tilde{b}_{\omega lm}) + H_0 \ . \tag{2.52}$$

This has the same form as the $D = 2$ case discussed in [25, 37], including both chiralities of the modes, as a result of the spherical symmetry of the spacetime. One clearly sees that the "inside" modes have negative energies for this hamiltonian.

Of course the simplicity of the hamiltonian (2.52) is somewhat illusory, since the specification of a good initial state, *e.g.* with a regularity condition at the horizon, is rather more complicated in this basis, as was clearly illustrated in the 2d case in [25, 37]. An alternate way to describe such a regular state is to work directly in terms of modes that are regular at the horizon, to which we now turn.

2.5 Regular modes and their evolution

To give a treatment of evolution respecting regularity at the horizon, it is most natural to consider a mode basis that is regular there. If we consider a general stationary slicing (2.12), a mode basis may be specified by giving pairs of functions $(\phi_A(x^i), \pi_A(x^i))$, with A a basis label, on those slices. These also provide Cauchy data for a corresponding solution that evolves forward from a given slice.

2.5.1 Properties of modes

We have found that the "in" energy eigenmodes are regular at the horizon, and so can be used to provide a regular basis, but the "up" and "inside" modes are singular there. However, as two-dimensional examples illustrate [25, 37], we should be able to also find a mode basis that is regular at the horizon by combining these latter two.

Specifically, working with initial data on a slice which may be chosen to be at $t = 0$, the space $\mathcal{H}_{in} = \mathrm{Span}\{(\tilde{\phi}_{\omega lm}(x^i, 0), \tilde{\pi}_{\omega lm}(x^i, 0))\}$ describes "in" modes. This is orthogonal to the spaces

$$\mathcal{H}_{up} = \mathrm{Span}\{(\phi_{\omega lm}(x^i, 0), \pi_{\omega lm}(x^i, 0))\}, \text{ and } \mathcal{H}_{inside} = \mathrm{Span}\{(\hat{\phi}_{\omega lm}(x^i, 0), \hat{\pi}_{\omega lm}(x^i, 0))\},$$

corresponding to the "up" and "inside" modes, where here the π's are derived from the corresponding solutions described in the previous section using (2.18). The orthogonality of the Cauchy data extends to orthogonality of the solutions. We combine elements of $\mathcal{H}_{up}$ and $\mathcal{H}_{inside}$ to give regular modes at the horizon.

Explicitly, we expect to be able to find regular modes which are determined by Cauchy data

$$\phi_{klm}(x^i,0) = \int d\omega(\beta^+_{k\omega l}\phi_{\omega lm} + \beta^-_{k\omega l}(-1)^m\phi^*_{\omega l,-m} + \hat\beta^+_{k\omega l}\hat\phi_{\omega lm} + \hat\beta^-_{k\omega l}(-1)^m\hat\phi^*_{\omega l,-m}) \ ,$$
$$\pi_{klm}(x^i,0) = \int d\omega(\beta^+_{k\omega l}\pi_{\omega lm} + \beta^-_{k\omega l}(-1)^m\pi^*_{\omega l,-m} + \hat\beta^+_{k\omega l}\hat\pi_{\omega lm} + \hat\beta^-_{k\omega l}(-1)^m\hat\pi^*_{\omega l,-m}) \ , \tag{2.53}$$

where k is a continuous quantum number, and where it is understood that the up and inside mode functions contain factors of $\theta(r-R)$ and $\theta(R-r)$, respectively. The regularity condition at the horizon enforces conditions on the Bogolubov coefficients $\beta^+, \beta^-, \hat\beta^+, \hat\beta^-$.

We can then think of the modes (ϕ_{klm}, π_{klm}) as spanning a space $\mathcal{H}^R_{up} \subset \mathcal{H}_{up} \oplus \mathcal{H}_{inside}$, which inherits its orthogonality to $\mathcal{H}_{in}$ from $\mathcal{H}_{up}$ and $\mathcal{H}_{inside}$. The corresponding set $u_{kl}(r)$ and their complex conjugates should form a complete basis of functions of r, and the functions (2.53) and their conjugates likewise a basis for Cauchy data of regular solutions. Of course, the corresponding solutions $\phi_{klm}(x^i,t)$ will then have non-trivial time dependence, as in the $D=2$ case in [25]. In $D>2$ dimensions the equation of motion also includes an effective potential, which leads to mixing between right and left moving modes and more complicated solutions.

It appears difficult to give explicit expressions for such regular up bases, and that they are most easily treated approximately. They can also be thought of as being specified by different characteristics. One is that a localized wavepacket superposition of such solutions increasingly localizes in the vicinity of $r=R$ in the far past, becoming singular in the infinite past, as is seen in the simpler 2d example [37]. We may alternately imagine defining the modes by specifying regular functions $\phi_{klm}(x^i)$, and then choosing corresponding $\pi_{klm}(x^i)$ so that the modes are orthogonal to those in $\mathcal{H}_{in}$ and satisfy an appropriate condition corresponding to a choice of "positive frequency" (in the nomenclature of Sec. 2.3.2), but this appears not to give conditions that are simple to solve.

Finally, these modes can be specified by requiring that they be regular at the horizon and that their evolution $u_{kl}(x^+, r)$ (compare (2.34)) be purely outgoing at $r = \infty$ for all x^+ or time.

Using a general such regular basis, the field and momentum operators can be expanded as

$$\phi(x^i) = \sum_{lm} \int_0^\infty \frac{dk}{4\pi k} \left(a_{klm}\phi_{klm} + \tilde{a}_{klm}\tilde{\phi}_{klm} + h.c. \right) \ ,$$
$$\pi(x^i) = \sum_{lm} \int_0^\infty \frac{dk}{4\pi k} \left(a_{klm}\pi_{klm} + \tilde{a}_{klm}\tilde{\pi}_{klm} + h.c. \right) \tag{2.54}$$

where both ϕ_{klm} and $\tilde{\phi}_{klm}$ are regular at the horizon.

2.5.2 Evolution of regular modes

Using (2.48) and (2.54), the Hamiltonian for regular modes takes the block diagonal form

$$H = \sum_{lm} \int \frac{dk}{4\pi}\frac{dk'}{4\pi} \Big[A_{lm}(k,k') a^\dagger_{klm} a_{k'lm} + B_{lm}(k,k') a^\dagger_{klm} a^\dagger_{k'l,-m} + \tilde{A}_{lm}(k,k') \tilde{a}^\dagger_{klm} \tilde{a}_{k'lm} + \tilde{B}_{lm}(k,k') \tilde{a}^\dagger_{klm} \tilde{a}^\dagger_{k'l,-m} + c.c. \Big] \ . \tag{2.55}$$

Here for a stationary slicing

$$A_{lm}(k,k') = \frac{1}{2kk'} (\phi_{klm}, i\partial_t \phi_{k'lm}) \tag{2.56}$$

and

$$B_{lm}(k,k') = \frac{1}{2kk'} (\phi_{klm}, i\partial_t \phi^*_{k'l,-m}) \ , \tag{2.57}$$

$\tilde{A}_{lm}(k,k')$ and $\tilde{B}_{lm}(k,k')$ are defined similarly for the in-modes, and *c.c.* denotes conjugation that doesn't change operator ordering.[6] Note that the Hamiltonian does not contain mixing terms between $\mathcal{H}_{in}$ and $\mathcal{H}_{up}^R$ due to the orthogonality of the basis modes. This is particularly clear when writing the Hamiltonian in terms of the Bogolubov coefficients. Using the expansion (2.53) and relation (2.49), the mixing terms reduce to inner products between orthogonal energy eigenmodes. The remaining nonzero terms of the regular Hamiltonian (2.55) are then characterized by the functions

$$A_{lm}(k,k') = \frac{1}{2kk'} \int d\omega 4\pi\omega^2 (\beta^{+*}_{k\omega l}\beta^{+}_{k'\omega l} + \beta^{-*}_{k\omega l}\beta^{-}_{k'\omega l} - \hat{\beta}^{+*}_{k\omega l}\hat{\beta}^{+}_{k'\omega l} - \hat{\beta}^{-*}_{k\omega l}\hat{\beta}^{-}_{k'\omega l}) \ , \tag{2.58}$$

and

$$B_{lm}(k,k') = \frac{(-1)^{-m}}{2kk'} \int d\omega 4\pi\omega^2 (\beta^{+*}_{k\omega l}\beta^{-*}_{k'\omega l} + \beta^{-*}_{k\omega l}\beta^{+*}_{k'\omega l} - \hat{\beta}^{+*}_{k\omega l}\hat{\beta}^{-*}_{k'\omega l} - \hat{\beta}^{-*}_{k\omega l}\hat{\beta}^{+*}_{k'\omega l}) \tag{2.59}$$

where $\tilde{A}_{lm}(k,k')$ and $\tilde{B}_{lm}(k,k')$ are defined similarly, but with only one set of coefficients $\tilde{\beta}^{+}$ and $\tilde{\beta}^{-}$. With specific choices of the mode functions the Bogolubov coefficients as well as the coefficients A and B can in principle be calculated.

Given the hamiltonian (2.55), the evolution is in principle well defined. In practice, evolution in such a regular description is more complicated than in the singular description in terms of energy eigenmodes. It has of course been of interest to establish that there *is* a regular description, as well as to understand aspects of its behavior. We will also explore its relation to the description using energy eigenmodes, and how properties of the evolving wavefunction can consequently be inferred.

[6]Reordering then yields a normal ordering constant.

2.6 Evolution for dynamic black holes and the "Hawking state"

In this section, we will extend the preceding discussion to consider evolution of quantum matter on a general time-dependent, spherically-symmetric BH background, corresponding for example to a BH that forms from collapse of a massive body, and will discuss some properties of the corresponding quantum state.

2.6.1 Geometry

Specifically, consider the general metric

$$ds^2 = -f(x^+, r)dx^{+2} + g(x^+, r)dx^+dr + r^2 d\Omega^2 \ , \tag{2.60}$$

where the null ingoing coordinate x^+ can be chose so that $g(x^+, r) \to 2$ at $r \to \infty$. This could represent the metric of a general collapsing matter distribution, as shown in Fig. 2.2; a specific case is the ingoing Vaidya solution with

$$f(x^+, r) = 1 - \frac{2M(x^+)}{r} \quad , \quad g(x^+, r) = 2 \ , \tag{2.61}$$

with an ingoing mass function $M(x^+)$.

If we wish to provide a slicing of a collapsing BH spacetime such as shown in Fig. 2.2 by Cauchy slices, those slices need to avoid the singularity at $r = 0$. We assume that slices in the far past, when matter is dilute, are increasingly close to Minkowski time slices. Later slices can remain Cauchy if they have "nice" behavior, with a minimal radius R_n. These slices may then be closed in the region prior to the singularity, as illustrated in the figure. If we suppose that these slices asymptote to Minkowski time slices, the portion

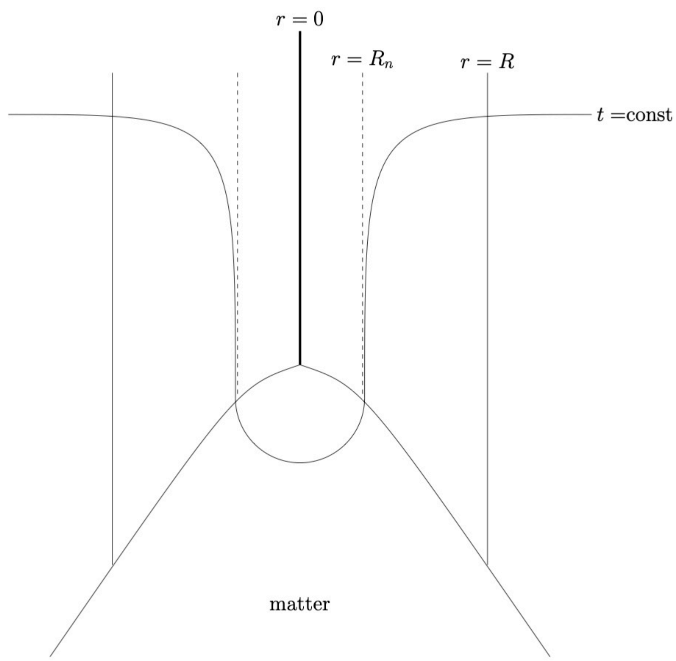

Figure 2.2: The geometry of a black hole formed from collapse, in an Eddington-Finkelstein diagram. Also shown is a slice that behaves like a nice slice in the vacuum region, which is then extended through the collapsing region to complete it to a Cauchy slice.

in the vacuum region needs to advance into the future with advancing time.

There are different ways to accomplish this, corresponding to different choices of coordinates or gauge, which arise from different choices of the functions $\mathcal{X}^+(t,x)$ and $\mathcal{X}^r(t,x)$ in (2.11). For example, these could be chosen so that the spatial metric q_{xx} on the slices of constant time t is time-dependent in the post-formation vacuum region, and might specifically undergo "stretching" as described in [51]. However, the description of the state in the post-formation region of interest is simpler if we instead use stationary slices, as in (2.12), in this region. Since the distance along the slices back to $r = 0$ must increase with time, these slices must undergo "stretching" in the early region, for example within or near the infalling matter region. In this section we will not focus on the latter behavior since it appears most relevant to the description of the "early" part of the BH state, and our interest will be in the part of the state corresponding to Hawking radiation and internal excitations at later times.

2.6.2 States

Given a slicing, we will define the “Hawking state” as the time-dependent state $|\psi(t)\rangle_H$ that arises from evolving the matter vacuum $|0\rangle$ at $t = -\infty$ to a future time t. Here, we focus on “spectator” matter that is different from the matter forming the BH. The Hawking state is then given by the expression

$$|\psi(t)\rangle_H = Te^{-i\int_{-\infty}^{t} dtH_\xi}|0\rangle \ , \tag{2.62}$$

with a hamiltonian as discussed in Sec. 2.3.2. This expression will implicitly depend on the choices of coordinates and mode bases, described in Sec. 2.3, used to define the picture.

The full time-dependent Hawking state (2.62) can be rather complicated, in part due to excitations created during the time-dependent BH formation phase. A simpler state that is sometimes considered is the Unruh state [52], which can be defined by working with the extended vacuum Schwarzschild solution, and evolving the vacuum defined with respect to the Kruskal coordinate X^- at the past horizon forward in time, via a similar procedure.

Indeed, if we compare the Hawking and Unruh states on a time slice that meets the horizon just after the transition to vacuum, such as shown in Fig. 2.2, they differ in some of the excitations escaping to infinity or falling into the BH. But, if we consider a much later slice, these excitations will have reached the asymptotic region, or reached the deep interior, either near $r = 0$, or with the kind of slicing we have described, the part of the slice at $r = R_n$. In contrast, the excitations being emitted from or falling into the BH near that later time are expected to be determined by the local short-distance structure of the state, which is the same for the Hawking and Unruh states.

In fact, these statements should extend to a more general state that behaves like vacuum near the horizon at short distances – the subsequent long time behavior is governed by this vacuum-like structure near the horizon. Specifically, we expect that any regular state has the same long-time behavior, once excitations that correspond to initial differences between states have escaped to near infinity or to the BH deep interior. The evolution of such a state can be examined at a more explicit level.

2.6.3 Evolution

To describe the evolution, first one makes a choice of slicing and coordinates along those slices, given by (2.11), or equivalently by specifying $t(x^+, r)$ and spatial coordinate $x(x^+, r)$, which we assume to be regular across the horizon. Next, choose a time t_0 which is taken so that the corresponding slice meets the horizon to the future of the collapsing matter as in Fig. 2.2. To begin, we wish to characterize what it means to be vacuum-like at this time near the horizon.

The vacuum-like structure can be characterizing by using the local relation to flat Minkowski geometry. In the vacuum region, the metric (2.60), (2.61) can be written in terms of Kruskal coordinates, defined for example in $D = 4$ by

$$X^\pm = \pm 2Re^{\pm x^\pm/2R} \ , \tag{2.63}$$

with corresponding extension across the horizon.[7] The vacuum metric then becomes

$$ds^2 = -\frac{R}{r}e^{1-\frac{r}{R}}dX^+dX^- + r^2 d\Omega^2 \ . \tag{2.64}$$

[7]For more discussion of these coordinates and the Rindler limit, see Appendix B.

The near horizon limit $|r-R| \ll R$ gives a "Rindler region"[53] in which the metric is locally $M^2 \times S^2$,

$$ds^2 \approx -dX^+dX^- + R^2 d\Omega^2 \ , \tag{2.65}$$

with corresponding 2d spacetime coordinates defined by $X^\pm = T \pm X$. In this near horizon limit and in these coordinates, a slice whose slice function $S(r)$ only varies on scales $\sim R$ will meet the horizon as a straight line. The regular basis (ϕ_{klm}, π_{klm}), $(\tilde{\phi}_{klm}, \tilde{\pi}_{klm})$ can then be chosen so that at high k, the corresponding solutions behave as

$$u_{kl} \approx e^{ikX - i\omega_k T} \quad , \quad \tilde{u}_{kl} \approx e^{-ikX - i\omega_k T} \tag{2.66}$$

in the Rindler region,[8] with $\omega_k^2 - k^2 = [l(l+1)+1]/R^2$. Then, with the field expansion (2.54), a state which is locally Minkowski is one satisfying $a_{klm}|\psi\rangle = \tilde{a}_{klm}|\psi\rangle = 0$ for the operators associated to $k \gg 1/R$, $l \gg 1$ modes in this region. Of course, modes with $k \lesssim 1/R$ can be excited in such a state.

The evolution of such a state is in general governed by the regular expression (2.55) for the hamiltonian. In the Rindler region this simplifies to give Minkowskian evolution, but this receives nontrivial corrections as excitations reach $|r-R| \sim R$. This nontrivial behavior causes excitations of the local vacuum. The details of this evolution depend on the detailed structure of the hamiltonian (2.55) and its corresponding evolution operator, which can be somewhat complicated. However, one thing that we do immediately learn in this description is that the transition to excited states takes place on scales with $|r-R| \sim R$, as was also found in 2d [25, 37]. This supports previous arguments [55] (see also the earlier related arguments [56, 57, 58]) that Hawking radiation "is produced" in a black hole atmosphere at these scales, as opposed to at ultrashort distances.

[8] A more careful treatment requires localization of the modes of the basis. This can be done by constructing wavepackets, for example as described in [2][54][37].

We have argued that the hamiltonian (2.55) gives a regular description of evolution on slices avoiding the singularity, but one with complicating features. To learn more about the state one would like a simpler description of this evolution. This is achieved by going to the energy eigenbasis of Sec. 2.4. While this description is inherently singular, as we have seen, it does furnish an effective way to more simply describe the evolution, due to the simplicity of the hamiltonian (2.52) in this basis. To describe the latter evolution, we must first rewrite a regular state $|\psi\rangle$ in this basis. This in principle follows from the Bogolubov transformation (2.53), which is however also complicated. But, as discussed, the long-time behavior is expected to follow from the local Minkowski structure near the horizon.

In particular, in the high-k Rindler region limit, the regular modes can be chosen to simplify to the form $\exp\{-ikX^{\pm}\}$. The locally right-moving modes $\exp\{-ikX^{-}\}$ there are related to the local (approximate) energy eigenmodes $\exp\{-i\omega x^{-}\}$, $\exp\{-i\omega \hat{x}^{-}\}$ by the same relation that relates Minkowski to Rindler modes, as seen from the coordinate transformation (2.63). Specifically, from this we find that the combinations

$$e^{-i\omega x^{-}} + e^{-2\pi R\omega} e^{i\omega \hat{x}^{-}} \quad , \quad e^{-i\omega \hat{x}^{-}} + e^{-2\pi R\omega} e^{i\omega x^{-}} \tag{2.67}$$

are analytic in the lower half complex X^{-} plane, and thus correspond to positive frequency modes in X^{-}. Then the corresponding operators

$$b_{\omega} - e^{-2\pi R\omega} \hat{b}_{\omega}^{\dagger} \quad , \quad \hat{b}_{\omega} - e^{-2\pi R\omega} b_{\omega}^{\dagger} \tag{2.68}$$

correspond to Minkowski annihilation operators, which should annihilate the state for large ω. Thus, for these high-wavenumber modes, the regular state has local description

$$|\psi\rangle \sim \sum_{\{n_\omega\}} e^{-2\pi R\int d\omega \omega n_\omega} |\{\hat{n}_\omega\}\rangle |\{n_\omega\}\rangle \tag{2.69}$$

in terms of occupation number eigenstates for the $\hat{b}_\omega$ and b_ω, and analogously for higher D. Just as with Minkowski space, this description is inherently singular. However, it is useful, and may for example be regulated with an appropriate short distance cutoff.

This description of the state is useful because it provides an effective intermediary to relate evolution of modes near the horizon to corresponding asymptotic modes.[9] Consider excited modes in the expression (2.69).[10] These are evolved by the simple hamiltonian (2.52), which vanishes for paired excitations, and a near-horizon wavepacket of such modes will evolve into a future wavepacket of the same modes. In the case of the "up" modes, associated to the $b_{\omega lm}$, the wavepackets will have an outgoing piece at infinity, with magnitude given in terms of the transmission coefficient for the effective potential fV of (2.41), and a reflected part that enters the BH.

Once we have used these modes as *intermediaries* to simplify the description of the state, we can alternately convert back into a regular mode basis. In fact, in the asymptotic region, the energy eigenmodes $u_{\omega l}$ should equate to corresponding regular modes, up to the factor of the transmission coefficients, since they are both governed by free Minkowski evolution. Specifically, asymptotically these modes take the form of flat space modes, with wavepackets that are linear superpositions of

$$\phi_{\omega lm} \sim T_{\omega l}\, j_l(kr) e^{-i\omega t} \frac{Y_{lm}}{r^{D/2-1}} \ , \tag{2.70}$$

[9]This can be seen even more explicitly in the two-dimensional example [37], which avoids complications such as reflection/transmission near the horizon.

[10]Again, a more precise version of this argument would use wavepackets to localize modes in both position and frequency.

where $T_{\omega l}$ is the relevant transmission coefficient. These can be regarded either as energy eigenmodes or as regular modes, in this region. In short, we convert to the energy eigenmode basis to simplify the evolution out to the asymptotic region, and then convert back to a regular basis using this relation between modes. In this fashion, the intermediaries provide a simple way to characterize the result of evolution of the regular expression for the state with the regular hamiltonian (2.55). It has a thermal spectrum at the expected temperature, *e.g.* $T = 1/4\pi R$ for $D = 4$, following from the form of (2.69). It also has the expected pairing and entanglement between quanta of Hawking radiation, and corresponding internal excitations of the BH, implied by the pairing in (2.69), along with the transmission factors. Again, this will be the generic long-time behavior, after transitory excitations, of states that are regular at the horizon.

We emphasize that in this discussion, the singular energy eigenmodes are *only* used as intermediary tool, and are not taken as part of a literal fundamental description of the state. This differs from a significant part of the literature, in which the singular energy eigenmodes are sometimes viewed as playing a more fundamental role; here, we stress, they are merely a convenient basis for some purposes. In practice, one way to work with a description of the state in terms of them is to introduce cutoffs in the description. But not regarding these modes as fundamental avoids the potential pathologies that arise if these modes are regarded as true physical excitations.

2.6.4 Internal evolution: nice slices and freezing

Evolution via a local quantum field theory hamiltonian, such as (2.28) or (2.30), has been argued to plausibly give a good approximate description of the complete physical evolution of excitations outside but near a BH, although one expects the need for important, but possibly small, corrections to ultimately restore unitarity [59, 60, 61, 19, 62].

On the other hand, one expects that the evolution of excitations inside the BH is likely to ultimately receive large corrections. Nonetheless, it seems of interest to better understand the leading field theory description of the internal evolution, as background and preparation for understanding its possible modifications.

Such evolution can be described on a family of Cauchy slices. As was noted in Sec. 2.2, slices that reach $r = 0$ are not Cauchy, and so a description on such slices must be supplemented by additional dynamics "at $r = 0$." But, evolution may be considered on a family of slices that avoid $r = 0$, and in particular on a family of nice slices that asymptote to a minimal radius $r = R_n$. We will describe some features of evolution on these slices. Our focus will be on the vacuum region of the BH, and we will consider stationary slices, as specified in (2.12) or (2.13), with a slice function chosen to asymptote to R_n.

As we have noted, one also needs choices of spatial coordinate and modes to describe evolution of the state. The use of r as a spatial coordinate on the constant-t slices leads to a coordinate system (t, r) that degenerates at $r = R_n$. This means it is preferable to use a more general spatial coordinate, $x(t, r)$.

The choice of a "stationary" coordinate $x(r)$ (say, with $x \to -\infty$ as $r \to R_n$) results in a nonzero shift N^x at large t, as the slices accumulate at $r = R_n$, although the lapse N vanishes in this limit. This implies a nontrivial contribution to the hamiltonian in this region, as seen for example from (2.20). This may be alternately understood by considering the form of the wavefunction solutions. For example, in such coordinates, the solutions (2.38) take the form

$$e^{-i\omega t} U_{\omega l m}(x, \Omega) \; , \tag{2.71}$$

and thus continue to have nontrivial time evolution as $t \to \infty$.

On the other hand, it is clear from the accumulation of slices at $r = R_n$ that the evolution of a state can be described as freezing [63][60] at this radius, as $t \to \infty$. This is best described with a choice of *non-stationary* coordinate along the slices. This choice can be specified through a more general relation $r(t, x)$, as in (2.11). Then, from the ADM form (2.14) of the metric, we find N to be unchanged from (2.15), and

$$q_{xx} = r'^2 q_{rr} \quad , \quad N^x = \frac{\dot{r} + N^r}{r'} \ , \tag{2.72}$$

with $r' = (\partial r/\partial x)_t$, $\dot{r} = (\partial r/\partial t)_x$. With such coordinates, the lapse, shift, and spatial metric are now explicitly time-dependent, also resulting in an explicitly time-dependent expression for the hamiltonian (2.20) or (2.28).

One way to exhibit the freezing behavior is if $x \to x^+$ as $r \to R_n$, which leads to both a vanishing lapse and shift as $r \to R_n$, and so vanishing hamiltonian density there. An example [37] is the coordinate $x = x^+ + g(r)$, with g vanishing as $r \to R_n$, although more generally we might instead like to use a time-dependent function $g(t, r)$ so that the coordinate x matches r asymptotically as $r \to \infty$, which is achieved if $g(r, t) \approx -t$ in this limit.

The freezing simplifies the description of the internal part of the state, since it no longer evolves, and thus gives one way of simply describing BH internal states in terms of this static appearance, in this approximation.

In these coordinates and this picture, the slices exhibit stretching behavior, rather than translating under a shift in t. For example, the distance along a slice from a given fixed x corresponding to a point near $r = R_n$ increases linearly in t. This arises from the time dependence of q_{xx} from (2.72), and may for example be concentrated in the vicinity of the horizon depending on the specific choice of $r(t, x)$. The explicit time dependence of the metric and hamiltonian in this gauge and picture introduces additional subtleties which we defer to future work, but which we expect may be resolved by connecting back to the underlying stationary description, also in analogy to [46].

2.7 Extensions: interactions, generalizing asymptotics, AdS and connection to $1/N$

The discussion of the bulk of this paper has been of a noninteracting theory such as (2.17), with flat asymptotic geometry, but it is expected that the quantum description extends both to interacting theories, and to more general, *e.g.* AdS, asymptotics, where there is also a connection with the large N limit of the AdS/CFT correspondence.

2.7.1 Evolution in interacting theories

The extension to interacting theories, and theories with higher-spin matter, is evident; beginning with a generalization of the action (2.17) to incorporate interactions and/or higher spin, the canonical approach yields a hamiltonian of the quadratic form (2.20), together with additional interaction terms. Canonical quantization proceeds from there via the canonical commutators, (2.21), or their higher-spin generalization. With a choice of basis of regular mode functions, and expansion in ladder operators analogous to (2.54), this results in a hamiltonian of the form (2.55), together with higher-order terms describing the interactions. While this can result in more complicated dynamics, the evolution of the state, *e.g.* as in (2.62), is in-principle concretely defined, modulo usual issues of renormalization, *etc.*

This observation illustrates two points. The first is that the present methods extend beyond Hawking's original derivation [2], which relied on use of the free propagating mode functions and so did not easily incorporate interactions. In evolution such as (2.62), the state continuously evolves in t according to the structure of the hamiltonian, without direct dependence on having solutions. It is important to have such a generalization, to treat Hawking radiation in interacting theories. The second point is that the specific choice of the mode functions is less important with this additional context. That is because a particular choice of modes may be motivated so to simplify evolution in the non-interacting case; a specific example is that of energy eigenstates, where the hamiltonian greatly simplified, to (2.52). However, once interactions are included, such simplifications are lost. In the interacting theory, it appears that different choices of regular mode bases won't make significant difference in the practical difficulty of describing the evolution of the state, and so fairly general choices can be considered.

This discussion also extends to include gravitational perturbations, and their couplings to perturbations of other fields. The full fluctuating metric may be expanded $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \kappa_D h_{\mu\nu}$, with $g_{\mu\nu}$ the metric of the BH background and $\kappa_D^2 = 32\pi G_D$. Then, the action and hamiltonian for the metric fluctuation $h_{\mu\nu}$ have a leading quadratic term similar to that for scalars (but also requiring gauge fixing), and interaction terms between $h_{\mu\nu}$ and the other fields, as well as self interactions of $h_{\mu\nu}$ at higher orders in the expansion in κ_D. By the steps just outlined, these interactions lead to an interacting hamiltonian generalizing (2.20), (2.55), which may be treated by similar methods, to determine the evolution of the state on a chosen set of slices.

2.7.2 Generalizing asymptotics, and AdS/Schwarzschild

Our main discussion has focussed on asymptotically flat spacetime. However, we can extend it to BHs with other asymptotics, using a slicing analogous to that described in Sec. 2.2.2 that is taken to similarly extend to the BH interior. Of course with general asymptotics, we may need to confront further subtleties associated with lack of a Killing vector corresponding to time translations. A prominent case with such a Killing vector is that of BHs in AdS. For example, the D-dimensional AdS/Schwarzschild solution for mass M takes the form (2.1), with

$$f(r) = 1 + \frac{r^2}{R_\Lambda^2} - \left(\frac{R_0}{r}\right)^{D-3} \quad ; \tag{2.73}$$

here R_Λ is the AdS radius, and

$$R_0^{D-3} = \frac{16\pi G_D M}{(D-2)A_{D-2}} \tag{2.74}$$

with A_{D-2} the area of the unit sphere. One may alternately use an ingoing null coordinate x^+ to rewrite the metric in the form (2.8), with f given by (2.73), and exhibit both exterior and interior of the BH. A trans-horizon slicing analogous to those of (2.10)-(2.13) may then be used to describe interior and exterior, and quantization may be performed analogous to the discussion of Sec. 2.3, resulting in hamiltonian evolution analogous to that of (2.62). In particular, we anticipate that the methods of this paper yield long-time thermal behavior for a large class of regular initial states. These would evolve similarly to the description of Sec. 2.6, with the additional feature that the AdS asymptotics behave like a reflecting cavity, and so Hawking excitations are reflected back towards the BH.

Specifically, we may describe the interacting hamiltonian and evolution perturbatively in Newton's constant G_D. The leading order evolution (also expanding in other couplings, if present) is hamiltonian evolution of free fields, including the graviton perturbations, by a hamiltonian analogous to (2.20), (2.55), on the AdS/Schwarzschild background. Couplings to gravitational perturbations, backreaction, dressing, *etc.* then arise at higher order in κ_D.

2.7.3 AdS/CFT and large-N description

Gravitational dynamics in AdS is conjectured to be equivalent to that of a "boundary" CFT [16]. In this context, it is interesting to explore the possible relation between the perturbative dynamics we have outlined, and the dynamics of the CFT. We focus on the "classic" example of AdS/CFT, with $AdS_5 \times S^5$ dynamics conjectured to be dual to $\mathcal{N}=4$ SU(N) super Yang-Mills on $S^3 \times \mathbb{R}$. The parameters are related by $(R_\Lambda/l_{10})^4 \sim N$, where $l_{10}^8 \sim G_{10}$ gives the ten-dimensional Planck length, and formulas here correctly include parameters but neglect numerical $\mathcal{O}(1)$ factors. Black holes with horizon radii $R \ll R_\Lambda$ are expected to be ten-dimensional localized objects in $AdS_5 \times S^5$; those with $R \gtrsim R_\Lambda$ are expected to behave as five-dimensional BHs (2.1), (2.73) that are uniform on S^5, and these two cases are expected to be connected by a Gregory-Laflamme transition [64]. The transition radius $R \sim R_\Lambda$ corresponds to a mass threshold $M \sim N^2/R_\Lambda$.

Thus, the case of AdS BHs, here with $D=5$, is strictly speaking only valid above this threshold. But in this case, the leading order dynamics, in a QFT description of the bulk, can be given by a free hamiltonian like (2.20), (2.55). And, this will receive perturbative corrections, order-by-order in the gravitational coupling. Given the relationship between parameters, and the relation $G_{10} \sim R_\Lambda^5 G_5$, this corresponds to an expansion in $\kappa_5 \sim R_\Lambda^{3/2}/N$.

As is known, this connects gravitational perturbation theory with the large N limit and $1/N$ expansion. A first question is what is held fixed as N is taken to be large. We will focus on the large N limit with AdS radius R_Λ held fixed, when then corresponds to the limit of G_{10} or G_5 becoming small. If we wish to consider BH states, then the preceding scaling tells us that we need to consider states whose energy scales up as N^2. However, if we consider for example fixing the temperature, *e.g.* as in [65], that is equivalent to holding the radius of the BH fixed and so the geometry (2.1), (2.73) is unchanged as N

increases.

The hamiltonian that describes BH excitations in the large N limit is then of the form described in the preceding subsections. Specifically, a candidate infinite-N hamiltonian H_∞ can be found by choosing a slicing for AdS-Schwarzschild like those described in Sec. 2.2.2, and specifically avoiding the singularity, and then deriving the corresponding hamiltonian (2.20) (or (2.55)) which is quadratic in each of the field perturbations that propogate on AdS.

Moreover, $1/N$ corrections to H_∞ correspond to the bulk interaction terms between these perturbations that arise in the perturbative expansion in κ_5.

Questions have recently been raised about the existence of a large N description of BHs in AdS/CFT in [66, 67],[65]. The present construction appears to begin to provide answers, and specifically to address the statement [65] that the literature doesn't contain a proposal for the hamiltonian for a BH in the large N limit.

Of course, what *is* expected to be true is the statement that the perturbative hamiltonian that is found this way does not give a complete description of the BH dynamics: specifically, there are good reasons to believe that this perturbative hamiltonian does not ultimately lead to unitary evolution, and one piece of that evidence is the fact that H_∞ can describe an infinite number of BH states.

For this reason, we expect the complete gravitational hamiltonian will be a corrected version of the perturbative hamiltonian H_{pert} we have just described:

$$H = H_{pert} + \Delta H \ . \tag{2.75}$$

We would obviously like to understand the structure of ΔH, and what phenomena it encodes; another pertinent question is its dependence on N.

It has previously been argued [61, 68, 19, 62] in the case of flat asymptotics that ΔH

has two important pieces: a piece ΔH_I containing interactions between the BH states and the BH's surroundings, necessary to transfer information or entanglement from the BH, and a piece ΔH_{BH} modifying the internal dynamics of the BH states. Simple forms of ΔH_I have been parameterized [19], and it is plausible that these corrections are small, even nonperturbatively so, in N. On the other hand, we might expect H_{pert} to receive large internal corrections in ΔH_{BH} corresponding to corrections to dynamics in the strong-curvature regime at the core of the BH. This is expected to be necessary, for example, to ensure a finite number of internal BH states. It is plausible that these corrections also yield chaotic internal behavior. Their dependence on N is less clear, but it is quite plausible that there are also important nonperturbative contributions here. A possible role for ensemble averages, like discussed in [65], is also less clear, unless the corrections to H_{pert} for example arise from baby universe emission [69, 70, 71, 72, 73, 74].

In such a plausible picture for ΔH, new chaotic dynamics is only associated with the deep interior dynamics of the BH; evolution in the near-horizon regime, both inside and outside the BH, may be close to that of LQFT, with only relatively small corrections that, for example, an infalling observer would perceive as innocuous. Further discussion of this picture, which may also lead to observational effects for BH observations [75, 76, 77][62], is given in the works cited above. Needless to say, it would be very interesting if such effects could be understood from, or even derived from, the AdS/CFT correspondence. Or, perhaps, they have a different explanation.

Chapter 3

Perturbative quantum evolution of the gravitational state and dressing in general backgrounds

3.1 Introduction and motivation

If there is a quantum-mechanical theory of gravity, the big challenges in its formulation include understanding the fundamental description of its quantum states and observables, as well as the nature of the unitary evolution on its Hilbert space. Approaches to this problem based on quantizing general relativity (GR) or related classical theories have run into vexing problems, initially nonrenormalizability[1] but likely more profoundly that of nonunitarity in the high-energy regime involving black holes. An alternative approach is to begin with the hypothesis that one is working with a quantum-mechanical theory, and investigate what mathematical structure of such a theory is necessary to describe gravity and consistently match the known and tested physics of local quantum fields propagating

[1]For a review with further references, see [78].

on a weakly-curved background, in the appropriate limits. This might be referred to as a "quantum-first" approach [79, 80, 81, 82].[2] One would like to understand the nature of the Hilbert space for gravity, and of its algebras of observables, symmetries, and unitary evolution law.

Such an approach does not argue for completely abandoning a perturbative quantization of GR. The match to the known and tested physics of local quantum field theory (LQFT) on a weakly curved background, the confirmed existence of gravitational waves, and the apparent approximate validity of strong-field classical solutions suggests that such a perturbative treatment gives at least *approximately* correct physics, in the weak-gravity regime, though one that is missing important effects in other regimes. One can view this as a "correspondence principle" for quantum gravity. What is interesting is that already in this limit, one encounters non-trivial new properties of quantum gravity that signal its departure from LQFT. This also raises the hope that, by better understanding this structure in the perturbative limit, one may infer key properties of the more basic structure of a fundamental theory of quantum gravity.

In particular, a significant part of the difficulty of gravity seems to stem from the form of its gauge symmetries. And, significant aspects of this non-trivial gauge structure appear to already be present at leading perturbative orders. This suggests that a useful starting point is simply to better understand this structure at these leading orders.

One aspect of this structure is the lack of local gauge invariant observables [86]. In short, any local observable clearly carries nontrivial Poincaré charge (in the example of flat asymptotics), since it doesn't commute with translation generators, and this must source an associated gravitational field that extends to infinity [30].

There are different approaches to constructing *non*local observables that respect gauge

[2] For related discussion, see [83, 84, 85]. Also note that if one can find a complete "holographic map," the AdS/CFT correspondence could be an approach to providing such structure.

invariance, typically in a "relational" fashion. One approach is to specify the position of a quantum operator relative to other quantum fields that vary in spacetime; we refer to the resulting operator as "field-relational" (or, observer-relational), and an example is provided by calculation of primordial perturbations in cosmology by referring to the time of reheating set by the inflaton in inflationary models. Another alternative is to construct relational observables by using position information from the gravitational sector; perturbatively, one can begin with a local observable, and "gravitationally dress" it to construct gauge-invariant operators that no longer commute at spacelike separation [80][29][30][32][82][33, 31].[3] In quantization of GR, the diffeomorphism symmetry is generated by the gravitational constraints, so in either approach, a test for gauge invariance of such operators is that they commute with the constraints.

Another key question for a quantum-mechanical theory is the structure of its states and their evolution. The observables both characterize states, and furnish a means of constructing gauge-invariant states: one can act with a quantum observable on a "vacuum" state to create a nontrivial state. The important question of evolution of these states, if it is unitary, can then be addressed by providing a hamiltonian.[4] In LQFT coupled to quantized GR, as we will review and further clarify, the hamiltonian and evolution are of course closely related to the constraints. In a closed universe, the hamiltonian is given by the constraints, and so formally vanishes on their solution; in a universe with asymptotic spatial infinity, there is an additional term in the hamiltonian, that is important for evolution. In particular, solving the quantum version of the hamiltonian

[3]To clarify a difference in terminology, the recent work [87] considers observables referred to an observer but calls them dressed, despite not having a gravitational component; here those would be referred to as field-relational. Earlier work related to dressed observables includes [88] and [89]; the first derived nontrivial commutators as arising from constraints, but didn't give the dressed operators, and the second focussed on deriving *commuting* operators. For related constructions in the cosmological setting, see [90].

[4]Even in LQFT, it is argued that the hamiltonian provides a more fundamental description of evolution than an action; see *e.g.* [91], sec. 2.2.

constraint is commonly referred to as solving the Wheeler-de Witt (WdW) equation, and is accomplished by gravitationally dressing undressed operators or states.[5] An additional subtlety (see below) is that physical states may only be annihilated by "half" of these constraints. The form of the evolution in the perturbative regime is expected to furnish clues about its nonperturbative completion.

Construction of the gravitational dressing, which can be explicitly treated at leading order in the gravitational coupling, is also relevant to the question of holography of gravity. A leading proposed explanation of holography in anti de Sitter space (AdS) is that it follows from the hamiltonian being a boundary term [93, 94] when the constraints are satisfied; for additional discussion, see [95, 96]. A closely similar argument is that momentum generators are also boundary terms, and so one can act both to translate a state to infinity, and to measure it, purely with operators at infinity [32]. There are related arguments that even perturbative observables at infinity can determine bulk states [92], but their sensitivity is exponentially suppressed [36]. These statements do rely on solution of the constraints, so implicitly on solution of the bulk dynamics [96].[6]

However these arguments are ultimately understood, it is clear that the dressing modifies the locality properties of the algebra of operators [80, 29]; it also apparently connects [98] to recent discussion of modification of the structure of von Neumann algebras from type III to II [67].

In short, at the perturbative level it appears that we can begin to learn important aspects of the structure of observables, states, and their evolution. This description is of course expected to miss crucial effects, particularly when treating strong gravitational configurations such as black holes.[7] But, a clearer understanding of the perturbative

[5]Ref. [92] also discusses perturbative solution of the WdW equation, but seems not to have realized that this is achieved by constructing gravitational dressing, nor recognized the relevance of preceding works on this subject.

[6]Entanglement wedge reconstruction appears to likewise assume solution of the constraints [97][96].

[7]For an approach to parameterizing such effects as departures from LQFT evolution, see [19, 62] and

structure is also expected to help provide a basis and background for understanding the role of modifications in the strong gravity context, if the complete theory respects the correspondence principle and is consistent with its weak gravity limit.

In the interest of such a deeper understanding of the interplay of evolution, gauge symmetry, the constraints, and gravitational dressing, in general contexts, this paper will investigate the perturbative structure of the hamiltonian and constraints, working perturbatively about a general background. The next section begins with a simplified derivation of the hamiltonian, exhibiting it either in a more conventional local form familiar from LQFT, or as a term proportional to the constraints plus a boundary term. Section three then outlines different approaches to perturbative quantization, and sets up a perturbative treatment of the constraints in what we refer to as "gauge-invariant canonical quantization." Section four gives a leading perturbative construction of operators commuting with the constraints, working about a general background, in terms of a construction of the gravitational dressing that generalizes [29][82][33, 31, 99]. Section five discusses construction of corresponding states, briefly discusses the form and characterization of their evolution, and illustrates application to the important cases of black holes and/or AdS spacetimes. Section six finishes with conclusions and further directions. Appendices illustrate basic features of the analogous treatment of electromagnetism, and show how the constraints generate gauge transformations.

references therein.

3.2 Action, hamiltonian, and boundary terms

3.2.1 Action and boundary terms

This section will review formulation of the action in Arnowitt-Deser-Misner (ADM) variables [20], and describe a simple approach to deriving the appropriate boundary terms. This paper focusses on quantization of Einstein gravity plus matter, perturbing around a general background. The usual starting point is the action[8]

$$S = \int d^D x \left(\frac{1}{16\pi G} \sqrt{|g|} R + \mathcal{L}_m \right) + S_\partial \tag{3.1}$$

in D spacetime dimensions, where G is Newton's constant, $\mathcal{L}_m$ is a matter lagrangian, and S_∂ is a boundary term. If a specific matter action is needed, the scalar theory with

$$\mathcal{L}_m = -\sqrt{|g|} \left[\frac{1}{2} (\nabla\phi)^2 + V(\phi) \right] \tag{3.2}$$

furnishes a useful example.

Since our focus will be on the evolving quantum state describing perturbations about a background, we would like to find a corresponding hamiltonian. We begin by introducing a foliation of the spacetime by slices labelled by time t, and with spatial coordinate x^i, with relation to general coordinates given by

$$x^\mu = \mathcal{X}^\mu(t, x^i) \ . \tag{3.3}$$

The displacement vector between points of equal x^i on nearby slices is given by

[8] We find it most convenient to work with expressions for lagrangians and hamiltonians that are densities.

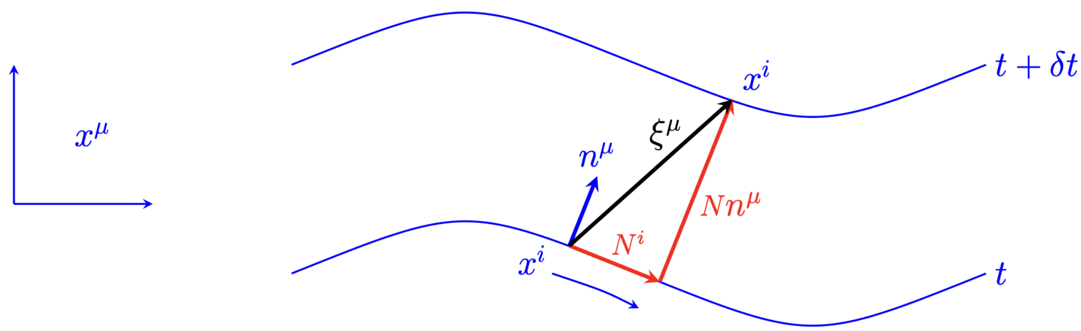

Figure 3.1: Shown are two members of a family of slices labelled by t. Points at the same spatial coordinate x^i are connected by the vector ξ^μ, which can be decomposed in terms of normal and tangential components to give the lapse and shift; vectors in the figure are scaled by an implicit δt.

$\xi^\mu = (\partial \mathcal{X}^\mu/\partial t)_{x^i}$, and can be decomposed into pieces normal and tangential to a slice,

$$\xi^\mu = Nn^\mu + N^\mu \ , \tag{3.4}$$

where N is the lapse, N^μ is the shift, and n^μ is the unit normal; in (t, x^i) coordinates $N^\mu = (0, N^i)$, and these quantities are illustrated in Fig. 3.1. In the coordinates (t, x^i) of the foliation the metric takes the ADM form

$$ds^2 = -N^2 dt^2 + q_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{3.5}$$

and the unit normal to the slices has components

$$n^\mu = \frac{1}{N}(1, -N^i) \ . \tag{3.6}$$

The gravitational lagrangian can then be derived in these variables, after introducing the extrinsic curvature of the slices,[9]

$$K_{ij} = \frac{1}{2N}\left(-\dot{q}_{ij} + D_i N_j + D_j N_i\right) \ , \tag{3.7}$$

with dot denoting $\partial/\partial t$, D_i the covariant derivative constructed from q, and latin indices raised/lowered with the spatial metric q. This lagrangian is given by [102, 103]

$$\sqrt{|g|}R = N\sqrt{q}\left[\left(K_{ij}K^{ij} - K^2\right) + R_q\right] - 2\partial_i\left(\sqrt{q}q^{ij}\partial_j N\right) + 2\partial_i\left(\sqrt{q}KN^i\right) - 2\partial_t(\sqrt{q}K) \ , \tag{3.8}$$

with $K = q^{ij}K_{ij}$ and R_q the scalar curvature of q. The total derivative terms become boundary terms in the action, which can be cancelled by S_∂,

$$S_\partial = \oint \frac{dtdA^i}{8\pi G}(\partial_i N - KN_i) + \int d^D x \frac{\partial_t(\sqrt{q}K)}{8\pi G} + S'_\partial \ . \tag{3.9}$$

An additional term S'_∂ is required, as is argued by requiring a well-defined variational principle in [104] or finiteness of the action in [105]. This can be described by introducing a background metric g_0, *e.g.* the Minkowski or anti de Sitter metrics, depending on boundary conditions, and writing the full metric as

$$g_{\mu\nu} = g_{0\mu\nu} + \Delta g_{\mu\nu} \quad , \quad q_{ij} = q_{0ij} + \Delta q_{ij} \ ; \tag{3.10}$$

it then takes the form

$$S'_\partial = -\oint \frac{dtdA^i}{16\pi G} N\left(D_0^j \Delta q_{ij} - \partial_i \Delta q\right) \ , \tag{3.11}$$

with dA^i the area element and $\Delta q = q_0^{ij}\Delta q_{ij}$. This can be checked (using equation (3.46)

[9] Note that there are differing sign conventions in the literature; *e.g.* [100, 101] differ by a sign.

below) to eliminate the problematic boundary terms in the variation of the action.

The resulting gravitational action can be written in terms of a local lagrangian,

$$S_g = \int d^D x \mathcal{L}_g = \int d^D x \frac{N\sqrt{q}}{16\pi G}\left[(K_{ij}K^{ij} - K^2) + R_q\right] + S'_\partial \ , \tag{3.12}$$

with S'_∂ rewritten as a volume integral. The structure of $\mathcal{L}_g$ can for example be investigated by using the relation [106, 107, 108]

$$\sqrt{q}R_q = \sqrt{q}q^{lm}\left(\gamma^i_{jl}\gamma^j_{im} - \gamma^i_{lm}\gamma^j_{ij}\right) + \partial_i\left[\sqrt{q}\left(q^{jk}\gamma^i_{jk} - q^{ij}\gamma^k_{jk}\right)\right] \ , \tag{3.13}$$

with γ^i_{jk} denoting the Christoffel symbols computed from the metric q. If the metric is expanded about a background solution as in (3.10), the linear terms vanish by the equations of motion of the background or cancellation with the boundary term, and quadratic and higher-order terms in the expansion of (3.13) give a lagrangian with quadratic contributions of the form $(\partial\Delta q)^2$, plus interaction terms.

3.2.2 Hamiltonian and constraints

Momenta conjugate to the spatial metric q are defined as

$$P^{ij} = \frac{\delta S_g}{\delta \dot{q}_{ij}} = -\frac{\sqrt{q}}{16\pi G}\left(K^{ij} - q^{ij}K\right) \ ; \tag{3.14}$$

we find it easiest to work with the form of these which are tensor *densities*. The momenta conjugate to N, N_i of course vanish, corresponding to the fact that the lapse and shift are Lagrange multipliers enforcing constraints. The gravitational action can then be rewritten in the canonical form

$$S_g = \int d^D x \left(P^{ij}\dot{q}_{ij} - \mathcal{H}_g\right) \ . \tag{3.15}$$

The hamiltonian is found by a straightforward calculation to be

$$\begin{aligned}\int dt H_g = \int d^D x \mathcal{H}_g \ &= \int d^D x (P^{ij}\dot{q}_{ij} - \mathcal{L}_g) \\ &= \int d^D x \left[\frac{16\pi G N}{\sqrt{q}}\left(P^{ij}P_{ij} - \frac{P^2}{D-2}\right) - \frac{N\sqrt{q}}{16\pi G}R_q + 2P^{ij}D_i N_j\right] - S'_\partial \ .\end{aligned} \tag{3.16}$$

The hamiltonian can be rewritten in different ways. First, as expected from the description of S_g given above, the expression $\mathcal{H}_g$ in (3.16) is quadratic in momenta and first derivatives of the metric perturbation. Alternately, (3.16) can be rewritten in terms of the Einstein tensor as

$$\int dt H_g = \int d^D x \left[-\frac{\sqrt{q}G_{nt}}{8\pi G} + 2D_i(P^{ij}N_j)\right] - S'_\partial, \tag{3.17}$$

where $G_{nt} = n^\mu \xi^\nu G_{\mu\nu}$. The matter hamiltonian likewise is given in terms of the stress tensor

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2 \tag{3.18}$$

as

$$\mathcal{H}_m = \sqrt{q}T_{nt} = \sqrt{q}\frac{N}{2}\left(\frac{\Pi^2}{q} + q^{ij}\partial_i\phi\partial_j\phi\right) + \Pi N^i \partial_i \phi \ , \tag{3.19}$$

where the densitized canonical momentum is $\Pi = \sqrt{q}\partial_n \phi$. Then, the full hamiltonian becomes

$$H = \int d^{D-1}x \, \mathcal{C}_\xi + H_\partial \ , \tag{3.20}$$

with

$$\mathcal{C}_\xi := \xi^\mu \mathcal{C}_\mu := \xi^\mu \sqrt{q}\left(-\frac{G_{\mu\nu}}{8\pi G} + T_{\mu\nu}\right) n^\nu \tag{3.21}$$

giving the usual gravitational constraints. The boundary contribution is, from (3.11) and (3.17),

$$H_\partial = \oint dA^i \left[\frac{N}{16\pi G}\left(D_0^j \Delta q_{ij} - \partial_i \Delta q\right) + 2\frac{P^{ij}}{\sqrt{q}} N_j\right] , \tag{3.22}$$

which is the expected boundary expression for gravity [104], $-N(\infty)P_0^{ADM} - N^i(\infty)P_i^{ADM}$. As is well known, if the constraints are satisfied,

$$\mathcal{C}_\mu = 0 , \tag{3.23}$$

then the hamiltonian becomes simply this boundary expression (3.22).

The interplay of the expressions (3.16) and (3.20) is worth noting, and can be summarized in

$$H = \int d^{D-1}x \left(N\mathcal{C}_n + N^i \mathcal{C}_i\right) + H_\partial = \int d^{D-1}x \left(\mathcal{H}_g + \mathcal{H}_m\right) \tag{3.24}$$

where we have used (3.4) relating ξ to n, and define $\mathcal{C}_n = n^\mu \mathcal{C}_\mu$. On the one hand, using the expansion of $\mathcal{L}_g$ described in connection with (3.13), the rightmost expression in (3.24) is of the general expected form for a field theory, with quadratic terms in momenta and derivatives of fields, as well as interaction terms. We can think of this as generating time evolution in the usual way. On the other hand, one can work with a solution of the constraints (3.23), in which case the hamiltonian reduces to the surface term H_∂. The later observation has been argued to be connected to the "holographic" property of gravity [93, 94, 95], but does rely [96] on first solving the constraints, which behave as equations of motion.

An important question is thus the role of the constraints and different forms for the hamiltonian in the quantum theory, as well as their possible corrections from a more complete quantum theory.

3.3 Quantization and perturbative expansion

3.3.1 Quantization, constraints, and gauge invariance

Our goal is to find a consistent quantum theory reproducing the preceding classical structure in the appropriate limits.[10] The canonical approach tells us to introduce canonical commutators,

$$[P^{ij}(x,t), q_{kl}(x',t)] = -i\delta^i_{(k}\delta^j_{l)}\delta^{D-1}(x-x') \ , \tag{3.25}$$

with normalization

$$\delta^i_{(k}\delta^j_{l)} = \frac{1}{2}\left(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k\right) \ . \tag{3.26}$$

Since N and N^i have vanishing conjugate momenta, they are taken to be c-numbers,[11] and (3.24) shows their role as Lagrange multipliers for the constraints. This means that these variables are not determined by the equations of motion, and their arbitrariness is part of the gauge symmetry. Gauge transformations acting on the canonical variables (q_{ij}, P^{ij}) (and (ϕ, Π)) are generated by $\mathcal{C}_n$ and $\mathcal{C}_i$, as described in Appendix D.

The Heisenberg equations of motion take the form

$$\partial_t q_{ij} = i[H, q_{ij}] \ , \tag{3.27}$$

[10] For an analogous discussion for QED, see Appendix C.

[11] Here we work on a "reduced" phase space; for further discussion see comments in Appendix D, and *e.g.* [109] or [110].

which reproduces the expression (3.7) for the extrinsic curvature, using (3.16), and

$$\partial_t P^{ij} = i[H, P^{ij}] \tag{3.28}$$

which gives the ij Einstein equations. These time derivatives are not gauge invariant, unlike the case of QED (see Appendix C), since the gauge transformations act non-trivially on q_{ij} and P^{ij}.

The next question is how to describe physical states. It is tempting to require $\mathcal{C}_\mu|\psi\rangle = 0$ for physical states, but this would then imply that for general operators O

$$\langle\psi|[\mathcal{C}_\mu, O]|\psi\rangle = 0 \tag{3.29}$$

conflicting with the preceding equations of motion. Thus, in order to correctly describe nontrivial evolution, the constraints should not be taken to vanish identically on the physical Hilbert space.

Multiple related ways to proceed have been studied in the literature, in each of which the question of locality becomes nontrivial. A brief summary is:

1. Dirac quantization. Here one introduces gauge-fixing conditions and solves these and the constraints, and also introduces a new Dirac bracket (or alternately re-defines operators) such that $[\mathcal{C}_\mu, O]_D = 0$. This appears to simplify commutators, but it is also true that solving the gauge conditions and constraints is nonlocal. This nonlocality is then "hidden" in the structure of the Dirac brackets; a simple example of this for QED is described in sec. IV.A.3 of [29].

2. Covariant gauge "fixing" (breaking). In this approach a gauge-violating term is added to the action, and then canonical commutators postulated for all components of the metric. This was used to study gauge-invariant operators in [29, 31]; the

gauge breaking term decouples for these. These operators are in general nonlocal, due to gravitational dressing, which is found by requiring vanishing commutators with the constraints [30].

3. BRST/BFV quantization [111, 112]. Here extra fields, including ghosts, are added; extra conditions are necessary as well.

4. Refined algebraic quantization, in which group averaging of states on an auxiliary kinematic space induces an inner product on the space of states satisfying the constraints [113, 114, 115, 116].

5. "Gauge invariant canonical quantization."

The latter approach appears to be distinct from approaches previously described in the literature; it is briefly described for QED in the Appendix C, and will be utilized here. While it is closely similar to covariant gauge breaking used in [29, 31], the constraints are separated into what may be called the positive and negative frequency parts, and the positive frequency constraint is taken to annihilate the vacuum state. Then, the constraints are found to commute with the gauge-invariant operators as usual, and also define the time evolution of the quantum state via the Hamiltonian. Specifically, one assumes a suitable decomposition of the constraints

$$\mathcal{C}_\mu(x) = \mathcal{C}^+_\mu(x) + \mathcal{C}^-_\mu(x) \tag{3.30}$$

into positive and negative frequency parts, and then imposes the physical state condition

$$\mathcal{C}^+_\mu(x)|\psi\rangle = 0 \tag{3.31}$$

for a $|\psi\rangle$ taken to be the vacuum state $|0\rangle$, on a given time slice. Since the constraints generate gauge transformations (see Appendix D), gauge-invariant operators are those satisfying

$$[\mathcal{C}_\mu(x), O] = 0 \ . \tag{3.32}$$

Such an operator creates a non-trivial state, $|\psi\rangle = O|0\rangle$, which also satisfies (3.31) by virtue of $[\mathcal{C}_\mu^+, O] = 0$.

A decomposition (3.30) is possible for example when perturbing about stationary backgrounds, using the corresponding Killing vector to define frequency. For perturbations about time-dependent backgrounds, there are additional complications [23, 44, 45, 24, 46] with such a picture which we will not address here. However, we note that the condition for gauge-invariant operators, (3.32), is independent of this decomposition, and so that should not play a direct role in their construction.

Suppose we consider evolution of states, in a Schrödinger picture, via the hamiltonian (3.24). Since the full constraints don't annihilate physical states, the time dependence of states will depend on gauge (here, choice of arbitrary $\mathbf{N} = (N, N^i)$); the gauge-dependent part of the change in the state for evolution for time δt via the hamiltonian (3.24) is

$$\delta_{\mathbf{N}}|\psi\rangle = i\delta t \int d^{D-1}x \left(N\mathcal{C}_n^- + N^i\mathcal{C}_i^- \right) |\psi\rangle \ . \tag{3.33}$$

However, this will be orthogonal to another physical state $|\psi'\rangle$. Likewise, if we consider evolution of a matrix element of a gauge invariant operator O,

$$\partial_t\langle\psi'|O|\psi\rangle = i\langle\psi'|[H, O]|\psi\rangle \ , \tag{3.34}$$

we find that this is also independent of the gauge-variant ($\mathbf{N}$-dependent) part of the hamiltonian (3.24). In short, while there is a gauge ambiguity in the states, that is not present in matrix elements of gauge-invariant operators. Of course, we find from the

Heisenberg equations (3.27) and (3.28) that evolution of matrix elements of q_{ij} and P^{ij} is gauge dependent.

3.3.2 Perturbative expansion

The remainder of this paper will primarily focus on a perturbative construction of states and operators like we have just described. Some analogous work has been done in the cosmological setting, including expanding the constraints to second order and related methods to find gauge invariant observables [117, 118]. However, they do not consider general backgrounds nor the general construction of gravitational dressing. The ADM decomposition and perturbation of the constraints has also been investigated in other special backgrounds, particularly asymptotically flat [26], AdS [27], and spacetimes with black holes [28]. Going beyond this work, we will consider a perturbative expansion of the ADM decomposition on an arbitrary classical background, and use the new expansions of the constraints to define the dressed operators to leading order by requiring that the operators commute with the constraints, as will be described in the subsequent sections. We begin with a classical metric $g_{\mu\nu} \leftrightarrow (N, N_i, q_{ij})$ satisfying Einstein's equations, including the constraints (3.23), possibly also with the stress tensor of a classical matter background ϕ_0. The corresponding quantum variables are denoted $\tilde{g}_{\mu\nu} \leftrightarrow (\tilde{N}, \tilde{N}_i, \tilde{q}_{ij})$ and $\tilde{\phi}$. Introducing the parameter $\kappa^2 = 32\pi G$, these may be expanded as

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu} \ , \ \tilde{\phi} = \phi_0 + \phi \ , \tag{3.35}$$

and likewise for $(\tilde{N}, \tilde{N}_i, \tilde{q}_{ij})$, in particular with

$$\tilde{q}_{ij} = q_{ij} + \kappa h_{ij} \ . \tag{3.36}$$

We will also expand $\tilde{P}^{ij}$ as

$$\tilde{P}^{ij} = P^{ij} + \frac{p^{ij}}{\kappa} \ , \tag{3.37}$$

in which case the canonical commutators (3.25) also take the simple form

$$[p^{ij}(x,t), h_{kl}(x',t)] = -i\delta^i_{(k}\delta^j_{l)}\delta^{D-1}(x-x') \ . \tag{3.38}$$

Notice, from (3.14), that for a non-trivial classical background, $P^{ij} \propto 1/\kappa^2$. We also expand $\tilde{\Pi} = \Pi_0 + \Pi$.

The explicit form of the constraints (temporarily written without tildes) is

$$0 = \mathcal{C}_n = \sqrt{q}\left(T_{nn} - \frac{4}{\kappa^2}G_{nn}\right) \tag{3.39}$$

and

$$0 = \mathcal{C}_i = \sqrt{q}\left(T_{ni} - \frac{4}{\kappa^2}G_{ni}\right) \ . \tag{3.40}$$

Here the pertinent components of the Einstein tensor are

$$-\frac{4}{\kappa^2}G_{nn} = -\frac{2}{\kappa^2}R_q + \frac{\kappa^2}{2q}\left(P^{ij}P_{ij} - \frac{P^2}{D-2}\right) \tag{3.41}$$

and

$$-\frac{4}{\kappa^2}\sqrt{q}G_{ni} = -2D_jP_i^j \ , \tag{3.42}$$

and those of the stress tensor are, from (3.19),

$$\sqrt{q}T_{nn} = \frac{\sqrt{q}}{2}\left(\frac{\Pi^2}{q} + q^{ij}\partial_i\phi\partial_j\phi\right) \quad , \quad \sqrt{q}T_{ni} = \Pi\partial_i\phi \ . \tag{3.43}$$

The expansions of the constraints (3.39) and (3.40) must then be found for the quantum perturbations h_{ij}, p^{ij} and ϕ, Π.

The hamiltonian constraint has the expansion

$$\tilde{\mathcal{C}}_n = \mathcal{C}_n + \frac{1}{2} q^{ij} \kappa h_{ij} \mathcal{C}_n - \frac{4}{\kappa^2} \sqrt{q} \delta_{\kappa h} G_{nn} + \sqrt{q} \left(\delta_\phi T_{nn} + \delta_{\kappa h} T_{nn} \right) + \sqrt{q} T^Q_{nn} + \sqrt{q} \breve{t}_{nn} \ . \quad (3.44)$$

Here $\delta_{\kappa h}$ and δ_ϕ denote first order variations, and we define a quantum stress tensor T^Q which collects terms that are quadratic and higher order in the variables ϕ, κh, coming from T_{nn}, as well as a gravitational stress tensor $\breve{t}$ that contains the quadratic and higher order terms in κh and p arising from the Einstein tensor term. Since the background satisfies Einstein's equations, the first two terms vanish. To find the third term and $\breve{t}$ we need the expansion of the Ricci scalar

$$R_{\tilde{q}} = R_q + \kappa \delta_h R_q + \delta^{\geq 2}_{\kappa h} R_q \ , \quad (3.45)$$

where the last term summarizes all higher-order terms in κh. Explicitly, the expansion of the Ricci scalar is well known

$$\delta_h R_q = D^i D^j h_{ij} - R^{ij}_q h_{ij} - D_i D^i (q^{kl} h_{kl}) := L^{ij} h_{ij} \ , \quad (3.46)$$

defining the second-order differential operator L^{ij}. We also need to find the expansion of the P-dependent term in an arbitrary classical background.

The terms $\delta_\phi T_{nn}$ and $\delta_{\kappa h} T_{nn}$ in (3.44) vanish in vacuum, but not with a nonzero background ϕ_0. They can be eliminated by passing to "perturbation picture" [37] or absorbed in T^Q. A matter background also leads to $h\phi$ and hh terms in T^Q. We will defer treatment of such a nontrivial background for future work and focus on the vacuum case.

In the vacuum case, working about a solution $\mathcal{C}_n = 0$, the preceding expansions then give

$$\tilde{\mathcal{C}}_n = \sqrt{q}\left(-\frac{2}{\kappa}L^{ij}h_{ij} + \frac{2}{\kappa}\mathcal{P}^{ij}h_{ij} - \frac{2}{\kappa\sqrt{q}}K_{ij}p^{ij} + T^Q_{nn} + \breve{t}_{nn}\right) \tag{3.47}$$

where K_{ij} is the extrinsic curvature of the slices in the background solution, related to the background P_{ij} by (3.14), and

$$\mathcal{P}^{ij} = \frac{\kappa^4}{2q}\left[P^{ik}P^j_k - \frac{PP^{ij}}{D-2} - \frac{q^{ij}}{2}\left(P^{kl}P_{kl} - \frac{P^2}{D-2}\right)\right] \ ; \tag{3.48}$$

recall that the classical P^{ij} is $\mathcal{O}(\kappa^{-2})$, so $\mathcal{P}^{ij}$ is $\mathcal{O}(\kappa^0)$.

Expansion of the constraint (3.40) is handled similarly, giving

$$\tilde{\mathcal{C}}_i = \mathcal{C}_i + \Pi_0\partial_i\phi + \Pi\partial_i\phi_0 - 2\kappa h_{ij}D_kP^{jk} - 2q_{ij}(\delta_{\kappa h}D_k)P^{jk} - \frac{2}{\kappa}q_{ij}D_kp^{jk} + \Pi\partial_i\phi + \sqrt{q}\breve{t}_{ni} \tag{3.49}$$

where again the quadratic and higher-order terms in h, p have been collected in $\breve{t}_{ni}$. Again restricting to the vacuum case, $\phi_0 = \Pi_0 = 0$, and using the statement that the background solves the constraints, this becomes

$$\tilde{\mathcal{C}}_i = -\frac{2}{\kappa}q_{ij}D_kp^{jk} - 2q_{ij}(\delta_{\kappa h}D_k)P^{jk} - 2\kappa h_{ij}D_kP^{jk} + \sqrt{q}(T^Q_{ni} + \breve{t}_{ni}) \ . \tag{3.50}$$

We will collect the terms linear in h by defining a linear differential operator $\mathcal{Q}$ by

$$\mathcal{Q}^{jk}_i h_{jk} = \kappa q_{ij}(\delta_{\kappa h}D_k)P^{jk} + \kappa^2 D_kP^{jk}h_{ij} \ . \tag{3.51}$$

Working about a classical background with $\mathcal{C}_n = \mathcal{C}_i = 0$, the $\kappa \to 0$ limit of the constraints gives the linear homogeneous equations

$$L^{ij}h_{ij} - \mathcal{P}^{ij}h_{ij} + \frac{K_{ij}}{\sqrt{q}}p^{ij} = 0 \tag{3.52}$$

and

$$D_j p_i^j + \mathcal{Q}_i^{jk} h_{jk} = 0 \ , \tag{3.53}$$

constraining linearized perturbations (h_{ij}, p^{ij}) about the solution, *i.e.* linearized gravitational waves. These are evolved by a quadratic hamiltonian, which may be found from the rightmost expression in (3.24), and which is expected to give evolution similar to that for other quantum fields, *e.g.* as treated in [34], such as Hawking production in a black hole background, *etc.*

At nonzero κ, the constraints become

$$L^{ij} h_{ij} - \mathcal{P}^{ij} h_{ij} + \frac{K_{ij}}{\sqrt{q}} p^{ij} = \frac{\kappa}{2} \left(T^Q_{nn} + \breve{t}_{nn} \right) \tag{3.54}$$

and

$$D_j p_i^j + \mathcal{Q}_i^{jk} h_{jk} = \frac{\kappa}{2} \sqrt{q} \left(T^Q_{ni} + \breve{t}_{ni} \right) \ . \tag{3.55}$$

In the classical theory the corresponding equations determine the perturbative "Coulomb fields" induced by matter, and at higher orders also incorporate the nonlinearities resulting from gravitational energy. At leading order in κ, the solutions to (3.54), (3.55) are of course highly nonunique, since a solution of the homogeneous equations (3.52), (3.53) may be added to any given solution.

3.4 Leading perturbative dressing

In the quantum theory, finding gauge-invariant operators O that commute with the constraints, (3.32), can be approached by perturbatively solving for operators that commute with (3.54), (3.55). Solutions can be found, beginning with an operator of the quantum field theory to which we couple gravity. This is done by gravitationally dressing that operator, as has been described to leading order in perturbation theory about flat space in [80][29][30, 32][82][33, 99] and about anti de Sitter space in [31]. Here we will extend those constructions to a more general background.

This gravitational dressing is most easily studied in the situation where the background satisfies $P^{ij} = 0$, corresponding to vanishing extrinsic curvature of the time slices of the background metric. This includes the case of flat and AdS backgrounds with standard time slicings. However, we would also like to consider evolution that for example perturbs about black hole solutions, either with flat or AdS asymptotics. For example in the case of the Schwarzschild solution, one may consider a general stationary slicing that is spherically symmetric [19][25][34],

$$x^+ = t + S(r) \ , \tag{3.56}$$

specified by a slicing function $S(r)$, where x^+ is the ingoing Eddington-Finkelstein coordinate. The nontrivial components of the extrinsic curvature of the slices are then given by

$$D_r N_r = \partial_r N_r - \gamma^r_{rr} N_r \quad , \quad D_\theta N_\theta = -\gamma^r_{\theta\theta} N_r \ , \tag{3.57}$$

and so vanish if and only if $N_r = 0$. The expression [19] $N_r = 1 - fS'$, with $-f$ the coefficient of dx^{+2} (see below), then implies this is true only for $S' = 1/f$, which is the case of Schwarzschild time slices. These lead to a singular basis for perturbations at the horizon, and as explained in [34] this can be avoided with a more general choice of slices. But, this therefore requires considering $P^{ij} \neq 0$; similar statements hold for the case of black holes in AdS.

The construction of [80][29][30, 32][82][33, 99] writes the linear order dressing of an underlying QFT operator O_0 as

$$O = e^{i\int d^{D-1}x\sqrt{q}V^\mu(x)(T_{n\mu}+\check{t}_{n\mu})}O_0 e^{-i\int d^{D-1}x\sqrt{q}V^\mu(x)(T_{n\mu}+\check{t}_{n\mu})}\ ; \tag{3.58}$$

as long as we work to linear order in κ the exponential is not strictly necessary, but is convenient and suggestive. (To leading order about a vacuum solution T^Q of (3.44) simplifies to T of matter perturbations; we have also included the stress tensor $\check{t}_{n\mu}$ for metric perturbations, in anticipation of the possibility that O_0 could also include such perturbations.) Here the dressing functions $V^\mu(x)$ are functionals of the metric perturbation which are fixed by the condition that the dressed operator O commute with the constraints.[12]

3.4.1 Vanishing background extrinsic curvature: $P^{ij} = 0$

We first consider this simplifying case. Specifically, generalizing the flat space construction [29, 99], we anticipate that $V^n(x)$ takes the form

$$V^n(x) = -\frac{\kappa}{2}L^{-1}_{ij}p^{ij} = \frac{\kappa}{2}\int d^{D-1}x'\check{h}_{ij}(x',x)p^{ij}(x')\ , \tag{3.59}$$

[12]Papers by Fröb, Lima, and collaborators [119, 120, 121, 122, 123, 124, 125, 126] have also studied construction of leading-order gravitationally-dressed observables. The equivalence of their approach is seen, *e.g.* in the case of dressing of a scalar field, by noting that if our dressing V^μ defines a map χ through $\chi(y) = y + V(y)$ (Lorentz indices suppressed), then the map $X(x)$ given for example in (3) of [125] is $X(x) = \chi^{-1}(x)$. Then the scalar version of (7) of that reference is the same form as $\phi(y+V(y))$ of ref. [29] eq. (33), and the transformation properties under diffeomorphisms of $X(x)$ that they give follow from those of V in [29]. This means that their dressing in eq. (8) corresponds to a special case of (39) of [29], up to a total derivative. However, this difference is important, since their (8) does not transform correctly under harmonic diffeomorphisms. The missing total derivative also appears to explain the claim of [123, 125] that they have observables commuting outside the light cone, in contradiction to the generic noncommutativity found in [29] and to the dressing theorem of [30].

where the inverse L^{-1} is given by a Green's function solution to

$$L^{ij}_{x'}\check{h}_{ij}(x',x) = -\frac{\delta^{D-1}(x'-x)}{\sqrt{q}} \ . \tag{3.60}$$

These Green's functions are highly non-unique, corresponding to the non-uniqueness of the perturbative classical solutions. Explicit examples of this nonuniqueness in dressings have been described for perturbations about a flat background [29]. Examples there include explicit expressions describing either line-like or Coulomb-like gravitational fields [29], and generalize to a broad class of instantaneous configurations of the field. This nonuniqueness extends here to more general backgrounds, and again corresponds to differences by homogeneous solutions, corresponding to source-free gravitational waves.[13] With such a Green function, the commutator with the constraint $\tilde{\mathcal{C}}_n$ is easily seen to give

$$[\tilde{\mathcal{C}}_n(x), V^n(x')] = i\delta^{D-1}(x'-x) + \mathcal{O}(\kappa) \ . \tag{3.61}$$

As a consequence, using the expression (3.58) and assuming the V_i term doesn't contribute (see below),

$$[\tilde{\mathcal{C}}_n(x), O] = 0 + \mathcal{O}(\kappa) \ , \tag{3.62}$$

with the commutator of the leading-order term from $(T+\check{t})$ in $\tilde{\mathcal{C}}_n$ being cancelled by the term arising from (3.61).[14]

To solve the momentum constraint (3.55), we consider dressing functions of the gen-

[13]This also means, as also described in [32, 33, 99] that soft charges are largely decorrelated with the quantum state of matter in a region. For the most part they depend on the arbitrary choice of gravitational dressing (which may be specified *e.g.* through imposition of boundary conditions), with the only necessary correlation through the total Poincaré charges of the matter state.

[14]Note that the leading order (in κ) transformation of the metric perturbations leads to terms that cancel the leading non-invariance of the operator $\mathcal{O}_0$ in (3.58). Higher-order terms in the transformation of the metric perturbation then contribute to the $\mathcal{O}(\kappa)$ terms here.

eral form

$$V^i(x) = \kappa \int d^{D-1}x' G^{ijk}(x', x) h_{jk}(x') \ , \tag{3.63}$$

and seek a solution of the equation

$$[\tilde{\mathcal{C}}_i(x), V^j(x')] = i\delta_i^j \delta^{D-1}(x' - x) + \mathcal{O}(\kappa) \ . \tag{3.64}$$

From the canonical commutators (3.38), we find this holds if

$$2q_{ij} D_k G^{ljk}(x, x') = \delta_i^l \delta^{D-1}(x - x') \ . \tag{3.65}$$

In a flat background, solutions are given by [99]

$$V^i(x) = \int d^{D-1}x' \check{h}^{jk}(x', x) \gamma^i_{jk} \ . \tag{3.66}$$

In more general backgrounds, the solutions are seen to correspond to Green functions for the equations for linearized metric perturbations, (3.54), (3.55), and therefore should exist once boundary conditions are specified to fix a specific solution. One can also easily check that

$$[\tilde{\mathcal{C}}_n(x), V^i(x')] = \mathcal{O}(\kappa) \quad , \quad [\tilde{\mathcal{C}}_j(x), V^n(x')] = \mathcal{O}(\kappa) \ . \tag{3.67}$$

Then, given the commutators (3.61), (3.64), (3.67), we find that the constraints have been solved to leading nontrivial order in κ by the dressed operators (3.58),

$$[\tilde{\mathcal{C}}_\mu(x), O] = 0 + \mathcal{O}(\kappa) \ . \tag{3.68}$$

3.4.2 $P^{ij} \neq 0$

Once one sees this structure, it is apparent how one can generalize to the case of background $P^{ij} \neq 0$. We now define

$$V^n(x) = \frac{\kappa}{2} \int d^{D-1}x' \left[\check{h}_{ij}(x', x) p^{ij}(x') - \check{p}^{ij}(x', x) h_{ij}(x')\right] \tag{3.69}$$

and

$$V^i(x) = \kappa \int d^{D-1}x' \left[G^{ijk}(x', x) h_{jk}(x') + H^i_{jk}(x', x) p^{jk}(x')\right] \tag{3.70}$$

where $\check{h}_{ij}$, $\check{p}^{ij}$, G^{ijk}, and H^i_{jk} are c-number functions. Then, the hamiltonian constraint gives

$$[\tilde{\mathcal{C}}_n(x), V^n(x')] = -i\sqrt{q}\left(L^{ij} - \mathcal{P}^{ij}\right) \check{h}_{ij}(x, x') - iK_{ij}\check{p}^{ij}(x, x') + \mathcal{O}(\kappa) \; ; \tag{3.71}$$

requiring this commutator to be of the form (3.61) then gives

$$\left(L^{ij} - \mathcal{P}^{ij}\right) \check{h}_{ij}(x, x') + \frac{K_{ij}}{\sqrt{q}} \check{p}^{ij}(x, x') = -\frac{\delta^{D-1}(x - x')}{\sqrt{q}} \; . \tag{3.72}$$

Leading order vanishing of the commutator of V^n with the momentum constraint likewise gives

$$D_j \check{p}^j_i(x, x') + \mathcal{Q}^{jk}_i \check{h}_{jk}(x, x') = 0 \; . \tag{3.73}$$

This generalizes the above Green function problem, and so should again have solutions for $\check{h}_{ij}$ and $\check{p}^{ij}$ by the relation to the classical problem of finding linearized solutions on the background.

Requiring the correct leading order commutators of the constraints with $V^i(x)$, (3.64) and (3.67), likewise gives the equations

$$2q_{ik}D_l G^{jkl}(x,x') - 2\mathcal{Q}_i^{kl} H^j_{kl}(x,x') = \delta_i^j \delta^{D-1}(x-x') \tag{3.74}$$

and

$$\left(L^{jk} - \mathcal{P}^{jk}\right) H^i_{jk}(x,x') - \frac{K_{jk}}{\sqrt{q}} G^{ijk}(x,x') = 0 \ , \tag{3.75}$$

which again corresponds to a Green function problem for linearized perturbations. Once $\check{h}_{ij}$, $\check{p}^{ij}$, G^{ijk}, and H^i_{jk} have been determined by solving these equations, together with specification of the homogenous part of the solution *e.g.* through boundary conditions, then (3.69) and (3.70), together with (3.58), give the dressed operator O to leading nontrivial order in κ.

Significant features of the role of gauge invariance can be seen from the leading order in κ construction of the dressed operators (3.58) given here. For example, the dressing modifies the commutators from those of the underlying LQFT operators O_0, such that operators associated with spacelike-separated regions generically no longer commute; examples can be given extending the discussion of [29]. Of course, to further understand the role and structure of the constraints and dressing, one would like to go beyond to higher orders in κ. One does expect further difficulties here, in particular associated to infinities and the need to regulate operators. We will leave further discussion of higher orders for future work, but will discuss some general features that already become apparent with these leading-order results.

3.5 Description of evolution

3.5.1 General structure

It is important to understand the general structure of evolution in quantum gravity, given the constraints of gauge invariance. The leading-order construction of gauge invariant operators, and the more general structure of the hamiltonian and constraints, already appear to provide significant guidance to this structure.

In particular, we have given a leading-order construction of gauge invariant observables O, commuting with the constraints (3.32). These then lead to states that evolve via the hamiltonian (3.24) of quantized general relativity, *e.g.* of the form

$$|\psi\rangle = O|0\rangle \ . \tag{3.76}$$

One can alternately construct dressed states directly from undressed states,

$$|\psi\rangle = e^{i\int d^{D-1}x\sqrt{q}V^\mu(x)(T_{n\mu}+\check{t}_{n\mu})}|\psi_0\rangle \ . \tag{3.77}$$

As a simple example, one could begin with the basic scalar field operator, $O_0 = \phi(x)$, and then construct the corresponding dressed operator O, given to leading order by (3.58), or equivalently [29] by $\Phi(x) = \phi(x^\mu + V^\mu(x))$. The resulting operator can be thought of as creating from the vacuum a quantum of the field ϕ, together with its corresponding gravitational field. As we have emphasized, the gravitational part of the operator is non-unique, corresponding to the fact that there are different possible gravitational field configurations dressing the particle, differing at leading order by free gravitational excitations.

Evolution can be thought of in two ways, related through the two expressions for the hamiltonian (3.24). Since the rightmost expression there is of the standard form of a LQFT hamiltonian, it determines evolution of the state by telling us how the ϕ and gravitational parts of the state evolve like standard quantum fields. This evolution is of course gauge dependent, through its dependence on N and N^i. We expect it to correspond to the quantum evolution of the matter state created by O, together with the quantum gravitational field created by the gravitational piece of the operator.

Or, one can describe evolution in terms of the middle expression in (3.24) which is written in terms of the constraints. The boundary hamiltonian H_∂ contributes to the time dependence of the state, because the gravitational dressing generically extends into the asymptotic region [30, 31]. One might have anticipated that the constraints $\mathcal{C}_\mu(x)$ annihilate the state, so that this is the only time dependence. This would incorporate the statement that the state "satisfies the Wheeler-DeWitt equation," since the constraint $\mathcal{C}_n(x)$ corresponds to the Wheeler-DeWitt operator. However, we have found that this would be inconsistent with the basic commutators and in particular with evolution such as described by the Heisenberg equations (3.27), (3.28). Instead the state is annihilated by "half" of the constraints (and of the Wheeler-DeWitt operator), (3.31). This implies that the constraint terms in (3.24) also contribute to gauge-dependent evolution of the state, though as we have argued above not to evolution of matrix elements of gauge-invariant operators. Alternately, transition amplitudes of the form

$$\langle\psi'|e^{-iHt}|\psi\rangle \tag{3.78}$$

will also exhibit time dependence.

3.5.2 Bubble evolution, cosmology, and field-relational observables

This raises the question of the description of evolution on slices that coincide at infinity, but not in a region in the interior of spacetime, so that the asymptotic lapse and shift vanish, implying $H_\partial = 0$. Then, the full hamiltonian commutes with the gauge-invariant operators O. This suggests that their evolution is trivial in such "bubble" evolution [127, 128]. This is also the case for closed cosmologies, with no boundary term. In both of these cases the hamiltonian is typically explicitly time-dependent, with additional subtleties [23, 44, 45, 24, 46]. Once again, we can anticipate that the physical states are annihilated by "half" of the constraints. An alternate way to then describe evolution is in terms of a different kind of relational observable that is not gravitationally dressed; an example of such a field-relational observable is

$$\int d^D x \sqrt{|g|} O_0(x) f(Z^I(x)) \tag{3.79}$$

where Z^I are D dynamical "locator" fields and $f(Z^I)$ is chosen so that in a particular state for these fields its support is localized near a particular point. An example is using the value of the inflaton field in inflation to localize in time to the reheating time; for further discussion (including of limitations to localization) see [129]. We leave further exploration of such evolution for future work.

3.5.3 Other specific examples

Beyond a flat space background, it is of interest to better understand gravitational evolution in other backgrounds such as those of black holes, AdS, or black holes in AdS.

The static cases can be subsumed in the line element

$$ds^2 = -f(r)dx^{+2} + 2dx^+dr + r^2d\Omega^2 \tag{3.80}$$

where

$$f(r) = 1 + \frac{r^2}{R_\Lambda^2} - \left(\frac{R_0}{r}\right)^{D-3} , \tag{3.81}$$

R_Λ is the AdS radius, and

$$R_0^{D-3} = \frac{16\pi G_D M}{(D-2)A_{D-2}} \tag{3.82}$$

with A_{D-2} the area of the unit sphere. Then, introducing a stationary slicing (3.56) given by a slice function $S(r)$ yields the ADM background solution [19]

$$N^2 = \frac{1}{S'(2-fS')} \quad , \quad N_r = 1 - fS' \quad , \quad q_{rr} = S'(2-fS') \quad , \tag{3.83}$$

and with angular components the standard round metric of radius r.

Evolution may then be described perturbatively about this solution, using the preceding general construction. Specifically, we may consider a dressed state (3.76) on a slice taken to be an initial slice. The evolution of this state can be described via either of the forms of the hamiltonian (3.24). The latter form in particular gives a standard description of field evolution, and so evolves the matter perturbation together with the perturbative gravitational field corresponding to its dressing in standard field theory fashion.[15]

In this way, one for example finds a perturbative expression for the bulk hamiltonian for an AdS black hole to leading order in κ, or in the language of the AdS/CFT correspondence, in $1/N$, also making connection with the discussion in [34] of this approach

[15]While Lorentzian evolution for black holes has been considered previously, see *e.g.* [130], the treatment outlined here extends to more general slicings than Schwarzschild, and including the black hole interior.

to defining such a hamiltonian. Alternately, by virtue of the constraints, the hamiltonian is related to a boundary hamiltonian as in (3.24). We defer more detailed investigation of this evolution to future work.

3.6 Conclusion and directions

In conclusion, we have shown that, starting from an ADM parameterization of the geometry and the corresponding construction of the hamiltonian, leading order perturbative gravitational states may be constructed and their evolution described. The states and evolution have a gauge symmetry, generated by the constraints, and perturbative solution of the constraints to construct gauge-invariant operators and states can be accomplished by gravitational dressing operators of an underlying field theory. Such a construction has been found to leading perturbative order about a general background, in terms of certain generalized Green functions of the given background, generalizing earlier constructions in a flat background [80][29][30, 32][82][33, 99]. The resulting gravitational part of the state is not uniquely determined, since it can be changed by addition of a piece corresponding to an arbitrary source-free propagating gravitational wave. The state gotten by acting with such a dressed operator on a vacuum state is then evolved by the hamiltonian, which may be described as a standard local QFT hamiltonian including the spin-two perturbative gravitational field, or alternately may be written in terms of a boundary hamiltonian, up to terms proportional to the constraints.

There are multiple directions for further work. Within the framework of local QFT, as noted above, we would like to better understand bubble evolution, on slices that match at infinity, and the related description of cosmological evolution, and the connection to other field-relational observables that are more useful in that context. There are also related issues that occur when the background slicing has an explicit time dependence

[23, 44, 45, 24, 46], which deserve to be more closely investigated. It also seems useful to have a more complete description of the evolving perturbative state of black holes, whether or not in AdS, *e.g.* generalizing [34] to include gravitational dynamics. And, the problem of solving the constraints connects directly to a leading argument for the origin of holography [93, 94, 95, 96], which is important to better understand. This also connects to the question of in what sense information can be localized in a gravitational theory, either because of the argued existence [93, 94] of a holographic map, or a similar argument [32] that states internal to AdS may be observed by boundary observables, *if* the constraints are solved.[16] This connects directly to the question of the extent to which subsystems may be defined [81, 82, 36, 131] in gravitational theories, whether exactly or approximately. We note that a perturbative description of the evolution like we have outlined is consistent with the perturbative solution of the constraints, and appears to describe a black hole with a growing number of internal states entangled with the exterior, and corresponding missing information if the black hole disappears at the end of evolution. Thus, while the perturbative solution of the constraints via the dressing does appear to provide some additional sensitivity to the black hole state, it does not appear to resolve the unitarity problem, in contrast to recent claims [92], and in particular does not obviously provide a mechanism for transfer of information from the black hole.

In clarifying these issues, understanding better the structure of higher-order solution to the constraints seems important (and it seems important to clarify the challenges to finding higher-order solutions). We would also like to better understand the structure of the algebras associated to dressed observables. It can be observed that the leading-order perturbative dressed observables, (3.58) and related expressions in [33], have similar

[16] Such information has even been argued to be accessible perturbatively [92], although only if one can measure exponentially small quantities [36].

structure to the observables in the crossed-product construction, argued [67][87] to convert type III von Neuman algebras into type II. Of course, the noncommutativity of perturbative observables associated with different regions [80, 29], due to the dressing, appears to be a likewise important modification of the underlying field theory structure; further investigation of these questions is in progress.

The modification of local algebras in gravity illustrate the general statement that locality is remarkably subtle in theories with structures like gauge symmetries. In QED or standard nonabelian gauge theories, observables found by dressing underlying matter operators are generically nonlocal, as with the gravitationally dressed operators discussed above. Put differently, the problem of solving the constraints is generically a nonlocal one, with a nonlocal solution. However, in gauge theories based on an internal group, locality is still realized since there also exist gauge-invariant operators that are local, such as Wilson loops or other neutral observables confined to a neighborhood. In gravity, the gauge symmetry is that of transformations including Poincaré symmetries; any local observable thus carries nontrivial charge, and so is nonlocal when its gravitational dressing is included [30]. In short, gauge theories with an internal symmetry appear to be barely local, but it is less clear what locality properties quantum gravity has.

There are also important directions that appear to go beyond the framework of local QFT. For one thing, if we consider evolution of a black hole, like that described in 3.5.3, that appears to lead to the breakdown of unitarity, also noted above, associated with the "black hole information paradox" or "unitarity crisis." This can be encapsulated in a "black hole theorem[73]": unitary evolution, and the statements that black holes behave like subsystems, that field configurations outside them evolve independently of the black hole internal state, and that they disappear at the end of their evolution, are inconsistent. We expect modification of both the structure of the Hilbert space and the hamiltonian as compared to those given by quantization of GR like that we have described. In a more complete description, we expect that the fundamental quantum variables are likely *not* those of fields moving on a background metric, with states labelled such as $|q_{ij}(\cdot), \phi(\cdot)\rangle$, but that these variables only give an approximate description of the states. An important question is what is the more accurate and complete description of the variables parameterizing the wavefunction. This, then, closely relates to the question of what are the fundamental observables, and the ultimate form of the hamiltonian and its interactions, as well as the symmetries of the theory.

If in a more complete description of the Hilbert space black holes still effectively behave as subsystems, the "black hole theorem" tells us that unitarity apparently requires interactions that go beyond the quantized GR/local QFT description, and specifically such that the evolution of the black hole exterior depends on the black hole internal state. An approach to parameterizing such interactions has been developed in [59, 60, 61][19, 62]. The present work serves as an even firmer foundation on which to describe their parameterization, if they can be regarded as corrections to the evolution governed by local QFT plus quantized GR. In short, one can describe corrections ΔH to the hamiltonian of (3.24), constrain their properties, and investigate their possible observational effects for example in electromagnetic or gravitational wave observations of the near-horizon regime [62]. The contributions to ΔH are plausibly small both far from the black hole and even in the near horizon region, but of course are expected not to be small in the deep black hole interior. More systematic analysis is planned for future work.

It is believed in a large segment of the quantum gravity community that a fundamental description like this may arise from a dual large-N gauge theory; if this is true, it is important to understand and characterize the departures from the bulk LQFT description that this implies, for example as corrections to the hamiltonian (3.24). But, it seems quite likely that the more fundamental description arises in connection with some other mathematical structure on Hilbert space [81, 82], which it is our goal to infer and further describe.

Chapter 4

Conclusion

This thesis explored quantum evolution in the BH setting and also in arbitrary backgrounds. By carefully studying the current standard description of QG, particularly in the semiclassical approximation, we can gain a better understanding of the underlying theory. Though this work does not directly address the question of restoring unitarity, a more complete examination could lead to clues about its modification. The Schrödinger picture of evolution, rather than the usual covariant approach, was used to make deeper connections with ordinary quantum mechanics. The ADM decomposition is complementary to the Schrödinger picture, and allows one to define the Hamiltonian of the system and consider the Hamiltonian dynamics.

In the Schrödinger picture, Hawking radiation was derived using the ADM decomposition. Building off of work done in the two dimensional setting, Hawking radiation was described in terms of both energy eigenmodes and arbitrary regular coordinates. This is achieved by introducing a foliation of "nice slices," which smoothly cross the BH horizon. These foliations still respect the time translation symmetry in the Schwarzschild spacetime, but remove the coordinate singularity at the event horizon. In each of these cases, the Hamiltonians which govern evolution of the system were found, and for the regular modes, the Hamiltonian is also regular everywhere.

Each Hamiltonian was used to define the "Hawking State," or the state described by the evolution of the the vacuum and matter outside the horizon. The first of these Hamiltonians corresponding to energy eigenmodes clearly shows the entanglement structure of the Hawking pairs produced in the vicinity of the horizon. The analysis of the Hawking state in regular coordinates indicates that the transplanckian modes are not the main contribution to the Hawking radiation, and are suppressed in favor of modes generated in an atmosphere of horizon length outside of the BH. The analysis is briefly repeated for the case of an AdS Schwarzschild BH.

The perturbative expansion of the Hamiltonian through the constraints was also explored. Gauge-invariant canonical quantization was introduced, which splits the constraints into positive and negative frequency parts. Having the vacuum be annihilated by the full constraints creates trivial evolution by the Heisenberg equation of motion, so the vacuum is annihilated by half the constraints. The constraint equations were expanded to second order in κ, in an arbitrary background spacetime with a scalar field.

Gravitationally dressed operators were discussed, which are built from the operators from local QFT. Suitable observables in such a system are gauge invariant, since gravity is a gauge theory, so operators that are gravitationally dressed must commute with the constraints. With the perturbative expansion of the constraints and operators, we found dressing functions which satisfy these equations. These are highly nonunique Green's functions, as in the previous cases in flat space, which satisfy the appropriate boundary conditions.

The study of QG in the Schrödinger picture could lead to a better description of the full theory. The Schrödinger picture more naturally follows the structure from quantum mechanics, so developing a version of QG in this approach as opposed to the Heisenberg picture is worthwhile. The role of the transplanckian modes, long a contentious point in the discussion of Hawking radiation, has been settled in all dimensions. Hawking radiation is produced in an atmosphere around the BH extending approximately a distance R outside of it. Furthermore, this work discussed the picture in AdS spacetimes, which has implications for AdS/CFT correspondence.

The construction of gauge invariant observables through gravitationally dressed operators also has implications for QG. This work extended the expansion of the constraints from flat backgrounds to discussions about a general background, but there are a number of other generalizations that can be calculated. The constraint equations, as well as the construction of the gravitational dressing, can be expanded to higher order in κ, to give more terms in the perturbative solutions for gravitationally dressed observables. Also, for both Hawking radiation and dressed observables, the analysis could be repeated with other types of fields.

There are several more outstanding questions and directions to go for new research relating to these topics discussed in the thesis. In the case of Hawking radiation, we considered mostly static spacetimes and eternal black holes, with some discussion of collapse. In reality, BHs are astrophysical objects formed through collapse in highly dynamical systems. There are many questions regarding dynamical evolution in the Schrödinger picture, in particular defining and finding the unitary operator that governs evolution in each of the previous sections. This operator was shown to exist as early as 1975 [132], but an explicit construction involves using the extended phase space [24]. The problems are wider than just defining an operator appropriate time-dependent gravitational collapse, as this unitary operator also describes the Schrödinger picture for expanding cosmological spacetimes.

Another direction would be defining an explicit parameterization of regular modes which extend across the horizon in the first paper mentioned. This has been done in the $D = 2$ dimensional case, which is simpler as there is the total absence of a potential in the equation describing the modes. In the higher dimensional case, it is easy to define incoming modes which are regular at the horizon, but modes which span the space of Hawking partners will involve highly non-trivial coordinates, rather than a simple transformation which respects the time translation symmetry. Though Kruskal coordinates are regular at the horizon, the modes are not orthogonal there, so a novel set of coordinates must be found.

Additionally, further work could explore the inclusion of nonlocal interactions to unitarize BH evolution. These interactions are expected to be small and characterized by perturbative corrections outside the BH, but could be larger for inside the BH, especially near the singularity. In the context of AdS/CFT correspondence, this could be extended from the point of view of the CFT, the study of which could indicate what terms must be added on the gravitational side. Such interactions between the BH and its surroundings that unitarize the evolution could be examined from either perspective.

Appendix A

Radial Equation and Heun Function for D=4

A more detailed picture of energy eigenmodes can be gained by further examining the equation of motion either in the form of (2.35) or (2.41). We will focus our discussion on the case of Eddington–Finkelstein coordinates, equation (2.35), to connect to the analysis in Sec. 2.4. The Schwarzschild coordinate solutions of (2.41) will be related by the transformation (2.40) to the solutions described in this appendix.

The ansatz $u(x^+, r) = re^{-i\omega x^+}y(r)$ can be used to rewrite the differential equation (2.35) as

$$y'' + p_0(r)y' + p_1(r)y = 0 \ , \tag{A.1}$$

where the prime denotes derivatives with respect to r,

$$p_0(r) = \frac{-2i\omega r^{D-2} + 2r^{D-3} + (D-5)R^{D-3}}{r(r^{D-3} - R^{D-3})}, \tag{A.2}$$

and

$$p_1(r) = \frac{1}{r^2\big(r^{D-3} - R^{D-3}\big)}\Big(- 2i\omega r^{D-2} - [l(l+D-3) + (D-2)(D-4)/4]\, r^{D-3} \\ + \big[(D-3) - (D/2-1)^2\big]\, R^{D-3}\Big). \tag{A.3}$$

It can be seen that in arbitrary dimension the differential equation has regular singular points at 0 and $Re^{2\pi in/(D-3)}$ for integers $n = 0, 1, ..., D-4$, and an irregular singular point at infinity. The solutions of (A.1) are unknown in arbitrary dimension, but by defining a rescaled spatial variable $x = r/R$, the $D = 4$ equation can be rewritten as

$$y'' + \Big(\frac{1}{x} + \frac{1-2i\omega R}{x-1} - 2i\omega R\Big)y' + \Big(\frac{-2i\omega Rx - l(l+1)}{x(x-1)}\Big)y = 0\ , \tag{A.4}$$

which is the confluent Heun equation;[1] the case of general D thus represents a generalization of the confluent Heun equation. A standard form for the confluent Heun equation is

$$y'' + \Big(\frac{\gamma}{z} + \frac{\delta}{z-1} + \epsilon\Big)y' + \Big(\frac{\alpha z - q}{z(z-1)}\Big)y = 0\ , \tag{A.5}$$

and we denote the solutions to (A.5) that satisfy the regularity condition $y = 1$ at the singular point $z = 0$ as as $\mathrm{HC}[q, \alpha, \gamma, \delta, \epsilon, z]$. This confluent Heun function is implemented in Mathematica as HeunC, with parameters as in (A.5).

The incoming and outgoing modes of interest can be specified by their behavior in the vicinity of the horizon. The three solutions used to define the basis outlined in Sec. 2.4 are found as follows. First, we let $x = 1-z$ in (A.4), and compare the resulting equation to (A.5); this gives the solution $\tilde{u}_{\omega l}$ regular at the horizon. For the up modes, we then substitute $y \to (-z)^{2i\omega R}y$ into the resulting equation, which gives the confluent Heun equation with different coefficients. The inside mode is found in the same fashion. This

[1]Some references discussing this equation and its relevance to BHs are [133, 134, 135, 136].

results in the the following explicit solutions.

- $\tilde{u}_{\omega l} = r\,\mathrm{HC}[l(l+1) + 2i\omega R, 2i\omega R, -2i\omega R + 1, 1, 2i\omega R, 1 - r/R]$ is the incoming mode, which is regular at the horizon. From (2.40), one sees that the corresponding function $g_{\omega l}$ is not regular.

- $u_{\omega l} = r(r/R-1)^{2i\omega R}\,\mathrm{HC}[l(l+1) - 4\omega^2 R^2, 2i\omega R - 4\omega^2 R^2, 2i\omega R+1, 1, 2i\omega R, 1-r/R]$ is the up mode solution, which is not regular at the horizon, and gives the outgoing Hawking mode.

- $\hat{u}^*_{\omega l} = r(1-r/R)^{2i\omega R}\,\mathrm{HC}[l(l+1) - 4\omega^2 R^2, 2i\omega R - 4\omega^2 R^2, 2i\omega R+1, 1, 2i\omega R, 1-r/R]$ is the inside mode solution. It is also not regular at the horizon, and corresponds to the internally trapped Hawking partner mode. It is defined inside the horizon, for $0 < r/R < 1$.

The analysis in Sec. 2.4 uses the asymptotic behavior $e^{-i\omega x^+}$ in the far past of the incoming solution near infinity, and $e^{-i\omega x^-}$ of the outgoing solution near the horizon, as well as $e^{-i\omega \hat{x}^-}$ of the corresponding partner inside the horizon.

Appendix B

Kruskal coordinates and Rindler region

While the coordinates (x^+, r) are useful for exhibiting the time translation symmetry, Kruskal coordinates $X^\pm$ are useful for exhibiting the Minkowski-like structure of the near-horizon Rindler region. Since the time translation symmetry becomes a scaling (boost) symmetry in these coordinates, this symmetry becomes less transparent in the equations of motion in these coordinates. In this appendix, we collect some basic results on this Kruskal description, in the example of $D = 4$.

As was described in the main text, the Kruskal coordinates are related to the Eddington–Finkelstein coordinates by

$$X^\pm = \pm 2Re^{\pm x^\pm/2R} , \tag{B.1}$$

with a continuation across the horizon in terms of $\hat{x}^-$ such that the vacuum metric, given in (2.64), is regular at the horizon. In the Rindler region $|r - R| \ll R$, or $|X^+X^-| \ll R^2$, the metric is well approximated as that of $M^2 \times S^2$, as seen in (2.65), with local Minkowski spacetime coordinates defined by $X^{\pm} = T \pm X$. The radial coordinate is related to Kruskal coordinates by

$$X^+X^- = 4R^2\left(1 - \frac{r}{R}\right)e^{r/R-1}\ , \tag{B.2}$$

or

$$\frac{r}{R} = 1 + W_0\left(-\frac{X^+X^-}{4R^2}\right)\ , \tag{B.3}$$

with W_0 the Lambert W function, showing that the boundary of the Rindler region is time-dependent in the local Minkowski coordinates.

The equation of motion may also be studied in these coordinates, and for a mode with definite angular momentum

$$\phi_{lm} = u_l \frac{Y_{lm}(\Omega)}{r^{D/2-1}} \tag{B.4}$$

becomes

$$\partial_{X^+}\partial_{X^-}u_l = -\frac{1}{4}\frac{R}{r}e^{-r/R+1}V_l(r)u_l \tag{B.5}$$

with $V_l(r)$ given by (2.37) together with (B.3). Notice that as a result of the latter equation, the effective potential is time-dependent in the locally Minkowski coordinates. While the general form of solutions appears less transparent in these coordinates, the solutions do simplify when restricted to the Rindler region. In this region, (B.5) becomes the 2d massive wave equation,

$$4\partial_{X^+}\partial_{X^-}u_l = -m_l^2 u_l\ , \tag{B.6}$$

with effective mass term

$$m_l^2 = \frac{l(l+1)+1}{R^2}\ . \tag{B.7}$$

This has a basis of solutions (see (2.66))

$$u_{kl} = e^{ikX - i\omega_k T} \quad , \quad \tilde{u}_{kl} = e^{-ikX - i\omega_k T} \tag{B.8}$$

with $\omega_k^2 = k^2 + m_l^2$. These are neither purely outgoing or ingoing, but do become purely outgoing or ingoing, respectively, in the large k limit. One may also compare the equations and solutions in the Eddington–Finkelstein coordinates; from (B.1), the equation (B.5) becomes

$$\partial_{x^+}\partial_{x^-} u_l = -\frac{1}{4}\left(1 - \frac{R}{r}\right) V_l(r) u_l \ , \tag{B.9}$$

and likewise inside the horizon, in terms of $\hat{x}^-$. In the Rindler region, the effective potential in these coordinates vanishes, resulting in solutions of the form $e^{-i\omega x^\pm}$.

Comparing these descriptions provides another way to compute the Bogolubov coefficients, in this high momentum, near horizon limit. The approximate solutions (B.8) are related to the energy eigenmodes (2.44) by

$$u_\omega = \int dk \big(\alpha^+_{\omega k} u_k + \alpha^-_{\omega k} u_k^*\big), \tag{B.10}$$

where the spherical indices have been suppressed to simplify the notation. The Bogolubov coefficients can be calculated by performing the Fourier transform in the usual way

$$\alpha^+_{\omega k} = \frac{1}{4\pi k}(u_k, u_\omega) = \frac{i}{4\pi k} \int dX^- \big(u_k^* \partial_{X^-} u_\omega - \partial_{X^-} u_k^* u_\omega\big), \tag{B.11}$$

where it is useful to take the inner product on a null surface with coordinate X^-. Similarly, the other coefficient is $\alpha^-_{\omega k} = -(u^*_k, u_\omega)/(4\pi k)$. The resulting integrals are

$$\begin{aligned} \alpha^+_{\omega k} &= \frac{1}{2\pi i k}\left(\frac{1}{2ikR}\right)^{2i\omega R}\Gamma(1+2i\omega R)\ , \\ \alpha^-_{\omega k} &= -\frac{1}{2\pi i k}\left(\frac{1}{-2ikR}\right)^{2i\omega R}\Gamma(1+2i\omega R)\ . \end{aligned} \tag{B.12}$$

This is of the same form as Hawking's result from [2], modulo conventions.

Appendix C

Gauge-invariant canonical quantization of electromagnetism

This appendix will illustrate aspects of the quantization of a gauge-invariant theory in the simpler context of QED, with particular focus on the "gauge-invariant canonical quantization" used for gravity in the main text.

The starting point is the gauge-invariant lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_m \ , \tag{C.1}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\mathcal{L}_m$ a matter lagrangian, for example that coupling to a fermion field,

$$\mathcal{L}_m = i\bar{\psi}(\partial_\mu + ieA_\mu)\gamma^\mu \psi - m\bar{\psi}\psi \ . \tag{C.2}$$

The momentum conjugate to A is

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu} \ . \tag{C.3}$$

As a result $\pi^0 = 0$. This can be implemented working with a reduced phase space, where A_0 is no longer treated as a canonical variable.[1] Spatial components of the momenta are the electric field,

$$\pi^i = \partial_0 A^i + \partial^i A^0 = -E^i \ . \tag{C.4}$$

Then the hamiltonian form of the Maxwell part of the action (C.1) becomes

$$S = \int d^4x \left(\pi^i \dot{A}_i - \mathcal{H}\right) \ , \tag{C.5}$$

with Maxwell hamiltonian

$$H = \int d^3x \mathcal{H} = \int d^3x \left(\frac{E^{i2}}{2} + \frac{B^{i2}}{2} + E^i \partial^i A^0\right) \tag{C.6}$$

and $B_i = \epsilon_{ijk} F^{jk}/2$. A constraint arises from varying A^0, which behaves like a Lagrange multiplier, giving

$$\partial_i E^i = j^0 \tag{C.7}$$

where we have included the contribution from the matter current. A^0 remains unfixed, and is taken to be arbitrary.

As described in the main text, there are different options for how to treat quantization: Dirac, covariant gauge "fixing" (breaking), BRST, refined algebraic, and what we will call gauge invariant canonical quantization, and examine here. The starting point for this is the canonical commutators,

$$[\pi^i(x,t), A_j(x',t)] = -[E^i(x,t), A_j(x',t)] = -i\delta^i_j \delta^3(x - x') \ , \tag{C.8}$$

[1] For certain gauges, additional care is needed here; an example is axial gauge, $A_z = 0$, as is further explored in *e.g.* [29].

Then, the constraint (C.7) generates the gauge transformations,

$$[\partial_i E^i(x), A_j(x')] = i\partial_j \delta^3(x - x') \ . \tag{C.9}$$

A_0 remains an arbitrary c-number function, which also behaves like a gauge parameter; the condition $\pi^0 = 0$ is implemented through independence of physical quantities on A_0. Eq. (C.9) also shows that vanishing of the constraint (*e.g.* consider $j^0 = 0$) on the Hilbert space would imply $\langle 0|[\partial_i E^i, A_j]|0\rangle = 0$ and be incompatible with the basic commutators, unless commutators are modified as in the Dirac approach.

The canonical commutation relations (C.8) can be represented in the usual fashion in terms of orthonormal polarization vectors $\epsilon_{i\lambda}(k)$ and annihilation/creation operators $a_{k\lambda}/a^\dagger_{k\lambda}$ as

$$\begin{aligned} A_i(x,0) &= \sum_\lambda \int \widetilde{dk}\, \left[\epsilon_{i\lambda}(k) a_{k\lambda} e^{ikx} + h.c.\right] + a_i(x) \\ E_i(x,0) &= \sum_\lambda \int \widetilde{dk}\, \left[ik\epsilon_{i\lambda}(k) a_{k\lambda} e^{ikx} + h.c.\right] \ , \end{aligned} \tag{C.10}$$

with

$$[a_{k\lambda}, a^\dagger_{k'\lambda'}] = (2\pi)^3 2|k| \delta_{\lambda\lambda'} \delta^3(k - k') \ , \tag{C.11}$$

$\widetilde{dk} = d^3k/(2\pi)^3 2|k|$, and $a_i(x)$ a c-number function arising from gauge invariance. General states can be constructed in the form $\prod(a^\dagger_{\kappa\lambda})|0\rangle$. However, the constraints (focussing on the free theory) are implemented as a physical state condition

$$\partial_i E^{i+}|\psi\rangle = 0 \tag{C.12}$$

in terms of the annihilation piece of $\partial_i E^i$, corresponding to $a_{3k}|\psi\rangle = 0$ in a standard choice of basis.

Evolution can be studied in Schrödinger or Heisenberg pictures. In the former, the evolution with H,

$$i\partial_t|\psi\rangle = H|\psi\rangle \ , \tag{C.13}$$

depends on the arbitrary A_0; for A_0 of compact support this results in a gauge-dependent piece of the evolution of the state,

$$\delta_{A_0}|\psi\rangle = i\delta t \int d^3x \partial_i E^{i-}(x) A_0(x)\,|\psi\rangle \ . \tag{C.14}$$

However, this piece is orthogonal to another physical state $|\psi'\rangle$, by (C.12). Moreover, consider evolution of the matrix element of an operator depending on the canonical variables E_i, A_i, but not on A_0,

$$\partial_t\langle\psi'|O|\psi\rangle = i\langle\psi'|[H,O]|\psi\rangle \ . \tag{C.15}$$

Gauge invariance of O also requires $[\partial_i E^i, O] = 0$, and in that case only the gauge-invariant (A^0 independent) part of H contributes to the evolution (C.15): evolution of matrix elements of such gauge-invariant operators is gauge invariant. In contrast, $\partial_t\langle\psi'|A_i|\psi\rangle$ is not gauge invariant.

Equivalently, evolution can likewise be described by converting to Heisenberg picture. The Heisenberg equations are

$$\partial_t E_i = i[H, E_i] = \nabla \times B_i - j_i \ , \tag{C.16}$$

and

$$\partial_t A_i = i[H, A_i] = -E_i + \partial_i A_0 \tag{C.17}$$

These are supplemented by the constraint (C.7), which as we have seen is *not* treated as an operator equation on physical states. From these equations we find that the evolution $\partial_t E_i$ and also $\partial_t B_i$ are independent of the arbitrary gauge parameter A_0. We can likewise consider gauge invariant operators built by dressing matter operators, and their evolution is also gauge-independent.

Appendix D

Gauge Transformations and diffeomorphisms

This appendix will discuss the role of the constraints in generating gauge transformations, in the canonical formalism used in the main text. For simplicity we will consider gravity coupled to the scalar field with lagrangian (3.2), which is also treated classically in [102]. Canonical data for the scalar field is the field ϕ and its canonical conjugate momentum Π. For the geometry, the phase space variables are $D-1$-dimensional spatial metric q_{ij} and the conjugate momentum P^{ij}, as well as the lapse and shift N, N^i, and their conjugate momenta. However, the latter momenta vanish for the Einstein action, analogously to the vanishing of π^0 in QED (see preceding appendix). As a result, one can commonly work on a reduced phase space where they are set to zero and where N and N^i are no longer treated as canonical variables, like with the electromagnetic case.[1] We will

[1]For further discussion of this see for example [109] or [110]. Note that as with QED, additional care is needed when imposing certain gauges, such as "axial" or Fefferman-Graham gauges, $h_{z\mu} = 0$.

then study the transformation generated by the general superposition of the constraints

$$C[\xi, \xi^i] = \int d^{D-1}x \left(\xi \mathcal{C}_n + \xi^i \mathcal{C}_i\right) \ , \tag{4.1}$$

acting on the reduced phase space. The explicit form of the constraints $\mathcal{C}_n$, $\mathcal{C}_i$ was given in eqs. (3.39), (3.40).

We begin by considering the action on matter.[2] The canonical commutators are

$$[\phi(x), \Pi(x')] = i\delta^{D-1}(x - x') \ , \tag{4.2}$$

where Π is the densitized canonical momentum, satisfying

$$\Pi = \sqrt{q}\partial_n \phi \ . \tag{4.3}$$

These commutators and the explicit form (3.39), (3.40) of the constraints give the commutator

$$i[C[\xi, \xi^i], \phi] = \xi \frac{\Pi}{\sqrt{q}} + \xi^i \partial_i \phi \ . \tag{4.4}$$

The second term is the action of a spatial diffeomorphism on ϕ. If we also use the Heisenberg equation of motion (4.3), we find that the full commutator becomes the Lie derivative with respect to the vector ξ^μ with components

$$\xi^\mu = (\xi, \xi^i) \ ; \tag{4.5}$$

explicitly

$$i[C[\xi, \xi^i], \phi] = \pounds_{\xi^\mu} \phi \ + \xi \left(\frac{\Pi}{\sqrt{q}} - \partial_n \phi \right) \ , \tag{4.6}$$

[2] For recent treatment of canonical quantization of matter on a general background, see [34].

and so the transformation generated by $C[\xi, \xi^i]$ can be identified as a general diffeomorphism for configurations satisfying the equations of motion.

One can likewise compute the commutator of the constraints with Π, which gives

$$i[C[\xi,\xi^i],\Pi] = \partial_i(\xi q^{ij}\sqrt{q}\partial_j\phi) + \partial_i(\xi^i\Pi) \tag{4.7}$$

$$= \partial_\mu(\xi n^\mu \Pi) + \partial_i(\xi^i\Pi) + \partial_\mu\left(\xi\sqrt{q}g^{\mu\nu}\partial_\nu\phi\right) + \partial_\mu[\xi n^\mu(\sqrt{q}\partial_n\phi - \Pi)] \tag{4.8}$$

Once again the second term gives a spatial diffeomorphism. If in addition ξ is identified with the lapse N, and the scalar field equations are satisfied, $C[\xi, \xi^i]$ also generates the action of a diffeomorphism, $\pounds_{\xi^\mu}\Pi$. In the case where ξ^i is also taken to be the shift, eqs. (4.6), (4.7) also give the time derivative defined via (3.4).

The transformations of the spatial metric q_{ij} and the conjugate momentum P^{ij} are similar in structure to those of the matter fields: using the canonical commutation relation (3.25) and the equations of motion, $C[\xi, \xi^i]$ generates diffeomorphisms. Beginning with the commutator of the metric, this results in an expression analogous to (4.4),

$$i[C[\xi,\xi^i],q_{kl}] = \frac{\kappa^2}{\sqrt{q}}\xi\left(P_{kl} - \frac{Pq_{kl}}{D-2}\right) + D_k\xi_l + D_l\xi_k \tag{4.9}$$

$$= -2\xi K_{kl} + D_k\xi_l + D_l\xi_k + \xi\left[2K_{kl} + \frac{\kappa^2}{\sqrt{q}}\left(P_{kl} - \frac{Pq_{kl}}{D-2}\right)\right] \ . \tag{4.10}$$

The terms involving ξ_i once again correspond to a spatial diffeomorphism. The last term, when set to zero, is the trace reverse of the relation of the conjugate momentum (3.14) to the extrinsic curvature, which is a Heisenberg equation of motion in the canonical description. If this equation is satisfied, the RHS of (4.10) is equal to

$$\pounds_{\xi^\mu} q_{kl} = \nabla_k\xi_l + \nabla_l\xi_k \ , \tag{4.11}$$

with the Lie derivative defined by using the D-dimensional expression $q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ for the spatial metric. This is the expected gauge transformation. If the vector ξ^μ is taken to be the time evolution vector (3.4), the equation (4.9) gives the equation of motion for q_{ij}, and can be solved for the expression for the extrinsic curvature (3.7) given in the main text.

Finally, for the ADM conjugate momentum, the commutator with the constraints gives

$$\begin{aligned} i[C[\xi,\xi^i],P^{kl}] = &\frac{2\sqrt{q}}{\kappa^2}(D^kD^l\xi - q^{kl}D^2\xi) - \frac{2\sqrt{q}}{\kappa^2}\xi\Big(R_q^{kl} - q^{kl}\frac{R_q}{2}\Big) \\ &- \frac{\kappa^2}{2\sqrt{q}}\xi\left[2P^{ki}P_i^l - 2\frac{P^{kl}P}{D-2} - \frac{1}{2}q^{kl}\left(P^{ij}P_{ij} - \frac{P^2}{D-2}\right)\right] + \frac{\sqrt{q}}{2}\xi S^{kl} \\ &+ \partial_i(\xi^i P^{kl}) - P^{ik}\partial_i\xi^l - P^{il}\partial_i\xi^k \ , \end{aligned} \tag{4.12}$$

where the tensor $S_{\mu\nu} = q_\mu^\lambda q_\nu^\sigma T_{\lambda\sigma}[\Pi,\phi]$ is the projection of the scalar stress energy tensor (3.18), written in terms of the canonical variables Π and ϕ, and has indices raised with the induced metric. Recalling that P^{kl} is a tensor density, the final line of (4.12) is once again the action of a spatial diffeomorphism, $\pounds_{\xi^i}P^{kl}$. The relationship between the remaining terms proportional to ξ and the normal component of the Lie derivative takes more work to illustrate. We begin by rewriting

$$\begin{aligned} i[C[\xi,\xi^i],P^{kl}] = \pounds_{\xi^i}P^{kl} + \Bigg\{&\frac{2\sqrt{q}}{\kappa^2}(D^kD^l\xi - q^{kl}D^2\xi) - \frac{2\sqrt{q}}{\kappa^2}\xi\Big(R_q^{kl} - q^{kl}\frac{R_q}{2}\Big) \\ &- \frac{\kappa^2}{2\sqrt{q}}\xi\left[2P^{ki}P_i^l - 2\frac{P^{kl}P}{D-2} - \frac{1}{2}q^{kl}\left(P^{ij}P_{ij} - \frac{P^2}{D-2}\right)\right] + \frac{\sqrt{q}}{2}\xi S^{kl}\Bigg\} \ . \end{aligned} \tag{4.13}$$

The Lie derivative in the normal direction can be defined by extending to tensors $P^{\mu\nu}$ and $K^{\mu\nu}$ on the full spacetime, with $K_{\mu\nu} = -q_\mu^\lambda q_\nu^\sigma \nabla_\lambda n_\sigma$. Then, when the Heisenberg equation (3.14) (replacing the Latin with Greek indices) holds, one can show

$$\begin{aligned}
\pounds_{\xi n^\lambda} P^{\mu\nu} = &\frac{2\sqrt{q}}{\kappa^2}\frac{\xi}{N}(D^\mu D^\nu N - q^{\mu\nu} D^2 N) - \frac{2\sqrt{q}}{\kappa^2}\xi\Big(R_q^{\mu\nu} - q^{\mu\nu}\frac{R_q}{2}\Big) \\
&- \frac{\kappa^2}{2\sqrt{q}}\xi\left[2P^{\mu\lambda}P_\lambda^\nu - 2\frac{P^{\mu\nu}P}{D-2} - \frac{1}{2}q^{\mu\nu}\left(P^{\lambda\rho}P_{\lambda\rho} - \frac{P^2}{D-2}\right)\right] \\
&+ \frac{2\sqrt{q}}{\kappa^2}\xi q^{\mu\lambda}q^{\nu\rho}\left(R_{\lambda\rho} - g_{\lambda\rho}\frac{R}{2}\right) \\
&- Nn^\mu P^{\nu\lambda} D_\lambda\left(\frac{\xi}{N}\right) - Nn^\nu P^{\mu\lambda} D_\lambda\left(\frac{\xi}{N}\right) \ ,
\end{aligned} \tag{4.14}$$

where we have also used the Gauss relation to simplify. Note that the first term is related to the acceleration, $a^\mu = n^\nu \nabla_\nu n^\mu = D^\mu \ln N$, and could be set to zero with the choice of Gaussian normal coordinates. The second to last line of (4.14) is proportional to the projected components of the D-dimensional Einstein tensor, and when the projected components of the Einstein equation hold can be replaced by $S^{\mu\nu}$. The last two terms have normal components to the surface. When the equations of motion hold, and if we again take $\xi = N$, the term of (4.13) in braces matches the Lie derivative with respect to Nn^μ, and so the RHS of (4.13) reduces to the Lie derivative with respect to ξ^μ. Additionally, the time evolution (3.28) for the ADM conjugate momentum may be found from (4.13) if ξ^μ is taken to be given in terms of the lapse and shift by (3.4).

In conclusion the gauge transformations generated by the constraints acting on the reduced phase space variables q_{ij}, P^{ij} correspond to the diffeomorphisms if the equations of motion hold, with ξ identified as the lapse.

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