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Growth and Convergence with a Normalized CES Production Function and Human Capital

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Growth and Convergence with a Normalized CES Production Function and Human Capital

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Gerald Eric Daniels Jr.

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ABSTRACT OF THE DISSERTATION

Growth and Convergence with a Normalized CES Production Function and Human Capital

by

Gerald Eric Daniels Jr.

Doctor of Philosophy, Graduate Program in Economics
University of California, Riverside, June 2013
Dr. R. Robert Russell, Co-Chairperson
Dr. Jang-Ting Guo, Co-Chairperson

Employing a neoclassical growth model with a constant elasticity of substitution production function, I examine the implications of assuming different values of the elasticity of substitution for the steady state growth path, growth threshold, and speed of convergence. Unlike earlier studies along these lines, I incorporate human capital, along with physical capital and raw labor, as a third input in the production function, thus eschewing the common assumption of "perfect substitutability" between human capital and labor inputs. I find that a higher elasticity of substitution leads to a higher steady state level for physical capital and human capital per effective unit of labor. For a high enough level, the elasticity of substitution can lead to permanent growth. Similarly, for a low enough level, the elasticity of substitution can lead to permanent decline. A higher elasticity of substitution can lead to a faster speed of convergence when the baseline level of capital per effective unit of labor is greater than the steady state level. I estimate the normalized production function and find estimates for the elasticity of substitution that range between 0.7331 and 0.82.

Models that employ the normalized aggregate production function have primarily focused on explaining the importance of the elasticity of substitution between factors for

production for exogenous and constant savings decisions. The balanced growth paths for these models are invariant to the level of the elasticity of substitution. Thus, relaxing the assumption of constant and exogenous savings rates for physical capital and human capital, I employ a neoclassical growth model with endogenous consumption and saving decisions to examine the effects of the level of the elasticity of substitution on the balanced growth rate and speed of converge. I find that the effects of the elasticity of substitution on the aforementioned depend upon the initial levels of the factors of production.

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Chapter 1

Growth and Convergence with a Normalized CES Production Function and Additive Human Capital

1.1 Introduction

The importance of human capital on economic growth has long been emphasized in the economic growth literature. Consider two classic works that emphasize human capital and its accumulation as important feature of economic development: Lucas et al. (1988) and Mankiw et al. (1992). These works construct neoclassical growth models that apply a Cobb–Douglas aggregate production function, for the goods market, and assume accumulation of human capital overtime. Although, there are many differences in the models, like many papers that include human capital, the application of the Cobb–Douglas aggregate production function is common to both. A drawback to the reliance on the Cobb–Douglas production function is that the elasticity of substitution between a pair of inputs is unity, inherently, and does not emphasize the importance of the level of elasticity of substitution for economic growth. Theoretical works of de La Grandville (1989), Klump and de La Grandville (2000), and Klump and Preissler (2000) have emphasized the importance of the level of elasticity of substitution for economic growth; in addition, recent empirical papers, such as Antràs (2004) and Klump et al. (2007), have stressed that the aggregate production function is not Cobb–Douglas, at least in the context of the United States. Since the assumption of unity for the elasticity of substitution is dubious, I relax it and apply an aggregate constant elasticity of substitution (CES, henceforth) production function to a neoclassical growth model that includes human capital, while eschewing "perfect substitutability" between human capital and labor inputs. Next, I examine the effects of the level of the elasticity of substitution on the balanced growth steady state and speed of convergence.

The use of the Cobb–Douglas aggregate production function has long been debated in the dynamic macroeconomic literature. Berndt (1976) has supported the use of the Cobb–Douglas specification as an aggregate production function, for U.S. data, while others, such as Antràs (2004), have argued that the United States is well described by aggregate Cobb–Douglas production function[1]. Furthermore, Antràs finds that the assumption of Hicks neutral technology biases the functional form of the aggregate production function. He goes on to suggest that the production function should assume CES with biased technology. de La Grandville (1989) and Klump and Preissler (2000) corroborate Antàs's analysis and find that a higher level of elasticity of substitution leads to a higher steady state level of capital per worker. Other works that support the use of the CES aggregate production

[1] Antràs (2004) explains that the U.S. aggregate production function can be misleading Cobb–Douglas by assuming Hicks neutral technology.

function as an alternative to the Cobb–Douglas have estimated values for the elasticity of substitution that typically range from 0.5 to 0.7[2], while, León-Ledesma et al. (2010) have identified the conditions under which identification of the elasticity of substitution and biased technological change is feasible and robust.

Focusing on the literature that suggests that the elasticity of substitution is not one, I assume a CES aggregate production function with three factors of production: physical capital, raw labor, and human capital. Unlike most studies that augment labor with human capital, following Mankiw et al. (1992), I include human capital into the production function as a third input. I avoid augmenting labor with human capital, to avoid the assumption of labor and human capital being "perfect substitutes" in production[3]. To examine the importance of the elasticity of substitution for an economic growth model that includes human capital, I apply the de La Grandville (1989) normalization approach to the CES production function. This approach facilitates the identification of the effects of elasticity of substitution on economic growth and convergence by defining several model parameters as functions of the elasticity of substitution.

The chapter proceeds as follows. Section 1.2 describes the normalization of the CES production function, followed by Section 1.3 where I apply the normalized CES to an extended neoclassical growth model. Section 1.4 examines the effects of the elasticity of substitution on the steady state levels of physical capital and human capital per effect unit of labor and growth thresholds which is a constraint that will provide perpetual growth (or permanent decline) for all steady state variables if violated. Section 1.5 examines the implied speed of convergence of this model. Lastly, I conclude and provide additional remarks.

[2]See Klump et al. (2007) for more details.

[3]Labor and human capital being "perfect substitutes" in production imply that increasing labor by one percent has the same effect on production as increasing human capital by one percent, see Barro and Sala-I-Martin (2004, p. 240).

1.2 CES Production Function with Additive Human Capital

The classic work of Solow (1956) examines long-run growth in relation to numerous specifications of the aggregate production function. One well known specification, proposed in his paper, is the CES production function. Solow's specification includes two factors of production, physical capital and labor, and does not include technological progress. To modify the CES aggregate production function to include three factors of production, I incorporate physical capital, human capital, and raw labor inputs in the production function. Avoiding the assumption of human capital and labor being "perfect substitutes" in production, the CES production function is given by the following[4]:

$$Y(t) = [\alpha K(t)^{\psi} + \beta H(t)^{\psi} + (1-\alpha-\beta)(A(t)L(t))^{\psi}]^{\frac{1}{\psi}} \tag{1.1}$$

where K denotes physical capital, H denotes human capital, and L is labor. The model parameters are α, β, and ψ, where $\psi = \frac{\sigma-1}{\sigma}$ is the substitution parameter and σ is the elasticity of substitution. I follow Mankiw et al. (1992) and assume labor and the level of technology at time t are determined by $L(t) = L_0 e^{nt}$ and $A(t) = A_0(t)e^{gt}$, respectively, where the initial level of labor and technology are given by L_0 and A_0, respectively. Also, the growth rates of labor and technology are assumed to be exogenous at population growth rate, n, and growth rate, g, respectively. I assume only labor augmenting technological change, for two purposes: First, this assumption is consistent with the three factor aggregate production function of Mankiw et al. (1992), which is a special case of (1.1) when $\sigma = 1$ ($\psi = 0$)[5]. Second, it is necessary for technology to be only labor augmenting for a steady state with constant growth rates[6].

[4]Assuming a more general specification for the aggregate production function could be useful for extending this model to include a varying elasticity of substitution.

[5]The Mankiw et al. (1992) functional form for the three-factor aggregate production function is given by $Y(t) = K(t)^{\alpha}H(t)^{\beta}[A(t)L(t)]^{1-\alpha-\beta}$.

[6]See the Barro and Sala-I-Martin (2004, p. 78–80) proof that technological progress must be labor augmenting.

Normalizing the CES

Building upon Solow's classic work, de La Grandville (1989), further delves into the CES production function in conjunction with economic growth. De la Grandville explains the importance of the elasticity of substitution by suggesting that the parameters of the CES could be endogenously influenced by the elasticity of substitution. This is achieved by writing the model parameters as a function of the elasticity of substitution and arbitrary baseline values for capital and labor—or the capital-labor ratio—and the marginal rate of technical substitution for the two inputs. This technique, in subsequent papers, is referred to as normalizing the CES production function[7]. Klump and Preissler (2000) apply the normalization approach developed by de La Grandville (1989). They find that for a neoclassical growth model a higher elasticity of substitution implies a higher steady state level of income per capita. Klump and de La Grandville (2000) and Klump and Preissler (2000) further develop this methodology, again for two factors of production, each with the goal of reinforcing the importance of the elasticity of substitution on economic growth.

Since normalizing the CES facilitates the identification of the effects of the elasticity of substitution, this paper applies de la Grandville's normalization of the CES to the three-factor production function, (1.1). Following Klump and Preissler (2000), I assume initial values, as baseline values, for physical capital, human capital, and labor denoted by K_0, H_0, and L_0, respectively. I assume initial values for the marginal rate of technical substitution for labor and physical capital, labor and human capital, and physical and human capital, denoted by μ_0, γ_0, ρ_0, respectively. Normalizing the parameters for the aggregate

[7]See Klump and Preissler (2000) and Klump et al. (2007)

production function, (1.1), yields the following[8]:

$$\begin{aligned}
\alpha(\sigma) &= \frac{\rho_0 K_0^{1-\psi} Y_0^{\psi}}{(\rho_0 K_0 + H_0) + \rho_0 \mu_0 L_0} = \pi_K^0 \left(\frac{Y_0}{K_0}\right)^{\psi}, \\
\beta(\sigma) &= \frac{H_0^{1-\psi} Y_0^{\psi}}{(\rho_0 K_0 + H_0) + \rho_0 \mu_0 L_0} = \pi_H^0 \left(\frac{Y_0}{H_0}\right)^{\psi}, \quad \text{and} \\
A_0(\sigma) &= \left(\frac{\rho_0 \mu_0 L_0^{1-\psi} Y_0^{\psi}}{(\rho_0 K_0 + H_0) - (\rho_0 K_0^{1-\psi} + H_0^{1-\psi}) Y_0^{\psi} + \rho_0 \mu_0 L_0}\right)^{\frac{1}{\psi}} \\
&= \left(\frac{\left(1 - (\pi_K^0 + \pi_H^0)\right)\left(\frac{Y_0}{L_0}\right)^{\psi}}{1 - \left(\pi_K^0 \left(\frac{Y_0}{K_0}\right)^{\psi} + \pi_H^0 \left(\frac{Y_0}{H_0}\right)^{\psi}\right)}\right)^{\frac{1}{\psi}}
\end{aligned}$$

where pi_K^0 and pi_H^0 are the factor share of physical and human capital at an arbitrary baseline value. Incorporating the normalized parameters, I rewrite the aggregate production function, (1.1), by replacing the model parameters, α and β, and the initial level of technology, A_0, by their normalized counterparts. Thus, the normalized CES aggregate production function is given by:

$$\begin{aligned}
Y(t) &= [\alpha(\sigma)K(t)^{\psi} + \beta(\sigma)H(t)^{\psi} + (1 - \alpha(\sigma) - \beta(\sigma))(\widetilde{A}(\sigma,t)L(t))^{\psi}]^{\frac{1}{\psi}} \quad \text{and} \\
y(t) &= [\alpha(\sigma)k(t)^{\psi} + \beta(\sigma)h(t)^{\psi} + (1 - \alpha(\sigma) - \beta(\sigma))]^{\frac{1}{\psi}},
\end{aligned} \tag{1.2}$$

where $\widetilde{A}(\sigma,t) = A_0(\sigma)e^{gt}$ is the normalized level of technology. Also, further transforming the production function, the normalized CES is written in terms of effective units, where physical capital, human capital, and output, all per effective unit of labor, are denoted by $k(t) = K(t)/A(t)L(t)$, $h(t) = H(t)/A(t)L(t)$ and $y(t) = Y(t)/A(t)L(t)$, respectively[9].

[8]See Appendix A.1 for the derivation.

[9]Klump and Preissler (2000) explain that the normalized CES represents a family of aggregate production functions that share the same baseline values.

1.3 Neoclassical Growth Model

I employ the normalized CES aggregate production function, (1.2), in a one sector economy that produces a homogenous good, Y, under perfect competition[10]. Following Mankiw et al. (1992), I assume exogenous savings rates s_k and s_h, where the marginal propensity to save $s = s_k + s_h$ and the total fraction of income saved is sY. The goods market is assumed to be in equilibrium when investment in each type of capital is equal to the savings for each type of capital ($s_kY = I_K$ and $s_hY = I_H$). As in Mankiw et al., both types of capital are assumed to accumulate in a similar fashion and depreciate at the same rate. Therefore, capital accumulation equations for physical capital and human capital per effective unit are determined by:

$$\dot{k}(t) = s_k y(t) - (n + g + \delta)k(t) \quad \text{and} \tag{1.3}$$

$$\dot{h}(t) = s_h y(t) - (n + g + \delta)h(t). \tag{1.4}$$

Notice, from equation (1.3), that an increase in physical capital per effective unit of labor occurs when savings in physical capital is greater than the break-even investment, $(n + g + \delta)k(t)$. Similarly, from equation (1.4) an increase in human capital per effective unit of labor occurs when savings in human capital are greater than the break-even investment, $(n + g + \delta)h(t)$.

For now, I focus on the balanced growth steady state with $\dot{k}(t) = \dot{h}(t) = 0$, where break-even investment for each type of capital per effective unit of labor is equal to its respective savings. The balanced growth steady state levels for physical capital, human capital, and output, all in terms of effective unit of labor, are identified by $k^\star$, $h^\star$, and $y^\star$,

[10] Firms problem $\pi = P \cdot F(K, H, L) - R_K \cdot K - R_H \cdot H - W \cdot L$ where P is the price index for all final goods and services, R_K rental rate of physical capital, R_H rental rate of human capital, and W wage rate

respectively, and the steady state level for each is determined by[11]:

$$
\begin{aligned}
k^{\star} &= \left(\frac{s_k^{\psi}(1-\alpha(\sigma)-\beta(\sigma))}{(n+g+\delta)^{\psi}-\alpha(\sigma)s_k^{\psi}-\beta(\sigma)s_h^{\psi}}\right)^{\frac{1}{\psi}} = \left(\frac{1-\alpha(\sigma)-\beta(\sigma)}{\alpha(\sigma)}\left(\frac{\pi_K^{\star}}{1-\pi_K^{\star}-\pi_H^{\star}}\right)\right)^{\frac{1}{\psi}}, \\
h^{\star} &= \left(\frac{s_h^{\psi}(1-\alpha(\sigma)-\beta(\sigma))}{(n+g+\delta)^{\psi}-\alpha(\sigma)s_k^{\psi}-\beta(\sigma)s_h^{\psi}}\right)^{\frac{1}{\psi}} = \left(\frac{1-\alpha(\sigma)-\beta(\sigma)}{\beta(\sigma)}\left(\frac{\pi_H^{\star}}{1-\pi_K^{\star}-\pi_H^{\star}}\right)\right)^{\frac{1}{\psi}}, \quad \text{and} \\
y^{\star} &= \left(\frac{(n+g+\delta)^{\psi}(1-\alpha(\sigma)-\beta(\sigma))}{(n+g+\delta)^{\psi}-\alpha(\sigma)s_k^{\psi}-\beta(\sigma)s_h^{\psi}}\right)^{\frac{1}{\psi}} = \left(\frac{1-\alpha(\sigma)-\beta(\sigma)}{1-\pi_K^{\star}-\pi_H^{\star}}\right)^{\frac{1}{\psi}}.
\end{aligned}
\tag{1.5}
$$

where factor shares for physical capital, $\pi_K^{\star}$, and human capital, $\pi_H^{\star}$, are given by:

$$\pi_K^{\star} = \alpha(\sigma)\left(\frac{k^{\star}}{y^{\star}}\right)^{\psi} = \alpha(\sigma)\left(\frac{s_k}{n+g+\delta}\right)^{\psi} \quad \text{and} \tag{1.6}$$

$$\pi_H^{\star} = \beta(\sigma)\left(\frac{h^{\star}}{y^{\star}}\right)^{\psi} = \beta(\sigma)\left(\frac{s_h}{n+g+\delta}\right)^{\psi}. \tag{1.7}$$

The existence and stability of the steady state require that all inputs are essential for production. Thus, all factors must receive a positive share of income; see Barro and Sala-I-Martin (2004, p. 19). Due to the concavity of $f(k,h)$, where $y \equiv f(k,h)$, and the linearity of $(n+g+\delta)k$ and $(n+g+\delta)h$ the steady state exists and is unique.

Elasticity of Substitution, Growth Thresholds and Long-term Growth Rates

Mankiw et al. (1992) guarantee the existence of a balanced growth steady state, for the three input Cobb–Douglas production function, by assuming $\alpha + \beta < 1$, which implies decreasing returns to all capital[12]. Since their production function is a limiting case of a CES production, (1.1), when $\sigma = 1$ ($\psi = 0$), the existence of a balanced growth steady state will also be guaranteed by $\alpha + \beta < 1$ and $\sigma = 1$. The same holds for the normalized production function. From (1.6) and (1.7), for $\sigma = 1$, the capital shares of income are given by $\alpha = \pi_K^{\star}$ and $\beta = \pi_H^{\star}$. Thus, the existence of the steady state, for elasticity of substitution equal to unity, implies $\pi_K^{\star} + \pi_H^{\star} < 1$. When $\sigma \neq 1$, the existence and stability of the steady state will hinge upon the parameters of the production function as well as

[11]See Appendix A.2 for derivations.

[12]For $\alpha + \beta = 1$ there is no steady state; see Mankiw et al. (1992, p. 417).

the rate depreciation of physical capital and human capital per effective units of labor and the saving rates for physical capital and human capital. Also, for suitable values for the elasticity of substitution, it is possible to remain or leave the domain of steady states, which I will refer to as the growth threshold[13]. I distinguish between cases of "high" elasticity of substitution, $\sigma > 1$, and low elasticity of substitution, $\sigma < 1$, to facilitate my analysis of the growth thresholds.

Before proceeding to the analysis, I must point out a criticism of Mankiw et al. (1992). Their model does not equate the net returns to human and physical capital. It has been argued by Barro and Sala-I-Martin (2004) that it is reasonable to think that households will invest in the capital that has the highest net return, therefore, the returns between the two types of capital should be equated. The result of this assumption is[14]

$$h(t) = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{1-\psi}} k(t). \tag{1.8}$$

After incorporating this restriction into the model, for a high elasticity of substitution, $\sigma > 1$, the existence condition can be determined from (1.3) and (1.8) and is given by:

$$\frac{(n+g+\delta)}{s_k} > \left\{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\right\}^{1/\psi} = \lim_{k\to\infty} f'(k) = \lim_{k\to\infty} f(k)/k, \tag{1.9}$$

$$\text{where} \quad \frac{d\left\{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\right\}^{1/\psi}}{d\sigma} > 0. \tag{1.10}$$

Figure A.1 depicts the condition for the existence of a steady state for a high elasticity of substitution. The existence condition, (1.9), is determined by the asymptote of $f(k)/k$ and the depreciation curve over the savings rate for physical capital, $(n+g+\delta)/s_k$[15]. The existence of the steady state for a high elasticity of substitution will not depend on

[13] Klump and Preissler (2000) provide a similar growth threshold for their model that includes two factors of production.

[14] (1.8) is from r = $R_H - \delta = R_K - \delta$ in equilibrium. Also, (1.8) requires that $\frac{s_k}{s_h} = \frac{H_0}{K_0}$, in the steady state.

[15] See Appendix A.3.

very low capital per effective unit of labor since the $\lim_{k\to 0} f(k)/k = \infty$. If the condition for the existence of the steady state, (1.9), is not upheld under high elasticity of substitution, perpetual growth will occur. Thus, when $\lim_{k\to\infty} f(k)/k$ is greater than the depreciation curve over the savings rate for physical capital, the long-term growth rates for physical and human capital per effective unit of labor are given by

$$\begin{aligned}
\frac{\dot{k}}{k} &= \lim_{k\to\infty} s_k f(k)/k - (n+g+\delta) \\
&= s_k \left\{ \alpha(\sigma) + \beta(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{\psi}{1-\psi}} \right\}^{1/\psi} - (n+g+\delta) > 0, \quad \text{and} \qquad (1.11) \\
\frac{\dot{h}}{h} &= \lim_{h\to\infty} s_k f(k)/k - (n+g+\delta) \\
&= s_h \left\{ \alpha(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{\psi}{1-\psi}} + \beta(\sigma) \right\}^{1/\psi} - (n+g+\delta) > 0. \qquad (1.12)
\end{aligned}$$

For a low elasticity of substitution $\sigma < 1$ the existence condition can also be determined from (1.3) and (1.8) and is given by

$$\frac{(n+g+\delta)}{s_k} < \left\{ \alpha(\sigma) + \beta(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{\psi}{1-\psi}} \right\}^{1/\psi} = \lim_{k\to 0} f'(k) = \lim_{k\to 0} f(k)/k. \qquad (1.13)$$

Figure A.2 depicts the condition for the existence of the steady state for a low elasticity of substitution; this condition depends on the intercept $f(k)/k$ and the break-even investment over capital, $(n+g+\delta)$. If condition (1.13) does not hold for a low elasticity of substitution, then a permanent decline will occur until the trivial steady state of $k^\star = h^\star = 0$ is reached.

Endogenous Growth

I have mentioned two possibilities for allowing for endogenous growth when I include the normalized aggregate CES production function in the neoclassical growth model described in Section 1.3. First, under unity of the elasticity of substitution, $\sigma = 1$, endogenous growth can occur for $\alpha + \beta = 1$. Under this assumption the steady-state does not exist. The second case, which allows for endogenous growth is when:

$$\sigma > 1 \quad \text{and} \quad \frac{(n+g+\delta)}{s_k} < \left\{ \alpha(\sigma) + \beta(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{\psi}{1-\psi}} \right\}^{1/\psi}.$$

For this case, the elasticity of substation does not effect the growth rates of physical capital, human capital, and output all per effective unit of labor[16]. Therefore, this model is insufficient to examine the effects of the elasticity of substitution for a model with endogenous growth and high elasticity of substitution.

1.4 Change in Elasticity of Substitution and the Steady State

The steady state levels of physical capital and human capital per effective unit of labor are both influenced by the elasticity of substitution. Klump and Preissler (2000, p. 49) have concluded "that within one family of (normalized) CES functions, an increase in the elasticity of substitution has a positive effect on the level of the steady state"; this is in the context of a two factor production function with physical capital and labor as inputs. When I consider effects of the elasticity of substitution on the normalized CES production function, (1.2), I find that—for both physical and human capital per effective unit of labor—an increase in the elasticity of substitution has positive effects on their steady state levels. Also, the effects of the elasticity of substitution on the steady state level of

[16]See Appendix A.7 for the proof.

physical and human capital per effective unit of labor are given by[17]:

$$\frac{dk^\star}{d\sigma} = \frac{-1}{\sigma^2}\frac{k^\star}{\psi^2}\left\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)\right.$$
$$\left. + \left(\frac{\ln y_0^\psi - \alpha(\sigma)\ln k_0^\psi - \beta(\sigma)\ln h_0^\psi}{y^{\star\psi}}\right) + \pi_K^\star \ln\frac{\pi_K^0}{\pi_K^\star} + \pi_H^\star \ln\frac{\pi_H^0}{\pi_H^\star}\right\} > 0 \quad \text{and} \tag{1.14}$$

$$\frac{dh^\star}{d\sigma} = \frac{-1}{\sigma^2}\frac{h^\star}{\psi^2}\left\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)\right.$$
$$\left. + \left(\frac{\ln y_0^\psi - \alpha(\sigma)\ln k_0^\psi - \beta(\sigma)\ln h_0^\psi}{y^{\star\psi}}\right) + \pi_K^\star \ln\frac{\pi_K^0}{\pi_K^\star} + \pi_H^\star \ln\frac{\pi_H^0}{\pi_H^\star}\right\} > 0, \tag{1.15}$$

where strict equality of the derivatives will hold for the steady state levels of human capital and physical capital per effective unit of labor not equal to their respective baseline values.

1.5 Speed of Convergence

To examine the speed with which an economy heads toward its steady state, I focus on the conditional convergence to the steady state, which assumes that an economy's steady state depends on its own model parameters such as the savings rates for both types of capital. From a log-linear approximation of output per effective unit of labor in the neighborhood of the balance growth steady state given by

$$\frac{\dot{y}(t)}{y(t)} = \frac{d\ln y(t)}{dt} \approx -\lambda \ln\left(y(t)/y^\star\right) \tag{1.16}$$

where

$$\lambda = (n+g+\delta)(1-\pi_K^\star-\pi_H^\star), \tag{1.17}$$

I derive the speed of convergence, λ, which explains the inverse relationship between the growth rate of output per effective unit of labor and its initial level. From the log-linear approximation in the neighborhood of the steady-state, I can examine the path and speed

[17] See Appendix A.4 for derivations.

with which the initial level output per effective unit of labor heads to the steady state. The effects of the elasticity of substitution on the speed of convergence for the normalized CES, (1.2), are determined by[18]

$$\frac{d\lambda}{d\sigma} = -\frac{(n+g+\delta)}{\sigma^2}\left(\pi_K^\star + \pi_H^\star\right)\ln\frac{y_0/k_0}{y^\star/k^\star}\begin{cases} >0 \iff k^\star < k_0 \\ =0 \iff k^\star = k_0 \\ <0 \iff k^\star > k_0 \end{cases} \tag{1.18}$$

Due to the curvature of the production function, from (1.18) it is clear that, when the baseline level of capital per effective unit of labor is greater than the steady state level, an increase in the elasticity of substitution leads to an increase in the speed of convergence. An alternative explanation is that when capital is relatively less scarce than effective labor when compared to the state steady levels then a higher elasticity of substitution leads to a higher speed of convergence.

Model	Speed of Convergence (λ)
CES with Human capital	$(n+g+\delta)(1-\pi_K^\star-\pi_H^\star)$
Mankiw et al. (1992)	$(n+g+\delta)(1-\alpha-\beta)$
Klump and Preissler (2000)	$(n+g+\delta)(1-\pi_K^\star)$

Table 1.1: Speed of convergences, a model comparaision

As mentioned in previous sections, the three factor Cobb-Douglas production function is a limiting case of the production function, (1.2). This is also the case for the the log-linear approximation of the growth rate of output per effect unit of labor for $\sigma = 1$ ($\psi = 0$). The corresponding speed of convergence for this model, as well as for the Mankiw et al. (1992) model, is determined by the depreciation curve and the labor share of income. Similarly, Klump and Preissler (2000) determine the speed of convergence for their two

[18]Please see Appendix A.5. Also these results are consistent with Klump and Preissler (2000) for their two factor CES

factor production function and examine the effects of the elasticity of substitution. Their specification is not a special case of (1.1) since they assume Hick's neutral technological progress. Although, their speed of convergence is not a special case of normalized CES production function, (1.2), there are analogous impacts of the depreciation curve and the labor share of income on the speed of convergence. To summarize the set of speed of convergence mentioned, I provide table 1.5.

1.6 Conclusions

This chapter employs a neoclassical growth model with a normalized aggregate two factor CES production function. Following (Lucas et al., 1988), I assume that human capital augments simple labor. The normalized aggregate CES production function includes the two factor Cobb-Douglas production function as a special case. By introducing the normalized CES in this context, I determine the comparative statics for a change in the elasticity of substitution for the balance growth path steady state variables, growth thresholds, and speed of convergence. This paper finds that constraints such that higher elasticity of substitution leads to a higher steady state level for the average product of capital, consumption-physical capital ratio, and the speed of convergence.

Appendix A

Appendix for Growth and Convergence with a Normalized CES Production Function and Additive Human Capital

A.1 Normalizing the CES with Human Capital

Marginal Products:

$$F_K(K(t), H(t), L(t)) = C(\cdot)^{1/\psi - 1}\, \alpha K(t)^{\psi - 1}, \tag{A.1}$$

$$F_H(K(t), H(t), L(t)) = C(\cdot)^{1/\psi - 1}\, \beta H(t)^{\psi - 1}, \quad \text{and} \tag{A.2}$$

$$F_L(K(t), H(t), L(t)) = C(\cdot)^{1/\psi - 1}\, (1 - \alpha - \beta) A(t)^{\psi} L(t)^{\psi - 1}. \tag{A.3}$$

Notation

Marginal Rate of Technical Substitution:

$C(\cdot) \equiv$	$\alpha(\sigma)K(t)^{\psi} + \beta(\sigma)H(t)^{\psi} + (1-\alpha(\sigma)-\beta(\sigma))(\widetilde{A}(\sigma,t)L(t))^{\psi}$,
K_0, H_0, L_0	Baseline values of Capital stocks and Labor,
$k_0 \equiv \frac{K_0}{A_0 L_0}$ and $h_0 \equiv \frac{H_0}{A_0 L_0}$	Physical- and human- capital per efficient unit, and
μ_0, γ_0, ρ_0	Marginal rate of technical substitution for L-K, L-H, and K-H respectively.

$$\frac{F_L(K_0,H_0,L_0)}{F_K(K_0,H_0,L_0)} = \frac{1-\alpha-\beta}{\alpha} A_0^{\psi} \left(\frac{K_0}{L_0}\right)^{1-\psi} = \mu_0, \tag{A.4}$$

$$\Leftrightarrow \alpha = (1-\beta)\frac{A_0^{\psi} K_0^{1-\psi}}{A_0^{\psi} K_0^{1-\psi} + \mu_0 L_0^{1-\psi}}, \tag{A.5}$$

$$\frac{F_L(K_0,H_0,L_0)}{F_H(K_0,H_0,L_0)} = \frac{1-\alpha-\beta}{\beta} A_0^{\psi} \left(\frac{H_0}{L_0}\right)^{1-\psi} = \gamma_0, \tag{A.6}$$

$$\Leftrightarrow \beta = (1-\alpha)\frac{A_0^{\psi} H_0^{1-\psi}}{A_0^{\psi} H_0^{1-\psi} + \gamma_0 L_0^{1-\psi}}, \quad \text{and} \tag{A.7}$$

$$\frac{F_K(K_0,H_0,L_0)}{F_H(K_0,H_0,L_0)} = \frac{\alpha}{\beta} \left(\frac{H_0}{K_0}\right)^{1-\psi} = \rho_0$$

$$\Leftrightarrow \beta = \frac{\alpha}{\rho_0} \left(\frac{H_0}{K_0}\right)^{1-\psi}. \tag{A.8}$$

From (A.4), (A.6), and (A.8)

$$\mu_0 = \frac{\gamma_0}{\rho_0} \tag{A.9}$$

i.e.

$$\frac{F_L(K_0,H_0,L_0)}{F_K(K_0,H_0,L_0)} = \frac{\frac{F_L(K_0,H_0,L_0)}{F_H(K_0,H_0,L_0)}}{\frac{F_K(K_0,H_0,L_0)}{F_H(K_0,H_0,L_0)}}.$$

Normalizing the Parameters

From (A.8) into (A.5),

$$\alpha = \frac{\rho_0 A_0^{\psi} K_0^{1-\psi}}{\rho_0 (A_0^{\psi} K_0^{1-\psi} + \mu_0 L_0^{1-\psi}) + A_0^{\psi} H_0^{1-\psi}}, \tag{A.10}$$

$$\beta = \frac{A_0^{\psi} H_0^{1-\psi}}{\rho_0 (A_0^{\psi} K_0^{1-\psi} + \mu_0 L_0^{1-\psi}) + A_0^{\psi} H_0^{1-\psi}}, \quad \text{and} \tag{A.11}$$

$$1-\alpha-\beta = \frac{\rho_0 \mu_0 L_0^{1-\psi}}{\rho_0 (A_0^{\psi} K_0^{1-\psi} + \mu_0 L_0^{1-\psi}) + A_0^{\psi} H_0^{1-\psi}} \tag{A.12}$$

Solving for the baseline level of technology, A_0, as a function of the elasticity of substitution, I find that:

$$
\begin{aligned}
Y_0 &= [\alpha K_0^{\psi} + \beta H_0^{\psi} + (1-\alpha-\beta)(A_0 L_0)^{\psi}]^{\frac{1}{\psi}} && \text{(A.13)} \\
Y_0^{\psi} &= \frac{\rho_0 A_0^{\psi} K_0 + A_0^{\psi} H_0 + \rho_0 \mu_0 A_0^{\psi} L_0}{\rho_0 (A_0^{\psi} K_0^{1-\psi} + \mu_0 L_0^{1-\psi}) + A_0^{\psi} H_0^{1-\psi}} \\
\Rightarrow \quad A_0 &= \left(\frac{\rho_0 \mu_0 L_0^{1-\psi} Y_0^{\psi}}{(\rho_0 K_0 + H_0) - (\rho_0 K_0^{1-\psi} + H_0^{1-\psi}) Y_0^{\psi} + \rho_0 \mu_0 L_0} \right)^{\frac{1}{\psi}} = A_0(\sigma). && \text{(A.14)}
\end{aligned}
$$

From (A.14) and (A.10), $\alpha(\sigma)$ can be written as

$$\alpha(\sigma) = \frac{\rho_0 K_0^{1-\psi} Y_0^{\psi}}{(\rho_0 K_0 + H_0) + \rho_0 \mu_0 L_0}. \tag{A.15}$$

From (A.14) and (A.11), $\beta(\sigma)$ can be written as

$$\beta(\sigma) = \frac{H_0^{1-\psi} Y_0^{\psi}}{(\rho_0 K_0 + H_0) + \rho_0 \mu_0 L_0}. \tag{A.16}$$

A.2 Neoclassical Growth Model

Assuming the normalized CES with human capital,

$$Y(t) = [\alpha(\sigma) K(t)^{\psi} + \beta(\sigma) H(t)^{\psi} + (1-\alpha(\sigma)-\beta(\sigma))(\widetilde{A}(\sigma,t) L(t))^{\psi}]^{\frac{1}{\psi}}.$$

Also, rewriting the production function in per efficient unit terms,

$$y(t) = [\alpha(\sigma) k(t)^{\psi} + \beta(\sigma) h(t)^{\psi} + (1-\alpha(\sigma)-\beta(\sigma))]^{\frac{1}{\psi}},$$

and the national income identity becomes

$$Y(t) = C(t) + I_K(t) + I_H(t),$$

where $C(t)$ is total consumption. The physical and human capital accumulation are given by:

$$
\begin{aligned}
\dot{K}(t) &= s_K Y(t) - \delta K_t, \quad \text{and} && \text{(A.17)} \\
\dot{H}(t) &= s_H Y(t) - \delta H_t && \text{(A.18)}
\end{aligned}
$$

Also,

$$L(t) = e^{nt} L_0.$$

The savings rates in the economy is determined by:

$$\begin{aligned} s(t) &= 1 - \frac{C(t)}{Yt} \\ &= 1 - \frac{c(t)}{yt}. \end{aligned}$$

The saving rate is decomposed into the fraction of saved for the accumulation of physical, s_k, and human capital, s_h, and the decomposition of s is determined by $s(t) = s_k(t) + s_h(t)$, where s_k. Also, the investment rate for for both type of capital are denoted by i_k, for physical capital, and i_h, for human capital. Therefore,

$$\begin{aligned} s_k(t) &= \frac{I_K(t)}{Y(t)} = i_k(t) \\ s_h(t) &= \frac{I_H(t)}{Y(t)} = i_h(t). \end{aligned}$$

Thus, consumption is determined by:

$$c(t) = (1 - s_k(t) - s_h(t))Y(t) \tag{A.19}$$

Following Solow (1956), I assume savings rates are constant for physical and human capital. Incorporating constant savings rates into the capital accumulation equations, (A.17) and (A.18), yield:

$$\begin{aligned}
\frac{\dot{K}(t)}{K(t)} &= \frac{I_K(t)}{K(t)} - \delta \\
&= \frac{\frac{I_K(t)}{Y(t)}}{\frac{K(t)}{Y(t)}} - \delta \\
&= s_k \frac{Y(t)}{K(t)} - \delta \quad \text{and} \\
\frac{\dot{H}(t)}{H(t)} &= s_h \frac{I_H(t)}{H(t)} - \delta \\
&= \frac{\frac{I_H(t)}{Y(t)}}{\frac{H(t)}{Y(t)}} - \delta \\
&= s_h \frac{Y(t)}{H(t)} - \delta.
\end{aligned}$$

The growth rate for physical and human capital per effective unit of labor are give by:

$$\begin{aligned}
\frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{\widetilde{A}}(\sigma,t)}{\widetilde{A}(\sigma,t)} - \frac{\dot{L}(t)}{L(t)} \\
&= \frac{\dot{K}(t)}{K(t)} - (n+g) \quad \text{and} \\
\frac{\dot{h}(t)}{h(t)} &= \frac{\dot{H}(t)}{H(t)} - \frac{\dot{\widetilde{A}}(\sigma,t)}{\widetilde{A}(\sigma,t)} - \frac{\dot{L}(t)}{L(t)} \\
&= \frac{\dot{H}(t)}{H(t)} - (n+g).
\end{aligned}$$

Therefore, in per effective unit of labor terms,

$$\begin{aligned}
\frac{\dot{k}(t)}{k(t)} &= s_k \frac{y(t)}{k(t)} - (n+g+\delta) \quad \text{and} \\
\frac{\dot{h}(t)}{h(t)} &= s_h \frac{y(t)}{h(t)} - (n+g+\delta).
\end{aligned}$$

Also, the physical and human capital accumulations are

$$\begin{aligned}\dot{k}(t) &= s_k y(t) - (n+g+\delta)k(t) \quad \text{and} \\ \dot{h}(t) &= s_h y(t) - (n+g+\delta)h(t).\end{aligned}$$

Steady State

In the steady state $\dot{k}(t) = 0$ and $\dot{h}(t) = 0$. From (1.3),

$$\begin{aligned}0 &= s_k y - (n+g+\delta)k \\ (n+g+\delta)k &= s_k y \\ (n+g+\delta)k &= s_k[\alpha(\sigma)k^{\psi} + \beta(\sigma)h^{\psi} + (1-\alpha(\sigma)-\beta(\sigma))]^{\frac{1}{\psi}} \\ k^{\psi}((n+g+\delta)^{\psi} - \alpha(\sigma)s_k^{\psi}) &= s_k^{\psi}[\beta(\sigma)h^{\psi} + (1-\alpha(\sigma)-\beta(\sigma))] \\ k^{\psi} &= \frac{s_k^{\psi}[\beta(\sigma)h^{\psi} + (1-\alpha(\sigma)-\beta(\sigma))]}{(n+g+\delta)^{\psi} - \alpha(\sigma)s_k^{\psi}}. \qquad (A.20)\end{aligned}$$

Also from (1.4),

$$\begin{aligned}0 &= s_h y - (n+g+\delta)h \\ (n+g+\delta)h &= s_h y \\ (n+g+\delta)h &= s_h[\alpha(\sigma)k^{\psi} + \beta(\sigma)h^{\psi} + (1-\alpha(\sigma)-\beta(\sigma))]^{\frac{1}{\psi}} \\ h^{\psi}((n+g+\delta)^{\psi} - \beta(\sigma)s_h^{\psi}) &= s_h^{\psi}[\alpha(\sigma)k^{\psi} + (1-\alpha(\sigma)-\beta(\sigma))] \\ h^{\psi} &= \frac{s_h^{\psi}[\alpha(\sigma)k^{\psi} + (1-\alpha(\sigma)-\beta(\sigma))]}{(n+g+\delta)^{\psi} - \beta(\sigma)s_h^{\psi}}. \qquad (A.21)\end{aligned}$$

From (A.20) and (A.21), the steady state levels of physical and human capital, both per effective unit of labor, are given by:

$$\begin{aligned}k^{\star} &= \left(\frac{s_k^{\psi}(1-\alpha(\sigma)-\beta(\sigma))}{(n+g+\delta)^{\psi} - \alpha(\sigma)s_k^{\psi} - \beta(\sigma)s_h^{\psi}}\right)^{\frac{1}{\psi}} \\ h^{\star} &= \left(\frac{s_h^{\psi}(1-\alpha(\sigma)-\beta(\sigma))}{(n+g+\delta)^{\psi} - \alpha(\sigma)s_k^{\psi} - \beta(\sigma)s_h^{\psi}}\right)^{\frac{1}{\psi}}.\end{aligned}$$

Elasticity of Substitution

When $\sigma \to 1$ ($\psi \to 0$)

$$\begin{aligned}
\lim_{\psi\to 0} \ln k^\star &= \lim_{\psi\to 0} \frac{1}{\psi} \ln\left(\frac{s_k^\psi(1-\alpha(\sigma)-\beta(\sigma))}{(n+g+\delta)^\psi - \alpha(\sigma)s_k^\psi - \beta(\sigma)s_h^\psi}\right) \\
\lim_{\psi\to 0} \ln h^\star &= \lim_{\psi\to 0} \frac{1}{\psi} \ln\left(\frac{s_h^\psi(1-\alpha(\sigma)-\beta(\sigma))}{(n+g+\delta)^\psi - \alpha(\sigma)s_k^\psi - \beta(\sigma)s_h^\psi}\right).
\end{aligned}$$

Applying L'Hôpital's rule

$$\begin{aligned}
k^\star &= \left(\frac{s_k^{1-\beta}s_h^\beta}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}} \\
h^\star &= \left(\frac{s_k^{\alpha}s_h^{1-\alpha}}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}}.
\end{aligned}$$

Determining the production function parameters in terms of the baseline values yield:

$$\lim_{\psi\to 0} \ln A_0 = \lim_{\psi\to 0} \frac{1}{\psi} \ln\left(\frac{\rho_0\mu_0 L_0^{1-\psi}Y_0^\psi}{(\rho_0K_0+H_0) - (\rho_0K_0^{1-\psi}+H_0^{1-\psi})Y_0^\psi + \rho_0\mu_0L_0}\right).$$

Applying L'Hôpital's rule:

$$\begin{aligned}
\lim_{\psi\to 0} \ln A_0 &= \lim_{\psi\to 0} -\ln L_0 + \ln Y_0 - \\
&\quad \frac{(\rho_0K_0^{1-\psi}\ln K_0 + H_0^{1-\psi}\ln H_0)Y_0^\psi - (\rho_0K_0^{1-\psi}+H_0^{1-\psi})Y_0^\psi \ln Y_0}{(\rho_0K_0+H_0) - (\rho_0K_0^{1-\psi}+H_0^{1-\psi})Y_0^\psi + \rho_0\mu_0L_0} \\
&= -\ln L_0 + \ln Y_0 + \frac{(\rho_0K_0+H_0)\ln Y_0 - (\rho_0K_0\ln K_0 + H_0\ln H_0)}{\rho_0\mu_0L_0} \\
\lim_{\psi\to 0} A_0 &= \frac{Y_0^{1+\frac{\rho_0K_0+H_0}{\rho_0\mu_0L_0}}}{L_0K_0^{\frac{\rho_0K_0}{\rho_0\mu_0L_0}}H_0^{\frac{H_0}{\rho_0\mu_0L_0}}},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\psi\to 0} \alpha &= \frac{\rho_0K_0}{\rho_0(K_0+\mu_0L_0)+H_0} \\
\lim_{\psi\to 0} \beta &= \frac{H_0}{\rho_0(K_0+\mu_0L_0)+H_0}.
\end{aligned}$$

A.3 Existence

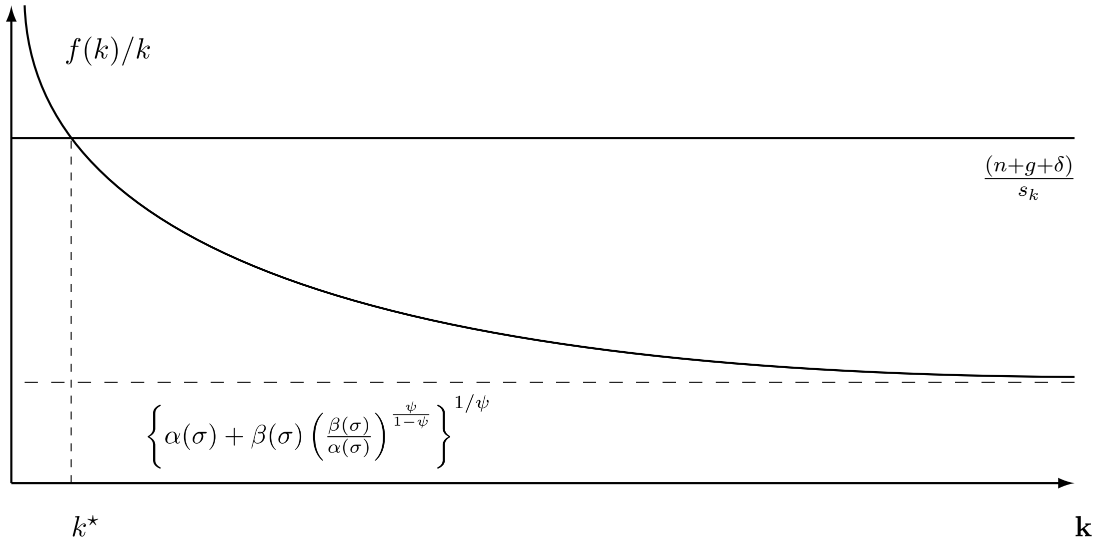

Figure A.1: $\sigma > 1$: threshold for the existence of a steady state with zero growth for human capital and physical capital both per effective unit of labor

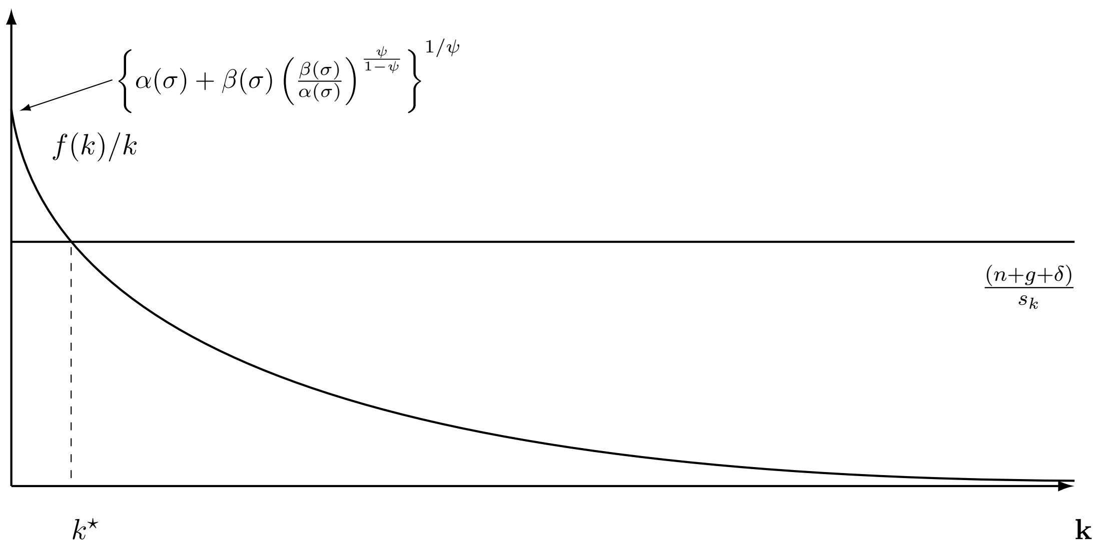

Figure A.2: $\sigma < 1$: threshold for the existence of steady state with zero growth for human capital and physical capital both per effective unit of labor

A.4 Elasticity of Substitution and the Steady State

$$\begin{aligned}\frac{d\alpha}{d\sigma} &= \frac{1}{\sigma^2}\left\{\frac{-\rho_0 K_0^{1-\psi}Y_0^{\psi}\ln K_0 + \rho_0 K_0^{1-\psi}Y_0^{\psi}\ln Y_0}{(\rho_0 K_0 + H_0) + \rho_0\mu_0 L_0}\right\} \\ &= \frac{-\alpha}{\sigma^2}(\ln K_0 - \ln Y_0) \\ &= \frac{-1}{\sigma^2}\frac{1}{\psi}\alpha\ln\left(\frac{k_0}{y_0}\right)^{\psi} \end{aligned} \tag{A.22}$$

$$\begin{aligned}\frac{d\beta}{d\sigma} &= \frac{1}{\sigma^2}\left\{\frac{-H_0^{1-\psi}Y_0^{\psi}\ln H_0 + H_0^{1-\psi}Y_0^{\psi}\ln Y_0}{(\rho_0 K_0 + H_0) + \rho_0\mu_0 L_0}\right\} \\ &= \frac{-\beta}{\sigma^2}(\ln H_0 - \ln Y_0) \\ &= \frac{-1}{\sigma^2}\frac{1}{\psi}\beta\ln\left(\frac{h_0}{y_0}\right)^{\psi} \end{aligned} \tag{A.23}$$

$$\begin{aligned}\frac{d\pi_k^{\star}}{d\sigma} &= \left(\frac{s_k}{n+g+\delta}\right)^{\psi}\frac{d\alpha(\sigma)}{d\sigma} + \frac{\alpha(\sigma)}{\sigma^2}\left(\frac{s_k}{n+g+\delta}\right)^{\psi}\ln\left(\frac{s_k}{n+g+\delta}\right) \\ &= \frac{1}{\psi}\frac{1}{\sigma^2}\alpha(\sigma)\left(\frac{s_k}{n+g+\delta}\right)^{\psi}\left\{-\ln\left(\frac{k_0}{y_0}\right)^{\psi} + \ln\left(\frac{s_k}{n+g+\delta}\right)^{\psi}\right\} \\ &= \frac{1}{\psi}\frac{1}{\sigma^2}\pi_k^{\star}\ln\left(\frac{\pi_k^{\star}}{\pi_k^0}\right) \end{aligned} \tag{A.24}$$

$$\begin{aligned}\frac{d\pi_h^{\star}}{d\sigma} &= \left(\frac{s_h}{n+g+\delta}\right)^{\psi}\frac{d\beta(\sigma)}{d\sigma} + \frac{\beta(\sigma)}{\sigma^2}\left(\frac{s_h}{n+g+\delta}\right)^{\psi}\ln\left(\frac{s_h}{n+g+\delta}\right) \\ &= \frac{1}{\psi}\frac{1}{\sigma^2}\beta(\sigma)\left(\frac{s_h}{n+g+\delta}\right)^{\psi}\left\{-\ln\left(\frac{h_0}{y_0}\right)^{\psi} + \ln\left(\frac{s_h}{n+g+\delta}\right)^{\psi}\right\} \\ &= \frac{1}{\psi}\frac{1}{\sigma^2}\pi_h^{\star}\ln\left(\frac{\pi_h^{\star}}{\pi_h^0}\right). \end{aligned} \tag{A.25}$$

Rewriting the steady state level of physical capital per effective unit of labor, I find

$$k^{\star} = e^{\frac{1}{\psi}\left\{\ln(1-\alpha(\sigma)-\beta(\sigma)) - \ln\alpha(\sigma) + \ln\pi_K^{\star} - \ln(1-\pi_K^{\star}-\pi_H^{\star})\right\}}$$

Therefore,

$$\frac{dk^\star}{d\sigma} = \frac{-1}{\sigma^2}\frac{1}{\psi^2}\left(\frac{k^\star}{1-\pi_K^\star-\pi_H^\star}\right)\Bigg\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\alpha(\sigma)-\beta(\sigma)}{1-\pi_K^\star-\pi_H^\star}\right)$$
$$-\left(\frac{1-\pi_K^\star-\pi_H^\star}{1-\alpha(\sigma)-\beta(\sigma)}\right)\left(\alpha(\sigma)\ln\left(\frac{k_0}{y_0}\right)^\psi+\beta(\sigma)\ln\left(\frac{h_0}{y_0}\right)^\psi\right)+\pi_K^\star\ln\frac{\pi_K^0}{\pi_K^\star}+\pi_H^\star\ln\frac{\pi_H^0}{\pi_H^\star}\Bigg\}.$$

From (A.13), it is clear that

$$\begin{aligned}1-\alpha(\sigma)-\beta(\sigma) &= y_0^\psi-\alpha(\sigma)k_0^\psi-\beta(\sigma)h_0^\psi\\ &= y_0^\psi(1-\pi_K^0-\pi_H^0).\end{aligned}$$

Thus,

$$\frac{dk^\star}{d\sigma} = \frac{-1}{\sigma^2}\frac{1}{\psi^2}\left(\frac{k^\star}{1-\pi_K^\star-\pi_H^\star}\right)\Bigg\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)$$
$$+\left(\frac{1-\pi_K^\star-\pi_H^\star}{y_0^\psi(1-\pi_K^0-\pi_H^0)}\right)\left(\ln y_0^\psi-\alpha(\sigma)\ln k_0^\psi-\beta(\sigma)\ln h_0^\psi\right)+\pi_K^\star\ln\frac{\pi_K^0}{\pi_K^\star}+\pi_H^\star\ln\frac{\pi_H^0}{\pi_H^\star}\Bigg\}. \tag{A.26}$$

Following Klump and Preissler (2000), I make use of the concavity of the natural logarithm and assume that $k^\star \neq k_0$, $h^\star \neq h_0$, $\pi_K^\star \neq \pi_K^0$, and $\pi_H^\star \neq \pi_H^0$. Therefore,

$$\begin{aligned}
ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right) &< \frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}-1=\frac{(1-\pi_K^0-\pi_H^0)-(1-\pi_K^\star-\pi_H^\star)}{1-\pi_K^\star-\pi_H^\star},\\
\ln k_0^\psi &< k_0^\psi-1,\\
\ln h_0^\psi &< h_0^\psi-1,\\
\ln\frac{\pi_K^0}{\pi_K^\star} &< \frac{\pi_K^0}{\pi_K^\star}-1=\frac{\pi_K^0-\pi_K^\star}{\pi_K^\star}, \quad \text{and}\\
\ln\frac{\pi_H^0}{\pi_H^\star} &< \frac{\pi_H^0}{\pi_H^\star}-1=\frac{\pi_H^0-\pi_H^\star}{\pi_H^\star}.
\end{aligned}$$

Multiplying the first expression by $1-\pi_K^\star-\pi_H^\star$ and the last two expression by $\pi_K^\star$ and $\pi_H^\star$, respectively. Would result in

$$(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)+\pi_K^\star\ln\frac{\pi_K^0}{\pi_K^\star}+\pi_H^\star\ln\frac{\pi_H^0}{\pi_H^\star}<(1-\pi_K^0-\pi_H^0)$$
$$-(1-\pi_K^\star-\pi_H^\star)+\pi_K^0-\pi_K^\star+\pi_H^0-\pi_H^\star=0.$$

Normalizing the production functions requires that a family production function share a common value and are distinguished by there σ only. Therefore,

$$y_0 = \left[\alpha(\sigma)k_0^{\psi} + \beta(\sigma)h_0^{\psi} + (1-\alpha(\sigma)-\beta(\sigma))\right]^{\frac{1}{\psi}} = k_0^{\alpha(1)}h_0^{\beta(1)}$$

The remaining expression may be written as

$$(\alpha(1)-\alpha(\sigma))\ln k_0^{\psi} + (\beta(1)-\beta(\sigma))\ln h_0^{\psi}$$

The natural concavity of logarithms would imply that[1]:

$$(\alpha(1)-\alpha(\sigma))\ln k_0^{\psi} + (\beta(1)-\beta(\sigma))\ln h_0^{\psi} <$$
$$(\alpha(1)-\alpha(\sigma))\ln k_0^{\psi} - (\alpha(1)-\alpha(\sigma)) + (\beta(1)-\beta(\sigma))\ln h_0^{\psi} - (\beta(1)-\beta(\sigma)).$$

Dividing both sides of the previous expression by y_0^{ψ}

$$\frac{1}{y_0^{\psi}}[(\alpha(1)-\alpha(\sigma))\ln k_0^{\psi} + (\beta(1)-\beta(\sigma))\ln h_0^{\psi}] <$$
$$\pi_K^0(1)+\pi_H^0(1)-\pi_K^0(\sigma)-\pi_H^0(\sigma)+(1-\pi_K^0(1)-\pi_H^0(1))-(1-\pi_K^0(\sigma)-\pi_H^0(\sigma))=0.$$

Thus, for $k^\star \neq k_0$ and $h^\star \neq h_0$,

$$\frac{dk^\star}{d\sigma} = \frac{-1}{\sigma^2}\frac{1}{\psi^2}\left(\frac{k^\star}{1-\pi_K^\star-\pi_H^\star}\right)\left\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)\right.$$
$$+\left(\frac{1-\pi_K^\star-\pi_H^\star}{y_0^{\psi}(1-\pi_K^0-\pi_H^0)}\right)\left(\ln y_0^{\psi} - \alpha(\sigma)\ln k_0^{\psi} - \beta(\sigma)\ln h_0^{\psi}\right)$$
$$\left.+\pi_K^\star\ln\frac{\pi_K^0}{\pi_K^\star} + \pi_H^\star\ln\frac{\pi_H^0}{\pi_H^\star}\right\} > 0.$$

Taking the derivative of $h^\star$ with respect to the elasticity of substitution, for $k^\star \neq k_0$ and

[1] Notice, $\alpha(1)-\alpha(\sigma) > 0$ and $\beta(1)-\beta(\sigma) > 0$ since the σ that maximizes α and β is σ equal to unity.

$h^\star \neq h_0$, leads to:

$$\frac{dh^\star}{d\sigma} = \frac{-1}{\sigma^2}\frac{1}{\psi^2}\left(\frac{h^\star}{1-\pi_K^\star-\pi_H^\star}\right)\Bigg\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)$$
$$+\left(\frac{1-\pi_K^\star-\pi_H^\star}{y_0^\psi(1-\pi_K^0-\pi_H^0)}\right)\left(\ln y_0^\psi - \alpha(\sigma)\ln k_0^\psi - \beta(\sigma)\ln h_0^\psi\right)$$
$$+\pi_K^\star \ln\frac{\pi_K^0}{\pi_K^\star} + \pi_H^\star \ln\frac{\pi_H^0}{\pi_H^\star}\Bigg\} > 0.$$

Lastly, taking the derivative of $y^\star$ with respect to the elasticity of substitution, for $k^\star \neq k_0$ and $h^\star \neq h_0$, leads to:

$$\frac{dy^\star}{d\sigma} = \frac{-1}{\sigma^2}\frac{1}{\psi^2}\left(\frac{y^\star}{1-\pi_K^\star-\pi_H^\star}\right)\Bigg\{(1-\pi_K^\star-\pi_H^\star)\ln\left(\frac{1-\pi_K^0-\pi_H^0}{1-\pi_K^\star-\pi_H^\star}\right)$$
$$+\left(\frac{1-\pi_K^\star-\pi_H^\star}{y_0^\psi(1-\pi_K^0-\pi_H^0)}\right)\left(\ln y_0^\psi - \alpha(\sigma)\ln k_0^\psi - \beta(\sigma)\ln h_0^\psi\right)$$
$$+\pi_K^\star \ln\frac{\pi_K^0}{\pi_K^\star} + \pi_H^\star \ln\frac{\pi_H^0}{\pi_H^\star}\Bigg\} > 0.$$

A.5 Elasticity of Substitution and Speed of Convergence

The growth rate of income per efficient unit is given by

$$\frac{\dot{y}(t)}{y(t)} = \pi_K\frac{\dot{k}(t)}{k(t)} + \pi_H\frac{\dot{h}(t)}{h(t)}. \tag{A.27}$$

Barro and Sala-I-Martin (2004) and other works have criticized Mankiw et al. (1992) for not assuming the two rates of returns between the two goods to be equal. This is explained by Barro and Sala-I-Martin (2004, pg. 59), that "[i]t is reasonable to think that households will invest in capital goods that delivers the higher return". Therefore, I will assume $R_K = R_H$ as a result $MP_K = MP_H$. This allows for the determination of output per efficient unit of labor by physical (or human) capital per efficient unit labor at time t. Thus,

$$k(t) = \left(\frac{y^\psi - (1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)^{\frac{1}{\psi}}. \tag{A.28}$$

Revisiting the growth rates of physical and human capital per efficient unit,

$$\begin{aligned}
\frac{\dot{k}(t)}{k(t)} &= s_k \frac{y(t)}{k(t)} - (n+g+\delta) \\
&= s_k y(t) \left(\frac{y(t)^{\psi} - (1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}} \right)^{\frac{-1}{\psi}} - (n+g+\delta) \\
&= s_k \left(\frac{1 - y(t)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}} \right)^{\frac{-1}{\psi}} - (n+g+\delta) \qquad \text{(A.29)} \\
\frac{\dot{h}(t)}{h(t)} &= s_h \frac{y(t)}{h(t)} - (n+g+\delta) \\
&= s_h \left(\frac{\beta}{\alpha}\right)^{\frac{-1}{1-\psi}} \frac{y(t)}{k(t)} - (n+g+\delta) \\
&= s_h \left(\frac{\beta}{\alpha}\right)^{\frac{-1}{1-\psi}} \left(\frac{1 - y(t)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}} \right)^{\frac{-1}{\psi}} - (n+g+\delta). \qquad \text{(A.30)}
\end{aligned}$$

Writing the physical and human capital shares of income as a function of y would result in

$$\begin{aligned}
\pi_K(y) &= \alpha(\sigma) y(t)^{-\psi} \left(\frac{y(t)^{\psi} - (1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}} \right) \\
&= \alpha(\sigma) \left(\frac{1 - y(t)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}} \right) \qquad \text{(A.31)} \\
\pi_H(y) &= \beta(\sigma) y(t)^{-\psi} \left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}} \left(\frac{y(t)^{\psi} - (1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}} \right) \\
&= \beta(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}} \left(\frac{1 - y(t)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}} \right). \qquad \text{(A.32)}
\end{aligned}$$

The growth rate of output (A.27) can be written as

$$\frac{\dot{y}(t)}{y(t)} = \alpha(s_k+s_h)\left(\frac{1-y(t)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)^{\frac{\psi-1}{\psi}}$$
$$- \left(\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\right)\left(\frac{1-y(t)^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)(n+g+\delta) \quad \text{(A.33)}$$

Taking the log-linear approximation of (A.33) in the neighborhood of the steady state[2]

$$\frac{\dot{y}(t)}{y(t)} \cong = -\lambda \log\left(y(t)/y^{\star}\right), \quad \text{(A.34)}$$

where

$$-\lambda = (\psi-1)\alpha(\sigma)(s_k+s_h)\left(\frac{1-(y^{\star})^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)^{\frac{-1}{\psi}}\left(\frac{(y^{\star})^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)$$
$$- \psi(y^{\star})^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))(n+g+\delta).$$

From (A.29) and (A.30), in the steady state

$$-\lambda = (\psi-1)\alpha(\sigma)\left(1+\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}}\right)(n+g+\delta)\left(\frac{(y^{\star})^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)$$
$$- \psi(y^{\star})^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))(n+g+\delta)$$
$$= (y^{\star})^{-\psi}(1-\alpha(\sigma)-\beta(\sigma))(n+g+\delta)\left((\psi-1)\left(\frac{\alpha(\sigma)\left(1+\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}}\right)}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)-\psi\right).$$

. Reducing further[3],

$$-\lambda = \left(\frac{(n+g+\delta)^{\psi}-\alpha(\sigma)s_k^{\psi}-\beta(\sigma)s_h^{\psi}}{(n+g+\delta)^{\psi-1}}\right)\left((\psi-1)\left(\frac{\alpha(\sigma)\left(1+\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}}\right)}{\alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}}\right)-\psi\right)$$
$$= -\left(\frac{(n+g+\delta)^{\psi}-\alpha(\sigma)s_k^{\psi}-\beta(\sigma)s_h^{\psi}}{(n+g+\delta)^{\psi-1}}\right)$$

[2]The equivalent exercise would be to rewrite the function in terms of logs ie $y = e^{\log(y)}$ and take a first order Taylor approximation

[3]This follows from $\alpha(\sigma)\left(1+\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}}\right) = \alpha(\sigma)+\beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}$

Therefore,

$$\lambda = (n+g+\delta)(1-\pi_K^\star-\pi_H^\star).$$

For the effects of changing the elasticity of substitution on the speed of convergence, I obtain the following

$$\begin{aligned}\frac{d\lambda}{d\sigma} &= \lambda = (n+g+\delta)\left(\frac{d\pi_K^\star}{d\sigma}+\frac{d\pi_H^\star}{d\sigma}\right)\\ &= -\frac{(n+g+\delta)}{\sigma^2}\left(\pi_K^\star+\pi_H^\star\right)\ln\frac{y_0/k_0}{y^\star/k^\star}. \end{aligned}\tag{A.35}$$

Since the production function is concave in k,

$$\frac{d\lambda}{d\sigma} = -\frac{(n+g+\delta)}{\sigma^2}\left(\pi_K^\star+\pi_H^\star\right)\ln\frac{y_0/k_0}{y^\star/k^\star}\begin{cases} >0 \iff k^\star < k_0\\ =0 \iff k^\star = k_0\\ <0 \iff k^\star > k_0\end{cases}\tag{A.36}$$

A.6 Elasticity of Substitution, Steady State, and Speed of Convergence

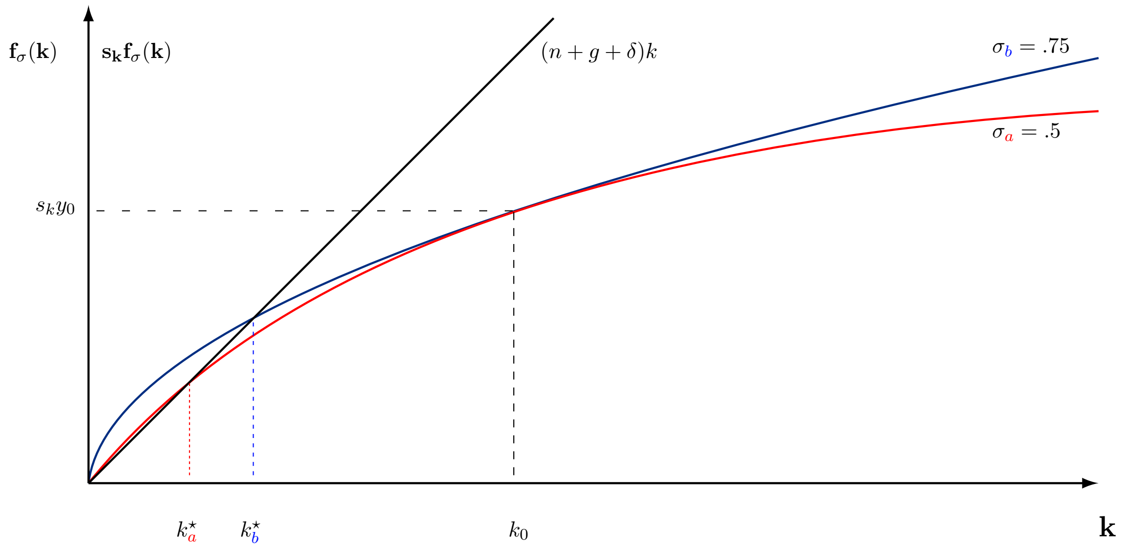

Figure A.3: Diagram for the effects of the elasticity of substitution on the steady state, when the initial level of capital per effective unit of labor is greater than the steady state level of capital per effective unit of labor.

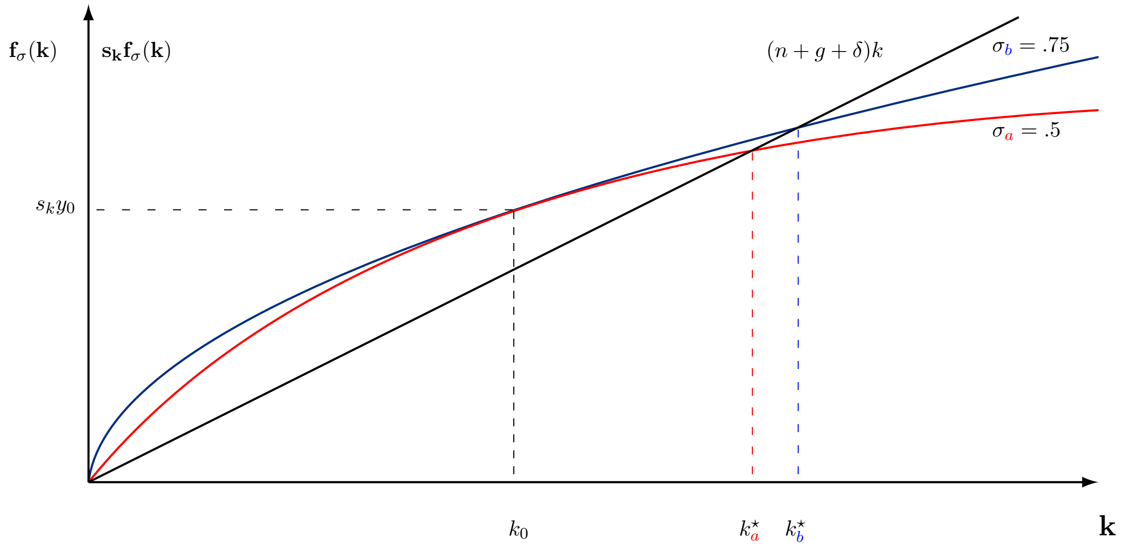

Figure A.4: Diagram for the effects of the elasticity of substitution on the steady state, when the initial level of capital per effective unit of labor is lower than the steady state level of capital per effective unit of labor.

A.7 Elasticity of Substitution and Endogenous Growth

When the inequality condition (1.9) derived for the high elasticity substitution is violated then this could lead to endogenous growth. Endogenous growth will occur for neoclassical growth model when

$$\sigma > 1 \quad \text{and} \quad \frac{(n+g+\delta)}{s_k} < \left\{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\right\}^{1/\psi}.$$

Balance growth would require that

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{y}(t)}{y(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{\dot{h}(t)}{h(t)}.$$

The average product of capital $\frac{y(t)}{k(t)}$, when there is steady state growth for physical and human capita per effective unit, is given by

$$\left\{\alpha(\sigma) + \beta(\sigma)\left(\frac{\beta(\sigma)}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}}\right\}^{1/\psi}.$$

The steady state growth rate of physical capital and human capital per effective unit of labor are give by the following

$$\xi_k^\star = \frac{\dot{k}(t)}{k(t)} = s_k \left\{ \alpha(\sigma) + \beta(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{\psi}{1-\psi}} \right\}^{1/\psi} - (n + g + \delta) \tag{A.37}$$

$$\xi_h^\star = \frac{\dot{h}(t)}{h(t)} = s_h \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{1}{1-\psi}} \left\{ \alpha(\sigma) + \beta(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{\psi}{1-\psi}} \right\}^{1/\psi} - (n + g + \delta). \tag{A.38}$$

where $\xi^\star$ denotes the steady state growth rate, and balance growth will imply that $\xi_k^\star = \xi_h^\star \equiv \xi^\star$.

From (A.27) and assuming steady state growth would imply

$$\frac{\dot{y}(t)}{y(t)} = \xi^\star \left\{ \alpha(\sigma) + \beta(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{\psi}{1-\psi}} \right\}^{\frac{1-\psi}{\psi}} . \tag{A.39}$$

Balance growth requires that

$$1 = \alpha(\sigma) + \beta(\sigma) \left(\frac{\beta(\sigma)}{\alpha(\sigma)} \right)^{\frac{\psi}{1-\psi}} . \tag{A.40}$$

For unitary elasticity of substitution, endogenous growth requires that $1 = \alpha(\sigma) + \beta(\sigma)$.

Chapter 2

Estimating the Elasticity of Substitution with a CES Production Function and Additive Human Capital

2.1 Introduction

Although the classic work of Solow (1956) and many others examine several functional forms for the aggregate production function, which are consistent with the neoclassical theory of economic growth, the workhorse for the dynamic macroeconomics literature continues to be the Cobb-Douglas production function for which the elasticity of substitution among physical capital, human capital, and labor is exactly unity. Furthermore, the long-standing discussion on the functional form of the aggregate production function with two factors of production, physical capital and labor, strongly suggests that the elasticity of substitution between these factors significantly differ from unity for most countries. In particular, works such as Antràs (2004) and Klump et al. (2007) estimate the level of elasticity of substitution, and both of these studies find that the elasticity of substitution between physical capital and labor is significantly different than unity for the United States. Antràs (2004) estimates the level of elasticity of substitution to range between .407 to .948, and Klump et al. (2007) estimate an elasticity of substitution around .5. More recently, Mallick (2012) provides country specific estimates for the elasticity of substitution, between physical capital of production, and suggests that the elasticity of substitution for most countries, in his study, significantly differ from unity. Many of the aforementioned works are motivated by de La Grandville (1989) and Klump and de La Grandville (2000), which provide a framework to explain the potency of the elasticity of substitution, between physical capital and labor on growth, on long-term per capita income.

Much of the dynamic macroeconomic literature includes human capital also relies on the Cobb-Douglas function to describe a countries technology for producing goods. As the discussion on the functional form of the aggregate production function for goods continues towards functions with nonunitary elasticities of substitution this would suggest that the elasticity of substitution among physical capital, human capital, and labor also differs from unity. To corroborate this claim, I estimate the normalized constant elasticity of substitution production function presented in Chapter 1 to determine if the level of elasticity of substitution between the previously mentioned factors of production significantly differ from unity. Upon estimating the normalized constant elasticity of substitution aggregate production function, I find an estimate for the elasticity of substitution that range between .72 to 81. Thus, under the assumption of constant elasticity of substitution among physical capital, human capital, and labor the elasticity of substitution is significantly below unity.

This chapter proceeds as follows. The second section discusses the per capita production function that is used to estimate the elasticity of substitution for the three-factor aggregate constant elasticity of substitution production function. Subsequently, in Section 2.3, I discuss the data sources and series transformations used for the estimates. This would be followed by Section 2.4 the empirical, results of my estimates. I then conclude and provide additional remarks.

2.2 The Model

I present a model in Chapter 1 that augments Solow (1956) to include human capital. In this chapter, I estimate the normalized constant elasticity of substitution from Chapter 1 that augments Solow (1956) to determine the level of the elasticity of substitution for cross-country data consistent with Mankiw et al. (1992). Following Mankiw et al. (1992), I assume exogenous and constant savings rates for physical and human capital, and assume that the growth rate of the population and technology are exogenous. The production of goods and services are assumed to be produced under perfect competition and by a constant elasticity of substitution aggregate production function, (1.1). The evolution of human and physical capital are determined by (1.4) and (1.3), respectively.

The aggregate per capita production is determined by the following technology:

$$\log \frac{Y_t}{L_t} = \log A_0(\sigma) + \left(\frac{\sigma}{\sigma - 1}\right) \log[\alpha(\sigma)(k^\star)^{\frac{\sigma-1}{\sigma}} + \beta(\sigma)(h^\star)^{\frac{\sigma-1}{\sigma}} + (1 - \alpha(\sigma) - \beta(\sigma))], \quad (2.1)$$

where $\alpha(\sigma)$, $\beta(\sigma)$, and $A_0(\sigma)$ are normalized parameters for the aggregate constant elasticity of production function (1.2) and $k^\star$ and $h^\star$ are steady values[1] of physical and human capital per effective units of labor.

[1] Derivations for the normalized parameters and steady state variables can be found in Chapter 1.2.

2.3 Data

The data are from the following sources: Penn World Table, Barro-Lee's Educational Attainment Dataset, United Nations, the World Bank, and Organisation for Economic Co-operation and Development. I determine the series for labor, output, physical capital investment, and physical capital from the Penn World Table, version 7.1[2], following Caselli (2005). The number of workers, L, is determined from the population growth rate, gross domestic product and gross domestic per worker. Both the gross domestic product and gross domestic product per worker are chain series at 2005 constant prices in international dollars. The level of output, Y, is given by the product of gross domestic product per worker and the number of workers, L. The aggregate physical capital investment, I_K, is derived from gross domestic product per capita and investment share of the gross domestic product per capita, where both series are determined by Laspeyres price index at 2005 constant prices in international dollars. The savings rate for physical capital is give by physical capital investment over output. The physical capital stock series for each country is estimated using the perpetual inventory equation:

$$K_t = I_t + (1 - \delta)K_{t-1}, \tag{2.2}$$

The initial level of capital, K_0, is given by $I_{K0}/(g_{I_K} + \delta)$ where δ is the depreciation rate of physical capital and g_{I_K} is the average geometric growth rate of physical capital investment between the first year of available data and 1970. Following the literature, the depreciation rate of physical capital, denoted by δ, is assumed to be 0.06.

To construct the human capital series, I employ the updated Barro and Lee (2010) Educational Attainment Dataset. Using this data, I follow Hall and Jones (1999) for the estimation of the human capital series and assume that the stock of human for country i at

[2]Heston et al. (2012)

time t is given by[3]:

$$H_{it} = \exp^{\phi(\varepsilon_{it})}, \tag{2.3}$$

where ϕ is a piece-wise linear function, with a zero intercept and slope of 0.134 through the fourth year of education, 0.101 for the next four years, and 0.068 for education beyond the eighth year.

Following Mankiw et al. (1992), I proxy for the savings rate for human capital by employing data from the United Nations Educational, Scientific and Cultural Organisation, Institute for Statistics, the United Nations, Department of Economic and Social Affairs, Population Division, and World Bank. To proxy for the savings rate, I require the fraction of the population that is enrolled in secondary school, this is expressed as a percentage of the official school-age population, and the fraction of the working-age population. I obtain the data for fraction of the population that is enrolled in secondary school from the United Nations Educational, Scientific and Cultural Organisation, Institute for Statistics. Using the series from United Nations, Department of Economic and Social Affairs, Population Division and World Bank, I divide the total working age population that is aged 15 by the total working-age population to obtain the fraction of the working-age population. Lastly, I multiply the fraction of the population that is enrolled in secondary school by the fraction of the working-age population.

Estimating the normalized constant elasticity of substitution function, (2.1), also requires baseline values for the factor prices. To proxy for the wage rate of raw labor, I employ the real minimum wage series from the Organisation for Economic Co-operation and Development. Following Diewert (2001, 2005), I use the ex post return method to determine the rental rate of physical capital. The normalized value for the rental rate of

[3]Hall and Jones (1999) is based on upon a survey of wage equations from Psacharopoulos (1994).

human capital is determined by the average rate of return to education, which is given by the derivative of the logarithm of human capital, (2.3), with respect to level of eduction. I use median values for the all normalized parameters for the last year available, year 2010, with the exception of returns to human capital. The growth rate of technology is calibrated to be 2% which is consistent with estimates for the growth rate of technology for the United States[4].

2.4 Results

The results of estimating equation (2.1), by nonlinear least squares, can be found in Table 2.4. I find values for the elasticity of substitution that are significantly below unity. The elasticity of substitution parameter is significant at the 1% level for the 1970 to 1985 sample period and 1970 to 2010 sample period as well for the data set used by Mankiw et al. (1992).

Nonlinear Least Squares - Estimation by Gauss-Newton.
Dependent Variable GDP per Capita

	Mankiw et al.	1985	2010
σ	0.72669	0.72631	0.81950
	(0.00004)	(0.00003)	(0.00036)
Usable Observations	106	102	121
Degrees of Freedom	105	101	120
Uncentered R^2	0.98261	0.98311	0.98383

Table 2.1: Estimates for the elasticity of substitution, σ. [*Note: Standard errors are in the parentheses.*]

The estimations of nonlinear production function, (2.1), can of course be sensitive to starting values set for the parameters. Therefore, to ensure that I find the global optimum, I performed a fine grid search around broad and plausible ranges for the elasticity of substi-

[4]See Mankiw et al. (1992) and Klump and Preissler (2000).

tution.

Solow (1956) employs a neoclassical growth model that explains that the per capita income in a given country converges to that country's steady state value. Thus, conditioning upon the determinants of the steady state, Solow argues that there is the existence of conditional convergence. Ergo, countries that have an income per effective units of labor closer to steady state value reach the steady faster. Allowing $y^\star$ to be the steady state level of income per effective worker, and $y(t)$ to be the level of output per effective worker at time t. Approximating around the steady state, the speed of convergence for the model derived in Chapter 1 is given by:

$$\frac{d\ln y(t)}{dt} = -\lambda \ln\left(y(t)/y^\star\right) \tag{2.4}$$

where

$$\lambda = (n + g + \delta)(1 - \pi_K^\star - \pi_H^\star). \tag{2.5}$$

Solving the differential (2.4) from time 0 to t, Mankiw et al. (1992) suggest that a natural model to study the speed of convergence is

$$\ln y(t) = (1 - \exp^{-\lambda t}) \ln y^\star + \exp^{-\lambda t} \ln y(0), \tag{2.6}$$

where $y(0)$ is the level of output per effective worker at some initial date. Subtracting the initial level of output per effective worker results in

$$\ln y(t) - \ln y(0) = (1 - \exp^{-\lambda t}) \ln y^\star - (1 - \exp^{-\lambda t}) \ln y(0). \tag{2.7}$$

I test for the convergence predictions of the Solow model, and begin by confirming the lack of unconditional convergence, and the results can be found in Table 2.4. In this table, I reproduce the failure of incomes to converge unconditionally, which is in line with the dynamic macroeconomic literature[5]. I find that the signs of the coefficients for the

[5]See Mankiw et al. (1992).

initial levels of income are correct, and the speed of convergence lambda is close to zero. Therefore, the results of this table suggest that there is little unconditional convergence.

Dependent Variable: log difference of GDP per capita 1960–1985 (MRW), 1970–1985, and 1970–2010			
	Mankiw et al.	1985	2010
CONSTANT	0.42328	0.40526	0.97394
	(0.37112)	(0.25143)	(0.42570)
$\ln(Y_{1960}/L_{1960})$	0.00336		
	(0.04825)	(—)	(—)
$\ln(Y_{1970}/L_{1970})$		-0.02213	-0.05296
	(—)	(0.02743)	(0.04644)
Observations	105	105	105
Degrees of Freedom	103	103	103
$\overline{R}^2$	-0.009661	-0.00337	0.00288
Implied λ	-0.00013	0.00149	0.00136

Table 2.2: Estimates for unconditional convergence. [*Note: Standard errors are in the parentheses. For Mankiw et al. (1992) the output is in terms of per working-age person.*]

The theoretical model presented in Chapter 1 would suggest that lower elasticity of substitution per effective units of labor led to a higher rate of convergence to the steady state, as all countries are assumed to initially to be below the steady value. In Chapter 1 section 1.18, Table 2.4 provides estimates for conditional convergence for the aforementioned data. For data provided by Mankiw et al. (1992) and the two data subsets of that follow from Section 2.3, I find a low rate of conditional convergence.

Nonlinear Least Squares - Estimation by Gauss-Newton **Dependent Variable: log difference of GDP per capita** **1960–1985 (MRW), 1970–1985, and 1970–2010**			
	Mankiw et al.	1985	2010
σ	0.72562	0.72606	0.82190
	(0.00002)	(0.00191)	(0.04170)
$1 - e^{-\lambda t}$	0.01286	0.02247	0.05293
	(0.04832)	(0.02747)	(0.04644)
Observations	105	105	105
Degrees of Freedom	103	103	103
Uncentered R^2	0.47262	0.27698	0.43754
Implied λ	.00052	.00152	0.00136

Table 2.3: Estimates for conditional convergence. [*Note: Standard errors are in the parentheses. For Mankiw et al. (1992) the output is in terms of per working-age person.*]

2.5 Conclusions

In this chapter, I estimate the normalized aggregate CES production function, (1.2), derived in Chapter 1. The literature on human capital in the aggregate production function typically ignores the elasticity of substitution between factors of production, as the aggregate production function in this literature is typically assumed to be Cobb-Douglas. I find estimates for the elasticity of substitution that are significantly below one which corroborates the claim that the elasticity of substitution form the aggregate production function with human capital differs from unity. In addition, when allowing for the production function to be given by the normalized CES, (1.2), I find a relatively low rate of conditional convergence. Furthermore, the findings of Chapters 1 and 2 emphasis that the elasticity of substitution is an important parameter for economic growth, as the parameter effects the per effective unit of labor steady state as well as the rate of convergence to said steady state.

Additional Remarks

There are many approaches to estimating an aggregate production function. For this study I have estimated the production function only, but this study could employ the supply-side systems approach of Marschak and Andrews Jr (1944), since it lends itself well to normalized production functions[6]. The supply-side approach is given by the estimation of the first order profit maximizing conditions and production function. Therefore, the supply-side system of the equations are:

$$\ln\left(\frac{Y(t)}{L(t)}\right) = \ln A(\sigma) + gt + \ln(n+g+\delta) + \frac{1}{\psi}\ln \quad \frac{1}{\psi}\ln\left(\frac{(1-\alpha(\sigma)-\beta(\sigma))}{\left((n+g+\delta)^{\psi} - \alpha(\sigma)s_k^{\psi} - \beta(\sigma)s_h^{\psi}\right)}\right) \tag{2.8}$$

$$\ln\left(\frac{R_K(t)K(t)}{P(t)Y(t)}\right) = \ln\alpha(\sigma) + \psi\ln\left(\frac{s_k}{n+g+\delta}\right) \tag{2.9}$$

$$\ln\left(\frac{R_H(t)H(t)}{P(t)Y(t)}\right) = \ln\beta(\sigma) + \psi\ln\left(\frac{s_h}{n+g+\delta}\right) \tag{2.10}$$

$$\ln\left(\frac{W(t)L(t)}{P(t)Y(t)}\right) = \ln\left((n+g+\delta)^{\psi} - \alpha(\sigma)s_k^{\psi} - \beta(\sigma)s_h^{\psi}\right) - \psi\ln(n+g+\delta). \tag{2.11}$$

The equations for the per capita production function, (2.8), and factor shares of income assume that physical capital and human capital, both per effective unit of labor, are in steady state and are derived by plugging (1.5) and (1.2). As the estimation of the supply system is superior for determine the bias technological change as well as the elasticity of substitution, σ, this approach may further enhance the empirical exercise in this chapter[7].

[6] León-Ledesma et al. (2010) explain that the supply-side approach is superior for identifying the elasticity of substitution and biased technological progress.

[7] See León-Ledesma et al. (2010).

Chapter 3

Endogenous Growth with a Normalized CES Production Function and Human Capital

3.1 Introduction

The degree to which factors of production are substitutable is an issue that is often suggested in the dynamic macroeconomic literature as not being of the first-order. The classic works of de La Grandville (1989) and Klump and de La Grandville (2000) establish a methodological approach, normalizing the production function, to determine the effects of the elasticity of substitution, between the factors of production, on the steady state level of income. In particular, Klump and de La Grandville establish that the steady state income is increasing in the elasticity of substitution for a neoclassical growth model with exogenous and constant savings rate for physical capital. To arrive at this conclusion, they determine the parameters of the aggregate constant elasticity of substitution production function as a function of arbitrary baseline values for the factors of production, output, and the marginal rate of technical substitution, and the elasticity of substitution.

The development of de La Grandville's normalization of the production function approach has lead to several papers such as Klump et al. (2007), León-Ledesma et al. (2010), and Mallick (2012), which have continue to draw attention to the importance of the elasticity of substitution parameter, σ, between physical capital and labor on economic growth. In spit of this, the elasticity of substitution between factors of production still continues to receive little attention in dynamic macroeconomic literature. This is particularly apparent in the economic growth literature with human capital. Classic works such as Lucas et al. (1988) and Mankiw et al. (1992) explain the importance of human capital for the development of an economy, though the dynamic macroeconomic literature that includes human capital typically does not make apparent the importance of the elasticity of substitution in this context. As Klump and de La Grandville suggest that the elasticity of substitution between physical capital and labor is important for economic growth models, it is natural suggest that the pairwise elasticity of substitution between these variables and human capital is also a potent variable for dynamic growth models.

The objective of this chapter is to contribute to the discussion on the potency of the elasticity of substitution on economic growth. In Chapter 1, I consider a neoclassical growth model that assumes exogenous and constant savings rates for human and physical capital. The assumption of exogenous and constant savings rates follows the works of Klump and de La Grandville, but provide little insight on the effects of elasticity of substitution between the factors of production on savings rate and the endogenous growth rate of output, human capital, and physical capital[1]. This chapter relaxes the assumption of exogenous saving

[1] See Appendix A.7.

rates for human capital and physical capital. I consider an economic growth model with physical and human capital and labor, which augments the classic work Lucas et al. (1988). I find that the steady state level of income is increasing in the elasticity of substitution when allowing for endogenous savings rates, and I find that the endogenous constant growth rates of the factors of production and output and the speed of convergence along a balance growth path is dependent are effected by changes in the elasticity of substitution.

The chapter proceeds as follows. Section 3.2 describes the normalization of the aggregate production function and the centralized economy. Section 3.3 examines the effects of the elasticity of substitution on balance growth path steady state levels of average product of capital, consumption-physical ratio and human-physical capital ratio. Section 3.4 examines the implied speed of convergence of this model, and how the elasticity affects the speed of convergence. Lastly, I conclude.

3.2 Dynamic General Equilibrium Model

I consider a dynamic general equilibrium model for a closed economy. The focus of this chapter is to analysis the effects of the elasticity of substitution amongst the pair of inputs, the inputs being physical capital, human capital, and labor, while relaxing the assumption of exogenous and constant savings rates to evaluate the effects of the level of the elasticity of substitution on the steady state level of income, physical capital, and human capital and the savings rates for both types of capital. This is unlike de La Grandville (1989), Klump and de La Grandville (2000), Klump and de La Grandville (2000), and Chapter 1, which assume the saving rate for capital are exogenous and constant. To simplify the analysis in this chapter, I employ a centralized economy model with endogenous savings rates for both physical and human capital.

Normalizing the CES Production Function

I assume that a homogenous good Y is produced by an aggregate production function given by:

$$F(K,HL,t) \equiv Y(t) = \Big(\alpha K(t)^{\psi} + (1-\alpha)(H(t)L(t))^{\psi}\Big)^{\frac{1}{\psi}}, \tag{3.1}$$

where physical capital is denoted by K, labor hours devoted to the production of the homogenous good Y are denoted by L, human capital is denoted by H, and the substitution parameter is denoted by ψ, which is a function of the elasticity of substitution, σ. The functional form for the substitution parameter is $\psi = \frac{\sigma-1}{\sigma}$. Human capital is assumed to enter the production function as labor augmenting, this is a tenuous for the model, but simplifies the numerical analysis. As in the previous chapters, I normalize the aggregate production function, (3.1), by the normalization approach introduced by de La Grandville (1989) and Klump and Preissler (2000). This approach facilitates the identification of the effects of the elasticity of substitution on the steady state values of output, physical capital, and human capital all per effective worker under the assumption of constant exogenous growth[2]. To normalize production function (3.1), I assume arbitrary baseline values for the factors of production, output for the homogenous good Y, and a baseline value for the marginal rate of technical substitution between the effective labor and physical capital denoted by $[F_{HL}/F_K]_0 = \gamma_0$. I normalize the parameter α, (3.1), from:

$$\left(\frac{F_{HL}}{F_K}\right)_0 = \frac{1-\alpha}{\alpha}\left(\frac{K_0}{H_0L_0}\right)^{1-\psi} = \gamma_0 \tag{3.2}$$

$$\Leftrightarrow \quad \alpha = \frac{K_0^{1-\psi}}{K_0^{1-\psi} + \gamma_0(H_0L_0)^{1-\psi}} = \frac{1}{1+\gamma_0(\hat{H}_0L_0)^{1-\psi}} = \alpha(\sigma). \tag{3.3}$$

Thus, the normalized aggregate production function is given by

$$F(K,HL,t) \equiv Y(t) = \Big(\alpha(\sigma)K(t)^{\psi} + (1-\alpha(\sigma))(L(t)H(t))^{\psi}\Big)^{\frac{1}{\psi}}, \tag{3.4}$$

[2]See Chapter 1, Klump and Preissler (2000), and Klump and Preissler (2000).

where the parameter α is now a function of the elasticity of substitution.

The Model

There are a unit measure of identical infinitely-lived households. Each agent is endowed with one unit of time that she can allocate to producing homogenous good $Y(t)$, or accumulating education (or human capital), denoted by $H(t)$. Consumption is assumed to be a stream $C(t)$, where $t \geq 0$, units of the homogenous good. Agents preferences over the consumption stream are given by:

$$\int_{t=0}^{\infty} e^{-\rho t} \left(\frac{C(t)^{1-\theta} - 1}{1-\theta} \right), \tag{3.5}$$

where the discount rate ρ and the rate of relative risk aversion θ are both positive.

The production of the homogenous good is divided into consumption, $C(t)$, and physical capital accumulation. The rate of change of the physical capital stock is denoted by $\dot{K}(t)$, and total output is determined by $C(t) + \dot{K}(t) + \delta K(t)$, where δ is the per period depreciation rate of physical capital. Production of the homogenous good depends upon the level of physical capital, time devoted to production of the homogenous good, level of human capital, and normalized parameter α according to:

$$C(t) + \dot{K}(t) + \delta K(t) = \Big(\alpha(\sigma) K(t)^{\psi} + (1 - \alpha(\sigma))(L(t)H(t))^{\psi} \Big)^{\frac{1}{\psi}}, \tag{3.6}$$

where the rate of change of physical capital is denoted by $\dot{K}(t)$, and the time devoted to the accumulation of physical capital is denoted by $L(t)$. For simplicity, I assume that the production of human capital depends upon a linear function[3], and the accumulation of human capita is determined by:

$$\dot{H}(t) = AH(t)(1 - L(t)) - \delta H(t), \tag{3.7}$$

[3]This follows Lucas et al. (1988) which is an adaptation of Uzawa (1965) and Rosen (1976).

where the technology parameter A and time devoted to the accumulation of human capital, $1 - L(t)$, are both positive.

The resource allocation problem faced by the centralized economy is to choose the optimal time path $C(t)$ for consumption and $1 - L(t)$ for time devoted to the accumulation of human capital. Given initial values for human capital, H_0, and physical capital, K_0, the optimal allocation for consumption, time devoted to human capital, physical capital, and human capital can be determined. The optimal allocation stream for the consumption and time devoted to accumulation of education are based on the maximization of agents's utility, (3.5), subject to technology (3.6) and accumulation of human capital, (3.7). Thus, the current-value Hamiltonian, $\mathcal{H}$, is

$$\mathcal{H}(C, K, H, L, \mu_1, \mu_2, t) = e^{-\rho t}\left(\frac{C^{1-\theta} - 1}{1-\theta}\right) + \mu_1 \left\{F(K, HL) - \delta K - C\right\} + \mu_2 \left\{AH(1 - L) - \delta H\right\} \quad (3.8)$$

where μ_1 is the shadow price for (3.6) and μ_2 is the shadow price for (3.7).

Along the balanced growth path, the ration of consumption to physical capital ratio and human to physical capital are constant over time in line with the stylized facts of long-term economic growth that were first summarized Kaldor (1966). The ratio of consumption to physical capital ratio is denoted by $\hat{C}(t)$ and human capital to physical capital ratio is denoted by $\hat{H}(t)$. For these ratios to be constant over time, consumption, physical capital, and human capital must all grow at the same rate, which I assume is constant. Along the balanced growth path, the ratios of consumption to physical capital and human capital to physical capital are:

$$L^{\star} = \frac{\theta-1}{\theta} + \frac{\delta(1-\theta)+\rho}{A\theta} \tag{3.9}$$

$$\hat{C}^{\star} = \left(\frac{A}{\alpha(\sigma))}\right)^{\frac{1}{1-\psi}} + A\left(\frac{\delta(1-\theta)+\rho}{A\theta} - \frac{1}{\theta}\right), \quad \text{and} \tag{3.10}$$

$$\hat{H}^{\star} = \left(\frac{\theta-1}{\theta} + \frac{\delta(1-\theta)+\rho}{A\theta}\right)^{-1} \left(\frac{\left(\frac{A}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}} - \alpha(\sigma)}{(1-\alpha(\sigma))}\right)^{\frac{1}{\psi}}. \tag{3.11}$$

Also, the average product of physical capital, along the balance growth path, is:

$$\hat{Y}^{\star} = \left(\frac{A}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}}. \tag{3.12}$$

3.3 Elasticity of Substitution and the Balance Growth Path

As the Cobb-Douglas production is typically employed as the workhorse of the dynamic economic literature, the effect of the elasticity of substitution, σ, are typically ignored for growth models that include human capital. When employing constant elasticity of substitution production function (3.4) in a model with endogenous savings rate, the elasticity of substitution will affect the balance growth rates of physical capital, human capital, and output. This is not the case for an augmented Solow model with constant and exogenous savings, as the average product of capital is invariant to the changes in the level of the the elasticity of substitution[4]. Relaxing the assumption of constant and exogenous savings rates for physical and human capital allows the average product of capital to be effected by elasticity of substitution according to[5]:

$$\frac{d\hat{Y}}{d\sigma} = \frac{\hat{Y}^{\star}}{\psi\sigma}\left[\ln \hat{Y}^{\star\psi} - (1-\alpha(\sigma))\ln(\hat{H}_0 L_0)^{\psi}\right] \begin{cases} < 0 \iff H_0 L_0 > \frac{1-\alpha(\sigma)}{2(1-\alpha(\sigma))-(1-\alpha(\sigma))\hat{H}^{\star}L^{\star}} \\ > 0 \iff H_0 L_0 < \frac{(2-\alpha(\sigma))\hat{H}^{\star}L^{\star}-(1-\alpha(\sigma))}{(\alpha(\sigma))+(1-\alpha(\sigma))\hat{H}^{\star}L^{\star}} \end{cases} \tag{3.13}$$

[4]See Appendix A.7.

[5]For the derivations of the effects of the elasticity of substitution on the balance growth steady state, see Appendix B.3.

The two cases listed in (3.13) are necessary conditions for how the elasticity of substitution effects the average product of physical capital along the balance growth path. The impact of the elasticity of substitution on the average product of capital depends upon the initial level of the effect units of labor. The elasticity of substitution also has the same effect on consumption per physical capital since:

$$\frac{d\hat{Y}}{d\sigma} = \frac{d\hat{C}}{d\sigma} \tag{3.14}$$

The effect of the elasticity of substitution on the human capital per physical capital, $\hat{H}$, is far more ambiguous[6]. Therefore, this study does not focus on the effects of the elasticity of substitution on this ratio.

3.4 Speed of Convergence and the Effects of σ

To examine the speed with which an economy heads toward the balance growth path steady, I take a log-linear approximation of the growth rate of the average product of capital, which is given by:

$$\frac{\dot{\hat{Y}}}{\hat{Y}} = -\lambda \log\left(\frac{\hat{Y}}{\hat{Y}^{\star}}\right) \tag{3.15}$$

where

$$\lambda \equiv \left[\hat{Y}^{\star} - 1\right]. \tag{3.16}$$

The parameter λ is the rate of convergence to the balanced growth steady state. Clearly, the speed of convergence is dependent upon the elasticity of substitution, and $\frac{d\hat{Y}}{d\sigma} = \frac{d\lambda}{d\sigma}$.

[6]I simulate effects of the elasticity of substitution on the human to physical capital ratio in Appendix B.3.

3.5 Conclusions

This chapter employs a neoclassical growth model with a normalized aggregate two factor CES production function. In particular, I followLucas et al. (1988), and employ a model that includes human capital and its accumulation. Furthermore, I assume that human capital augments simple labor. Unlike most models that employ the normalized CES, I allow the savings for capital to be endogenously determined. Thus, allowing the elasticity of substitution to affect the endogenous growth rate of the average product of physical capital and human capital per physical capital. As a result, a change in the level of the elasticity of substitution, along the balance growth path, can increase/decrease the average product of capital, human capital per physical capital, and speed of convergence, ceteris paribus.

Appendix B

Appendix for Endogenous Growth with a Normalized CES Production Function and Human Capital

B.1 First Order Conditions

$$C \quad : \quad C^{-\theta}e^{-\rho t} = \mu_1 \tag{B.1}$$

$$K \quad : \quad \left(\alpha(\sigma)\frac{Y^{1-\psi}}{K^{1-\psi}} - \delta\right) = \frac{-\dot{\mu_1}}{\mu_1} \tag{B.2}$$

$$H \quad : \quad \left(\frac{\mu_1}{\mu_2}\right)(1-\alpha(\sigma))\frac{Y^{1-\psi}}{H^{1-\psi}L^{-\psi}} + A(1-L) - \delta = \frac{-\dot{\mu_2}}{\mu_2} \tag{B.3}$$

$$L \quad : \quad \frac{A}{(1-\alpha(\sigma))}\frac{(HL)^{1-\psi}}{Y^{1-\psi}} = \frac{\mu_1}{\mu_2} \tag{B.4}$$

F.O.C. Manipulation

Plugging (B.4) into (B.3) would result in

$$A - \delta = \frac{-\dot{\mu_2}}{\mu_2}. \tag{B.5}$$

Taking the logarithm of (B.1) and derivative with respect to t would result in

$$\theta \frac{\dot{C}}{C} + \rho = \frac{-\dot{\mu_1}}{\mu_1} \tag{B.6}$$

Next, from plugging (B.1) into (B.4) would result in

$$\frac{(1-\alpha(\sigma))}{A} \left(\frac{Y}{HL}\right)^{1-\psi} e^{-\rho t} C^{-\theta} = \mu_2. \tag{B.7}$$

Taking the logarithm of (B.7) and derivative with respect to t would result in

$$(1-\psi)\left(\frac{\dot{Y}}{Y} - \frac{\dot{H}}{H} - \frac{\dot{L}}{L}\right) - \rho - \theta \frac{\dot{C}}{C} = \frac{\dot{\mu_2}}{\mu_2}. \tag{B.8}$$

So, (B.5) and (B.8) imply that

$$\frac{\dot{C}}{C} = \frac{1}{\theta}\left\{A - \delta - \rho + (1-\psi)\left(\frac{\dot{Y}}{Y} - \frac{\dot{H}}{H} - \frac{\dot{L}}{L}\right)\right\} \tag{B.9}$$

From (B.6) and (B.2), I find that

$$\frac{\dot{C}}{C} = \frac{1}{\theta}\left(\alpha(\sigma)\left(\frac{Y}{K}\right)^{1-\psi} - \delta - \rho\right). \tag{B.10}$$

Balance Growth Path (BGP)

To facilitate the dynamic analysis, I assume $\hat{H}(t) \equiv H(t)/K(t)$ and $\hat{C}(t) \equiv C(t)/K(t)$. From the difference of growth rate of human capital, derived from (3.7), and growth rate of physical capital, derived from (3.6), the growth rate of $\hat{H}(t)$ is written as

$$\frac{\dot{\hat{H}}}{\hat{H}} = A(1-L) - \left(\alpha(\sigma) + (1-\alpha(\sigma))(\hat{H}L)^{\psi}\right)^{\frac{1}{\psi}} + \hat{C}. \tag{B.11}$$

The growth rate of $\hat{C}$ may be determined by the difference of (B.10) and growth rate of capital, derived from (3.6). Thus,

$$\frac{\dot{\hat{C}}}{\hat{C}} = \left(\alpha(\sigma) + (1-\alpha(\sigma))(\hat{H}L)^{\psi}\right)^{\frac{1}{\psi}} \left\{ \left(\frac{\alpha(\sigma)\left(\alpha(\sigma) + (1-\alpha(\sigma))(\hat{H}L)^{\psi}\right)^{-1}}{\theta} \right) - 1 \right\} + \hat{C} - \frac{1}{\theta}\left(\delta(1-\theta)+\rho\right) \quad \text{(B.12)}$$

To determine the growth rate of labor, I take the logarithm of (B.4) and the derivative with respect to t yielding

$$\left(\frac{\alpha(\sigma)(1-\psi)}{(\alpha(\sigma)+(1-\alpha(\sigma))(\hat{H}L)^{\psi})}\right)\left(\frac{\dot{\hat{H}}}{\hat{H}} + \frac{\dot{L}}{L}\right) = \frac{\dot{\mu}_1}{\mu_1} - \frac{\dot{\mu}_2}{\mu_2}. \quad \text{(B.13)}$$

Plugging in (B.2) for $\frac{\dot{\mu}_1}{\mu_1}$ and (B.5) for $\frac{\dot{\mu}_2}{\mu_2}$, I find that

$$\left(\frac{\alpha(\sigma)(1-\psi)}{(\alpha(\sigma)+(1-\alpha(\sigma))(\hat{H}L)^{\psi})}\right)\left(\frac{\dot{\hat{H}}}{\hat{H}} + \frac{\dot{L}}{L}\right) = -\alpha(\sigma)\left(\alpha(\sigma)+(1-\alpha(\sigma))(\hat{H}L)^{\psi}\right)^{\frac{1-\psi}{\psi}} + A.$$

Reducing further,

$$\left(\frac{\dot{\hat{H}}}{\hat{H}} + \frac{\dot{L}}{L}\right) = \frac{-1}{1-\psi}\left(\alpha(\sigma)+(1-\alpha(\sigma))(\hat{H}L)^{\psi}\right)^{\frac{1}{\psi}} + \frac{A}{\alpha(\sigma)(1-\psi)}(\alpha(\sigma)+(1-\alpha(\sigma))(\hat{H}L)^{\psi}).$$

Isolating the growth rate of labor leads to

$$\frac{\dot{L}}{L} = \frac{-1}{1-\psi}\left(\alpha(\sigma)+(1-\alpha(\sigma))(\hat{H}L)^{\psi}\right)^{\frac{1}{\psi}} + \frac{A}{\alpha(\sigma)(1-\psi)}(\alpha(\sigma)+(1-\alpha(\sigma))(\hat{H}L)^{\psi}) - \frac{\dot{\hat{H}}}{\hat{H}}.$$

Therefore, the growth rate of labor is determined by

$$\frac{\dot{L}}{L} = \frac{-\psi}{1-\psi}\left(\alpha(\sigma)+(1-\alpha(\sigma))(\hat{H}L)^{\psi}\right)^{\frac{1}{\psi}} + \frac{A}{\alpha(\sigma)(1-\psi)}(\alpha(\sigma)+(1-\alpha(\sigma))(\hat{H}L)^{\psi}) - A(1-L) - \hat{C}. \quad \text{(B.14)}$$

Steady State

On a balanced-growth path, the steady state implies $\dot{C}/C = \dot{Y}/Y = \dot{K}/K = \dot{H}/H = g$. I will assume that g is constant, which would imply that all previously mentioned variables growth at a constant rate. Constant time devoted to production of the homogenous good Y would imply that $\dot{L}/L = 0$. Therefore, from (B.9) expression for the optimal growth rate of consumption is given by:

$$\frac{\dot{C}}{C} = \frac{A - \rho - \delta}{\theta}. \tag{B.15}$$

Since $\frac{A-\rho-\delta}{\theta} > 0$ this will ensure that g is strictly positive. The growth rate of human capital is determined by

$$\frac{\dot{H}}{H} = A(1 - L) - \delta.$$

Along a balance growth path, the steady state stock of labor is given by

$$L^{\star} = \frac{\theta - 1}{\theta} + \frac{\delta(1 - \theta) + \rho}{A\theta}. \tag{B.16}$$

Along a balance growth path, $\frac{\dot{\hat{H}}}{\hat{H}} = 0$ which implies

$$\left(\alpha(\sigma) + (1 - \alpha(\sigma))(\hat{H}L)^{\psi}\right)^{\frac{1}{\psi}} = A(1 - L) + \hat{C}. \tag{B.17}$$

Also, $\frac{\dot{L}}{L} = 0$

$$A(1-L)+\hat{C} = \frac{-\psi}{1-\psi}\left(\alpha(\sigma) + (1 - \alpha(\sigma))(\hat{H}L)^{\psi}\right)^{\frac{1}{\psi}} + \frac{A}{\alpha(\sigma)(1-\psi)}(\alpha(\sigma)+(1-\alpha(\sigma))(\hat{H}L)^{\psi}). \tag{B.18}$$

Plugging (B.17) into (B.18)

$$\begin{aligned}
A(1 - L) + \hat{C} &= \left(\frac{-\psi}{1-\psi}\right)(A(1 - L) + \hat{C}) + \frac{A}{\alpha(\sigma)(1-\psi)}(A(1 - L) + \hat{C})^{\psi} \\
\left(\frac{1}{1-\psi}\right)(A(1 - L) + \hat{C}) &= \frac{A}{\alpha(\sigma)(1-\psi)}(A(1 - L) + \hat{C})^{\psi} \\
(A(1 - L) + \hat{C})^{1-\psi} &= \frac{A}{\alpha(\sigma)} \\
\hat{C}^{\star} &= \left(\frac{A}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}} - A(1 - L^{\star}). \qquad \text{(B.19)}
\end{aligned}$$

Plugging in the steady state level of labor, (B.16), provides

$$\hat{C}^{\star} = \left(\frac{A}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}} + A\left(\frac{\delta(1 - \theta) + \rho}{A\theta} - \frac{1}{\theta}\right). \tag{B.20}$$

From (B.17),

$$\hat{H}^{\star} = \frac{1}{L}\left(\frac{(A(1-L)+\hat{C})^{\psi} - \alpha(\sigma)}{1-\alpha(\sigma)}\right)^{\frac{1}{\psi}}. \tag{B.21}$$

Plugging in (B.16) and (B.19) into (B.21) yields

$$\hat{H}^{\star} = \left(\frac{\theta-1}{\theta} + \frac{\delta(1-\theta)+\rho}{A\theta}\right)^{-1}\left(\frac{\left(\frac{A}{\alpha(\sigma)}\right)^{\frac{\psi}{1-\psi}} - \alpha(\sigma)}{(1-\alpha(\sigma))}\right)^{\frac{1}{\psi}}. \tag{B.22}$$

The average product of capital, $\hat{Y} \equiv Y/K$, is

$$\hat{Y} = \left(\alpha(\sigma) + (1-\alpha(\sigma))(\hat{H}L)^{\psi}\right)^{\frac{1}{\psi}} \tag{B.23}$$

Along the balanced growth path, the steady state of the average product of capital is

$$\hat{Y}^{\star} = \left(\frac{A}{\alpha(\sigma)}\right)^{\frac{1}{1-\psi}} \tag{B.24}$$

B.2 Speed of Convergence

Taking the logarithm of (B.23) and the derivative with respect to t and plugging in (B.23), (B.11), and (B.14) yields

$$\frac{\dot{\hat{Y}}}{\hat{Y}} = \frac{1}{1-\psi}\left(1-\alpha(\sigma)\hat{Y}^{-\psi}\right)\left[\left(\frac{A}{\alpha(\sigma)}\right)\hat{Y}^{\psi} - \hat{Y}\right] \tag{B.25}$$

After Log linearizing (B.25) in the neighborhood of the balance growth steady state, I find the approximation for $\hat{Y}$ to be

$$\frac{\dot{\hat{Y}}}{\hat{Y}} = -\lambda \log\left(\frac{\hat{Y}}{\hat{Y}^{\star}}\right) \tag{B.26}$$

where

$$\lambda \equiv \left[\hat{Y}^{\star} - 1\right] \tag{B.27}$$

B.3 Effects of the Elasticity of Substitution

$$
\begin{aligned}
\frac{d\alpha(\sigma)}{d\sigma} &= \left(\frac{1}{\sigma^2}\right)\frac{\gamma_0(\hat{H}_0L_0)^{1-\psi}\ln\hat{H}_0L_0}{(1+\gamma_0(\hat{H}_0L_0)^{1-\psi})^2} \\
&= \frac{\alpha(\sigma)(1-\alpha(\sigma))}{\psi\sigma^2}\ln{(\hat{H}_0L_0)^{\psi}} \qquad \text{(B.28)}
\end{aligned}
$$

$$
\frac{dL^\star}{d\sigma} = 0 \qquad \text{(B.29)}
$$

$$
\frac{d\hat{H}^\star}{d\sigma} = \frac{\hat{H}^\star}{\psi^2\sigma^2}\left[\ln\left(\frac{\hat{H}_0L_0}{\hat{H}^\star L^\star}\right)^{\psi} + \frac{\sigma\hat{Y}^{\star\psi}}{(1-\alpha(\sigma))(\hat{H}^\star L^\star)^{\psi}}\left(\ln\hat{Y}^{\star\psi} - (1-\alpha(\sigma))\ln(\hat{H}_0L_0)^{\psi}\right)\right] \qquad \text{(B.30)}
$$

$$
\begin{aligned}
\frac{d\hat{Y}^\star}{d\sigma} &= \hat{Y}^\star\left[\ln\left(\frac{A}{\alpha(\sigma)}\right) - \frac{\sigma}{\alpha(\sigma)}\frac{d\alpha(\sigma)}{d\sigma}\right] \\
&= \hat{Y}^\star\left[\ln\left(\frac{A}{\alpha(\sigma)}\right) - \frac{(1-\alpha(\sigma))}{\sigma}\ln\hat{H}_0L_0\right] \\
&= (1-\psi)\hat{Y}^\star\left[\ln\hat{Y}^\star - (1-\alpha(\sigma))\ln(\hat{H}_0L_0)\right] \qquad \text{(B.31)}
\end{aligned}
$$

A closer examination of a change in the elasticity of substitution:

$$
(1-\psi)\hat{Y}^\star > 0
$$

By concavity of the natural logarithm

$$
\ln\hat{Y}^\star < \hat{Y}^\star - 1 \qquad \text{(B.32)}
$$

$$
\ln(\hat{H}_0L_0)^{-1} < \frac{1}{(\hat{H}_0L_0)} - 1 \qquad \text{(B.33)}
$$

$$
\qquad \text{(B.34)}
$$

Adding (B.32) and (B.38) multiplied by $(1 - \alpha(\sigma))$:

$$\ln \hat{Y}^{\star^{\psi}} + (1-\alpha(\sigma))\ln(\hat{H}_0 L_0)^{-\psi} < \hat{Y}^{\star} - 1 + \frac{1-\alpha(\sigma)}{(\hat{H}_0 L_0)} - (1-\alpha(\sigma)) \tag{B.35}$$

Furthermore,

$$\begin{aligned} 0 &> \hat{Y}^{\star} - 1 + \frac{1-\alpha(\sigma)}{(\hat{H}_0 L_0)} - (1-\alpha(\sigma)) \\ \hat{H}_0 L_0 &> \frac{1-\alpha(\sigma)}{2(1-\alpha(\sigma)) - (1-\alpha(\sigma))\hat{H}^{\star} L^{\star}} \qquad \text{(B.36)} \end{aligned}$$

If (B.36) holds then increasing the elasticity of substitution will increase the average product of capital $\hat{Y}$ at the steady state. This (B.36) is a necessary condition, but is not a sufficient condition.

Equation B.31 may be written as:

$$\frac{d\hat{Y}^{\star}}{d\sigma} = -(1-\psi)\hat{Y}^{\star}\left[\ln \hat{Y}^{\star^{-1}} + (1-\alpha(\sigma))\ln(\hat{H}_0 L_0)\right] \tag{B.37}$$

By concavity of the natural logarithm

$$\begin{aligned} \ln \hat{Y}^{\star^{-\psi}} &< \frac{1}{\hat{Y}^{\star^{-1}}} - 1 \qquad \text{(B.38)} \\ \ln(\hat{H}_0 L_0)^{\psi} &< (\hat{H}_0 L_0) - 1 \qquad \text{(B.39)} \\ & \qquad \text{(B.40)} \end{aligned}$$

Adding (B.32) and (B.38) multiplied by $(1 - \alpha(\sigma))$:

$$\ln \hat{Y}^{\star^{-1}} + (1-\alpha(\sigma))\ln(\hat{H}_0 L_0) < \frac{1}{\hat{Y}^{\star^{-1}}} - 1 + (1-\alpha(\sigma))(\hat{H}_0 L_0) - (1-\alpha(\sigma)) \tag{B.41}$$

Furthermore,

$$\begin{aligned} 0 &> \frac{1}{\hat{Y}^{\star^{-1}}} - 1 + (1-\alpha(\sigma))(\hat{H}_0 L_0) - (1-\alpha(\sigma)) \\ H_0 L_0 &< \frac{(2-\alpha(\sigma))\hat{H}^{\star} L^{\star} - (1-\alpha(\sigma))}{(\alpha(\sigma)) + (1-\alpha(\sigma))\hat{H}^{\star} L^{\star}} \qquad \text{(B.42)} \end{aligned}$$

If (B.42) holds then increasing the elasticity of substitution will increase the average product of capital $\hat{Y}$ at the steady state. This (B.42) is also a necessary condition, but is not a sufficient condition.

$$\frac{d\hat{Y}^{\star}}{d\sigma} = \frac{d\hat{C}^{\star}}{d\sigma} = \frac{d\lambda}{d\sigma} \tag{B.43}$$

Balance Growth Path Simulation

To examine the potential effects of changing the elasticity of substitution on the average product of physical capital, the speed of convergence, and human capital per physical capital, I use the United States, 2010, values for human capital, physical capital, raw labor, price of physical capital, and the average wage in constant international dollars. The initial for all variables are from the sources mentioned in Section 2.3. The values for the level of relative risk aversion, θ, and the depreciation factor are two and .99, which corresponds to a 4% real interest rate for the United States. The calibration of the two previous mentioned variables are standard and follow the dynamic macroeconomic literature. Using the 2010 values for the factors of production and their corresponding prices and calibrated values, Table B.1 simulates the effects of changing the elasticity of substitution on the average product of physical capital and human capital per physical capital.

Balance Growth Path Simulation		
σ	$\frac{d\hat{Y}^\star}{d\sigma}$	$\frac{d\hat{H}^\star}{d\sigma}$
0.250	0.000	5.622
0.350	0.000	1.726
0.450	0.000	1.636
0.550	0.002	1.573
0.650	0.036	1.563
0.750	0.291	1.848
0.850	1.408	3.322
0.950	4.911	8.096
1.050	14.397	20.476
1.150	39.890	51.355
1.250	110.170	131.665

Table B.1: Simulates the effects of changing the level of elasticity of substitution on the average product of capital and human capital per physical capital along the balance growth path.

Chapter 4

Conclusion

Chapter 1 employs a neoclassical growth model with a normalized aggregate three factor CES production function. Following (Mankiw et al., 1992), I assume that human capital and simple labor are not "perfect substitutes" in production. The normalized aggregate CES production function includes the three factor Cobb-Douglas production function of Mankiw et al. (1992) as a special case. By introducing the normalized CES in this context, I determine the comparative statics for a change in the elasticity of substitution for steady state variables, growth thresholds, and speed of convergence. This paper finds that a higher elasticity of substitution leads to a higher steady state level for human capital and physical capital per effective unit of labor. Also, a high-enough, level of the elasticity of substitution can lead to permanent growth. Lastly, a higher elasticity of substitution will lead to a higher speed of convergence when the baseline level of capital per effective unit of labor is greater than the steady state level.

In Chapter 2, I estimate the normalized aggregate CES production function, (1.2), derived in Chapter 1. I find estimates for the elasticity of substitution that are significantly below one corroborating many of the estimates for the aggregate production function. The literature on human capital in the aggregate production function typically understates the importance of the elasticity of substitution in economic growth models by assuming a Cobb-Douglas function. Furthermore, the finding of Chapter 1 and 2 suggest that the elasticity of substitution is an important parameter for economic growth. When allowing for the production function to be given by the normalized CES, (1.2), I find little evidence of conditional or unconditional convergence.

Chapter 3 employs a neoclassical growth model with a normalized aggregate two factor CES production function. In particular, I followLucas et al. (1988), and employ a model that includes human capital and its accumulation. Furthermore, I assume that human capital augments simple labor. Unlike most models that employ the normalized CES, I allow the savings for capital to be endogenously determined. Thus, allowing the elasticity of substitution to affect the endogenous growth rate of the average product of physical capital and human capital per physical capital. As a result, a change in the level of the elasticity of substitution, along the balance growth path, can increase/decrease the average product of capital, human capital per physical capital, and speed of convergence, ceteris paribus.

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