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Canonical Forms of the Equation of Transfer*

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1. Introduction. The conventional form of the equation of transfer in radiative transfer theory is tailored principally to fit the needs of theoretical investigations. The quantities appearing in the equation, while readily measurable, do not allow the equation to express their interconnections in a way which is helpful to the intuition of the experimenter. The purpose of this note is to present a reformulation of the equation of transfer which appears to be of help in the task of collating and understanding the experimentally obtained properties of the optical medium. Quite interestingly, this reformulation appears to hold the key to the solution of one of the long-standing theoretical problems of this field, namely the problem of the existence and form of the asymptotic radiance distribution in an arbitrary medium with arbitrary external lighting conditions. The solution of this problem, in turn, supplies a raft of rules and laws about the behaviour of the light field in optically deep media which promise to be of additional help to the experimenter in understanding his data, and in applying them to practical problems. In this note, we will be concerned primarily with the motivation for the reformulation of the transfer equation, and with the details of the deriva-

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tion of the equation. The discussion will be limited to some immediate practical examples of its use. The complete discussion of the solution of the asymptotic radiance distribution problem is reserved for a subsequent note.

2. Motivation. The motivation for the reformulation stems primarily from the experimental studies of the light field in natural hydrosols such as oceans, lakes, and harbors. In these studies the experimenter wants to find out as much as possible about the following three topics:

(i) The absolute amount of light in a given direction at a given depth.

(ii) The directional distribution of the light at a given depth, i.e., the relative amounts in each direction.

(iii) The depth below which the directional distribution of the light has become essentially fixed, and hence below which the amount of the light falls off at an essentially fixed exponential rate.

The most detailed information about these topics is obtained by means of the radiometric quantity called radiance (or specific intensity) and denoted by $N(z, \theta, \phi)$. This gives the number of watts of radiant flux crossing a unit area at depth z in a unit solid angle about the direction (θ, ϕ) of the area's normal. Radiance is measured by means of a Gershun tube.

In the absence of detailed knowledge of the values $N(z,\theta,\phi)$, important insight into problem (i) is obtained by measuring the down and upwelling irradiances $H(z,-)$, and $H(z,+)$ respectively, at each depth z. These quantities are measured by flat lambert collecting surfaces oriented horizontally. For example, $H(z,-)$ represents the amount of radiant flux flowing downward in all directions across a unit area at depth z. Countless measurements in the past have shown a consistent tendency for both irradiances $H(z,-)$ and $H(z,+)$ to behave very nearly in an exponential manner with depth, so that, ideally,

$$K(\cdot,\pm) = \frac{-1}{H(\cdot,\pm)} \frac{dH(\cdot,\pm)}{dz} = \text{constant function.}$$

Observe that there is a K-function for each stream. These are generally different. Their difference is observed experimentally and understood theoretically. Furthermore, careful studies uncover inevitable departures from this ideal, that is to say, each K-function is not a constant function over all depths. These departures, far from being annoying irregularities, more than not supply valuable and unexpected insight into the local structure of the light field in real optical media. This is illustrated, for example, in reference 1. Some theoretical connections between these K-functions may be found in reference 2.

For a full solution of problem (ii), detailed measurements of the radiance distributions are unavoidable. However, in the absence of knowledge of the values $N(z, \theta, \phi)$, there exists excellent substitutes in the form of the so-called <u>distribution</u> functions:[2]

$$D(\cdot, \pm) = \frac{h(\cdot, \pm)}{H(\cdot, \pm)},$$

where $h(z, \pm)$ are the scalar irradiances induced by the upwelling and downwelling streams of radiation at depth z . For example, $h(z,-)$ is measured by a spherical lambert collecting surface suitably shielded from the upwelling stream of flux. $h(z,-)$ gives the number of watts incident per unit area on the surface of a small sphere. The radiation that gives rise to this incident flux comes from all directions of the upper hemisphere. $h(z,-)$ measures the <u>downwelling scalar irradiance</u>. A similar set of statements applies to $h(z,+)$ which represents the <u>upwelling scalar irradiance</u>. The utility of the distribution functions stems from the following fact: the scalar irradiances are insensitive to the directional distribution of the flux; they thereby serve to give the absolute amount of flux converging on a point from their respective hemispheres. On the other hand, the irradiances are measurably sensitive to the directional distribution of the incident flux. Thus, for example, if a flashlight were shone down onto a flat lambert collecting surface from various directions but always from a fixed distance, the irradiance $H(z,-)$ would exhibit an essentially cosine response to this variation in the directional distribution of the incident flux. On the other hand, the quantity $h(z,-)$, were the flat collector replaced by the appropriate spherical collector, would remain the same for all positions of the searchlight. The quotient of these two readings thus serves to eliminate the unwanted absolute magnitude aspects of the light field and retains only the desired relative magnitudes. The reason for dividing a little h by a big H and not the other way around simply stems from the theoretical considerations in which the present quotient most naturally arises. To give an indication of how the D-values reflect the directional character of the flux distribution, we observe that if the radiance distribution were uniform over an hemisphere, then D would equal 2. In the other extreme, where the flux is collimated and incident at an angle θ with respect to the normal to the flat collector, then D would equal the secant of θ. In real situations where the directional character could be anything between these extremes, the D-functions take correspondingly intermediate values.

With the advent of more sophisticated experimental techniques, it has become possible, in analogy to the irradiance case, to determine the quantities:

$$K(z,\theta,\phi) = \frac{-1}{N(z,\theta,\phi)} \frac{dN(z,\theta,\phi)}{dz},$$

over great ranges of depths and all directions. No confusion should result from the continued use of the letter K, for $K(z,\theta,\phi)$ will always necessarily exhibit three variables, while those for irradiance will always necessarily have only one variable, namely depth. Retaining the letter K serves to point up their inherent similarities. The function $K(\cdot,\theta,\phi)$ characterizes the depth rate of change of the radiance values in the fixed direction (θ,ϕ).

The preceding discussion has served to draw out the following set of quantities which are of principal importance in experimental studies of the light field in natural hydrosols: $D(z,-)$, $D(z,+)$, $K(z,-)$, $K(z,+)$, and $K(z,\theta,\phi)$. The main radiometric concepts associated with these quantities are $H(z,-)$, $H(z,+)$, $h(z,-)$, $h(z,+)$, and $N(z,\theta,\phi)$. We now go on to show how these various constructs can be incorporated into the equation of transfer. We will begin with the simplest but perhaps the most important of the reformulations, namely the canonical equation of transfer for radiance. While the momentum of the mathematical discussion is still high, we will allow it to carry us through to the generalized canonical form. The final section will pause for a survey of what comes into view after the smoke of the derivations has cleared away.

3. <u>The Canonical Form of the Transfer Equation</u>. The general form of the equation of transfer for an arbitrary optical medium is:

$$\underset{\sim}{\xi} \cdot \nabla N(\underset{\sim}{x},\underset{\sim}{\xi}) = -\alpha(\underset{\sim}{x}) N(\underset{\sim}{x},\underset{\sim}{\xi}) + N_*(\underset{\sim}{x},\underset{\sim}{\xi}) + N_\eta(\underset{\sim}{x},\underset{\sim}{\xi})$$

in which

$$N_*(\underset{\sim}{x},\underset{\sim}{\xi}) = \int_{\Xi} N(\underset{\sim}{x},\underset{\sim}{\xi}') \, \sigma(\underset{\sim}{x};\underset{\sim}{\xi}';\underset{\sim}{\xi}) \, d\Omega(\underset{\sim}{\xi})$$

defines the path function N_*, and in which N_η is the emission function. Ξ is the collection of all unit vectors (the unit sphere). We will at first limit ourselves to emission-free ($N_\eta \equiv 0$) plane-parallel media, these being the conventional models for the geometric settings of natural hydrosols. The equation then may be written as:

$$-\cos\theta \frac{dN(z,\theta,\phi)}{dz} = -\alpha(z)\,N(z,\theta,\phi) + N_*(z,\theta,\phi).$$

We now use the definition of $K(z,\theta,\phi)$, insert this into the above equation of transfer, and perform a simple rearrangement. The result is the desired canonical transfer equation:

$$N(z,\theta,\phi) = \frac{N_*(z,\theta,\phi)}{\left[\alpha(z) + \cos\theta\, K(z,\theta,\phi)\right]}.$$

4. The General Canonical Form. It is possible to subsume all the practical radiometric concepts discussed above under the notion of the generalized irradiance function defined by:

$$H(\underset{\sim}{x},\underset{\sim}{\Omega},\Xi_0) \equiv \underset{\sim}{\Omega} \cdot \underset{\sim}{H}(\underset{\sim}{x},\Xi_0),$$

where

$$\underset{\sim}{H}(\underset{\sim}{x},\Xi_0) \equiv \int_{\Xi_0} \underset{\sim}{\xi}\, N(\underset{\sim}{x},\underset{\sim}{\xi})\, d\Omega(\underset{\sim}{\xi}).$$

$H(\underset{\sim}{x}, \underset{\sim}{n}, \Xi_0)$ is the quantity measured by a general Gershun tube whose photosensitive plate is exposed to the radiometric environment over the arbitrary but fixed subset of directions Ξ_0 of the unit sphere Ξ (Figure 1 (b)). The measurement takes place at point $\underset{\sim}{x}$ in the medium and the unit inward normal $\underset{\sim}{n}$ to the plate gives its orientation in space. The associated <u>generalized scalar irradiance</u> is defined by:

$$h(\underset{\sim}{x}, \Xi_0) \equiv \int_{\Xi_0} N(\underset{\sim}{x}, \underset{\sim}{\xi})\, d\Omega(\underset{\sim}{\xi}) ,$$

and is measured in practice by a spherical collector exposed to the same subset of directions Ξ_0 of Ξ . In analogy to the conventional definition of the distribution functions, we have the <u>generalized distribution function</u>

$$D(\underset{\sim}{x}, \underset{\sim}{n}, \Xi_0) = \frac{h(\underset{\sim}{x}, \Xi_0)}{H(\underset{\sim}{x}, \underset{\sim}{n}, \Xi_0)} .$$

It follows, for example, that the conventional irradiances and radiances are defined by

$$H(\underset{\sim}{x}, +) = H(\underset{\sim}{x}, \underset{\sim}{n}, \Xi_+) ,$$

where (Figure 1 (c)) Ξ_+ is the hemisphere of directions defined by

$\Xi_+ = \{\underset{\sim}{\xi} : \underset{\sim}{\xi} \cdot \underset{\sim}{n} \geq 0 \}$, and $\underset{\sim}{n}$ is set equal to $\underset{\sim}{k}$, the unit upward normal.

Furthermore (Figure 1 (a))

$$N(\underset{\sim}{x}, \underset{\sim}{\Omega}) = N(\underset{\sim}{x}, \underset{\sim}{\xi}) = \lim_{\Xi_0 \to \underset{\sim}{\Omega} = \underset{\sim}{\xi}} \frac{H(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0)}{\Omega(\Xi_0)} .$$

The experimentally useful functions $K(\cdot, \pm)$ and $K(\cdot, \theta, \phi)$ are evidentally special cases of the general function $K(\cdot, \underset{\sim}{\Omega}, \Xi_0)$ defined by the following logarithmic divergence relation:

$$K(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0) \equiv \frac{-1}{H(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0)} \nabla \cdot \underset{\approx}{H}(\underset{\sim}{x}, \Xi_0).$$

Starting with the general equation of transfer and integrating each side over the arbitrary but fixed subset Ξ_0 of Ξ , we have:

$$\nabla \cdot \underset{\approx}{H}(\underset{\sim}{x}, \Xi_0) = -\alpha(\underset{\sim}{x}) h(\underset{\sim}{x}, \Xi_0) + \int_{\Xi_0} N_*(\underset{\sim}{x}, \underset{\sim}{\xi})\, d\Omega(\underset{\sim}{\xi}) .$$

Dividing each side by $H(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0)$ and employing the preceding definitions, the result is

$$K(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0) = \alpha(\underset{\sim}{x}) D(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0) - \frac{1}{H(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0)} \int_{\Xi_0} N_*(\underset{\sim}{x}, \underset{\sim}{\xi})\, d\Omega(\underset{\sim}{\xi}) .$$

The final arrangement yields

$$D(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0) H(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0) = \frac{\int_{\Xi_0} N_*(\underset{\sim}{x}, \underset{\sim}{\xi})\, d\Omega(\underset{\sim}{\xi})}{\left[\alpha(\underset{\sim}{x}) - \frac{K(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0)}{D(\underset{\sim}{x}, \underset{\sim}{\Omega}, \Xi_0)} \right]} ;$$

which is the desired general form.

5. Observations and Notes. We summarize below some of the uses to which the canonical form of the transfer equation has been put by the research staff at the Scripps Institution of Oceanography's Visibility Laboratory.

(i) An experimental determination of the path function N_* has been made without explicit knowledge of the volume scattering function σ which appears in the equation of transfer. These determinations were made via the canonical form of the transfer equation using tabulations of N, α, and K obtained from experimental data.[3]

(ii) The experimental determination of the volume scattering function has been made from knowledge of the radiance distributions only. The customary use of special instrumentation, namely nephelometers (σ-meters), was sidestepped.[4]

(iii) The canonical form of the transfer equation provides a convenient and natural representation of the size and shape of the radiance distributions $N(z,\cdot,\cdot)$ at any depth in terms of N_*, K, and α. For example, by setting $\theta = \pi/2$, we obtain $N(z,\pi/2,\phi) = N_*(z,\pi/2,\phi)/\alpha(z) \equiv N_q(z,\pi/2,\phi)$, the horizontal equilibrium radiance. For all other directions of sight, the radiance may be interpreted as the equilibrium radiance $N_q(z,\theta,\phi)$ in the direction (θ, ϕ) modified by the elliptical factor $\left[1+\cos\theta \frac{K(z,\theta,\phi)}{\alpha(z)}\right]^{-1}$ for that direction.

(iv) We now have a conceptually convenient criterion for the existence of the asymptotic radiance distribution by means of the function $K(\cdot, \theta, \phi)$. For example, Figure 2 shows a plot of $K(\cdot, \theta, \phi)$ for various choices of directions (θ, ϕ). The depths considered range from $z = 0$ at the surface to $z_o = 72$ meters. The value of the volume attenuation function was found to be constant and of magnitude $\alpha = 0.402$/meter. Hence the optical depth to which these measurements apply is $\alpha z_o = 28.5$, approximately. The graphs show that at about an optical depth of 30, the function $K(z, \cdot, \cdot)$ is relatively constant over the unit sphere. This means that the radiance values in all directions are decreasing at about the same rate. In other words, the <u>shape</u> of the radiance distribution appears to be relatively constant at this and greater depths. Thus we may say that the asymptotic radiance distribution for this medium appears to have been reached at a depth of 72 meters, or about 28.5 optical depths. The above information was derived from J. E. Tyler's clear sunny sky Lake Pend Oreille measurements. The associated wavelength for all quantities is 478 mμ.

(v) The problem of the existence of an asymptotic radiance distribution can now be stated in terms of the function $K(\cdot\,,\cdot\,,\cdot)$. Furthermore, the shape of the asymptotic radiance distribution, when such a distribution exists, is conveniently formulated by means of the canonical equation. When this is done, we obtain a two-dimensional Fredholm integral equation in which the Fredholm determinant and its minors are constructed from the constants α, K, and the <u>arbitrary</u> function σ. Details of solution are reserved for a later note. In the case of isotropic scattering, the canonical form shows directly that an asymptotic radiance distribution, if it exists, must be of a very special shape, namely that of a prolate spheroid with vertical axis and eccentricity K/α. In the case of non-isotropic scattering, the form of the asymptotic radiance distribution is governed accordingly by the shape of σ. For media that exhibit marked forward scattering, the asymptotic distribution is more narrow in the upper lobe and less narrow in the lower lobe than the prolate ellipsoid (the upper lobe refers to downwelling flux, the lower lobe to upwelling flux). The prolate ellipsoid may conveniently serve as a standard ellipsoid to which all other measurements or solutions for the asymptotic radiance distribution in a particular medium may be referred for comparison.

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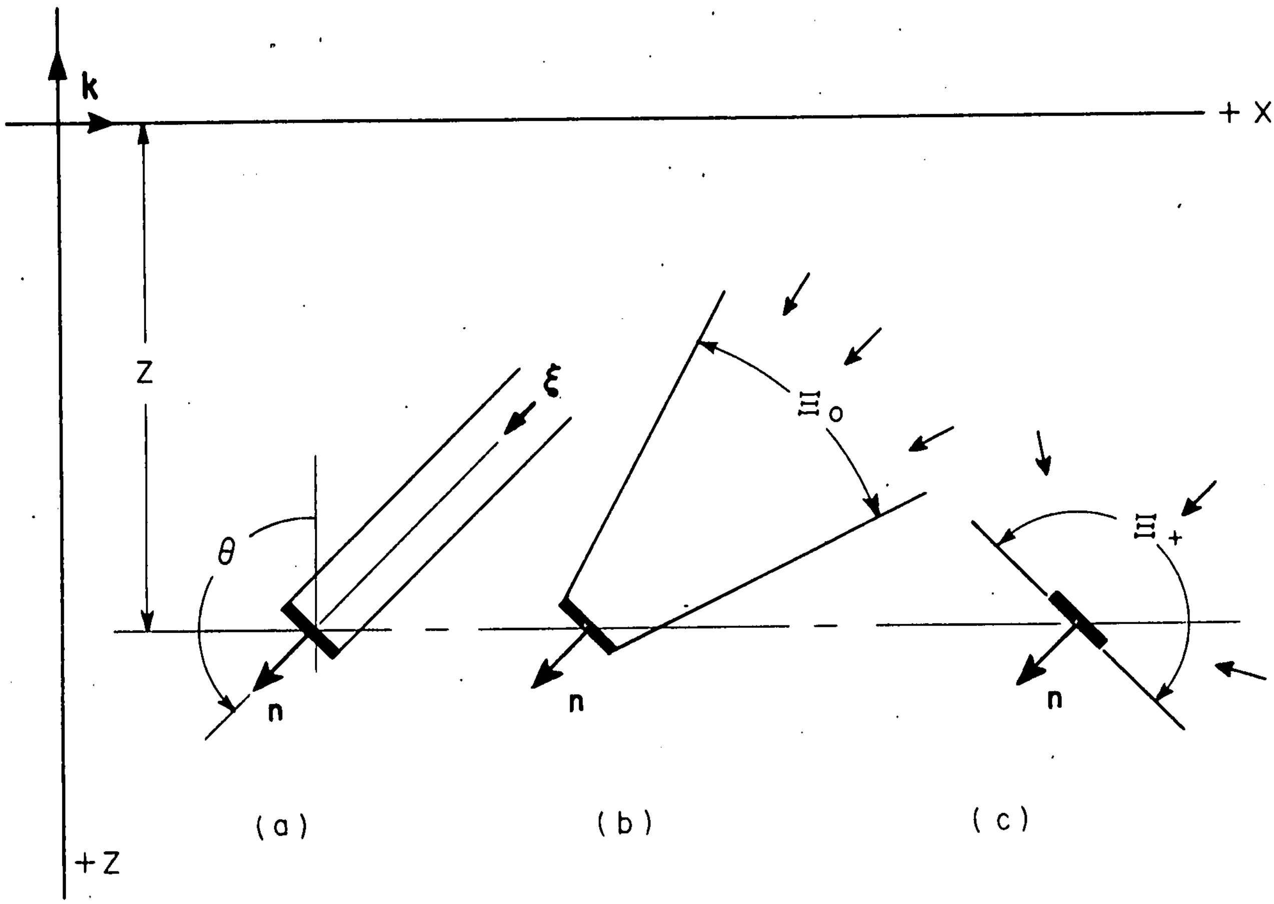

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Figure 1

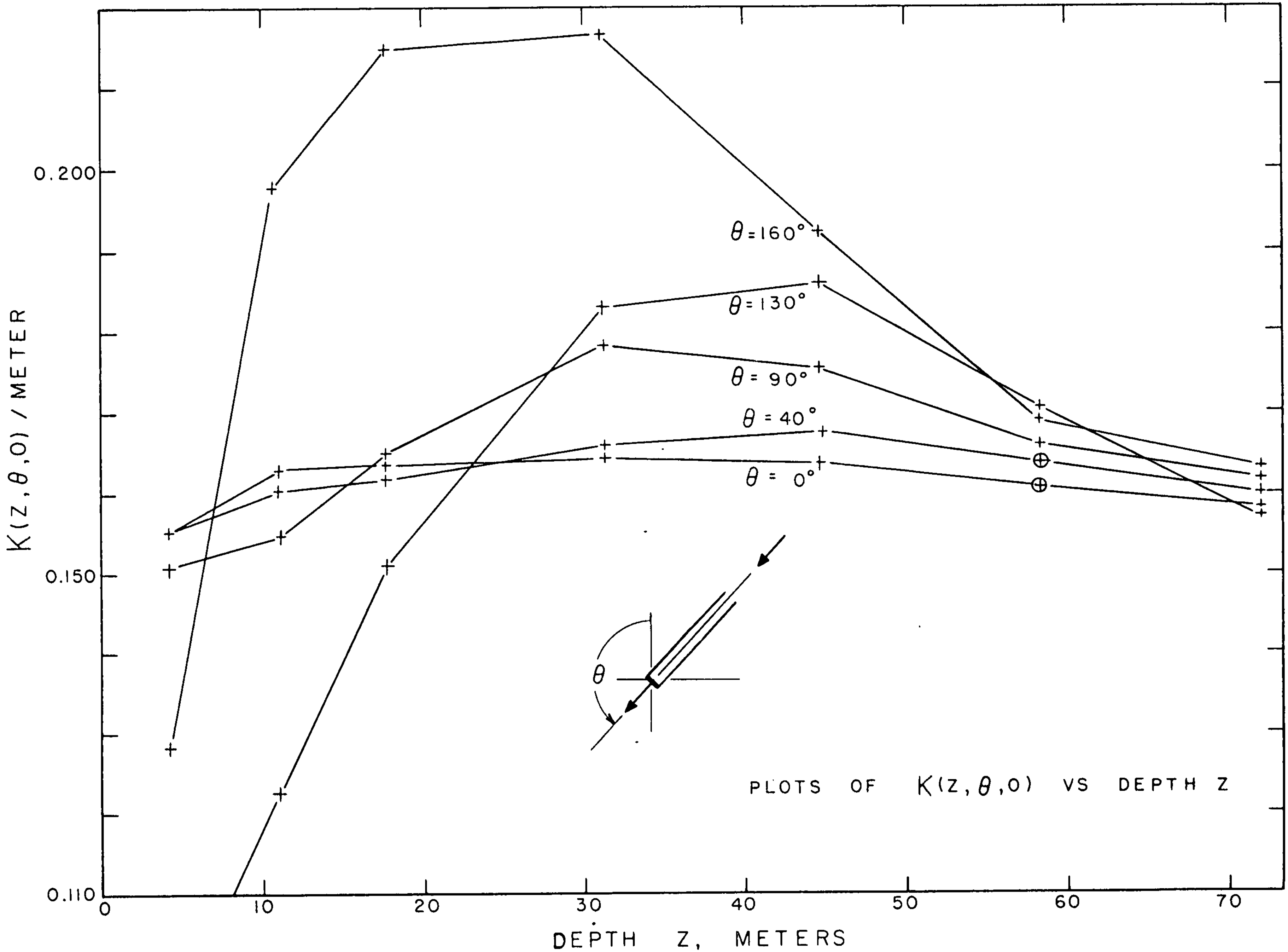

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Figure 2